A Short Proof of Euler–Poincaré Formula

Petr Hliněný

Masaryk University, Faculty of Informatics, Brno, Czech Republic
hlineny@fi.muni.cz

Supported by research grant GAČR 20-04567S of the Czech Science Foundation.

Abstract: "$V - E + F = 2$", the famous Euler's polyhedral formula, has a natural generalization to convex polytopes in every finite dimension, also known as the Euler–Poincaré Formula. We provide another short inductive combinatorial proof of the general formula. Our proof is self-contained and it does not use shellability of polytopes.

1 Introduction

In this paper we follow the standard terminology of polytopes theory, such as Ziegler [**7**]. We consider *convex polytopes*, defined as a convex hull of finitely many points, in the d-dimensional Euclidean space for an arbitrary $d \in \mathbb{N}$, $d \geq 1$. We shortly say a polytope to mean a convex polytope. A landmark discovery in the history of combinatorial investigation of polytopes was famous Euler's formula, stating that for any 3-dimensional polytope with v vertices, e edges and f faces, $v - e + f = 2$ holds. This finding was later generalized, in every dimension d, to what is nowadays known as (generalized) Euler's relation or Euler–Poincaré formula, as follows.

For instance, in dimension $d = 1$ we have $v = 2$, which can be rewritten as $v - 1 = 1$, and in dimension $d = 2$ we have got $v - e = 0$ or $v - e + 1 = 1$. Similarly, the $d = 3$ case can be rewritten as $v - e + f - 1 = 1$. Note that the '1' left of '=' stands in these expressions for the polytope itself. In general, the following holds:

Theorem 1 ("Euler–Poincaré formula"; Schläfli [**5**] 1852)**.** *Let P be a convex polytope in $\mathbb{R}^d$, and denote by f^c, $c \in \{0, 1, \dots, d\}$, the numbers of faces of P of dimension c. Then*

$$f^0 - f^1 + f^2 - \cdots + (-1)^d f^d = 1. \tag{1}$$

We refer to classical textbooks of Grünbaum [**3**] and Ziegler [**7**] for a closer discussion of the interesting history of this formula and of the difficulties associated with its proof. Here we just briefly remark that all the historical attempts to prove the formula in a combinatorial way, starting from Schläfli, implicitly assumed validity of a special property called *shellability* of a polytope. However, the shellability of any polytope was formally established only in 1971 by Bruggesser and Mani [**1**].

We provide a new short and self-contained inductive combinatorial proof of (1) which does not assume shellability of polytopes.

2 New Combinatorial Proof

Our proof of Theorem 1 proceeds by induction on the dimension $d \geq 1$. Note that validity of (1) is trivial for $d = 1, 2$, and hence it is enough to show the following:

Lemma 2. *Let $k \geq 2$ and P be a polytope of dimension $k + 1$. Assume that* (1) *holds for any polytope of dimension $d \in \{k - 1, k\}$. Then* (1) *holds for P (with $d = k + 1$).*

Research Perspectives CRM Barcelona, Summer 20XX, vol. XX, in Trends in Mathematics
Springer-Birkhäuser, Basel

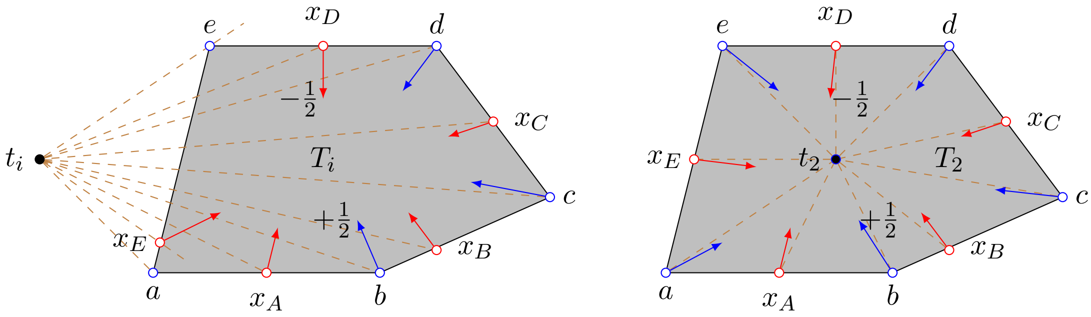

FIGURE 1. Proof of Lemma 2: a facet in a 3-dimensional polytope P ($k = 2$). Each vertex of P initially gets charge of 1 and each edge -1. Consider, e.g., a facet T_i of P which is a pentagon with vertices a, b, c, d, e and sides (edges) A, B, C, D, E in order. Let t_i be the point in which the plane of T_i intersects the line q (see in the proof). On the left of the picture ($t_i \notin T_i$, for $i \geq 3$), we have that the vertices b, c, d send charge of $\frac{1}{2}$ to T_i by the rule (3), while a, e are not sending to T_i. On the right ($t_i \in T_i$, $i = 1, 2$), all the vertices a, b, c, d, e send charge of $\frac{1}{2}$ to T_i. In both cases, every side A, B, C, D, E sends charge of $-\frac{1}{2}$ to T_i. Consequently, on the left T_i ends up with charge -1 (compare to (5)), while on the right with charge 0 (cf.(4)).

Proof. Recall that f^c, $c \in \{0, 1, \ldots, k+1\}$, denote the numbers of faces of P of dimension c. The only (improper) face of dimension $k+1$ is P itself, and the faces of dimension k are the *facets* of P. Our goal is to prove

$$f^0 - f^1 + f^2 - \cdots + (-1)^{k-1} f^{k-1} + (-1)^k f^k + (-1)^{k+1} f^{k+1} = 1,$$

or equivalently, since $f^{k+1} = 1$,

$$f^0 - f^1 + f^2 - \cdots + (-1)^{k-1} f^{k-1} = 1 + (-1)^k (1 - f^k). \tag{2}$$

We choose arbitrary two facets T_1, T_2 of P (distinct, but not necessarily disjoint) and two points $t_1 \in T_1$ and $t_2 \in T_2$ in their relative interior, such that the straight line $q = \overline{t_1 t_2}$ passing through t_1, t_2 is in a general position with respect to P. In particular, we demand that *no* nontrivial line segment lying in a face of P of dimension $c \leq k-1$ is coplanar with q. We also denote by $T_3, \ldots, T_{f^k}$ the remaining facets of P, in any order.

In the proof we use a *discharging* argument, an advanced variant of the double-counting method in combinatorics. To every face F of P of dimension $0 \leq c \leq k-1$, we assign *charge* of value $(-1)^c$ (the facets start with no charge). Hence the *total change* initially assigned to all faces of P equals the left-hand side of (2).

Now we *discharge* all the assigned charge from those faces to the facets of P (which initially have no charge). The discharging rule is only one and very simple. Consider a facet T_i of P, $1 \leq i \leq f^k$. Let $t_i \in q$ denote the unique point which is the intersection of the line q with the support hyperplane of T_i. This is a sound definition of t_i according to a general position of q, and it is consistent with the choice of t_1, t_2 above. Consider further any proper face F of T_i (so F is a face of P as well and is of dimension $0 \leq c \leq k-1$), and choose a fixed point x_F in the relative interior of F (note that $x_v = v$ if v is a vertex of P).

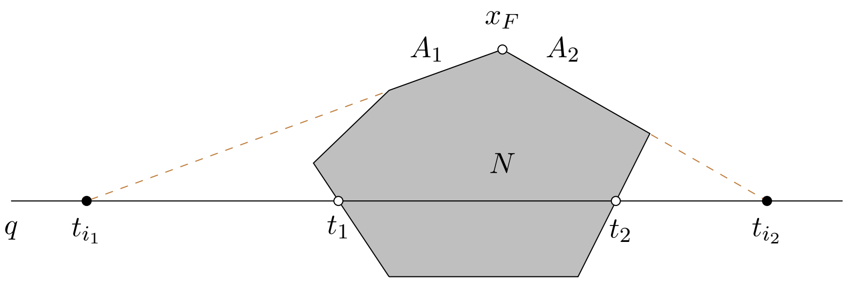

FIGURE 2. Proof of Lemma 2: A polygon N which is the intersection of the polytope P with the plane spanning q and a point x_F of a face F. The two sides A_1, A_2 of N incident to x_F determine the two unique facets T_{i_1}, T_{i_2} of P that F sends charge to.

Our discharging rule reads (see in Figure 1):

(3) *The face F sends half of its initial charge, i.e. $\frac{1}{2}(-1)^c$, to the facet T_i if, and only if, the straight line passing through x_F and t_i intersects the relative interior of T_i.*

Note that we will be finished if we prove that, after applying the discharging rule, (i) every face of P of dimension $\leq k-1$ ends up with charge 0, and (ii) the total charge of the facets of P sums up to the right-hand side of (2).

For the task (i), consider any face F of P of dimension $c \leq k-1$ and the point x_F chosen in F above. Let L denote the plane determined by the line $q = \overline{t_1 t_2}$ and the point $x_F \notin q$. Then $N := P \cap L$ is a convex polygon. See Figure 2. We claim that x_F must be a vertex of N: indeed, if x_F belonged to a relative interior of a side A_0 of N, then $A_0 \subseteq F$ and A_0 would be coplanar with q, contradicting our assumption of a general position of q. Consequently, x_F is incident to two sides A_1, A_2 of N, and there exist facets T_{i_1}, T_{i_2} of P, $1 \leq i_1 \neq i_2 \leq f^k$, such that $A_j = T_{i_j} \cap L$ for $j = 1, 2$. Observe that the support line of A_j intersects q precisely in t_{i_j} (which has been defined as the intersection of the support hyperplane of T_{i_j} with q).

Moreover, since A_j is coplanar with q, by our assumption of a general position of q it cannot happen that A_j is contained in a face of dimension $\leq k-1$. Consequently, A_j (except its ends) belongs to the relative interior of T_{i_j}, and T_{i_j} is a unique such face for A_j. Hence, taking this argument for $j = 1, 2$, we see that F sends away by (3) exactly two halves of its initial charge, ending up with charge 0.

For the task (ii), let f_i^c, where $c \in \{0, 1, \ldots, k\}$ and $i \in \{1, \ldots, f^k\}$, denote the number of faces of T_i of dimension c. We first look at the two special facets T_i, $i = 1, 2$ (Figure 1 right). Since $t_i \in T_i$ in this case, by (3) T_i receives charge from every of its proper faces. Using (1) for T_i, which is of dimension k, we thus get that the total charge T_i ends up with, is

$$\frac{1}{2}\left(f_i^0 - f_i^1 + \cdots + (-1)^{k-1} f_i^{k-1}\right) = \frac{1}{2}\left(1 - (-1)^k f_i^k\right) = \frac{1}{2}\left(1 - (-1)^k\right). \tag{4}$$

Second, consider a facet T_i where $i \geq 3$. Let H_i be the support hyperplane of T_i. Then $\{t_i\} = H_i \cap q$ and $t_i \notin T_i$. We restrict ourselves to the affine space formed by H_i, and denote by S_i a projection of T_i from the point t_i onto a suitable hyperplane within H_i. Since t_i is in a general position with respect to T_i (which is implied by a general position of q), the following holds: every proper face of S_i is the image of an equivalent face of T_i (of the same dimension!). Furthermore, by convexity, a face F of T_i has no image among the faces of S_i if, and only if, the line through x_F and t_i intersects the relative interior of T_i. See also Figure 1 left.

Consequently, as directed by (3), T_i receives charge precisely from those of its faces F which do not have an image among the proper faces of S_i (in particular, T_i receives charge from all of its faces of dimension $k-1$). Denote by g_i^c the number of faces of S_i of dimension $c \leq k-1$, and notice that $f_i^k = g_i^{k-1} = 1$. Hence, precisely, T_i receives $\frac{1}{2}(-1)^{k-1}$ of charge from each of its f_i^{k-1} faces of dimension $k-1$, and $\frac{1}{2}(-1)^c$ from $f_i^c - g_i^c$ of its faces of dimension $0 \leq c \leq k-2$. Summing together, and using (1) for T_i (of dimension k) and for S_i (of dimension $k-1$), we get

$$\begin{aligned} \frac{1}{2}(-1)^{k-1}f_i^{k-1} \quad + \quad \frac{1}{2}\sum_{c=0}^{k-2}(-1)^c(f_i^c - g_i^c) &= \frac{1}{2}\sum_{c=0}^{k-1}(-1)^c f_i^c - \frac{1}{2}\sum_{c=0}^{k-2}(-1)^c g_i^c \\ &= \frac{1}{2}\left(1-(-1)^k f_i^k\right) - \frac{1}{2}\left(1-(-1)^{k-1}g_i^{k-1}\right) = -(-1)^k. \end{aligned} \tag{5}$$

Since the total charge is not changed (only redistributed), we get that (the left-hand side of) (2) must equal the sum of (4) over $i = 1, 2$ and of (5) over $i = 3, \ldots, f^k$, leading to

$$f^0 - f^1 + f^2 - \cdots + (-1)^{k-1}f^{k-1} = 2\cdot\frac{1}{2}\left(1-(-1)^k\right) - (f^k - 2)\cdot(-1)^k = 1 + (-1)^k(1 - f^k),$$

and thus finishing the proof of (2) for P. □

3 Final Remarks

We have shown a full proof of the Euler–Poincaré formula (1) with only simple, combinatorial and elementary geometric arguments. Our proof has been in parts inspired by a proof of basic Euler's formula via angles [**2**, "Proof 8: Sum of Angles"], and by Welzl's probabilistic proof [**6**] of Gram's equation. Although, the resulting exposition of the proof does not resemble either of those; in fact, it might look like a generalization of a discharging proof [**2**, "Proof 6: Electrical Charge"], but that was not our way to the result . Lastly, we remark that the underlying idea of our proof can be expressed also in an alternative, more geometric way, such as the exposition in the preprint [**4**].

References

[1] Heinz Bruggesser and Peter Mani. Shellable decompositions of cells and spheres. *Math. Scand.*, 29:197–205, 1971.

[2] David Eppstein. The geometry junkyard: Twenty proofs of Euler's formula. `https://www.ics.uci.edu/~eppstein/junkyard/euler/`, 2016.

[3] Branko Grünbaum. *Convex polytopes.* Graduate texts in mathematics. Springer, New York, Berlin, London, 2003.

[4] Petr Hliněný. A short proof of Euler–Poincaré formula. *arXiv*, abs/1612.01271, 2016.

[5] Ludwig Schläfli. *Theorie der vielfachen Kontinuität [1852].* (in German). First publication 1901 Graf, J. H., ed. Republished by Cornell University Library historical math monographs, Zürich, Basel: Georg & Co., 2010.

[6] Emo Welzl. Gram's equation - A probabilistic proof. In *Results and Trends in Theoretical Computer Science*, volume 812 of *Lecture Notes in Computer Science*, pages 422–424. Springer, 1994.

[7] Günter M. Ziegler. *Lectures on polytopes.* Graduate texts in mathematics. Springer, New York, 1995.