Can dense graphs be “sparse”?

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Presenting results obtained with J. Gajarský [MSc. thesis, and arXiv],
and with R. Ganian, J. Nešetřil, J. Obdržálek, P. Ossona de Mendez, R. Ramadurai [MFCS 12].

1 Introduction: Tree-likeness

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- Solving algorithmic problems;
  - e.g., running DP algorithms on decompositions,
  - and proving *algorithmic metatheorems*.

Tree-likeness

- two basic kinds on undirected graphs. . .

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$\sim$ branch-width	$\sim$ rank-width

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  - [NEW] kernelization for *MSO model checking* on trees of bd. height $\rightarrow$ elementary FPT algorithm (faster than Courcelle).

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MSO_2 model checking in FPT, but nonelementary in ϕ [Courcelle]	MSO_2 model checking in *elementary* FPT wrt. ϕ [NEW], by the previous kernelization – extending, e.g., [Lampis 2010] with vertex cover

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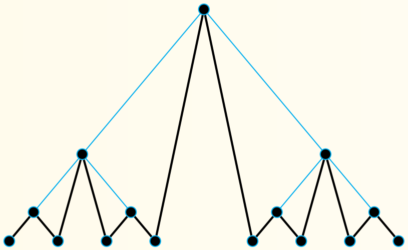
MSO_1 model checking in FPT, nonelem. [Courcelle, Makowsky, Rotics]	MSO_1 model checking in elementary FPT, again by the prev.

3 What is Tree-depth?

[Nešetřil, Ossona de Mendez]



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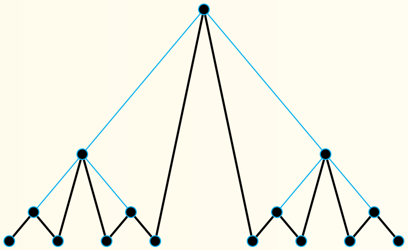

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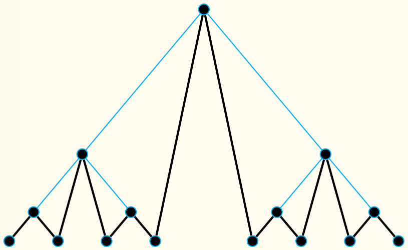

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- Asympt. equivalent to not having long paths as subgraphs.

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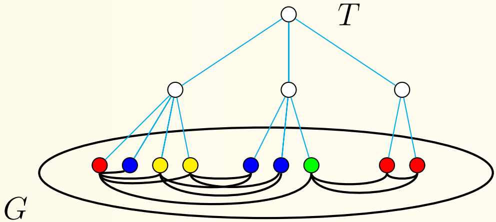

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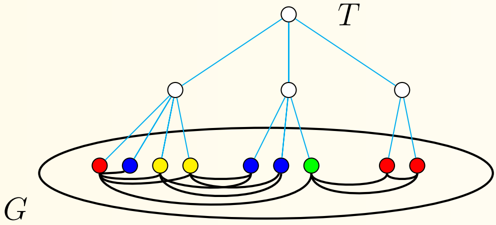

- An (m-coloured) *tree model* T of height d,
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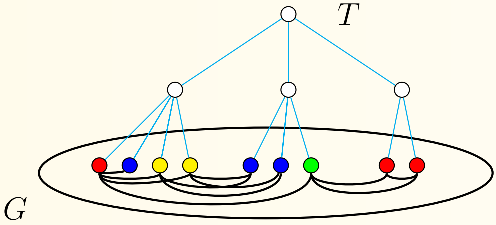

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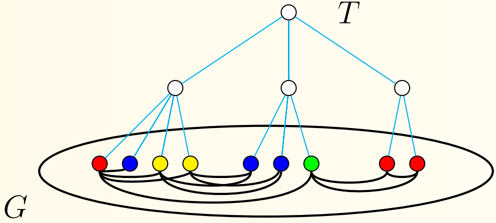

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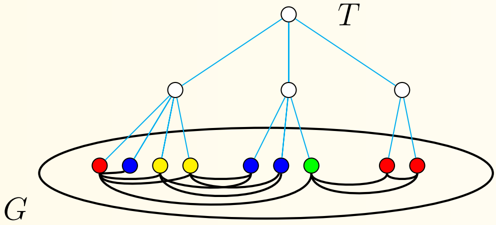

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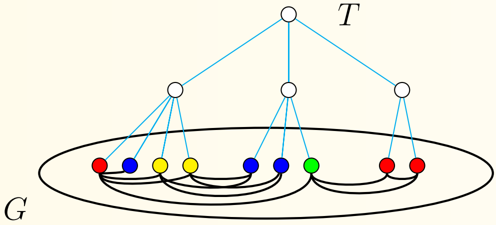

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4 The "Shrubs MSO Theorem"

Theorem [NEW]. For a given tree T of fixed height, there is a bounded-size subtree $T' \subseteq T$ such that $\boldsymbol{T} \models \boldsymbol{\varrho} \iff \boldsymbol{T'} \models \boldsymbol{\varrho}$, for any MSO ϱ.

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Shrubs; another consequence

Inspired by very recent...

Theorem [Elberfeld, Grohe, and Tantau – LICS 2012].
The following are equivalent on hereditary (monotone) graph classes $\mathcal{G}$:

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Theorem [NEW]. On hereditary graph classes of bounded shrub-depth, expressive powers of *FO logic and MSO_1 logic coincide*.

Conjecture. The previous Theorem can be reversed.

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why the *complement* of a sparse class is not sparse?

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* somewhere dense * $\quad$ if $\quad$ * somewhere "logically dense" *

$\mathcal{G} \nabla j$ $\qquad$ $\mathcal{G} \,\overline{\nabla}\, j$

contains all graphs for some $j < \infty$.

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- $\mathcal{G}$ is **somewhere FO dense** $\leftrightarrow$ $\exists j$: $\mathcal{G} \bar{\bar{\nabla}}\, j$ contains all graphs,

then "*FO sparsity*" $\equiv$ nowhere FO dense.

FO sparsity examples

For better understanding...

Graph class	FO resolution $\triangledown j$
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heredit. n-subdiv. of K_n, $n \in \mathbb{N}$	???, but not all graphs
heredit. 10^{10}-subd. of K_n, $n \in \mathbb{N}$	all graphs, for $j > 10^{10}$ (dense)
somewhere dense and monotone	all graphs, eventually
planar, or nowhere dense	**???**, are those nowhere FO dense?

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  related to traditional sparsity notions as
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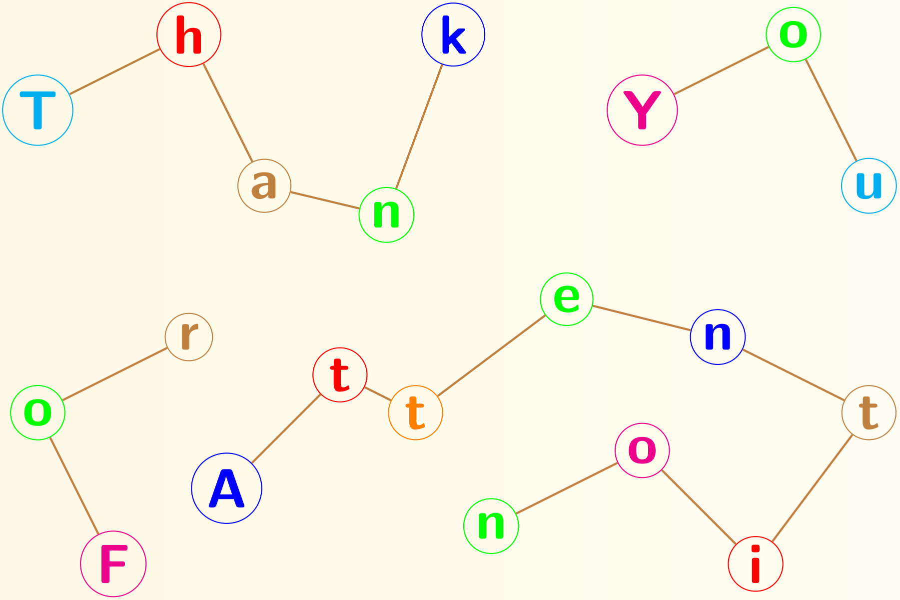
  - low *shrub-depth colouring* – can this be more than just shallow interpretation in bounded expansion classes?


T h a n k
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