How “Good” Digraph Width Measures
Do / Can We Have?

Petr Hliněný*

Robert Ganian
Jan Obdržálek

Joachim Kneis
Alexander Langer
Daniel Meister
Peter Rossmanith
Somnath Sikdar

FI MU Brno

RWTH Aachen

*Corresponding author
1 How to Measure Graph “Width”

**Tree-width** (Robertson and Seymour) — a real success story:

- FPT algorithms for many problems, incl. all MSO₂
- structurally nice, FPT computable, *just great!*
- related to (even nicer) branch-width
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- but not subgraph or minor-monotone
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- XP algorithms for Hamiltonian path or \(k\)-path (linkage) problems
- technically difficult, not many efficient algorithms...
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Recent additions
• an explosion of new directed measures in the past decade...
giving finer resolution for better algorithmic applications?
Directed measures: briefly (and chronologically)...

Cycle rank, —— directed path-width, dir. tree-width, $D$-width, entanglement, DAG-width, Kelly-width, DFVS-number, bi-rank-width, K-width, DAG-depth
Directed measures: briefly (and chronologically) . . .


\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
Probl. \  \Param. & K-width & DAG-depth & DAG-width & Cycle-rank & DFVS-num. & DAGs & Bi-rank-width \\
\hline
\hline
c-Path (§4.4) & FPT & FPT & XP* & XP* & XP & P & FPT \\
\hline
k-Path (§4.4) & para-NPC & para-NPC & NPC & NPC & NPC & NPC & para-NPC \\
\hline
DiDS (§4.5) & para-NPC & para-NPC & NPC & NPC & NPC & NPC & FPT \\
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DiSTP (§4.5) & para-NPC & para-NPC & NPC & NPC & NPC & NPC & FPT \\
\hline
MaxLOB (§4.6) & para-NPC & para-NPC & para-NPC & para-NPC & para-NPC & para-NPC & para-NPC \\
\hline
MinLOB (§4.6) & para-NPC & para-NPC & para-NPC & para-NPC & para-NPC & para-NPC & para-NPC \\
\hline
c-MinLOB (§4.6) & XP & FPT & XP & XP & P & open & XP / W[2]-hard \\
\hline
MaxDiCut (§4.7) & para-NPC & para-NPC & NPC & NPC & NPC & NPC & XP / W[2]-hard \\
\hline
c-OCN (§4.8) & para-NPC & para-NPC & NPC & NPC & NPC & NPC & FPT \\
\hline
DFVS (§4.9) & open & open & para-NPC & para-NPC & para-NPC & para-NPC & FPT \\
\hline
Kernel (§4.9) & para-NPC & para-NPC & para-NPC & para-NPC & para-NPC & para-NPC & FPT \\
\hline
\hline
ϕ-MSO \(_1\)MC (§4.2) & para-NPH & para-NPH & NPH & NPH & NPH & NPH & FPT \\
\hline
ϕ-LTLMC (§4.10) & p.-coNPH & p.-coNPH & coNPH & coNPH & coNPH & coNPH & para-coNPH \\
\hline
Parity (§4.10) & XP & XP & XP & XP & P & XP & XP \\
\hline
\hline
\end{tabular}

References: 

\cite{[JHRST01]} \cite{[LKM08]} \cite{[GH010]} \cite{[FGLS09]} \cite{[EIS76]} \cite{[GW06]} \cite{[GDK09]} \cite{[GRK09]} \cite{[FGLS10]} \cite{[CD06]} \cite{[K008]} \cite{[CLL+08]} \cite{[VL76]} \cite{[CMR00]} \cite{[BDHK06]} \cite{[Obd07]}.

\[ FPT \simeq \text{runtime } O(f(k) \cdot n^c) \]

\[ \text{XP} \simeq \text{runtime } O(n^{f(k)}) \]
2 What are these Directed Width Measures

DAG – directed *acyclic* graph (the *simplest* class ???)

[Diagram showing a directed acyclic graph with a simple structure and a more complex structure]

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**DAG-width** – how many cops catch a *visible robber*
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Cycle rank (60’s!) – how “deep” to remove vertices to become acyclic
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**DAG-depth** – how many cop moves are needed to catch a *visible robber*, related to the longest directed path.
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and slightly different sort...

**Clique-width** – same def. for undirected and directed:

Minimum number of *labels* to build the graph using

– create a (labeled) vertex,
– make disjoint union,
– relabel all *i*'s to *j*,
– and add all arcs from label *i* to *j*. 
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**Bi-rank-width** (Kanté) – related to clique-width / rank-width;

i.e. the branch-width of the *bi-cutrank* function on the vertex set.
How these measures compare

<table>
<thead>
<tr>
<th>Graph family</th>
<th>DAG-depth</th>
<th>K-width</th>
<th>DFVS-number</th>
<th>cycle-rank</th>
<th>DAG-width</th>
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<td>...</td>
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<td>3</td>
</tr>
</tbody>
</table>
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Very good: DAG-width, Kelly-width, DAG-depth

- having nice cops-and-robber *game characterizations*
- *monotone* under taking subgraphs and some restricted form of *arc contractions*
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**and Bad:** clique-width, bi-rank-width

- subgraphs can have much higher width,
  e.g. the complete graph (bidirected) has small width while its subgraphs are complex
- still, not so bad since related to so called *vertex minors*
4 and Algorithmic Usefulness

<table>
<thead>
<tr>
<th>Probl. \ Param.</th>
<th>K-width</th>
<th>DAG-depth</th>
<th>DAG-width</th>
<th>Cycle-rank</th>
<th>DFVS-num.</th>
<th>DAGs</th>
<th>Bi-rank-width</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$-Path ($\S 4.4$)</td>
<td>FPT</td>
<td>FPT</td>
<td>XP$^a$ $\dagger$</td>
<td>XP$^a$ $\dagger$</td>
<td>XP$^a$ $\dagger$</td>
<td>P$^a$</td>
<td>FPT</td>
</tr>
<tr>
<td>$k$-Path ($\S 4.4$)</td>
<td>para-NPC</td>
<td>para-NPC</td>
<td>NPC$^e$</td>
<td>NPC$^e$</td>
<td>NPC$^e$</td>
<td>NPC$^e$</td>
<td>para-NPC$^f$</td>
</tr>
<tr>
<td>DiDS ($\S 4.5$)</td>
<td>para-NPC</td>
<td>para-NPC</td>
<td>NPC</td>
<td>NPC</td>
<td>NPC</td>
<td>NPC</td>
<td>FPT</td>
</tr>
<tr>
<td>DistP ($\S 4.5$)</td>
<td>para-NPC</td>
<td>para-NPC</td>
<td>NPC</td>
<td>NPC</td>
<td>NPC</td>
<td>NPC</td>
<td>FPT</td>
</tr>
<tr>
<td>MaxLOB ($\S 4.6$)</td>
<td>para-NPC</td>
<td>para-NPC</td>
<td>NPC</td>
<td>NPC</td>
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</tr>
<tr>
<td>MinLOB ($\S 4.6$)</td>
<td>para-NPC</td>
<td>para-NPC</td>
<td>para-NPC$^g$</td>
<td>para-NPC$^g$</td>
<td>para-NPC$^g$</td>
<td>para-NPC$^g$</td>
<td>$p^h$ (open)</td>
</tr>
<tr>
<td>MaxDiCut ($\S 4.7$)</td>
<td>para-NPC$^b$</td>
<td>para-NPC$^b$</td>
<td>NPC$^b$</td>
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<td>NPC$^b$</td>
<td>XP$^c$/W$^b[2]$-hard$^i$</td>
</tr>
<tr>
<td>$c$-OCN ($\S 4.8$)</td>
<td>para-NPC</td>
<td>para-NPC</td>
<td>NPC$^k$</td>
<td>NPC$^k$</td>
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<td>FPT</td>
</tr>
<tr>
<td>DFVS ($\S 4.9$)</td>
<td>open</td>
<td>open</td>
<td>para-NPC$^l$</td>
<td>para-NPC$^l$</td>
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<td>para-NPC$^l$</td>
<td>FPT</td>
</tr>
<tr>
<td>Kernel ($\S 4.9$)</td>
<td>para-NPC$^n$</td>
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<td>para-NPC$^l,n$</td>
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</tr>
<tr>
<td>$\phi$-MSO$_1$MC ($\S 4.2$)</td>
<td>para-NPH</td>
<td>para-NPH</td>
<td>NPH</td>
<td>NPH</td>
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<td>NPH</td>
<td>FPT$^p$</td>
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<tr>
<td>$\phi$-LTLmc ($\S 4.10$)</td>
<td>p.-coNPH</td>
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<td>Parity ($\S 4.10$)</td>
<td>XP$^q$ $\dagger$</td>
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<td>P</td>
<td>XP$^r$ $\dagger$</td>
</tr>
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</table>

References $^a$[JRST01] $^b$LKM08 $^c$[GHO10] $^d$[FGLS09] $^e$[EIS76] $^f$[GW06] $^g$DGK09 $^h$GRK09 $^i$[FGLS10] $^k$CD06 $^l$[KO08] $^m$[CLL+08] $^n$vL76 $^p$CMR00 $^q$BDH06 $^r$[Obsd07].

FPT $\simeq$ runtime $O(f(k) \cdot n^c)$
NPC $\simeq$ lik. no efficient alg. at all
XP $\simeq$ runtime $O(n^{f(k)})$
W$[i]$-hard $\simeq$ lik. no better than XP alg.
Conclusions from the Table...

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- classical digraph problems like dominating set, Steiner tree, max-/min-LOB (outbranching), oriented colouring, etc. are still **NP-hard** for the measures
- positive algorithmic results seem **rather incidental**, e.g. Hamiltonian path and related, or some particular algorithms parametrized by the DFVS number
5 Can we do better?

The contrast: So far we have got no directed measure that is structurally nice and algorithmically useful at the same time!
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OK, but we want a directed measure that is

NOT tree-width bounding!
The Question, II:

Can we have an *algorithmically useful* measure of digraphs that is not tree-width bounding and *monotone on subgraphs* (i.e. “structural”)?
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What about add. monotonicity under *butterfly contractions* (minors)?

NO, this does not help to dismiss the "crazy" measure either...
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  - let $V_3$ be the subset of vertices with $> 2$ neighbours;
  - arc $\vec{a}$ is *2-contractible* if
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    - no new dir. path between vert. of $V_3$ after contraction of $\vec{a}$
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**Powerfulness - why undirected MSO$_1$?**
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**Powerfulness - why undirected MSO$_1$?**

- A useful width measure should **not only incidentally solve** a few problems, but a whole rich class (a *framework*).
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  - $\implies$ **undirected MSO$_1$ is the least common denominator!**
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I.e., for every undirected $G$, one can efficiently orient (in XP time) the edges of $G$ such that the width is (approximately) optimal over all orientations of $G$. 
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- excessive info. — even knowing a graph is 3-colourable, there is no efficient way to find a colouring (this measure is cheating!)
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  Our point of view is *algorithmic*, and so the only possibility here to give up is the **structural condition**!
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- Hence, for algorithmically useful directed measures, we can not require nice structural properties at the same time, and thus . . .
- Bi-rank-width is a really good dir. measure – the best we (can) have?
THANK YOU FOR YOUR ATTENTION