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Part A.

Set Theory

where $\mu_o := \mathrm{cf}(\mu) \leq \mu$. We construct a strong μ_o-chain $(\mathfrak{N}_\alpha)_{\alpha<\mu_o}$ where each $\mathfrak{N}_\alpha \leq_{\mathcal{K}} \mathfrak{M}_{f(\alpha)}$ has size $|N_\alpha| = \mu^+$ and, for all $\alpha < \mu_o$, we have

$$U \cap M_{f(\alpha+1)} \subseteq N_{\alpha+1} \subseteq M_{f(\alpha+1)} .$$

We define $\mathfrak{N}_\alpha$ by induction on α. We start with an arbitrary structure $\mathfrak{N}_o \leq_{\mathcal{K}} \mathfrak{M}$ of size $|N_o| = \mu^+$. For limit ordinals γ, we set $\mathfrak{N}_\gamma := \bigcup_{\alpha<\gamma} \mathfrak{N}_\alpha$.

For the successor step, suppose that $\mathfrak{N}_\alpha$ has already been defined. We construct a weak μ^+-chain $(\mathfrak{B}_\beta)_{\beta<\mu^+}$ with $|B_\beta| = \mu^+$ as follows. We start with an arbitrary structure $\mathfrak{B}_o \leq_{\mathcal{K}} \mathfrak{M}_{f(\alpha+1)}$ of size $|B_o| = \mu^+$ such that $N_\alpha \cup (U \cap M_{f(\alpha+1)}) \subseteq B_o$. Then we use Lemma 2.13 to inductively define $\mathfrak{B}_\beta$, for $o < \beta < \mu^+$. Since $\mathcal{K}$ is μ^+-Galois stable, we can choose all $\mathfrak{B}_\beta$ of size $|B_\beta| = \mu^+$. Since $\mathfrak{M}_{f(\beta+1)}$ is μ^{++}-Galois saturated, we can further choose $\mathfrak{B}_\beta$ such that $\mathfrak{B}_\beta \leq_{\mathcal{K}} \mathfrak{M}_{f(\beta+1)}$. Let $\mathfrak{N}_{\alpha+1} := \bigcup_{\beta<\mu^+} \mathfrak{B}_\beta$ be the limit. By Lemma 3.3, $\mathfrak{N}_{\alpha+1}$ is μ^{++}-universal over $\mathfrak{B}_o \geq_{\mathcal{K}} \mathfrak{N}_\alpha$.

We have constructed a strong μ_o-chain $(\mathfrak{N}_\alpha)_{\alpha<\mu_o}$ whose limit $\mathfrak{A} := \bigcup_{\alpha<\mu_o} \mathfrak{N}_\alpha$ has size $|A| = \mu_o \otimes \mu^+ = \mu^+$. Since $|N_o| = \mu^+$ it follows by Theorem 4.11 that $\mathfrak{A}$ is Galois saturated. Consequently, $\mathfrak{p}$ is realised in $\mathfrak{A} \leq_{\mathcal{K}} \mathfrak{N}$. $\square$

Part F.

Independence and Forking

F1. Geometries

1. Dependence relations

We have seen that a vector space or an algebraically closed field (of a given characteristic) is uniquely determined by, respectively, its dimension and its transcendence degree. In this chapter we try to generalise these two results. We investigate first-order theories whose models are uniquely determined by some kind of dimension. We start by introducing an abstract notion of dimension. As for vector spaces and algebraically closed fields, this notion is based on a closure operator. With these tools in hand we can then prove categoricity results for certain theories. Our first application will be Theorem 4.13, which states that two models of the same dimension are isomorphic.

Definition 1.1. (a) A *dependence relation* on a set A is a system $D \subseteq \wp(A)$ with the property that

$$X \in D \quad \text{iff} \quad X_0 \in D \text{ for some nonempty finite } X_0 \subseteq X\,.$$

A subset $X \subseteq A$ is *D-dependent* if $X \in D$. Otherwise, it is called *D-independent.*

(b) An element $a \in A$ *D-depends* on a set $X \subseteq A$ if $a \in X$ or there is an D-independent subset $I \subseteq X$ such that $I \cup \{a\}$ is D-dependent. The set of all elements D-depending on X is denoted by $\langle\!\langle X \rangle\!\rangle_D$.

(c) A dependence relation D on A is *transitive* if $\langle\!\langle \langle\!\langle X \rangle\!\rangle_D \rangle\!\rangle_D = \langle\!\langle X \rangle\!\rangle_D$, for all $X \subseteq A$.

Remark. Note that, if I is D-independent then we have $a \in \langle\!\langle I \rangle\!\rangle_D$ if and only if $I \cup \{a\}$ is D-dependent.

Example. (a) Let $\mathfrak{V}$ be a $\mathfrak{K}$-vector space. Then

$$D := \{ X \subseteq V \mid X \text{ is linearly dependent} \}$$

is a transitive dependence system on V.

(b) Let $\mathfrak{K}$ be a field. Then

$$D := \{ X \subseteq K \mid X \text{ is algebraically dependent} \}$$

is a transitive dependence system on K.

(c) Let $\mathfrak{G} = \langle V, E \rangle$ be an undirected graph. Then

$$D := \{ X \subseteq E \mid E \text{ contains a cycle} \}$$

is a transitive dependence system on E.

Lemma 1.2. *Let D be a transitive dependence relation on A. The function $c : X \mapsto \langle\!\langle X \rangle\!\rangle_D$ is a finitary closure operator with the exchange property.*

Proof. By definition c is finitary. To show that it is a closure operator note that we have $X \subseteq c(X)$ since all elements of X D-depend on X. As D is transitive we further have $c(c(X)) = c(X)$. Finally, if $X \subseteq Y$ then every element D-depending on X also D-depends on Y. Hence, $c(X) \subseteq c(Y)$.

For the exchange property, suppose that $b \in c(X \cup \{a\}) \setminus c(X)$. Then there is a D-independent subset $I \subseteq X \cup \{a\}$ with $I \cup \{b\} \in D$. Let $I_0 := I \setminus \{a\}$. Note that $I' := I_0 \cup \{b\}$ is D-independent since, otherwise, we would have $b \in c(I_0) \subseteq c(X)$. Therefore, $I' \cup \{a\} \in D$ implies that $a \in c(I') \subseteq c(X \cup \{b\})$, as desired. □

Lemma 1.3. *Let D be a transitive dependence relation, I a D-independent set, and $I_0 \subseteq I$. If $a \in \langle\!\langle I \rangle\!\rangle_D \setminus \langle\!\langle I_0 \rangle\!\rangle_D$ then there exists an element $b \in I \setminus I_0$ such that $I' := (I \setminus \{b\}) \cup \{a\}$ is D-independent and $b \in \langle\!\langle I' \rangle\!\rangle_D$.*

Proof. Since $a \in \langle\!\langle I \rangle\!\rangle_D$ there is some D-independent subset $J \subseteq I$ such that $J \cup \{a\} \in D$. Choose J minimal. Since $a \notin \langle\!\langle I_0 \rangle\!\rangle_D$ we have $J \nsubseteq I_0$. Fix some element $b \in J \setminus I_0$ and set $J' := J \setminus \{b\}$ and $I' := I \setminus \{b\}$. By minimality of J we have $J' \cup \{a\} \notin D$. Consequently, $b \in \langle\!\langle J' \cup \{a\} \rangle\!\rangle_D \subseteq \langle\!\langle I' \cup \{a\} \rangle\!\rangle_D$.

It remains to prove that $I' \cup \{a\}$ is D-independent. For a contradiction, suppose that $I' \cup \{a\} \in D$. Then $a \in \langle\!\langle I' \rangle\!\rangle_D$. Since D is transitive it follows that $b \in \langle\!\langle I' \cup \{a\} \rangle\!\rangle_D \subseteq \langle\!\langle I' \rangle\!\rangle_D$. Consequently, $I = I' \cup \{b\}$ is not D-independent. Contradiction. □

We can characterise transitive dependence systems in terms of closure operators with the exchange property.

Proposition 1.4. (a) *If c is a finitary closure operator on A with the exchange property, then*

$$D := \{ X \subseteq A \mid \text{there is some } a \in X \text{ with } a \in c(X \setminus \{a\}) \}$$

is a transitive dependence relation with $c(X) = \langle\!\langle X \rangle\!\rangle_D$, for all X.

(b) *A subset $D \subseteq \mathscr{P}(A)$ is a transitive dependence relation if and only if the function $c : X \mapsto \langle\!\langle X \rangle\!\rangle_D$ is a finitary closure operator with the exchange property.*

Proof. (a) To show that D is a dependence relation let $X \in D$. We have to find a finite subset $X_0 \subseteq X$ with $X_0 \in D$. By definition, there is some element $a \in X$ with $a \in c(X \setminus \{a\})$. Since c is finitary it follows that there is some $X_0 \subseteq X \setminus \{a\}$ with $a \in c(X_0)$. Consequently, $X_0 \cup \{a\} \in D$.

It remains to show that D is transitive and that $c(X) = \langle\!\langle X \rangle\!\rangle_D$. We start with the latter. Let $a \in c(X)$ and choose a minimal subset $X_0 \subseteq X$ with $a \in c(X_0)$. Then there is no $b \in X_0$ with $b \in c(X_0 \setminus \{b\})$ since, otherwise, $c(X_0) = c(X_0 \setminus \{b\})$ and X_0 would not be minimal. It follows that X_0 is D-independent while $X_0 \cup \{a\}$ is not. Consequently, we have $a \in \langle\!\langle X \rangle\!\rangle_D$.

Conversely, suppose that $a \in \langle\!\langle X \rangle\!\rangle_D$. Then there is a D-independent subset $I \subseteq X$ with $I \cup \{a\} \in D$. Hence, we can find an element $b \in I \cup \{a\}$ such that $b \in c((I \cup \{a\}) \setminus \{b\})$. If $b = a$ then we have $a \in c(I) \subseteq c(X)$, as desired. Otherwise, let $I_0 := I \setminus \{b\}$. Since I is D-independent we have $b \notin c(I_0)$. Therefore, $b \in c(I_0 \cup \{a\}) \setminus c(I_0)$ implies that $a \in c(I_0 \cup \{b\}) \subseteq c(X)$.

Finally, note that $c \circ c = c$ implies that D is transitive.

(b) ($\Rightarrow$) was already proved in Lemma 1.2. ($\Leftarrow$) By (a), we only have to show that, if D and D' are sets such that $\langle\!\langle X \rangle\!\rangle_D = \langle\!\langle X \rangle\!\rangle_{D'}$, for all $X \subseteq A$, then we have $D = D'$. By symmetry, suppose that there is a set $X \in D \setminus D'$. Then there is a finite nonempty subset $X_0 \subseteq X$ with $X_0 \in D \setminus D'$. Choose X_0 such that its size is minimal and fix some element $a \in X_0$. By minimality we have $X_0 \setminus \{a\} \notin D$. This implies that $a \in \langle\!\langle X_0 \setminus \{a\} \rangle\!\rangle_D$. But $X_0 \notin D'$ implies $X_0 \setminus \{a\} \notin D'$. Therefore, $a \notin \langle\!\langle X_0 \setminus \{a\} \rangle\!\rangle_{D'} = \langle\!\langle X_0 \setminus \{a\} \rangle\!\rangle_D$. A contradiction. □

We can use this proposition to translate between dependence relations and closure operators. In the following we will use the terminology for both interchangeably, e.g., we will speak of independent sets with respect to a closure operator.

Using dependence relations or, equivalently, closure operators with the exchange property, we can introduce bases and dimensions as for vector spaces.

Definition 1.5. Let D be a dependence relation on A. A set $X \subseteq A$ is *D-spanning* if $\langle\!\langle X \rangle\!\rangle_D = A$. A *$D$-basis* is a D-spanning set which is D-independent.

Lemma 1.6. *Let D be a transitive dependence relation on A and $X \subseteq A$ a set. The following statements are equivalent:*

(1) *X is a maximal D-independent set.*

(2) *X is a minimal D-spanning set.*

(3) *X is a D-basis.*

Proof. (1) $\Rightarrow$ (2) Let X be maximal D-independent and suppose that there is some element $a \in A \setminus \langle\!\langle X \rangle\!\rangle_D$. Since X is D-independent we have $X \cup \{a\} \notin D$ and X is not maximal.

(2) $\Rightarrow$ (3) Let X be minimal D-spanning. For a contradiction suppose that $X \in D$. Let $X_0 \subseteq X$ be a minimal subset with $X_0 \in D$ and fix some element $a \in X_0$. By minimality, $I := X_0 \setminus \{a\}$ is D-independent. Hence, $a \in \langle\!\langle I \rangle\!\rangle_D \subseteq \langle\!\langle X \setminus \{a\} \rangle\!\rangle_D$. By transitivity, it follows that $\langle\!\langle X \setminus \{a\} \rangle\!\rangle_D = \langle\!\langle X \rangle\!\rangle_D = A$. This contradicts the minimality of X.

(3) $\Rightarrow$ (1) Every D-basis X is D-independent. If X were not maximal, we could find an element $a \in A \setminus X$ such that $X \cup \{a\}$ were D-independent. But this would imply that $a \notin \langle\!\langle X \rangle\!\rangle_D = A$. A contradiction. □

Once we have shown that all bases have the same cardinality, we obtain a well-defined notion of dimension.

Lemma 1.7 (Exchange Lemma). *Let D be a transitive dependence relation on A. If I is D-independent and S is D-spanning then there exists a subset $S_0 \subseteq S$ with $I \cap S_0 = \varnothing$ such that $I \cup S_0$ is a D-basis.*

Proof. The set $F := \{ J \mid J \text{ is } D\text{-independent with } I \subseteq J \subseteq I \cup S \}$ is inductively ordered by $\subseteq$ since $\bigcup C \in D$ would imply that there is a finite subset $C_0 \subseteq C$ with $\bigcup C_0 \in D$. Consequently, F has a maximal element B. By maximality, every element of $S \setminus B$ D-depends on B. Hence, $S \subseteq \langle\!\langle B \rangle\!\rangle_D$ implies that $\langle\!\langle B \rangle\!\rangle_D \supseteq \langle\!\langle S \rangle\!\rangle_D = A$, and B is a D-basis. Setting $S_0 := B \setminus I$ yields the desired subset of S. □

Lemma 1.8. *Let D be a transitive dependence relation on A. If I, J are D-independent sets with $J \subseteq \langle\!\langle I \rangle\!\rangle_D$ then $|J| \leq |I|$.*

Proof. Since D induces a transitive dependence relation on $\langle\!\langle I \rangle\!\rangle_D$ we may assume that $A = \langle\!\langle I \rangle\!\rangle_D$ and that I is a D-basis.

First, suppose that J is finite. We prove the claim by induction on $|J \setminus I|$. If $J \subseteq I$ then there is nothing to do. Hence, suppose that there is some element $a \in J \setminus I$, and set $H := I \cap J$. Since J is D-independent we have $a \in \langle\!\langle I \rangle\!\rangle_D \setminus \langle\!\langle H \rangle\!\rangle_D$. By Lemma 1.3, we can find an element $b \in I \setminus H$ such that $I_0 := (I \setminus \{b\}) \cup \{a\}$ is D-independent and $b \in \langle\!\langle I_0 \rangle\!\rangle_D$. By transitivity of D it follows that $J \subseteq \langle\!\langle I \rangle\!\rangle_D \subseteq \langle\!\langle I_0 \cup \{b\} \rangle\!\rangle_D = \langle\!\langle I_0 \rangle\!\rangle_D$. Since $|J \setminus I_0| < |J \setminus I|$ we can apply the induction hypothesis to conclude that $|J| \leq |I_0| = |I|$.

It remains to consider the case that J is infinite. If I were finite, we could choose a subset $J_0 \subseteq J$ of size $|J_0| = |I| + 1$. This would contradict the finite case proved above. Hence, I is also infinite. Since the operator $X \mapsto \langle\!\langle X \rangle\!\rangle_D$ is finitary we have

$$J \subseteq \bigcup \{ \langle\!\langle I_0 \rangle\!\rangle_D \mid I_0 \subseteq I \text{ is finite} \} .$$

If $I_0 \subseteq I$ is finite, we have seen above that $|J \cap \langle\!\langle I_0 \rangle\!\rangle_D| \leq |I_0|$. Consequently,

$$J = \bigcup \{ J \cap \langle\!\langle I_0 \rangle\!\rangle_D \mid I_0 \subseteq I \text{ is finite} \}$$

implies that

$$|J| \leq \sum \big\{ |J \cap \langle\!\langle I_0 \rangle\!\rangle_D| \bigm| I_0 \subseteq I \text{ is finite} \big\} \leq |I|^{<\omega} = |I| . \qquad \square$$

Theorem 1.9. *Let D be a transitive dependence relation on A.*

(a) *For every D-independent set I and every D-spanning set $S \supseteq I$ there exists a D-basis B with $I \subseteq B \subseteq S$.*

(b) *There exists a D-basis and all D-bases have the same cardinality*

Proof. (a) follows from Lemma 1.7.

(b) The existence of a D-basis follows from (a) by setting $I := \varnothing$ and $S := A$. The fact that two bases have the same cardinality follows from Lemma 1.8. $\square$

2. *Matroids and geometries*

It will be convenient to work with closure operators instead of dependence relations.

Definition 2.1. Let Ω be a set.

(a) A *matroid* is a pair $\langle \Omega, \mathrm{cl} \rangle$ where cl is a finitary closure operator on Ω with the exchange property.

(b) A matroid $\langle \Omega, \mathrm{cl} \rangle$ is a *geometry* if it satisfies

$$\mathrm{cl}(\varnothing) = \varnothing \quad \text{and} \quad \mathrm{cl}(\{a\}) = \{a\} , \quad \text{for every } a \in \Omega .$$

(c) Let $\langle \Omega, \mathrm{cl} \rangle$ be a matroid. For $U, I \subseteq \Omega$, we say that I is *independent over U* if

$$a \notin \mathrm{cl}(U \cup (I \setminus \{a\})) , \quad \text{for all } a \in I .$$

We call I *independent* if it is independent over the empty set.

A *basis* of a set $X \subseteq \Omega$ is an independent set $I \subseteq X$ with $\mathrm{cl}(I) \supseteq X$. The *dimension* of X is the cardinality of any basis of X. We denote it by $\dim_{\mathrm{cl}}(X)$. Similarly, we define a basis of X *over a set U* as a maximal set $I \subseteq X$ that is independent over U. The dimension $\dim_{\mathrm{cl}}(X/U)$ of X *over U* is the cardinality of any such set.

Example. Let $f : A \to B$ be a function and define

$$c(X) := f^{-1}[f[X]] , \quad \text{for } X \subseteq A .$$

Then $\langle A, c \rangle$ forms a matroid.

Remark. With any matroid $\langle \Omega, \mathrm{cl} \rangle$ we can associate the lattice $\langle \mathrm{fix}\,\mathrm{cl}, \subseteq \rangle$ of all closed sets and the closure space $\langle \Omega, \mathrm{fix}\,\mathrm{cl} \rangle$.

Exercise 2.1. Let $\langle \Omega, \mathrm{cl} \rangle$ be a matroid, $X \subseteq \Omega$, and let $C \subseteq \mathrm{fix}\,\mathrm{cl}$ be a maximal chain of closed sets such that $A \subseteq \mathrm{cl}(X)$, for all $A \in C$. Prove that $|C| = \dim_{\mathrm{cl}}(X) \oplus 1$.

Definition 2.2. Let $\mathfrak{V}$ be a vector space over a skew field $\mathfrak{S}$.

(a) The *linear matroid* associated with $\mathfrak{V}$ is the matroid $\langle V, \mathrm{cl} \rangle$ where $\mathrm{cl}(X) := \langle\!\langle X \rangle\!\rangle_{\mathfrak{V}}$ is the linear subspace spanned by X.

(b) The *affine geometry* associated with $\mathfrak{V}$ is the matroid $\langle V, \mathrm{cl} \rangle$ where

$$\mathrm{cl}(X) := \{ s_0 x_0 + \dots + s_{n-1} x_{n-1} \mid n < \omega,\ s_i \in S,\ x_i \in X \text{ with } s_0 + \dots + s_{n-1} = 1 \} .$$

Example. Let $\mathfrak{V}$ be a vector space and let $x, y \in V$ be linearly independent. In the linear matroid the closure of $\{x, y\}$ is the plain through x, y, and the zero vector 0. In the affine geometry the closure of $\{x, y\}$ is the line through x and y.

Remark. (a) The linear matroid is not a geometry since $\mathrm{cl}\,\varnothing = \{0\} \neq \varnothing$. Furthermore, the usual dimension of a linear subspace $U \subseteq V$ coincides with its dimension $\dim_{\mathrm{cl}}(U)$ in the linear matroid as defined above.

(b) The affine geometry $\langle V, \mathrm{cl}\rangle$ associated with a vector space $\mathfrak{V}$ is really a geometry. But note that the usual affine dimension of an affine subspace $U \subseteq V$ is one less than its dimension $\dim_{\mathrm{cl}}(U)$ in the affine geometry as defined above.

The dimension function of a matroid has the following basic properties. In fact, we will show below that every function of this kind arises from a matroid.

Definition 2.3. Let Ω be a set. A function $\dim : \wp(\Omega) \times \wp(\Omega) \to \mathrm{Cn}$ is a *geometric dimension function* if, for all sets $A, B, U, V \subseteq \Omega$, the following conditions are satisfied:

(1) $\dim(A/U) \leq |A \smallsetminus U|$.

(2) $\dim(A \cup U/U) = \dim(A/U)$.

(3) $A \subseteq B$ and $U \subseteq V$ implies $\dim(A/V) \leq \dim(B/U)$.

(4) If, for some ordinal γ, $(A_\alpha)_{\alpha<\gamma}$ is an increasing chain of sets $A_\alpha \subseteq \Omega$, then

$$\dim(A_{<\gamma}/U) = \sum_{\alpha<\gamma} \dim(A_\alpha/U \cup A_{<\alpha})\,,$$

where $A_{<\alpha} := \bigcup_{\beta<\alpha} A_\beta$.

(5) For every element $a \in \Omega$ with $\dim(a/U) = 0$, there is a finite subset $U_0 \subseteq U$ such that $\dim(a/U_0) = 0$.

First, let us show that the dimension function of a matroid has these properties.

Proposition 2.4. *The dimension function* $\dim_{\mathrm{cl}}$ *associated with a matroid* $\langle \Omega, \mathrm{cl}\rangle$ *is a geometric dimension function.*

Proof. We have to check five conditions.

(1) If I is a basis of A over U, then $I \subseteq A \smallsetminus U$. Hence, $\dim_{\mathrm{cl}}(A/U) = |I| \leq |A \smallsetminus U|$.

(2) Every basis of $A \cup U$ over U is also a basis of A over U.

(3) Every set $I \subseteq A$ that is independent over V is also independent over U. Hence, $|I| \leq \dim_{\mathrm{cl}}(B/U)$.

(4) Let I be a basis of U. We define an increasing sequence of sets $(J_\alpha)_{\alpha<\gamma}$ such that J_α is a basis of $U \cup A_\alpha$ with $I \subseteq J_\alpha$. We proceed by induction on $\alpha < \gamma$. Suppose that we have already defined J_β, for all $\beta < \alpha$. Set $J_{<\alpha} := I \cup \bigcup_{\beta<\alpha} J_\beta$. By inductive hypothesis, $J_{<\alpha}$ is a basis of $U \cup A_{<\alpha}$. We can use Theorem 1.9 to extend $J_{<\alpha}$ to a basis J_α of $U \cup A_\alpha$. It follows that $B_\alpha := J_\alpha \smallsetminus J_{<\alpha}$ is a basis of A_α over $U \cup A_{<\alpha}$ and $J_{<\gamma} \smallsetminus I$ is a basis of $A_{<\gamma}$ over U. Hence,

$$\dim_{\mathrm{cl}}(A_{<\gamma}/U) = |J_{<\gamma} \smallsetminus I| = \sum_{\alpha<\gamma} |B_\alpha| = \sum_{\alpha<\gamma} \dim_{\mathrm{cl}}(A_\alpha/U \cup A_{<\alpha})\,.$$

(5) If $\dim_{\mathrm{cl}}(a/U) = 0$ then $a \in \mathrm{cl}(U)$. Since cl has finite character, there is a finite subset $U_0 \subseteq U$ such that $a \in \mathrm{cl}(U_0)$. This implies $\dim_{\mathrm{cl}}(a/U_0) = 0$. □

Before proving that, conversely, every geometric dimension function arises from a matroid, let us collect some immediate consequences of the definition of a dimension function.

Lemma 2.5. *Let* $\dim : \wp(\Omega) \times \wp(\Omega) \to \mathrm{Cn}$ *be a geometric dimension function.*

(a) $\dim(A \cup B/U) = \dim(A/U \cup B) \oplus \dim(B/U)$

(b) *If* $(a_\alpha)_{\alpha<\kappa}$ *is an enumeration of* A *then*

$$\dim(A/U) = \sum_{\alpha<\kappa} \dim(a_\alpha/U \cup A_{<\alpha})\,,$$

where $A_{<\alpha} := \{\, a_\beta \mid \beta < \alpha \,\}$.

Proof. (a) Considering the two-element increasing sequence $B \subseteq A \cup B$, it follows from the axioms of a geometric dimension function that

$$\begin{aligned}\dim(A \cup B/U) &= \dim(A \cup B/U \cup B) \oplus \dim(B/U)\\ &= \dim\big(A \cup B \cup (U \cup B) \bigm/ U \cup B\big) \oplus \dim(B/U)\\ &= \dim(A/U \cup B) \oplus \dim(B/U)\,.\end{aligned}$$

(b) By (a) and the axioms of a geometric dimension function, we have

$$\begin{aligned}\dim(A/U) &= \sum_{\alpha<\kappa} \dim\big(\{a_\alpha\} \cup A_{<\alpha} \bigm/ U \cup A_{<\alpha}\big)\\ &= \sum_{\alpha<\kappa} \big[\dim(a_\alpha/U \cup A_{<\alpha}) \oplus \dim(A_{<\alpha}/U \cup A_{<\alpha})\big]\\ &= \sum_{\alpha<\kappa} \dim(a_\alpha/X \cup A_{<\alpha}) \oplus 0\,.\end{aligned}$$

□

Proposition 2.6. *Let* $\dim : \wp(\Omega) \times \wp(\Omega) \to \mathrm{Cn}$ *be a geometric dimension function. For* $X \subseteq \Omega$*, we define*

$$\mathrm{cl}(X) := \{\, a \in \Omega \mid \dim(a/X) = 0 \,\}\,.$$

Then $\langle \Omega, \mathrm{cl} \rangle$ *is a matroid such that* $\dim_{\mathrm{cl}} = \dim$.

Proof. First, let us show that cl is a closure operator. Note that, for every $a \in X$, $\dim(a/X) \leq |\{a\} \smallsetminus X| = 0$ implies that $a \in \mathrm{cl}(X)$. Consequently, $X \subseteq \mathrm{cl}(X)$.

For monotonicity, assume that $X \subseteq Y$ and let $a \in \mathrm{cl}(X)$. Then

$$\dim(a/Y) \leq \dim(a/X) = 0 \quad \text{implies} \quad a \in \mathrm{cl}(Y)\,.$$

It remains to show that $\mathrm{cl}(\mathrm{cl}(X)) = X$. Let $a \in \mathrm{cl}(\mathrm{cl}(X))$. Then $\dim(a/\mathrm{cl}(X)) = 0$. Furthermore, $\dim(b/X) = 0$ for each $b \in \mathrm{cl}(X)$. Let $(b_\alpha)_{\alpha<\kappa}$ be an enumeration of $\mathrm{cl}(X)$ and set $B_{<\alpha} := \{\, b_\beta \mid \beta < \alpha \,\}$. Then $B_{<\kappa} = \mathrm{cl}(X)$ and, by Lemma 2.5 (b), it follows that

$$\dim(B_{<\kappa}/X) = \sum_{\alpha<\kappa} \dim(b_\alpha/X \cup B_{<\alpha}) \leq \sum_{\alpha<\kappa} \dim(b_\alpha/X) = 0\,.$$

Consequently, Lemma 2.5 (a) implies

$$\begin{aligned}\dim(a/X) &\leq \dim\big(\mathrm{cl}(X) \cup \{a\} \bigm/ X\big)\\ &= \dim(a/\mathrm{cl}(X)) \oplus \dim(\mathrm{cl}(X)/X) = 0 \oplus 0\,,\end{aligned}$$

as desired.

We have shown that cl is a closure operator. To prove that it has finite character, suppose that $a \in \mathrm{cl}(X)$. Then $\dim(a/X) = 0$. Hence, there is a finite subset $X_0 \subseteq X$ such that $\dim(a/X_0) = 0$. This implies $a \in \mathrm{cl}(X_0)$.

It remains to check that cl has the exchange property. Suppose that $b \in \mathrm{cl}(U \cup \{a\}) \smallsetminus \mathrm{cl}(U)$. Then $\dim(b/U \cup \{a\}) = 0$. Since $b \notin \mathrm{cl}(U)$, we have $\dim(b/U) = 1$. Hence,

$$\begin{aligned}&\dim(a/U \cup \{b\}) \oplus 1\\ =\;& \dim(a/U \cup \{b\}) \oplus \dim(b/U)\\ =\;& \dim(ab/U)\\ =\;& \dim(b/U \cup \{a\}) \oplus \dim(a/U) = \dim(a/U) \leq 1\end{aligned}$$

implies that $\dim(a/U \cup \{b\}) = 0$. Consequently, $a \in \mathrm{cl}(U \cup \{b\})$.

We have shown that $\langle \Omega, \mathrm{cl} \rangle$ is a matroid. To conclude the proof, we must check that $\dim_{\mathrm{cl}} = \dim$. We proceed in two steps. First, we show that $\dim(I/U) = |I|$ for every set I that is cl-independent over U. Let I be such a set. By definition of cl, it follows that

$$\dim(a/U \cup (I \smallsetminus \{a\})) = 1\,, \quad \text{for every } a \in I\,.$$

Set $\kappa := |I|$ and let $(a_\alpha)_{\alpha<\kappa}$ be an enumeration of I. Setting $I_{<\alpha} := \{\, a_\beta \mid \beta < \alpha \,\}$ it follows from Lemma 2.5 (b) that

$$\begin{aligned}\dim(I/U) &= \sum_{\alpha<\kappa} \dim(a_\alpha/U \cup I_{<\alpha})\\ &\geq \sum_{\alpha<\kappa} \dim(a_\alpha/U \cup (I \smallsetminus \{a_\alpha\})) = \kappa\,.\end{aligned}$$

Therefore, $\dim(I/U) \leq |I \smallsetminus U| \leq \kappa$ implies $\dim(I/U) = \kappa$.

Finally, we prove that $\dim(\mathrm{cl}(X)/U) = \dim(X/U)$, for every set X. Let $(a_\alpha)_{\alpha<\kappa}$ be an enumeration of $\mathrm{cl}(X)$ and set $A_{<\alpha} := \{ a_\beta \mid \beta < \alpha \}$. Then

$$\begin{aligned}
\dim(\mathrm{cl}(X)/U) &= \dim(\mathrm{cl}(X)/X) \oplus \dim(X/U) \\
&= \sum_{\alpha<\kappa} \dim(a_\alpha/X \cup A_{<\alpha}) \oplus \dim(X/U) \\
&\leq \sum_{\alpha<\kappa} \dim(a_\alpha/X) \oplus \dim(X/U) \\
&= 0 \oplus \dim(X/U)\,.
\end{aligned}$$

To prove that $\dim_{\mathrm{cl}}(X/U) = \dim(X/U)$, let I be a cl-basis of X over U. Then $\dim(I/U) \leq \dim(X/U) \leq \dim(\mathrm{cl}(I)/U) = \dim(I/U)$ implies that

$$\dim_{\mathrm{cl}}(X/U) = |I| = \dim(I/U) = \dim(X/U)\,. \qquad \square$$

Note that it follows from Proposition 2.6 that a dimension function is uniquely determined by the set of all pairs A, U such that $\dim(A/U) = 0$.

Corollary 2.7. *Let $d, d' : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \to \mathrm{Cn}$ be two geometric dimension functions. If*

$$d(A/U) = 0 \quad \text{iff} \quad d'(A/U) = 0\,, \quad \textit{for all } A, U \subseteq \Omega\,,$$

then $d = d'$.

Proof. According to Proposition 2.6, we can associate with d and d' matroids $\langle \Omega, c\rangle$ and $\langle \Omega, c'\rangle$, respectively. Since $d(A/U) = 0$ if, and only if, $d'(A/U) = 0$, it follows that $c = c'$. Hence,

$$d = \dim_c = \dim_{c'} = d'\,. \qquad \square$$

3. Modular geometries

There is a general construction turning an arbitrary matroid into a geometry.

Definition 3.1. Let $\langle \Omega, \mathrm{cl}\rangle$ be a matroid and $U \subseteq \Omega$. The *localisation* of $\langle \Omega, \mathrm{cl}\rangle$ at U is the matroid $\langle \Omega, \mathrm{cl}\rangle_{(U)} := \langle \Omega_{(U)}, \mathrm{cl}_{(U)}\rangle$ where

$$\begin{aligned}
\Omega_{(U)} &:= \{ \mathrm{cl}(U \cup \{a\}) \mid a \in \Omega \smallsetminus \mathrm{cl}(U) \}\,, \\
\mathrm{cl}_{(U)}(X) &:= \{ L \in \Omega_{(U)} \mid L \subseteq \mathrm{cl}(U \cup \textstyle\bigcup X) \}\,.
\end{aligned}$$

Lemma 3.2. *Every localisation of a matroid is a geometry.*

Exercise 3.1. Prove the preceding lemma.

Definition 3.3. Let $\mathfrak{V}$ be a vector space over a skew field $\mathfrak{S}$. The *projective geometry* associated with $\mathfrak{V}$ is the localisation $\langle V, \mathrm{cl}\rangle_{(0)}$ of the linear matroid at the subspace $\{0\}$.

Remark. This coincides with the usual definition of a projective space: the points are the lines $L \subseteq V$ through the origin.

Lemma 3.4. *Let $\langle \Omega, \mathrm{cl}\rangle$ be a matroid, $U, X \subseteq \Omega$ sets, and $\langle \Omega_{(U)}, \mathrm{cl}_{(U)}\rangle$ the localisation at U. Let*

$$X_{(U)} := \{ \mathrm{cl}(U \cup \{x\}) \mid x \in X \smallsetminus \mathrm{cl}(U) \}$$

be the image of X in $\Omega_{(U)}$.

$$\dim_{\mathrm{cl}}(X/U) = \dim_{\mathrm{cl}_{(U)}}(X_{(U)})\,.$$

Proof. Let I be a basis of X over U. Then $I \cap \mathrm{cl}(U) = \varnothing$. Hence, if we can show that

$$I_{(U)} := \{ \mathrm{cl}(U \cup \{a\}) \mid a \in I \}$$

is a basis of $X_{(U)}$, then $|I_{(U)}| = |I|$ and the claim follows.

For $x \in X$, let $L_x := \mathrm{cl}(U \cup \{x\})$. To show that $I_{(U)}$ is independent, suppose that there is some $a \in I$ such that

$$\begin{aligned}
L_a &\in \mathrm{cl}_{(U)}(I_{(U)} \smallsetminus \{L_a\}) \\
&= \{ L \in \Omega_{(U)} \mid L \subseteq \mathrm{cl}\big(U \cup \textstyle\bigcup (I_{(U)} \smallsetminus \{L_a\})\big) \}\,.
\end{aligned}$$

Since $a \in L_a$ it follows that

$$a \in \mathrm{cl}\big(U \cup \bigcup(I_{(U)} \smallsetminus \{L_a\})\big) \subseteq \mathrm{cl}(U \cup (I \smallsetminus \{a\}))\,.$$

Hence, I is not independent over U. A contradiction.

It remains to show that $X_{(U)} \subseteq \mathrm{cl}_{(U)}(I_{(U)})$. Let $L_x \in X_{(U)}$. Then

$$U \cup \{x\} \subseteq U \cup X \subseteq \mathrm{cl}(U \cup I) \quad \text{implies} \quad L_x \in \mathrm{cl}_{(U)}(I_{(U)})\,. \qquad \square$$

Some special properties of affine and projective geometries are worth singling out.

Definition 3.5. Let $\langle \Omega, \mathrm{cl}\rangle$ be a matroid.

(a) $\langle \Omega, \mathrm{cl}\rangle$ is *modular* if the lattice $\langle \mathrm{fix}\,\mathrm{cl}, \subseteq\rangle$ of its closed sets is modular. The matroid is *locally modular* if all of its localisations at a single point $a \in \Omega$ are modular.

(b) $\langle \Omega, \mathrm{cl}\rangle$ is *disintegrated* if $\mathrm{cl}(X) = X$, for all $X \subseteq \Omega$.

(c) $\langle \Omega, \mathrm{cl}\rangle$ is *locally finite* if the closure of every finite set is finite.

(d) A *morphism* between matroids is a continuous function between the corresponding closure spaces.

(e) $\langle \Omega, \mathrm{cl}\rangle$ is *homogeneous* if, for every finite set $U \subseteq \Omega$ and all $a, b \in \Omega \smallsetminus \mathrm{cl}(U)$, there is an isomorphism $\pi : \Omega \to \Omega$ with $\pi \restriction \mathrm{cl}(U) = \mathrm{id}$ and $\pi(a) = b$.

We have defined modularity of a matroid in terms of the corresponding lattice of closed sets. The next lemma lists some equivalent conditions on the matroid itself.

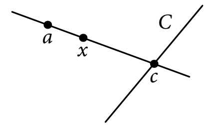
Lemma 3.6. *Let $\langle \Omega, \mathrm{cl}\rangle$ be a matroid. The following statements are equivalent:*

(1) *$\langle \Omega, \mathrm{cl}\rangle$ is modular.*

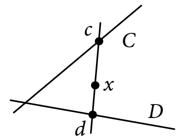
(2) *For all finite $X, Y \subseteq \Omega$, we have*

$$\dim_{\mathrm{cl}}(X \cap Y) + \dim_{\mathrm{cl}}(X \cup Y) = \dim_{\mathrm{cl}}(X) + \dim_{\mathrm{cl}}(Y)\,.$$

(3) *For all closed sets $C \subseteq \Omega$ and every pair of elements $a, x \in \Omega$ with $x \in \mathrm{cl}(C \cup \{a\})$, there exists an element $c \in C$ with $x \in \mathrm{cl}(\{a, c\})$.*



(4) *For all closed sets $C, D \subseteq \Omega$ and every element $x \in \mathrm{cl}(C \cup D)$, there exist elements $c \in C$ and $d \in D$ with $x \in \mathrm{cl}(\{c, d\})$.*



Proof. (1) $\Rightarrow$ (2) We have $\dim_{\mathrm{cl}}(X) = \dim_{\mathrm{cl}}(\mathrm{cl}(X))$ and the latter dimension coincides with the height of $\mathrm{cl}(X)$ in the lattice $\langle \mathrm{fix}\,\mathrm{cl}, \subseteq\rangle$. Consequently, the equation follows from the modular law (Theorem B2.5.5).

(2) $\Rightarrow$ (3) If $a \in C$, we can take $c := x$ and, if $x \in \mathrm{cl}(a)$, we can take an arbitrary $c \in C$. Hence, suppose that $a \notin C \cup \mathrm{cl}(a)$ and choose a finite set $C_0 \subseteq C$ with $x \in \mathrm{cl}(C_0 \cup \{a\})$. Then (2) implies that

$$\begin{aligned}\dim(C_0 \cap \mathrm{cl}(a, x)) &= \dim(C_0) + \dim(a, x) - \dim(C_0 \cup \{a, x\})\\ &= \dim(C_0) + 2 - (\dim(C_0) + 1) = 1\,.\end{aligned}$$

Hence, there is some $c \in C_0 \cap \mathrm{cl}(a, x)$. By the exchange property it follows that $x \in \mathrm{cl}(a, c)$, as desired.

(3) $\Rightarrow$ (4) Since cl has finite character, there are finite sets $C_0 \subseteq C$ and $D_0 \subseteq D$ such that $x \in \mathrm{cl}(C_0 \cup D_0)$. We prove the claim by induction on $|C_0|$. If $C_0 = \varnothing$ then $x \in \mathrm{cl}(D_0) \subseteq D$ and we are done. Suppose that $C_0 = A \cup \{a\}$. Since $x \in \mathrm{cl}(A \cup D_0 \cup \{a\})$, we can use (3) to find some $b \in \mathrm{cl}(A \cup D_0)$ with $x \in \mathrm{cl}(\{a, b\})$. By inductive hypothesis, there are $a' \in A$ and $d \in D_0$ such that $b \in \mathrm{cl}(\{a', d\})$. Hence, $x \in \mathrm{cl}(\{a, a', d\})$ and, applying (3) again, we can find some $c \in \mathrm{cl}(\{a, a'\}) \subseteq C$ with $x \in \mathrm{cl}(\{c, d\})$.

(4) ⇒ (1) Let $A, B, C \subseteq \Omega$ be closed sets with $A \subseteq B$. We have to show that $\mathrm{cl}(A \cup (B \cap C)) = B \cap \mathrm{cl}(A \cup C)$. According to Lemma B2.2.6, one inclusion holds in every lattice. For the other one, let $x \in B \cap \mathrm{cl}(A \cup C)$. By (4) there are elements $a \in A$ and $c \in C$ with $x \in B \cap \mathrm{cl}(\{a, c\})$. If $x \in \mathrm{cl}(a)$ then $x \in A$ and we are done. Hence, suppose that $x \notin \mathrm{cl}(a)$. By the exchange property, it then follows that $c \in \mathrm{cl}(\{a, x\}) \subseteq \mathrm{cl}(A \cup B) = B$. Hence, $c \in B \cap C$ and $x \in \mathrm{cl}(\{a, c\}) \subseteq \mathrm{cl}(A \cup (B \cap C))$. □

Disintegrated, projective, and affine geometries frequently appear in model theory. The next lemma lists some of their properties.

Lemma 3.7. *Disintegrated geometries and projective geometries are modular and homogeneous. Affine geometries are locally modular and homogeneous, but not modular if the dimension is at least* 3.

Proof. To show that a disintegrated geometry $\langle \Omega, \mathrm{cl} \rangle$ is modular, one only has to check that

$$X \subseteq Y \quad \text{implies} \quad X \cup (Y \cap Z) = Y \cap (X \cup Z)\,.$$

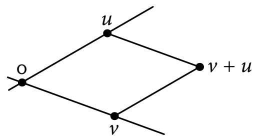
To show that it is homogeneous, let $U \subseteq \Omega$ and $a, b \in \Omega \setminus U$. The bijection $h : \Omega \to \Omega$ exchanging a and b and fixing every other element of Ω is continuous.

Suppose that $\langle \Omega, \mathrm{cl} \rangle$ is the projective geometry associated with a vector space $\mathfrak{V}$. Modularity follows from Lemma B6.4.5. For homogeneity, let $U \subseteq \Omega$ be finite and $a, b \notin \mathrm{cl}(U)$. Let $\langle V, \mathrm{cl}_\wedge \rangle$ be the corresponding linear matroid. For every element $x \in \Omega$ there is a non-zero vector $\hat{x} \in V$ such that $x = \mathrm{cl}_\wedge(\hat{x})$. Fix a basis B of $\hat{U} := \mathrm{cl}_\wedge(\{\, \hat{x} \mid x \in U \,\})$. Since $\hat{a}, \hat{b} \notin \hat{U}$, there exists a linear map $h : V \to V$ fixing B and interchanging $\hat{a}$ and $\hat{b}$. The function $\Omega \to \Omega$ induced by h is the desired continuous mapping.

Suppose that $\langle \Omega, \mathrm{cl} \rangle$ is the affine geometry associated with a vector space $\mathfrak{V}$ and let $a \in \Omega$. Then $\langle \Omega, \mathrm{cl} \rangle_{(a)} \cong \langle \Omega, \mathrm{cl} \rangle_{(0)}$ and the latter geometry is isomorphic to the projective geometry associated with $\mathfrak{V}$. Since we have just seen that such geometries are modular, it follows that $\langle \Omega, \mathrm{cl} \rangle$ is locally modular.

To show that it is not modular let $u, v \in V$ be linearly independent vectors. Then $\mathrm{cl}(0) \subseteq \mathrm{cl}(0, u)$ but

$$\begin{aligned}
\mathrm{cl}\big(\mathrm{cl}(0) \cup (\mathrm{cl}(0, u) \cap \mathrm{cl}(v, v + u))\big) &= \mathrm{cl}(\mathrm{cl}(0) \cup \varnothing) \\
&= \mathrm{cl}(0)\,, \\
\text{and} \qquad \mathrm{cl}(0, u) \cap \mathrm{cl}\big(\mathrm{cl}(0) \cup \mathrm{cl}(v, v + u)\big) &= \mathrm{cl}(0, u) \cap \mathrm{cl}(0, u, v) \\
&= \mathrm{cl}(0, u)\,.
\end{aligned}$$



For homogeneity, let $U \subseteq \Omega$ be finite and $a, b \notin \mathrm{cl}(U)$ distinct elements. If $U = \varnothing$ and a and b are both non-zero, we can take some linear map $h : V \to V$ interchanging a and b. This map is continuous.

If $U = \varnothing$ and $a = 0$, we first apply a translation f that maps both a and b to non-zero vectors. Then we can use a linear map h as above. The composition $f^{-1} \circ h \circ f$ is the desired continuous map.

Note that there is one case where such a translation f does not exists. If $\mathfrak{V}$ has only two elements. Then $V = \{a, b\}$ and the function interchanging a and b is continuous.

It remains to consider the case that $U \neq \varnothing$. Fix some $x \in U$. By applying a suitable translation f, we can assume that $x = 0 \in U$. Hence, $\mathrm{cl}(U)$ is a linear subspace of $\mathfrak{V}$. Let B be a basis of $\mathrm{cl}(U)$ and let $h : V \to V$ be a linear map fixing B and interchanging a and b. Then $f^{-1} \circ h \circ f$ is the desired continuous map. □

Algebraically closed fields provide examples of geometries that are not locally modular.

Proposition 3.8. *Let $\mathfrak{K}$ be an algebraically closed field of infinite transcendence degree and let $\langle K, \mathrm{cl} \rangle$ be the matroid where* cl *maps a set $X \subseteq K$ to its algebraic closure.*

(a) $\langle K, \mathrm{cl}\rangle$ *is homogeneous.*

(b) *No localisation of* $\langle K, \mathrm{cl}\rangle$ *at a finite set is modular.*

Proof. (a) follows by Corollary B6.5.31.

(b) We consider the localisation $\langle K, \mathrm{cl}\rangle_{(U)}$ at a finite set $U \subseteq K$. Let $n := \dim_{\mathrm{cl}}(U)$. Since $\mathfrak{K}$ has infinite transcendence degree, there are elements a, b, c, d that are algebraically independent over U. Set $x := (a-c)/(b-d)$ and $y := a - bx$, and let

$$A := \mathrm{cl}(a, b, U) \quad \text{and} \quad B := \mathrm{cl}(x, y, U)\,.$$

Then $\mathrm{cl}(A \cup B) = \mathrm{cl}(a, b, x, U)$ has dimension $n+3$, while A and B both have dimension $n+2$. To show that $\langle K, \mathrm{cl}\rangle_{(U)}$ is not modular it is sufficient to prove that the dimension of $A \cap B$ is different from $n+1$.

In fact, we claim that $A \cap B = \mathrm{cl}(U)$ and, hence, the dimension is n. Clearly, we have $U \subseteq A \cap B$. Conversely, consider an element $z \in A \cap B$. By (a), there exists an automorphism $\pi \in \mathrm{Aut}\,\mathfrak{K}$ that fixes B pointwise and maps a tø c. It follows that $\pi(b) = \pi((a-y)/x) = (c-y)/x = d$. Consequently, $z \in B$ implies $\pi(z) = z$, and $z \in A = \mathrm{cl}(a, b, U)$ implies $z = \pi(z) \in \mathrm{cl}(c, d, U)$. Thus,

$$z \in \mathrm{cl}(a, b, U) \cap \mathrm{cl}(c, d, U) = \mathrm{cl}(U)\,. \qquad \square$$

We conclude this section with the following characterisation of homogeneous, locally finite geometries.

Theorem 3.9 (Cherlin, Mills, Zil'ber). *Let* $\langle \Omega, \mathrm{cl}\rangle$ *be a homogeneous, locally finite geometry of infinite dimension. Then exactly one of the following cases holds:*

(1) $\langle \Omega, \mathrm{cl}\rangle$ *is disintegrated.*

(2) $\langle \Omega, \mathrm{cl}\rangle$ *is isomorphic to a projective geometry over a finite field.*

(3) $\langle \Omega, \mathrm{cl}\rangle$ *is isomorphic to an affine geometry over a finite field.*

4. Strongly minimal sets

Having introduced geometries we are interested in first-order theories where the algebraic closure operator forms such a geometry.

Definition 4.1. Let $\mathfrak{M}$ be a structure and $S \subseteq M^n$ an infinite M-definable relation.

(a) We call S *minimal* if, for every M-definable subset $X \subseteq S$, either X, or $S \setminus X$ is finite. A formula $\varphi(\bar{x}; \bar{c})$ with $\bar{c} \subseteq M$ is *minimal* if the relation $\varphi(\bar{x}; \bar{c})^{\mathfrak{M}}$ it defines is minimal.

(b) A relation S, or a formula $\varphi(\bar{x}; \bar{c})$, is *strongly minimal,* if it is minimal in every elementary extension of $\mathfrak{M}$.

Example. (a) Let $\mathfrak{E} = \langle E, \sim\rangle$ be a structure where $\sim$ is an equivalence relation with infinitely many classes each of which is infinite. For every $a \in E$, the formula $x \sim a$ is strongly minimal.

(b) Let $\mathfrak{K}$ be an algebraically closed field. Every definable set $X \subseteq K$ is a boolean combination of solution sets of polynomials. Hence, every such set is either finite or cofinite. Therefore, K is strongly minimal.

(c) In $\mathfrak{A} = \langle \omega, \leq\rangle$ the set ω is minimal, but not strongly minimal since, in every elementary extension $\mathfrak{B} \succ \mathfrak{A}$ we can pick an element $c \in B \setminus \omega$ such that $(x \leq c)^{\mathfrak{B}}$ and $(x > c)^{\mathfrak{B}}$ are both infinite.

We are mainly interested in strongly minimal relations. As the next lemma shows, we can find such a relation by looking for a minimal relation in an $\aleph_0$-saturated structure.

Lemma 4.2. *Every minimal relation in an* $\aleph_0$*-saturated structure* $\mathfrak{M}$ *is strongly minimal.*

Proof. Let $\varphi(\bar{x}; \bar{c})$ be a minimal formula with parameters $\bar{c} \subseteq M$. To show that φ is strongly minimal we consider an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ and a formula $\psi(\bar{x}; \bar{d})$ with parameters $\bar{d} \subseteq N$. For a contradiction, suppose that both sets

$$\varphi(\bar{x}; \bar{c})^{\mathfrak{N}} \cap \psi(\bar{x}; \bar{d})^{\mathfrak{N}} \quad \text{and} \quad \varphi(\bar{x}; \bar{c})^{\mathfrak{N}} \setminus \psi(\bar{x}; \bar{d})^{\mathfrak{N}}$$

are infinite.

Since $\mathfrak{M}$ is $\aleph_0$-saturated we can find a tuple $\bar{d}' \subseteq M$ with $\operatorname{tp}(\bar{d}'/\bar{c}) = \operatorname{tp}(\bar{d}/\bar{c})$. For every $n < \omega$, we have

$$\mathfrak{N} \vDash \exists^n \bar{x}[\varphi(\bar{x};\bar{c}) \wedge \psi(\bar{x};\bar{d})] \wedge \exists^n \bar{x}[\varphi(\bar{x};\bar{c}) \wedge \neg\psi(\bar{x};\bar{d})]$$

which implies that

$$\mathfrak{N} \vDash \exists^n \bar{x}[\varphi(\bar{x};\bar{c}) \wedge \psi(\bar{x};\bar{d}')] \wedge \exists^n \bar{x}[\varphi(\bar{x};\bar{c}) \wedge \neg\psi(\bar{x};\bar{d}')] .$$

It follows that all these formulae also hold in $\mathfrak{M}$. Consequently, both sets $\varphi(\bar{x};\bar{c})^{\mathfrak{M}} \cap \psi(\bar{x};\bar{d}')^{\mathfrak{M}}$ and $\varphi(\bar{x};\bar{c})^{\mathfrak{M}} \setminus \psi(\bar{x};\bar{d}')^{\mathfrak{M}}$ are infinite. A contradiction. □

The reason for studying strongly minimal sets is the fact that the algebraic closure operator has the exchange property for these sets.

Lemma 4.3. *Let $\mathfrak{M}$ be a structure and $S \subseteq M^n$ a minimal set. The restriction of* acl *to S forms a matroid.*

Proof. We have already seen in Lemma E2.1.2 that acl is a finitary closure operator. Hence, it remains to check that it has the exchange property.

Suppose that $\bar{a} \subseteq \operatorname{acl}(U \cup \bar{b}) \setminus \operatorname{acl}(U)$ for $\bar{a}, \bar{b} \in S$. We have to show that $\bar{b} \subseteq \operatorname{acl}(U \cup \bar{a})$. There exists a formula $\varphi(\bar{x};\bar{y})$ over U such that $\varphi^{\mathfrak{M}}(\bar{x};\bar{b})$ is a finite set containing $\bar{a}$. Set $m := |\varphi^{\mathfrak{M}}(\bar{x};\bar{b})|$ and let $\psi(\bar{y})$ be the formula stating that there are exactly m tuples $\bar{x} \in S$ such that $\mathfrak{M} \vDash \varphi(\bar{x};\bar{y})$. If $\psi^{\mathfrak{M}}(\bar{y})$ is finite, $\mathfrak{M} \vDash \psi(\bar{b})$ implies that $\bar{b} \subseteq \operatorname{acl}(U)$. Consequently, we have $\bar{a} \subseteq \operatorname{acl}(U)$. A contradiction.

Hence, the set $\psi^{\mathfrak{M}}(\bar{y})$ is infinite. If $(\varphi(\bar{a};\bar{y}) \wedge \psi(\bar{y}))^{\mathfrak{M}}$ is finite then $\bar{b} \subseteq \operatorname{acl}(U \cup \bar{a})$ and we are done. For a contradiction, suppose that this set if infinite. Since S is minimal it follows that the complement $S \cap \neg(\varphi(\bar{a};\bar{y}) \wedge \psi(\bar{y}))^{\mathfrak{M}}$ is finite. Let $k < \aleph_0$ be its cardinality and let $\vartheta(\bar{x})$ be the formula stating that there are exactly k elements $\bar{y} \in S$ that do not satisfy $\varphi(\bar{x};\bar{y}) \wedge \psi(\bar{y})$. If $\vartheta(\bar{x})^{\mathfrak{M}}$ is finite then $\bar{a} \subseteq \operatorname{acl}(U)$. A contradiction.

Hence, $\vartheta(\bar{x})^{\mathfrak{M}}$ is infinite and we can choose $m + 1$ distinct elements $\bar{a}_0, \dots, \bar{a}_m \in \vartheta(\bar{x})^{\mathfrak{M}}$. The set

$$B := \bigcap_{i \leq m} [\varphi(\bar{a}_i;\bar{y}) \wedge \psi(\bar{y})]^{\mathfrak{M}}$$

is a finite intersection of cofinite sets and, therefore, cofinite itself. In particular, it is nonempty and we can find some element $\bar{b}^* \in B$. It follows that

$$\mathfrak{M} \vDash \varphi(\bar{a}_i;\bar{b}^*) , \quad \text{for all } i \leq m .$$

Consequently, $|\varphi^{\mathfrak{M}}(\bar{x};\bar{b}^*)| > m$. But this implies that $\mathfrak{M} \nvDash \psi(\bar{b}^*)$. A contradiction. □

The geometry of a strongly minimal relation is closely related to its logical properties. For instance, we shall show below that all independent sets are totally indiscernible with the same type. But first, let us collect some technical properties of strongly minimal relations.

Lemma 4.4. *Let $\varphi(\bar{x};\bar{c})$ be a strongly minimal formula with parameters $\bar{c}$. Let $\bar{s}$ be the sorts of the variables $\bar{x}$.*

(a) *If $\bar{d}$ is a tuple with $\operatorname{tp}(\bar{d}) = \operatorname{tp}(\bar{c})$ then $\varphi(\bar{x};\bar{d})$ is also strongly minimal.*

(b) *For every set $U \supseteq \bar{c}$, there exists a unique nonalgebraic type $\mathfrak{p} \in S^{\bar{s}}(U)$ with $\varphi \in \mathfrak{p}$.*

Proof. (a) For every formula $\psi(\bar{x};\bar{a})$ with parameters $\bar{a} \subseteq \mathbb{M}$, we have to show that exactly one of

$$(\varphi(\bar{x};\bar{d}) \wedge \psi(\bar{x};\bar{a}))^{\mathbb{M}} \quad \text{and} \quad (\varphi(\bar{x};\bar{d}) \wedge \neg\psi(\bar{x};\bar{a}))^{\mathbb{M}}$$

is finite. Since $\operatorname{tp}(\bar{d}) = \operatorname{tp}(\bar{c})$ there is an automorphism π of $\mathbb{M}$ with $\pi(\bar{d}) = \bar{c}$. Let $\bar{b} := \pi(\bar{a})$. As $\varphi(\bar{x};\bar{c})$ is strongly minimal, exactly one of

$$(\varphi(\bar{x};\bar{c}) \wedge \psi(\bar{x};\bar{b}))^{\mathbb{M}} \quad \text{and} \quad (\varphi(\bar{x};\bar{c}) \wedge \neg\psi(\bar{x};\bar{b}))^{\mathbb{M}}$$

is finite. Since

$$\pi\big[(\varphi(\bar{x};\bar{d}) \wedge \psi(\bar{x};\bar{a}))^{\mathbb{M}}\big] = (\varphi(\bar{x};\bar{c}) \wedge \psi(\bar{x};\bar{b}))^{\mathbb{M}},$$
$$\text{and}\quad \pi\big[(\varphi(\bar{x};\bar{d}) \wedge \neg\psi(\bar{x};\bar{a}))^{\mathbb{M}}\big] = (\varphi(\bar{x};\bar{c}) \wedge \neg\psi(\bar{x};\bar{b}))^{\mathbb{M}},$$

the claim follows.

(b) Let $\mathfrak{M}$ be an $\aleph_0$-saturated model containing U and set

$$\mathfrak{p} := \big\{\, \psi \,\big|\, \psi \text{ a formula over } U \text{ such that } (\varphi \wedge \psi)^{\mathfrak{M}} \text{ is infinite} \,\big\}.$$

Since φ is strongly minimal, it follows that

$$\psi \in \mathfrak{p} \quad\text{iff}\quad \neg\psi \notin \mathfrak{p}, \quad \text{for every formula } \psi \text{ over U}.$$

Hence, $\mathfrak{p}$ is a complete type over U containing φ. Clearly, $\mathfrak{p}$ is nonalgebraic since, if there were some algebraic formula $\psi \in \mathfrak{p}$, then $\varphi \wedge \psi$ were also algebraic, in contradiction to the definition of $\mathfrak{p}$.

Suppose that $\mathfrak{q} \in S^{\bar{s}}(U)$ is another nonalgebraic type containing φ. To show that $\mathfrak{q} \subseteq \mathfrak{p}$, consider $\psi \in \mathfrak{q}$. Then $\varphi \wedge \psi \in \mathfrak{q}$ and, by assumption, this formula is nonalgebraic. By definition of $\mathfrak{p}$ it follows that $\psi \in \mathfrak{p}$. □

Lemma 4.5. *Let $\varphi(\bar{x})$ be a strongly minimal formula over a set U of parameters. Let $\bar{s}$ be the sorts of the variables $\bar{x}$, and let $\mathfrak{p} \in S^{\bar{s}}(U)$ be the unique nonalgebraic type containing φ.*

(a) *$\mathfrak{p}$ is isolated if, and only if, $\varphi^{\mathbb{M}}$ contains only finitely many tuples in* $\mathrm{acl}(U)$.

(b) *Let $V \supseteq U$ and let $\mathfrak{q} \in S^{\bar{s}}(V)$ be the unique nonalgebraic extension of $\mathfrak{p}$. If $\mathfrak{p}$ is isolated, so is $\mathfrak{q}$.*

Proof. (a) Let $R := \{\, \bar{a} \in \varphi^{\mathbb{M}} \mid \bar{a} \subseteq \mathrm{acl}(U) \,\}$. For ($\Leftarrow$), suppose that $R = \{\bar{a}_0, \dots, \bar{a}_{n-1}\}$ is finite. For each $i < n$, we fix an algebraic formula ψ_i over U such that $\mathbb{M} \vDash \psi_i(\bar{a}_i)$. It follows that $\psi := \bigvee_{i<n}[\psi_i \wedge \varphi]$ is a formula over U defining R. We claim that $\varphi \wedge \neg\psi$ isolates $\mathfrak{p}$.

Since $\mathfrak{p}$ is nonalgebraic, we have $\psi \notin \mathfrak{p}$. Therefore, $\varphi \wedge \neg\psi \in \mathfrak{p}$. Conversely, let $\mathfrak{q}$ be an arbitrary complete type over U containing $\varphi \wedge \neg\psi$. If $\mathfrak{q}$ is nonalgebraic, it coincides with $\mathfrak{p}$, by Lemma 4.4 (b), and we are done. Therefore, we may assume that $\mathfrak{q}$ contains an algebraic formula ϑ. Then each of the finitely many realisations of $\mathfrak{q}$ is in $\mathrm{acl}(U)$. Consequently, $\mathfrak{q}^{\mathbb{M}} \subseteq R$, which implies that $\psi \in \mathfrak{q}$. A contradiction.

($\Rightarrow$) For a contradiction, suppose that there is some $\psi(\bar{x}) \in \mathfrak{p}$ isolating $\mathfrak{p}$, but R is infinite. Let Γ be the set of all algebraic formulae over U. As $\mathfrak{p}$ is the unique nonalgebraic type in $S^{\bar{s}}(U)$ containing φ, the set

$$\{\varphi \wedge \neg\psi\} \cup \{\, \neg\vartheta \mid \vartheta \in \Gamma \,\}$$

is inconsistent. Hence, there are finitely many formula $\vartheta_0, \dots, \vartheta_{n-1} \in \Gamma$ such that

$$T(U) \cup \{\varphi, \neg\vartheta_0, \dots, \neg\vartheta_{n-1}\} \vDash \psi.$$

Since R is infinite and all ϑ_i are algebraic, there is some element

$$\bar{a} \in R \smallsetminus (\vartheta_0^{\mathbb{M}} \cup \dots \cup \vartheta_{n-1}^{\mathbb{M}}) \subseteq (\varphi \wedge \neg\vartheta_0 \wedge \dots \wedge \neg\vartheta_{n-1})^{\mathbb{M}} \subseteq \psi^{\mathbb{M}}.$$

But $\mathrm{tp}(\bar{a}/U) \neq \mathfrak{p}$ since the former type is algebraic, while the latter one is not. Consequently, ψ does not isolate $\mathfrak{p}$. A contradiction.

(b) follows immediately from (a). □

Proposition 4.6. *Let $\mathfrak{M}$ be a structure, $U \subseteq M$, and suppose that $S \subseteq M^k$ a U-definable minimal relation. If $\bar{a}, \bar{b} \in S^n$ are finite tuples each of which is independent over U, then*

$$\mathrm{tp}(\bar{a}/U) = \mathrm{tp}(\bar{b}/U).$$

Proof. We prove the claim by induction on n. For $n = 0$ there is nothing to do. Suppose that we have already proved the claim for n-tuples and let $\bar{a}c \in S^{n+1}$ and $\bar{b}d \in S^{n+1}$ be both independent over U. By inductive hypothesis, we have $\mathrm{tp}(\bar{a}/U) = \mathrm{tp}(\bar{b}/U)$. Let $\psi(\bar{x}, y)$ be a formula over U such that

$$\mathfrak{M} \vDash \psi(\bar{a}, c).$$

Since $c \notin \operatorname{acl}(U \cup \bar{a})$ it follows that the set $S \cap \psi(\bar{a}, y)^{\mathfrak{M}}$ is infinite and its complement $S \smallsetminus \psi(\bar{a}, y)^{\mathfrak{M}}$ is finite. Furthermore, $\operatorname{tp}(\bar{a}/U) = \operatorname{tp}(\bar{b}/U)$ implies that

$$\big|S \smallsetminus \psi(\bar{b}, y)^{\mathfrak{M}}\big| = \big|S \smallsetminus \psi(\bar{a}, y)^{\mathfrak{M}}\big| < \aleph_0 .$$

Hence, $d \notin \operatorname{acl}(U \cup \bar{b})$ implies that $\mathfrak{M} \vDash \psi(\bar{b}, d)$. □

Corollary 4.7. *Let $\mathfrak{M}$ be a structure, $U \subseteq M$ a set of parameters, and $S \subseteq M$ a U-definable minimal set. Every U-independent set $A \subseteq S$ is totally indiscernible over U.*

Proof. Let $\bar{a}, \bar{b} \in [A]^n$. Then $\bar{a}$ and $\bar{b}$ are U-independent and, therefore, they have the same type over U. □

We have seen that we can use geometric methods to study models containing minimal sets. Let us turn to prove the existence of minimal sets.

Lemma 4.8. *Let T be a $\aleph_0$-stable theory over a countable signature Σ, $\mathfrak{M} \vDash T$ infinite, $\vartheta(\bar{x})$ a formula over M, and let $\kappa \leq |\vartheta^{\mathfrak{M}}|$ be an infinite cardinal. There exists a formula $\varphi(\bar{x})$ over M such that $\varphi^{\mathfrak{M}} \subseteq \vartheta^{\mathfrak{M}}$, $|\varphi^{\mathfrak{M}}| \geq \kappa$ and, for every formula $\psi(\bar{x})$ over M, we either have*

$$\big|(\varphi \wedge \psi)^{\mathfrak{M}}\big| < \kappa \quad \text{or} \quad \big|(\varphi \wedge \neg\psi)^{\mathfrak{M}}\big| < \kappa .$$

Proof. For a contradiction, suppose that there is no such φ. We construct a family $(\varphi_w)_{w \in 2^{<\omega}}$ of formulae over M such that, for all $w \in 2^{<\omega}$, we have

$$\varphi_w^{\mathfrak{M}} \subseteq \vartheta^{\mathfrak{M}}, \quad \big|\varphi_w^{\mathfrak{M}}\big| \geq \kappa \quad \text{and} \quad \varphi_{w0}^{\mathfrak{M}} \cap \varphi_{w1}^{\mathfrak{M}} = \varnothing .$$

We start with $\varphi_{\langle\rangle} := \vartheta$. Then $\varphi_{\langle\rangle}^{\mathfrak{M}} = \vartheta^{\mathfrak{M}}$ and $\big|\varphi_{\langle\rangle}^{\mathfrak{M}}\big| \geq \kappa$. For the inductive step, suppose that we have already defined φ_w. By assumption, there is a formula ψ over M such that

$$\big|(\varphi_w \wedge \psi)^{\mathfrak{M}}\big| \geq \kappa \quad \text{and} \quad \big|(\varphi_w \wedge \neg\psi)^{\mathfrak{M}}\big| \geq \kappa .$$

We set $\varphi_{w0} := \varphi_w \wedge \psi$ and $\varphi_{w1} := \varphi_w \wedge \neg\psi$.

Having defined $(\varphi_w)_w$, let $U \subseteq M$ be the set of all parameters appearing in some φ_w. Then U is countable and the family $(\varphi_w)_{w \in 2^{<\omega}}$ is an embedding of $2^{<\omega}$ into $\mathrm{FO}^{\bar{s}}[\Sigma_U]/T$, where $\bar{s}$ are the sorts of $\bar{x}$. By Lemma B5.7.3, it follows that $|S^{\bar{s}}(U)| > \aleph_0$. A contradiction to $\aleph_0$-stability. □

Corollary 4.9. *Let T be a $\aleph_0$-stable theory over a countable signature Σ. Every infinite model of T contains a minimal relation.*

Proof. This follows from the preceding lemma for $\vartheta(x) :=$ true and $\kappa := \aleph_0$. □

We can use minimal sets to define isomorphisms between models.

Lemma 4.10. *Every elementary function $f_0 : A \to B$ can be extended to a elementary function $f : \operatorname{acl}(A) \to \operatorname{acl}(\operatorname{rng} f_0)$ that is bijective.*

Proof. W.l.o.g. we may assume that $B = \operatorname{rng} f_0$. Let F be the set of all elementary functions $g : C \to D$ such that $A \subseteq C \subseteq \operatorname{acl}(A)$ and $g \restriction A = f_0$. Then $\langle F, \subseteq \rangle$ is inductively ordered. Hence, it has a maximal element $f : C \to D$. We claim that f is the desired function.

First of all, every elementary function is injective. For surjectivity, suppose that $b \in \operatorname{acl}(B) \smallsetminus D$. Since $b \in \operatorname{acl}(D)$, we can use Lemma E3.1.3 to find a formula $\varphi(x; \bar{d})$ with parameters $\bar{d} \subseteq D$ isolating $\operatorname{tp}(b/D)$. Since $\operatorname{tp}(b/D)$ is algebraic, φ must be an algebraic formula. Fixing $\bar{c} \subseteq C$ such that $f(\bar{c}) = \bar{d}$ it follows that

$$f[\varphi(x; \bar{c})^{\mathbb{M}}] \subseteq \varphi(x; \bar{d})^{\mathbb{M}} \quad \text{and} \quad \big|\varphi(x; \bar{c})^{\mathbb{M}}\big| = \big|\varphi(x; \bar{d})^{\mathbb{M}}\big| .$$

Consequently, there exists some element $a \in \varphi(x; \bar{c})^{\mathbb{M}} \smallsetminus C$. Furthermore, $\varphi(x; \bar{c})$ isolates $\operatorname{tp}(a/C)$. Hence, $f[\operatorname{tp}(a/C)] = \operatorname{tp}(b/D)$ and we have $f \cup \{\langle a, b\rangle\} \in F$. This contradicts the maximality of f.

It remains to prove that $C = \operatorname{acl}(A)$. Suppose that there exists an element $a \in \operatorname{acl}(A) \smallsetminus C$. Then $\operatorname{tp}(a/C)$ is isolated and, as above, we can find an element b such that $f \cup \{\langle a, b\rangle\} \in F$. Again a contradiction. □

Corollary 4.11. *Let T be a theory, $\varphi(x)$ a strongly minimal formula, and $\mathfrak{A}$ and $\mathfrak{B}$ models of T. If $\dim(\varphi^{\mathfrak{A}}) = \dim(\varphi^{\mathfrak{B}})$, there exists a bijective elementary map $f : \mathrm{acl}(\varphi^{\mathfrak{A}}) \to \mathrm{acl}(\varphi^{\mathfrak{B}})$.*

Proof. Fix bases I and J of, respectively, $\varphi^{\mathfrak{A}}$ and $\varphi^{\mathfrak{B}}$. By assumption, $|I| = |J|$. Let $f_0 : I \to J$ be a bijection. By Corollary 4.6, it follows that f_0 is elementary. Hence, we can use Lemma 4.10 to extend f_0 to an elementary map $f : \mathrm{acl}(I) \to \mathrm{acl}(J)$. Since $\mathrm{acl}(I) = \mathrm{acl}(\varphi^{\mathfrak{A}})$ and $\mathrm{acl}(J) = \mathrm{acl}(\varphi^{\mathfrak{B}})$, this is the desired map. □

We can apply the results on minimal sets to study theories where every model consists of a minimal set. In fact, it is sufficient that every model is generated by a minimal set.

Definition 4.12. Let T be a complete first-order theory.

(a) T is *strongly minimal* if the formula $x = x$ is strongly minimal.

(b) T is *almost strongly minimal* if there exists a strongly minimal formula $\varphi(x; \bar{c})$ with parameters $\bar{c}$ such that $\mathrm{tp}(\bar{c})$ is isolated and

$$\mathrm{acl}(\varphi^{\mathfrak{M}} \cup \bar{c}) = M\,, \quad \text{for every model } \mathfrak{M} \text{ of } T(\bar{c})\,.$$

Example. The theories DAG and ACF_p are strongly minimal.

Theorem 4.13. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of an almost strongly minimal theory T and let $\varphi(x; \bar{c})$ be the corresponding strongly minimal formula. Then*

$$\mathfrak{A} \cong \mathfrak{B} \quad \text{iff} \quad \dim(\varphi^{\mathfrak{A}}/\bar{c}) = \dim(\varphi^{\mathfrak{B}}/\bar{c})\,.$$

Proof. $(\Rightarrow)$ is trivial and $(\Leftarrow)$ follows from Corollary 4.11. □

Corollary 4.14. *Every almost strongly minimal theory T is κ-categorical, for all $\kappa > |T|$.*

Proof. Let $\varphi(x; \bar{c})$ be the strongly minimal formula associated with T and let $\mathfrak{A}$ and $\mathfrak{B}$ be models of T of the same size $|A| = |B| > |T|$. Since

$\mathrm{tp}(\bar{c})$ is isolated, there are tuples $\bar{a} \subseteq A$ and $\bar{b} \subseteq B$ realising $\mathrm{tp}(\bar{c})$. Fix bases $I \subseteq A$ and $J \subseteq B$ of $\varphi^{\mathfrak{A}}$ over $\bar{a}$ and of $\varphi^{\mathfrak{B}}$ over $\bar{b}$, respectively. Then

$$\dim(\varphi^{\mathfrak{A}}/\bar{a}) = |I| = |\mathrm{acl}(I)| = |A| \quad \text{and} \quad \dim(\varphi^{\mathfrak{B}}/\bar{b}) = \dots = |B|\,.$$

By Theorem 4.13, it follows that $\mathfrak{A} \cong \mathfrak{B}$. □

5. *Vaughtian pairs and the Theorem of Morley*

In this section we shall prove the Theorem of Morley which states that a countable first-order theory T that is κ-categorical, for *some* uncountable cardinal κ, is λ-categorical, for *all* uncountable cardinals λ. We have already seen in Theorem E6.3.16 that such a theory is necessarily $\aleph_0$-stable. It follows that every uncountable model is saturated. Note that, according to Lemma E1.2.17, we have $|\varphi^{\mathfrak{M}}| < \aleph_0$ or $|\varphi^{\mathfrak{M}}| = |M|$, for every saturated model $\mathfrak{M}$ of T and every formula φ. In fact, we will show below that a $\aleph_0$-stable theory T is uncountably categorical if, and only if, this property holds for *all* uncountable models $\mathfrak{M}$.

Definition 5.1. Let T be a first-order theory.

(a) A *Vaughtian pair* for T consists of two models $\mathfrak{A} \prec \mathfrak{B}$ of T such that, for some formula $\varphi(\bar{x})$ over A, $\varphi^{\mathfrak{A}}$ is infinite and $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}$.

(b) The *size* of a Vaughtian pair $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is the tuple $\langle \kappa, \lambda \rangle$ where $\kappa := |A|$ and $\lambda := |B|$.

(c) If $\mathfrak{A} \leq \mathfrak{B}$ are structures, we denote by $\langle \mathfrak{B}, A \rangle$ the expansion of $\mathfrak{B}$ by a new unary predicate P with value A.

Example. Let $\mathfrak{A} = \langle A, \sim \rangle$ where $\sim$ is an equivalence relation on A and let $\mathfrak{B} \succ \mathfrak{A}$. Then $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is a Vaughtian pair if, and only if, there is some $a \in A$ whose equivalence class

$$[a]_\sim := \{\, b \in B \mid b \sim a \,\}$$

is infinite and contained in A.

In the first part of this section we will study constructions of Vaughtian pairs. The goal is Lemma 5.8 which states that a countable theory with a Vaughtian pair cannot be κ-categorical for an uncountable cardinal κ. In the second part of the section, we will then investigate minimal sets in theories without Vaughtian pairs.

We will use the following lemma to construct new Vaughtian pairs from a given one.

Lemma 5.2. *Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}' \subseteq \mathfrak{B}'$ are structures such that $\langle \mathfrak{B}, A \rangle \equiv \langle \mathfrak{B}', A' \rangle$.*

(a) *$\mathfrak{A} \preceq \mathfrak{B}$ if, and only if, $\mathfrak{A}' \preceq \mathfrak{B}'$.*

(b) *Let $\varphi(\bar{x}, \bar{y})$ be a formula and $\bar{a} \subseteq A$ and $\bar{a}' \subseteq A'$ tuples such that $\langle \mathfrak{B}, A, \bar{a} \rangle \equiv \langle \mathfrak{B}', A', \bar{a}' \rangle$. Then $\varphi(\bar{x}, \bar{a})$ is a witness for $\langle \mathfrak{A}, \mathfrak{B} \rangle$ being Vaughtian if, and only if, $\varphi(\bar{x}, \bar{a}')$ is a witness for $\langle \mathfrak{A}', \mathfrak{B}' \rangle$ being Vaughtian.*

Proof. (a) By symmetry, it is sufficient to prove one direction. For every formula $\psi(\bar{x})$, $\mathfrak{A} \preceq \mathfrak{B}$ implies

$$\langle \mathfrak{B}, A \rangle \vDash (\forall \bar{x}. \textstyle\bigwedge_i P x_i)[\psi(\bar{x}) \leftrightarrow \psi^{(P)}(\bar{x})] \,,$$

where $\psi^{(P)}$ is the relativisation of ψ to P. Hence, all these formulae also hold in $\langle \mathfrak{B}', A' \rangle$. This implies that $\mathfrak{A}' \preceq \mathfrak{B}'$.

(b) Suppose that $\varphi(\bar{x}, \bar{a})$ witnesses that $\langle \mathfrak{B}, A \rangle$ is Vaughtian. By (a) and the fact that

$$\langle \mathfrak{B}, A \rangle \vDash \exists x \neg P x \,,$$

it follows that $\mathfrak{A}' \prec \mathfrak{B}'$. Furthermore, for every $n < \omega$,

$$\langle \mathfrak{B}, A \rangle \vDash \exists^n \bar{x} \varphi(\bar{x}, \bar{a}) \wedge \forall \bar{x}[\varphi(\bar{x}, \bar{a}) \rightarrow \textstyle\bigwedge_i P x_i] \,.$$

Hence, the tuple $\bar{a}'$ satisfies these formulae in $\langle \mathfrak{B}', A' \rangle$. Consequently, $\varphi(\bar{x}, \bar{a}')^{\mathfrak{A}'}$ is infinite and $\varphi(\bar{x}, \bar{a}')^{\mathfrak{A}'} = \varphi(\bar{x}, \bar{a}')^{\mathfrak{B}'}$. □

The aim of the following sequence of results is Proposition 5.7 below which states that, given an arbitrary Vaughtian pair, we can construct a pair of size $\langle \kappa, \aleph_0 \rangle$, for every infinite cardinal κ.

Lemma 5.3. *Let T be a complete first-order theory. If there is a Vaughtian pair for T, then there are Vaughtian pairs for T of size $\langle \kappa, \kappa \rangle$, for every $\kappa \geq |T|$.*

Proof. Let $\mathfrak{A} \prec \mathfrak{B}$ be a Vaughtian pair for T and let $\varphi(\bar{x})$ be the corresponding formula with parameters $\bar{a} \subseteq A$. Since $\varphi^{\mathfrak{A}}$ is infinite, we can use the Compactness Theorem to construct an elementary extension $\langle \mathfrak{B}_1, A_1 \rangle \succeq \langle \mathfrak{B}, A \rangle$ such that $|\varphi^{\mathfrak{A}_1}| \geq \kappa$. By the Theorem of Löwenheim and Skolem, we can choose an elementary substructure $\langle \mathfrak{B}_0, A_0 \rangle \preceq \langle \mathfrak{B}_1, A_1 \rangle$ with $|B_0| = \kappa$, $|A_0| = \kappa$, and $\bar{a} \subseteq A_0$. By Lemma 5.2, it follows that $\mathfrak{A}_0 \prec \mathfrak{B}_0$ is a Vaughtian pair. □

Proposition 5.4. *Let T be a countable complete first-order theory. For every pair $\mathfrak{A}_0 \preceq \mathfrak{B}_0$ of countable models of T there exist countable homogeneous models $\mathfrak{A} \preceq \mathfrak{B}$ of T such that $\langle \mathfrak{B}_0, A_0 \rangle \preceq \langle \mathfrak{B}, A \rangle$ and $\mathfrak{A}$ and $\mathfrak{B}$ realise the same types in $S^{<\omega}(T)$.*

Proof. We start by proving the following claims.

(a) For every finite subset $U \subseteq A_0$ and every type $\mathfrak{p} \in S^{<\omega}(U)$, there exists a countable extension $\langle \mathfrak{B}, A \rangle \succeq \langle \mathfrak{B}_0, A_0 \rangle$ such that $\mathfrak{p}$ is realised in $\mathfrak{A} := \mathfrak{B}|_A$.

(b) For every finite subset $U \subseteq B_0$ and every type $\mathfrak{p} \in S^{<\omega}(U)$, there exists a countable extension $\langle \mathfrak{B}, A \rangle \succeq \langle \mathfrak{B}_0, A_0 \rangle$ such that $\mathfrak{p}$ is realised in $\mathfrak{B}$.

(c) There exists a countable extension $\langle \mathfrak{B}, A \rangle \succeq \langle \mathfrak{B}_0, A_0 \rangle$ such that $\mathfrak{A} := \mathfrak{B}|_A$ realises every type over a finite subset $U \subseteq A_0$ that is realised in $\mathfrak{B}_0$.

(a) We set

$$\Phi := \Delta \cup \{ \varphi^{(P)} \mid \varphi \in \mathfrak{p} \} \,,$$

where Δ is the elementary diagram of $\langle\mathfrak{B}_0, A_0\rangle$. To show that Φ is satisfiable, we consider finitely many formulae $\varphi_0(\bar{x}), \dots, \varphi_{n-1}(\bar{x}) \in \mathfrak{p}$. Since $\mathfrak{p}$ is a type and, hence, finitely satisfiable in every model of T, we have $\mathfrak{A}_0 \vDash \exists\bar{x} \bigwedge_{i<n} \varphi_i(\bar{x})$, which implies that

$$\langle\mathfrak{B}_0, A_0\rangle \vDash \exists\bar{x} \bigwedge_{i<n} \varphi_i^{(P)}(\bar{x}) .$$

Consequently, Φ is finitely satisfiable. Fix a countable model $\langle\mathfrak{B}, A, \bar{a}\rangle$ of Φ. Then $\langle\mathfrak{B}_0, A_0\rangle \preceq \langle\mathfrak{B}, A\rangle$ and $\bar{a} \subseteq A$ realises $\mathfrak{p}$.

(b) This claim follows immediately from compactness and the Theorem of Löwenheim and Skolem.

(c) Let $(\mathfrak{p}_\alpha)_{\alpha<\omega}$ be an enumeration of all types over a finite set $U \subseteq A_0$ that are realised in $\mathfrak{B}_0$. We can use (a) to construct an increasing chain $\langle\mathfrak{B}_\alpha, A_\alpha\rangle_{\alpha<\omega}$ of countable models starting with $\langle\mathfrak{B}_0, A_0\rangle$ such that $\mathfrak{A}_{\alpha+1} := \mathfrak{B}_{\alpha+1}|_{A_{\alpha+1}}$ realises $\mathfrak{p}_\alpha$. The union $\langle\mathfrak{B}, A\rangle := \bigcup_{\alpha<\omega} \langle\mathfrak{B}_\alpha, A_\alpha\rangle$ is the desired extension of $\langle\mathfrak{B}_0, A_0\rangle$.

To prove the proposition we construct a chain $\langle\mathfrak{B}_\alpha, A_\alpha\rangle_{\alpha<\omega}$ of countable models starting with $\langle\mathfrak{B}_0, A_0\rangle$ as follows.

(1) For indices of the form $\alpha = 3n$, we use (c) to find a countable extension $\langle\mathfrak{B}_{\alpha+1}, A_{\alpha+1}\rangle \succeq \langle\mathfrak{B}_\alpha, A_\alpha\rangle$ such that every type over a finite set $U \subseteq A_\alpha$ that is realised in $\mathfrak{B}_\alpha$ is realised in $\mathfrak{A}_{\alpha+1}$.

(2) For indices $\alpha = 3n + 1$, we iterate (a) to find a countable extension $\langle\mathfrak{B}_{\alpha+1}, A_{\alpha+1}\rangle \succeq \langle\mathfrak{B}_\alpha, A_\alpha\rangle$ such that, for all tuples $\bar{a}, \bar{b} \in A_\alpha^{<\omega}$ with $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ and every element $c \in A_\alpha$, there is an element $d \in A_{\alpha+1}$ such that $\mathrm{tp}(\bar{a}c) = \mathrm{tp}(\bar{b}d)$.

(3) For $\alpha = 3n + 2$, we use (b), amalgamation, and the Theorem of Löwenheim and Skolem to find an extension $\langle\mathfrak{B}_{\alpha+1}, A_{\alpha+1}\rangle \succeq \langle\mathfrak{B}_\alpha, A_\alpha\rangle$ such that, for all tuples $\bar{a}, \bar{b} \in B_\alpha^{<\omega}$ with $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ and every element $c \in B_\alpha$, there is an element $d \in B_{\alpha+1}$ such that $\mathrm{tp}(\bar{a}c) = \mathrm{tp}(\bar{b}d)$.

The limit $\langle\mathfrak{B}, A\rangle := \bigcup_{\alpha<\omega} \langle\mathfrak{B}_\alpha, A_\alpha\rangle$ is a countable elementary extension of $\langle\mathfrak{B}_0, A_0\rangle$. Furthermore, by (1), the structures $\mathfrak{A} := \mathfrak{B}|_A$ and $\mathfrak{B}$ realise the same types in $S^{<\omega}(T)$. Finally, (2) and (3) ensure that $\mathfrak{A}$ and $\mathfrak{B}$ are homogeneous. □

Proposition 5.5. *Let T be a countable complete first-order theory. If there is a Vaughtian pair for T, then there is a Vaughtian pair for T of size $\langle\aleph_0, \aleph_1\rangle$.*

Proof. By Lemma 5.3 and Proposition 5.4, we can find a Vaughtian pair $\mathfrak{A} \prec \mathfrak{B}$ for T of size $\langle\aleph_0, \aleph_0\rangle$ such that $\mathfrak{A}$ and $\mathfrak{B}$ are homogeneous and realise the same types. By Theorem E1.1.9, this implies that $\mathfrak{A} \cong \mathfrak{B}$. Let φ be a formula over A such that $\varphi^{\mathfrak{A}}$ is infinite and $\varphi^{\mathfrak{B}} = \varphi^{\mathfrak{A}}$.

We construct an elementary chain $(\mathfrak{M}_\alpha)_{\alpha<\aleph_1}$ of models of T such that, for every $\alpha < \aleph_1$, we have

$$\varphi^{\mathfrak{M}_\alpha} = \varphi^{\mathfrak{A}} \quad \text{and} \quad \langle\mathfrak{M}_{\alpha+1}, M_\alpha\rangle \cong \langle\mathfrak{B}, A\rangle .$$

Note that, in particular, every $\mathfrak{M}_\alpha$ is isomorphic to $\mathfrak{A}$.

We start with $\mathfrak{M}_0 := \mathfrak{B}$. For the successor step, suppose that we have already defined $\mathfrak{M}_\alpha \cong \mathfrak{A}$. We choose an elementary extension $\mathfrak{M}_{\alpha+1} \succeq \mathfrak{M}_\alpha$ such that $\langle\mathfrak{M}_{\alpha+1}, M_\alpha\rangle \cong \langle\mathfrak{B}, A\rangle$. Then $\varphi^{\mathfrak{M}_{\alpha+1}} = \varphi^{\mathfrak{M}_\alpha} = \varphi^{\mathfrak{A}}$.

For limit ordinals δ, we set $\mathfrak{M}_\delta := \bigcup_{\alpha<\delta} \mathfrak{M}_\alpha$. Then $\varphi^{\mathfrak{M}_\delta} = \bigcup_{\alpha<\delta} \varphi^{\mathfrak{M}_\alpha} = \varphi^{\mathfrak{A}}$. To show that $\mathfrak{M}_\delta \cong \mathfrak{A}$ it is sufficient to prove that $\mathfrak{M}_\delta$ is homogeneous and that it realises the same types as $\mathfrak{A}$. For homogeneity, suppose that $\bar{a}, \bar{b} \in M_\delta^{<\omega}$ and $c \in M_\delta$ are elements such that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. Then there is some $\alpha < \delta$ such that $\bar{a}, \bar{b}, c \subseteq M_\alpha$. As $\mathfrak{M}_\alpha \cong \mathfrak{A}$ is homogeneous, there is some $d \in M_\alpha \subseteq M_\delta$ such that $\mathrm{tp}(\bar{a}c) = \mathrm{tp}(\bar{b}d)$.

Clearly, every type realised in $\mathfrak{A}$ is realised in $\mathfrak{M}_\delta \succeq \mathfrak{A}$. Conversely, let $\mathfrak{p} \in S^{<\omega}(T)$ be realised in $\mathfrak{M}_\delta$. Then there is some $\bar{a} \in M_\delta^{<\omega}$ with $\mathrm{tp}(\bar{a}) = \mathfrak{p}$. Let $\alpha < \delta$ be an index such that $\bar{a} \subseteq M_\alpha$. Then $\mathfrak{p}$ is realised in $\mathfrak{M}_\alpha \cong \mathfrak{A}$.

Having defined $(\mathfrak{M}_\alpha)_\alpha$ we set $\mathfrak{N} := \bigcup_{\alpha<\aleph_1} \mathfrak{M}_\alpha$. Then $|N| = \aleph_1$ and $\varphi^{\mathfrak{N}} = \varphi^{\mathfrak{A}}$. Hence, $\mathfrak{A} \prec \mathfrak{N}$ is the desired Vaughtian pair of size $\langle\aleph_0, \aleph_1\rangle$. □

Lemma 5.6. *Let T be a complete $\aleph_0$-stable theory over a countable signature. Every uncountable model $\mathfrak{M}$ of T has a proper elementary extension $\mathfrak{N} \succ \mathfrak{M}$ such that every countable type $\mathfrak{p}$ realised in $\mathfrak{N}$ is already realised in $\mathfrak{M}$.*

Proof. By Lemma 4.8 there exists a formula $\varphi(\bar{x})$ over M such that $|\varphi^{\mathfrak{M}}| \geq \aleph_1$ and we have either

$$|(\varphi \wedge \psi)^{\mathfrak{M}}| \leq \aleph_0 \quad \text{or} \quad |(\varphi \wedge \neg\psi)^{\mathfrak{M}}| \leq \aleph_0\,,$$

for every formula $\psi(\bar{x})$ over M. Let $\bar{s}$ be the sorts of the variables $\bar{x}$ and set

$$\mathfrak{p} := \{\, \psi(\bar{x}) \in \mathrm{FO}^{\bar{s}}[\Sigma_M] \mid (\varphi \wedge \psi)^{\mathfrak{M}} \text{ is uncountable} \,\}\,.$$

Note that, for $\psi_0, \dots, \psi_{n-1} \in \mathfrak{p}$, we have

$$\Big|(\varphi \wedge \bigvee_{i<n} \neg\psi_i)^{\mathfrak{M}}\Big| = \big|(\varphi \wedge \neg\psi_0)^{\mathfrak{M}} \cup \dots \cup (\varphi \wedge \neg\psi_{n-1})^{\mathfrak{M}}\big| \leq \aleph_0\,,$$

which implies that $\bigwedge_{i<n} \psi_i \in \mathfrak{p}$. Hence, $(\bigwedge_i \psi_i)^{\mathfrak{M}} \neq \emptyset$ and $\mathfrak{p}$ is finitely satisfiable. Furthermore, by choice of φ, we have $\psi \in \mathfrak{p}$ or $\neg\psi \in \mathfrak{p}$, for every formula $\psi(\bar{x})$ over M. Therefore, $\mathfrak{p}$ is a complete type.

Let $\mathfrak{M}_+ \succeq \mathfrak{M}$ be an elementary extension containing a finite tuple $\bar{a} \in M_+^{\bar{s}}$ realising $\mathfrak{p}$. By Theorem E3.4.14, there exists a model $\mathfrak{M} \preceq \mathfrak{N} \preceq \mathfrak{M}_+$ that is atomic over $M \cup \bar{a}$.

To show that $\mathfrak{N}$ has the desired property, we consider a countable type $\Phi(\bar{y})$ over M that is realised by some finite tuple $\bar{b} \in N^{<\omega}$. Since $\mathfrak{N}$ is atomic over $M \cup \bar{a}$, there exists a formula $\chi(\bar{y}, \bar{a})$ over M isolating $\mathrm{tp}(\bar{b}/M)$. Then $\mathfrak{N} \vDash \chi(\bar{b}, \bar{a})$ implies

$$\exists \bar{y}\chi(\bar{y}, \bar{x}) \in \mathfrak{p}$$
and $$\forall \bar{y}[\chi(\bar{y}, \bar{x}) \to \vartheta(\bar{y})] \in \mathfrak{p}\,, \quad \text{for all } \vartheta(\bar{y}) \in \mathrm{tp}(\bar{b}/M) \supseteq \Phi\,.$$

Hence, the set

$$\Gamma := \{\exists \bar{y}\chi(\bar{y}, \bar{x})\} \cup \{\, \forall \bar{y}[\chi(\bar{y}, \bar{x}) \to \vartheta(\bar{y})] \mid \vartheta(\bar{x}) \in \Phi \,\}$$

is a countable subset of $\mathfrak{p}$. Furthermore, if a tuple $\bar{a}' \in M^{\bar{s}}$ realises Γ then we have

$$\mathfrak{M} \vDash \exists \bar{y}\chi(\bar{y}, \bar{a}')$$

and every $\bar{b}' \subseteq M$ with $\mathfrak{M} \vDash \chi(\bar{b}', \bar{a}')$ realises Φ. Let $\psi_0, \psi_1, \dots$ be an enumeration of Γ. By choice of $\mathfrak{p}$, we have

$$|\varphi^{\mathfrak{M}}| > \aleph_0 \quad \text{and} \quad |(\varphi \wedge \neg(\psi_0 \wedge \dots \wedge \psi_n))^{\mathfrak{M}}| \leq \aleph_0\,, \quad \text{for all } n\,.$$

It follows that $(\varphi \wedge \neg \bigwedge \Gamma)^{\mathfrak{M}} = \bigcup_{n<\omega} (\varphi \wedge \neg \bigwedge_{i<n} \psi_i)^{\mathfrak{M}}$ is countable and

$$(\varphi \wedge \bigwedge \Gamma)^{\mathfrak{M}} = \varphi^{\mathfrak{M}} \setminus (\varphi \wedge \neg \bigwedge \Gamma)^{\mathfrak{M}}$$

is uncountable. Hence, there are uncountably many $\bar{a}' \in M^{\bar{s}}$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}') \wedge \bigwedge \Gamma(\bar{a}')\,.$$

As we have seen above, this implies that $\mathfrak{M}$ contains a realisation of Φ. □

Proposition 5.7. *Let T be an $\aleph_0$-stable, countable, complete first-order theory. If there is a Vaughtian pair for T, then there are Vaughtian pairs for T of size $\langle \aleph_0, \kappa \rangle$, for every uncountable cardinal κ.*

Proof. By Proposition 5.5, there is a Vaughtian pair $\mathfrak{A} \prec \mathfrak{B}$ for T of size $\langle \aleph_0, \aleph_1 \rangle$. Let φ be a formula over A such that $\varphi^{\mathfrak{A}}$ is infinite and $\varphi^{\mathfrak{B}} = \varphi^{\mathfrak{A}}$. Starting with $\mathfrak{M}_0 := \mathfrak{B}$, we construct a strictly increasing elementary chain $(\mathfrak{M}_\alpha)_{\alpha<\kappa}$ such that $\varphi^{\mathfrak{M}_\alpha} = \varphi^{\mathfrak{A}}$, for all α.

As usual, we take unions $\mathfrak{M}_\delta := \bigcup_{\alpha<\delta} \mathfrak{M}_\alpha$ for limit ordinals δ. For the successor step, suppose that $\mathfrak{M}_\alpha$ has already been defined. We apply Lemma 5.6 to find a proper elementary extension $\mathfrak{M}_{\alpha+1} \succ \mathfrak{M}_\alpha$ that realises the same countable types as $\mathfrak{M}_\alpha$. In particular, $\mathfrak{M}_{\alpha+1}$ does not realise the type

$$\{\varphi(x)\} \cup \{\, x \neq c \mid c \in \varphi^{\mathfrak{M}_\alpha} \,\}\,.$$

Therefore, $\varphi^{\mathfrak{M}_{\alpha+1}} = \varphi^{\mathfrak{M}_\alpha} = \varphi^{\mathfrak{A}}$.

Let $\mathfrak{N} := \bigcup_{\alpha<\kappa} \mathfrak{M}_\alpha$ be the union of the chain and choose an elementary substructure $\mathfrak{A} \prec \mathfrak{C} \preceq \mathfrak{N}$ of size $|C| = \kappa$. Then $\mathfrak{A} \prec \mathfrak{C}$ is the desired Vaughtian pair of size $\langle \aleph_0, \kappa \rangle$. □

We can use this proposition to show that uncountably categorical theories do not have Vaughtian pairs.

Lemma 5.8. *Let T be a countable complete first-order theory with infinite models. If T is κ-categorical, for some uncountable cardinal κ, then T has no Vaughtian pairs.*

Proof. For a contradiction, suppose that T is a κ-categorical theory with a Vaughtian pair. By Theorem E6.3.16, T is $\aleph_0$-stable. Hence, we can use Proposition 5.7 to find a Vaughtian pair $\mathfrak{A} < \mathfrak{B}$ of size $\langle \aleph_0, \kappa \rangle$. Let φ be a formula such that $\varphi^{\mathfrak{A}}$ is infinite and $\varphi^{\mathfrak{B}} = \varphi^{\mathfrak{A}}$. By Theorem E1.2.16, T has a saturated model $\mathfrak{C}$ of size κ. But $\mathfrak{B} \not\cong \mathfrak{C}$ since we have $|\varphi^{\mathfrak{C}}| = \kappa$ by Lemma E1.2.17. This contradicts κ-categoricity. □

Next we study minimal formulae in theories without Vaughtian pairs. First, we show that such a theory is graduated which, according to Theorem D1.2.15, is equivalent to admitting elimination of the quantifier $\exists^{\aleph_0}$.

Lemma 5.9. *Suppose that T is a theory without Vaughtian pairs. Let $\mathfrak{M}$ be a model of T and $\varphi(\bar{x}; \bar{y})$ a formula over M. There exists a number $n < \omega$, such that, for all $\bar{c} \subseteq M$,*

$$\left|\varphi(\bar{x}; \bar{c})^{\mathfrak{M}}\right| > n \quad \text{implies} \quad \left|\varphi(\bar{x}; \bar{c})^{\mathfrak{M}}\right| \geq \aleph_0 .$$

Proof. Suppose that such a number n does not exist. Then we can find, for every $n < \omega$, parameters $\bar{c}_n \subseteq M$ with

$$n < \left|\varphi(\bar{x}; \bar{c}_n)\right| < \aleph_0 .$$

Let P be a new unary predicate and let $\Phi(\bar{y})$ be the set of formulae containing the following statements:

- P induces a proper elementary substructure;
- $\bigwedge_i P y_i$;
- there are infinitely many tuples $\bar{x}$ such that $\varphi(\bar{x}; \bar{y})$;
- $\forall \bar{x}[\varphi(\bar{x}; \bar{y}) \rightarrow \bigwedge_i P x_i]$.

To see that $T \cup \Phi(\bar{y})$ is satisfiable, we fix an extension $\mathfrak{N} > \mathfrak{M}$. Since $\varphi(\bar{x}; \bar{c}_n)^{\mathfrak{M}}$ is finite, we have $\varphi(\bar{x}; \bar{c}_n)^{\mathfrak{N}} = \varphi(\bar{x}; \bar{c}_n)^{\mathfrak{M}}$. For every finite subset $\Phi_0 \subseteq \Phi$, we can therefore choose n large enough such that

$$\langle \mathfrak{N}, M \rangle \vDash T \cup \Phi_0(\bar{c}_n) .$$

Let $\langle \mathfrak{B}, A, \bar{c} \rangle$ be a model of $T \cup \Phi$. Then $\mathfrak{A} := \mathfrak{B}|_A < \mathfrak{B}$ are models of T and $\varphi(\bar{x}; \bar{c})^{\mathfrak{A}} = \varphi(\bar{x}; \bar{c})^{\mathfrak{B}}$ is infinite. Hence, $\mathfrak{A} < \mathfrak{B}$ is a Vaughtian pair. A contradiction. □

Corollary 5.10. *In a theory T without Vaughtian pairs, every minimal formula is strongly minimal.*

Proof. Let $\mathfrak{M}$ be a model of T and $\varphi(\bar{x})$ a minimal formula over M. For a contradiction, suppose that $\varphi(\bar{x})$ is not strongly minimal. Then we can find an extension $\mathfrak{N} > \mathfrak{M}$ and a formula $\psi(\bar{x}; \bar{c})$ with parameters $\bar{c} \subseteq N$ such that

$$\varphi(\bar{x})^{\mathfrak{N}} \cap \psi(\bar{x}; \bar{c})^{\mathfrak{N}} \quad \text{and} \quad \varphi(\bar{x})^{\mathfrak{N}} \setminus \psi(\bar{x}; \bar{c})^{\mathfrak{N}}$$

are both infinite. By Lemma 5.9 there exists a number $n < \omega$ such that, for all models $\mathfrak{A}$ and all $\bar{a} \subseteq A$,

$$\begin{aligned} &\left|\varphi(\bar{x})^{\mathfrak{A}} \cap \psi(\bar{x}; \bar{a})^{\mathfrak{A}}\right| > n \quad \text{implies} \quad \left|\varphi(\bar{x})^{\mathfrak{A}} \cap \psi(\bar{x}; \bar{a})^{\mathfrak{A}}\right| \geq \aleph_0 , \\ \text{and} \quad &\left|\varphi(\bar{x})^{\mathfrak{A}} \setminus \psi(\bar{x}; \bar{a})^{\mathfrak{A}}\right| > n \quad \text{implies} \quad \left|\varphi(\bar{x})^{\mathfrak{A}} \setminus \psi(\bar{x}; \bar{a})^{\mathfrak{A}}\right| \geq \aleph_0 . \end{aligned}$$

By minimality of φ, it follows that

$$\mathfrak{M} \vDash \forall \bar{y}\left[\left|\varphi(\bar{x})^{\mathfrak{M}} \cap \psi(\bar{x}; \bar{y})^{\mathfrak{M}}\right| \leq n \vee \left|\varphi(\bar{x})^{\mathfrak{M}} \setminus \psi(\bar{x}; \bar{y})^{\mathfrak{M}}\right| \leq n\right] .$$

Since $\mathfrak{M} \preceq \mathfrak{N}$, the same formula also holds in $\mathfrak{N}$. A contradiction. □

Corollary 5.11. *Let T be a countable, complete, $\aleph_0$-stable theory without Vaughtian pairs and let $\mathfrak{M}_0$ be the prime model of T. There exists a strongly minimal formula $\varphi(x)$ over M_0.*

Proof. We use Corollary 4.9 to find a minimal formula $\varphi(x)$ over M_0. By Corollary 5.10, this formula is strongly minimal. □

Lemma 5.12. *Let T be a theory without Vaughtian pairs, $\mathfrak{B}$ a model of T, and let $\varphi(\bar{x};\bar{c})$ be a strongly minimal formula with parameters $\bar{c} \subseteq B$.*

(a) *If $\mathfrak{A} \prec \mathfrak{B}$ is a proper elementary substructure with $\bar{c} \subseteq A$, then $\varphi^{\mathfrak{A}} \subset \varphi^{\mathfrak{B}}$.*

(b) $\dim(\varphi^{\mathfrak{B}}) = |B|$.

(c) *If T is $\aleph_0$-stable then $\mathfrak{B}$ is prime over $\varphi^{\mathfrak{B}} \cup \bar{c}$.*

Proof. (a) $\mathfrak{A} \prec \mathfrak{B}$ implies $\varphi^{\mathfrak{A}} \subseteq \varphi^{\mathfrak{B}}$. Furthermore, if $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}$, then $\mathfrak{A} \prec \mathfrak{B}$ would be a Vaughtian pair.

(b) Let I be a basis of $\varphi^{\mathfrak{B}}$. If $|I| < |B|$ then we can use the Theorem of Löwenheim and Skolem to find an elementary substructure $\mathfrak{A} \prec \mathfrak{B}$ of size $|A| = |I|$ with $I \cup \bar{c} \subseteq A$. It follows that $\varphi^{\mathfrak{B}} \subseteq \operatorname{acl}(I) \subseteq A$. Hence, $\varphi^{\mathfrak{B}} = \varphi^{\mathfrak{A}}$ in contradiction to (a).

(c) Since T is $\aleph_0$-stable there exists, according to Theorem E3.4.14 a unique prime model $\mathfrak{M}$ over $\varphi^{\mathfrak{B}} \cup \bar{c}$. W.l.o.g. we may assume that $\mathfrak{M} \preceq \mathfrak{B}$. Since $\varphi^{\mathfrak{B}} \cup \bar{c} \subseteq M \subseteq B$ it follows by (a) that $M = B$, as desired. □

Lemma 5.13. *Let T be a countable, complete first-order theory with infinite models. Suppose that there exists a strongly minimal formula $\varphi(x;\bar{c})$ such that*

- $\operatorname{tp}(\bar{c})$ *is isolated,*
- *every model $\mathfrak{M}$ of $T(\bar{c})$ is prime over $\varphi^{\mathfrak{M}} \cup \bar{c}$,*
- *no model $\mathfrak{M}$ of $T(\bar{c})$ has a proper elementary substructure $\mathfrak{A} \prec \mathfrak{M}$ such that $\varphi^{\mathfrak{M}} \subseteq A$.*

Then

$$\dim(\varphi^{\mathfrak{A}}/\bar{c}) = \dim(\varphi^{\mathfrak{B}}/\bar{c}) \quad \textit{implies} \quad \mathfrak{A} \cong \mathfrak{B}\,,$$

for all models $\mathfrak{A}, \mathfrak{B}$ of $T(\bar{c})$.

Proof. Set $S := \varphi(\bar{x};\bar{c})^{\mathfrak{A}}$ and $S' := \varphi(\bar{x};\bar{c})^{\mathfrak{B}}$. Since $\dim(S) = \dim(S')$ we can use Corollary 4.11 to find an elementary bijection $h_0 : S \to S'$. As $\mathfrak{A}$ and $\mathfrak{B}$ are models of $T(\bar{c})$, we can extend h_0 to an elementary map $h_1 : S \cup \bar{c} \to S' \cup \bar{c}$. Because $\mathfrak{A}$ is prime over $S \cup \bar{c}$, we can extend this map h_1 to an elementary map $h : \mathfrak{A} \to \mathfrak{B}$. We claim that h is surjective and, therefore, the desired isomorphism.

For a contradiction, suppose otherwise. Then we obtain a proper elementary substructure $\mathfrak{B}_0 := f[\mathfrak{A}] \prec \mathfrak{B}$ with $S' \cup \bar{c} = \operatorname{rng} h_1 \subseteq B_0$. But $\mathfrak{B}$ is prime over $S' \cup \bar{c}$. A contradiction. □

Theorem 5.14 (Morley). *Let T be a countable, complete first-order theory with infinite models. The following statements are equivalent:*

(1) *T is κ-categorical, for some uncountable cardinal κ.*

(2) *T is κ-categorical, for every uncountable cardinal κ.*

(3) *T is $\aleph_0$-stable and it has no Vaughtian pairs.*

(4) *There exists a strongly minimal formula $\varphi(x;\bar{c})$ such that*
   - $\operatorname{tp}(\bar{c})$ *is isolated,*
   - *every model $\mathfrak{M}$ of $T(\bar{c})$ is prime over $\varphi^{\mathfrak{M}} \cup \bar{c}$,*
   - *no model $\mathfrak{M}$ of $T(\bar{c})$ has a proper elementary substructure $\mathfrak{A} \prec \mathfrak{M}$ such that $\varphi^{\mathfrak{M}} \subseteq A$.*

Proof. (2) ⇒ (1) is trivial.

(1) ⇒ (3) follows by Theorem E6.3.16 and Lemma 5.8.

(3) ⇒ (4) Let T be an $\aleph_0$-stable theory without Vaughtian pairs. By Theorem E3.4.14, T has a prime model $\mathfrak{M}_0$. We can use Corollary 5.11 to find a strongly minimal formula $\varphi(x;\bar{c})$ with parameters $\bar{c} \subseteq M_0$. Since prime models are atomic, the type of $\bar{c} \subseteq M_0$ is isolated. The remaining two claims of (4) follow by Lemma 5.12 (a) and (c), respectively.

(4) ⇒ (2) Let κ be an uncountable cardinal. To show that T is κ-categorical, we consider two models $\mathfrak{A}$ and $\mathfrak{B}$ of size κ. Since $\operatorname{tp}(\bar{c})$ is isolated there are tuples $\bar{a} \subseteq A$ and $\bar{b} \subseteq B$ realising $\operatorname{tp}(\bar{c})$. Thus, $\langle \mathfrak{A}, \bar{a} \rangle$ and $\langle \mathfrak{B}, \bar{b} \rangle$ are models of $T(\bar{c})$. Set $S := \varphi(\bar{x};\bar{a})^{\mathfrak{A}}$ and $S' := \varphi(\bar{x};\bar{b})^{\mathfrak{B}}$.

Since $\mathfrak{A}$ and $\mathfrak{B}$ have no proper elementary substructures containing, respectively, $S \cup \bar{a}$ and $S' \cup \bar{b}$, it follows by the Theorem of Löwenheim and Skolem that

$$\dim(S) = |A| = |B| = \dim(S') .$$

Consequently, we can use Lemma 5.13 to show that $\mathfrak{A} \cong \mathfrak{B}$. □

F2. Ranks and forking

1. Morley rank and Δ-rank

We have seen that each model of an uncountably categorical theory is governed by a strongly minimal set and that we can define a geometry on such a set. Unfortunately, for most theories we cannot find actual geometries. But there is a large class of theories where we have something slightly weaker. In this chapter we study the kind of combinatorial structure that will serve as our substitute for a geometry.

We start by defining certain ranks that provide a weak notion of dimension. Guided by the observation that, for a strongly-minimal formula φ over a model $\mathfrak{M}$, the Cantor-Bendixson rank of the set $\langle\varphi\rangle$ in $\mathfrak{S}^{\bar{s}}(M)$ is equal to 1, we take a look at the Cantor-Bendixson rank of type spaces. Let us first describe how to compute the Cantor-Bendixson rank in $\mathfrak{S}_\Delta(U)$ by using the equality of Cantor-Bendixson rank and partition rank.

Lemma 1.1. *Let Δ be a set of formulae, U a set of parameters, and let Δ^+_U be the set of all finite boolean combinations of formulae of the form $\psi(\bar{x};\bar{c})$ with $\psi(\bar{x};\bar{y}) \in \Delta$ and $\bar{c} \subseteq U$.*

For an arbitrary formula φ over U and an ordinal $\alpha > 0$, we have

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi\rangle_{\mathfrak{S}_\Delta(U)}\big) \geq \alpha$$

if, and only if, for all ordinals $\beta < \alpha$, there are formulae $\psi_i \in \Delta^+_U$, for $i < \omega$, such that

$$\begin{aligned} &\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi \wedge \psi_i\rangle_{\mathfrak{S}_\Delta(U)}\big) \geq \beta\,, && \text{for every } i\,,\\ \text{and}\quad &\psi_i^{\mathbb{M}} \cap \psi_k^{\mathbb{M}} = \varnothing\,, && \text{for all } i \neq k\,. \end{aligned}$$

Proof. Note that, by definition of $\mathfrak{S}_\Delta(U)$ and Lemma C3.3.5,

$$\begin{aligned}\mathfrak{S}_\Delta(U) &= \mathfrak{S}_{\Delta_U^-}\big(\mathrm{FO}[\Sigma_U, X]/T(U)\big)\\ &\cong \mathfrak{S}_{\Delta_U^+}\big(\mathrm{FO}[\Sigma_U, X]/T(U)\big) = \mathfrak{S}_{\Delta^+}(U)\,,\end{aligned}$$

where Δ^+ is the set of all finite boolean combinations of formulae in Δ. Therefore, we may w.l.o.g. work in $\mathfrak{S}_{\Delta^+}(U)$. Set $C := \langle\varphi\rangle_{\mathfrak{S}_{\Delta^+}(U)}$ and let $\mathfrak{S}_C$ be the subspace of $\mathfrak{S}_{\Delta^+}(U)$ induced by C. According to Corollary B5.7.10, we have

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi\rangle_{\mathfrak{S}_{\Delta^+}(U)}\big) = \mathrm{rk}_{\mathrm{P}}(C/\mathrm{clop}(\mathfrak{S}_C))\,.$$

Furthermore,

$$\mathrm{rk}_{\mathrm{P}}(C/\mathrm{clop}(\mathfrak{S}_C)) \geq \alpha$$

if, and only if, for all $\beta < \alpha$, there are clopen sets $D_i \in \mathrm{clop}(\mathfrak{S}_C)$, for $i < \omega$, such that

$$\mathrm{rk}_{\mathrm{P}}(D_i/\mathrm{clop}(\mathfrak{S}_C)) \geq \beta \quad\text{and}\quad D_i \cap D_k = \varnothing\,,\quad \text{for } i \neq k\,.$$

Hence, it is sufficient to show that this latter condition is equivalent to the existence of formulae $\psi_i \in \Delta_U^+$, for $i < \omega$, such that

$$\begin{aligned}&\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi \wedge \psi_i\rangle_{\mathfrak{S}_\Delta(U)}\big) \geq \beta\,, && \text{for every } i\,,\\ \text{and}\quad &\psi_i^{\mathbb{M}} \cap \psi_k^{\mathbb{M}} = \varnothing\,, && \text{for all } i \neq k\,.\end{aligned}$$

($\Leftarrow$) Given formulae ψ_i, we set $D_i := \langle\varphi \wedge \psi_i\rangle_{\mathfrak{S}_{\Delta^+}(U)}$. By Corollaries B5.7.10 and B5.7.13, it follows that

$$\begin{aligned}\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi \wedge \psi_i\rangle_{\mathfrak{S}_{\Delta^+}(U)}\big) &= \mathrm{rk}_{\mathrm{P}}(D_i/\mathrm{clop}(D_i))\\ &= \mathrm{rk}_{\mathrm{P}}(D_i/\mathrm{clop}(\mathfrak{S}_C)) \geq \beta\,,\end{aligned}$$

as desired.

($\Rightarrow$) By Lemma B5.7.11, the clopen sets D_i are of the form

$$D_i = C \cap \langle\psi'_i\rangle_{\mathfrak{S}_{\Delta^+}(U)} = \langle\varphi \wedge \psi'_i\rangle_{\mathfrak{S}_{\Delta^+}(U)}\,,$$

for formulae $\psi'_i \in \Delta_U^+$. Setting

$$\psi_i := \psi'_i \wedge \bigwedge_{k<i} \neg\psi'_k$$

we obtain formulae $\psi_i \in \Delta_U^+$ such that

$$\psi_i^{\mathbb{M}} \cap \psi_k^{\mathbb{M}} = \varnothing\,,\quad \text{for } i \neq k\,.$$

Furthermore, $D_i \cap D_k = \varnothing$, for $k < i$, implies that

$$D_i = D_i \setminus (D_0 \cup \dots \cup D_{i-1}) = \langle\varphi \wedge \psi_i\rangle_{\mathfrak{S}_{\Delta^+}(U)}\,.$$

The claim follows since, by Corollaries B5.7.10 and B5.7.13,

$$\begin{aligned}\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi \wedge \psi_i\rangle_{\mathfrak{S}_{\Delta^+}(U)}\big) &= \mathrm{rk}_{\mathrm{P}}(D_i/\mathrm{clop}(D_i))\\ &= \mathrm{rk}_{\mathrm{P}}(D_i/\mathrm{clop}(\mathfrak{S}_C)) \geq \beta\,.\end{aligned}$$

□

When using the Cantor-Bendixson rank to define the dimension of a definable relation, we have first to choose a set Δ of formulae and a set U of parameters to know which type space $\mathfrak{S}_\Delta(U)$ to consider. Let us take a look at what happens to the Cantor-Bendixson rank when we change these two sets. First of all, the dependence is monotone: if we enlarge the set of formulae or the set of parameters, the rank either increases, or it stays the same.

Lemma 1.2. *Let Δ, Γ be sets of formulae, U, V sets of parameters, and Φ a set of formulae over U. Then*

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\Phi\rangle_{\mathfrak{S}_\Delta(U)}\big) \leq \mathrm{rk}_{\mathrm{CB}}\big(\langle\Phi\rangle_{\mathfrak{S}_{\Delta\cup\Gamma}(U\cup V)}\big)\,.$$

Proof. Let Δ_U^- be the sets of all formulae of the form $\psi(\bar{x};\bar{c})$ or $\neg\psi(\bar{x};\bar{c})$ with $\psi \in \Delta$ and $\bar{c} \subseteq U$, and let $\Delta\Gamma_{UV}^-$ be the corresponding set of formulae for $\Delta \cup \Gamma$ and $U \cup V$. The statement follows from Lemma B5.7.14 since

$$\mathfrak{S}(i)^{-1}\big[\langle\Phi\rangle_{\mathfrak{S}_\Delta(U)}\big] = \langle\Phi\rangle_{\mathfrak{S}_{\Delta\cup\Gamma}(U\cup V)}\,,$$

where $i : \Delta_U^- \to \Delta\Gamma_{UV}^-$ is the inclusion map. □

If the set of parameters is an $\aleph_0$-saturated model, the Cantor-Bendixson rank does not change anymore.

Lemma 1.3. *Let Δ be a set of formulae and $\varphi(\bar{x};\bar{y})$ a single formula. If $\mathfrak{A}$ and $\mathfrak{B}$ are $\aleph_0$-saturated structures with $\langle\mathfrak{A},\bar{a}\rangle \equiv \langle\mathfrak{B},\bar{b}\rangle$, then*

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi(\bar{x};\bar{a})\rangle_{\mathfrak{S}_\Delta(A)}\big) = \mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi(\bar{x};\bar{b})\rangle_{\mathfrak{S}_\Delta(B)}\big)\,.$$

Proof. By symmetry it is sufficient to prove that

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi(\bar{x};\bar{a})\rangle_{\mathfrak{S}_\Delta(A)}\big) \geq \alpha$$

implies

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi(\bar{x};\bar{b})\rangle_{\mathfrak{S}_\Delta(B)}\big) \geq \alpha\,.$$

We proceed by induction on α. For $\alpha = 0$ there is nothing to do. Since the limit step follows immediately from the inductive hypothesis, we may therefore assume that $\alpha = \beta + 1$. If

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi(\bar{x};\bar{a})\rangle_{\mathfrak{S}_\Delta(A)}\big) \geq \beta + 1\,,$$

we can use Lemma 1.1 to find formulae $\psi_n(\bar{x};\bar{c}^n) \in \Delta^+_A$, for $n < \omega$, with $\bar{c}^n \subseteq A$ such that

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi(\bar{x};\bar{a}) \wedge \psi_n(\bar{x};\bar{c}^n)\rangle_{\mathfrak{S}_\Delta(A)}\big) \geq \beta\,,$$

$$\text{and}\quad \mathfrak{A} \vDash \neg[\psi_m(\bar{x};\bar{c}^m) \wedge \psi_n(\bar{x};\bar{c}^n)]\,,\quad \text{for } m \neq n\,.$$

Since $\mathfrak{A} \sqsubseteq^{\aleph_0}_{\mathrm{FO}} \mathfrak{B}$, we can inductively find tuples $\bar{d}^n \subseteq B$, for $n < \omega$, such that

$$\langle\mathfrak{A},\bar{a}\bar{c}^0\dots\bar{c}^n\rangle \equiv \langle\mathfrak{B},\bar{b}\bar{d}^0\dots\bar{d}^n\rangle\,,\quad \text{for all } n < \omega\,.$$

This implies that

$$\mathfrak{B} \vDash \neg[\psi_m(\bar{x};\bar{d}^m) \wedge \psi_n(\bar{x};\bar{d}^n)]\,,\quad \text{for } m \neq n\,.$$

By inductive hypothesis, we furthermore have

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi(\bar{x};\bar{b}) \wedge \psi_n(\bar{x};\bar{d}^n)\rangle_{\mathfrak{S}_\Delta(B)}\big) \geq \beta\,,\quad \text{for all } n\,.$$

Consequently, Lemma 1.1 implies that

$$\mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi(\bar{x};\bar{b})\rangle_{\mathfrak{S}_\Delta(B)}\big) \geq \beta + 1\,. \qquad \square$$

It follows that there is a limit of the Cantor-Bendixson rank for increasing sets of parameters. This limit is called the *Δ-rank* of the theory.

Definition 1.4. (a) Let Δ be a set of formulae and $\varphi(\bar{x};\bar{c})$ an FO-formula with parameters $\bar{c} \subseteq \mathbb{M}$. The *$\Delta$-rank* of φ is

$$\mathrm{rk}_\Delta(\varphi(\bar{x};\bar{c})) := \mathrm{rk}_{\mathrm{CB}}\big(\langle\varphi(\bar{x};\bar{c})\rangle_{\mathfrak{S}_\Delta(M)}\big)\,,$$

where $\mathfrak{M} \preceq \mathbb{M}$ is an arbitrary $\aleph_0$-saturated model with $\bar{c} \subseteq M$.

(b) Let $\bar{s}$ be a tuple of sorts and let $\varphi(\bar{x};\bar{c})$ be an FO-formula with parameters $\bar{c} \subseteq \mathbb{M}$. The *Morley rank* of φ is

$$\mathrm{rk}^{\bar{s}}_{\mathrm{M}}(\varphi(\bar{x};\bar{c})) := \mathrm{rk}_\Delta(\varphi(\bar{x};\bar{c}))\,,$$

where Δ is the set of all first-order formulae $\psi(\bar{x};\bar{y})$ where the variables $\bar{x}$ have sorts $\bar{s}$.

(c) For a set of formulae $\Phi(\bar{x})$ (possibly with parameters) we define

$$\mathrm{rk}_\Delta(\Phi) := \min\{\,\mathrm{rk}_\Delta(\varphi) \mid \Phi \vDash \varphi\,\}\,,$$
$$\mathrm{rk}^{\bar{s}}_{\mathrm{M}}(\Phi) := \min\{\,\mathrm{rk}^{\bar{s}}_{\mathrm{M}}(\varphi) \mid \Phi \vDash \varphi\,\}\,.$$

For $\bar{a} \in \mathbb{M}^{\bar{s}}$ and $U \subseteq \mathbb{M}$, we set

$$\mathrm{rk}_\Delta(\bar{a}/U) := \mathrm{rk}_\Delta(\mathrm{tp}(\bar{a}/U))\,,$$
$$\mathrm{rk}_{\mathrm{M}}(\bar{a}/U) := \mathrm{rk}^{\bar{s}}_{\mathrm{M}}(\mathrm{tp}(\bar{a}/U))\,.$$

Remark. (a) Note that, by Lemmas 1.2 and 1.3, the definitions of $\mathrm{rk}_\Delta(\varphi)$ and $\mathrm{rk}^{\bar{s}}_{\mathrm{M}}(\varphi)$ do not depend on the choice of $\mathfrak{M}$. According to Theorem C3.4.5 (b), they also do not depend on what we consider the free variables of the formula φ. But note that, by Lemma 1.2, we have $\mathrm{rk}^{\bar{s}}_{\mathrm{M}}(\varphi) \leq$

$\mathrm{rk}_{\mathrm{M}}^{\bar{t}}(\varphi)$, for $\bar{s} \subseteq \bar{t}$. This inequality can be strict. An example is given by the formula $x = x$ with respect to the theory of infinite structures with empty signature. Then $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(x = x) = |\bar{s}|$.

(b) If $\mathfrak{p}$ is a complete type over an $\aleph_0$-saturated model $\mathfrak{M}$, it follows by Theorem B5.7.8 and Corollary B5.7.9 that

$$\mathrm{rk}_\Delta(\mathfrak{p}) = \mathrm{rk}_{\mathrm{CB}}(\mathfrak{p}/\mathfrak{S}_\Delta(M))\,.$$

Example. Consider the theory T of structures of the form $\langle A, \sim \rangle$, where $\sim$ is an equivalence relation on A with infinitely many classes, all of which are infinite. For $a \in \mathbb{M}$ and a model $\mathfrak{M} \prec \mathbb{M}$, we have

$$\mathrm{rk}_{\mathrm{M}}(a/M) = \begin{cases} 0 & \text{if } a \in M\,, \\ 1 & \text{if } a \notin M \text{ and } a \sim b \text{ for some } b \in M\,, \\ 2 & \text{otherwise}\,. \end{cases}$$

Exercise 1.1. Show that $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi) = 1$, for every strongly minimal formula $\varphi(\bar{x})$.

Exercise 1.2. Let T be the theory of structures of the form $\langle A, \sim \rangle$, where $\sim$ is an equivalence relation on A with infinitely many classes, all of which are infinite. Determine the possible values of $\mathrm{rk}_{\mathrm{M}}(ab/M)$, for two elements $a, b \in \mathbb{M}$ and a model $\mathfrak{M} \prec \mathbb{M}$.

Let us collect some basic properties of the Δ-rank of a formula.

Lemma 1.5. *Let T be a theory and φ, ψ formulae.*

(a) *$T \cup \{\varphi\} \vDash \psi$ implies* $\mathrm{rk}_\Delta(\varphi) \leq \mathrm{rk}_\Delta(\psi)$.

(b) $\mathrm{rk}_\Delta(\varphi \vee \psi) = \max\{\mathrm{rk}_\Delta(\varphi), \mathrm{rk}_\Delta(\psi)\}$.

(c) *If Δ contains the formula $x = y$, then* $\mathrm{rk}_\Delta(\varphi) = 0$ *if, and only if, φ is algebraic and consistent with T.*

Proof. (a) follows from Lemma B2.5.10, (b) from Lemma B2.5.11, and (c) follows immediately from the definition. □

Exercise 1.3. Show that $\mathrm{rk}_\Delta(\varphi \wedge \psi) \leq \min\{\mathrm{rk}_\Delta(\varphi), \mathrm{rk}_\Delta(\psi)\}$, and that this inequality may be strict.

Lemma 1.6. *Let $\bar{a}, \bar{b} \subseteq \mathbb{M}$ be tuples and $U, V \subseteq \mathbb{M}$ sets of parameters.*

(a) $\mathrm{rk}_\Delta(\bar{a}/U) \leq \mathrm{rk}_{\Delta \cup \Gamma}(\bar{a}/U)$.

(b) $\mathrm{rk}_\Delta(\bar{a}/U) \geq \mathrm{rk}_\Delta(\bar{a}/U \cup V)$.

(c) *There exists a finite subset $U_0 \subseteq U$ with* $\mathrm{rk}_\Delta(\bar{a}/U) = \mathrm{rk}_\Delta(\bar{a}/U_0)$.

Proof. (a) follows immediately from Lemma 1.2.

(b) By definition of the Δ-rank of a type, we have

$$\begin{aligned} \mathrm{rk}_\Delta(\bar{a}/U) &= \min\{\,\mathrm{rk}_\Delta(\varphi) \mid \varphi \in \mathrm{tp}(\bar{a}/U)\,\} \\ &\geq \min\{\,\mathrm{rk}_\Delta(\varphi) \mid \varphi \in \mathrm{tp}(\bar{a}/U \cup V)\,\} \\ &= \mathrm{rk}_\Delta(\bar{a}/U \cup V)\,. \end{aligned}$$

(c) Fix a formula $\varphi \in \mathrm{tp}(\bar{a}/U)$ such that $\mathrm{rk}_\Delta(\varphi) = \mathrm{rk}_\Delta(\bar{a}/U)$. Let $U_0 \subseteq U$ be the finite set of parameters from φ. Then $\varphi \in \mathrm{tp}(\bar{a}/U_0)$ implies

$$\mathrm{rk}_\Delta(\bar{a}/U_0) \leq \mathrm{rk}_\Delta(\varphi) = \mathrm{rk}_\Delta(\bar{a}/U) \leq \mathrm{rk}_\Delta(\bar{a}/U_0)\,,$$

where the last inequality holds by (b). □

For theories where it is defined, the Morley rank is usually better behaved than the Δ-rank. Let us collect some of its properties, in particular with respect to strongly minimal sets. First of all note that, using the equivalence of the Morley rank of a formula and its partition rank, we can define a notion of degree.

Definition 1.7. The *Morley degree* $\deg_{\mathrm{M}}^{\bar{s}}(\varphi)$ of a formula φ is the maximal number $m < \omega$ such that there are formulae $\psi_0, \dots, \psi_{m-1}$ of rank $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\psi_i) = \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi)$ such that $\psi_i^{\mathbb{M}} \cap \psi_k^{\mathbb{M}} = \varnothing$, for $i \neq k$. If such a number m does not exist, we set $\deg_{\mathrm{M}}^{\bar{s}}(\varphi) := \infty$.

Remark. It follows by Lemma B2.5.16 that

$$\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi) < \infty \quad \text{implies} \quad \deg_{\mathrm{M}}^{\bar{s}}(\varphi) < \infty\,.$$

Exercise 1.4. Show that a formula $\varphi(\bar{x})$ is strongly minimal if, and only if, $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi) = 1$ and $\deg_{\mathrm{M}}^{\bar{s}}(\varphi) = 1$.

For types there is a related notion of degree: the number of *free* extensions.

Definition 1.8. Let $\mathfrak{p} \subseteq \mathfrak{q}$ be (partial) types with free variables of sort $\bar{s}$. We say that $\mathfrak{q}$ is a *Morley-free extension* of $\mathfrak{p}$ if $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\mathfrak{q}) = \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\mathfrak{p})$.

Lemma 1.9. *Let $\mathfrak{p}$ be a (partial) type over U and suppose that $U \subseteq V$.*

(a) *$\mathfrak{p}$ has a Morley-free extension $\mathfrak{q} \in S^{\bar{s}}(V)$.*

(b) *If $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\mathfrak{p}) < \infty$, then $\mathfrak{p}$ has only finitely many Morley-free extensions in $S^{\bar{s}}(V)$.*

Proof. Choose an $\aleph_0$-saturated model $\mathfrak{M}$ containing V.

(a) First suppose that $\alpha := \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\mathfrak{p}) < \infty$. According to Lemma B5.5.15, the closed set $\langle \mathfrak{p} \rangle_{\mathfrak{S}^{\bar{s}}(M)}$ contains some type $\mathfrak{r}$ with

$$\mathrm{rk}_{\mathrm{CB}}(\mathfrak{r}/\mathfrak{S}^{\bar{s}}(M)) = \mathrm{rk}_{\mathrm{CB}}(\langle \mathfrak{p} \rangle_{\mathfrak{S}^{\bar{s}}(M)}) = \alpha\,.$$

Set $\mathfrak{q} := \mathfrak{r}|_V$. Then $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{r}$ implies

$$\alpha = \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\mathfrak{p}) \geq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\mathfrak{q}) \geq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\mathfrak{r}) = \mathrm{rk}_{\mathrm{CB}}(\mathfrak{r}/\mathfrak{S}^{\bar{s}}(M)) = \alpha\,.$$

Consequently, $\mathfrak{q}$ is the desired extension of $\mathfrak{p}$.

It remains to consider the case where $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\mathfrak{p}) = \infty$. Then

$$\mathrm{rk}_{\mathrm{CB}}(\langle \mathfrak{p} \rangle_{\mathfrak{S}^{\bar{s}}(M)}) = \infty$$

implies that there is some $\mathfrak{r} \in \langle \mathfrak{p} \rangle_{\mathfrak{S}^{\bar{s}}(M)}$ with $\mathrm{rk}_{\mathrm{CB}}(\mathfrak{r}/\mathfrak{S}^{\bar{s}}(M)) = \infty$. As above, it follows that $\mathfrak{q} := \mathfrak{r}|_V$ is the desired Morley-free extension of $\mathfrak{p}$ over V.

(b) Let $\alpha := \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\mathfrak{p})$. By (a), every type $\mathfrak{q} \in \langle \mathfrak{p} \rangle_{\mathfrak{S}^{\bar{s}}(V)}$ of rank α has an extension $\mathfrak{r} \in \langle \mathfrak{p} \rangle_{\mathfrak{S}^{\bar{s}}(M)}$ of the same rank. These extensions are obviously distinct, for different types $\mathfrak{q}$. The claim follows since, according to Lemma B5.5.15, the set $\langle \mathfrak{p} \rangle_{\mathfrak{S}^{\bar{s}}(M)}$ contains only finitely many types $\mathfrak{r}$ with $\mathrm{rk}_{\mathrm{CB}}(\mathfrak{r}/\mathfrak{S}^{\bar{s}}(M)) = \alpha$. □

Corollary 1.10. *For every formula $\varphi(\bar{x})$ over a set U, there exists some $\bar{a} \in \varphi^{\mathbb{M}}$ with $\mathrm{rk}_{\mathrm{M}}(\bar{a}/U) = \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi)$, where $\bar{s}$ are the sorts of $\bar{x}$.*

Proof. By Lemma 1.9, there exists a type $\mathfrak{q} \in S^{\bar{s}}(U)$ with $\{\varphi\} \subseteq \mathfrak{q}$ and $\mathrm{rk}_{\mathrm{M}}(\mathfrak{q}) = \mathrm{rk}_{\mathrm{M}}(\varphi)$. Every tuple $\bar{a}$ realising $\mathfrak{q}$ has the desired properties. □

The following lemmas show that the notion of Morley rank generalises the dimension of a strongly minimal set. We start by showing that the Morley rank increases with the length of a tuple and that elements in the algebraic closure do not increase the rank.

Lemma 1.11. *Let T be a first-order theory and let $\varphi(\bar{x}, \bar{y})$ be a formula with free variables $\bar{x}$ and $\bar{y}$ of sorts $\bar{s}$ and $\bar{t}$, respectively. Then*

$$\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\exists \bar{y}\varphi) \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi)\,.$$

Proof. We prove by induction on α that

$$\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\exists \bar{y}\varphi) \geq \alpha \quad \text{implies} \quad \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi) \geq \alpha\,.$$

For $\alpha = 0$, it is sufficient to note that the consistency of $\exists \bar{y}\varphi$ implies the one of φ. Hence, suppose that $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\exists \bar{y}\varphi) \geq \alpha$, for some $\alpha > 0$, and let $\beta < \alpha$. By Lemma 1.1, there are formulae $\psi_k(\bar{x})$, for $k < \omega$, such that

$$\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\exists \bar{y}\varphi \wedge \psi_k) \geq \beta \quad \text{and} \quad \psi_i^{\mathbb{M}} \cap \psi_k^{\mathbb{M}} = \varnothing\,, \quad \text{for all } i \neq k\,.$$

Note that, if $T \vDash \neg\exists \bar{y}\,\text{true}$, then $\exists \bar{y}\varphi$ is inconsistent with T. Hence, $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\exists \bar{y}\varphi) = -1 \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi)$ and we are done. Consequently, we may assume that $T \vDash \exists \bar{y}\,\text{true}$. We therefore have

$$\exists \bar{y}\varphi(\bar{x}, \bar{y}) \wedge \psi_k(\bar{x}) \equiv \exists \bar{y}[\varphi(\bar{x}, \bar{y}) \wedge \psi_k(\bar{x})] \quad \text{modulo } T.$$

It follows by inductive hypothesis that

$$\beta \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\exists \bar{y} \varphi \wedge \psi_k) = \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\exists \bar{y}(\varphi \wedge \psi_k)) \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi \wedge \psi_k)\,.$$

Since this holds for every β, it follows by Lemma 1.1 that $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi) \geq \alpha$. □

Lemma 1.12. *Let $\bar{a} \in \mathbb{M}^{\bar{s}}$ and $\bar{b} \in \mathbb{M}^{\bar{t}}$ be finite tuples and $U \subseteq \mathbb{M}$ a set of parameters.*

(a) $\mathrm{rk}_{\mathrm{M}}(\bar{a}/U) \leq \mathrm{rk}_{\mathrm{M}}(\bar{a}\bar{b}/U)$.

(b) $\mathrm{rk}_{\Delta}(\bar{a}/\operatorname{acl}(U)) = \mathrm{rk}_{\Delta}(\bar{a}/U)$.

(c) $\mathrm{rk}_{\mathrm{M}}(\bar{a}c/U) = \mathrm{rk}_{\mathrm{M}}(\bar{a}/U)$, *for all $c \in \operatorname{acl}(U \cup \bar{a})$.*

Proof. (a) Let $\alpha := \mathrm{rk}_{\mathrm{M}}(\bar{a}\bar{b}/U)$. By definition, there is a formula $\varphi(\bar{x}, \bar{y})$ over U such that $\mathbb{M} \vDash \varphi(\bar{a}, \bar{b})$ and $\mathrm{rk}_{\mathrm{M}}^{\bar{s}\bar{t}}(\varphi) = \mathrm{rk}_{\mathrm{M}}(\bar{a}\bar{b}/U)$. Then $\exists \bar{y} \varphi \in \operatorname{tp}(\bar{a}/U)$ implies, by Lemma 1.11, that

$$\mathrm{rk}_{\mathrm{M}}(\bar{a}/U) \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\exists \bar{y} \varphi) \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi) \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}\bar{t}}(\varphi) = \mathrm{rk}_{\mathrm{M}}(\bar{a}\bar{b}/U)\,,$$

as desired.

(b) It follows by Lemma 1.6 that $\mathrm{rk}_{\mathrm{M}}(\bar{a}/\operatorname{acl}(U)) \leq \mathrm{rk}_{\mathrm{M}}(\bar{a}/U)$. For a contradiction, suppose that this inequality is strict. Then there is some formula $\varphi(\bar{x}; \bar{c}) \in \operatorname{tp}(\bar{a}/\operatorname{acl}(U))$ such that $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi(\bar{x}; \bar{c})) < \mathrm{rk}_{\mathrm{M}}(\bar{a}/U)$. Since $\bar{c}$ is algebraic over U, we know by Lemma E3.1.3 that $\operatorname{tp}(\bar{c}/U)$ is isolated. Let $\psi(\bar{y})$ be a formula over U isolating this type and set

$$\vartheta(\bar{x}) := \exists \bar{y}[\varphi(\bar{x}; \bar{y}) \wedge \psi(\bar{y})]\,.$$

Then $\vartheta(\bar{x}) \in \operatorname{tp}(\bar{a}/U)$ implies, by Lemmas 1.5 and 1.11, that

$$\mathrm{rk}_{\mathrm{M}}(\bar{a}/U) \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\vartheta) \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi \wedge \psi) \leq \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\varphi) < \mathrm{rk}_{\mathrm{M}}(\bar{a}/U)\,.$$

A contradiction.

(c) We have just seen in (a) that $\mathrm{rk}_{\mathrm{M}}(\bar{a}c/U) \geq \mathrm{rk}_{\mathrm{M}}(\bar{a}/U)$. For the converse inequality, we prove by induction on α that, for elements $c \in \operatorname{acl}(U \cup \bar{a})$,

$$\mathrm{rk}_{\mathrm{M}}(\bar{a}c/U) \geq \alpha \quad \text{implies} \quad \mathrm{rk}_{\mathrm{M}}(\bar{a}/U) \geq \alpha\,.$$

For $\alpha = 0$, note that $\mathrm{rk}_{\mathrm{M}}(\bar{a}/U) \geq 0$ since $\operatorname{tp}(\bar{a}/U)$ is satisfiable. For limit ordinals α, the claim follows immediately by the inductive hypothesis. For the successor step, let

$$\mathrm{rk}_{\mathrm{M}}(\bar{a}c/U) \geq \alpha + 1$$

and, for a contradiction, suppose that $\mathrm{rk}_{\mathrm{M}}(\bar{a}/U) \leq \alpha$. Fix a formula $\varphi(\bar{x}) \in \operatorname{tp}(\bar{a}/U)$ over U with minimal rank. Since $c \in \operatorname{acl}(\bar{a}/U)$, there is a formula $\chi(\bar{x}, y)$ over U such that $\chi(\bar{a}, y)^{\mathbb{M}}$ is a finite set containing c. Let $m := |\chi(\bar{a}, y)^{\mathbb{M}}|$ and set

$$\vartheta(\bar{x}, y) := \varphi(\bar{x}) \wedge \chi(\bar{x}, y) \wedge \neg \exists^{m+1} y \chi(\bar{x}, y)\,.$$

Since $\vartheta \in \operatorname{tp}(\bar{a}c/U)$ we have $\mathrm{rk}_{\mathrm{M}}^{\bar{s}u}(\vartheta) \geq \mathrm{rk}_{\mathrm{M}}(\bar{a}c/U) \geq \alpha + 1$, where u is the sort of c. By Lemma 1.1, there are formulae ψ_n, for $n < \omega$, such that $\mathrm{rk}_{\mathrm{M}}^{\bar{s}u}(\vartheta \wedge \psi_n) \geq \alpha$ and $\psi_i^{\mathbb{M}} \cap \psi_k^{\mathbb{M}} = \varnothing$, for $i \neq k$. Set

$$\eta_n := \exists y(\vartheta \wedge \psi_n) \quad \text{and} \quad \eta_I := \bigwedge_{i \in I} \eta_i\,, \quad \text{for } I \subseteq \omega\,.$$

First, let us show that $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\eta_n) \geq \alpha$. By Lemma 1.10, there exists a tuple $\bar{b}d \in (\vartheta \wedge \psi_n)^{\mathbb{M}}$ such that $\mathrm{rk}_{\mathrm{M}}(\bar{b}d/U) = \mathrm{rk}_{\mathrm{M}}^{\bar{s}u}(\vartheta \wedge \psi_n)$. Then $d \in \operatorname{acl}(\bar{b})$ and, by inductive hypothesis,

$$\mathrm{rk}_{\mathrm{M}}(\bar{b}d/U) = \mathrm{rk}_{\mathrm{M}}^{\bar{s}u}(\vartheta \wedge \psi_n) \geq \alpha \quad \text{implies} \quad \mathrm{rk}_{\mathrm{M}}(\bar{b}/U) \geq \alpha\,.$$

Since $\eta_n \in \operatorname{tp}(\bar{b}/U)$, it follows that $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\eta_n) \geq \alpha$.

Furthermore, for every set $I \subseteq \omega$ of size $|I| > m$, the formula η_I is unsatisfiable since $\mathbb{M} \vDash \eta_I(\bar{b})$ implies that there are elements $d_i \in \mathbb{M}$, for $i \in I$, such that $\mathbb{M} \vDash \vartheta_i(\bar{b}, d_i)$. But, since $|\vartheta(\bar{b}, y)|^{\mathbb{M}} \leq m$ there must be indices $i < k$ in I such that $d_i = d_k$. Hence, $\bar{b}d_i$ satisfies $\psi_i \wedge \psi_k$, which contradicts our choice of the formulae ψ_n, $n < \omega$.

In particular, $\mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\eta_I) = -1 < \alpha$, for large enough sets I. The set

$$F := \big\{ I \subseteq \omega \bigm| \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\eta_I) \geq \alpha \text{ and there is no } J \supset I \text{ with } \mathrm{rk}_{\mathrm{M}}^{\bar{s}}(\eta_J) \geq \alpha \big\}$$

is infinite, since every $I \in F$ is finite and, for each $n < \omega$, there is some $I \in F$ with $n \in I$. Fix countably many distinct sets $I_0, I_1, \dots \in F$ and set

$$\xi_n := \eta_{I_n} \wedge \bigwedge_{i<n} \neg\eta_{I_i} \,.$$

By definition of F, $i \neq k$ implies $I_i \nsubseteq I_k$. Therefore, $I_i \cup I_k \notin F$ and

$$\mathrm{rk}^{\bar s}_{\mathrm{M}}(\eta_{I_i} \wedge \eta_{I_k}) = \mathrm{rk}^{\bar s}_{\mathrm{M}}(\eta_{I_i \cup I_k}) < \alpha\,, \quad \text{for } i \neq k\,.$$

By Lemma 1.5, this implies that

$$\mathrm{rk}^{\bar s}_{\mathrm{M}}(\eta_{I_i} \wedge \textstyle\bigvee_{k<i} \eta_{I_k}) = \mathrm{rk}^{\bar s}_{\mathrm{M}}(\bigvee_{k<i}(\eta_{I_i} \wedge \eta_{I_k})) < \alpha\,.$$

Since $\mathrm{rk}^{\bar s}_{\mathrm{M}}(\eta_{I_i}) = \alpha$, it therefore follows that

$$\mathrm{rk}^{\bar s}_{\mathrm{M}}(\xi_i) = \mathrm{rk}^{\bar s}_{\mathrm{M}}(\eta_{I_i} \wedge \neg \textstyle\bigvee_{k<i} \eta_{I_k}) \geq \alpha\,.$$

Note that $\xi_i \vDash \exists y \vartheta \vDash \varphi$ implies $\mathrm{rk}^{\bar s}_{\mathrm{M}}(\varphi \wedge \xi_i) \geq \mathrm{rk}^{\bar s}_{\mathrm{M}}(\xi_i) \geq \alpha$. As $\xi_i^{\mathbb{M}} \cap \xi_k^{\mathbb{M}} = \emptyset$, for $i \neq k$, it therefore follows by Lemma 1.1 that

$$\alpha < \mathrm{rk}^{\bar s}_{\mathrm{M}}(\varphi) = \mathrm{rk}_{\mathrm{M}}(\bar a/U) \leq \alpha\,.$$

A contradiction. □

Corollary 1.13. *Let $\varphi(\bar x)$ and $\psi(\bar y)$ be formulae with parameters and let $\bar s$ and $\bar t$ by the sorts of, respectively, $\bar x$ and $\bar y$. If there exists a parameter-definable surjective function $f : \varphi^{\mathbb{M}} \to \psi^{\mathbb{M}}$ such that $f^{-1}(\bar b)$ is finite, for every $\bar b \in \psi^{\mathbb{M}}$, then*

$$\mathrm{rk}^{\bar s}_{\mathrm{M}}(\varphi) = \mathrm{rk}^{\bar t}_{\mathrm{M}}(\psi)\,.$$

Proof. Let $U \subseteq \mathbb{M}$ be a set of parameters such that φ and ψ are over U and f is definable over U. By assumption, every $\bar a \in \varphi^{\mathbb{M}}$ is algebraic over $U \cup \{f(\bar a)\}$. Since $f(\bar a)$ is algebraic over $U \cup \bar a$, it follows by Lemma 1.12 that

$$\mathrm{rk}_{\mathrm{M}}(\bar a/U) = \mathrm{rk}_{\mathrm{M}}(\bar a f(\bar a)/U) = \mathrm{rk}_{\mathrm{M}}(f(\bar a)/U)\,.$$

We can use Corollary 1.10 to find tuples $\bar a \in \varphi^{\mathbb{M}}$ and $\bar b \in \psi^{\mathbb{M}}$ with

$$\mathrm{rk}_{\mathrm{M}}(\bar a/U) = \mathrm{rk}^{\bar s}_{\mathrm{M}}(\varphi) \quad \text{and} \quad \mathrm{rk}_{\mathrm{M}}(\bar b/U) = \mathrm{rk}^{\bar t}_{\mathrm{M}}(\psi)\,.$$

Then $\psi \in \mathrm{tp}(f(\bar a)/U)$ implies

$$\mathrm{rk}^{\bar t}_{\mathrm{M}}(\psi) \geq \mathrm{rk}_{\mathrm{M}}(f(\bar a)/U) = \mathrm{rk}_{\mathrm{M}}(\bar a/U) = \mathrm{rk}^{\bar s}_{\mathrm{M}}(\varphi)\,.$$

Conversely, by surjectivity of f, there is some $\bar c \in f^{-1}(\bar b)$. Therefore,

$$\mathrm{rk}^{\bar s}_{\mathrm{M}}(\varphi) \geq \mathrm{rk}_{\mathrm{M}}(\bar c/U) = \mathrm{rk}_{\mathrm{M}}(\bar b/U) = \mathrm{rk}^{\bar t}_{\mathrm{M}}(\psi)\,.$$ □

Finally, we are able to show that, in a strongly minimal set, the Morley rank of a finite tuple coincides with its dimension.

Theorem 1.14. *Let $\varphi(x)$ be a strongly minimal formula over U.*

$$\mathrm{rk}_{\mathrm{M}}(\bar a/U) = \dim_{\mathrm{acl}}(\bar a/U)\,, \quad \textit{for all finite tuples } \bar a \subseteq \varphi^{\mathbb{M}}.$$

Proof. Let $\bar a_0 \subseteq \bar a$ be an acl-basis of $\bar a$ over U. Then $|\bar a_0| = \dim_{\mathrm{acl}}(\bar a/U)$ and it follows by Lemma 1.12 that

$$\mathrm{rk}_{\mathrm{M}}(\bar a/U) = \mathrm{rk}_{\mathrm{M}}(\bar a_0/U)\,.$$

Hence, it is sufficient to prove that $\mathrm{rk}_{\mathrm{M}}(\bar a_0/U) = |\bar a_0|$. W.l.o.g. we may assume that $\bar a_0 = \bar a$, i.e., $\bar a$ is independent over U. We prove the claim by induction on $m := |\bar a|$. Let $\bar s$ be the sorts of $\bar a$.

First, suppose that $m = 1$, i.e., $\bar a = a_0$ and $\bar s = s_0$. As $\mathrm{tp}(a_0/U)$ contains the strongly minimal formula $\varphi(x)$, we have $\mathrm{rk}_{\mathrm{M}}(a_0/U) \leq \mathrm{rk}^{s_0}_{\mathrm{M}}(\varphi) = 1$. Conversely, $a_0 \notin \mathrm{acl}(U)$ implies that $\mathrm{tp}(a_0/U)$ is non-algebraic. Hence, for every formula $\psi(x) \in \mathrm{tp}(a_0/U)$, the set $\psi^{\mathbb{M}}$ is infinite and, therefore, $\mathrm{rk}^{s_0}_{\mathrm{M}}(\psi) \geq 1$.

For the inductive step, suppose that $m > 1$. We start by showing that $\mathrm{rk}_{\mathrm{M}}(\bar a/U) \geq m$. Note that $|\mathrm{acl}(A)| \leq |T|$, for every countable set A, while $|\varphi^{\mathbb{M}}| = |\mathbb{M}| > |T|$. Therefore, $\dim_{\mathrm{acl}}(\varphi^{\mathbb{M}}) > \aleph_0$ and we can fix a countably infinite set $I = \{ b^n_i \mid n < \omega,\ i < m \} \subseteq \varphi^{\mathbb M}$ that is independent over U. Setting $\bar b^n := \langle b^n_0, \dots, b^n_{m-1} \rangle$, it follows by Proposition F1.4.6 that

$$\mathrm{tp}(\bar b^n/U) = \mathrm{tp}(\bar a/U)\,, \quad \text{for every } n < \omega\,.$$

Let $I_0 := \{ b^n_0 \mid n < \omega \}$. Lemma F1.3.4 (a) implies that

$$\dim_{\mathrm{acl}}(\bar b^n/U \cup I_0) = \dim_{\mathrm{acl}}(\bar b^n/U \cup \{b^n_0\}) = m - 1\,.$$

By inductive hypothesis it therefore follows that

$$\mathrm{rk}_{\mathrm M}(\bar b^n/U \cup I_0) = m - 1\,.$$

Let $\vartheta(\bar x) \in \mathrm{tp}(\bar a/U)$ be a formula with $\mathrm{rk}^{\bar s}_{\mathrm M}(\vartheta) = \mathrm{rk}_{\mathrm M}(\bar a/U)$ and set $\psi_n(\bar x) := x_0 = b^n_0$. Then $\vartheta \wedge \psi_n \in \mathrm{tp}(\bar b^n/U \cup I_0)$ implies that

$$\mathrm{rk}^{\bar s}_{\mathrm M}(\vartheta \wedge \psi_n) \geq \mathrm{rk}_{\mathrm M}(\bar b^n/U \cup I_0) \geq m - 1\,.$$

Since $\psi^{\mathbb M}_i \cap \psi^{\mathbb M}_k = \emptyset$, for $i \neq k$, it follows by Lemma 1.1 that

$$\mathrm{rk}_{\mathrm M}(\bar a/U) = \mathrm{rk}^{\bar s}_{\mathrm M}(\vartheta) > \mathrm{rk}^{\bar s}_{\mathrm M}(\vartheta \wedge \psi_n) \geq m - 1\,.$$

It remains to prove that $\mathrm{rk}_{\mathrm M}(\bar a/U) \leq m$. Let $\mathfrak M$ be an $\aleph_0$-saturated model containing U. According to Proposition F1.4.6, every tuple $\bar c$ that is independent over M has the same type over U as $\bar a$. Replacing $\bar a$ by $\bar c$ we may therefore w.l.o.g. assume that $\bar a$ is independent over M. Fix a formula $\vartheta \in \mathrm{tp}(\bar a/U)$ such that $\mathrm{rk}^{\bar s}_{\mathrm M}(\vartheta) = \mathrm{rk}_{\mathrm M}(\bar a/U)$. For a contradiction, suppose that $\mathrm{rk}^{\bar s}_{\mathrm M}(\vartheta) > m$. Then, by Lemma 1.1, there are formulae ψ_i, $i < \omega$, such that $\mathrm{rk}^{\bar s}_{\mathrm M}(\vartheta \wedge \psi_i) \geq m$ and $\psi^{\mathbb M}_i \cap \psi^{\mathbb M}_k = \emptyset$, for $i \neq k$. By Lemma 1.3 and the definition of Morley rank, we can choose the formulae ψ_i over M. Since the sets $\psi^{\mathbb M}_i$ are disjoint, there is some index i such that $\bar a \notin \psi^{\mathbb M}_i$. Consequently, there exists a formula $\psi := \psi_i$ over M such that $\neg\psi \in \mathrm{tp}(\bar a/M)$ and $\mathrm{rk}^{\bar s}_{\mathrm M}(\psi) \geq \mathrm{rk}^{\bar s}_{\mathrm M}(\vartheta \wedge \psi) \geq m$.

By Corollary 1.10, there exists a tuple $\bar b \in \psi^{\mathbb M}$ with $\mathrm{rk}_{\mathrm M}(\bar b/M) = \mathrm{rk}^{\bar s}_{\mathrm M}(\psi)$. Since $\mathrm{tp}(\bar b/M) \neq \mathrm{tp}(\bar a/M)$, Proposition F1.4.6 implies that $\bar b$ is

not independent over M. Let $\bar b_0 \subseteq \bar b$ be an acl-basis of $\bar b$ over M. By Lemma 1.12 and inductive hypothesis, it follows that

$$\begin{aligned} m \leq \mathrm{rk}^{\bar s}_{\mathrm M}(\psi) = \mathrm{rk}^{\bar s}_{\mathrm M}(\bar b/M) &= \mathrm{rk}^{\bar s}_{\mathrm M}(\bar b_0/M) \\ &= \dim_{\mathrm{acl}}(\bar b_0/M) = |\bar b_0| < m\,, \end{aligned}$$

a contradiction. □

2. *Independence relations*

Besides closure operators and dimensions, a matroid can also be characterised in terms of a so-called *independence relation.* This characterisation is the easiest to generalise to the geometry-like configurations appearing in model theory. In this section we introduce independence relations and show that they give an alternative characterisation of matroids. In the next section, we then present the generalisation used in model theory.

Definition 2.1. Let cl be a closure operator on the set Ω. The *independence relation* $\overset{\mathrm{cl}}{\mathop{\vee\!\!\!\mid}}$ associated with cl is the ternary relation between sets $A, B, U \subseteq \Omega$ that is defined by

$$A \overset{\mathrm{cl}}{\mathop{\vee\!\!\!\mid}}_U B \quad :\text{iff} \quad \begin{array}{l} \text{every set } I \subseteq B \text{ that is independent over } U \\ \text{is also independent over } U \cup A\,. \end{array}$$

Example. Let $\mathfrak V$ be a vector space, $A, B, U \subseteq V$ subspaces with $U \subseteq A, B$, and let cl be the closure operator mapping a set $X \subseteq V$ to the subspace $\langle\!\langle X \rangle\!\rangle_{\mathfrak V}$ spanned by X. Then

$$A \overset{\mathrm{cl}}{\mathop{\vee\!\!\!\mid}}_U B \quad \text{iff} \quad A \cap B = U\,.$$

In the abstract, the properties of an independence relation $\overset{\mathrm{cl}}{\mathop{\vee\!\!\!\mid}}$ are given by the following axioms.

Definition 2.2. Let Ω be a set and let $A \surd_U B$ be a ternary relation on subsets $A, B, U \subseteq \Omega$.

(a) $\surd$ is an *abstract independence relation* if it satisfies the following conditions:

(MON) *Monotonicity.* If $A_0 \subseteq A$ and $B_0 \subseteq B$ then

$$A \surd_U B \quad \text{implies} \quad A_0 \surd_U B_0 .$$

(NOR) *Normality.*

$$A \surd_U B \quad \text{implies} \quad A \cup U \surd_U B \cup U .$$

(LRF) *Left Reflexivity.*

$$A \surd_A B , \quad \text{for all } A, B \subseteq \Omega .$$

(LTR) *Left Transitivity.* If $A_0 \subseteq A_1 \subseteq A_2$ then

$$A_2 \surd_{A_1} B \quad \text{and} \quad A_1 \surd_{A_0} B \quad \text{implies} \quad A_2 \surd_{A_0} B .$$

(FIN) *Finite Character.*

$$A \surd_U B \quad \text{iff} \quad A_0 \surd_U B \quad \text{for all finite } A_0 \subseteq A .$$

(b) A *geometric independence relation* is an abstract independence relation $\surd$ that satisfies the following additional conditions:

(SYM) *Symmetry.*

$$A \surd_U B \quad \text{implies} \quad B \surd_U A .$$

(BMON) *Base Monotonicity.*

$$A \surd_U B \cup C \quad \text{implies} \quad A \surd_{U \cup C} B \cup C .$$

(SRB) *Strong Right Boundedness.* Let γ be an ordinal and let $(U_\alpha)_{\alpha \leq \gamma}$ be a strictly increasing chain of subsets $U_\alpha \subseteq \Omega$. If $A \not\surd_{U_\alpha} U_{\alpha+1}$, for all $\alpha < \gamma$, then $|\gamma| \leq |A|$.

(c) We call an abstract independence relation *symmetric, base monotone,* or *strongly right bounded* if it satisfies the corresponding axiom. Frequently, we will use the symbol $\downarrow$ to denote symmetric independence relations.

Example. (a) Let Ω be a set. For $A, B, U \subseteq \Omega$, we set

$$A \overset{\circ}{\surd}_U B \quad :\text{iff} \quad A \subseteq U .$$

$\overset{\circ}{\surd}$ is an abstract independence relation on Ω that satisfies (BMON) and (SRB), but not (SYM). Moreover, it is minimal in the sense that $\overset{\circ}{\surd} \subseteq \surd$, for every abstract independence relation $\surd$ on Ω.

(b) Let Ω be a set. For $A, B, U \subseteq \Omega$, define

$$A \downarrow^{\circ}_U B \quad :\text{iff} \quad A \cap B \subseteq U .$$

Then $\downarrow^{\circ}$ is a geometric independence relation. It is minimal in the sense that $\downarrow^{\circ} \subseteq \downarrow$, for every symmetric independence relation on Ω. Note that $\downarrow^{\circ} = \overset{\mathrm{cl}}{\surd}$, where $\mathrm{cl} : X \mapsto X$ is the trivial closure operator on Ω.

(c) Let $\mathfrak{G} = \langle V, E \rangle$ be an undirected graph. For $A, B, U \subseteq V$, we define

$$A \downarrow^{\mathrm{sep}}_U B \quad :\text{iff} \quad \text{every path connecting an element of } A \text{ to an element of } B \text{ contains an element of } U .$$

Then $\downarrow^{\mathrm{sep}}$ is an abstract independence relation that is symmetric and base monotone.

As most axioms are immediate we only check left transitivity. Suppose, for a contradiction, that $A_2 \downarrow^{\mathrm{sep}}_{A_1} B$ and $A_1 \downarrow^{\mathrm{sep}}_{A_0} B$, but $A_2 \not\downarrow^{\mathrm{sep}}_{A_0} B$. Then there exists a path π from some vertex $a_2 \in A_2$ to some $b \in B$ such that π does not contain an element of A_0. Since $A_2 \downarrow^{\mathrm{sep}}_{A_1} B$, this path contains a vertex $a_1 \in A_1$. Let π' be the subpath of π connecting a_1 to b.

Since $A_1 \mathop{\downarrow}^{\mathrm{sep}}_{A_0} B$, this subpath contains a vertex of A_0. Hence, so does π. A contradiction.

(d) Let $\mathfrak{X} = \langle X, d \rangle$ be a metric space. For $A, B, U \subseteq X$, we define

$$A \mathop{\downarrow}^{d}_{U} B \quad :\text{iff} \quad \text{for all } a \in A \text{ and } b \in B \text{ there is some } c \in U \text{ such that } d(a,b) = d(a,c) + d(c,b)\,.$$

Again, $\mathop{\downarrow}^d$ is a symmetric abstract independence relation.

Note that, for (undirected) trees, this definition generalises that in (c). Given a tree T, we define the distance between two vertices $u, v \in T$ as the length of the unique path between u and v. The independence relation $\mathop{\downarrow}^d$ corresponding to this metric coincides with $\mathop{\downarrow}^{\mathrm{sep}}$ from (c) since the equation $d(u,v) = d(u,w) + d(w,v)$ implies that w is a vertex on the path from u to v.

Exercise 2.1. Given an abstract independence relation $\surd$, we define the relation

$$A \overset{b}{\surd}_U B \quad :\text{iff} \quad A \surd_{UB_0} B\,, \quad \text{for all } B_0 \subseteq B\,.$$

Prove that $\overset{b}{\surd}$ is a base monotone abstract independence relation.

Let us collect some immediate consequences of the axioms of an abstract independence relation. In proofs we will usually use the axioms (MON), (NOR), and (LRF) tacitly, while all uses of other axioms will be explicit. The first two lemmas contain versions of the left transitivity axiom that are frequently more convenient to use. The third lemma presents an infinite version of left transitivity.

Lemma 2.3. *Let $\surd$ be an abstract independence relation.*

$$A \surd_{U \cup C} B \quad \textit{and} \quad C \surd_U B \quad \textit{implies} \quad A \cup C \surd_U B\,.$$

Proof. By (NOR), we have $A \cup U \cup C \surd_{U \cup C} B$ and $C \cup U \surd_U B$. By (LTR) it follows that $A \cup U \cup C \surd_U B$. □

Lemma 2.4. *Let $\surd$ be a base monotone abstract independence relation.*

$$A \surd_U B \cup C \quad \textit{and} \quad C \surd_U B \quad \textit{implies} \quad A \cup C \surd_U B\,.$$

Proof. By (BMON), $A \surd_U B \cup C$ implies $A \surd_{U \cup C} B \cup C$. Since $C \surd_U B$, it follows by Lemma 2.3 and monotonicity that $A \cup C \surd_U B$. □

Lemma 2.5. *Let $\surd$ be an abstract independence relation.*

(a) *If $(A_i)_{i \in I}$ is an increasing chain of sets with $A_i \surd_U B$, for all $i \in I$, then $\bigcup_{i \in I} A_i \surd_U B$.*

(b) *If γ is an ordinal and $(A_\alpha)_{\alpha < \gamma}$ an increasing chain of sets with $A_\alpha \surd_{U \cup \bigcup_{i<\alpha} A_i} B$, for all $\alpha < \gamma$, then $\bigcup_{\alpha<\gamma} A_\alpha \surd_U B$.*

Proof. (a) By (FIN) it is sufficient to show that $C \surd_U B$, for all finite $C \subseteq \bigcup_{i \in I} A_i$. Hence, let $C \subseteq \bigcup_{i \in I} A_i$ be finite. As $(A_i)_{i \in I}$ is increasing, there exists an index $i \in I$ such that $C \subseteq A_i$. Consequently, $A_i \surd_U B$ implies that $C \surd_U B$.

(b) We prove the claim by induction on γ. For $\gamma = 0$, we have $\varnothing \surd_U B$ by (LRF). For the inductive step, suppose that $\bigcup_{i<\alpha} A_i \surd_U B$, for all $\alpha < \gamma$. By (a) it follows that $\bigcup_{\alpha<\gamma} \bigcup_{i<\alpha} A_i \surd_U B$. If γ is a limit ordinal, then $\bigcup_{\alpha<\gamma} \bigcup_{i<\alpha} A_i = \bigcup_{\alpha<\gamma} A_\alpha$ and we are done. Hence, suppose that $\gamma = \beta + 1$. Then

$$A_\beta \surd_{U \cup \bigcup_{i<\beta} A_i} B \quad \text{and} \quad \bigcup_{i<\beta} A_i \surd_U B$$

implies, by Lemma 2.3, that $A_\beta \surd_U B$. □

We will show that geometric independence relations are precisely those associated with a matroid. The easy direction is to show that every matroid induces a geometric independence relation. As a first step, let us see which axioms hold if we do not assume the exchange property.

Lemma 2.6. *The independence relation* $\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ *associated with a finitary closure operator* cl *on* Ω *is an abstract independence relation.*

Proof. We have to check five axioms.

(MON) Suppose that $A \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$ and let $A_0 \subseteq A$ and $B_0 \subseteq B$. To show that $A_0 \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B_0$, consider a subset $I \subseteq B_0$ that is independent over U. Since $A \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$, I is also independent over $U \cup A$. In particular, it is independent over $U \cup A_0$.

(NOR) Suppose that $A \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$. To show that $A \cup U \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B \cup U$, consider a set $I \subseteq B \cup U$ that is independent over U. Then $I \subseteq B$ and $A \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$ implies that I is independent over $U \cup A$.

(LRF) Trivially, if $I \subseteq B$ is independent over A, then it is independent over A.

(LTR) Suppose that $A_2 \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{A_1} B$ and $A_1 \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{A_0} B$, for $A_0 \subseteq A_1 \subseteq A_2$. If I is independent over A_0, it is independent over A_1 and, hence, also over A_2.

(FIN) Suppose that $A \not\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$. We have to find a finite set $A_0 \subseteq A$ such that $A_0 \not\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$. By assumption, there is a set $I \subseteq B$ that is independent over U, but not over $U \cup A$. Hence, there is some element $b \in I$ such that $b \in \mathrm{cl}(U \cup A \cup (I \smallsetminus \{b\}))$. We choose a finite subset $A_0 \subseteq A$ such that $b \in \mathrm{cl}(U \cup A_0 \cup (I \smallsetminus \{b\}))$. Since I is independent over U, but not over $U \cup A_0$, it follows that $A_0 \not\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$. □

To show that, for a matroid $\langle \Omega, \mathrm{cl} \rangle$, the relation $\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ is a geometric independence relation, we start with a technical lemma.

Lemma 2.7. *Let* $\langle \Omega, \mathrm{cl} \rangle$ *be a matroid and let* $I, J \subseteq \Omega$ *be sets that are both independent over* U. *If* I *is independent over* $U \cup J$, *then* J *is independent over* $U \cup I$.

Proof. Suppose that J is not independent over $U \cup I$. Then there is some $b \in J$ such that

$$b \in \mathrm{cl}(U \cup I \cup (J \smallsetminus \{b\})) \smallsetminus \mathrm{cl}(U \cup (J \smallsetminus \{b\}))\,.$$

By the exchange property, there is some $a \in I$ such that

$$a \in \mathrm{cl}(U \cup (I \smallsetminus \{a\}) \cup J)\,.$$

Consequently, I is not independent over $U \cup J$. □

Proposition 2.8. *The relation* $\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ *associated with a matroid* $\langle \Omega, \mathrm{cl} \rangle$ *is a geometric independence relation.*

Proof. We have already seen in Lemma 2.6 that $\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ is an abstract independence relation. Hence, it remains to check the following three axioms.

(SYM) Suppose that $A \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$. To show that $B \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U A$, consider a set $I \subseteq A$ that is independent over U. Let J be a basis of B over U. By assumption, J is independent over $U \cup A$. Hence, it follows by Lemma 2.7 that I is independent over $U \cup J$ and, therefore, over $U \cup B$.

(BMON) Since we have already shown (SYM), it is sufficient to prove that $A \cup C \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$ implies $A \cup C \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{U \cup C} B$. Thus, suppose that $A \cup C \mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_U B$. If $I \subseteq B$ is independent over $U \cup C$, it is also independent over U and, hence, over $U \cup A \cup C$.

(SRB) Let $(U_\alpha)_{\alpha \leq \gamma}$ be a strictly increasing sequence with $A \not\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{U_\alpha} U_{\alpha+1}$, for all $\alpha < \gamma$. By induction on α, we construct a decreasing chain $(I_\alpha)_{\alpha \leq \gamma}$ of subsets $I_\alpha \subseteq A$ such that I_α is a basis of A over U_α. We start with an arbitrary basis I_0 of A over U_0. For the inductive step, suppose that we have already defined I_β for all $\beta < \alpha$. For I_α we choose a maximal subset of $\bigcap_{\beta < \alpha} I_\beta$ that is independent over U_α.

Since $A \not\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{U_\alpha} U_{\alpha+1}$ we can find a set $J \subseteq U_{\alpha+1}$ that is independent over U_α, but not over $U_\alpha \cup A$. By Lemma 2.7 it follows that I_α is not independent over $U_\alpha \cup J \subseteq U_{\alpha+1}$. Therefore, each inclusion $I_{\alpha+1} \supset I_\alpha$ is strict. It follows that $|\gamma| \leq |I_0| \leq |A|$. □

Our next aim is to show that every geometric independence relation arises from a matroid. As motivation for the definition below, let us explain how one can recover the closure operation cl from the independence relation $\mathrel{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ associated with it.

Lemma 2.9. *Let* $\overset{\mathrm{cl}}{\surd}$ *be the independence relation associated with a closure operator* cl *on* Ω *and let* $a \in \Omega$ *and* $A, B, U \subseteq \Omega$.

(a) $$a \in \mathrm{cl}(U) \quad \text{iff} \quad a \overset{\mathrm{cl}}{\surd}_U a$$
$$\text{iff} \quad a \overset{\mathrm{cl}}{\surd}_{U\cup C} B \quad \textit{for all } B, C \subseteq \Omega\,.$$

(b) $$A \subseteq \mathrm{cl}(U \cup B) \quad \text{iff} \quad B \overset{\mathrm{cl}}{\surd}_U C \Rightarrow A \overset{\mathrm{cl}}{\surd}_U C \quad \textit{for all } C \subseteq \Omega\,.$$

Proof. (a) First, suppose that $a \in \mathrm{cl}(U)$. We claim that $a \overset{\mathrm{cl}}{\surd}_{U\cup C} B$, for all $B, C \subseteq \Omega$. Fix B and C and let $I \subseteq B$ be independent over $U \cup C$. Then I is independent over $\mathrm{cl}(U \cup C)$ and, therefore, over $U \cup \{a\} \subseteq \mathrm{cl}(U \cup C)$.

If $a \overset{\mathrm{cl}}{\surd}_{U\cup C} B$, for all B, C, then, trivially, $a \overset{\mathrm{cl}}{\surd}_U a$.

Hence, it remains to show that $a \overset{\mathrm{cl}}{\surd}_U a$ implies $a \in \mathrm{cl}(U)$. Suppose that $a \overset{\mathrm{cl}}{\surd}_U a$. Since the set $\{a\}$ is not independent over $U \cup \{a\}$, it follows that $\{a\}$ is not independent over U. Hence, $a \in \mathrm{cl}(U)$.

(b) ($\Rightarrow$) Suppose that $A \subseteq \mathrm{cl}(U \cup B)$ and $B \overset{\mathrm{cl}}{\surd}_U C$. To show that $A \overset{\mathrm{cl}}{\surd}_U C$, consider a set $I \subseteq C$ that is independent over U. Then I is also independent over $U \cup B$ and, hence, over $\mathrm{cl}(U \cup B)$. In particular, I is independent over $U \cup A \subseteq \mathrm{cl}(U \cup B)$.

($\Leftarrow$) Suppose that $A \nsubseteq \mathrm{cl}(U \cup B)$ and fix an element $a \in A \setminus \mathrm{cl}(U \cup B)$. Then $B \overset{\mathrm{cl}}{\surd}_U a$ since $\varnothing$ and $\{a\}$ are both independent over U and independent over $U \cup B$. But $A \overset{\mathrm{cl}}{\not\surd}_U a$ since $\{a\}$ is independent over U, but not over $U \cup A$. $\square$

We use the characterisation in (a) to associate a closure operator with an arbitrary abstract independence relation $\surd$.

Definition 2.10. Let $\surd$ be an abstract independence relation on the set Ω. For $U \subseteq \Omega$, we define

$$\mathrm{cl}_\surd(U) := \{\, a \in \Omega \mid a \surd_{U\cup C} B \text{ for all } B, C \subseteq \Omega \,\}\,.$$

Let us start by proving that this definition results in a closure operator. The main technical argument is contained in the following lemma.

Lemma 2.11. *Let* $\surd$ *be an abstract independence relation on the set* Ω.

$$A \subseteq \mathrm{cl}_\surd(U) \quad \text{iff} \quad A \surd_{U\cup C} B \quad \textit{for all } B, C \subseteq \Omega\,.$$

Proof. ($\Leftarrow$) Let $a \in A$. Then $a \surd_{U\cup C} B$, for all sets B, C. Consequently, $a \in \mathrm{cl}_\surd(U)$.

($\Rightarrow$) By (FIN), it is sufficient to prove the claim for finite sets A. We proceed by induction on $|A|$. For $A = \varnothing$ and arbitrary sets $B, C \subseteq \Omega$, $U \cup C \surd_{U\cup C} B$ implies that $\varnothing \surd_{U\cup C} B$, as desired.

Hence, suppose that $A = A_0 \cup \{a\}$ and that we have already shown that $A_0 \surd_{U\cup C} B$, for all sets B, C. Given $B, C \subseteq \Omega$, it follows that $A_0 \surd_{U\cup C\cup\{a\}} B$ and $a \surd_{U\cup C} B$ which, by Lemma 2.3, implies that $A_0 \cup \{a\} \surd_{U\cup C} B$. $\square$

Corollary 2.12. *Let* $\surd$ *be an abstract independence relation on the set* Ω.

$$\mathrm{cl}_\surd(U) \surd_{U\cup C} B\,, \quad \textit{for all } B, C, U \subseteq \Omega\,.$$

Proposition 2.13. *Let* $\surd$ *be an abstract independence relation on the set* Ω. *Then* $\mathrm{cl}_\surd$ *is a closure operator on* Ω.

Proof. To show that $U \subseteq \mathrm{cl}_\surd(U)$, consider $a \in U$ and $B, C \subseteq \Omega$. Then $U \cup C \surd_{U\cup C} B$ implies $a \surd_{U\cup C} B$. Hence, $a \in \mathrm{cl}_\surd(U)$.

For monotonicity, let $U \subseteq V$ and suppose that $a \surd_{U\cup C} B$, for all $B, C \subseteq \Omega$. Given $B, C \subseteq \Omega$, we have $a \surd_{U\cup V\cup C} B$. Hence, $\mathrm{cl}_\surd(U) \subseteq \mathrm{cl}_\surd(V)$.

To show that $\mathrm{cl}_\surd(\mathrm{cl}_\surd(U)) = \mathrm{cl}_\surd(U)$, fix an element $a \in \mathrm{cl}_\surd(\mathrm{cl}_\surd(U))$ and sets $B, C \subseteq \Omega$. Then

$$a \surd_{\mathrm{cl}_\surd(U)\cup\mathrm{cl}_\surd(U\cup C)} B\,.$$

Since we have already shown that $\mathrm{cl}_\surd$ is monotone, we have $\mathrm{cl}_\surd(U) \subseteq \mathrm{cl}_\surd(U \cup C)$ and it follows that $a \surd_{\mathrm{cl}_\surd(U\cup C)} B$. Furthermore, according to Corollary 2.12, $\mathrm{cl}_\surd(U \cup C) \surd_{U\cup C} B$. By Lemma 2.3 and monotonicity, it therefore follows that $a \surd_{U\cup C} B$. Hence, $a \in \mathrm{cl}_\surd(U)$. $\square$

For symmetric independence relations we have the following desirable relationship to the associated closure operator.

Lemma 2.14. *Let $\downarrow$ be an abstract independence relation on the set Ω satisfying* (SYM) *and* (BMON).

$$A \downarrow_U B \quad \text{iff} \quad \mathrm{cl}_\downarrow(A) \downarrow_{\mathrm{cl}_\downarrow(U)} \mathrm{cl}_\downarrow(B)\,, \quad \text{for all } A, B, U \subseteq \Omega\,.$$

Proof. ($\Leftarrow$) By Corollary 2.12, we have $\mathrm{cl}_\downarrow(U) \downarrow_U \mathrm{cl}_\downarrow(B)$. Therefore, $\mathrm{cl}_\downarrow(A) \downarrow_{\mathrm{cl}_\downarrow(U)} \mathrm{cl}_\downarrow(B)$ implies $\mathrm{cl}_\downarrow(A) \downarrow_U \mathrm{cl}_\downarrow(B)$, by Lemma 2.3. Hence, the claim follows by (MON).

($\Rightarrow$) Suppose that $A \downarrow_U B$. Then $A \cup U \downarrow_U B$. We have shown in Corollary 2.12 that $\mathrm{cl}_\downarrow(A \cup U) \downarrow_{A \cup U} B$. Using (LTR) we see that $\mathrm{cl}_\downarrow(A \cup U) \downarrow_U B$. By symmetry, it follows in exactly the same way that $\mathrm{cl}_\downarrow(A \cup U) \downarrow_U \mathrm{cl}_\downarrow(B \cup U)$. Hence, we can use (BMON) and (MON) to show that $\mathrm{cl}_\downarrow(A) \downarrow_{\mathrm{cl}_\downarrow(U)} \mathrm{cl}_\downarrow(B)$. □

If an abstract independence relation $\surd$ is induced by a closure operator, we obtain this operator back if we form $\mathrm{cl}_{\surd}$.

Lemma 2.15. $\mathrm{cl} = \mathrm{cl}_{\overset{\mathrm{cl}}{\surd}}$, *for every finitary closure operator* cl.

Proof. By definition of $\mathrm{cl}_{\overset{\mathrm{cl}}{\surd}}$ and Lemma 2.9,

$$\begin{aligned} a \in \mathrm{cl}_{\overset{\mathrm{cl}}{\surd}}(U) \quad &\text{iff} \quad a \overset{\mathrm{cl}}{\surd}_{U \cup C} B \text{ for all sets } B, C \\ &\text{iff} \quad a \in \mathrm{cl}(U)\,. \end{aligned}$$ □

Remark. Note that, in general, the dual statement does not hold: there are distinct independence relations inducing the same closure operator.

For a geometric independence relation $\downarrow$, we not only obtain a closure operator, but even a matroid. Again, we begin with two technical lemmas.

Lemma 2.16. *Let $\downarrow$ be a geometric independence relation. Then*

$$a \not\downarrow_U B \quad \text{iff} \quad a \in \mathrm{cl}_\downarrow(U \cup B) \setminus \mathrm{cl}_\downarrow(U)\,.$$

Proof. ($\Leftarrow$) Suppose that $a \in \mathrm{cl}_\downarrow(U \cup B)$ and $a \downarrow_U B$. We have to show that $a \in \mathrm{cl}_\downarrow(U)$. Hence, let $C, D \subseteq \Omega$ be arbitrary sets. Then $a \downarrow_{U \cup B} C \cup D$ and $a \downarrow_U B$ implies, by Lemma 2.3 and symmetry, that $a \downarrow_U C \cup D$. Consequently, we have $a \downarrow_{U \cup C} D$ by (BMON).

($\Rightarrow$) Suppose that $a \not\downarrow_U B$. Then $a \notin \mathrm{cl}_\downarrow(U)$. For a contradiction, assume that there are sets C, D such that $a \not\downarrow_{U \cup B \cup C} D$. Then (MON) implies

$$a \not\downarrow_U U \cup B \cup C \quad \text{and} \quad a \not\downarrow_{U \cup B \cup C} U \cup B \cup C \cup D\,.$$

By (SRB), it follows that $2 \leq |\{a\}| = 1$. A contradiction. □

Lemma 2.17. *Let $\downarrow$ be a geometric independence relation on Ω. For all $a \in \Omega$ and $B \subseteq \Omega$, there exists a finite set $B_0 \subseteq B$ such that $a \downarrow_{B_0} B$.*

Proof. We prove the claim by induction on $\kappa := |B|$. For $\kappa < \aleph_0$, we have $a \downarrow_B B$ by (LRF) and symmetry. Hence, suppose that $\kappa \geq \aleph_0$. Let $(b_\alpha)_{\alpha<\kappa}$ be an enumeration of B and set $B_\alpha := \{\, b_i \mid i < \alpha \,\}$, for $\alpha \leq \kappa$. If $a \downarrow_\varnothing B$, we are done. Otherwise, let α be the minimal ordinal such that $a \not\downarrow_\varnothing B_\alpha$. By Lemma 2.16, it follows that $a \in \mathrm{cl}_\downarrow(B_\alpha)$. Consequently, $a \downarrow_{B_\alpha} B$. Note that $\alpha < \kappa$ since $a \downarrow_\varnothing B_\beta$ for all $\beta < \kappa$ would imply, by Lemma 2.5 and symmetry, that $a \downarrow_\varnothing B$. Hence $|B_\alpha| = |\alpha| < \kappa$, and we can apply the inductive hypothesis to find a finite set $U \subseteq B_\alpha$ with $a \downarrow_U B_\alpha$. Consequently, it follows by (LTR) and symmetry that $a \downarrow_U B$. □

Proposition 2.18. *If $\downarrow$ is a geometric independence relation on the set Ω, then $\langle \Omega, \mathrm{cl}_\downarrow \rangle$ is a matroid.*

Proof. We have already seen in Proposition 2.13 that $\mathrm{cl}_\downarrow$ is a closure operator. Hence, it remains to check that it has finite character and the exchange property.

For finite character, suppose that $a \in \mathrm{cl}_\downarrow(U)$. By Lemma 2.17 we can find a finite set $U_0 \subseteq U$ such that $a \downarrow_{U_0} U$. For all sets B, C it follows by $a \downarrow_U B \cup C$, Lemma 2.3, and (SYM) that $a \downarrow_{U_0} B \cup C$. Hence, (BMON) implies $a \downarrow_{U_0 \cup C} B$ and we have $a \in \mathrm{cl}_\downarrow(U_0)$.

It remains to check the exchange property. Suppose that

$$b \in \mathrm{cl}_{\downarrow}(U \cup \{a\}) \smallsetminus \mathrm{cl}_{\downarrow}(U)\,.$$

By Lemma 2.16, it follows that $b \not\downarrow_U a$. By symmetry, we have $a \not\downarrow_U b$ and we can use Lemma 2.16 again to conclude that

$$a \in \mathrm{cl}_{\downarrow}(U \cup \{b\}) \smallsetminus \mathrm{cl}_{\downarrow}(U)\,. \qquad \square$$

The next lemma, together with Lemma 2.15, shows that the operation $\mathrm{cl} \mapsto \overset{\mathrm{cl}}{\surd}$ is a bijective function from the class of all matroids to the class of all geometric independence relations. Its inverse is given by the function $\downarrow\ \mapsto \mathrm{cl}_{\downarrow}$.

Lemma 2.19. *If $\downarrow$ is a geometric independence relation then $\overset{\mathrm{cl}_{\downarrow}}{\surd} = \downarrow$.*

Proof. ($\supseteq$) Suppose that $A \overset{\mathrm{cl}_{\downarrow}}{\not\surd}_U B$. We have to show that $A \not\downarrow_U B$. By assumption, there exists a set $I \subseteq B$ that is $\mathrm{cl}_{\downarrow}$-independent over U, but not over $U \cup A$. Fix an element $b \in I$ such that $b \in \mathrm{cl}_{\downarrow}(U \cup A \cup I_0)$ where $I_0 := I \smallsetminus \{b\}$. Since $b \notin \mathrm{cl}_{\downarrow}(U \cup I_0)$, it follows by Lemma 2.16 that $b \not\downarrow_{U \cup I_0} A$. By monotonicity, this implies that $B \not\downarrow_{U \cup I_0} A$. Hence, we can use symmetry and (BMON) to deduce that $A \not\downarrow_U B$.

($\subseteq$) By (FIN) and symmetry, it is sufficient to show that $A \overset{\mathrm{cl}_{\downarrow}}{\surd}_U B$ implies $A \downarrow_U B$, for all finite sets A, B. Furthermore, we may assume by Lemmas 2.14 and 2.15 that A and B are $\mathrm{cl}_{\downarrow}$-independent over U. Hence, suppose that $A \overset{\mathrm{cl}_{\downarrow}}{\surd}_U B$ for finite sets A and B that are $\mathrm{cl}_{\downarrow}$-independent over U. We prove by induction on $|B|$ that $B \downarrow_U A$. If $B = \varnothing$, then $U \downarrow_U A$ implies $\varnothing \downarrow_U A$. Hence, suppose that $B = B_0 \cup \{b\}$ and that we have already shown that $B_0 \downarrow_U A$. Since B is $\mathrm{cl}_{\downarrow}$-independent over U, it is also $\mathrm{cl}_{\downarrow}$-independent over $U \cup A$. Hence, $b \notin \mathrm{cl}_{\downarrow}(U \cup A \cup B_0)$ and Lemma 2.16 implies that $b \downarrow_{U \cup B_0} A$. Together with $B_0 \downarrow_U A$ it follows by Lemma 2.3 that $B_0 \cup \{b\} \downarrow_U A$. $\square$

We conclude this section with a characterisation of modularity in terms of the independence relation $\overset{\mathrm{cl}}{\surd}$.

Proposition 2.20. *A matroid $\langle \Omega, \mathrm{cl} \rangle$ is modular if, and only if,*

$$A \overset{\mathrm{cl}}{\surd}_{\mathrm{cl}(A) \cap \mathrm{cl}(B)} B\,, \quad \textit{for all } A, B \subseteq \Omega\,.$$

Proof. ($\Rightarrow$) Suppose that $\langle \Omega, \mathrm{cl} \rangle$ is modular and let $A, B \subseteq \Omega$. We have to show that $A \overset{\mathrm{cl}}{\surd}_{\mathrm{cl}(A) \cap \mathrm{cl}(B)} B$. By Lemmas 2.14 and 2.15, we may assume that A and B are closed sets. Hence, let A and B be closed and $I \subseteq B$ independent over $A \cap B$. Let $I_0 \subseteq I$ be a basis of I over A and set $C_0 := \mathrm{cl}(I_0)$ and $C := \mathrm{cl}(I)$. We have to show that $I_0 = I$. Note that

$$\mathrm{cl}(C_0 \cup A) = \mathrm{cl}(I_0 \cup A) = \mathrm{cl}(I \cup A) = \mathrm{cl}(C \cup A)\,.$$

By Lemma B2.2.9, it follows that

$$C = \mathrm{cl}(C_0 \cup (C \cap A)) = \mathrm{cl}(I_0 \cup (C \cap A))\,.$$

Hence, I_0 is a basis of C over $C \cap A$. Since $I \supseteq I_0$ is independent over $C \cap A$, it follows that $I = I_0$ and I is independent over A.

($\Leftarrow$) Suppose that $A \overset{\mathrm{cl}}{\surd}_{\mathrm{cl}(A) \cap \mathrm{cl}(B)} B$, for all $A, B \subseteq \Omega$. To show that $\langle \Omega, \mathrm{cl} \rangle$ is modular it is sufficient, by Lemma B2.2.9, to prove that

$$\mathrm{cl}(A \cup C) = \mathrm{cl}(B \cup C) \quad \text{implies} \quad \mathrm{cl}(A \cup (B \cap C)) = B\,,$$

for all closed sets $A, B, C \subseteq \Omega$ with $A \subseteq B$. Hence, fix closed sets $A, B, C \subseteq \Omega$ with $A \subseteq B$ and $\mathrm{cl}(A \cup C) = \mathrm{cl}(B \cup C)$. Choose a maximal set $I \subseteq A$ that is independent over C. Then $\mathrm{cl}(I \cup C) = \mathrm{cl}(A \cup C) = \mathrm{cl}(B \cup C)$ and I is a basis of $B \cup C$ over C. We claim that $B \subseteq \mathrm{cl}(I \cup (B \cap C))$. Suppose otherwise. Then there is some element $b \in B \smallsetminus \mathrm{cl}(I \cup (B \cap C))$. Since $b \in B \subseteq \mathrm{cl}(I \cup C)$ and $b \notin \mathrm{cl}(I \cup (B \cap C))$, it follows that $I \cup \{b\}$ is independent over $B \cap C$, but not over C. Hence, $C \overset{\mathrm{cl}}{\not\surd}_{B \cap C} B$. A contradiction.

We have shown that $B \subseteq \mathrm{cl}(I \cup (B \cap C))$. It follows that

$$B \subseteq \mathrm{cl}(I \cup (B \cap C)) \subseteq \mathrm{cl}(A \cup (B \cap C)) \subseteq B\,,$$

as desired. $\square$

Corollary 2.21. *Let $\langle \Omega, \mathrm{cl} \rangle$ be a modular matroid. Then*

$$A \sqrt{}^{\mathrm{cl}}_U B \quad \text{iff} \quad \mathrm{cl}(A \cup U) \cap \mathrm{cl}(B \cup U) = \mathrm{cl}(U)\,.$$

Proof. ($\Leftarrow$) According to Proposition 2.20, we have

$$A \cup U \sqrt{}^{\mathrm{cl}}_{\mathrm{cl}(A \cup U) \cap \mathrm{cl}(B \cup U)} B \cup U\,.$$

If $\mathrm{cl}(A \cup U) \cap \mathrm{cl}(B \cup U) = \mathrm{cl}(U)$, then

$$A \cup U \sqrt{}^{\mathrm{cl}}_{\mathrm{cl}(U)} B \cup U \quad \text{implies} \quad A \sqrt{}^{\mathrm{cl}}_U B\,,$$

by Lemma 2.14.

($\Rightarrow$) Suppose that $A \sqrt{}^{\mathrm{cl}}_U B$. By Lemma 2.14, it follows that

$$\mathrm{cl}(A \cup U) \sqrt{}^{\mathrm{cl}}_{\mathrm{cl}(U)} \mathrm{cl}(B \cup U)\,.$$

For a contradiction, suppose that there is some element

$$c \in (\mathrm{cl}(A \cup U) \cap \mathrm{cl}(B \cup U)) \smallsetminus \mathrm{cl}(U)\,.$$

Then $\{c\}$ is independent over $\mathrm{cl}(U)$, but not over $\mathrm{cl}(A \cup U)$. Hence, $\mathrm{cl}(A \cup U) \not\sqrt{}^{\mathrm{cl}}_{\mathrm{cl}(U)} \mathrm{cl}(B \cup U)$. A contradiction. □

3. *Preforking relations*

We would like to define an independence relation using Δ-rank or Morley rank as our notion of dimension. In general, the resulting relation will not be a geometric independence relation but something slightly weaker, called a *forking relation.* In this section, we introduce the abstract framework for forking relations and we will present several examples of such relations. To simplify notation, we will frequently omit union symbols and just write AB instead of $A \cup B$.

Definition 3.1. Let T be a complete first-order theory and suppose that $A \sqrt{}_U B$ is a ternary relation that is defined on the class of all small subsets $A, B, U \subseteq \mathbb{M}$.

(a) The relation $\sqrt{}$ is a *preforking relation* for T if it is an abstract independence relation that satisfies (BMON) and the following two axioms:

(INV) *Invariance.* $ABU \equiv_\varnothing A'B'U'$ implies that

$$A \sqrt{}_U B \quad \text{iff} \quad A' \sqrt{}_{U'} B'\,.$$

(DEF) *Definability.* If $A \not\sqrt{}_U B$, there are finite tuples $\bar a \subseteq A$ and $\bar b \subseteq B$ and a formula $\varphi(\bar x, \bar x') \in \mathrm{tp}(\bar a \bar b / U)$ such that

$$\bar a' \not\sqrt{}_U \bar b\,, \quad \text{for all } \bar a' \in \varphi(\bar x, \bar b)^{\mathbb{M}}\,.$$

(b) The relation $\sqrt{}$ is a *forking relation* if it is a preforking relation that satisfies the following additional axiom:

(EXT) *Extension.* If $A \sqrt{}_U B_0$ and $B_0 \subseteq B_1$ then there is some A' with

$$A' \equiv_{UB_0} A \quad \text{and} \quad A' \sqrt{}_U B_1\,.$$

We are mostly interested in symmetric forking relations since many properties of geometric independence relations can be generalised to them. Unfortunately, there are first-order theories were no nontrivial symmetric forking relations exist. On the other hand there are always several natural preforking relations and below we will see that every preforking relation can be used to define a corresponding forking relation, although not necessarily a symmetric one.

Remark. The intersection of an arbitrary family of preforking relations is again a preforking relation. It follows that the class of all preforking relations on a structure $\mathbb{M}$ forms a complete partial order.

Examples

Before proceeding let us collect several examples. We start with a trivial one.

Example. The trivial relation $\surd$ with $A \surd_U B$, for all sets A, B, U, is a symmetric forking relation.

Exercise 3.1. Prove that the relation

$$A \mathop{\downarrow}^{\mathrm{o}}_U B \quad :\text{iff} \quad A \cap B \subseteq U$$

is a symmetric preforking relation.

More interesting are the following three examples. The second one has historically been used to develop stability theory.

Definition 3.2. For $\bar{a}, A, B, U \subseteq \mathbb{M}$, we define

$$\begin{array}{lll} A \overset{\mathrm{at}}{\surd}_U B & :\text{iff} & \text{for every finite } \bar{a} \subseteq A, \\ & & \operatorname{tp}(\bar{a}/UB) \text{ is isolated by a formula over } U. \\ \bar{a} \overset{\mathrm{df}}{\surd}_U B & :\text{iff} & \operatorname{tp}(\bar{a}/UB) \text{ is definable over } U. \\ A \overset{\mathrm{s}}{\surd}_U B & :\text{iff} & \bar{b} \equiv_U \bar{b}' \ \Rightarrow\ \bar{b} \equiv_{UA} \bar{b}', \quad \text{for all } \bar{b}, \bar{b}' \subseteq B. \end{array}$$

If $\bar{a} \overset{\mathrm{s}}{\surd}_U B$, we say that the type $\operatorname{tp}(\bar{a}/UB)$ is *invariant* over U. Otherwise, it *splits* over U.

Lemma 3.3.

(a) $\overset{\mathrm{at}}{\surd} \subseteq \overset{\mathrm{df}}{\surd} \subseteq \overset{\mathrm{s}}{\surd}$

(b) $\overset{\mathrm{at}}{\surd}$ *is an abstract independence relation that satisfies* (INV) *and* (BMON).

(c) $\overset{\mathrm{df}}{\surd}$ *is an abstract independence relation that satisfies* (INV) *and* (BMON).

(d) $\overset{\mathrm{s}}{\surd}$ *is a preforking relation.*

Proof. (a) Suppose that $A \overset{\mathrm{at}}{\surd}_U B$ and let $\bar{a}$ be an enumeration of A. To show that $A \overset{\mathrm{df}}{\surd}_U B$, consider a formula $\varphi(\bar{x}; \bar{b}) \in \operatorname{tp}(\bar{a}/UB)$. Let $\bar{a}_0 \subseteq \bar{a}$

be the finite tuple of elements mentioned in φ. By assumption, there is a formula $\psi(\bar{x})$ over U isolating $\operatorname{tp}(\bar{a}_0/UB)$. It follows that

$$\delta(\bar{y}) := \forall \bar{x}[\psi(\bar{x}) \rightarrow \varphi(\bar{x}; \bar{y})]$$

is a φ-definition of $\operatorname{tp}(\bar{a}/UB)$.

For the second inclusion, suppose that $A \overset{\mathrm{df}}{\surd}_U B$. Let $\bar{b}, \bar{b}' \subseteq B$ be tuples with $\bar{b} \not\equiv_{UA} \bar{b}'$. We have to show that $\bar{b} \not\equiv_U \bar{b}'$. Fix a formula $\varphi(\bar{x}; \bar{a}, \bar{c})$ with parameters $\bar{a} \subseteq A$ and $\bar{c} \subseteq U$ such that

$$\mathbb{M} \vDash \varphi(\bar{b}; \bar{a}, \bar{c}) \wedge \neg\varphi(\bar{b}'; \bar{a}, \bar{c}) .$$

By assumption, $\operatorname{tp}(\bar{a}/UB)$ has a φ-definition $\delta(\bar{x})$ over U. It follows that $\mathbb{M} \vDash \delta(\bar{b}) \wedge \neg\delta(\bar{b}')$. Consequently, $\bar{b} \not\equiv_U \bar{b}'$.

(b) (INV) and (FIN) follow immediately from the definition.

(MON) Suppose that $A \overset{\mathrm{at}}{\surd}_U B$ and let $A_0 \subseteq A$, $B_0 \subseteq B$. For $\bar{a} \subseteq A_0$ we know that $\operatorname{tp}(\bar{a}/UB)$ is isolated by a formula over U. Hence, so is $\operatorname{tp}(\bar{a}/UB_0)$.

(NOR) Suppose that $A \overset{\mathrm{at}}{\surd}_U B$. Let $\bar{a} \subseteq A \cup U$ be finite. Then $\bar{a} = \bar{a}' \cup \bar{c}$ for $\bar{a}' \subseteq A$ and $\bar{c} \subseteq U$. Furthermore, $\operatorname{tp}(\bar{a}'/UB)$ is isolated by a formula $\varphi(\bar{x})$ over U and $\operatorname{tp}(\bar{c}/UB)$ is isolated by the formula $\bar{x} = \bar{c}$. Consequently, $\operatorname{tp}(\bar{a}'\bar{c}/UB)$ is isolated by $\psi(\bar{x}, \bar{x}') := \varphi(\bar{x}) \wedge \bar{x}' = \bar{c}$.

(LRF) If $\bar{a} \subseteq A$ is finite then $\operatorname{tp}(\bar{a}/AB)$ is isolated by the formula $\bar{x} = \bar{a}$. Hence, $A \overset{\mathrm{at}}{\surd}_A B$.

(LTR) Suppose that $A_2 \overset{\mathrm{at}}{\surd}_{A_1} B$ and $A_1 \overset{\mathrm{at}}{\surd}_{A_0} B$ for $A_0 \subseteq A_1 \subseteq A_2$. Let $\bar{a} \subseteq A_2$ be finite. Then $\operatorname{tp}(\bar{a}/A_1B)$ is isolated by a formula $\varphi(\bar{x}; \bar{c})$ with parameters $\bar{c} \subseteq A_1$. Furthermore, $\operatorname{tp}(\bar{c}/A_0B)$ is isolated by a formula $\psi(\bar{x})$ over A_0. By Lemma E3.1.5, it follows that $\operatorname{tp}(\bar{a}\bar{c}/A_0B)$ is isolated by the formula $\varphi(\bar{x}; \bar{z}) \wedge \psi(\bar{z})$. Therefore, $\operatorname{tp}(\bar{a}/A_0B)$ is isolated by the formula $\exists \bar{z}[\varphi(\bar{x}; \bar{z}) \wedge \psi(\bar{z})]$.

(BMON) Suppose that $A \overset{\mathrm{at}}{\surd}_U BC$. For every $\bar{a} \subseteq A$, $\operatorname{tp}(\bar{a}/UBC)$ is isolated by a formula over U and, hence, by a formula over $U \cup C$.

(c) (INV) follows immediately from the definition.

(MON) Suppose that $\bar a \mathrel{\overset{\mathrm{df}}{\surd}}_U B$. If $\bar a_0 \subseteq \bar a$ and $B_0 \subseteq B$ then

$$\mathrm{tp}(\bar a_0/UB_0) \subseteq \mathrm{tp}(\bar a/UB)$$

and every φ-definition of the latter type is also a φ-definition of the former one.

(NOR) Suppose that $\mathrm{tp}(\bar a/B\bar c)$ is definable over $\bar c$. To find the desired $\varphi(\bar x, \bar x'; \bar y)$-definition of $\mathrm{tp}(\bar a\bar c/B\bar c)$ over $\bar c$, let $\psi(\bar y, \bar y'; \bar c)$ be a $\varphi(\bar x; \bar y', \bar y)$-definition of $\mathrm{tp}(\bar a/B\bar c)$ over $\bar c$. For $\bar b \subseteq B \cup \bar c$ it follows that

$$\mathbb{M} \vDash \varphi(\bar a, \bar c; \bar b) \quad \text{iff} \quad \mathbb{M} \vDash \psi(\bar b, \bar c; \bar c)\,.$$

Hence, $\psi(\bar y, \bar c; \bar c)$ is a φ-definition of $\mathrm{tp}(\bar a\bar c/B)$ over $\bar c$.

(LRF) Note that $\varphi(\bar a; \bar y)$ is a $\varphi(\bar x; \bar y)$-definition of $\mathrm{tp}(\bar a/B \cup \bar a)$. Hence, $\mathrm{tp}(\bar a/B\bar a)$ is definable over $\bar a$.

(LTR) Suppose that $\bar a_0\bar a_1\bar a_2 \mathrel{\overset{\mathrm{df}}{\surd}}_{\bar a_0\bar a_1} B$ and $\bar a_0\bar a_1 \mathrel{\overset{\mathrm{df}}{\surd}}_{\bar a_0} B$. For every formula $\varphi(\bar x_0, \bar x_1, \bar x_2; \bar y)$, there exist

- a φ-definition $\psi(\bar y; \bar a_0, \bar a_1)$ of $\mathrm{tp}(\bar a_0\bar a_1\bar a_2/B\bar a_0\bar a_1)$ over $\bar a_0\bar a_1$, and
- a $\psi(\bar y; \bar x_0, \bar x_1)$-definition $\vartheta(\bar y; \bar a_0)$ of $\mathrm{tp}(\bar a_0\bar a_1/B\bar a_0)$ over $\bar a_0$.

For $\bar b \subseteq B \cup \bar a_0$, we have

$$\begin{aligned} \mathbb{M} \vDash \varphi(\bar a_0, \bar a_1, \bar a_2; \bar b) \quad &\text{iff} \quad \mathbb{M} \vDash \psi(\bar b; \bar a_0, \bar a_1) \\ &\text{iff} \quad \mathbb{M} \vDash \vartheta(\bar b; \bar a_0)\,. \end{aligned}$$

Hence, ϑ is a φ-definition of $\mathrm{tp}(\bar a_0\bar a_1\bar a_2/B\bar a_0)$ over $\bar a_0$.

(BMON) Clearly, every φ-definition of $\mathrm{tp}(\bar a/UBC)$ over U is also a φ-definition of $\mathrm{tp}(\bar a/UBC)$ over $U \cup C$.

(FIN) Since each formula $\varphi(\bar x) \in \mathrm{tp}(\bar a/UB)$ contains only finitely many variables from $\bar x$, it follows that $\mathrm{tp}(\bar a/UB)$ is definable over U if, and only if, $\mathrm{tp}(\bar a_0/UB)$ is definable over U, for all finite $\bar a_0 \subseteq \bar a$.

(d) (INV) follows immediately from the definition.

(MON) Suppose that $A \mathrel{\overset{\mathrm{s}}{\surd}}_U B$ and let $A_0 \subseteq A$ and $B_0 \subseteq B$. For $\bar b, \bar b' \subseteq B_0$ it follows that

$$\bar b \equiv_U \bar b' \quad \text{implies} \quad \bar b \equiv_{UA} \bar b' \quad \text{implies} \quad \bar b \equiv_{UA_0} \bar b'\,.$$

Hence, $A_0 \mathrel{\overset{\mathrm{s}}{\surd}}_U B_0$.

(NOR) Suppose that $A \mathrel{\overset{\mathrm{s}}{\surd}}_U B$. If $\bar b, \bar b' \subseteq B \cup U$ are tuples such that $\bar b \equiv_U \bar b'$, then there are tuples $\bar b_0, \bar b'_0 \subseteq B$ and $\bar c \subseteq U$ such that $\bar b = \bar b_0 \cup \bar c$ and $\bar b' = \bar b'_0 \cup \bar c$. It follows that

$$\begin{aligned} \bar b \equiv_U \bar b' \quad &\text{implies} \quad \bar b_0 \equiv_U \bar b'_0 \\ &\text{implies} \quad \bar b_0 \equiv_{UA} \bar b'_0 \\ &\text{implies} \quad \bar b_0\bar c \equiv_{UA} \bar b'_0\bar c \quad \text{implies} \quad \bar b \equiv_{UA} \bar b'\,. \end{aligned}$$

Consequently, $AU \mathrel{\overset{\mathrm{s}}{\surd}}_U BU$.

(LRF) Since, trivially, $\bar b \equiv_A \bar b'$ implies $\bar b \equiv_A \bar b'$, we have $A \mathrel{\overset{\mathrm{s}}{\surd}}_A B$, for all sets A and B.

(LTR) Suppose that $A_2 \mathrel{\overset{\mathrm{s}}{\surd}}_{A_1} B$ and $A_1 \mathrel{\overset{\mathrm{s}}{\surd}}_{A_0} B$, for $A_0 \subseteq A_1 \subseteq A_2$. For $\bar b, \bar b' \subseteq B$ it follows that

$$\bar b \equiv_{A_0} \bar b \quad \text{implies} \quad \bar b \equiv_{A_1} \bar b \quad \text{implies} \quad \bar b \equiv_{A_2} \bar b\,.$$

as desired.

(BMON) Suppose that $A \mathrel{\overset{\mathrm{s}}{\surd}}_U BC$. Let $\bar b, \bar b' \subseteq B \cup C$ be tuples such that $\bar b \not\equiv_{UAC} \bar b'$. We claim that $\bar b \not\equiv_{UC} \bar b'$. There exists a formula $\varphi(\bar x; \bar a, \bar c, \bar d)$ with parameters $\bar a \subseteq A$, $\bar c \subseteq C$, and $\bar d \subseteq U$ such that

$$\mathbb{M} \vDash \varphi(\bar b; \bar a, \bar c, \bar d) \land \neg\varphi(\bar b'; \bar a, \bar c, \bar d)\,.$$

Consequently, $\bar b\bar c \not\equiv_{UA} \bar b'\bar c$. Since $A \mathrel{\overset{\mathrm{s}}{\surd}}_U BC$ it follows that $\bar b\bar c \not\equiv_U \bar b'\bar c$. As $\bar c \subseteq C$ this means that $\bar b \not\equiv_{UC} \bar b'$, as desired.

(DEF) Suppose that $A \mathrel{\overset{\mathrm{s}}{\not\surd}}_U B$. Then there exist tuples $\bar b, \bar b' \subseteq B$ such that $\bar b \equiv_U \bar b'$ and $\bar b \not\equiv_{UA} \bar b'$. Fix a formula $\varphi(\bar x, \bar y)$ over U and a tuple

$\bar a \subseteq A$ such that

$$\mathbb{M} \vDash \varphi(\bar a, \bar b) \wedge \neg\varphi(\bar a, \bar b')\,.$$

For every tuple $\bar a' \subseteq \mathbb{M}$ it follows that

$$\mathbb{M} \vDash \varphi(\bar a', \bar b) \wedge \neg\varphi(\bar a', \bar b') \quad \text{implies} \quad \bar a' \overset{\mathrm{s}}{\not\surd}_U \bar b \bar b'\,. \qquad \square$$

Let us mention that, in general, $\overset{\mathrm{df}}{\surd}$ and $\overset{\mathrm{at}}{\surd}$ are no preforking relations.

Example. (a) The relation $\overset{\mathrm{df}}{\surd}$ is not definable. As a counterexample, consider the theory T of dense linear orders. Note that T has quantifier elimination. Let $a \in \mathbb{R} \smallsetminus \mathbb{Q}$ be an irrational number. Then $\mathrm{tp}(a/\mathbb{Q})$ is not definable over $\mathbb{Q}$. Consider a formula $\varphi(x;\bar b) \in \mathrm{tp}(a/\mathbb{Q})$ with rational parameters $b_0 < \dots < b_{n-1}$. By enlarging the tuple $\bar b$ we may assume that there is some index i such that $b_i < a < b_{i+1}$. It follows that $\langle \mathbb{R}, \leq\rangle \vDash \varphi(a';\bar b)$, for all $a' \in (b_i, b_{i+1})$. But for $a' \in (b_i, b_{i+1}) \cap \mathbb{Q}$ the type $\mathrm{tp}(a'/\mathbb{Q})$ is definable over $\mathbb{Q}$. This contradicts (DEF).

(b) The relation $\overset{\mathrm{at}}{\surd}$ is not definable. As a counterexample, consider the theory T of the structure $\langle \mathbb{R}, s\rangle$ where $s(x) = x + 1$. Note that $\mathrm{tp}(a/b)$ is isolated if, and only if, $a = b + k$, for some $k \in \mathbb{Z}$. In particular $\mathrm{tp}(\frac{1}{2}/0)$ is not isolated. Using an Ehrenfeucht-Fraïssé argument, one can show that, for every formula $\varphi(x;y)$ with $\langle \mathbb{R}, s\rangle \vDash \varphi(\frac{1}{2};0)$, there exists a number $a \in \mathbb{R}$ such that $\langle \mathbb{R}, s\rangle \vDash \varphi(b;0)$, for all $b \geq a$. But, for $b \in \mathbb{N}$, the type $\mathrm{tp}(b/0)$ is isolated by the formula $x = s^b(0)$.

Let us take a look at the closure operators associated with these relations. In each case, we obtain the definable closure.

Lemma 3.4. $\mathrm{cl}_{\overset{\mathrm{df}}{\surd}} = \mathrm{cl}_{\overset{\mathrm{at}}{\surd}} = \mathrm{cl}_{\overset{\mathrm{s}}{\surd}} = \mathrm{dcl}$

Proof. Note that $\overset{\mathrm{at}}{\surd} \subseteq \overset{\mathrm{df}}{\surd} \subseteq \overset{\mathrm{s}}{\surd}$ implies $\mathrm{cl}_{\overset{\mathrm{at}}{\surd}} \subseteq \mathrm{cl}_{\overset{\mathrm{df}}{\surd}} \subseteq \mathrm{cl}_{\overset{\mathrm{s}}{\surd}}$. Hence, we only need to prove that $\mathrm{dcl} \subseteq \mathrm{cl}_{\overset{\mathrm{at}}{\surd}}$ and $\mathrm{cl}_{\overset{\mathrm{s}}{\surd}} \subseteq \mathrm{dcl}$.

For the first inclusion, note that every formula defining a over U isolates $\mathrm{tp}(a/UBC)$. Hence, $a \in \mathrm{dcl}(U)$ implies $a \overset{\mathrm{at}}{\surd}_{UC} B$, for all B, C.

For the second inclusion, consider an element $a \notin \mathrm{dcl}(U)$. By Theorem E2.1.6, there exists an automorphism $\pi \in \mathbb{M}_U$ with $\pi(a) \neq a$. Setting $a' := \pi(a)$ it follows that $a \equiv_U a'$ and $a \not\equiv_{Ua} a'$. Hence, $a \overset{\mathrm{s}}{\not\surd}_U aa'$ and $a \notin \mathrm{cl}_{\overset{\mathrm{s}}{\surd}}(U)$. $\square$

We conclude this section with the remark that, for forking relations, the definition of the closure operator $\mathrm{cl}_{\surd}$ can be simplified.

Lemma 3.5. *If $\surd$ is a forking relation, then*

$$a \surd_U a \quad \textit{implies} \quad a \surd_{UC} B \quad \textit{for all } B, C\,.$$

Proof. Suppose that $a \surd_U a$ and let B, C be arbitrary sets. By (EXT), there exists an element $a' \equiv_{Ua} a$ with $a' \surd_U BC$. It follows that $a' = a$. Therefore, (BMON) implies $a \surd_{UC} B$. $\square$

Finitely satisfiable types

Let us take a look at some consequences of the definability axiom (DEF). First, note that, by invariance, we can extend every preforking relation from subsets of $\mathbb{M}$ to types.

Definition 3.6. Let $\surd$ be a preforking relation and $B, U \subseteq \mathbb{M}$.

(a) A partial type $\Phi(\bar x)$ over B *$\surd$-forks* over U if

$$\bar a \not\surd_U B\,, \quad \text{for all } \bar a \in \Phi^{\mathbb{M}}.$$

Similarly, we say that a single formula $\varphi(\bar x)$ over B $\surd$-forks over U, if the type $\{\varphi\}$ does.

(b) A type $\mathfrak{p}$ over B is *$\surd$-free* over U if it does not $\surd$-fork over U.

(c) For complete types $\mathfrak{p} \in S^{\bar s}(U)$ and $\mathfrak{q} \in S^{\bar s}(UB)$, we say that $\mathfrak{q}$ is a *$\surd$-free extension* of $\mathfrak{p}$ if

$$\mathfrak{p} \subseteq \mathfrak{q} \quad \text{and} \quad \mathfrak{q} \text{ is } \surd\text{-free over } U\,.$$

We denote this fact by $\mathfrak{p} \leq_{\surd} \mathfrak{q}$.

Remark. (a) By (INV), we have $\bar a \sqrt{}_U B$ if, and only if, $\mathrm{tp}(\bar a/UB)$ is $\sqrt{}$-free over U.

(b) By (DEF), a complete type $\mathfrak{p}$ $\sqrt{}$-forks over U if, and only if, some formula $\varphi(\bar x) \in \mathfrak{p}$ $\sqrt{}$-forks over U.

Lemma 3.7. *Let $\sqrt{}$ be a preforking relation. The set*

$$F^{\bar s}_{\sqrt{}}(A/U) := \{\, \mathfrak{p} \in S^{\bar s}(A) \mid \mathfrak{p} \text{ is } \sqrt{}\text{-free over } U \,\}$$

is a closed subset of $\mathfrak{S}^{\bar s}(A)$.

Proof. Let

$$\Phi := \{\, \neg\varphi \mid \varphi \text{ a formula over } A \text{ that } \sqrt{}\text{-forks over } U \,\}.$$

Then $\Phi \subseteq \mathfrak{p}$, for every $\mathfrak{p} \in F^{\bar s}_{\sqrt{}}(A/U)$, while (DEF) implies that $\Phi \nsubseteq \mathfrak{p}$, for every type $\mathfrak{p}$ that $\sqrt{}$-forks over U. Hence,

$$F^{\bar s}_{\sqrt{}}(A/U) := \langle \Phi \rangle_{\mathfrak{S}^{\bar s}(A)}\,. \qquad \square$$

Let us treat in more detail one important forking relation that is connected with the definability axiom. It is based on the notion of a finitely satisfiable type.

Definition 3.8. A type $\mathfrak{p}$ is *finitely satisfiable* in a set U if, for every formula $\varphi(\bar x; \bar c) \in \mathfrak{p}$, there is some tuple $\bar a \subseteq U$ with $\mathbb{M} \vDash \varphi(\bar a; \bar c)$. We write

$$\bar a \overset{\mathrm{u}}{\sqrt{}}_U B \quad :\text{iff} \quad \mathrm{tp}(\bar a/U \cup B) \text{ is finitely satisfiable in } U\,.$$

Example. Let T be the theory of dense linear orders. For a single element $a \in \mathbb{M}$ and sets $U, B \subseteq \mathbb{M}$, we have $a \overset{\mathrm{u}}{\sqrt{}}_U B$ if, and only if, at least one of the following conditions is satisfied:

- $a \in U$, or
- $\Uparrow a \cap U \neq \varnothing$ and, for every $b \in \Uparrow a \cap (U \cup B)$, there is some $c \in \Uparrow a \cap U$ with $c \leq b$, or
- $\Downarrow a \cap U \neq \varnothing$ and, for every $b \in \Downarrow a \cap (U \cup B)$, there is some $c \in \Downarrow a \cap U$ with $c \geq b$.

We shall prove that $\overset{\mathrm{u}}{\sqrt{}}$ is the least preforking relation and that it is, in fact, a forking relation. Before doing so, let us give an alternative characterisation of finitely satisfiable types in terms of ultrafilters. (The letter 'u' in $\overset{\mathrm{u}}{\sqrt{}}$ stands for 'ultrafilter'.)

Definition 3.9. Let T be a theory over the signature Σ, let $U, B \subseteq \mathbb{M}$ be sets, and $\mathfrak{u}$ an ultrafilter over $U^{\bar s}$, for some tuple $\bar s$ of sorts. The *average type* of $\mathfrak{u}$ over B is the set

$$\mathrm{Av}(\mathfrak{u}/B) := \{\, \varphi(\bar x) \in \mathrm{FO}^{\bar s}[\Sigma_B] \mid U^{\bar s} \cap \varphi(\bar x)^{\mathbb{M}} \in \mathfrak{u} \,\}\,.$$

Lemma 3.10. *Let T be a complete first-order theory and $\mathfrak{u}$ an ultrafilter over $U^{\bar s}$. Then $\mathrm{Av}(\mathfrak{u}/B)$ is a complete type over B that is finitely satisfiable in U.*

Proof. We start by showing that $\mathrm{Av}(\mathfrak{u}/B)$ is a type. For a contradiction, suppose that $T \cup \mathrm{Av}(\mathfrak{u}/B)$ is unsatisfiable. Then there exist a finite subset $\Phi \subseteq \mathrm{Av}(\mathfrak{u}/B)$ such that $T \vDash \neg \bigwedge \Phi$. By definition of $\mathrm{Av}(\mathfrak{u}/B)$,

$$U^{\bar s} \cap \varphi^{\mathbb{M}} \in \mathfrak{u}\,, \quad \text{for all } \varphi \in \Phi\,.$$

As ultrafilters are closed under finite intersections, it follows that

$$U^{\bar s} \cap (\bigwedge \Phi)^{\mathbb{M}} \in \mathfrak{u}\,.$$

In particular, $(\bigwedge \Phi)^{\mathbb{M}} \neq \varnothing$. Hence, $T \vDash \exists \bar x \bigwedge \Phi$. A contradiction.

Moreover, $\mathrm{Av}(\mathfrak{u}/B)$ is complete since, for every formula $\varphi(\bar x)$ over B,

$$\begin{aligned} \varphi(\bar x) \in \mathrm{Av}(\mathfrak{u}/B) \quad &\text{iff} \quad U^{\bar s} \cap \varphi(\bar x)^{\mathbb{M}} \in \mathfrak{u} \\ &\text{iff} \quad U^{\bar s} \setminus \varphi(\bar x)^{\mathbb{M}} \notin \mathfrak{u} \\ &\text{iff} \quad U^{\bar s} \cap \neg\varphi(\bar x)^{\mathbb{M}} \notin \mathfrak{u} \\ &\text{iff} \quad \neg\varphi(\bar x) \notin \mathrm{Av}(\mathfrak{u}/B)\,. \end{aligned}$$

Finally, to show that $\mathrm{Av}(\mathfrak{u}/B)$ is finitely satisfiable in U, note that $\varphi(\bar{x}) \in \mathrm{Av}(\mathfrak{u}/B)$ implies $U^{\bar{s}} \cap \varphi(\bar{x})^{\mathbb{M}} \in \mathfrak{u}$. In particular, this set is not empty. Hence, there is some $\bar{a} \in U^{\bar{s}}$ satisfying $\varphi(\bar{x})$. $\square$

Lemma 3.11. *A type $\mathfrak{p} \in S^{\bar{s}}(B)$ is finitely satisfiable in U if, and only if, $\mathfrak{p} = \mathrm{Av}(\mathfrak{u}/B)$, for some ultrafilter $\mathfrak{u}$ over $U^{\bar{s}}$.*

Proof. $(\Leftarrow)$ follows by Lemma 3.10. For $(\Rightarrow)$, suppose that $\mathfrak{p}$ is finitely satisfiable in U. We start by showing that the set

$$\mathfrak{u}_0 := \big\{\, U^{\bar{s}} \cap \varphi(\bar{x})^{\mathbb{M}} \;\big|\; \varphi(\bar{x}) \in \mathfrak{p} \,\big\}\,.$$

has the finite intersection property. Let

$$U^{\bar{s}} \cap \varphi_0(\bar{x})^{\mathbb{M}}, \dots, U^{\bar{s}} \cap \varphi_n(\bar{x})^{\mathbb{M}} \in \mathfrak{u}_0\,, \quad \text{for } \varphi_0, \dots, \varphi_n \in \mathfrak{p}\,.$$

Since $\mathfrak{p}$ is closed under conjunction, it follows that $\varphi_0 \wedge \dots \wedge \varphi_n \in \mathfrak{p}$. As $\mathfrak{p}$ is finitely satisfiable in U,

$$\begin{aligned} &(U^{\bar{s}} \cap \varphi_0(\bar{x}))^{\mathbb{M}} \cap \dots \cap (U^{\bar{s}} \cap \varphi_n(\bar{x}))^{\mathbb{M}} \\ &\quad = U^{\bar{s}} \cap (\varphi_0(\bar{x}) \wedge \dots \wedge \varphi_n(\bar{x}))^{\mathbb{M}} \neq \emptyset\,, \end{aligned}$$

as desired.

By Corollary B2.4.10, there exists an ultrafilter $\mathfrak{u} \supseteq \mathfrak{u}_0$ over $U^{\bar{s}}$. Since, for every formula φ over B,

$$U^{\bar{s}} \cap \varphi(\bar{x})^{\mathbb{M}} \in \mathfrak{u} \quad \text{iff} \quad U^{\bar{s}} \cap \varphi(\bar{x})^{\mathbb{M}} \in \mathfrak{u}_0\,,$$

it follows that

$$\begin{aligned} \mathrm{Av}(\mathfrak{u}/B) &= \big\{\, \varphi(\bar{x}) \;\big|\; U^{\bar{s}} \cap \varphi(\bar{x})^{\mathbb{M}} \in \mathfrak{u} \,\big\} \\ &= \big\{\, \varphi(\bar{x}) \;\big|\; U^{\bar{s}} \cap \varphi(\bar{x})^{\mathbb{M}} \in \mathfrak{u}_0 \,\big\} = \{\, \varphi(\bar{x}) \mid \varphi \in \mathfrak{p} \,\} = \mathfrak{p}\,, \end{aligned}$$

as desired. $\square$

Using this characterisation of finite satisfiable types, we can prove that $\overset{\mathfrak{u}}{\surd}$ is a forking relation.

Proposition 3.12. *$\overset{\mathfrak{u}}{\surd}$ is a forking relation.*

Proof. (INV) follows immediately from the definition.

(MON) If $\mathrm{tp}(\bar{a}_0\bar{a}_1/UB)$ is finitely satisfiable in U and $B_0 \subseteq B$, then $\mathrm{tp}(\bar{a}_0/UB_0)$ is finitely satisfiable in U.

(NOR) If $\mathrm{tp}(\bar{a}/\bar{c}B)$ is finitely satisfiable in $\bar{c}$ then so is $\mathrm{tp}(\bar{a}\bar{c}/\bar{c}B)$.

(LRF) Clearly, $\mathrm{tp}(\bar{a}/B\bar{a})$ is finitely satisfiable in $\bar{a}$.

(LTR) Suppose that $\mathrm{tp}(\bar{a}_0\bar{a}_1\bar{a}_2/\bar{a}_0\bar{a}_1 B)$ is finitely satisfiable in $\bar{a}_0\bar{a}_1$ and $\mathrm{tp}(\bar{a}_0\bar{a}_1/\bar{a}_0 B)$ is finitely satisfiable in $\bar{a}_0$. If $\mathbb{M} \vDash \varphi(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{b})$, for $\bar{b} \subseteq \bar{a}_0 B$, there exists a tuple $\bar{a}'_2 \subseteq \bar{a}_0\bar{a}_1$ such that $\mathbb{M} \vDash \varphi(\bar{a}_0, \bar{a}_1, \bar{a}'_2, \bar{b})$. Suppose that $\bar{a}'_2 = \bar{a}'_0\bar{a}'_1$ with $\bar{a}'_0 \subseteq \bar{a}_0$ and $\bar{a}'_1 \subseteq \bar{a}_1$. Then there are tuples $\bar{c}_1, \bar{c}'_1 \subseteq \bar{a}_0$ with $\mathbb{M} \vDash \varphi(\bar{a}_0, \bar{c}_1, \bar{a}'_0\bar{c}'_1, \bar{b})$. Hence, $\mathrm{tp}(\bar{a}_0\bar{a}_1\bar{a}_2/\bar{a}_0 B)$ is finitely satisfiable in $\bar{a}_0$.

(BMON) Obviously, if $\mathrm{tp}(\bar{a}/UBC)$ is finitely satisfiable in U, it is also finitely satisfiable in $U \cup C$.

(DEF) Suppose that $\mathrm{tp}(\bar{a}/UB)$ is not finitely satisfiable in U. Then there is some formula $\varphi(\bar{x};\bar{b}) \in \mathrm{tp}(\bar{a}/UB)$ such that $\mathbb{M} \nvDash \varphi(\bar{a}';\bar{b})$, for all $\bar{a}' \subseteq U$. It follows that $\mathrm{tp}(\bar{a}'/U\bar{b})$ is not finitely satisfiable in U, for every tuple $\bar{a}'$ that satisfies $\varphi(\bar{x};\bar{b})$.

(EXT) Suppose that the type $\mathfrak{p} := \mathrm{tp}(\bar{a}/UB_0)$ is finitely satisfiable in U and let $B_1 \supseteq B_0$. According to Lemma 3.11 there exists an ultrafilter $\mathfrak{u}$ such that $\mathfrak{p} = \mathrm{Av}(\mathfrak{u}/UB_0)$. Let $\bar{a}'$ be a realisation of $\mathrm{Av}(\mathfrak{u}/UB_1)$. Then $\mathrm{tp}(\bar{a}'/UB_0) = \mathrm{Av}(\mathfrak{u}/UB_0) = \mathfrak{p}$ and $\mathrm{tp}(\bar{a}'/UB_1) = \mathrm{Av}(\mathfrak{u}/UB_1)$ is finitely satisfiable in U. $\square$

Our next aim is to show that $\overset{\mathfrak{u}}{\surd}$ is the least preforking relation.

Theorem 3.13 (Adler). *$\overset{\mathfrak{u}}{\surd} \subseteq \surd$, for every preforking relation $\surd$.*

Proof. For a contradiction, suppose that $A \overset{\mathfrak{u}}{\surd}_U B$ but $A \not\surd_U B$. By (DEF), there are a formula $\varphi(\bar{x}, \bar{y})$ over U and tuples $\bar{a} \subseteq A$ and $\bar{b} \subseteq B$ such that $\mathbb{M} \vDash \varphi(\bar{a}, \bar{b})$ and

$$\bar{a}' \not\surd_U \bar{b}\,, \quad \text{for all } \bar{a}' \in \varphi(\bar{x}, \bar{b})^{\mathbb{M}}\,.$$

Since $\operatorname{tp}(\bar a/BU)$ is finitely satisfiable in U, there is some tuple $\bar c \subseteq U$ with $\mathbb{M} \vDash \varphi(\bar c, \bar b)$. Consequently, $\bar c \not\mathrel{\overset{u}{\surd}}_U \bar b$ which, by (MON), implies that $U \not\mathrel{\overset{u}{\surd}}_U B$. A contradiction to (LRF). $\square$

As a corollary we obtain the following result which, in the terminology introduced below, states that the relation $\overset{u}{\surd}$ is *left local*. Below we will extend this result to all preforking relations.

Lemma 3.14. *Let T be a complete first-order theory. For all $\bar a, B \subseteq \mathbb{M}$, there is a set $U \subseteq \bar a$ of size $|U| \leq |T| \oplus |B|$ such that* $\operatorname{tp}(\bar a/UB)$ *is finitely satisfiable in U.*

Proof. We construct an increasing sequence $U_0 \subseteq U_1 \subseteq \dots$ of sets $U_n \subseteq \bar a$ with $|U_n| \leq |T| \oplus |B|$ as follows. We start with $U_0 := \emptyset$. For the inductive step suppose that we have already constructed $U_n \subseteq \bar a$. For every formula $\varphi(\bar x; \bar b) \in \operatorname{tp}(\bar a/BU_n)$, let $\bar c_\varphi \subseteq \bar a$ be the elements of $\bar a$ that are mentioned in $\varphi(\bar x)$. Note that $\bar c_\varphi$ is finite. Let U_{n+1} be the set obtained from U_n by adding all these tuples $\bar c_\varphi$. Then $|U_{n+1}| \leq |T| \oplus |B| \oplus |U_n| \leq |T| \oplus |B|$.

Setting $U := \bigcup_{n<\omega} U_n$ it follows that $\operatorname{tp}(\bar a/UB)$ is finitely satisfiable in U. Furthermore, $|U| \leq |T| \oplus |B|$. $\square$

Let us conclude this section with a remark about sets where $\overset{u}{\surd}$ is right reflexive.

Lemma 3.15. *Let T be a complete first-order theory. A subset $M \subseteq \mathbb{M}$ is the universe of a model of T if, and only if, $A \mathrel{\overset{u}{\surd}}_M M$, for all sets A.*

Proof. ($\Rightarrow$) Let $\mathfrak{M} \preceq \mathbb{M}$ be a model of T and $\bar a \subseteq \mathbb{M}$ a tuple. To show that $\bar a \mathrel{\overset{u}{\surd}}_M M$, consider a formula $\varphi(\bar x) \in \operatorname{tp}(\bar a/M)$. Then $\mathbb{M} \vDash \exists \bar x \varphi$ implies $\mathfrak{M} \vDash \exists \bar x \varphi$. Hence, there is some $\bar c \subseteq M$ with $\mathfrak{M} \vDash \varphi(\bar c)$.

($\Leftarrow$) Suppose that $A \mathrel{\overset{u}{\surd}}_M M$ for all sets A. We prove that M satisfies the Tarski-Vaught Test. Let $\varphi(x)$ be a formula over M such that $\mathbb{M} \vDash \exists x \varphi(x)$. We fix an element $a \in \mathbb{M}$ with $\mathbb{M} \vDash \varphi(a)$. Since $a \mathrel{\overset{u}{\surd}}_M M$, there is some element $c \in M$ with $\mathbb{M} \vDash \varphi(c)$. By Theorem C2.2.5, it follows that $M \preceq \mathbb{M}$. Consequently, M is a model of T. $\square$

Local character and forking sequences

In the remainder of this section we study preforking relations with a property called *local character*. In the next section, we will prove that having local character is equivalent to being symmetric.

Definition 3.16. A ternary relation $\surd$ has *local character* if it satisfies the following two axioms:

(LLOC) *Left Locality.* There exists some cardinal κ such that, for all sets A and B, there is a subset $A_0 \subseteq A$ of size $|A_0| < \kappa \oplus |B|^+$ with $A \surd_{A_0} B$.

(RLOC) *Right Locality.* There exists a cardinal κ such that, for all sets A and B, there is a subset $B_0 \subseteq B$ of size $|B_0| < \kappa \oplus |A|^+$ with $A \surd_{B_0} B$.

If $\surd$ is right local, we denote by $\operatorname{loc}(\surd)$ the least cardinal κ such that $\surd$ satisfies the condition in (RLOC). Similarly, $\operatorname{loc}_0(\surd)$ the least cardinal κ such that $\surd$ satisfies the above condition for *finite* sets A. If $\surd$ is not right local, we set $\operatorname{loc}(\surd) := \infty$ and $\operatorname{loc}_0(\surd) := \infty$.

We start by proving that every preforking relation is left local.

Proposition 3.17. *Let T be a complete first-order theory and let $\surd$ be a preforking relation. For all sets $A, B \subseteq \mathbb{M}$, there exists a subset $A_0 \subseteq A$ of size $|A_0| \leq |T| \oplus |B|$ such that*

$$A \surd_{A_0} B .$$

Proof. Let A and B be sets. By Lemma 3.14, there is a set $A_0 \subseteq A$ of size $|A_0| \leq |T| \oplus |B|$ such that $A \mathrel{\overset{u}{\surd}}_{A_0} B$. By Theorem 3.13, this implies that $A \surd_{A_0} B$. $\square$

Corollary 3.18. *Let T be a complete first-order theory and let $\downarrow$ a symmetric preforking relation. Then* $\operatorname{loc}(\downarrow) \leq |T|^+$.

The two parameters $\operatorname{loc}_0(\surd)$ and $\operatorname{loc}(\surd)$ are nearly the same. They can only differ if the first one is a singular cardinal.

Definition 3.19. For a cardinal κ, we denote by κ^{reg} the minimal regular cardinal with $\kappa^{\text{reg}} \geq \kappa$, that is,

$$\kappa^{\text{reg}} := \begin{cases} \kappa & \text{if } \kappa \text{ is regular,} \\ \kappa^+ & \text{if } \kappa \text{ is singular.} \end{cases}$$

Lemma 3.20. *Let* $\surd$ *be an abstract independence relation that satisfies* (BMON) *and* (RLOC). *Then*

$$\text{loc}_0(\surd) \leq \text{loc}(\surd) \leq \text{loc}_0(\surd)^{\text{reg}}.$$

Proof. The lower bound follows immediately from the definitions. For the upper bound, let $\kappa := \text{loc}_0(\surd)^{\text{reg}}$ and consider sets $A, B \subseteq \mathbb{M}$. We have to find a set $U \subseteq A$ of size $|U| < \kappa \oplus |A|^+$ with $A \surd_U B$.

For every finite set $A_0 \subseteq A$, we choose a set $U(A_0) \subseteq B$ of size $|U(A_0)| < \text{loc}_0(\surd) \leq \kappa$ such that

$$A_0 \surd_{U(A_0)} B.$$

Setting $U := \bigcup \{ U(A_0) \mid A_0 \subseteq A \text{ finite} \}$ it follows by (BMON) that

$$A_0 \surd_U B, \quad \text{for all finite } A_0 \subseteq A.$$

By (FIN), this implies $A \surd_U B$. Since the cardinal $\kappa \oplus |A|^+$ is regular, we furthermore have

$$|U| \leq \sum_{A_0 \subseteq A \text{ finite}} |U(A_0)| < \kappa \oplus |A|^+. \qquad \square$$

We can characterise preforking relations with local character in terms of so-called *forking chains.*

Definition 3.21. Let $\surd$ be a preforking relation.

(a) Let $A, U \subseteq \mathbb{M}$ be sets. A sequence of finite sets $(B_\alpha)_{\alpha<\gamma}$ is a $\surd$-*forking chain* for A over U if

$$A \not\surd_{UB[<\alpha]} B_\alpha, \quad \text{for every } \alpha < \gamma,$$

where we have set $B[<\alpha] := \bigcup_{\beta<\alpha} B_\beta$. The ordinal γ is the *length* of the chain.

(b) We denote by $\text{fc}(\surd)$ the least cardinal κ such that no finite set A has a $\surd$-forking chain over $\varnothing$ of length κ. If such a cardinal does not exist, we set $\text{fc}(\surd) := \infty$.

In the theorem below we show that the cardinal $\text{fc}(\surd)$ is closely related to the parameter $\text{loc}(\surd)$. As we will apply these results in a later chapter to relations that are not preforking relations, we state them in a slightly more general setting.

Definition 3.22. A ternary relation $\surd$ has *strong finite character* if it satisfies the following axiom:

(SFIN) *Strong Finite Character.*

$$A \surd_U B \quad \text{iff} \quad A_0 \surd_U B_0 \quad \text{for all finite } A_0 \subseteq A \text{ and } B_0 \subseteq B.$$

Remark. Note that every preforking relation has strong finite character since (SFIN) follows from (FIN) and (DEF).

The following lemma contains the key argument of the translation between $\text{fc}(\surd)$ and $\text{loc}(\surd)$.

Lemma 3.23. *Let* $\surd$ *be an abstract independence relation that satisfies* (BMON) *and* (SFIN), *let* κ *be an infinite cardinal and* $A \subseteq \mathbb{M}$.

(a) *If there exists some set* B *such that* $A \not\surd_U B$, *for all* $U \subseteq B$ *of size* $|U| < \kappa$, *then there is a* $\surd$-*forking chain for* A *over* $\varnothing$ *of length* κ.

(b) *If* κ *is regular and every set* B *has a subset* $U \subseteq B$ *of size* $|U| < \kappa$ *with* $A \surd_U B$, *then there is no* $\surd$-*forking chain for* A *over* $\varnothing$ *of length* κ.

Proof. (a) We construct the desired $\surd$-forking chain $(B_\alpha)_{\alpha<\kappa}$ by induction on α. Suppose that we have already defined B_α, for all $\alpha < \beta$. Then

$$\begin{aligned} & \big|B[<\beta]\big| < \aleph_0 \leq \kappa, && \text{for } \beta < \omega, \\ \text{and} \quad & \big|B[<\beta]\big| \leq \aleph_0 \otimes |\beta| < \kappa, && \text{for } \omega \leq \beta < \kappa. \end{aligned}$$

In both cases it follows that $A \not\surd_{B[<\beta]} B$. Hence, we can use (SFIN) to find a finite set $B_\beta \subseteq B$ with $A \not\surd_{B[<\beta]} B_\beta$.

(b) Let $(B_\alpha)_{\alpha<\kappa}$ a sequence of finite sets of length κ. By assumption, there exists a set $U \subseteq B[<\kappa]$ of size $|U| < \kappa$ such that

$$A \surd_U B[<\kappa]\,.$$

As κ is regular, there is some index $\alpha < \kappa$ with $U \subseteq B[<\alpha]$. By (BMON) and (MON) it follows that

$$A \surd_{B[<\alpha]} B_\alpha\,.$$

Consequently, $(B_\alpha)_{\alpha<\kappa}$ is no $\surd$-forking chain for A over $\varnothing$. ☐

Proposition 3.24. *Let $\surd$ be an abstract independence relation satisfying* (BMON) *and* (SFIN). *Then*

$$\mathrm{loc}_0(\surd) \leq \mathrm{fc}(\surd) \leq \mathrm{loc}_0(\surd)^{\mathrm{reg}}.$$

Proof. For the lower bound, consider a finite set A and an arbitrary set B. If there were no set $U \subseteq B$ of size $|U| < \mathrm{fc}(\surd)$ with $A \surd_U B$, we could use Lemma 3.23 (a) to construct a $\surd$-forking chain for A over $\varnothing$ of length $\mathrm{fc}(\surd)$. A contradiction.

For the upper bound, consider a finite set A. Then Lemma 3.23 (b) implies that there is no $\surd$-forking chain for A over $\varnothing$ of length $\mathrm{loc}_0(\surd)^{\mathrm{reg}}$. ☐

Theorem 3.25. *For a preforking relation $\surd$, the following statements are equivalent:*

(1) *$\surd$ has local character.*

(2) *$\surd$ is right local.*

(3) *For every set A, there exists a cardinal κ such that there is no $\surd$-forking chain for A over $\varnothing$ of length κ.*

(4) *There exists a cardinal κ such that, for every finite set A, there is no $\surd$-forking chain for A over $\varnothing$ of length κ.*

Proof. (1) ⇔ (2) ⇔ (4) follow by Propositions 3.17 and 3.24, respectively.

(2) ⇒ (3) Given a set A, it follows by Lemma 3.23 (b) that there is no $\surd$-forking chain for A over $\varnothing$ of length $\kappa := \mathrm{loc}(\surd)^+ \oplus |A|^+$.

(3) ⇒ (4) For every type $\mathfrak{p} \in S^{<\omega}(\varnothing)$, fix a tuple $\bar a_\mathfrak{p}$ realising $\mathfrak{p}$. By (3), there are cardinals $\kappa_\mathfrak{p}$ such that there are no $\surd$-forking chains for $\bar a_\mathfrak{p}$ over $\varnothing$ of length $\kappa_\mathfrak{p}$. We claim that the cardinal

$$\kappa := \sup\{\,\kappa_\mathfrak{p} \mid \mathfrak{p} \in S^{<\omega}(\varnothing)\,\}$$

has the desired properties. Let $\bar a$ be a finite tuple and $(B_\alpha)_{\alpha<\kappa}$ a sequence of finite sets of length κ. Then $\bar a \equiv_\varnothing \bar a_\mathfrak{p}$, for $\mathfrak{p} := \mathrm{tp}(\bar a)$, and there exists an automorphism π with $\pi(\bar a) = \bar a_\mathfrak{p}$. Since $\kappa \geq \kappa_\mathfrak{p}$, there is an index $\alpha < \kappa$ such that

$$\bar a_\mathfrak{p} \surd_{\pi[B[<\alpha]]} \pi[B_\alpha]\,.$$

By invariance, it follows that $\bar a \surd_{B[<\alpha]} B_\alpha$. Hence, $(B_\alpha)_{\alpha<\kappa}$ is not a $\surd$-forking chain for $\bar a$ over $\varnothing$. ☐

4. Forking relations

In this section we consider the special properties of forking relations that follow form the extension axiom. We start by presenting a canonical way to turn every preforking relation into a forking relation.

Definition 4.1. Let $\surd$ be a preforking relation. We define a relation $\overset{*}{\surd}$ by

$$A \overset{*}{\surd}_U B \quad :\text{iff} \quad \text{for every set } C \subseteq \mathbb{M} \text{ there is some set } A' \subseteq \mathbb{M} \text{ with } A' \equiv_{UB} A \text{ and } A' \surd_U BC\,.$$

Remark. Note that $\overset{*}{\surd} \subseteq \surd$. Furthermore, by Proposition 4.5 below it will follow that $\overset{*}{\surd} = \surd$ if, and only if, $\surd$ is a forking relation. Consequently, the operation $\surd \mapsto \overset{*}{\surd}$ is a so-called *kernel operator*, the dual of a closure operator:

$$\overset{*}{\surd} \subseteq \surd\,, \qquad \overset{**}{\surd} = \overset{*}{\surd}\,, \qquad \text{and} \qquad \surd_0 \subseteq \surd_1 \;\Rightarrow\; \overset{*}{\surd}_0 \subseteq \overset{*}{\surd}_1\,.$$

Before proving that $\overset{*}{\surd}$ is a forking relation, we present two alternative definitions. The first one characterises such relations in terms of global types.

Definition 4.2. A *global type* is a complete type over $\mathbb{M}$.

Proposition 4.3. *Let $\surd$ be a preforking relation and $\bar a, U, B \subseteq \mathbb{M}$. Then*

$$\bar a \overset{*}{\surd}_U B \quad \text{iff} \quad \operatorname{tp}(\bar a/UB) \textit{ can be extended to a global type that is } \surd\textit{-free over } U\,.$$

Proof. $(\Leftarrow)$ Let $\mathfrak{p} \supseteq \operatorname{tp}(\bar a/UB)$ be a global type that is $\surd$-free over U. To show that $\bar a \overset{*}{\surd}_U B$, consider a set $C \subseteq \mathbb{M}$. Choosing some tuple $\bar a'$ realising $\mathfrak{p} \restriction UBC$, we have $\bar a' \equiv_{UB} \bar a$ and $\bar a' \surd_U BC$.

$(\Rightarrow)$ Suppose that $\bar a \overset{*}{\surd}_U B$ and set

$$\Phi(\bar x) := \operatorname{tp}(\bar a/UB) \cup \{\, \neg\varphi(\bar x) \mid \varphi \text{ a formula over } \mathbb{M} \text{ that } \surd\text{-forks over } U \,\}\,.$$

We start by proving that Φ is satisfiable. Let $\Phi_0 \subseteq \Phi$ be finite. Then

$$\Phi_0 \equiv \{\psi(\bar x), \neg\varphi_0(\bar x;\bar c_0), \dots, \neg\varphi_n(\bar x;\bar c_n)\}\,,$$

for some $\psi \in \operatorname{tp}(\bar a/UB)$ and formulae $\varphi_i(\bar x;\bar c_i)$ that $\surd$-fork over U. Since $\bar a \overset{*}{\surd}_U B$, there exists a tuple $\bar a' \equiv_{UB} \bar a$ such that $\bar a' \surd_U B\bar c_0 \dots \bar c_n$. Then $\bar a'$ satisfies Φ_0.

Hence, Φ is satisfiable and there exists a global type $\mathfrak{p} \supseteq \Phi$. We claim that $\mathfrak{p}$ is $\surd$-free over U. For a contradiction, suppose that $\mathfrak{p} \vDash \varphi(\bar x)$, for some formula φ that $\surd$-forks over U. Then $\neg\varphi \in \Phi \subseteq \mathfrak{p}$. A contradiction. □

The second characterisation considers forking relations in terms of types and formulae. The key here is that the formulae ψ_i below might have parameters that do not appear in Φ.

Lemma 4.4. *Let $\surd$ be a preforking relation. A partial type Φ $\overset{*}{\surd}$-forks over U if, and only if, for some $n < \omega$, there are formulae $\psi_0, \dots, \psi_{n-1}$ with parameters such that*

$$\Phi(\bar x) \vDash \bigvee_{i<n} \psi_i(\bar x) \quad \textit{and each } \psi_i \ \surd\textit{-forks over } U\,.$$

Proof. $(\Leftarrow)$ Fix a tuple $\bar a \in \Phi^{\mathbb{M}}$ and let B be a set such that Φ is a partial type over B. For a contradiction, suppose that $\bar a \overset{*}{\surd}_U B$. We choose a set C containing the parameters of every formula ψ_i. By definition of $\overset{*}{\surd}$, there is some tuple $\bar a' \equiv_{UB} \bar a$ such that $\bar a' \surd_U BC$. Since $\Phi \vDash \bigvee_i \psi_i$, we have $\mathbb{M} \vDash \psi_i(\bar a')$, for some $i < n$. As ψ_i $\surd$-forks over U, it follows that $\bar a' \not\surd_U BC$. A contradiction.

$(\Rightarrow)$ Suppose that Φ $\overset{*}{\surd}$-forks over U and let B be some set such that Φ is a partial type over B. By definition of $\overset{*}{\surd}$, there exists, for every tuple $\bar a \in \Phi^{\mathbb{M}}$, some set $C_{\bar a}$ such that

$$\bar a' \not\surd_U BC_{\bar a}\,, \qquad \text{for all } \bar a' \equiv_{UB} \bar a\,.$$

By (DEF), we can find a formula $\psi_{\bar a}(\bar x, \bar b_{\bar a}, \bar c_{\bar a})$ with parameters $\bar b_{\bar a} \subseteq B$ and $\bar c_{\bar a} \subseteq C_{\bar a}$ such that

$$\mathbb{M} \vDash \psi_{\bar a}(\bar a, \bar b_{\bar a}, \bar c_{\bar a}) \quad \text{and} \quad \psi_{\bar a}(\bar x, \bar b_{\bar a}, \bar c_{\bar a}) \ \surd\text{-forks over } U\,.$$

Consequently, the set

$$\Phi(\bar x) \cup \big\{\, \neg\psi_{\bar a}(\bar x, \bar b_{\bar a}, \bar c_{\bar a}) \bigm| \bar a \in \Phi^{\mathbb{M}} \,\big\}$$

is inconsistent. By compactness, we can therefore find finitely many tuples $\bar a_0, \dots, \bar a_{n-1}$ such that

$$\Phi(\bar x) \vDash \bigvee_{i<n} \psi_{\bar a_i}(\bar x, \bar b_{\bar a_i}, \bar c_{\bar a_i})$$

and each formula $\psi_{\bar a_i}(\bar x, \bar b_{\bar a_i}, \bar c_{\bar a_i})$ $\surd$-forks over U. □

Next we prove that the operations $\surd \mapsto \overset{*}{\surd}$ turns every preforking relation into a forking relation.

Proposition 4.5. *If $\surd$ is a preforking relation then $\overset{*}{\surd}$ is a forking relation.*

Proof. (INV) follows easily from the definition.

(MON) Suppose that $A_0A_1 \overset{*}{\surd}_U B$ and let $B_0 \subseteq B$. To show that $A_0 \overset{*}{\surd}_U B_0$ let $C \subseteq \mathbb{M}$. By definition of $\overset{*}{\surd}$, there are sets A'_0 and A'_1 with $A'_0A'_1 \equiv_{UB} A_0A_1$ and $A'_0A'_1 \surd_U BC$. This implies that $A'_0 \equiv_{UB_0} A_0$ and $A'_0 \surd_U BC$.

(NOR) Suppose that $A \overset{*}{\surd}_U B$. To show that $AU \overset{*}{\surd}_U BU$, let $C \subseteq \mathbb{M}$. There is some set A' such that $A' \equiv_{UB} A$ and $A' \surd_U BCU$. It follows by (NOR) that $A'U \surd_U BCU$. Since $A'U \equiv_{UB} AU$ the claim follows.

(LRF) For all sets $A, B, C \subseteq \mathbb{M}$, we have $A \surd_A BC$. Hence, $A \overset{*}{\surd}_A B$.

(LTR) Suppose that $A_2 \overset{*}{\surd}_{A_1} B$ and $A_1 \overset{*}{\surd}_{A_0} B$ for $A_0 \subseteq A_1 \subseteq A_2$. To show that $A_2 \overset{*}{\surd}_{A_0} B$ let $C \subseteq \mathbb{M}$. There exists a set A'_1 with $A'_1 \equiv_{A_0B} A_1$ and $A'_1 \surd_{A_0} BC$. Let A'_2 be some set such that $A'_1A'_2 \equiv_{A_0B} A_1A_2$. By (INV) it follows that $A'_2 \overset{*}{\surd}_{A'_1} B$. Therefore, there exists a set A''_2 with $A''_2 \equiv_{A'_1B} A'_2$ and $A''_2 \surd_{A'_1} BC$. By (LTR) it follows that $A''_2 \surd_{A_0} BC$, as desired.

(BMON) Suppose that $A \overset{*}{\surd}_U BC$. To show that $A \overset{*}{\surd}_{UC} BC$, let $D \subseteq \mathbb{M}$. There is a set A' with $A' \equiv_{UBC} A$ such that $A' \surd_U BCD$. By (BMON) it follows that $A' \surd_{UC} BCD$.

(EXT) Suppose that $A \overset{*}{\surd}_U B$ and let $\bar{a}$ be an enumeration of A. By Proposition 4.3, there exists some global type $\mathfrak{p} \supseteq \mathrm{tp}(\bar{a}/UB)$ that is $\surd$-free over U. Given a set $C \subseteq \mathbb{M}$, we choose some tuple $\bar{a}'$ realising $\mathfrak{p} \restriction UBC$. Then $\bar{a}' \equiv_{UB} \bar{a}$ and $\mathrm{tp}(\bar{a}'/UBC) = \mathfrak{p} \restriction UBC$ has the global extension $\mathfrak{p}$, which is $\surd$-free over U. Hence, Proposition 4.3 implies that $\bar{a}' \overset{*}{\surd}_U BC$.

(DEF) Suppose that $\bar{a} \overset{*}{\not\surd}_U B$. Then there is a set $C \subseteq \mathbb{M}$ such that $\bar{a}' \not\surd_U BC$ for all tuples $\bar{a}' \equiv_{UB} \bar{a}$. Let Φ be the set of all formulae $\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/UBC)$ that $\surd$-fork over U. Since $\surd$ is definable, it follows by choice of C that the set

$$\mathrm{tp}(\bar{a}/UB) \cup \{ \neg\varphi \mid \varphi \in \Phi \}$$

is inconsistent. Hence, there is some formula $\psi(\bar{x};\bar{b}) \in \mathrm{tp}(\bar{a}/UB)$ such that

$$\psi(\bar{x};\bar{b}) \vDash \bigvee \Phi .$$

We claim that $\mathbb{M} \vDash \psi(\bar{a}';\bar{b})$ implies $\bar{a}' \overset{*}{\not\surd}_U \bar{b}$. Suppose otherwise. Then there exists a tuple $\bar{a}''$ such that $\bar{a}'' \equiv_{U\bar{b}} \bar{a}'$ and $\bar{a}'' \surd_U BC$. But there is some formula $\varphi \in \Phi$ with $\mathbb{M} \vDash \varphi(\bar{a}'')$. By definition of Φ this implies that $\bar{a}'' \not\surd_U BC$. A contradiction. □

Lemma 4.6. $\mathrm{cl}_{\surd} = \mathrm{cl}_{\overset{*}{\surd}}$, *for every preforking relation* $\surd$.

Proof. Note that $\overset{*}{\surd} \subseteq \surd$ implies $\mathrm{cl}_{\overset{*}{\surd}} \subseteq \mathrm{cl}_{\surd}$. Conversely, suppose that $a \notin \mathrm{cl}_{\overset{*}{\surd}}(U)$. Then there are sets B and C such that $a \overset{*}{\not\surd}_{UC} B$. Hence, we can find a set D such that $a' \not\surd_{UC} BD$, for all $a' \equiv_{UCB} a$. In particular, we have $a \not\surd_{UC} BD$, which implies that $a \notin \mathrm{cl}_{\surd}(U)$. □

Exercise 4.1. Let $\surd$ be a preforking relation. Prove that, if $\overset{*}{\surd}$ is right local, then so is $\surd$.

To check whether a forking relation is contained in another one, we can frequently use the following lemma.

Lemma 4.7. *Let $\overset{0}{\surd}$ be a relation satisfying* (EXT) *and let $\overset{1}{\surd}$ be a relation satisfying* (INV) *and* (MON). *If, for all sets B and U, there exists some set C such that*

$$A \overset{0}{\surd}_U BC \quad \textit{implies} \quad A \overset{1}{\surd}_U BC , \quad \textit{for all sets } A ,$$

then $\overset{0}{\surd} \subseteq \overset{1}{\surd}$.

Proof. Suppose that $A \sqrt[0]{}_U B$. By assumption, we can find a set C such that

$$A \sqrt[0]{}_U BC \quad \text{implies} \quad A \sqrt[1]{}_U BC\,, \quad \text{for all sets } A\,.$$

By (EXT), there is some set $A' \equiv_{UB} A$ such that $A' \sqrt[0]{}_U BC$. By choice of C, it follows that $A' \sqrt[1]{}_U BC$. Consequently, (MON) and (INV) imply that $A \sqrt[1]{}_U B$. □

Morley sequences

The aim of this section is to introduce the notion of a basis for an arbitrary forking relation. Since, in general, forking relations are not symmetric, these bases are ordered. To simplify notation we write $\bar a[<k]$, for a sequence $(\bar a_i)_{i\in I}$, to denote the set $\bigcup_{i<k} \bar a_i$.

Definition 4.8. Let $\surd$ be a preforking relation and $\mathfrak p \in S^{\bar s}(U \cup B)$ a type.

(a) A $\surd$-*Morley sequence* for $\mathfrak p$ over U is an indiscernible sequence $(\bar a_i)_{i\in I}$ over $U \cup B$ such that every $\bar a_i$ realises $\mathfrak p$ and

$$\bar a_i \ \surd_U \ \bar a[<i]\,, \quad \text{for all } i \in I\,.$$

We call $(\bar a_i)_{i\in I}$ a $\surd$-*Morley sequence over* U if it is a $\surd$-Morley sequence for $\operatorname{tp}(\bar a_i/U)$ over U.

(b) A *reverse* $\surd$-*Morley sequence* for $\mathfrak p$ over U is an indiscernible sequence $(\bar a_i)_{i\in I}$ over $U \cup B$ such that every $\bar a_i$ realises $\mathfrak p$ and

$$\bar a[<i] \ \surd_U \ \bar a_i\,, \quad \text{for all } i \in I\,.$$

Remark. If $(\bar a_i)_{i\in I}$ is a $\surd$-Morley sequence for $\mathfrak p$ over U, then it follows by (FIN), Lemma 2.4, and induction, that

$$\bar a[I_1] \ \surd_U \ \bar a[I_0]\,, \quad \text{for all } I_0, I_1 \subseteq I \text{ with } I_0 < I_1\,.$$

For symmetric preforking relations, we obtain the following stronger result.

Lemma 4.9. *Let $\downarrow$ be a symmetric preforking relation and $(\bar a_i)_{i\in I}$ a sequence such that*

$$\bar a_i \downarrow_U \bar a[<i]\,, \quad \text{for all } i \in I\,.$$

Then

$$\bar a[K] \downarrow_U \bar a[L]\,, \quad \text{for all disjoint } K, L \subseteq I\,.$$

Proof. By (FIN), it is sufficient to prove the claim for finite sets K and L. We do so by induction on $|K \cup L|$. If both sets are empty, the claim follows by (NOR). Otherwise, let $k := \max(K \cup L)$. By (SYM), we may assume without loss of generality that $k \in K$. Set $K_0 := K \smallsetminus \{k\}$. By inductive hypothesis, we have

$$\bar a[K_0] \downarrow_U \bar a[L]\,.$$

Furthermore,

$$\bar a_k \downarrow_U \bar a[<k] \quad \text{implies} \quad \bar a_k \downarrow_U \bar a[K_0]\bar a[L]\,.$$

Consequently, it follows by Lemma 2.4 that

$$\bar a_k \bar a[K_0] \downarrow_U \bar a[L]\,.$$ □

We can use the extension axiom to construct Morley sequences.

Proposition 4.10. *Let $\surd$ be a forking relation. If $\bar a \ \surd_U \ B$ then there is a $\surd$-Morley sequence $(\bar a_n)_{n<\omega}$ for $\operatorname{tp}(\bar a/UB)$ over U.*

Proof. Set $\lambda := |T| \oplus |U| \oplus |B| \oplus |\bar a| \oplus \aleph_0$ and let $\kappa > \beth_{2^\lambda}$. First, we construct a sequence $(\bar c_\alpha)_{\alpha<\kappa}$ of tuples realising $\operatorname{tp}(\bar a/UB)$ such that

$$\bar c_\alpha \ \surd_U \ B\bar c[<\alpha]\,, \quad \text{for all } \alpha < \kappa\,.$$

By induction, suppose that we have already defined $\bar c_\beta$, for all $\beta < \alpha$. Since $\bar a \ \surd_U \ B$, we can use (EXT) to find a tuple $\bar c_\alpha \equiv_{UB} \bar a$ such that $\bar c_\alpha \ \surd_U \ B\bar c[<\alpha]$.

Having constructed $(\bar{c}_\alpha)_{\alpha<\kappa}$, we use Theorem E5.3.7 to find an indiscernible sequence $(\bar{a}_n)_{n<\omega}$ over $U \cup B$ such that, for every $n < \omega$, there are indices $\alpha_0 < \dots < \alpha_{n-1} < \kappa$ with

$$\bar{a}_0 \dots \bar{a}_{n-1} \equiv_{UB} \bar{c}_{\alpha_0} \dots \bar{c}_{\alpha_{n-1}} .$$

By (INV) and (MON) it follows that $\bar{a}_n \sqrt_U B\bar{a}[<n]$. Hence, $(\bar{a}_n)_{n<\omega}$ is the desired $\sqrt{}$-Morley sequence. $\square$

Corollary 4.11. *Let $\downarrow$ be a symmetric forking relation. For every tuple $\bar{a}$, every set U, and every linear order I, there exists a $\downarrow$-Morley sequence $(\bar{a}_i)_{i\in I}$ for $\mathrm{tp}(\bar{a}/U)$ over U.*

Proof. As $\downarrow$ is symmetric, we have $\bar{a} \downarrow_U U$. Therefore, we can use Proposition 4.10 to find a $\downarrow$-Morley sequence $(\bar{c}_n)_{n<\omega}$ for $\mathrm{tp}(\bar{a}/U)$ over U. By compactness and (FIN), it follows that there also exists a $\downarrow$-Morley sequence $(\bar{a}_i)_{i\in I}$ for $\mathrm{tp}(\bar{a}/U)$ over U that is indexed by I. $\square$

Lemma 4.12. *Let $\sqrt{}$ be a forking relation and let $\mathfrak{p}$ be a type over $U \cup B$. If there exists a $\sqrt{}$-Morley sequence $(\bar{c}_n)_{n<\omega}$ for $\mathfrak{p}$ over U, then there exists a reverse $\sqrt{}$-Morley sequence $(\bar{a}_n)_{n<\omega}$ for $\mathfrak{p}$ over U.*

Proof. Let $(\bar{c}_n)_{n<\omega}$ be a $\sqrt{}$-Morley sequence for $\mathfrak{p}$ over U. By compactness, there exists a sequence $(\bar{a}_n)_{n<\omega}$ such that

$$\bar{a}_0 \dots \bar{a}_n \equiv_{UB} \bar{c}_n \dots \bar{c}_0 , \quad \text{for all } n < \omega .$$

By definition of a Morley sequence we have

$$\bar{c}_n \sqrt_U \bar{c}_0 \dots \bar{c}_{n-1} .$$

Hence (INV) implies that

$$\bar{a}_i \sqrt_U \bar{a}_{i+1} \dots \bar{a}_n , \quad \text{for all } i < n < \omega .$$

Repeatedly applying Lemma 2.4 it follows that

$$\bar{a}_0 \dots \bar{a}_{n-1} \sqrt_U \bar{a}_n , \quad \text{for every } n < \omega .$$ $\square$

The following lemma can be used in some cases to construct a reverse $\sqrt{}$-Morley sequence out of an indiscernible sequence.

Lemma 4.13. *Let $\sqrt{}$ be a preforking relation and let I, J be linear orders such that I has no maximal element. If $(\bar{a}_i)_{i\in I+J}$ is indiscernible over U then $(\bar{a}_j)_{j\in J}$ is a reverse $\sqrt{}$-Morley sequence over $U \cup \bar{a}[I]$.*

Proof. Clearly, $(\bar{a}_j)_{j\in J}$ is indiscernible over $U \cup \bar{a}[I]$. To show that it is a reverse $\sqrt{}$-Morley sequence over $U \cup \bar{a}[I]$, it is sufficient, by (FIN), to prove that

$$\bar{a}_{j_0} \dots \bar{a}_{j_{k-1}} \sqrt_{U\bar{a}[I]} \bar{a}_{j_k} , \quad \text{for all } j_0 < \dots < j_k \text{ in } J ,\ k < \omega .$$

Hence, consider indices $j_0 < \dots < j_k$ in J. By indiscernibility and the fact that I has no maximal element, we can find, for every finite set $I_0 \subseteq I$, indices $i_0 < \dots < i_{k-1}$ in I such that

$$\bar{a}_{j_0} \dots \bar{a}_{j_{k-1}} \bar{a}_{j_k} \equiv_{U\bar{a}[I_0]} \bar{a}_{i_0} \dots \bar{a}_{i_{k-1}} \bar{a}_{j_k} .$$

It follows that $\mathrm{tp}(\bar{a}_{j_0} \dots \bar{a}_{j_{k-1}} / U \cup \bar{a}[I] \cup \bar{a}_{j_k})$ is finitely satisfiable in $U \cup \bar{a}[I]$. Consequently,

$$\bar{a}_{j_0} \dots \bar{a}_{j_{k-1}} \sqrt[\mathrm{u}]{}_{U\cup\bar{a}[I]} \bar{a}_{j_k} \quad \text{implies} \quad \bar{a}_{j_0} \dots \bar{a}_{j_{k-1}} \sqrt_{U\cup\bar{a}[I]} \bar{a}_{j_k} ,$$

as desired. $\square$

For preforking relations that are contained in the splitting relation $\sqrt[\mathrm{s}]{}$, we no not need to check for indiscernibility when proving that a given sequence is a Morley sequence.

Lemma 4.14. *Let $\alpha = (\bar{a}_i)_{i\in I}$ and $\beta = (\bar{b}_i)_{i\in I}$ be two sequences and $U \subseteq \mathbb{M}$ a set of parameters.*

(a) *If $\bar{b}_i \equiv_{U\bar{a}[<i]} \bar{a}_i$ and $\bar{b}_i \sqrt[\mathrm{s}]{}_U \bar{a}[<i]\bar{b}[<i]$, for all $i \in I$, then $\alpha \equiv_U \beta$.*

(b) *If $\bar{a}_j \equiv_{U\bar{a}[<i]} \bar{a}_i$ and $\bar{a}_i \sqrt[\mathrm{s}]{}_U \bar{a}[<i]$, for all $i \leq j$ in I, then α is indiscernible over U.*

Proof. (a) We prove by induction on $n < \omega$ that

$$\bar{a}[\bar{\imath}] \equiv_U \bar{b}[\bar{\imath}], \quad \text{for all } \bar{\imath} \in [I]^n.$$

For $n = 0$, the claim is trivial. For the inductive step, suppose that we have already proved it for n and consider a tuple of indices $\bar{\imath} \in [I]^{n+1}$. Setting $\bar{\imath}' := i_0 \dots i_{n-1}$ we have

$$\bar{a}[\bar{\imath}'] \equiv_U \bar{b}[\bar{\imath}'] \quad \text{and} \quad \bar{b}_{i_n} \overset{s}{\surd}_U \bar{a}[\bar{\imath}']\bar{b}[\bar{\imath}'],$$

which implies that $\bar{a}[\bar{\imath}'] \equiv_{U\bar{b}_{i_n}} \bar{b}[\bar{\imath}']$. Since $\bar{b}_{i_n} \equiv_{U\bar{a}[<i_n]} \bar{a}_{i_n}$, it follows that

$$\bar{a}[\bar{\imath}']\bar{a}_{i_n} \equiv_U \bar{a}[\bar{\imath}']\bar{b}_{i_n} \equiv_U \bar{b}[\bar{\imath}']\bar{b}_{i_n}.$$

(b) We have to prove that

$$\bar{a}[\bar{\imath}] \equiv_U \bar{a}[\bar{\jmath}], \quad \text{for all } \bar{\imath}, \bar{\jmath} \in [I]^n,\ n < \omega.$$

Hence, let $\bar{\imath}, \bar{\jmath} \in [I]^n$. First, we consider the case where $i_s \leq j_s$, for all $s < n$. Then we have

$$\bar{a}_{j_s} \equiv_{U\bar{a}_{i_0}\dots\bar{a}_{i_{s-1}}} \bar{a}_{i_s} \quad \text{and} \quad \bar{a}_{j_s} \overset{s}{\surd}_U \bar{a}_{i_0}\dots\bar{a}_{i_{s-1}}\bar{a}_{j_0}\dots\bar{a}_{j_{s-1}},$$

for all $s < n$. Consequently, it follows by (a) that $\bar{a}[\bar{\imath}] \equiv_U \bar{a}[\bar{\jmath}]$.

For the general case, let $\bar{\imath}, \bar{\jmath} \in [I]^n$ be arbitrary. We set

$$k_s := \max\{i_s, j_j\}, \quad \text{for } s < n.$$

Then $\bar{k} \in [I]^n$ and it follows by the special case considered above that $\bar{a}[\bar{\imath}] \equiv_U \bar{a}[\bar{k}] \equiv_U \bar{a}[\bar{\jmath}]$. □

As an application of Morley sequences we show that, for forking relations, right locality and symmetry are equivalent. One direction is based on the following two lemmas.

Lemma 4.15. *Let $\surd$ be a right local forking relation, $B, U \subseteq \mathbb{M}$ sets, and let $\kappa \geq \mathrm{loc}(\surd) \oplus |B|^+$ be a regular cardinal. For every reverse $\surd$-Morley sequence $(\bar{a}_i)_{i<\kappa}$ over U, there exists an index $\alpha < \kappa$ such that*

$$B\bar{a}[<\beta] \surd_U \bar{a}_\beta, \quad \text{for all } \alpha \leq \beta < \kappa.$$

Proof. By (RLOC), there exists a set $U_0 \subseteq U \cup \bar{a}[<\kappa]$ of size

$$|U_0| < \mathrm{loc}(\surd) \oplus |B|^+ \leq \kappa$$

such that

$$B \surd_{U_0} U\bar{a}[<\kappa].$$

Set $I := \{ i < \kappa \mid \bar{a}_i \cap U_0 \neq \varnothing \}$. Then $|I| < \kappa$ and, by regularity of κ, there exists an index $\alpha < \kappa$ that is larger than every element of I. For $\alpha \leq \beta < \kappa$, it follows by (BMON) and monotonicity that $B \surd_{U\bar{a}[<\beta]} \bar{a}_\beta$. Since $(\bar{a}_i)_{i<\kappa}$ is a reverse $\surd$-Morley sequence, we furthermore have $\bar{a}[<\beta] \surd_U \bar{a}_\beta$. By Lemma 2.3, it follows that $B\bar{a}[<\beta] \surd_U \bar{a}_\beta$. □

Lemma 4.16. *Let $\surd$ be a right local preforking relation. If there exists a reverse $\surd$-Morley sequence $(\bar{a}_n)_{n<\omega}$ for $\mathrm{tp}(\bar{a}/BU)$ over U then $B \surd_U \bar{a}$.*

Proof. Set $\kappa := |B|^+ \oplus \mathrm{loc}(\surd)^+$ and let $(\bar{a}_n)_{n<\omega}$ be a reverse $\surd$-Morley sequence. By compactness, we can extend $(\bar{a}_n)_{n<\omega}$ to an indiscernible sequence $(\bar{a}_i)_{i<\kappa}$ over $B \cup U$ of length κ. By (FIN) and (INV) it follows that

$$\bar{a}[<\alpha] \surd_U \bar{a}_\alpha, \quad \text{for all } \alpha < \kappa.$$

Hence, $(\bar{a}_i)_{i<\kappa}$ is a reverse $\surd$-Morley sequence. By Lemma 4.15, there is some index $\alpha < \kappa$ with $B \surd_U \bar{a}_\alpha$. As $\bar{a}_\alpha \equiv_{UB} \bar{a}$, we can use (INV) to conclude that $B \surd_U \bar{a}$. □

Theorem 4.17 (Adler). *A forking relation $\surd$ is right local if, and only if, it is symmetric.*

Proof. ($\Leftarrow$) follows by Corollary 3.18.

($\Rightarrow$) If $\bar a \mathop{\surd}_U B$, we can use Proposition 4.10 and Lemma 4.12 to construct a reverse $\surd$-Morley sequence of $\operatorname{tp}(\bar a/UB)$ over U. Therefore, it follows by Lemma 4.16 that $B \mathop{\surd}_U \bar a$. □

F3. *Simple theories*

1. *Dividing and forking*

In this section we introduce the central forking relation of model theory, which is simply called *forking*.

Definition 1.1. Let T be a first-order theory, U a set of parameters, and $k < \omega$.

(a) We say that a set Φ of formulae over U is *k-inconsistent* (with respect to T) if $T(U) \cup \Phi_0$ is inconsistent, for every subset $\Phi_0 \subseteq \Phi$ of size $|\Phi_0| \geq k$.

(b) A formula $\varphi(\bar x;\bar c)$ with parameters $\bar c$ *k-divides* over U if there exists a sequence $(\bar c_n)_{n<\omega}$ such that

- $\bar c_n \equiv_U \bar c$, for all $n < \omega$, and
- the set $\{\, \varphi(\bar x;\bar c_n) \mid n < \omega \,\}$ is k-inconsistent.

We say that $\varphi(\bar x;\bar c)$ *divides* over U if it k-divides over U, for some $k < \omega$.

(c) A set Φ of formulae *divides* over U if $T(\mathbb{M}) \cup \Phi \models \varphi$, for some formula φ that divides over U. We define

$$\bar a \mathop{\surd}^{\mathrm{d}}_U B \quad :\text{iff} \quad \operatorname{tp}(\bar a/UB) \text{ does not divide over } U.$$

(d) A set Φ of formulae *forks* over U if there are finitely many formulae $\varphi_0, \dots, \varphi_{n-1}$ such that

$$T(\mathbb{M}) \cup \Phi \models \varphi_0 \lor \dots \lor \varphi_{n-1}$$

and each φ_i divides over U. We define

$$\bar a \mathop{\surd}^{\mathrm{f}}_U B \quad :\text{iff} \quad \operatorname{tp}(\bar a/UB) \text{ does not fork over } U.$$

Example. (a) Consider the structure $\langle\mathbb{Q},<\rangle$ and let $b<c$ be rational numbers. The formula $\varphi(x;b,c) := b<x \wedge x<c$ divides over the set $U := \{ a \in \mathbb{Q} \mid a < b \}$ since we can choose numbers b_n and c_n such that $b \leq b_0 < c_0 < b_1 < c_1 < \dots$. Then $b_n c_n \equiv_U bc$ and the set $\{ b_n < x \wedge x < c_n \mid n < \omega \}$ is 2-inconsistent.

(b) We consider the tree $\langle A^{<\omega}, \preceq\rangle$ where A is an infinite set. Fix a vertex $u_0 \in A^{<\omega}$, an element $a \in A$, and set $u := u_0 a$. The formula $\varphi(x;u) := u \preceq x$ divides over the set $U := \{ v \in A^{<\omega} \mid u_0 \npreceq v \}$ since, fixing distinct elements $b_n \in A$, for $n<\omega$, we can set $c_n := ub_n$. Then $c_n \equiv_U u$ and $\{ c_n \preceq x \mid n < \omega \}$ is 2-inconsistent.

Remark. Note that, if a formula φ divides over U and $\psi \vDash \varphi$, then ψ also divides over U. It follows that a formula φ divides over U if, and only if, the set $\{\varphi\}$ divides over U. Furthermore, if a set Φ divides over U, then there exists a finite subset $\Phi_0 \subseteq \Phi$ such that the formula $\bigwedge \Phi_0$ divides over U. In particular, a complete type $\mathfrak{p}$ divides over U if, and only if, some formula $\varphi \in \mathfrak{p}$ divides over U. The same holds for forking.

Below we will prove that $\overset{\mathrm{d}}{\not\!\downarrow}$ is a preforking relation and $\overset{\mathrm{f}}{\not\!\downarrow}$ the associated forking relation. Before doing so, let us give an alternative characterisation of dividing in terms of indiscernible sequences.

Lemma 1.2. *Let $\varphi(\bar{x};\bar{y})$ be a formula and $\bar{c}, U \subseteq \mathbb{M}$. The following statements are equivalent:*

(1) *$\varphi(\bar{x};\bar{c})$ divides over U.*

(2) *There exists an indiscernible sequence $(\bar{c}_n)_{n<\omega}$ over U such that $\bar{c}_0 = \bar{c}$ and the set $\{ \varphi(\bar{x};\bar{c}_n) \mid n < \omega \}$ is k-inconsistent, for some $k<\omega$.*

(3) *There exists an indiscernible sequence $(\bar{c}_n)_{n<\omega}$ over U such that $\bar{c}_0 = \bar{c}$ and the set*

$$T(\textstyle\bigcup_{n<\omega} \bar{c}_n) \cup \{ \varphi(\bar{x};\bar{c}_n) \mid n < \omega \}$$

is inconsistent.

Proof. (2) ⇒ (1) and (2) ⇒ (3) are trivial.

(3) ⇒ (2) Let $(\bar{c}_n)_{n<\omega}$ be an indiscernible sequence over U with $\bar{c}_0 = \bar{c}$ such that

$$T(\textstyle\bigcup_{n<\omega} \bar{c}_n) \cup \{ \varphi(\bar{x};\bar{c}_n) \mid n < \omega \}$$

is inconsistent. Then there exists a finite subset $I \subseteq \omega$ such that

$$T(\textstyle\bigcup_{n\in I} \bar{c}_n) \cup \{ \varphi(\bar{x};\bar{c}_n) \mid n \in I \}$$

is inconsistent. Let $n_0 < \dots < n_{k-1}$ be an enumeration of I. For every k-tuple of indices $i_0 < \dots < i_{k-1}$, $\bar{c}[\bar{\imath}] \equiv_U \bar{c}[\bar{n}]$ implies that

$$T(\bar{c}_{i_0} \dots \bar{c}_{i_{k-1}}) \cup \{\varphi(\bar{x};\bar{c}_{i_0}), \dots, \varphi(\bar{x};\bar{c}_{i_{k-1}})\}$$

is inconsistent. Hence, $\{ \varphi(\bar{x};\bar{c}_n) \mid n < \omega \}$ is k-inconsistent.

(1) ⇒ (2) Suppose that $\varphi(\bar{x};\bar{c})$ divides over U. Then there exists a sequence $(\bar{c}_n)_{n<\omega}$ such that $\bar{c}_n \equiv_U \bar{c}$ and $\{ \varphi(\bar{x};\bar{c}_n) \mid n < \omega \}$ is k-inconsistent, for some k. By Proposition E5.3.6, there exists an indiscernible sequence $(\bar{d}_n)_{n<\omega}$ over U with

$$\mathrm{Av}((c_n)_n/U) \subseteq \mathrm{Av}((d_n)_n/U)\,.$$

In particular, $\mathrm{tp}(\bar{c}/U) \subseteq \mathrm{Av}((d_n)_n/U)$ and

$$\neg\exists\bar{z}[\varphi(\bar{z};\bar{x}_0) \wedge \dots \wedge \varphi(\bar{z};\bar{x}_{k-1})] \in \mathrm{Av}((d_n)_n/U)\,.$$

Consequently, $\bar{d}_0 \equiv_U \bar{c}$ and the set $\{ \varphi(\bar{x};\bar{d}_n) \mid n < \omega \}$ is k-inconsistent. Fixing an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ with $\pi(\bar{d}_0) = \bar{c}$, we obtain a sequence $(\pi(\bar{d}_n))_{n<\omega}$ with the desired properties. □

Exercise 1.1. Prove that a formula $\varphi(\bar{x};\bar{c})$ divides over a set U if, and only if, it divides over some model $M \supseteq U$. (*Hint.* Use Lemma E5.3.11.)

Lemma 1.3. *The following statements are equivalent:*

(1) $\bar{a} \overset{\mathrm{d}}{\not\!\downarrow}_U \bar{b}$

(2) *For every infinite indiscernible sequence* $(\bar{b}_i)_{i \in I}$ *over* U *with* $\bar{b} = \bar{b}_i$, *for some* i, *there exists a tuple* $\bar{a}' \equiv_{U\bar{b}} \bar{a}$ *such that* $(\bar{b}_i)_{i \in I}$ *is indiscernible over* $U \cup \bar{a}'$.

(3) *For every indiscernible sequence* $(\bar{b}_n)_{n<\omega}$ *over* U *with* $\bar{b} = \bar{b}_0$, *there is some* $\bar{a}' \equiv_{U\bar{b}} \bar{a}$ *such that*

$$\bar{b}_m \equiv_{U\bar{a}'} \bar{b}_n\,, \quad \text{for all } m, n < \omega\,.$$

Proof. (2) ⇒ (3) is trivial.

(3) ⇒ (1) Suppose that $\bar{a} \mathop{\not\downarrow}^{\mathrm{d}}_U \bar{b}$. By Lemma 1.2, we can find a formula $\varphi(\bar{x};\bar{c}) \in \mathrm{tp}(\bar{a}/U\bar{b})$ and an indiscernible sequence $(\bar{c}_n)_{n<\omega}$ over U such that $\bar{c}_n \equiv_U \bar{c}$ and $\{\, \varphi(\bar{x};\bar{c}_n) \mid n < \omega \,\}$ is k-inconsistent, for some $k < \omega$. By adding and permuting free variables of φ, we may assume that $\bar{c}_n = \bar{b}_n\bar{d}$ where $\bar{d} \subseteq U$ and $\bar{b}_n \equiv_U \bar{b}$, for all n. Finally, applying an automorphism of $\mathbb{M}$, we may assume that $\bar{b}_0 = \bar{b}$.

To show that (3) fails, consider a tuple $\bar{a}' \equiv_{U\bar{b}} \bar{a}$. Then

$$\mathbb{M} \vDash \varphi(\bar{a}';\bar{b}_0\bar{d})\,,$$

but the k-inconsistency of $\{\, \varphi(\bar{x};\bar{b}_n\bar{d}) \mid n < \omega \,\}$ implies that there is some $n < k$ with

$$\mathbb{M} \nvDash \varphi(\bar{a}';\bar{b}_n\bar{d})\,.$$

Consequently, $\bar{b}_n \not\equiv_{U\bar{a}'} \bar{b}_0$.

(1) ⇒ (3) Consider an indiscernible sequence $(\bar{b}_n)_{n<\omega}$ over U with $\bar{b} = \bar{b}_0$ and suppose that there is no such tuple $\bar{a}'$. Then the set

$$\mathrm{tp}(\bar{a}/U\bar{b}) \cup \{\, \varphi(\bar{x};\bar{b}_i) \leftrightarrow \varphi(\bar{x};\bar{b}_j) \mid i, j < \omega \text{ and } \varphi(\bar{x};\bar{y}) \text{ a formula over } U \,\}$$

is inconsistent. This set is equivalent to the union

$$\bigcup_{n<\omega} \mathfrak{p}(\bar{x}, \bar{b}_n)\,, \quad \text{where} \quad \mathfrak{p}(\bar{x}, \bar{x}') := \mathrm{tp}(\bar{a}\bar{b}/U)\,.$$

By compactness, we can therefore find a finite subset $\Phi \subseteq \mathfrak{p}$ and indices $n_0 < \dots < n_{k-1} < \omega$ such that

$$T \cup \Phi(\bar{x}, \bar{b}_{n_0}) \cup \dots \cup \Phi(\bar{x}, \bar{b}_{n_{k-1}})$$

is inconsistent. Setting $\varphi := \bigwedge \Phi$ it follows by indiscernibility that

$$T \vDash \neg\exists\bar{x}[\varphi(\bar{x}, \bar{b}_{i_0}) \wedge \dots \wedge \varphi(\bar{x}, \bar{b}_{i_{k-1}})]\,,$$

for every increasing tuple $i_0 < \dots < i_{k-1}$. Hence, $\{\, \varphi(\bar{x}, \bar{b}_n) \mid n < \omega \,\}$ is k-inconsistent and φ divides over U. Consequently, $\bar{a} \mathop{\not\downarrow}^{\mathrm{d}}_U \bar{b}$.

(3) ⇒ (2) Let $(\bar{b}_i)_{i \in I}$ be an infinite indiscernible sequence over U with $\bar{b}_{i_0} = \bar{b}$, for some $i_0 \in I$. Setting

$$\Psi := \big\{\, \psi(\bar{x};\bar{b}[\bar{\imath}]) \leftrightarrow \psi(\bar{x};\bar{b}[\bar{k}]) \;\big|\; \psi \text{ a formula over } U \text{ and } \operatorname{ord}(\bar{\imath}) = \operatorname{ord}(\bar{k}) \,\big\}\,,$$

it is sufficient to prove that $\mathrm{tp}(\bar{a}/U\bar{b}) \cup \Psi$ is satisfiable.

Fix a dense linear order $J \supseteq I$ without end points. Using Lemma E5.3.9, we can extend $(\bar{b}_i)_{i \in I}$ to an indiscernible sequence $(\bar{b}_i)_{i \in J}$ over U. By (3) and compactness, there exists a tuple $\bar{a}' \equiv_{U\bar{b}} \bar{a}$ such that

$$\bar{b}_i \equiv_{U\bar{a}'} \bar{b}_j\,, \quad \text{for all } i, j \in J\,.$$

To show that $\mathrm{tp}(\bar{a}/U\bar{b}) \cup \Psi$ is satisfiable, let $\Psi_0 \subseteq \Psi$ be finite and let $I_0 \subseteq I$ be the finite set of all indices i such that Ψ_0 contains the constants $\bar{b}_i$. By the Theorem of Ramsey, there exist an order embedding $h_0 : I_0 \to J$ such that the sequence $(\bar{b}_{h(i)})_{i \in I_0}$ is indiscernible over $U \cup \bar{a}'$ with respect to the formulae in Ψ_0. We extend $h_0 : I_0 \to J$ to an order embedding $h : I_0 \cup \{i_0\} \to J$. There exists an automorphism $\pi \in \operatorname{Aut} \mathbb{M}_U$ mapping $\bar{b}_{h(i)}$ to $\bar{b}_i$, for $i \in I_0 \cup \{i_0\}$. Then the tuple $\pi(\bar{a}')$ satisfies $\bigcup_{i \in I_0 \cup \{i_0\}} \mathrm{tp}(\bar{a}/U\bar{b}_i) \cup \Psi_0$. In particular, it satisfies $\mathrm{tp}(\bar{a}/U\bar{b}) \cup \Psi_0$. □

Remark. Comparing the statement in (2) above with Lemma E5.3.11, we see that, when $\bar{a} \mathop{\downarrow}^{\mathrm{d}}_U B$, we can choose $\bar{a}' \equiv_{UB} \bar{a}$ while, in general, we only find $\bar{a}' \equiv_U \bar{a}$.

Example. (a) Consider the structure $\langle\mathbb{Q},<\rangle$ and let $b<a<c$ be elements. Then $bc \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\varnothing} a$ but $a \mathop{\overset{\mathrm{d}}{\not\smile\!\!\!\!\!\!|}}_{\varnothing} bc$. In particular, $\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}$ is not symmetric.

We have already seen above that $\varphi(x;b,c) := b<x \wedge x<c$ divides over $\downarrow b$ and, hence, also over the empty set. Consequently, $a \mathop{\overset{\mathrm{d}}{\not\smile\!\!\!\!\!\!|}}_{\varnothing} bc$. To show that $bc \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\varnothing} a$, let $(a_i)_{i<\omega}$ be an indiscernible sequence over $\varnothing$. Choose elements b' and c' such that $b'<a<c'$ and $b'<a_i<c'$, for all $i<\omega$. Then $b'c' \equiv_a bc$ and $(a_i)_{i<\omega}$ is indiscernible over $\{b',c'\}$. By Lemma 1.3, it follows that $bc \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\varnothing} a$.

(b) Let $\langle A,\sim\rangle$ be a structure where $\sim$ is an equivalence relation with infinitely many classes all of which are infinite. Fix elements $a,b\in A$ and a set $U\subseteq A$. Then

$$a \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{U} b \quad\text{iff}\quad \begin{array}{l}\{a\}\cap\{b\}\subseteq U \text{ and,}\\ a\nsim b \text{ or there is some } c\in U \text{ with } b\sim c\,.\end{array}$$

Let us show next that $\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}$ is a preforking relation, that $\overset{\mathrm{f}}{\smile\!\!\!\!\!\!|}$ is the corresponding forking relation, and that acl is the closure operator associated with them.

Proposition 1.4. *$\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}$ is a preforking relation.*

Proof. Throughout the proof we will tacitly make use of the characterisation of $\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}$ from Lemma 1.3.

(INV) follows immediately from the definition.

(MON) Suppose that $\bar a_0\bar a_1 \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{U} B$ and let $B_0\subseteq B$. For a contradiction, suppose that $\bar a_0 \mathop{\overset{\mathrm{d}}{\not\smile\!\!\!\!\!\!|}}_{U} B_0$. Then we can find a formula $\varphi\in\mathrm{tp}(\bar a_0/UB_0)$ that divides over U. Hence, $\varphi\in\mathrm{tp}(\bar a_0\bar a_1/UB)$ implies that $\bar a_0\bar a_1 \mathop{\overset{\mathrm{d}}{\not\smile\!\!\!\!\!\!|}}_{U} B$. A contradiction.

(NOR) Suppose that $\bar a \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar c} \bar b$. To show that $\bar a\bar c \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar c} \bar b\bar c$, let $(\bar b_n\bar c_n)_{n<\omega}$ be an indiscernible sequence over $\bar c$ with $\bar b_0\bar c_0=\bar b\bar c$. Then $\bar c_n=\bar c$, for all n. Since $\bar a \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar c} \bar b$, there is a tuple $\bar a'\equiv_{\bar b\bar c}\bar a$ such that $(\bar b_n)_{n<\omega}$ is indiscernible over $\bar a'\bar c$. Hence, $(\bar b_n\bar c)_{n<\omega}$ is also indiscernible over $\bar a'\bar c$. As $\bar a'\bar c\equiv_{\bar b\bar c}\bar a\bar c$, the claim follows.

(LRF) Let $\bar a,\bar b$ be tuples. To show that $\bar a \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar a} \bar b$ it is sufficient to note that every indiscernible sequence $(\bar b_n)_{n<\omega}$ over $\bar a$ is also indiscernible over $\bar a\cup\bar a$.

(LTR) Suppose that $\bar a_0\bar a_1\bar a_2 \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar a_0\bar a_1} \bar b$ and $\bar a_0\bar a_1 \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar a_0} \bar b$. Let $(\bar b_n)_{n<\omega}$ be an infinite indiscernible sequence over $\bar a_0$ such that $\bar b_0=\bar b$. We have to find tuples

$$\bar a_0''\bar a_1''\bar a_2''\equiv_{\bar a_0\bar b}\bar a_0\bar a_1\bar a_2$$

such that $(\bar b_n)_{n<\omega}$ is indiscernible over $\bar a_0''\bar a_1''\bar a_2''$. Since $\bar a_0\bar a_1 \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar a_0} \bar b$, there are tuples $\bar a_0'\bar a_1'\equiv_{\bar a_0\bar b}\bar a_0\bar a_1$ such that $(\bar b_n)_{n<\omega}$ is indiscernible over $\bar a_0'\bar a_1'$. Let $\bar a_2'$ be a tuple such that

$$\bar a_0'\bar a_1'\bar a_2'\equiv_{\bar a_0\bar b}\bar a_0\bar a_1\bar a_2\,.$$

Then $\bar a_0'\bar a_1'\bar a_2' \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar a_0'\bar a_1'} \bar b$ and there are tuples

$$\bar a_0''\bar a_1''\bar a_2''\equiv_{\bar a_0'\bar a_1'\bar b}\bar a_0'\bar a_1'\bar a_2'$$

such that $(\bar b_n)_{n<\omega}$ is indiscernible over $\bar a_0''\bar a_1''\bar a_2''$. Since

$$\bar a_0''=\bar a_0 \quad\text{and}\quad \bar a_0''\bar a_1''\bar a_2''\equiv_{\bar a_0\bar b}\bar a_0\bar a_1\bar a_2$$

the claim follows.

(BMON) Suppose that $\bar a \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar c} \bar b\bar d$. To show that $\bar a \mathop{\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}}_{\bar c\bar d} \bar b$, let $(\bar b_n)_{n<\omega}$ be a sequence of indiscernibles over $\bar c\bar d$ with $\bar b_0=\bar b$. Then $(\bar b_n\bar d)_{n<\omega}$ is indiscernible over $\bar c$. Consequently, there is some tuple $\bar a'\equiv_{\bar c\bar b\bar d}\bar a$ such that $(\bar b_n\bar d)_{n<\omega}$ is indiscernible over $\bar a'\bar c$. It follows that $(\bar b_n)_{n<\omega}$ is indiscernible over $\bar a'\bar c\bar d$.

(DEF) Suppose that $\bar a \mathop{\overset{\mathrm{d}}{\not\smile\!\!\!\!\!\!|}}_{U} B$. Then there exists a formula $\varphi(\bar x;\bar b)\in\mathrm{tp}(\bar a/UB)$ that divides over U. For every $\bar a'\in\varphi(\bar x;\bar b)^{\mathbb{M}}$ it follows that $\mathrm{tp}(\bar a'/U\bar b)$ divides over U. $\square$

Before proving that $\overset{\mathrm{f}}{\smile\!\!\!\!\!\!|}$ is the forking relation associated with $\overset{\mathrm{d}}{\smile\!\!\!\!\!\!|}$, let us show that forking satisfies the axiom (EXT) even for incomplete types.

Lemma 1.5. *A partial type Φ over $U \cup C$ forks over U if, and only if, every complete type $\mathfrak{p} \in \langle \Phi \rangle$ forks over U.*

Proof. Clearly, if Φ forks over U, then so does every type containing Φ. Conversely, suppose that every $\mathfrak{p} \in \langle \Phi \rangle$ forks over U. For each $\mathfrak{p} \in \langle \Phi \rangle$, we fix a formula $\varphi_{\mathfrak{p}} \in \mathfrak{p}$ that forks over U. By compactness,

$$\langle \Phi \rangle = \{ \mathfrak{p} \mid \mathfrak{p} \in \langle \Phi \rangle \} \subseteq \bigcup_{\mathfrak{p} \in \langle \Phi \rangle} \langle \varphi_{\mathfrak{p}} \rangle$$

implies that there are finitely many types $\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1} \in \langle \Phi \rangle$ such that

$$\langle \Phi \rangle \subseteq \langle \varphi_{\mathfrak{p}_0} \rangle \cup \dots \cup \langle \varphi_{\mathfrak{p}_{n-1}} \rangle .$$

Consequently, $\Phi \vDash \varphi_{\mathfrak{p}_0} \vee \dots \vee \varphi_{\mathfrak{p}_{n-1}}$ and Φ forks over U. $\square$

Proposition 1.6. $\overset{\mathrm{f}}{\surd} = {}^*(\overset{\mathrm{d}}{\surd})$

Proof. $(\subseteq)$ To prove that $\overset{\mathrm{f}}{\surd} \subseteq {}^*(\overset{\mathrm{d}}{\surd})$, note that $\overset{\mathrm{f}}{\surd} \subseteq \overset{\mathrm{d}}{\surd}$ and that the operation $\surd \mapsto \overset{*}{\surd}$ is monotone. Therefore, it is sufficient to prove that $\overset{\mathrm{f}}{\surd} = {}^*(\overset{\mathrm{f}}{\surd})$, i.e., that $\overset{\mathrm{f}}{\surd}$ satisfies (EXT). Hence, suppose that $\bar{a} \overset{\mathrm{f}}{\surd}_U B$ and let C be an arbitrary set. By Lemma 1.5, there exists a complete type $\mathfrak{p}$ over $U \cup B \cup C$ that contains $\mathrm{tp}(\bar{a}/UB)$ and that does not fork over U. Fix a realisation $\bar{a}'$ of $\mathfrak{p}$. Then $\bar{a}' \equiv_{UB} \bar{a}$ and $\bar{a}' \overset{\mathrm{f}}{\surd}_U BC$.

$(\supseteq)$ Suppose that $\bar{a} \not\overset{\mathrm{f}}{\surd}_U B$. Then we can find finitely many formulae $\varphi_0(\bar{x}; \bar{c}_0), \dots, \varphi_{n-1}(\bar{x}; \bar{c}_{n-1})$ that each divide over U and such that

$$\mathrm{tp}(\bar{a}/UB) \vDash \varphi_0(\bar{x}; \bar{c}_0) \vee \dots \vee \varphi_{n-1}(\bar{x}; \bar{c}_{n-1}) .$$

For every tuple $\bar{a}' \equiv_{UB} \bar{a}$, there is some $i < n$ such that $\mathbb{M} \vDash \varphi_i(\bar{a}'; \bar{c}_i)$. Consequently,

$$\bar{a}' \not\overset{\mathrm{d}}{\surd}_U B\bar{c}_0 \dots \bar{c}_{n-1} , \quad \text{for all } \bar{a}' \equiv_{UB} \bar{a} .$$

Hence, $\bar{a}\ {}^*(\overset{\mathrm{d}}{\surd})_U\ B$ does not hold. $\square$

Corollary 1.7. *$\overset{\mathrm{f}}{\surd}$ is a forking relation.*

Lemma 1.8. $\mathrm{cl}_{\overset{\mathrm{f}}{\surd}} = \mathrm{cl}_{\overset{\mathrm{d}}{\surd}} = \mathrm{acl}$

Proof. By Lemma F2.4.6, it is sufficient to prove that $\mathrm{cl}_{\overset{\mathrm{d}}{\surd}} = \mathrm{acl}$.

For one inclusion, let $a \notin \mathrm{acl}(U)$. Then there exists an indiscernible sequence $(a_n)_{n<\omega}$ over U with $a_0 = a$ and $a_i \neq a_k$, for $i \neq k$. Since a is the only element realising $\mathrm{tp}(a/Ua)$ and $(a_n)_n$ is not indiscernible over $U \cup \{a\}$ it follows by Lemma 1.3 that $a \not\overset{\mathrm{d}}{\surd}_U a$.

Conversely, suppose that there are sets B, C such that $a \not\overset{\mathrm{d}}{\surd}_{UC} B$. By Lemma 1.2, we can find a formula $\varphi(x; \bar{c}) \in \mathrm{tp}(a/UCB)$ and an indiscernible sequence $(\bar{c}_n)_{n<\omega}$ such that $\bar{c}_0 = \bar{c}$ and $\{ \varphi(x; \bar{c}_n) \mid n < \omega \}$ is k-inconsistent, for some k. For every $n < \omega$, fix an element a_n such that $a_n \bar{c}_n \equiv_U a\bar{c}$. Since $\mathbb{M} \vDash \varphi(a_n; \bar{c}_n)$ and $\{ \varphi(x; \bar{c}_n) \mid n < \omega \}$ is k-inconsistent, there exists an infinite subset $I \subseteq \omega$ such that $a_i \neq a_j$, for distinct $i, j \in I$. As each a_n satisfies $\mathrm{tp}(a/U)$ it follows that $a \notin \mathrm{acl}(U)$. $\square$

At first sight, the definition of $\overset{\mathrm{d}}{\surd}$ might seem rather ad-hoc. The following result indicates that $\overset{\mathrm{d}}{\surd}$ plays a rather distinguished role: it is the largest preforking relation that is contained in every symmetric forking relation.

Theorem 1.9. $\overset{\mathrm{d}}{\surd} \subseteq \downarrow$, *for every symmetric forking relation* $\downarrow$.

Proof. Suppose that $\bar{a} \overset{\mathrm{d}}{\surd}_U \bar{b}$. Since $\downarrow$ is symmetric, (LRF) implies that $B \downarrow_U U$. Therefore, we can use Proposition F2.4.10 and Lemma F2.4.12 to construct a reverse $\downarrow$-Morley sequence $(\bar{b}_n)_{n<\omega}$ for $\mathrm{tp}(\bar{b}/U)$ over U. By (INV) we may assume that $\bar{b}_0 = \bar{b}$. Since $\bar{a} \overset{\mathrm{d}}{\surd}_U \bar{b}$ there is a tuple $\bar{a}' \equiv_{U\bar{b}} \bar{a}$ such that $(\bar{b}_n)_{n<\omega}$ is indiscernible over $U\bar{a}'$. Hence, $(\bar{b}_n)_{n<\omega}$ is a reverse $\downarrow$-Morley sequence for $\mathrm{tp}(\bar{b}/U)$ over $U\bar{a}'$. Since $\downarrow$ is right local, it follows by Lemma F2.4.16 that $\bar{a}' \downarrow_U \bar{b}$. By invariance we obtain $\bar{a} \downarrow_U \bar{b}$. $\square$

Remark. In the next section we will show that there are theories where $\overset{\mathrm{d}}{\surd}$ is symmetric and a forking relation. For such theories, $\overset{\mathrm{d}}{\surd}$ is the largest preforking relation that is contained in every symmetric forking relation.

To conclude this section we compare $\mathop{\not\downarrow}^{\mathrm{d}}$ and $\mathop{\not\downarrow}^{\mathrm{f}}$ with the preforking relations introduced in Section F2.3. First, let us introduce the forking relation associated with the splitting relation $\mathop{\not\downarrow}^{\mathrm{s}}$.

Definition 1.10. $\mathop{\not\downarrow}^{\mathrm{i}} := {}^*(\mathop{\not\downarrow}^{\mathrm{s}})$.

Lemma 1.11. $\mathop{\not\downarrow}^{\mathrm{i}} \subseteq \mathop{\not\downarrow}^{\mathrm{d}}$

Proof. Suppose that $\bar a \mathop{\not\downarrow}^{\mathrm{i}}_U B$. To show that $\bar a \mathop{\not\downarrow}^{\mathrm{d}}_U B$, consider a formula $\varphi(\bar x;\bar c) \in \operatorname{tp}(\bar a/UB)$ and let $(\bar c_n)_{n<\omega}$ be a sequence such that $\bar c_n \equiv_U \bar c$, for all n. We have to show that the set $\{\,\varphi(\bar x;\bar c_n) \mid n < \omega\,\}$ is not k-inconsistent for any k.

There is a tuple $\bar a' \equiv_{UB} \bar a$ such that

$$\bar a' \mathop{\not\downarrow}^{\mathrm{s}}_U B\bar c[<\omega]\,.$$

Hence, $\varphi(\bar x;\bar c) \in \operatorname{tp}(\bar a'/UB\bar c[<\omega])$ implies that

$$\varphi(\bar x;\bar c_n) \in \operatorname{tp}(\bar a'/UB\bar c[<\omega])\,, \quad \text{for all } n\,.$$

Consequently, $\bar a'$ satisfies $\{\,\varphi(\bar x;\bar c_n) \mid n < \omega\,\}$ and this set is not k-inconsistent. □

Proposition 1.12. $\mathop{\not\downarrow}^{\mathrm{u}} \subseteq \mathop{\not\downarrow}^{\mathrm{i}} \subseteq \mathop{\not\downarrow}^{\mathrm{f}} \subseteq \mathop{\not\downarrow}^{\mathrm{d}}$

Proof. The inclusions $\mathop{\not\downarrow}^{\mathrm{u}} \subseteq \mathop{\not\downarrow}^{\mathrm{i}} \subseteq \mathop{\not\downarrow}^{\mathrm{d}}$ follow from Theorem F2.3.13 and the preceding lemma, respectively. Since the operation $\mathop{\not\downarrow} \mapsto \mathop{\not\downarrow}^{*}$ is monotone and idempotent, we further have $\mathop{\not\downarrow}^{\mathrm{i}} = {}^*(\mathop{\not\downarrow}^{\mathrm{i}}) \subseteq {}^*(\mathop{\not\downarrow}^{\mathrm{d}}) = \mathop{\not\downarrow}^{\mathrm{f}}$. □

2. *Simple theories and the tree property*

The aim of this section is to characterise those theories where the relation $\mathop{\not\downarrow}^{\mathrm{f}}$ is symmetric. In the same way as stable theories are characterised by the absence of the order property, we will present a combinatorial property causing $\mathop{\not\downarrow}^{\mathrm{f}}$ to be non-symmetric.

Definition 2.1. A first-order theory T is *simple* if $\mathop{\not\downarrow}^{\mathrm{f}}$ is symmetric. For simple theories we will write $\mathop{\downarrow}^{\mathrm{f}}$ and $\mathop{\downarrow}^{\mathrm{d}}$ instead of $\mathop{\not\downarrow}^{\mathrm{f}}$ and $\mathop{\not\downarrow}^{\mathrm{d}}$. In later chapters, where $\mathop{\downarrow}^{\mathrm{f}}$ will be the only forking relation under consideration, we will frequently drop the superscript and just write $\downarrow$.

Before giving a combinatorial characterisation of simple theories, let us note some special properties of the relation $\mathop{\downarrow}^{\mathrm{f}}$ in such theories. It follows from Theorem 1.9 that, for complete types in simple theories, forking and dividing is the same. According to the next lemma this is also true for partial types.

Lemma 2.2. *Let T be a simple theory, $\Phi(\bar x;\bar y)$ a set of formulae over U, and $\bar c \subseteq \mathbb{M}$. The following statements are equivalent:*

(1) *$\Phi(\bar x;\bar c)$ forks over U.*

(2) *$\Phi(\bar x;\bar c)$ divides over U.*

(3) *For every $\mathop{\downarrow}^{\mathrm{f}}$-Morley sequence $(\bar c_n)_{n<\omega}$ for $\operatorname{tp}(\bar c/U)$ over U, the set $\bigcup_{i<\omega}\Phi(\bar x;\bar c_n)$ is inconsistent.*

Proof. (2) ⇒ (1) follows immediately from the definition of forking.

(3) ⇒ (2) Let $(\bar c_n)_{n<\omega}$ be a $\mathop{\downarrow}^{\mathrm{f}}$-Morley sequence for $\operatorname{tp}(\bar c/U)$ over U. Applying a U-automorphism we can ensure that $\bar c_0 = \bar c$. By assumption, $\bigcup_{n<\omega}\Phi(\bar x;\bar c_n)$ is inconsistent. Using compactness, we obtain a finite subset $\Phi_0 \subseteq \Phi$ such that $\bigcup_{n<\omega}\Phi_0(\bar x;\bar c_n)$ is inconsistent. Set $\varphi := \bigwedge\Phi_0$. By Lemma 1.2, it follows that $\varphi(\bar x;\bar c)$ divides over U. Since $\Phi(\bar x;\bar c) \vDash \varphi(\bar x;\bar c)$, so does $\Phi(\bar x;\bar c)$.

(1) ⇒ (3) Suppose that $(\bar c_n)_{n<\omega}$ is a $\mathop{\downarrow}^{\mathrm{f}}$-Morley sequence for $\operatorname{tp}(\bar c/U)$ over U such that the set $\bigcup_{n<\omega}\Phi(\bar x;\bar c_n)$ is consistent. Fix a regular cardinal $\kappa \geq \operatorname{loc}(\mathop{\downarrow}^{\mathrm{f}}) \oplus |\bar x|^+$. By compactness, there exists a $\mathop{\downarrow}^{\mathrm{f}}$-Morley sequence $(\bar c_i)_{i<\kappa}$ for $\operatorname{tp}(\bar c/U)$ over U such that $\bigcup_{i<\kappa}\Phi(\bar x;\bar c_i)$ is consistent. Let $\bar a$ be a tuple satisfying this set. By Lemma F2.4.15, we can find an index $\alpha < \kappa$ such that

$$\bar a\bar c[<\alpha] \mathop{\downarrow}^{\mathrm{f}}_U \bar c_\alpha\,.$$

Consequently, $\Phi(\bar{x};\bar{c}_\alpha)$ does not fork over U. By (INV), the same holds for $\Phi(\bar{x};\bar{c})$. $\square$

Next, we present an improved version of Lemma 1.3.

Proposition 2.3 (Kim)**.** *Let T be a simple theory. The following statements are equivalent.*

(1) $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_U \bar{b}$

(2) $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_U \bar{b}$

(3) *For every infinite $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$-Morley sequence $(\bar{b}_i)_{i\in I}$ for $\mathrm{tp}(\bar{b}/U)$ over U there exists a tuple $\bar{a}' \equiv_{U\bar{b}} \bar{a}$ such that $(\bar{b}_i)_{i\in I}$ is a $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$-Morley sequence over $U \cup \bar{a}'$.*

(4) *For some $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$-Morley sequence $(\bar{b}_i)_{i<\omega}$ for $\mathrm{tp}(\bar{b}/U)$ over U there exists a tuple $\bar{a}' \equiv_{U\bar{b}} \bar{a}$ such that $(\bar{b}_i)_{i<\omega}$ is a $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$-Morley sequence over $U \cup \bar{a}'$.*

Proof. (1) $\Leftrightarrow$ (2) has already been shown in Lemma 2.2 and (1) $\Rightarrow$ (3) is a special case of Lemma 1.3.

(3) $\Rightarrow$ (4) is trivial since we have seen in Corollary F2.4.11 that, for symmetric forking relations, Morley sequences always exist.

(4) $\Rightarrow$ (2) Let $(\bar{b}_i)_{i<\omega}$ be a $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$-Morley sequence for $\mathrm{tp}(\bar{b}/U)$ over $U\cup\bar{a}'$, for some $\bar{a}' \equiv_{U\bar{b}} \bar{a}$. Set $\mathfrak{p}(\bar{x},\bar{x}') := \mathrm{tp}(\bar{a}\bar{b}/U)$. Then $\bar{a}'$ realises $\mathfrak{p}(\bar{x},\bar{b})$. Hence, $\bar{a}'$ is a realisation of $\bigcup_{i<\omega}\mathfrak{p}(\bar{x},\bar{b}_i)$ and it follows by Lemma 2.2 that $\mathfrak{p}(\bar{x},\bar{b})$ does not fork over U. $\square$

Right locality

Note that, if the relation $\sqrt[\mathrm{f}]{}$ is right local, then $\sqrt[\mathrm{f}]{} \subseteq \sqrt[\mathrm{d}]{}$ implies that $\sqrt[\mathrm{d}]{}$ is also right local. (This is also a consequence of Lemma 2.2.) In this section we will prove that the converse is also true: if $\sqrt[\mathrm{d}]{}$ is right local, then so is $\sqrt[\mathrm{f}]{}$. Recall the notion of a $\sqrt{}$-forking chain introduced in Section F2.3.

Definition 2.4. (a) We call $\sqrt[\mathrm{d}]{}$-forking chains and $\sqrt[\mathrm{f}]{}$-forking chains *dividing chains* and *forking chains,* respectively.

(b) A *specification* of a dividing chain $(\bar{b}_\alpha)_{\alpha<\gamma}$ for $\bar{a}$ over U is a sequence $\langle\varphi_\alpha, k_\alpha\rangle_{\alpha<\gamma}$ of pairs consisting of a formula $\varphi_\alpha(\bar{x};\bar{y}_\alpha)$ and a natural number k_α such that, for all $\alpha<\gamma$,

$$\mathbb{M} \vDash \varphi_\alpha(\bar{a};\bar{b}_\alpha) \quad \text{and} \quad \varphi_\alpha(\bar{x};\bar{b}_\alpha)\ k_\alpha\text{-divides over } U \cup \bar{b}[<\alpha]\,.$$

Similarly, a *specification* of a forking chain $(\bar{b}_\alpha)_{\alpha<\gamma}$ for $\bar{a}$ over U is a sequence $\langle\varphi_\alpha, \bar{\psi}_\alpha, \bar{k}_\alpha, m_\alpha\rangle_{\alpha<\gamma}$, where φ_α is a formula, m_α a natural number, $\bar{\psi}_\alpha$ an m_α-tuple of formulae, and $\bar{k}_\alpha$ is an m_α-tuple of natural numbers such that, for all $\alpha<\gamma$,

$$\mathbb{M} \vDash \varphi_\alpha(\bar{a};\bar{b}_\alpha)$$

and there are tuples $\bar{d}_0,\dots,\bar{d}_{m_\alpha-1}$ such that

$$\varphi_\alpha(\bar{x};\bar{b}_\alpha) \vDash \psi_{\alpha,0}(\bar{x},\bar{d}_0) \vee \dots \vee \psi_{\alpha,m_\alpha-1}(\bar{x},\bar{d}_{m_\alpha-1})$$

and each $\psi_{\alpha,i}(\bar{x},\bar{d}_i)$ $k_{\alpha,i}$-divides over $U\cup\bar{b}[<\alpha]$.

(c) A dividing chain is *uniform* if it has a specification $\langle\varphi_\alpha, k_\alpha\rangle_{\alpha<\gamma}$ where

$$\varphi_\alpha = \varphi_\beta \quad \text{and} \quad k_\alpha = k_\beta\,, \quad \text{for all } \alpha,\beta<\gamma\,.$$

Similarly, we say that a forking chain is *uniform* if it has a specification $\langle\varphi_\alpha, \bar{\psi}_\alpha, \bar{k}_\alpha, m_\alpha\rangle_{\alpha<\gamma}$ where

$$\varphi_\alpha = \varphi_\beta\,, \quad m_\alpha = m_\beta\,, \quad \psi_{\alpha,i} = \psi_{\beta,i}\,, \quad k_{\alpha,i} = k_{\beta,i}\,,$$

for all $\alpha,\beta<\gamma$ and $i<m_\alpha$.

Note that, according to Theorem F2.3.25, $\sqrt[\mathrm{d}]{}$ is not right local if, and only if, there are arbitrarily long dividing chains. The same holds for $\sqrt[\mathrm{f}]{}$ and forking chains. Our aim is therefore to show that, if a theory has arbitrarily long forking chains, then there are also arbitrarily long dividing chains. We start with the observation that any subsequence of a forking chain is again a forking chain. As a consequence we can use the Pigeon Hole Principle to construct uniform forking chains.

Lemma 2.5. *Let γ be an ordinal and $I \subseteq \gamma$.*

(a) *If $(\bar b_\alpha)_{\alpha<\gamma}$ is a dividing chain for $\bar a$ over U with the specification $\langle \varphi_\alpha, k_\alpha \rangle_{\alpha<\gamma}$, then $(\bar b_\alpha)_{\alpha \in I}$ is a dividing chain for $\bar a$ over U with specification $\langle \varphi_\alpha, k_\alpha \rangle_{\alpha \in I}$.*

(b) *If $(\bar b_\alpha)_{\alpha<\gamma}$ is a forking chain for $\bar a$ over U with the specification $\langle \varphi_\alpha, \bar\psi_\alpha, \bar k_\alpha, m_\alpha \rangle_{\alpha<\gamma}$, then $(\bar b_\alpha)_{\alpha \in I}$ is a forking chain for $\bar a$ over U with specification $\langle \varphi_\alpha, \bar\psi_\alpha, \bar k_\alpha, m_\alpha \rangle_{\alpha \in I}$.*

Proof. (a) Fix $\alpha \in I$ and set $B := \bigcup\{ \bar b_\beta \mid \beta \in I,\ \beta < \alpha \}$. It is sufficient to show that $\varphi_\alpha(\bar x; \bar b_\alpha)$ k_α-divides over $U \cup B$. This follows from the definition of dividing and the fact that $\varphi_\alpha(\bar x; \bar b_\alpha)$ k_α-divides over the superset $U \cup \bar b[<\alpha] \supseteq U \cup B$.

(b) follows analogously. □

Corollary 2.6. *Let $\kappa > |T|$ be a cardinal. If there exists a forking chain for $\bar a$ over U of length κ, then there also exists a uniform forking chain for $\bar a$ over U of length κ.*

Proof. Let $(\bar b_\alpha)_{\alpha<\kappa}$ be a forking chain for $\bar a$ over U with specification $\langle \varphi_\alpha, \bar\psi_\alpha, \bar k_\alpha, m_\alpha \rangle_{\alpha<\kappa}$. Since there are at most $|T| < \kappa$ formulae over $\varnothing$, there exist a subset $I \subseteq \kappa$ of size $|I| = \kappa$, formulae φ, $\bar\psi$, and numbers m, $\bar k$ such that

$$\varphi_\alpha = \varphi\,, \quad m_\alpha = m\,, \quad \psi_{\alpha,i} = \psi_i\,, \quad k_{\alpha,i} = k_i\,,$$

for all $\alpha < \kappa$ and $i < m$. By Lemma 2.5, the subsequence $(\bar b_\alpha)_{\alpha \in I}$ is a uniform forking chain for $\bar a$ over U. □

The key property of dividing which allows us to turn forking chains into dividing chains is contained in the following lemma.

Lemma 2.7. *Suppose that the formula $\varphi(\bar x; \bar b)$ k-divides over a set U. For every set $C \subseteq \mathbb{M}$, there is some tuple $\bar b' \equiv_U \bar b$ such that $\varphi(\bar x; \bar b')$ k-divides over $U \cup C$.*

Proof. By Lemma 1.2, there exists an indiscernible sequence $(\bar b_n)_{n<\omega}$ over U such that $\bar b_0 = \bar b$ and the set $\{ \varphi(\bar x; \bar b_n) \mid n < \omega \}$ is k-inconsistent. Using Lemma E5.3.11, we can find a set $C' \equiv_U C$ such that $(\bar b_n)_{n<\omega}$ is indiscernible over $U \cup C'$. Let $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ be an automorphism with $\pi[C'] = C$, and set $\bar b'_n := \pi(\bar b_n)$. Then $(\bar b'_n)_{n<\omega}$ is indiscernible over $U \cup C$ and the set $\{ \varphi(\bar x; \bar b'_n) \mid n < \omega \}$ is k-inconsistent. By Lemma 1.2, it follows that $\varphi(\bar x; \bar b'_0)$ k-divides over $U \cup C$. Since $\bar b'_0 \equiv_U \bar b_0 = \bar b$, the claim follows. □

Corollary 2.8. *Let $(\bar b_i)_{i<n}$ be a dividing chain for $\bar a$ over U with finite length. For every set $C \subseteq \mathbb{M}$, there exist tuples*

$$\bar a' \bar b'_0 \dots \bar b'_{n-1} \equiv_U \bar a \bar b_0 \dots \bar b_{n-1}$$

such that $(\bar b'_i)_{i<n}$ is a dividing chain for $\bar a'$ over $U \cup C$ with the same specification as $(\bar b_i)_{i<n}$.

Proof. Let $\langle \varphi_i, k_i \rangle_{i<n}$ be a specification of $(\bar b_i)_{i<n}$. We prove the claim by induction on n. For $n = 0$, there is nothing to do. Hence, suppose that $n > 0$. We can use Lemma 2.7 to find a tuple $\bar b'_0 \equiv_U \bar b_0$ such that $\varphi_0(\bar x; \bar b'_0)$ k_0-divides over $U \cup C$. Let $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ be an automorphism with $\pi(\bar b_0) = \bar b'_0$. Then $(\pi(\bar b_i))_{0<i<n}$ is a dividing chain for $\pi(\bar a)$ over $U \cup \bar b'_0$. Applying the inductive hypothesis to it, we obtain tuples

$$\bar a' \bar b'_1 \dots \bar b'_{n-1} \equiv_{U \bar b'_0} \pi(\bar a)\pi(\bar b_1) \dots \bar\pi(b_{n-1})$$

such that $(\bar b'_i)_{0<i<n}$ is a dividing chain for $\bar a'$ over $U \cup C \cup \bar b'_0$. Since

$$\bar a' \bar b'_0 \bar b'_1 \dots \bar b'_{n-1} \equiv_U \pi(\bar a) \bar b'_0 \pi(\bar b_1) \dots \bar\pi(b_{n-1}) \equiv_U \bar a \bar b_0 \bar b_1 \dots \bar b_{n-1}\,,$$

it follows that $(\bar b'_i)_{i<n}$ is the desired dividing chain for $\bar a'$ over $U \cup C$. □

In order to turn a forking chain into a dividing chain, we iterate the following construction.

Lemma 2.9. *Let $(\bar b_i)_{i<n}$ be a dividing chain for $\bar a$ over $U \cup C$ with a finite length n and with the specification $\langle \varphi_i, k_i\rangle_{i<n}$. If*

$$\mathrm{tp}(\bar a/UC) \vDash \vartheta_0(\bar x;\bar d_0) \vee \dots \vee \vartheta_{m-1}(\bar x;\bar d_{m-1})\,,$$

where each $\vartheta_j(\bar x;\bar d_j)$ l_j-divides over U, then there exist an index $j < m$ and a tuple $\bar d' \equiv_U \bar d_j$ such that $\bar d', \bar b_0, \dots, \bar b_{n-1}$ is a dividing chain for $\bar a$ over U with specification

$$\langle \vartheta_j, l_j\rangle, \langle \varphi_0, k_0\rangle, \dots, \langle \varphi_{n-1}, k_{n-1}\rangle\,.$$

Proof. We prove the claim by induction on n. For $n = 0$, pick an index j such that $\mathbb{M} \vDash \vartheta_j(\bar a;\bar d_j)$. Then $\bar d_j$ is a dividing chain for $\bar a$ over U with specification $\langle \vartheta_j, l_j\rangle$. Hence, suppose that $n > 0$. By Corollary 2.8, there exist tuples

$$\bar a'\bar b'_0 \dots \bar b'_{n-1} \equiv_{UC} \bar a\bar b_0 \dots \bar b_{n-1}$$

such that $(\bar b'_i)_{i<n}$ is a dividing chain for $\bar a'$ over $U \cup C \cup \bar d_0 \dots \bar d_{n-1}$. Since $\bar a' \equiv_{UC} \bar a$, there is some index $j < m$ such that

$$\mathbb{M} \vDash \vartheta_j(\bar a';\bar d_j)\,.$$

It follows that $\bar d_j, \bar b'_0, \dots, \bar b'_{n-1}$ is a dividing chain for $\bar a'$ over U with specification

$$\langle \vartheta_j, l_j\rangle, \langle \varphi_0, k_0\rangle, \dots, \langle \varphi_{n-1}, k_{n-1}\rangle\,.$$

Fix a tuple $\bar d'$ such that

$$\bar a\bar d'\bar b_0 \dots \bar b_{n-1} \equiv_U \bar a'\bar d_j\bar b'_0 \dots \bar b'_{n-1}\,.$$

Then $\bar d', \bar b_0, \dots, \bar b_{n-1}$ is the desired dividing chain. □

Corollary 2.10. *Let $(\bar b_i)_{i<n}$ be a uniform forking chain for $\bar a$ over U with specification $\langle \varphi, \bar\psi, \bar k, m\rangle_{i<n}$. There exists a function $g : [n] \to [m]$ and a dividing chain $(\bar b'_i)_{i<n}$ for $\bar a$ over U with specification*

$$\langle \psi_{g(0)}, k_{g(0)}\rangle, \dots, \langle \psi_{g(n-1)}, k_{g(n-1)}\rangle.$$

Proof. We prove the claim by induction on n. For $n = 0$, there is nothing to do. Hence, suppose that $n > 0$. Applying the inductive hypothesis to the subchain $(\bar b_i)_{0<i<n}$ we obtain a dividing chain $(\bar b'_i)_{0<i<n}$ for $\bar a$ over $U \cup \bar b_0$ with specification

$$\langle \psi_{g(1)}, k_{g(1)}\rangle, \dots, \langle \psi_{g(n-1)}, k_{g(n-1)}\rangle.$$

Since $\mathbb{M} \vDash \varphi(\bar a;\bar b_0)$ and

$$\varphi(\bar x;\bar b_0) \vDash \psi_0(\bar x;\bar d_0) \vee \dots \vee \psi_{m-1}(\bar x;\bar d_{m-1})\,,$$

for suitable $\bar d_0, \dots, \bar d_{m-1}$, we can use Lemma 2.9 to find an index $j < m$ and a tuple $\bar b'_0 \equiv_U \bar d_j$ such that $(\bar b'_i)_{i<n}$ is a dividing chain for $\bar a$ over U with specification

$$\langle \psi_j, k_j\rangle, \langle \psi_{g(1)}, k_{g(1)}\rangle, \dots, \langle \psi_{g(n-1)}, k_{g(n-1)}\rangle.$$ □

Starting from a sufficiently long forking chain, we have constructed arbitrarily long finite dividing chains. According to the next lemma, this is sufficient to obtain dividing chains of every ordinal length.

Lemma 2.11. *Let φ be a formula and $k < \omega$ a number. If, for each $n < \omega$, there exists a uniform dividing chain for $\bar a$ over U of length n with specification $\langle \varphi, k\rangle_{i<n}$, then, for every ordinal γ, we can find a uniform dividing chain for $\bar a$ over U of length γ with specification $\langle \varphi, k\rangle_{\alpha<\gamma}$.*

Proof. Let γ be an ordinal. We define the following set of formulae with variables $\bar x$, $\bar y^\alpha$, $\bar z^\alpha_i$, for $\alpha < \gamma$ and $i < \omega$.

$$\begin{aligned}
\Phi := {}& \bigl\{\, \varphi(\bar x;\bar y^\alpha) \bigm| \alpha < \gamma \,\bigr\} \\
&\cup \bigl\{\, \psi(\bar z^\alpha_i;\bar y^{\beta_0}, \dots, \bar y^{\beta_{n-1}}) \leftrightarrow \psi(\bar y^\alpha;\bar y^{\beta_0}, \dots, \bar y^{\beta_{n-1}}) \bigm| \\
&\qquad \psi \text{ a formula over } U,\ i, n < \omega, \text{ and} \\
&\qquad \beta_0 < \dots < \beta_{n-1} < \alpha < \gamma \,\bigr\} \\
&\cup \bigl\{\, \neg\exists\bar x[\varphi(\bar x;\bar z^\alpha_{i_0}) \wedge \dots \wedge \varphi(\bar x;\bar z^\alpha_{i_{k-1}})] \bigm| \\
&\qquad \alpha < \gamma,\ i_0 < \dots < i_{k-1} < \omega \,\bigr\}\,.
\end{aligned}$$

Note that, if $\bar a$, $\bar b^\alpha$, and $\bar c^\alpha_i$, for $\alpha < \gamma$ and $i < n$, satisfy Φ, then

$$\bar c^\alpha_i \equiv_{U\bar b[<\alpha]} \bar b^\alpha$$

and the set $\{\, \varphi(\bar x; \bar c^\alpha_i) \mid i < \omega \,\}$ is k-inconsistent. Hence, the formula $\varphi(\bar x; \bar b^\alpha)$ k-divides over $U\bar b[<\alpha]$. Consequently, $(\bar b^\alpha)_{\alpha<\gamma}$ is a dividing chain for $\bar a$ over U with specification $\langle \varphi, k\rangle_{\alpha<\gamma}$.

It therefore remains to show that Φ is satisfiable. Let $\Phi_0 \subseteq \Phi$ be finite and let $I \subseteq \gamma$ be the finite set of indices α such that Φ_0 contains some of the variables $\bar y^\alpha$ or $\bar z^\alpha_i$, for $i < \omega$. Choose a uniform dividing chain $(\bar b_i)_{i<n}$ for $\bar a$ over U of length $n := |I|$. We can satisfy Φ_0 by interpreting $\bar x$ by $\bar a$, $\bar y^\alpha$ by the corresponding $\bar b_i$, and $\bar z^\alpha_i$ by tuples witnessing the fact that $\varphi(\bar x; \bar b_i)$ k-divides over $U \cup \bar b[<i]$. By the Compactness Theorem, it follows that Φ is satisfiable. ☐

Combining the results of this section, we have proved that, if $\sqrt[\mathrm{f}]{\ }$ is not right local, then neither is $\sqrt[\mathrm{d}]{\ }$.

Theorem 2.12. *Let T be a complete first-order theory. The following statements are equivalent:*

(1) $\sqrt[\mathrm{d}]{\ }$ *is right local.*

(2) $\sqrt[\mathrm{f}]{\ }$ *is right local.*

(3) *There is no dividing chain of length $|T|^+$.*

Proof. (2) ⇒ (1) If $\sqrt[\mathrm{f}]{\ }$ is right local, then T is simple. Hence, it follows by Lemma 2.2 that $\sqrt[\mathrm{d}]{\ } = \sqrt[\mathrm{f}]{\ }$. In particular, $\sqrt[\mathrm{d}]{\ }$ is right local.

(1) ⇒ (3) If there are arbitrarily long dividing chains, it follows by Theorem F2.3.25 that $\sqrt[\mathrm{d}]{\ }$ is not right local.

(3) ⇒ (2) Suppose that $\sqrt[\mathrm{f}]{\ }$ is not right local and set $\kappa := |T|^+$. By Theorem F2.3.25, there exists a forking chain of length κ for a suitable tuple $\bar a$ over the empty set $\varnothing$. Using Corollary 2.6 we obtain a uniform forking chain of the same length. Let $\langle \varphi, \bar\psi, \bar k, m\rangle_{\alpha<\kappa}$ be its specification. According to Corollary 2.10, there exists, for every $n < \omega$, a dividing chain of length n with specification $\langle \vartheta_i, l_i\rangle_{i<n}$, where $\vartheta_i \in \bar\psi$ and $l_i \in \bar k$, for every $i < n$.

By the Pigeon Hole Principle and Lemma 2.5, we can find a formula $\vartheta \in \bar\psi$ and a number $l \in \bar k$ such that, for every $n < \omega$, there exists a uniform dividing chain of length n with specification $\langle \vartheta, l\rangle_{i<n}$. Consequently, it follows from Lemma 2.11 that there exist arbitrarily long dividing chains. ☐

The tree property

The following combinatorial property characterises simple theories in the same way as the order property characterises stable theories.

Definition 2.13. Let T be a first-order theory. A formula $\varphi(\bar x; \bar y)$ has the *tree property* if there exists a family $(\bar c_\eta)_{\eta\in\omega^{<\omega}}$ of parameters and a number $k < \omega$ such that

- for every $\beta \in \omega^\omega$, the set $\{\, \varphi(\bar x; \bar c_\eta) \mid \eta \prec \beta \,\}$ is consistent and
- for every $\eta \in \omega^{<\omega}$, the set $\{\, \varphi(\bar x; \bar c_{\eta i}) \mid i < \omega \,\}$ is k-inconsistent.

Exercise 2.1. Prove that, in the theory of dense linear orders, the formula $\varphi(x; y_0, y_1) := y_0 < x \wedge x < y_1$ has the tree property.

Before proving that a theory is simple if, and only if, no formula has the tree property, let us note that the tree property implies the order property.

Lemma 2.14. *Every formula with the tree property has the order property.*

Proof. Let $(\bar c_\eta)_{\eta\in\omega^{<\omega}}$ be a family witnessing the tree property of the formula $\varphi(\bar x; \bar y)$. For every $\beta \in \omega^\omega$, we choose a tuple $\bar a_\beta$ satisfying $\{\, \varphi(\bar x; \bar c_\eta) \mid \eta \prec \beta \,\}$. To prove that φ has the order property it is sufficient to find indices $\eta_0 \prec \eta_1 \prec \dots$ in $\omega^{<\omega}$ and a sequence $(\beta_n)_{n<\omega}$ in ω^ω such that $\eta_n \prec \beta_n$ and

$$\mathbb{M} \vDash \varphi(\bar a_{\beta_i}; \bar c_{\eta_k}) \quad \text{iff} \quad i \leq k\,.$$

We proceed by induction on n, starting with $\eta_0 := \langle\rangle$ and an arbitrary $\beta_0 \in \omega^\omega$. For the inductive step, suppose that η_n and β_n are already defined. The k-inconsistency of $\{ \varphi(\bar{x}; \bar{c}_{\eta_n i}) \mid i < \omega \}$ implies that, for each $m \leq n$, there are only finitely many $i < \omega$ such that

$$\mathbb{M} \vDash \varphi(\bar{a}_{\beta_m}; \bar{c}_{\eta_n i}) .$$

Hence, there is some $i < \omega$ such that

$$\mathbb{M} \vDash \neg\varphi(\bar{a}_{\beta_m}; \bar{c}_{\eta_n i}) , \quad \text{for all } m \leq n .$$

We set $\eta_{n+1} := \eta_n i$, for such an index i, and we choose some $\beta_{n+1} \in \omega^\omega$ such that $\eta_{n+1} \prec \beta_{n+1}$. Then $\eta_m \prec \beta_{n+1}$ implies that

$$\mathbb{M} \vDash \varphi(\bar{a}_{\beta_{n+1}}; \bar{c}_{\eta_m}) , \quad \text{for all } m \leq n+1 . \qquad \square$$

To show that simple theories are exactly those where no formula has the tree property, we introduce a generalised form of the tree property.

Definition 2.15. Let κ be a cardinal, γ an ordinal, $(\varphi_\alpha)_{\alpha<\gamma}$ a sequence of formulae, and $(k_\alpha)_{\alpha<\gamma}$ a sequence of numbers.

(a) A family $(\bar{c}_\eta)_{\eta\in\kappa^{\leq\gamma}}$ of tuples $\bar{c}_\eta \subseteq \mathbb{M}$ is a *dividing κ-tree* with *specification* $\langle\varphi_\alpha, k_\alpha\rangle_{\alpha<\gamma}$ if

- for each $\beta \in \kappa^\gamma$, the set $\{ \varphi_\alpha(\bar{x}; \bar{c}_{\beta\restriction(\alpha+1)}) \mid \alpha < \gamma \}$ is consistent,
- for each $\eta \in \kappa^{<\gamma}$, the set $\{ \varphi_{|\eta|}(\bar{x}; \bar{c}_{\eta\alpha}) \mid \alpha < \kappa \}$ is $k_{|\eta|}$-inconsistent.

We call γ the *height* of the dividing κ-tree.

(b) A dividing κ-tree $(\bar{c}_\eta)_{\eta\in\kappa^{\leq\gamma}}$ with specification $\langle\varphi_\alpha, k_\alpha\rangle_{\alpha<\gamma}$ is *uniform* if

$$\varphi_\alpha = \varphi_\beta \quad \text{and} \quad k_\alpha = k_\beta , \quad \text{for all } \alpha, \beta < \gamma .$$

Remark. Note that a formula $\varphi(\bar{x}; \bar{y})$ has the tree property if, and only if, there exists a uniform dividing ω-tree of height ω with specification $\langle\varphi, k\rangle_{n<\omega}$, for some $k < \omega$.

Lemma 2.16. *Let $\kappa > |T|$ be a cardinal. If there exists a dividing ω-tree of height κ, then there also exists an uniform dividing ω-tree of height ω.*

Proof. Let $(\bar{b}_\eta)_{\eta\in\omega^{<\kappa}}$ be a dividing ω-tree of height κ and let $\langle\varphi_\alpha, k_\alpha\rangle_{\alpha<\kappa}$ be its specification. Since $\kappa > |T|$, there exist a subset $I \subseteq \kappa$ of size $|I| = \kappa$, a formula φ_*, and a number $k_* < \omega$ such that

$$\varphi_\alpha = \varphi_* \quad \text{and} \quad k_\alpha = k_* , \quad \text{for all } \alpha \in I .$$

Choose a strictly increasing map $h : \omega \to I$. We inductively define an embedding $g : \omega^{<\omega} \to \omega^{<\kappa}$ as follows. We start with $g(\langle\rangle) := \langle\rangle$. If $g(\eta)$ is already defined, we choose some $\zeta \in \omega^{<\kappa}$ with $g(\eta) \preceq \zeta$ and $|\zeta| = h(|\eta|)$, and we set $g(\eta i) := \zeta i$, for $i < \omega$.

We claim that the family $(\bar{b}_{g(\eta)})_{\eta\in\omega^{<\omega}}$ is a uniform dividing ω-tree of height ω. By construction, the set $\{ \varphi_*(\bar{x}; \bar{b}_{g(\eta n)}) \mid n < \omega \}$ is k_*-inconsistent, for every $\eta \in \omega^{<\omega}$. Furthermore, for each $\beta \in \omega^\omega$, we can choose some $\beta' \in \omega^{<\kappa}$ with

$$\beta' \succeq g(\beta \restriction \alpha) , \quad \text{for all } \alpha < \omega ,$$

and we see that

$$\{ \varphi_*(\bar{x}; \bar{b}_{g(\eta)}) \mid \eta \prec \beta \} \subseteq \{ \varphi_\alpha(\bar{x}; \bar{b}_{\beta'\restriction(\alpha+1)}) \mid \alpha < \gamma \}$$

is consistent. $\square$

The following lemma contains the main technical argument we use to relate the tree property to dividing.

Lemma 2.17. *The following statements are equivalent:*

(1) *There exists a dividing ω-tree of height γ.*

(2) *There exists a dividing chain of length γ.*

Proof. (1) ⇒ (2) Set $\kappa := (2^{|T|\oplus|\gamma|})^+$. If there is a dividing ω-tree, we can use the Compactness Theorem to construct a dividing κ-tree $(\bar{b}_\eta)_{\eta\in\kappa^{\leq\gamma}}$.

Let $\langle \varphi_\alpha, k_\alpha \rangle_{\alpha<\gamma}$ be its specification. We define an embedding $h : \kappa^{\leq\gamma} \to \kappa^{\leq\gamma}$ as follows. We start with $h(\langle\rangle) := \langle\rangle$. If $|\eta|$ is a limit ordinal, we set

$$h(\eta) := \sup \{ h(\zeta) \mid \zeta < \eta \} .$$

For the successor step, we proceed as follows. Suppose that the value of $h(\eta)$ is already defined. Let $\bar{s}$ be the sorts of $\bar{b}_{\eta 0}$. As $|S^{\bar{s}}(\bigcup_{\zeta\leq\eta} \bar{b}_\zeta)| < \kappa$ there exists a subset $I \subseteq \kappa$ of size $|I| = \kappa$ such that

$$\bar{b}_{\eta i} \equiv_{\bigcup_{\zeta\leq\eta} \bar{b}_\zeta} \bar{b}_{\eta k} , \quad \text{for all } i, k \in I .$$

We fix a bijection $g : \kappa \to I$ and we set $h(\eta i) := h(\eta)g(i)$.

Having defined the embedding h, we fix some $\beta \in \kappa^{<\omega}$ and we set $\bar{c}_\alpha := \bar{b}_{h(\beta\restriction(\alpha+1))}$, for $\alpha < \gamma$. We claim that the sequence $(\bar{c}_\alpha)_{\alpha<\gamma}$ is a dividing chain for some $\bar{a}$ over $\varnothing$ with specification $\langle \varphi_\alpha, k_\alpha \rangle_{\alpha<\gamma}$.

Set $\beta' := \sup \{ h(\beta \restriction \alpha) \mid \alpha < \gamma \}$ and choose some tuple $\bar{a}$ satisfying

$$\{ \varphi_\alpha(\bar{x}; \bar{b}_{\beta'\restriction(\alpha+1)}) \mid \alpha < \gamma \} .$$

Then

$$\begin{aligned} \{ \varphi_\alpha(\bar{x}; \bar{c}_\alpha) \mid \alpha < \gamma \} &= \{ \varphi_\alpha(\bar{x}; \bar{b}_{h(\beta\restriction(\alpha+1))}) \mid \alpha < \gamma \} \\ &= \{ \varphi_\alpha(\bar{x}; \bar{b}_{\beta'\restriction(\alpha+1)}) \mid \alpha < \gamma \} , \end{aligned}$$

implies that

$$\mathbb{M} \vDash \varphi_\alpha(\bar{a}; \bar{c}_\alpha) , \quad \text{for all } \alpha < \gamma .$$

It therefore remains to show that $\varphi_\alpha(\bar{x}; \bar{c}_\alpha)$ k_α-divides over $\bar{c}[<\alpha]$. Let $\bar{a}_n := \bar{b}_{h((\beta\restriction\alpha)n)}$, for $n < \omega$. Then $\bar{a}_n \equiv_{\bar{c}[<\alpha]} \bar{b}_{h(\beta\restriction(\alpha+1))} = \bar{c}_\alpha$ and the set $\{ \varphi_\alpha(\bar{x}; \bar{a}_n) \mid n < \omega \}$ is k_α-inconsistent.

(2) ⇒ (1) Given a dividing chain $(\bar{c}_\alpha)_{\alpha<\gamma}$ for $\bar{a}$ over U with specification $\langle \varphi_\alpha, k_\alpha \rangle_{\alpha<\gamma}$, we construct a dividing ω-tree $(\bar{b}_\eta)_{\eta\in\omega^{\leq\gamma}}$ with the additional property that, for every $\eta \in \omega^{\leq\gamma}$,

$$(\bar{b}_{\eta\restriction(\alpha+1)})_{\alpha<|\eta|} \equiv_\varnothing (\bar{c}_\alpha)_{\alpha<|\eta|} .$$

If $\eta = \langle\rangle$ or if $|\eta|$ is a limit ordinal, we can choose an arbitrary tuple $\bar{b}_\eta$, since the definition of a dividing ω-tree places no constraint on such tuples. Hence, it remains to consider the successor step. Suppose that $\bar{b}_\eta$ has already been defined and set $\alpha := |\eta|$. Since

$$(\bar{b}_{\eta\restriction(i+1)})_{i<\alpha} \equiv_\varnothing (\bar{c}_i)_{i<\alpha} .$$

there exists some $\bar{b}'$ such that

$$(\bar{b}_{\eta\restriction(i+1)})_{i<\alpha}\bar{b}' \equiv_\varnothing (\bar{c}_i)_{i<\alpha}\bar{c}_\alpha .$$

Since $\varphi_\alpha(\bar{x}; \bar{c}_\alpha)$ k_α-divides over $U \cup \bar{c}[<\alpha]$, we can find a sequence $(\bar{c}'_n)_{n<\omega}$ such that $\bar{c}'_n \equiv_{U\bar{c}[<\alpha]} \bar{c}_\alpha$ and $\{ \varphi_\alpha(\bar{x}; \bar{c}'_n) \mid n < \omega \}$ is k_α-inconsistent. By choice of $\bar{b}'$, we can therefore find a sequence $(\bar{b}'_n)_{n<\omega}$ such that

$$\bar{b}'_n \equiv_{\bigcup_{i<\alpha} \bar{b}_{\eta\restriction(i+1)}} \bar{b}'$$

and $\{ \varphi_\alpha(\bar{x}; \bar{b}'_n) \mid n < \omega \}$ is k_α-inconsistent. We set $\bar{b}_{\eta i} := \bar{b}'_i$, for $i < \omega$.

To see that the family $(\bar{b}_\eta)_{\eta\in\omega^{\leq\gamma}}$ constructed in this way is a dividing ω-tree, note that, for each $\beta \in \omega^\gamma$, $(\bar{b}_{\eta\restriction(\alpha+1)})_{\alpha<\gamma} \equiv_\varnothing (\bar{c}_\alpha)_{\alpha<\gamma}$ implies that the set $\{ \varphi_\alpha(\bar{x}; \bar{b}_{\beta\restriction(\alpha+1)}) \mid \alpha < \gamma \}$ is consistent. □

Using these two lemmas, we obtain the following characterisation of simple theories.

Theorem 2.18. *Let T be a complete first-order theory. The following statements are equivalent:*

(1) *T is simple.*

(2) *$\overset{\mathrm{d}}{\surd}$ is right local.*

(3) *No formula has the tree property.*

(4) *There is no dividing chain of length $|T|^+$.*

(5) *For some cardinal κ, there is no dividing chain of length κ.*

Proof. (4) ⇒ (5) is trivial and (1) ⇔ (2) ⇔ (4) was already shown in Theorem 2.12.

(5) ⇒ (3) Suppose that there exists a formula $\varphi(\bar{x};\bar{y})$ with the tree property. Fix a family $(\bar{c}_\eta)_{\eta\in\omega^{<\omega}}$ and a number $k<\omega$ witnessing this fact.

For every cardinal κ, we will construct a dividing chain of length κ. Given κ, we use compactness to find a family $(\bar{b}_\eta)_{\eta\in\omega^{<\kappa}}$ such that

- for every $\beta\in\omega^\kappa$, the set $\{\,\varphi(\bar{x};\bar{b}_\eta) \mid \eta\prec\beta\,\}$ is consistent and
- for every $\eta\in\omega^{<\kappa}$, the set $\{\,\varphi(\bar{x};\bar{b}_{\eta i}) \mid i<\omega\,\}$ is k-inconsistent.

In particular, $(\bar{b}_\eta)_{\eta\in\omega^{<\kappa}}$ is a uniform dividing ω-tree of height κ. Hence, we can use Lemma 2.17 to obtain a dividing chain of length κ. A contradiction.

(3) ⇒ (4) Suppose that there exists a dividing chain of length $\kappa:=|T|^+$. We will show that some formula has the tree property. By Lemma 2.17, there exists a dividing ω-tree $(\bar{b}_\eta)_{\eta\in\omega^{<\kappa}}$ of height κ. Hence, we can use Lemma 2.16 to obtain a uniform dividing ω-tree $(\bar{b}'_\eta)_{\eta\in\omega^{<\omega}}$ of height ω. Let $\langle\varphi,k\rangle_{n<\omega}$ be its specification. Then the formula φ has the tree property. A contradiction. ☐

Corollary 2.19. *Every stable theory is simple.*

Proof. This follows by Theorem 2.18 and Lemma 2.14. ☐

Corollary 2.20. *A theory T is simple if, and only if, T^{eq} is simple.*

Proof. Clearly, if φ has the tree property with respect to T, it also has the tree property with respect to T^{eq}. Conversely, if φ has the tree property with respect to T^{eq} we can use Proposition E2.2.10 to construct a formula φ' that has the tree property with respect to T. ☐

Finally, we show that no simple theory has the strict order property. Consequently, all simple theories that are not stable have the independence property.

Proposition 2.21. *No simple theory has the strict order property.*

Proof. Suppose that the formula $\varphi(\bar{x};\bar{y})$ has the strict order property. We will show that the formula

$$\psi(\bar{x};\bar{y}_0\bar{y}_1) := \neg\varphi(\bar{x};\bar{y}_0)\wedge\varphi(\bar{x};\bar{y}_1)$$

has the tree property. By compactness, there exists a sequence $(\bar{c}_i)_{i\in\mathbb{Q}}$ such that

$$\varphi(\bar{x};\bar{c}_i)^{\mathbb{M}}\subset\varphi(\bar{x};\bar{c}_k)^{\mathbb{M}}, \quad \text{for all } i<k\,.$$

We define two functions $\lambda,\rho:\omega^{<\omega}\to\mathbb{Q}$ such that $\lambda(\eta)<\rho(\eta)$, for all η. We proceed by induction on $\eta\in\omega^{<\omega}$ starting with $\lambda(\langle\rangle):=0$ and $\rho(\langle\rangle):=1$. If $\lambda(\eta)<\rho(\eta)$ are already defined, we choose a strictly increasing sequence $\lambda(\eta)<z_0<z_1<\dots<\rho(\eta)$ and we set $\lambda(\eta i):=z_i$ and $\rho(\eta i):=z_{i+1}$, for $i<\omega$.

Having defined λ and ρ, we set $\bar{b}_\eta:=\bar{c}_{\lambda(\eta)}\bar{c}_{\rho(\eta)}$, for $\eta\in\omega^{<\omega}$. To show that this family witnesses the tree property of ψ, note that

$$\psi(\bar{x};\bar{b}_\eta)^{\mathbb{M}}=\varphi(\bar{x};\bar{c}_{\rho(\eta)})^{\mathbb{M}}\setminus\varphi(\bar{x};\bar{c}_{\lambda(\eta)})^{\mathbb{M}}.$$

Hence,

$$\begin{aligned} &\psi(\bar{x};\bar{b}_\eta)^{\mathbb{M}}\subseteq\psi(\bar{x};\bar{b}_\zeta)^{\mathbb{M}}, && \text{for } \eta\leq\zeta\,,\\ \text{and}\quad &\psi(\bar{x};\bar{b}_\eta)^{\mathbb{M}}\cap\psi(\bar{x};\bar{b}_\zeta)^{\mathbb{M}}=\varnothing, && \text{for incomparable } \eta \text{ and } \zeta\,. \end{aligned}$$

Consequently, the set $\{\,\psi(\bar{x};\bar{b}_{\eta i}) \mid i<\omega\,\}$ is 2-inconsistent, for every η. Furthermore, for every $\beta\in\omega^\omega$, we can use compactness and the fact that $\psi(\bar{x};\bar{b}_\eta)^{\mathbb{M}}\neq\varnothing$, for all η, to show that $\{\,\psi(\bar{x};\bar{b}_\eta) \mid \eta\prec\beta\,\}$ is satisfiable. ☐

Strongly minimal theories

We conclude this section by considering the example of strongly minimal theories. Note that such theories are stable and, hence, simple. We will show that, for strongly minimal theories, the relations $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ coincide. One of the inclusions holds in general.

Lemma 2.22. *If $\surd$ is a forking relation, then $\surd \subseteq {}^{\mathrm{cl}_\surd}\surd$.*

Proof. Suppose that $A \surd_U B$. To show that $A \; {}^{\mathrm{cl}_\surd}\surd_U \; B$, consider a set $I \subseteq B$ that is not $\mathrm{cl}_\surd$-independent over $U \cup A$. We have to show that I is not $\mathrm{cl}_\surd$-independent over U. There exists an element $b \in I$ such that $b \in \mathrm{cl}_\surd(UAI_0)$ where $I_0 := I \setminus \{b\}$. Consequently, $b \surd_{UAI_0} B$. By (BMON), $A \surd_U B$ implies $A \surd_{UI_0} B$. Hence, it follows by Lemma F2.2.3 that $Ab \surd_{UI_0} B$. In particular, we have $b \surd_{UI_0} b$ which, by Lemma F2.3.5, implies that $b \in \mathrm{cl}_\surd(UI_0)$. Therefore, I is not $\mathrm{cl}_\surd$-independent over U. ☐

The converse is given by the following lemma.

Lemma 2.23. *Let T be a simple theory and $\mathbb{S}$ a U-definable strongly minimal set. Then*

$$A \; {}^{\mathrm{acl}}\surd_U \; B \quad \textit{implies} \quad A \mathop{\downarrow}^{\mathrm{f}}_U B\,, \quad \textit{for all } A, B, U \subseteq \mathbb{S}\,.$$

Proof. Recall that we have shown in Lemma F1.4.3 that $\langle \mathbb{S}, \mathrm{acl} \rangle$ forms a matroid. By (DEF), it is sufficient to prove the claim for finite sets A and B. Hence, suppose that A and B are finite sets with $A \; {}^{\mathrm{acl}}\surd_U \; B$. We choose bases $I \subseteq A$ and $J \subseteq B$ of, respectively, A over U and B over U, and enumerations $\bar a$ of I and $\bar b$ of J. Then $\bar a \; {}^{\mathrm{acl}}\surd_U \; \bar b$. Since $\bar b$ is independent over U, it follows that it is also independent over $U \cup \bar a$. Hence, $\bar a \bar b$ is independent over U.

To show that $\bar a \mathop{\downarrow}^{\mathrm{f}}_U \bar b$, let $(\bar b_n)_{n<\omega}$ be an indiscernible sequence over U with $\bar b_0 = \bar b$. Note that the union $\bar b[<\omega]$ is independent over U. We choose a tuple $\bar a' \subseteq \mathbb{S}$ such that $|\bar a'| = |\bar a|$ and $\bar a'$ is independent over $U \cup \bar b[<\omega]$. According to Proposition F1.4.6, we have $\bar a' \equiv_{U\bar b} \bar a$. Since $\bar b[<\omega]$ is independent over $U \cup \bar a'$, it follows by the same proposition that the sequence $(\bar b_n)_{n<\omega}$ is indiscernible over $U \cup \bar a'$. By Lemma 1.3, it follows that $\bar a \; {}^{\mathrm{d}}\surd_U \; \bar b$. Since T is simple, this implies that $\bar a \mathop{\downarrow}^{\mathrm{f}}_U \bar b$. Hence, we can use Lemma F2.2.14 to show that $\mathrm{acl}(\bar a U) \mathop{\downarrow}^{\mathrm{f}}_U \mathrm{acl}(\bar b U)$. By monotonicity, it follows that $A \mathop{\downarrow}^{\mathrm{f}}_U B$. ☐

Corollary 2.24. *For a strongly minimal theory T, we have ${}^{\mathrm{acl}}\surd = \mathop{\downarrow}^{\mathrm{f}} = \mathop{\downarrow}^{\mathrm{d}}$. In particular, T is simple and $\mathop{\downarrow}^{\mathrm{f}}$ is a geometric independence relation.*

Proof. First, note that, according to Lemma F1.4.3, $\langle \mathbb{M}, \mathrm{acl} \rangle$ is a matroid. Hence, it follows from Proposition F2.2.8 that ${}^{\mathrm{acl}}\surd$ is a geometric independence relation. We have seen in Corollary F1.4.14 that a strongly minimal theory T is κ-categorical, for every $\kappa > |T|$. Consequently, it follows by Theorem E6.3.16 that T is stable. Using Corollary 2.19, we see that T is simple. Therefore, the equality ${}^{\mathrm{acl}}\surd = \mathop{\downarrow}^{\mathrm{f}} = \mathop{\downarrow}^{\mathrm{d}}$ follows the two preceding lemmas. ☐

Exercise 2.2. Prove that, in an arbitrary theory, ${}^{\mathrm{acl}}\surd$ satisfies (INV) and (DEF).

F4. Theories without the independence property

1. Honest definitions

Alternation numbers

We have seen in Proposition E5.4.2 that the independence property can be characterised by counting the number of segments of sets of the form $[\![\varphi(\bar{a}_i)]\!]_{i\in I}$ for an indiscernible sequence $(\bar{a}_i)_{i\in I}$. In this section we will use this characterisation to derive various properties of theories without the independence property. We start by setting up the required combinatorial machinery.

Definition 1.1. Let $\varphi(\bar{x})$ be a formula over $\mathbb{M}$.

(a) The *φ-alternation number* $\mathrm{alt}_\varphi(\alpha)$ of a sequence $\alpha = (\bar{a}_i)_{i\in I}$ is the maximal number $n < \omega$ such that there are indices $\bar{k} \in [I]^{n+1}$ with

$$\mathbb{M} \vDash \varphi(\bar{a}_{k_i}) \leftrightarrow \neg\varphi(\bar{a}_{k_{i+1}}) \,, \quad \text{for all } i < n \,.$$

If this maximum does not exist, we set $\mathrm{alt}_\varphi(\alpha) := \infty$.

(b) The *alternation rank* of φ is

$$\mathrm{rk}_{\mathrm{alt}}(\varphi) := \max\big\{ \mathrm{alt}_\varphi(\alpha) \bigm| \alpha \text{ an indiscernible sequence in } \mathbb{M} \big\} \,.$$

If this maximum does not exist, we set $\mathrm{rk}_{\mathrm{alt}}(\varphi) := \infty$.

(c) A sequence $\alpha = (\bar{a}_i)_{i\in I}$ is *maximally φ-alternating over U* if it is indiscernible over U and

$$\mathrm{alt}_\varphi(\alpha) = \mathrm{alt}_\varphi(\alpha\beta) < \infty \,,$$

for every extension $\alpha\beta$ of α that is still indiscernible over U.

Using these notions, we can characterise the independence property as follows.

Proposition 1.2. *Let $\varphi(\bar{x};\bar{y})$ be a formula without parameters and let $U \subseteq \mathbb{M}$. The following statements are equivalent.*

(1) *$\varphi(\bar{x};\bar{y})$ does not have the independence property.*

(2) $\mathrm{rk}_{\mathrm{alt}}(\varphi(\bar{x};\bar{c})) < \infty$, *for all $\bar{c} \subseteq \mathbb{M}$.*

(3) *There exists some number $n < \omega$ such that*

$$\mathrm{rk}_{\mathrm{alt}}(\varphi(\bar{x};\bar{c})) \leq n\,, \quad \textit{for all } \bar{c} \subseteq \mathbb{M}\,.$$

(4) $\mathrm{alt}_{\varphi(\bar{x};\bar{c})}(\alpha) < \infty$, *for every indiscernible sequence α over U and every tuple $\bar{c} \subseteq \mathbb{M}$.*

(5) *Let $\bar{c} \subseteq \mathbb{M}$. Every indiscernible sequence α over U has an extension $\alpha\beta$ that is maximally $\varphi(\bar{x};\bar{c})$-alternating over U.*

Proof. (3) ⇒ (2) is trivial.

(2) ⇒ (5) Suppose that $\mathrm{rk}_{\mathrm{alt}}(\varphi(\bar{x};\bar{c})) < \infty$ and let α be an indiscernible sequence over U. We construct a maximally $\varphi(\bar{x};\bar{c})$-alternating extension of α by induction on the difference

$$\mathrm{rk}_{\mathrm{alt}}(\varphi(\bar{x};\bar{c})) - \mathrm{alt}_{\varphi(\bar{x};\bar{c})}(\alpha)\,.$$

If α is already maximally $\varphi(\bar{x};\bar{c})$-alternating, there is nothing to do. Hence, suppose otherwise. Then we can find some extension $\alpha\beta$ that is indiscernible over U such that $\mathrm{alt}_{\varphi(\bar{x};\bar{c})}(\alpha\beta) > \mathrm{alt}_{\varphi(\bar{x};\bar{c})}(\alpha)$. By inductive hypothesis, this sequence has an extension $\alpha\beta\gamma$ that is maximally $\varphi(\bar{x};\bar{c})$-alternating over U.

(5) ⇒ (4) Let $\alpha\beta$ be a maximally $\varphi(\bar{x};\bar{c})$-alternating extension of α over U. Then $\mathrm{alt}_{\varphi(\bar{x};\bar{c})}(\alpha) \leq \mathrm{alt}_{\varphi(\bar{x};\bar{c})}(\alpha\beta) < \infty$.

(4) ⇒ (1) Suppose that $\varphi(\bar{x};\bar{y})$ has the independence property. By Proposition E5.4.2, there exists an indiscernible sequence $\alpha = (\bar{a}_n)_{n<\omega}$ and a tuple $\bar{c}$ such that

$$\mathbb{M} \vDash \varphi(\bar{a}_n;\bar{c}) \quad \text{iff} \quad n \text{ is even.}$$

Hence, $\mathrm{alt}_{\varphi(\bar{x};\bar{c})}(\alpha) = \infty$.

(1) ⇒ (3) Suppose that, for every number $n < \omega$, there exists some tuple $\bar{c} \subseteq \mathbb{M}$ such that $\mathrm{rk}_{\mathrm{alt}}(\varphi(\bar{x};\bar{c})) > n$. We claim that φ has the independence property. Let Ψ be a set of formulae stating that the sequence $(\bar{x}_i)_{i<\omega}$ is indiscernible and set

$$\Phi := \Psi \cup \{\, \varphi(\bar{x}_{2i};\bar{y}) \mid i < \omega \,\} \cup \{\, \neg\varphi(\bar{x}_{2i+1};\bar{y}) \mid i < \omega \,\}\,.$$

We will show that Φ is satisfiable. Then there exists an indiscernible sequence $(\bar{a}_i)_{i<\omega}$ and a tuple $\bar{b}$ such that

$$\mathbb{M} \vDash \varphi(\bar{a}_i;\bar{b}) \quad \text{iff} \quad i \text{ is even,}$$

and it follows by Proposition E5.4.2 that φ has the independence property.

Thus, let $\Phi_0 \subseteq \Phi$ be finite. Then there exists a number $n < \omega$ such that all variables occurring in Φ_0 are among $\bar{x}_0, \dots, \bar{x}_{2n-1}$. By assumption, we can find a tuple $\bar{c}$ and an indiscernible sequence $\alpha = (\bar{a}_i)_{i\in I}$ such that

$$\mathrm{alt}_{\varphi(\bar{x};\bar{c})}(\alpha) \geq 2n\,.$$

We choose indices $\bar{m} \in [I]^{2n+1}$ such that

$$\mathbb{M} \vDash \varphi(\bar{a}_{m_i};\bar{c}) \leftrightarrow \neg\varphi(\bar{a}_{m_{i+1}};\bar{c})\,, \quad \text{for all } i < 2n\,.$$

Depending on whether or not $\mathbb{M} \vDash \varphi(\bar{a}_{m_0};\bar{c})$, it follows that either the sequence $(\bar{a}_{m_i})_{0\leq i<2n}$ or the sequence $(\bar{a}_{m_i})_{1\leq i<2n+1}$ satisfies Φ_0 together with the tuple $\bar{c}$. □

Below we will frequently make use of the following consequence of this characterisation.

Corollary 1.3. *Let T be a theory without the independence property and let Δ be a finite set of formulae over $\mathbb{M}$. Every indiscernible sequence α over U has an extension $\alpha\beta$ that is maximally φ-alternating over U, for all $\varphi \in \Delta$.*

Proof. Let α be indiscernible over U. We construct the desired extension by induction on $|\Delta|$. If $\Delta = \varnothing$, we can take the sequence α itself. Hence, we may assume that there is some formula $\varphi \in \Delta$. Suppose that $\varphi(\bar{x}) = \varphi_0(\bar{x};\bar{c})$ where $\bar{c} \subseteq \mathbb{M}$ and $\varphi_0(\bar{x};\bar{y})$ is a formula without parameters. As $\varphi_0(\bar{x};\bar{y})$ does not have the independence property, it follows by Proposition 1.2 that α has a maximally φ-alternating extension $\alpha\beta$. By inductive hypothesis, this sequence has an extension $\alpha\beta\gamma$ that is maximally ψ-alternating, for every $\psi \in \Delta \setminus \{\varphi\}$. Since $\mathrm{alt}_\varphi(\alpha\beta) \leq \mathrm{alt}_\varphi(\alpha\beta\gamma)$, this extension is also maximally φ-alternating. Hence, $\alpha\beta\gamma$ is the desired extension of α. □

Honest definitions

Stable theories have the property that every set $\mathbb{A} \subseteq \mathbb{M}$ is self-contained as far as definable relations are concerned, that is, all parameter-definable relations $\mathbb{R} \subseteq \mathbb{A}^{\bar{s}}$ are definable with parameters from $\mathbb{A}$ itself. In this section, we will prove that theories without the independence property have a similar, but weaker property: the parameters are not necessarily in the set $\mathbb{A}$, but in some elementary extension. We start by taking a look at the stable case.

Definition 1.4. A set $\mathbb{A} \subseteq \mathbb{M}$ is *stably embedded* if, for every parameter-definable relation $\mathbb{R} \subseteq \mathbb{M}^{\bar{s}}$, there is a formula $\varphi(\bar{x})$ over $\mathbb{A}$ such that

$$\mathbb{R} \cap \mathbb{A}^{\bar{s}} = \varphi^{\mathbb{M}} \cap \mathbb{A}^{\bar{s}} .$$

Proposition 1.5. *In a stable theory, every set $\mathbb{A} \subseteq \mathbb{M}$ is stably embedded.*

Proof. Let $\psi(\bar{x};\bar{c})$ be a formula with parameters $\bar{c} \subseteq \mathbb{M}$. As T is stable, it follows by Theorem C3.5.17 that the type $\mathrm{tp}(\bar{c}/\mathbb{A})$ is definable over $\mathbb{A}$. Consequently, there exists a formula $\delta_\psi(\bar{y})$ over $\mathbb{A}$ such that

$$\mathbb{M} \vDash \delta_\psi(\bar{a}) \quad \text{iff} \quad \mathbb{M} \vDash \psi(\bar{a};\bar{c}) .$$

This implies that $\psi(\bar{x};\bar{c})^{\mathbb{M}} \cap \mathbb{A}^{\bar{s}} = \delta_\psi(\bar{x})^{\mathbb{M}} \cap \mathbb{A}^{\bar{s}}$. □

For theories with the independence property, we need to consider elementary extensions of the given structure to find the desired parameters. Alternatively, we can also use the following finitary version of stable embeddedness.

Definition 1.6. An *honest definition* of a relation $\mathbb{R} \subseteq \mathbb{M}^{\bar{s}}$ over a set U is a formula $\varphi(\bar{x};\bar{y})$ without parameters such that, for every finite $U_0 \subseteq U$, there is some tuple $\bar{c} \subseteq U$ with

$$\mathbb{R} \cap U_0^{\bar{s}} \subseteq \varphi(\bar{x};\bar{c})^{\mathbb{M}} \cap U^{\bar{s}} \subseteq \mathbb{R} \cap U^{\bar{s}} .$$

Example. The set $\mathbb{Q}$ of rationals is not stably embedded in $\langle \mathbb{R}, \leq \rangle$. For instance, for the parameter-definable relation $(0,\sqrt{2})$, there is no formula $\varphi(x)$ over $\mathbb{Q}$ with

$$\varphi^{\mathbb{R}} \cap \mathbb{Q} = (0,\sqrt{2}) \cap \mathbb{Q} .$$

But $(0,\sqrt{2})$ *does* have an honest definition over $\mathbb{Q}$. For every finite subset $A \subseteq (0,\sqrt{2})$, we have

$$(0,\sqrt{2}) \cap A \subseteq \varphi(x;a,b)^{\mathbb{R}} \cap \mathbb{Q} \subseteq (0,\sqrt{2}) \cap \mathbb{Q} ,$$

where $\varphi(x;y,z) := y \leq x \wedge x \leq z$ and a and b are, respectively, the minimal and the maximal element of A.

Below we will prove that these two weaker version of stable embeddedness are equivalent and that they hold in theories without the independence property. The key argument is contained in the following lemma.

Lemma 1.7. *Let $\kappa > |T|$ be a cardinal and let $\langle \mathfrak{M}, C \rangle \preceq \langle \mathfrak{M}_+, C_+ \rangle$ be structures where the former one has size $|M| < \kappa$ and the latter one is κ-saturated. For all sets $A, B \subseteq M_+$ of size $|A|, |B| < \kappa$ with $A \sqrt[u]{}_C B$, there exists some $A' \subseteq C_+$ such that $A' \equiv_B A$.*

Proof. Let $\bar{a} = (a_i)_{i<\lambda}$ be an enumeration of A and let $\mathbb{C} \subseteq \mathbb{M}$ be a set such that $\langle \mathbb{M}, \mathbb{C} \rangle \succeq \langle \mathfrak{M}_+, C_+ \rangle$. Set

$$\Phi(\bar{x}) := \mathrm{Th}(\langle \mathbb{M}, \mathbb{C} \rangle) \cup \mathrm{tp}(\bar{a}/B) \cup \{ Px_i \mid i < \lambda \} ,$$

where the type $\mathrm{tp}(\bar{a}/B)$ is taken with respect to the structure $\mathbb{M}$ and P is the predicate symbol of $\langle \mathbb{M}, \mathbb{C} \rangle$ corresponding to the set $\mathbb{C}$. If $\Phi(\bar{x})$ is satisfiable, it follows by κ-saturation of $\langle \mathfrak{M}_+, C_+ \rangle$ that there is some tuple $\bar{a}' \subseteq M_+$ with $\langle \mathfrak{M}_+, C_+ \rangle \vDash \Phi(\bar{a}')$. By definition of Φ, this implies that $\bar{a}' \subseteq C_+$ and $\bar{a}' \equiv_B \bar{a}$. Hence, it remains to prove that Φ is satisfiable.

Let $\Phi_0 \subseteq \Phi$ be finite. Then

$$\Phi_0(\bar{x}) \equiv \psi \wedge \varphi(\bar{x}) \wedge \bigwedge_{i \in I} P x_i \,,$$

for suitable formulae $\psi \in \mathrm{Th}(\langle \mathbb{M}, \mathbb{C} \rangle)$, $\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/B)$, and some finite set $I \subseteq \lambda$. Since $\bar{a} \overset{\mathrm{u}}{\sqrt{}}_C B$, we can find some tuple $\bar{a}' \subseteq C \subseteq C_+$ with $\mathbb{M} \vDash \varphi(\bar{a}')$. Consequently,

$$\langle \mathfrak{M}_+, C_+ \rangle \vDash \psi \wedge \varphi(\bar{a}') \wedge \bigwedge_{i \in I} P a_i' \,,$$

and $\bar{a}'$ satisfies $\Phi_0(\bar{x})$. □

A second technical ingredient we need in the proof below is the notion of a type *generating* a sequence.

Definition 1.8. Let $\mathfrak{p}$ be a type. A sequence $(\bar{a}_i)_{i \in I}$ is *generated by* $\mathfrak{p}$ *over* U if $\bar{a}_i$ realises $\mathfrak{p} \upharpoonright U\bar{a}[<i]$, for all $i \in I$.

Exercise 1.1. Prove that, for every type $\mathfrak{p} \in S^{\bar{s}}(\mathbb{M})$ and every small index set I, there is some sequence $(\bar{a}_i)_{i \in I}$ generated by $\mathfrak{p}$.

When using a suitable type, the generated sequence is automatically a Morley sequence.

Lemma 1.9. *Let $\sqrt{}$ be a preforking relation and $\mathfrak{p}$ a global type that is $\sqrt{}$-free over U. Every sequence generated by $\mathfrak{p}$ over a set $U \cup C$ is a $\sqrt{}$-Morley sequence for $\mathfrak{p} \upharpoonright UC$ over U.*

The existence of honest definitions turns out to being equivalent to not having the independence property.

Theorem 1.10. *Let $\varphi(\bar{x})$ be a formula over $\mathbb{M}$ and let $\bar{s}$ be the sorts of $\bar{x}$. The following statements are equivalent:*

(1) $\mathrm{rk}_{\mathrm{alt}}(\varphi) < \infty$.

(2) *For every set $C \subseteq \mathbb{M}$, there is a honest definition of $\varphi^{\mathbb{M}}$ over C.*

(3) *For every model $\mathfrak{M}$ containing the parameters of φ, every set $C \subseteq M$ of parameters, and every $(|T| \oplus |M|)^+$-saturated elementary extension $\langle \mathfrak{M}_+, C_+ \rangle \succeq \langle \mathfrak{M}, C \rangle$, there exists a formula $\varphi_+(\bar{x})$ over C_+ such that*

$$\varphi(\bar{x})^{\mathbb{M}} \cap C^{\bar{s}} \subseteq \varphi_+(\bar{x})^{\mathbb{M}} \cap C_+^{\bar{s}} \subseteq \varphi(\bar{x})^{\mathbb{M}} \cap C_+^{\bar{s}}.$$

Proof. (3) ⇒ (2) Fix a model $\mathfrak{M}$ containing the parameters of φ, a set $C \subseteq M$, and a $(|T| \oplus |M|)^+$-saturated elementary extension $\langle \mathfrak{M}_+, C_+ \rangle \succeq \langle \mathfrak{M}, C \rangle$. By (3), there is some formula $\varphi_+(\bar{x}; \bar{c})$ with parameters $\bar{c} \subseteq C_+$ such that

$$\varphi(\bar{x})^{\mathbb{M}} \cap C^{\bar{s}} \subseteq \varphi_+(\bar{x}; \bar{c})^{\mathbb{M}} \cap C_+^{\bar{s}} \subseteq \varphi(\bar{x})^{\mathbb{M}} \cap C_+^{\bar{s}}.$$

We claim that $\varphi_+(\bar{x}; \bar{y})$ is a honest definition of $\varphi^{\mathbb{M}}$ over C. Let $C_0 \subseteq C$ be finite. Then

$$\begin{aligned} \langle \mathfrak{M}_+, C_+ \rangle \vDash & \bigwedge_{\bar{a} \in C_0^{\bar{s}}} [\varphi_+(\bar{a}; \bar{c}) \leftrightarrow \varphi(\bar{a})] \\ & \wedge (\forall \bar{x}. \textstyle\bigwedge_i P x_i)[\varphi_+(\bar{x}; \bar{c}) \to \varphi(\bar{x})] \,. \end{aligned}$$

Consequently,

$$\begin{aligned} \langle \mathfrak{M}, C \rangle \vDash (\exists \bar{y}. \textstyle\bigwedge_i P y_i) \Big[& \bigwedge_{\bar{a} \in C_0^{\bar{s}}} [\varphi_+(\bar{a}; \bar{y}) \leftrightarrow \varphi(\bar{a})] \\ & \wedge (\forall \bar{x}. \textstyle\bigwedge_i P x_i)[\varphi_+(\bar{x}; \bar{y}) \to \varphi(\bar{x})] \Big] \,, \end{aligned}$$

and there is some tuple $\bar{c}' \subseteq C$ such that

$$\varphi^{\mathbb{M}} \cap C_0^{\bar{s}} \subseteq \varphi_+(\bar{x}; \bar{c}')^{\mathbb{M}} \cap C^{\bar{s}} \subseteq \varphi^{\mathbb{M}} \cap C^{\bar{s}}.$$

(2) ⇒ (1) For a contradiction, suppose that $\mathrm{rk}_{\mathrm{alt}}(\varphi(\bar{x})) = \infty$ but $\varphi^{\mathbb{M}}$ has honest definitions over all sets $C \subseteq \mathbb{M}$. By compactness there

exists an indiscernible sequence $\alpha = (\bar{a}_n)_{n<\omega}$ such that $\mathrm{alt}_\varphi(\alpha) = \infty$. Omitting some elements of α we may assume that

$$\mathbb{M} \vDash \varphi(\bar{a}_n) \quad \text{iff} \quad n \text{ is even.}$$

Let $\psi(\bar{x};\bar{y})$ be an honest definition of $\varphi^{\mathbb{M}}$ over the set $C := \bar{a}[<\omega]$ and let $C_0 := \bar{a}[<2k+2]$ where $k := |\bar{y}|$. By assumption, there is some tuple $\bar{c} \subseteq C$ such that

$$\varphi^{\mathbb{M}} \cap C_0^{\bar{s}} \subseteq \psi(\bar{x};\bar{c})^{\mathbb{M}} \cap C^{\bar{s}} \subseteq \varphi^{\mathbb{M}} \cap C^{\bar{s}}.$$

Fix some tuple $\bar{j} \in [\omega]^k$ such that $\bar{c} \subseteq \bar{a}[\bar{j}]$. Then there is some index $i < 2k+1$ such that

$$\mathrm{ord}(i\bar{j}) = \mathrm{ord}((i+1)\bar{j}) .$$

Consequently,

$$\mathbb{M} \vDash \psi(\bar{a}_i;\bar{c}) \leftrightarrow \psi(\bar{a}_{i+1};\bar{c}) .$$

If i is even, then

$$\psi(\bar{x};\bar{c})^{\mathbb{M}} \cap C^{\bar{s}} \subseteq \varphi^{\mathbb{M}} \cap C^{\bar{s}} \quad \text{implies} \quad \bar{a}_{i+1} \notin \psi(\bar{x};\bar{c})^{\mathbb{M}},$$

while $$\varphi^{\mathbb{M}} \cap C_0^{\bar{s}} \subseteq \psi(\bar{x};\bar{c})^{\mathbb{M}} \cap C^{\bar{s}} \quad \text{implies} \quad \bar{a}_i \in \psi(\bar{x};\bar{c})^{\mathbb{M}}.$$

A contradiction. In the case where i is odd, we can show in the same way that $\bar{a}_i \notin \psi(\bar{x};\bar{c})^{\mathbb{M}}$ and $\bar{a}_{i+1} \in \psi(\bar{x};\bar{c})^{\mathbb{M}}$.

(1) ⇒ (3) Let $F \subseteq S^{\bar{s}}(M_+)$ be the set of all types over M_+ that are finitely satisfiable in C and let $F_\varphi := F \cap \langle\varphi\rangle$ be the subset of those types containing φ. As $\mathrm{rk}_{\mathrm{alt}}(\varphi) < \infty$, we can choose, for every type $\mathfrak{p} \in F$, a sequence $\alpha_\mathfrak{p} \subseteq C_+$ that is generated by $\mathfrak{p}$ over C and such that $\mathrm{alt}_{\varphi(\bar{x})}(\alpha_\mathfrak{p})$ is maximal (among all such sequences in C_+).

Let $\bar{a}' \subseteq C_+$ be a tuple realising $\mathfrak{p} \upharpoonright C\alpha_\mathfrak{p}$, for some $\mathfrak{p} \in F$. We claim that

$$\mathbb{M} \vDash \varphi(\bar{a}') \quad \text{iff} \quad \varphi(\bar{x}) \in \mathfrak{p} .$$

By Lemma 1.7, there is some $\bar{a}'' \in C_+^{\bar{s}}$ realising $\mathfrak{p} \upharpoonright M\alpha_\mathfrak{p}\bar{a}'$. Then the sequence $\alpha_\mathfrak{p}\bar{a}'\bar{a}''$ is generated by $\mathfrak{p}$ over C and our choice of $\alpha_\mathfrak{p}$ implies that

$$\mathrm{alt}_\varphi(\alpha_\mathfrak{p}\bar{a}'\bar{a}'') = \mathrm{alt}_\varphi(\alpha_\mathfrak{p}) .$$

As φ is over M, it follows by choice of $\bar{a}''$ that

$$\mathbb{M} \vDash \varphi(\bar{a}') \quad \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}'') \quad \text{iff} \quad \varphi(\bar{x}) \in \mathfrak{p} ,$$

as desired.

For types $\mathfrak{p} \in F_\varphi$, the claim we have just proved implies that

$$\mathrm{Th}(\langle \mathbb{M}_M, \mathbb{C}\rangle) \cup \mathfrak{p} \upharpoonright C\alpha_\mathfrak{p} \cup \{Px_0, \dots, Px_{n-1}\} \vDash \varphi(\bar{x}) ,$$

where $\bar{x} = x_0 \dots x_{n-1}$, $\mathbb{C}$ is a set such that $\langle \mathbb{M}, \mathbb{C}\rangle \succeq \langle \mathfrak{M}_+, C_+\rangle$, and P is the predicate symbol corresponding to $\mathbb{C}$. Therefore, we can use compactness to find a formula $\vartheta_\mathfrak{p}(\bar{x}) \in \mathfrak{p} \upharpoonright C\alpha_\mathfrak{p}$ such that

$$\mathrm{Th}(\langle \mathbb{M}_M, \mathbb{C}\rangle) \cup \{\vartheta_\mathfrak{p}(\bar{x}), Px_0, \dots, Px_{n-1}\} \vDash \varphi(\bar{x}) .$$

Note that $\vartheta_\mathfrak{p} \in \mathfrak{p}$ implies $\mathfrak{p} \in \langle\vartheta_\mathfrak{p}\rangle$. Hence,

$$F_\varphi \subseteq \bigcup_{\mathfrak{p} \in F_\varphi} \langle\vartheta_\mathfrak{p}\rangle .$$

By Lemma F2.3.7, F is a closed set. Hence, so is $F_\varphi = F \cap \langle\varphi(\bar{x})\rangle$. As closed sets in Hausdorff spaces are compact, it follows that there exists a finite subset $F_0 \subseteq F_\varphi$ such that

$$F_\varphi \subseteq \bigcup_{\mathfrak{p} \in F_0} \langle\vartheta_\mathfrak{p}\rangle .$$

We claim that

$$\varphi_+(\bar{x}) := \bigvee_{\mathfrak{p} \in F_0} \vartheta_\mathfrak{p}$$

is the desired formula.

Consider a tuple $\bar{a} \in C^{\bar{s}}$ with $\mathbb{M} \vDash \varphi(\bar{a})$. Then $\mathfrak{p} := \mathrm{tp}(\bar{a}/M_+)$ is trivially finitely satisfiable in C. Hence, $\mathfrak{p} \in F_\varphi$ and we have $\vartheta_\mathfrak{q} \in \mathfrak{p}$, for some $\mathfrak{q} \in F_0$. This implies that $\varphi_+(\bar{x}) \in \mathfrak{p}$. Consequently,

$$\varphi(\bar{x})^{\mathbb{M}} \cap C^{\bar{s}} \subseteq \varphi_+(\bar{x})^{\mathbb{M}} \cap C^{\bar{s}} \subseteq \varphi_+(\bar{x})^{\mathbb{M}} \cap C_+^{\bar{s}}.$$

For the second inclusion, let $\bar{a} \in C_+^{\bar{s}}$ be a tuple with $\mathbb{M} \vDash \vartheta_\mathfrak{p}(\bar{a})$, for some $\mathfrak{p} \in F_0$. Then we have $\mathbb{M} \vDash \varphi(\bar{a})$, by choice of $\vartheta_\mathfrak{p}$. Hence,

$$\varphi_+(\bar{x})^{\mathbb{M}} \cap C_+^{\bar{s}} \subseteq \varphi(\bar{x})^{\mathbb{M}} \cap C_+^{\bar{s}}. \qquad \square$$

As a corollary, we obtain the following weak variant of stable embeddedness for theories without the independence property.

Corollary 1.11. *For every model $\mathfrak{M}$, every set $C \subseteq M$, and every formula $\varphi(\bar{x})$ over M with $\mathrm{rk}_{\mathrm{alt}}(\varphi) < \infty$, there exists an elementary extension $\langle \mathfrak{M}_+, C_+ \rangle \succeq \langle \mathfrak{M}, C \rangle$ and a formula $\varphi_+(\bar{x})$ over C_+ such that*

$$\varphi(\bar{x})^{\mathbb{M}} \cap C^{\bar{s}} = \varphi_+(\bar{x})^{\mathbb{M}} \cap C^{\bar{s}}\,.$$

Another convenient consequence of Theorem 1.10 is contained in the proposition below. Again we isolate the main argument in a lemma.

Lemma 1.12. *Let T be a theory without the independence property and κ an infinite cardinal. Let $\mathfrak{M}$ be a model of T of size $|M| < \kappa$, $B \subseteq M$ a set, and $\langle \mathfrak{M}_+, B_+ \rangle \succeq \langle \mathfrak{M}, B \rangle$ a κ-saturated elementary extension. For every set $C \subseteq M$, there exists a set $U \subseteq B_+$ of size $|U| \leq |T| \oplus |C|$ such that*

$$\bar{b} \equiv_U \bar{b}' \quad \textit{implies} \quad \bar{b} \equiv_C \bar{b}'\,, \quad \textit{for all } \bar{b}, \bar{b}' \subseteq B\,.$$

Proof. For every formula $\varphi(\bar{x})$ over C, we use Theorem 1.10 to find a formula φ_+ over B_+ such that

$$\varphi(\bar{x})^{\mathbb{M}} \cap B^{\bar{s}} \subseteq \varphi_+(\bar{x})^{\mathbb{M}} \cap (B_+)^{\bar{s}} \subseteq \varphi(\bar{x})^{\mathbb{M}} \cap (B_+)^{\bar{s}}.$$

Let $U \subseteq B_+$ be a set of size $|U| \leq |T| \oplus |C|$ containing the parameters of each of these formulae φ_+.

To show that U has the desired properties, consider tuples $\bar{b}, \bar{b}' \subseteq B$ with $\bar{b} \equiv_U \bar{b}'$. For every formula $\varphi(\bar{x})$ over C and every finite set I of indices, it follows that

$$\begin{aligned} \mathbb{M} \vDash \varphi(\bar{b}|_I) \quad &\text{iff} \quad \mathbb{M} \vDash \varphi_+(\bar{b}|_I) \\ &\text{iff} \quad \mathbb{M} \vDash \varphi_+(\bar{b}|'_I) \quad \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{b}|'_I)\,. \end{aligned}$$

Consequently, $\bar{b} \equiv_C \bar{b}'$. $\square$

Proposition 1.13. *Let T be a theory without the independence property, $\mathfrak{M}$ a model of T, and $B \subseteq M$. Then there exists an elementary extension $\langle \mathfrak{M}_+, B_+ \rangle \succeq \langle \mathfrak{M}, B \rangle$ such that, for every set $A \subseteq M$, there exists a set $U \subseteq B_+$ of size $|U| \leq |T| \oplus |A|$ with $A \sqrt[\mathrm{s}]{}_U\, B_+$.*

Proof. We iterate the preceding lemma. Let $\langle \mathfrak{M}_+, B_+ \rangle$ be the union of an elementary chain $\langle \mathfrak{M}_n, B_n \rangle_{n<\omega}$ starting with $\langle \mathfrak{M}_0, B_0 \rangle := \langle \mathfrak{M}, B \rangle$ where each $\langle \mathfrak{M}_{n+1}, B_{n+1} \rangle \succeq \langle \mathfrak{M}_n, B_n \rangle$ is $(|T| \oplus |M_n|)^+$-saturated. We inductively construct a sequence $(U_n)_{n<\omega}$ of sets $U_n \subseteq B_{n+1}$ of size $|U_n| \leq |T| \oplus |A|$ as follows. Suppose that we have already defined $U_0, \dots, U_{n-1} \subseteq B_n \subseteq M_n$. By Lemma 1.12, there exists some set $U_n \subseteq B_{n+1}$ of size

$$|U_n| \leq |T| \oplus |A| \oplus |U_0| \oplus \dots \oplus |U_{n-1}| = |T| \oplus |A|$$

such that

$$\bar{b} \equiv_{U_n} \bar{b}' \quad \text{implies} \quad \bar{b} \equiv_{A \cup U_0 \cup \dots \cup U_{n-1}} \bar{b}'\,, \quad \text{for all } \bar{b}, \bar{b}' \subseteq B_n\,.$$

Set $U := \bigcup_{n<\omega} U_n$ and let $\bar{b}, \bar{b}' \subseteq B_+$ be finite tuples with $\bar{b} \equiv_U \bar{b}'$. Then $|U| \leq |T| \oplus |A|$ and there is some $k < \omega$ such that $\bar{b}, \bar{b}' \subseteq B_k$. It follows that

$$\bar{b} \equiv_{A \cup U_0 \cup \dots \cup U_{n-1}} \bar{b}'\,, \quad \text{for all } n \geq k\,.$$

Consequently, $\bar{b} \equiv_{AU} \bar{b}'$, as desired.

For infinite tuples $\bar{b}, \bar{b}' \subseteq B_+$ with $\bar{b} \equiv_U \bar{b}'$, it therefore follows that

$$\bar{b}|_I \equiv_U \bar{b}'|_I \quad \text{implies} \quad \bar{b}|_I \equiv_{AU} \bar{b}'|_I\,, \quad \text{for all finite sets } I\,.$$

Consequently, $\bar{b} \equiv_{AU} \bar{b}'$. $\square$

Convex equivalence relations

As an application we study the structure of indiscernible sequences in theories without the independence property.

Definition 1.14. Let $\mathfrak{I} = \langle I, \leq \rangle$ be a linear order and $\sim$ an equivalence relation on I.

(a) $\sim$ is *convex* if

$$i \sim j \quad \text{implies} \quad i \sim k \quad \text{for all } i \leq k \leq j\,.$$

(b) $\sim$ is *finite* if it has only finitely many classes.

(c) The *intersection number* $\mathrm{in}(\sim)$ of a convex equivalence relation $\sim$ is the least cardinal κ such that $\sim$ can be written as an intersection of κ finite convex equivalence relations.

(d) For tuples $\bar{\imath}, \bar{\jmath} \in I^{<\omega}$, we set

$$\bar{\imath} \sim \bar{\jmath} \quad :\text{iff} \quad \mathrm{ord}(\bar{\imath}) = \mathrm{ord}(\bar{\jmath}) \quad \text{and} \quad i_s \sim j_s \quad \text{for all } s\,.$$

(e) For a subset $C \subseteq I$ and tuples $\bar{\imath}, \bar{\jmath} \subseteq I$, we define

$$\bar{\imath} \equiv^{\mathrm{o}}_C \bar{\jmath} \quad :\text{iff} \quad \mathfrak{I}, \bar{\imath}\bar{c} \equiv^{\mathrm{o}} \mathfrak{I}, \bar{\jmath}\bar{c} \quad \text{where } \bar{c} \text{ is an enumeration of } C\,.$$

Let us note that the relation $\equiv^{\mathrm{o}}_C$ is convex and that its definition for tuples is consistent with the notation introduced in (d) above.

Lemma 1.15. *$\equiv^{\mathrm{o}}_C$ is a convex equivalence relation with $\mathrm{in}(\equiv^{\mathrm{o}}_C) \leq |C|$ that satisfies*

$$\bar{\imath} \equiv^{\mathrm{o}}_C \bar{\jmath} \quad :\text{iff} \quad \mathrm{ord}(\bar{\imath}) = \mathrm{ord}(\bar{\jmath}) \quad \textit{and} \quad i_s \equiv^{\mathrm{o}}_C j_s \quad \textit{for all } s\,.$$

Proof. For the bound on the intersection number, note that

$$\equiv^{\mathrm{o}}_C = \bigcap_{c \in C} \equiv^{\mathrm{o}}_{\{c\}}\,.$$

The other claims are straightforward. □

The statement of the preceding lemma has a weak converse: every convex equivalence relation can be obtained as a coarsening of a relation of the form $\equiv^{\mathrm{o}}_C$.

Lemma 1.16. *Let $\sim$ be a convex equivalence relation on a linear order I and J a complete linear order containing I. Then there exists a set $C \subseteq J$ of size $|C| \leq \mathrm{in}(\sim) \oplus \aleph_0$ such that the restriction of $\equiv^{\mathrm{o}}_C$ to I refines $\sim$.*

Proof. Set $\kappa := \mathrm{in}(\sim) \oplus \aleph_0$ and let F be a set of finite convex equivalence relations of size $|F| \leq \kappa$ such that $\sim = \bigcap F$. We set

$$\begin{aligned} C := {} & \{\, \inf E \mid E \text{ an } \approx\text{-class for some } \approx \in F \,\} \\ & \cup \{\, \sup E \mid E \text{ an } \approx\text{-class for some } \approx \in F \,\}\,, \end{aligned}$$

where we take the infima and suprema in the ordering J. Then $|C| \leq |F| \otimes \aleph_0 \leq \kappa$ and the restriction of $\equiv^{\mathrm{o}}_C$ to I refines $\sim$. □

Theorem 1.17. *Let T be a theory without the independence property and $\alpha = (\bar{a}^i)_{i \in I}$ an indiscernible sequence over U. For every set $C \subseteq \mathbb{M}$, there exist a linear order $J \supseteq I$, an indiscernible sequence $\alpha_+ = (\bar{a}^j)_{j \in J}$ over U with $\alpha_+ \upharpoonright I = \alpha$, and a subset $K \subseteq J$ of size $|K| \leq |T| \oplus |C|$ such that*

$$\bar{\imath} \equiv^{\mathrm{o}}_K \bar{\jmath} \quad \textit{implies} \quad \bar{a}[\bar{\imath}] \equiv_{UC} \bar{a}[\bar{\jmath}]\,, \quad \textit{for all } \bar{\imath}, \bar{\jmath} \in [J]^{<\omega}.$$

Proof. Let $\mathfrak{M}$ be a model containing $U \cup C \cup \alpha$. Suppose that the sequence α consists of γ-tuples $\bar{a}^i = (a^i_k)_{k<\gamma}$ and set

$$\begin{aligned} P &:= U \cup \{\, a^i_k \mid i \in I,\ k < \gamma \,\}\,, \\ E &:= \{\, \langle a^i_k, a^i_l \rangle \mid i \in I,\ k, l < \gamma \,\}\,, \\ F &:= \{\, \langle a^i_k, a^j_k \rangle \mid i, j \in I,\ k < \gamma \,\}\,, \\ R &:= \{\, \langle a^i_k, a^j_l \rangle \mid i < j \text{ in } I,\ k, l < \gamma \,\}\,. \end{aligned}$$

Fix an $|M|^+$-saturated elementary extension

$$\langle \mathfrak{M}_+, P_+, U_+, E_+, F_+, R_+ \rangle \succeq \langle \mathfrak{M}, P, U, E, F, R \rangle\,.$$

Using the relations E_+, F_+, and R_+, we see that there are a linear order $I_+ \supseteq I$, an ordinal $\gamma_+ \geq \gamma$, and a family $(b^i_k)_{i \in I_+, k < \gamma_+}$ of elements such that

- $P_+ = U_+ \cup \{ b^i_k \mid i \in I_+,\ k < \gamma_+ \}$,
- $b^i_k = a^i_k$, for $i \in I$ and $k < \gamma$,
- the sequence $(\bar b^i)_{i \in I_+}$ consisting of $\bar b^i := (b^i_k)_{k < \gamma_+}$, $i \in I_+$, is indiscernible over U_+.

By Lemma 1.12, we can find a set $W \subseteq P_+$ of size $|W| \leq |T| \oplus |C|$ such that

$$\bar a \equiv_W \bar a' \quad \text{implies} \quad \bar a \equiv_C \bar a', \quad \text{for all } \bar a, \bar a' \subseteq P.$$

We claim that the sequence $\alpha' := (\bar b^i|_\gamma)_{i \in I_+}$ and the set

$$K := \{ i \in I_+ \mid \bar b^i \cap W \neq \varnothing \}$$

have the desired properties. Consider tuples $\bar\imath, \bar\jmath \in [I_+]^{<\omega}$ with $\bar\imath \equiv^{\mathrm{o}}_K \bar\jmath$ and let $\bar k$ be an enumeration of K. Since $(\bar b^i)_{i \in I_+}$ is indiscernible over U, it follows that

$$\bar\imath \equiv^{\mathrm{o}}_K \bar\jmath \quad \Rightarrow \quad \mathfrak{I}_+, \bar\imath\bar k \equiv^{\mathrm{o}} \mathfrak{I}_+, \bar\jmath\bar k \quad \Rightarrow \quad \bar b[\bar\imath\bar k] \equiv_U \bar b[\bar\jmath\bar k].$$

Fix an enumeration $\bar c$ of U. Since $\bar a[\bar\imath], \bar a[\bar\jmath], \bar c \subseteq P$, it follows by choice of W that

$$\bar a[\bar\imath]\bar c \equiv_W \bar a[\bar\jmath]\bar c \quad \text{implies} \quad a[\bar\imath]\bar c \equiv_C \bar a[\bar\jmath]\bar c.$$

Hence, $\bar a[\bar\imath] \equiv_{UC} \bar a[\bar\jmath]$ and the claim follows. □

Corollary 1.18. *Let T be a theory without the independence property and $\alpha = (\bar a^i)_{i \in I}$ an indiscernible sequence over U. For every set $C \subseteq \mathbb{M}$, there exists a convex equivalence relation $\approx$ on I with* $\mathrm{in}(\approx) \leq |T| \oplus |C|$ *such that*

$$\bar\imath \approx \bar\jmath \quad \textit{implies} \quad \bar a[\bar\imath] \equiv_{UC} \bar a[\bar\jmath].$$

Proof. Let $\alpha' = (\bar a^j)_{j \in J}$ and $K \subseteq J$ be the sequence and the set obtained from Theorem 1.17. We claim that the restriction $\approx$ of $\equiv^{\mathrm{o}}_K$ to I has the desired properties. By Lemma 1.15, $\approx$ is convex and

$$\mathrm{in}(\approx) \leq |K| \leq |T| \oplus |C|.$$

Consider tuples $\bar\imath, \bar\jmath \subseteq I$ with $\bar\imath \approx \bar\jmath$. Then

$$\mathrm{ord}(\bar\imath) = \mathrm{ord}(\bar\jmath) \quad \text{and} \quad i_s \approx j_s \quad \text{for all } s,$$

and it follows by Lemma 1.15 that $\bar\imath \equiv^{\mathrm{o}}_K \bar\jmath$. By choice of α' and K, this implies that $\bar a[\bar\imath] \equiv_{UC} \bar a[\bar\jmath]$. □

Corollary 1.19. *Let T be a theory without the independence property, $\alpha = (\bar a^i)_{i \in I}$ an indiscernible sequence over U, and $C \subseteq \mathbb{M}$ a set of parameters. If* $\mathrm{cf}\, I > |T| \oplus |C|$, *then there exists an index $k \in I$ such that the subsequence $(\bar a_i)_{i \geq k}$ is indiscernible over $U \cup C \cup \bar a[<k]$.*

Proof. Let $\alpha' = (\bar a^j)_{j \in J}$ and $K \subseteq J$ be the sequence and the set obtained from Theorem 1.17. Since $\mathrm{cf}\, I > |K|$, there exists some index $k \in J \setminus K$ that is greater than all elements of K. This index has the desired properties. □

2. Lascar invariant types

As forking is less well-behaved in non-simple theories, we need additional tools to investigate theories without the independence properties.

Lascar strong types

We start by studying the question of when two tuples $\bar a$, $\bar b$ can appear as elements of the same indiscernible sequence.

Definition 2.1. For two tuples $\bar a$ and $\bar b$, we write

$$\bar a \approx^{\mathrm{ls}}_U \bar b \quad :\text{iff} \quad \text{there is some indiscernible sequence } (\bar c_n)_{n<\omega} \text{ over } U \text{ such that } \bar c_0 = \bar a \text{ and } \bar c_1 = \bar b.$$

We denote the transitive closure of $\approx^{\mathrm{ls}}_U$ by $\equiv^{\mathrm{ls}}_U$. If $\bar a \equiv^{\mathrm{ls}}_U \bar b$, we say that $\bar a$ and $\bar b$ have the same *Lascar strong type* over U.

Remark. Clearly, $\bar a \equiv^{\mathrm{ls}}_U \bar b$ implies $\bar a \equiv_U \bar b$.

Example. If $b \in \mathrm{acl}(Ua)$, then $a \approx^{\mathrm{ls}}_U b$ iff $a = b$.

Exercise 2.1. Prove that $\approx^{\mathrm{ls}}_U$ is reflexive and symmetric, but in general not transitive.

Let us start by giving an alternative characterisation of the relation $\approx^{\mathrm{ls}}_U$ in terms of formulae that are *chain-bounded.*

Definition 2.2. A formula $\varphi(\bar x, \bar y)$ where $\bar x$ and $\bar y$ have the same sorts is *chain-bounded* if there exists a number $n < \omega$ such that

$$\mathbb{M} \vDash \neg\exists \bar x_0 \cdots \exists \bar x_n \bigwedge_{0 \le i < k \le n} \varphi(\bar x_i, \bar x_k)\,.$$

Remark. Let $\varphi(\bar x, \bar y)$ be a formula where $\bar x$ and $\bar y$ both have sorts $\bar s$. By compactness, it follows that the formula φ is not chain-bounded if, and only if, for every strict linear order $\langle I, < \rangle$, there exist a homomorphism $\langle I, < \rangle \to \langle \mathbb{M}^{\bar s}, \varphi^{\mathbb{M}} \rangle$.

Example. If $\chi(\bar x, \bar y) \in \mathrm{FE}^{\bar s}(U)$, then $\neg\chi(\bar x, \bar y)$ is chain-bounded.

Lemma 2.3. *The following statements are equivalent:*

(1) $\bar a \approx^{\mathrm{ls}}_U \bar b$

(2) $\bar a \approx^{\mathrm{ls}}_C \bar b\,,$ *for all finite $C \subseteq U$.*

(3) $\bar a \approx^{\mathrm{ls}}_M \bar b\,,$ *for some model $M \supseteq U$.*

(4) *For every set C, there exists some set $C' \equiv_U C$ such that $\bar a \approx^{\mathrm{ls}}_{UC'} \bar b$.*

(5) $\mathbb{M} \vDash \neg\varphi(\bar a, \bar b)\,,$ *for every chain-bounded formula φ over U.*

(6) $\bigcup_{0 \le i < k < \omega} \mathfrak{p}(\bar x_i, \bar x_k)$ *is satisfiable, where $\mathfrak{p}(\bar x, \bar x') := \mathrm{tp}(\bar a \bar b / U)$.*

Proof. (4) ⇒ (3) Fix an arbitrary model $\mathfrak{M}$ containing U. By (4), there is some $M' \equiv_U M$ such that $\bar a \approx^{\mathrm{ls}}_{M'} \bar b$.

(3) ⇒ (1) ⇒ (2) If $(\bar c_i)_{i<\omega}$ is an indiscernible sequence over a model $M \supseteq U$ with $\bar c_0 = \bar a$ and $\bar c_1 = \bar b$, then $(\bar c_i)_{i<\omega}$ is also indiscernible over U.

Similarly, if $(\bar c_i)_{i<\omega}$ is indiscernible over U, it is also indiscernible over every subset $C \subseteq U$.

(2) ⇒ (5) Consider a chain-bounded formula $\varphi(\bar x, \bar y)$ over U. Fix a finite set $C \subseteq U$ such that φ is over C. Since $\bar a \approx^{\mathrm{ls}}_C \bar b$, there exists an indiscernible sequence $(\bar c_n)_{n<\omega}$ over C such that $\bar c_0 = \bar a$ and $\bar c_1 = \bar b$. If $\mathbb{M} \vDash \varphi(\bar a, \bar b)$, then φ would not be chain-bounded since indiscernibility would imply that

$$\mathbb{M} \vDash \varphi(\bar c_i, \bar c_k)\,, \quad \text{for all } i < k < \omega\,.$$

Therefore, $\mathbb{M} \vDash \neg\varphi(\bar a, \bar b)$.

(5) ⇒ (6) Suppose that $\bigcup_{0 \le i < k < \omega} \mathfrak{p}(\bar x_i, \bar x_k)$ is inconsistent. By compactness, there exists a number $n < \omega$ and a finite subset $\Phi \subseteq \mathfrak{p}$ such that $\bigcup_{0 \le i < k < n} \Phi(\bar x_i, \bar x_k)$ is inconsistent. Setting $\varphi(\bar x, \bar x') := \bigwedge \Phi$ we have

$$\mathbb{M} \vDash \neg\exists \bar x_0 \cdots \exists \bar x_{n-1} \bigwedge_{0 \le i < k < n} \varphi(\bar x_i, \bar x_k)\,.$$

Hence, φ is chain-bounded formula, and $\varphi \in \mathfrak{p}$ implies $\mathbb{M} \nvDash \neg\varphi(\bar a, \bar b)$.

(6) ⇒ (4) Let $(\bar c_n)_{n<\omega}$ be a sequence satisfying $\bigcup_{0 \le i < k < \omega} \mathfrak{p}(\bar x_i, \bar x_k)$. By Proposition E5.3.6, there exists an indiscernible sequence $(\bar d_n)_{n<\omega}$ over U with

$$\mathrm{Av}((\bar c_n)_{n<\omega}/U) \subseteq \mathrm{Av}((\bar d_n)_{n<\omega}/U)\,.$$

Since $\mathfrak{p}(\bar x_0, \bar x_1) \subseteq \mathrm{Av}((\bar c_n)_n/U)$, the sequence $(\bar d_n)_{n<\omega}$ also satisfies $\bigcup_{0 \le i < k < \omega} \mathfrak{p}(\bar x_i, \bar x_k)$. In particular, $\bar d_0 \bar d_1 \equiv_U \bar a \bar b$ and there exists an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ such that $\pi(\bar d_0) = \bar a$ and $\pi(\bar d_1) = \bar b$. We can use Lemma E5.3.11 to find a set $C' \equiv_U C$ such that $(\pi(\bar d_n))_{n<\omega}$ is indiscernible over $U \cup C'$. It follows that $\bar a \approx^{\mathrm{ls}}_{UC'} \bar b$. □

Our next goal is to show that, for a model $\mathfrak{M}$, the relation $\equiv^{\mathrm{ls}}_M$ coincides with $\equiv_M$. We start with a technical lemma.

Lemma 2.4. *If $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ are chain-bounded, then so is $\varphi \lor \psi$.*

Proof. Suppose that $\varphi \lor \psi$ is not chain-bounded. Then there exists a sequence $(\bar{c}_n)_{n<\omega}$ such that

$$\mathbb{M} \vDash (\varphi \lor \psi)(\bar{c}_i, \bar{c}_k)\,, \quad \text{for all } i < k < \omega\,.$$

By the Theorem of Ramsey, we can find an infinite subset $I \subseteq \omega$ such that

$$\mathbb{M} \vDash \varphi(\bar{c}_i, \bar{c}_k)\,, \quad \text{for all } i < k \text{ in } I\,,$$

or

$$\mathbb{M} \vDash \psi(\bar{c}_i, \bar{c}_k)\,, \quad \text{for all } i < k \text{ in } I\,.$$

In the first case, φ is not chain-bounded; in the second case, ψ is not chain-bounded. ☐

Proposition 2.5. *For a model $\mathfrak{M}$, the following statements are equivalent:*

(1) $\bar{a} \equiv^{\mathrm{ls}}_M \bar{b}$

(2) $\bar{a} \equiv_M \bar{b}$

(3) $\bar{a} \approx^{\mathrm{ls}}_M \bar{c} \approx^{\mathrm{ls}}_M \bar{b}$, *for some $\bar{c}$.*

(4) *There exist tuples $\bar{c}_0, \bar{c}_1, \dots$ such that the sequences $\bar{a}, \bar{c}_0, \bar{c}_1, \bar{c}_2, \dots$ and $\bar{b}, \bar{c}_0, \bar{c}_1, \bar{c}_2, \dots$ are both indiscernible over M.*

(5) $\mathbb{M} \vDash \exists \bar{y}[\neg\varphi(\bar{a}, \bar{y}) \land \neg\varphi(\bar{b}, \bar{y})]$, *for every chain-bounded formula $\varphi(\bar{x}, \bar{y})$ over M.*

Proof. (3) ⇒ (1) is trivial.

(1) ⇒ (2) By definition of $\equiv^{\mathrm{ls}}_M$, there are tuples $\bar{c}_0, \dots, \bar{c}_n$ such that

$$\bar{a} = \bar{c}_0 \approx^{\mathrm{ls}}_M \cdots \approx^{\mathrm{ls}}_M \bar{c}_n = \bar{b}\,.$$

For each $k < n$, there is an indiscernible sequence $(\bar{d}^k_i)_{i<\omega}$ over M with $\bar{d}^k_0 = \bar{c}_k$ and $\bar{d}^k_1 = \bar{c}_{k+1}$. Consequently, $\bar{c}_k \equiv_M \bar{c}_{k+1}$ and the claim follows.

(2) ⇒ (4) Suppose that $\bar{a} \equiv_M \bar{b}$. By Lemma F2.3.15, we have $\bar{a} \sqrt[\mathrm{u}]{}_M M$. As $\sqrt[\mathrm{u}]{}$ is a forking relation, the type $\mathrm{tp}(\bar{a}/M)$ has some $\sqrt[\mathrm{u}]{}$-free extension $\mathfrak{p} \in S^{\bar{s}}(\mathbb{M})$. We construct a sequence $\beta = (\bar{c}_n)_{n<\omega}$ by inductively choosing a tuple $\bar{c}_n$ realising $\mathfrak{p} \restriction M\bar{a}\bar{a}'\bar{c}[<n]$. Since $\sqrt[\mathrm{u}]{} \subseteq \sqrt[\mathrm{s}]{}$, the type $\mathfrak{p}$ is invariant over M and the sequences $\alpha := \bar{a}\beta$ and $\alpha' := \bar{a}'\beta$ both satisfy the conditions of Lemma F2.4.14 (b). Hence, they are indiscernible over M.

(4) ⇒ (3) Suppose that $\bar{a}, \bar{c}_0, \bar{c}_1, \bar{c}_2, \dots$ and $\bar{b}, \bar{c}_0, \bar{c}_1, \bar{c}_2, \dots$ are indiscernible sequences over M. Then

$$\bar{a} \approx^{\mathrm{ls}}_M \bar{c}_0 \quad \text{and} \quad \bar{b} \approx^{\mathrm{ls}}_M \bar{c}_0\,,$$

and the claim follows by symmetry of $\approx^{\mathrm{ls}}_M$.

(2) ⇒ (5) Suppose that $\bar{a} \equiv_M \bar{b}$. Let $\varphi(\bar{x}, \bar{y})$ be a chain-bounded formula over M and let n be the minimal number such that

$$\mathbb{M} \vDash \neg\exists \bar{x}_0 \cdots \exists \bar{x}_n \bigwedge_{0 \le i < k \le n} \varphi(\bar{x}_i, \bar{x}_k)\,.$$

Then

$$\mathbb{M} \vDash \exists \bar{x}_0 \cdots \exists \bar{x}_{n-1} \bigwedge_{0 \le i < k < n} \varphi(\bar{x}_i, \bar{x}_k)\,.$$

As the same formula holds in $\mathfrak{M}$, there are tuples $\bar{c}_0, \dots, \bar{c}_{n-1}$ in M such that

$$\mathfrak{M} \vDash \bigwedge_{0 \le i < k < n} \varphi(\bar{c}_i, \bar{c}_k)\,.$$

By choice of n, there is an index $k < n$ such that $\mathbb{M} \nvDash \varphi(\bar{a}, \bar{c}_k)$. Since $\bar{a} \equiv_M \bar{b}$ we also have $\mathbb{M} \nvDash \varphi(\bar{b}, \bar{c}_k)$. Consequently,

$$\mathbb{M} \vDash \neg\varphi(\bar{a}, \bar{c}_k) \land \neg\varphi(\bar{b}, \bar{c}_k)\,.$$

(5) ⇒ (3) Set

$$\Phi(\bar{y}) := \big\{ \neg\varphi(\bar{a}, \bar{y}) \land \neg\varphi(\bar{b}, \bar{y}) \bigm| \varphi(\bar{x}, \bar{y}) \text{ a chain-bounded formula over } M \big\}\,.$$

If there is a tuple $\bar{c}$ satisfying Φ, then it follows from Lemma 2.3 that

$$\bar{a} \approx^{\mathrm{ls}}_M \bar{c} \quad \text{and} \quad \bar{b} \approx^{\mathrm{ls}}_M \bar{c}\,.$$

Hence, it remains to show that $T(\mathbb{M}) \cup \Phi$ is satisfiable. Let $\Phi_0 \subseteq \Phi$ be finite. Then there are chain-bounded formulae $\varphi_0, \dots, \varphi_{n-1}$ over M such that

$$\Phi_0 = \{\, \neg\varphi_i(\bar{a}, \bar{y}) \wedge \neg\varphi_i(\bar{b}, \bar{y}) \mid i < n \,\}\,.$$

By Lemma 2.4 the disjunction $\psi := \varphi_0 \vee \dots \vee \varphi_{n-1}$ is also chain-bounded. Therefore, (5) implies that there is some tuple $\bar{c}$ with

$$\mathbb{M} \vDash \neg\psi(\bar{a}, \bar{c}) \wedge \neg\psi(\bar{b}, \bar{c})\,.$$

Consequently, $\bar{c}$ satisfies $T(\mathbb{M}) \cup \Phi_0$. By compactness, it follows that $T(\mathbb{M}) \cup \Phi$ is satisfiable. □

Finally we provide several characterisations of the relation $\equiv^{\mathrm{ls}}_U$ for arbitrary sets U. One of them is in terms of bounded equivalence relations, where boundedness is an analog to the notion of chain-boundedness, but for the complement of the relation.

Definition 2.6. Let $\mathbb{R} \subseteq \mathbb{M}^{\bar{s}} \times \mathbb{M}^{\bar{s}}$ be a relation.

(a) $\mathbb{R}$ is *U-invariant* if

$$\bar{a}\bar{b} \equiv_U \bar{a}'\bar{b}' \quad \text{implies} \quad \langle \bar{a}, \bar{b}\rangle \in \mathbb{R} \Leftrightarrow \langle \bar{a}', \bar{b}'\rangle \in \mathbb{R}\,.$$

(b) $\mathbb{R}$ is *co-chain-bounded* if there exists a small cardinal κ such that, for every sequence $\alpha = (\bar{a}_i)_{i<\kappa}$ in $\mathbb{M}^{\bar{s}}$, there are indices $i < j$ with $\langle \bar{a}_i, \bar{a}_j\rangle \in \mathbb{R}$. A co-chain-bounded equivalence relation is simply called *bounded.*

Before concentrating on equivalence relations, let us first give several characterisations of co-chain-boundedness for arbitrary relations.

Proposition 2.7. *Let $\mathbb{R} \subseteq \mathbb{M}^{\bar{s}} \times \mathbb{M}^{\bar{s}}$ be a U-invariant relation. The following statements are equivalent.*

(1) *$\mathbb{R}$ is co-chain-bounded.*

(2) $\approx^{\mathrm{ls}}_U \subseteq \mathbb{R}$

(3) *For every indiscernible sequence $(\bar{a}_n)_{n<\omega}$ over U with $\bar{a}_n \in \mathbb{M}^{\bar{s}}$, we have $\langle \bar{a}_i, \bar{a}_j\rangle \in \mathbb{R}$, for all $i < j < \omega$.*

Proof. (2) ⇒ (3) Let $(\bar{a}_n)_{n<\omega}$ be an indiscernible sequence over U. For every pair of indices $i < j < \omega$, we obtain an indiscernible sequence $\bar{a}_i, \bar{a}_j, \bar{a}_{j+1}, \dots$ over U, which witnesses that $\bar{a}_i \approx^{\mathrm{ls}}_U \bar{a}_j$. By (2), this implies that $\langle \bar{a}_i, \bar{a}_j\rangle \in \mathbb{R}$.

(3) ⇒ (2) Let $\bar{a} \approx^{\mathrm{ls}}_U \bar{b}$. By definition, there exists an indiscernible sequence $(\bar{c}_n)_{n<\omega}$ over U with $\bar{c}_0 = \bar{a}$ and $\bar{c}_1 = \bar{b}$. Hence, it follows by (3) that $\langle \bar{c}_0, \bar{c}_1\rangle \in \mathbb{R}$.

(1) ⇒ (3) Let $\mathbb{R}$ be co-chain-bounded and let κ be the corresponding cardinal. For a contradiction, suppose that there exists an indiscernible sequence $\alpha = (\bar{a}_n)_{n<\omega}$ such that $\langle \bar{a}_i, \bar{a}_j\rangle \notin \mathbb{R}$, for some $i < j$. We extend α to an indiscernible sequence $(\bar{a}_i)_{i<\kappa}$ of length κ. By U-invariance, it follows that $\langle \bar{a}_i, \bar{a}_j\rangle \notin \mathbb{R}$, for all $i < j < \kappa$. This contradicts our choice of κ.

(3) ⇒ (1) Suppose that $\mathbb{R}$ is not co-chain-bounded. Then there exists a sequence $(\bar{a}_i)_{i<\kappa}$ of length $\kappa := \beth_{\lambda^+}$ where $\lambda := 2^{|T| \oplus |U| \oplus |\bar{s}|}$ such that

$$\langle \bar{a}_i, \bar{a}_j\rangle \notin \mathbb{R}\,, \quad \text{for all } i < j < \kappa\,.$$

We can use Theorem E5.3.7 to find an indiscernible sequence $(\bar{b}_n)_{n<\omega}$ over U such that, for every $\bar{\imath} \in [\omega]^{<\omega}$, there is some $\bar{\jmath} \in [\kappa]^{<\omega}$ with

$$\bar{b}[\bar{\imath}] \equiv_U \bar{a}[\bar{\jmath}]\,.$$

By U-invariance, it follows that $\langle \bar{b}_i, \bar{b}_j\rangle \notin \mathbb{R}$, for all $i < j < \omega$. This contradicts (3). □

For equivalence relations, we obtain the following characterisation.

Proposition 2.8. *Let $\approx$ be a U-invariant equivalence relation on $\mathbb{M}^{\bar{s}}$. The following statements are equivalent:*

(1) $\approx$ *is bounded.*

(2) $\approx$ *has at most* $2^{|T| \oplus |U| \oplus |\bar{s}|}$ *classes.*

(3) $\equiv^{\mathrm{ls}}_U \subseteq \approx$

(4) *For every indiscernible sequence* $(\bar{a}_n)_{n<\omega}$ *over* U *with* $\bar{a}_n \in \mathbb{M}^{\bar{s}}$, *we have* $\bar{a}_i \approx \bar{a}_j$, *for all* $i, j < \omega$.

(5) *For every model* $\mathfrak{M}$ *containing* U,

$$\bar{a} \equiv_M \bar{b} \quad \text{implies} \quad \bar{a} \approx \bar{b}\,, \qquad \text{for all } \bar{a}, \bar{b} \in \mathbb{M}^{\bar{s}}.$$

Proof. (2) $\Rightarrow$ (1) is trivial, and the equivalence (1) $\Leftrightarrow$ (4) has already been proved in Proposition 2.7. The equivalence (1) $\Leftrightarrow$ (3) also follows by Proposition 2.7 since $\approx$ is an equivalence relation and $\equiv^{\mathrm{ls}}_U$ is the transitive closure of $\approx^{\mathrm{ls}}_U$. Consequently, we have

$$\equiv^{\mathrm{ls}}_U \subseteq \approx \qquad \text{iff} \qquad \approx^{\mathrm{ls}}_U \subseteq \approx\,.$$

(4) $\Rightarrow$ (5) Suppose that $\bar{a} \equiv_M \bar{b}$. By Proposition 2.5 (4), we can find a sequence $\gamma = (\bar{c}_n)_{n<\omega}$ such that $\bar{a}\gamma$ and $\bar{b}\gamma$ are both indiscernible over M. By (4), this implies that $\bar{a} \approx \bar{c}_0 \approx \bar{b}$.

(5) $\Rightarrow$ (2) Fix a model $\mathfrak{M}$ containing U of size $|M| \leq |T| \oplus |U|$. Then $\equiv_M \subseteq \approx$ implies that $\approx$ has at most as many classes as $\equiv_M$. The latter number is $|S^{\bar{s}}(M)| \leq 2^{|T| \oplus |M| \oplus |\bar{s}|} = 2^{|T| \oplus |U| \oplus |\bar{s}|}$. $\square$

Corollary 2.9. *Let* $U \subseteq \mathbb{M}$.

(a) $\approx^{\mathrm{ls}}_U$ *is the finest relation that is co-chain-bounded and* U*-invariant.*

(b) $\equiv^{\mathrm{ls}}_U$ *is the finest equivalence relation that is bounded and* U*-invariant.*

Over arbitrary sets U, we can characterise the relation $\equiv^{\mathrm{ls}}_U$ as follows.

Proposition 2.10. *Let* $\bar{a}, \bar{b} \in \mathbb{M}^{\bar{s}}$ *and* $U \subseteq \mathbb{M}$. *The following statements are equivalent:*

(1) $\bar{a} \equiv^{\mathrm{ls}}_U \bar{b}$

(2) $\bar{a} \approx \bar{b}$, *for every equivalence relation* $\approx$ *on* $\mathbb{M}$ *that is bounded and* U*-invariant.*

(3) *There are tuples* $\bar{c}_0, \dots, \bar{c}_n$ *and models* $M_0, \dots, M_{n-1} \supseteq U$, *for some* $n < \omega$, *such that*

$$\bar{a} = \bar{c}_0 \equiv_{M_0} \bar{c}_1 \equiv_{M_1} \cdots \equiv_{M_{n-2}} \bar{c}_{n-1} \equiv_{M_{n-1}} \bar{c}_n = \bar{b}\,.$$

(4) *There are models* $M_0, \dots, M_{n-1} \supseteq U$, *for some* $n < \omega$, *and automorphisms* $\pi_i \in \mathrm{Aut}\,\mathbb{M}_{M_i}$ *such that*

$$\bar{b} = (\pi_{n-1} \circ \cdots \circ \pi_0)(\bar{a})\,.$$

Proof. (3) $\Leftrightarrow$ (4) follows from the fact that $\bar{c}_i \equiv_{M_i} \bar{c}_{i+1}$ if, and only if, there exists some automorphism $\pi_i \in \mathrm{Aut}\,\mathbb{M}_{M_i}$ with $\bar{c}_{i+1} = \pi(\bar{c}_i)$.

(1) $\Rightarrow$ (2) follows by Proposition 2.8 (3).

(2) $\Rightarrow$ (3) Let $\sim^*$ be the transitive closure of the relation

$$\bar{c} \sim \bar{d} \quad :\text{iff} \quad \bar{c} \equiv_M \bar{d}\,, \quad \text{for some model } \mathfrak{M} \text{ containing } U\,.$$

This relation is clearly U-invariant. Furthermore, it is bounded since it satisfies property (4) of Proposition 2.8. By (2), it follows that $\bar{a} \sim^* \bar{b}$.

(3) $\Rightarrow$ (1) By Proposition 2.5, there are tuples $\bar{d}_i$, for $i < n$, such that

$$\bar{c}_i \approx^{\mathrm{ls}}_{M_i} \bar{d}_i \approx^{\mathrm{ls}}_{M_i} \bar{c}_{i+1}\,.$$

According to Lemma 2.3 this implies that

$$\bar{c}_i \approx^{\mathrm{ls}}_U \bar{d}_i \approx^{\mathrm{ls}}_U \bar{c}_{i+1}\,, \quad \text{for all } i < n\,.$$

Hence, $\bar{a} = \bar{c}_0 \equiv^{\mathrm{ls}}_U \bar{c}_n = \bar{b}$. $\square$

Two tuples are said to have the same *strong type* over a set U if they are elementarily equivalent over $\mathrm{acl}^{\mathrm{eq}}(U)$. The next result shows that having the same Lascar strong type implies having the same strong type.

Corollary 2.11. $\bar{a} \equiv^{\mathrm{ls}}_U \bar{b}$ *implies* $\bar{a} \equiv_{\mathrm{acl}^{\mathrm{eq}}(U)} \bar{b}\,.$

Proof. Suppose that $\bar a \equiv^{\mathrm{ls}}_U \bar b$. We can use Proposition 2.10 to find tuples $\bar c_0, \dots, \bar c_n$ and models $M_0, \dots, M_{m-1} \supseteq U$ such that

$$\bar a = \bar c_0 \equiv_{M_0} \cdots \equiv_{M_{n-1}} \bar c_n = \bar b\,.$$

This implies that

$$\bar a = \bar c_0 \equiv_{M_0^{\mathrm{eq}}} \cdots \equiv_{M_{n-1}^{\mathrm{eq}}} \bar c_n = \bar b\,.$$

Since $\mathrm{acl}^{\mathrm{eq}}(U) \subseteq M_i^{\mathrm{eq}}$, for all i, it follows that

$$\bar a = \bar c_0 \equiv_{\mathrm{acl}^{\mathrm{eq}}(U)} \cdots \equiv_{\mathrm{acl}^{\mathrm{eq}}(U)} \bar c_n = \bar b\,. \qquad \square$$

We conclude our investigation of Lascar strong types by two technical results. The first one shows that the relation $\approx^{\mathrm{ls}}_U$ satisfies a restricted form of the back-and-forth property.

Lemma 2.12. *If $\bar a \approx^{\mathrm{ls}}_U \bar b$ and $\bar c \;{}^{\mathrm{d}}\!\!\surd_{U\bar a}\; \bar b$, there exists a tuple $\bar d$ such that $\bar a\bar c \approx^{\mathrm{ls}}_U \bar b\bar d$.*

Proof. Let $(\bar a_i)_{i<\omega}$ be an indiscernible sequence over U with $\bar a_0 = \bar a$ and $\bar a_1 = \bar b$. Since the subsequence $(\bar a_i)_{0<i<\omega}$ is indiscernible over $U \cup \bar a$ and $\bar c \;{}^{\mathrm{d}}\!\!\surd_{U\bar a}\; \bar b$, we can use Lemma F3.1.3 to find an element $\bar c' \equiv_{U\bar a\bar b} \bar c$ such that $(\bar a_i)_{0<i<\omega}$ is indiscernible over $U\bar a\bar c'$. Applying an $U\bar a\bar b$-automorphism mapping $\bar c'$ to $\bar c$, we obtain an indiscernible sequence $(\bar a'_i)_{0<i<\omega}$ over $U\bar a\bar c$ such that

$$(\bar a'_i)_{0<i<\omega} \equiv_{U\bar a\bar b} (\bar a_i)_{0<i<\omega}\,.$$

Replacing $\bar a_i$ by $\bar a'_i$, for $0 < i < \omega$, we may therefore assume that the sequence $(\bar a_i)_{0<i<\omega}$ is indiscernible over $U\bar a\bar c$.

For every $i < \omega$, we choose an automorphism $\pi_i \in \mathrm{Aut}\,\mathbb{M}_U$ such that $\pi_i(\bar a_n) = \bar a_{n+i}$, for all n, and we set $\bar c_i := \pi_i(\bar c)$. Since $(\bar a_i)_{0<i<\omega}$ is indiscernible over $U\bar a\bar c$, it follows that

$$\bar c\bar a\bar b \equiv_U \bar c\bar a\bar a_n \equiv_U \bar c_i\bar a_i\bar a_{n+i}\,, \qquad \text{for all } i < \omega \text{ and } 0 < n < \omega\,.$$

By Proposition E5.3.6, there exists an indiscernible sequence $(\bar c'_i\bar a'_i)_{i<\omega}$ over U such that

$$\mathrm{Av}((\bar c_i\bar a_i)_{i<\omega}/U) \subseteq \mathrm{Av}((\bar c'_i\bar a'_i)_{i<\omega}/U)\,.$$

In particular, we have

$$\bar c'_i\bar a'_i\bar a'_{n+i} \equiv_U \bar c_i\bar a_i\bar a_{n+i} \equiv_U \bar c\bar a\bar b\,.$$

Let σ be an U-automorphism such that $\sigma(\bar c'_0) = \bar c$, $\sigma(\bar a'_0) = \bar a$, and $\sigma(\bar a'_1) = \bar b$. The tuple $\bar d := \sigma(\bar c'_1)$ has the desired properties. $\square$

The second observation contains a strengthening of the extension axiom.

Lemma 2.13. *Let $\surd$ be a forking relation and suppose that $\bar a \surd_U U$. For every set B, there exists a tuple $\bar a' \approx^{\mathrm{ls}}_U \bar a$ such that $\bar a' \surd_U B$.*

Proof. Since $\bar a \surd_U U$, we can use Proposition F2.4.10 to construct a $\surd$-Morley sequence $(\bar a_n)_{n<\omega}$ for $\mathrm{tp}(\bar a/U)$ over U. Applying a suitable automorphism we may assume that $\bar a_0 = \bar a$. Since $\bar a[>0] \surd_U \bar a_0$, there exists a sequence $\alpha' \equiv_{U\bar a_0} \bar a[>0]$ such that $\alpha' \surd_U B\bar a_0$. Let $\alpha' = (\bar a'_i)_{0<i<\omega}$. As $\bar a_0\alpha'$ is indiscernible over U, we have $\bar a_0 \approx^{\mathrm{ls}}_U \bar a'_1$. Since $\bar a'_1 \surd_U B$, the claim follows. $\square$

Lascar invariance

To study theories without the independence property, we introduce variants of the relations ${}^{\mathrm{s}}\!\!\surd$ and ${}^{\mathrm{i}}\!\!\surd$ that are based on Lascar strong types instead of elementary equivalence.

Definition 2.14. For $A, B, U \subseteq \mathbb{M}$ we define

$$\begin{aligned}
A \;{}^{\mathrm{q}}\!\!\surd_U\; B \quad &:\text{iff} \quad \bar b \approx^{\mathrm{ls}}_U \bar b' \Rightarrow \bar b \approx^{\mathrm{ls}}_{UA} \bar b' \quad \text{for all } \bar b, \bar b' \subseteq B\,,\\
A \;{}^{\mathrm{ls}}\!\!\surd_U\; B \quad &:\text{iff} \quad \bar b \equiv^{\mathrm{ls}}_U \bar b' \Rightarrow \bar b \equiv_{UA} \bar b' \quad \text{for all } \bar b, \bar b' \subseteq B\,,\\
A \;{}^{\mathrm{li}}\!\!\surd_U\; B \quad &:\text{iff} \quad A\; {}^*({}^{\mathrm{ls}}\!\!\surd)_U\; B\,.
\end{aligned}$$

If $\bar a \mathop{\vee\!\!\!/}^{\mathrm{ls}}_U B$, we say that $\mathrm{tp}(\bar a/UB)$ is *Lascar invariant* over U.

Note that $\mathop{\vee\!\!\!/}^{\mathrm{s}} \subseteq \mathop{\vee\!\!\!/}^{\mathrm{ls}} \subseteq \mathop{\vee\!\!\!/}^{\mathrm{q}}$. Unfortunately, the relation $\mathop{\vee\!\!\!/}^{\mathrm{ls}}$ is not a preforking relation since it fails transitivity. But $\mathop{\vee\!\!\!/}^{\mathrm{q}}$ is. Hence, in order to show that $\mathop{\vee\!\!\!/}^{\mathrm{li}}$ is a forking relation, we will prove below that $\mathop{\vee\!\!\!/}^{\mathrm{li}} = {}^*(\mathop{\vee\!\!\!/}^{\mathrm{q}})$.

Exercise 2.2. Prove that $\mathop{\vee\!\!\!/}^{\mathrm{ls}}$ satisfies all axioms of a preforking relation except for (LTR).

Before turning to $\mathop{\vee\!\!\!/}^{\mathrm{li}}$, we take a look at the relation $\mathop{\vee\!\!\!/}^{\mathrm{q}}$.

Lemma 2.15. *$\mathop{\vee\!\!\!/}^{\mathrm{q}}$ is a preforking relation.*

Proof. (INV) follows immediately from the definition.

(MON) Suppose that $A \mathop{\vee\!\!\!/}^{\mathrm{q}}_U B$ and let $A_0 \subseteq A$ and $B_0 \subseteq B$. For tuples $\bar b, \bar b' \subseteq B_0 \subseteq B$, we have

$$\bar b \approx^{\mathrm{ls}}_U \bar b' \;\Rightarrow\; \bar b \approx^{\mathrm{ls}}_{UA} \bar b' \;\Rightarrow\; \bar b \approx^{\mathrm{ls}}_{UA_0} \bar b' .$$

(BMON) Suppose that $A \mathop{\vee\!\!\!/}^{\mathrm{q}}_U BC$ and let $\bar b, \bar b' \subseteq B$. Fixing an enumeration $\bar c$ of C, we have

$$\bar b \approx^{\mathrm{ls}}_{UC} \bar b' \;\Rightarrow\; \bar b\bar c \approx^{\mathrm{ls}}_U \bar b'\bar c \;\Rightarrow\; \bar b\bar c \approx^{\mathrm{ls}}_{UA} \bar b'\bar c \;\Rightarrow\; \bar b \approx^{\mathrm{ls}}_{UCA} \bar b' .$$

(NOR) Suppose that $A \mathop{\vee\!\!\!/}^{\mathrm{q}}_U B$. To show that $AU \mathop{\vee\!\!\!/}^{\mathrm{q}}_U BU$, consider tuples $\bar b, \bar b' \subseteq U \cup B$ with $\bar b \approx^{\mathrm{ls}}_U \bar b'$. Reordering $\bar b$ and $\bar b'$, we may assume that $\bar b = \bar b_0\bar c$ and $\bar b' = \bar b'_0\bar c$ for $\bar b_0, \bar b'_0 \subseteq B$ and $\bar c \subseteq U$. Consequently,

$$\bar b_0\bar c \approx^{\mathrm{ls}}_U \bar b'_0\bar c \;\Rightarrow\; \bar b_0 \approx^{\mathrm{ls}}_U \bar b'_0 \;\Rightarrow\; \bar b_0 \approx^{\mathrm{ls}}_{UA} \bar b'_0 \;\Rightarrow\; \bar b_0\bar c \approx^{\mathrm{ls}}_{UA} \bar b'_0\bar c .$$

(LRF) To show that $A \mathop{\vee\!\!\!/}^{\mathrm{q}}_A B$, let $\bar b, \bar b' \subseteq B$. Since, trivially,

$$\bar b \approx^{\mathrm{ls}}_A \bar b' \quad \text{implies} \quad \bar b \approx^{\mathrm{ls}}_A \bar b' ,$$

the claim follows.

(LTR) Suppose that $A_2 \mathop{\vee\!\!\!/}^{\mathrm{q}}_{A_1} B$ and $A_1 \mathop{\vee\!\!\!/}^{\mathrm{q}}_{A_0} B$ for $A_0 \subseteq A_1 \subseteq A_2$. To show that $A_2 \mathop{\vee\!\!\!/}^{\mathrm{q}}_{A_0} B$, consider two tuples $\bar b, \bar b' \subseteq B$. Then

$$\bar b \approx^{\mathrm{ls}}_{A_0} \bar b' \quad\Rightarrow\quad \bar b \approx^{\mathrm{ls}}_{A_1} \bar b' \quad\Rightarrow\quad \bar b \approx^{\mathrm{ls}}_{A_2} \bar b' .$$

(FIN) Suppose that $A_0 \mathop{\vee\!\!\!/}^{\mathrm{q}}_U B$, for all finite $A_0 \subseteq A$. To show that $A \mathop{\vee\!\!\!/}^{\mathrm{q}}_U B$, consider two tuples $\bar b, \bar b' \subseteq B$. Then

$$\bar b \approx^{\mathrm{ls}}_U \bar b' \quad \text{implies} \quad \bar b \approx^{\mathrm{ls}}_{UA_0} \bar b' , \quad \text{for all finite } A_0 \subseteq A .$$

By Lemma 2.3, it follows that $\bar b \approx^{\mathrm{ls}}_{UA} \bar b'$.

(DEF) Suppose that $\bar a \not\mathop{\vee\!\!\!/}^{\mathrm{q}}_U B$. Then there are tuples $\bar b, \bar b' \subseteq B$ such that

$$\bar b \approx^{\mathrm{ls}}_U \bar b' \quad \text{and} \quad \bar b \not\approx^{\mathrm{ls}}_{U\bar a} \bar b' .$$

By Lemma 2.3, there exists some formula $\varphi(\bar x, \bar y; \bar z)$ over U such that $\varphi(\bar x, \bar y; \bar a)$ is chain-bounded and $\mathbb{M} \vDash \varphi(\bar b, \bar b'; \bar a)$. Let n be the minimal number such that

$$\mathbb{M} \vDash \neg\exists\bar x_0 \cdots \exists\bar x_{n-1} \bigwedge_{0 \le i < k < n} \varphi(\bar x_i, \bar x_k; \bar a) ,$$

and set

$$\psi(\bar z) := \varphi(\bar b, \bar b'; \bar z) \wedge \neg\exists\bar x_0 \cdots \exists\bar x_{n-1} \bigwedge_{0 \le i < k < n} \varphi(\bar x_i, \bar x_k; \bar z) .$$

If $\bar a'$ is a tuple satisfying $\psi(\bar x)$, then $\varphi(\bar x, \bar y; \bar a')$ is chain-bounded and it follows by Lemma 2.3 that $\bar b \not\approx^{\mathrm{ls}}_{U\bar a'} \bar b'$. Hence, $\bar a' \not\mathop{\vee\!\!\!/}^{\mathrm{q}}_U B$. $\square$

There is also a characterisation of $\mathop{\vee\!\!\!/}^{\mathrm{q}}$ in terms of indiscernible sequences, which is obtained by simply replacing the relation $\approx^{\mathrm{ls}}_U$ by its definition.

Lemma 2.16. *$A \mathop{\vee\!\!\!/}^{\mathrm{q}}_U B$ if, and only if, for every indiscernible sequence $(\bar b_i)_{i<\omega}$ over U with $\bar b_0, \bar b_1 \subseteq B$, we can find some indiscernible sequence $(\bar b'_i)_{i<\omega}$ over $U \cup A$ with $\bar b'_0 = \bar b_0$ and $\bar b'_1 = \bar b_1$.*

Proof. ($\Leftarrow$) To show that $A \overset{\mathrm{q}}{\surd}_U B$, consider two tuples $\bar b, \bar b' \subseteq B$ with $\bar b \approx^{\mathrm{ls}}_U \bar b'$. Then there is some indiscernible sequence $(\bar c_i)_{i<\omega}$ over U with $\bar c_0 = \bar b$ and $\bar c_1 = \bar b'$. By assumption, we can find an indiscernible sequence $(\bar c'_i)_{i<\omega}$ over $U \cup A$ with $\bar c'_0 = \bar c_0$ and $\bar c'_1 = \bar c_1$. This implies that $\bar b = \bar c'_0 \approx^{\mathrm{ls}}_{U\cup A} \bar c'_1 = \bar b'$.

($\Rightarrow$) Suppose that $A \overset{\mathrm{q}}{\surd}_U B$ and let $(\bar b_i)_{i<\omega}$ be an indiscernible sequence over U with $\bar b_0, \bar b_1 \subseteq B$. Then $\bar b_0 \approx^{\mathrm{ls}}_U \bar b_1$, which implies that $\bar b_0 \approx^{\mathrm{ls}}_{UA} \bar b_1$. Consequently, there is some indiscernible sequence $(\bar b'_i)_{i<\omega}$ over $U \cup A$ with $\bar b'_0 = \bar b_0$ and $\bar b'_1 = \bar b_1$. ☐

Before proving that $\overset{\mathrm{li}}{\surd}$ is a forking relation, we collect several different characterisations of this relation. We start with the following one.

Lemma 2.17. *$A \overset{\mathrm{li}}{\surd}_U B$ if, and only if, for every finite set of indiscernible sequences $\alpha_0, \dots, \alpha_{n-1}$ over U, there exists a set $A' \equiv_{UB} A$ such that each α_i is indiscernible over $U \cup A'$.*

Proof. ($\Rightarrow$) Suppose that $A \overset{\mathrm{li}}{\surd}_U B$ and let $\alpha_0, \dots, \alpha_{n-1}$ be indiscernible over U. W.l.o.g. we may assume that each α_i is indexed by a dense order I_i. By definition of $\overset{\mathrm{li}}{\surd}$, there exists a set $A' \equiv_{UB} A$ such that

$$A' \overset{\mathrm{ls}}{\surd}_U B\alpha_0 \dots \alpha_{n-1}\,.$$

We claim that each sequence α_i is indiscernible over $U \cup A'$. Suppose that $\alpha_i = (\bar a^i_j)_{j\in I_i}$. By Lemma E5.3.12, it is sufficient to prove that

$$\bar a^i[\bar k] \equiv_{UA'} \bar a^i[\bar l]\,, \quad \text{for all } \bar k, \bar l \in [I_i]^n \text{ such that } \bar k = \bar u s \bar v \text{ and } \bar l = \bar u t \bar v \text{ with } s < t\,.$$

Given $\bar u, \bar v, s, t$, we fix a strictly increasing function $g : \omega \to I_i$ such that

$$g(0) = s\,, \quad g(1) = t\,, \quad \text{and} \quad g(j) < \bar v\,, \quad \text{for all } j < \omega\,.$$

The sequence $(\bar a^i[\bar u g(j) \bar v])_{j<\omega}$ witnesses that

$$\bar a^i[\bar u s \bar v] \approx^{\mathrm{ls}}_U \bar a^i[\bar u t \bar v]\,.$$

Therefore, $A' \overset{\mathrm{ls}}{\surd}_U B\alpha_0 \dots \alpha_{n-1}$ implies that $\bar a^i[\bar u s \bar v] \equiv_{UA'} \bar a^i[\bar u t \bar v]$.

($\Leftarrow$) Let $\bar a$, B, and U be sets such that, for all indiscernible sequences $\alpha_0, \dots, \alpha_{n-1}$ over U, there is some tuple $\bar a' \equiv_{UB} \bar a$ such that each α_i is indiscernible over $U \cup \bar a'$. To show that $\bar a \overset{\mathrm{li}}{\surd}_U B$, consider some set $C \subseteq \mathbb{M}$. We have to find some tuple $\bar a' \equiv_{UB} \bar a$ such that $\bar a' \overset{\mathrm{ls}}{\surd}_U BC$. To do so, it is sufficient to prove that the set

$$\begin{aligned}\Phi(\bar x) := {}& \mathrm{tp}(\bar a/UB) \\ & \cup \{\, \varphi(\bar x; \bar b) \leftrightarrow \varphi(\bar x; \bar b') \mid \bar b, \bar b' \subseteq UBC\,,\ \bar b \equiv^{\mathrm{ls}}_U \bar b' \,\}\end{aligned}$$

is satisfiable. Hence, consider a finite subset $\Phi_0 \subseteq \Phi$. Then there are formulae $\varphi_0(\bar x; \bar y_0), \dots, \varphi_n(\bar x; \bar y_n)$ and parameters $\bar b_0, \bar b'_0, \dots, \bar b_n, \bar b'_n \subseteq U \cup B \cup C$ such that $\bar b_i \equiv^{\mathrm{ls}}_U \bar b'_i$, for all $i \leq n$, and

$$\Phi_0 \subseteq \mathrm{tp}(\bar a/UB) \cup \{\, \varphi_i(\bar x; \bar b_i) \leftrightarrow \varphi_i(\bar x; \bar b'_i) \mid i \leq n \,\}\,.$$

For each $i \leq n$, we fix a finite sequence $\bar c^i_0 \approx^{\mathrm{ls}}_U \cdots \approx^{\mathrm{ls}}_U \bar c^i_{m(i)}$ with $\bar c^i_0 = \bar b_i$ and $\bar c^i_{m(i)} = \bar b'_i$ and, for every $j < m(i)$, we choose an indiscernible sequence β^i_j over U starting with the tuples $\bar c^i_j$ and $\bar c^i_{j+1}$. By assumption, there exists a tuple $\bar a' \equiv_{UB} \bar a$ such that every β^i_j is indiscernible over $U \cup \bar a'$. This implies that

$$\bar c^i_j \approx^{\mathrm{ls}}_{U\bar a'} \bar c^i_{j+1}\,.$$

Hence, $\bar b_i \equiv^{\mathrm{ls}}_{U\bar a'} \bar b'_i$, which implies that $\bar b_i \equiv_{U\bar a'} \bar b'_i$. Consequently, $\bar a'$ realises Φ_0. ☐

It follows that $\overset{\mathrm{li}}{\surd}$ is the coarsest forking relation that preserves indiscernibility.

Proposition 2.18. *Let $\surd$ be a forking relation. Then $\surd \subseteq \overset{\mathrm{li}}{\surd}$ if, and only if, whenever β is an indiscernible sequence over some set U and $A \surd_U \beta$, then β is indiscernible over $U \cup A$.*

Proof. ($\Rightarrow$) Suppose that $\sqrt{} \subseteq \sqrt[\mathrm{li}]{}$ and that $A \sqrt{}_U \beta$, for some indiscernible sequence β over U. Then $A \sqrt[\mathrm{li}]{}_U \beta$ and we can use Lemma 2.17 to find a set $A' \equiv_{U\beta} A$ such that β is indiscernible over $U \cup A'$. Since $A'\beta \equiv_U A\beta$, it follows that β is also indiscernible over $U \cup A$.

($\Leftarrow$) To show that $\sqrt{} \subseteq \sqrt[\mathrm{li}]{}$, suppose that $A \sqrt{}_U B$. We use the characterisation of Lemma 2.17 to prove that $A \sqrt[\mathrm{li}]{}_U B$. Hence, consider indiscernible sequences $\alpha_0, \dots, \alpha_{n-1}$ over U. By (EXT), there exists a set $A' \equiv_{UB} A$ such that

$$A' \sqrt{}_U B\alpha_0 \dots \alpha_{n-1} .$$

By assumption, $A' \sqrt{}_U \alpha_i$ implies that α_i is indiscernible over $U \cup A'$. $\square$

We also need the following technical lemma about the splitting relation $\sqrt[\mathrm{s}]{}$.

Lemma 2.19. *Let $\bar a \sqrt[\mathrm{s}]{}_U M$ where $\mathfrak{M}$ is a κ-saturated model and $U \subseteq M$ a set of size $|U| < \kappa$. For every set C, there exists a unique extension of $\mathrm{tp}(\bar a/M)$ over $M \cup C$ that is $\sqrt[\mathrm{s}]{}$-free over U.*

Proof. For uniqueness, suppose that there are two extension $\mathfrak{p}$ and $\mathfrak{p}'$ of $\mathrm{tp}(\bar a/M)$ over $C \supseteq M$ that are both $\sqrt[\mathrm{s}]{}$-free over U. Fix realisations $\bar b$ and $\bar b'$ of these two types and consider a finite tuple $\bar c \subseteq C$. Since $\mathfrak{M}$ is κ-saturated, we can find some tuple $\bar d \subseteq M$ with $\bar d \equiv_U \bar c$. Then

$$\bar b \sqrt[\mathrm{s}]{}_U C\,, \quad \bar b' \sqrt[\mathrm{s}]{}_U C\,, \quad \text{and} \quad \bar c \equiv_U \bar d$$

implies $\bar c \equiv_{U\bar b} \bar d$ and $\bar c \equiv_{U\bar b'} \bar d$. Furthermore,

$$\bar b \equiv_M \bar a \equiv_M \bar b' \quad \text{implies} \quad \bar b \equiv_{U\bar d} \bar b' .$$

Consequently,

$$\bar b\bar c \equiv_U \bar b\bar d \equiv_U \bar b'\bar d \equiv_U \bar b'\bar c .$$

Hence, $\bar b \equiv_{U\bar c} \bar b'$, for all finite $\bar c \subseteq C$, which implies that $\bar b \equiv_{UC} \bar b'$. Consequently, $\mathfrak{p} = \mathrm{tp}(\bar b/C) = \mathrm{tp}(\bar b'/C) = \mathfrak{p}'$.

It remains to prove the existence of a $\sqrt[\mathrm{s}]{}$-free extension. As $\mathfrak{M}$ is κ-saturated, it realises every type over U. Hence, there exists a function $g : C^{<\omega} \to M^{<\omega}$ such that

$$g(\bar c) \equiv_U \bar c\,, \quad \text{for all } \bar c \in C^{<\omega}.$$

We claim that

$$\mathfrak{p} := \big\{\, \varphi(\bar x; \bar c) \;\big|\; \varphi(\bar x; \bar y) \text{ a formula, } \bar c \in C^{<\omega},\ \mathbb{M} \vDash \varphi(\bar a; g(\bar c)) \,\big\}$$

is the desired type.

Let us start by showing that the set $\mathfrak{p}$ is satisfiable. Consider finitely many formulae $\varphi_0(\bar x; \bar c_0), \dots, \varphi_n(\bar x; \bar c_n) \in \mathfrak{p}$ and set $\bar c := \bar c_0 \dots \bar c_n$ and $\bar d := g(\bar c_0) \dots g(\bar c_n)$. By definition of $\mathfrak{p}$, we have

$$\mathbb{M} \vDash \varphi_0(\bar a; g(\bar c_0)) \wedge \dots \wedge \varphi_n(\bar a; g(\bar c_n)) .$$

By κ^+-saturation of $\mathfrak{M}$, there exists a tuple $\bar b \subseteq M$ with $\bar b \equiv_U \bar c$. Then

$$g(\bar c) \equiv_U \bar c \equiv_U \bar b \quad \text{and} \quad \bar a \sqrt[\mathrm{s}]{}_U M \quad \text{implies} \quad g(\bar c) \equiv_{U\bar a} \bar b .$$

Choosing some tuple $\bar a'$ such that $\bar a\bar b \equiv_U \bar a'\bar c$, it follows that

$$\bar a g(\bar c) \equiv_U \bar a\bar b \equiv_U \bar a'\bar c .$$

Suppose that $g(\bar c) = \bar d_0 \dots \bar d_n$. Then

$$\mathbb{M} \vDash \varphi_i(\bar a; g(\bar c_i)) \quad \text{and} \quad \bar a \sqrt[\mathrm{s}]{}_U M \quad \text{implies} \quad \mathbb{M} \vDash \varphi_i(\bar a; \bar d_i) .$$

By choice of $\bar a'$, it follows that

$$\mathbb{M} \vDash \varphi_0(\bar a'; \bar c_0) \wedge \dots \wedge \varphi_n(\bar a'; \bar c_n) .$$

Thus, $\bar a'$ is the desired tuple satisfying every $\varphi_i(\bar x; \bar c_i)$.

Furthermore, note that $\mathfrak{p}$ is a complete type over C since, for every formula $\varphi(\bar x;\bar c)$ with parameters $\bar c \subseteq C$, we have

$$\begin{aligned} \varphi(\bar x;\bar c) \in \mathfrak{p} \quad &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar a; g(\bar c)) \\ &\text{iff} \quad \mathbb{M} \nvDash \neg\varphi(\bar a; g(\bar c)) \quad \text{iff} \quad \neg\varphi(\bar x;\bar c) \notin \mathfrak{p}\,. \end{aligned}$$

To see that $\mathfrak{p}$ is $\sqrt[\mathrm{s}]{}$-free over U, consider two tuples $\bar c, \bar c' \subseteq C$ such that $\bar c \equiv_U \bar c'$. Then

$$g(\bar c) \equiv_U \bar c \equiv_U \bar c' \equiv_U g(\bar c') \quad \text{and} \quad \bar a \sqrt[\mathrm{s}]{}_U M$$

implies that $g(\bar c) \equiv_{U\bar a} g(\bar c')$. For a formula $\varphi(\bar x;\bar y)$ over U, it follows that

$$\begin{aligned} \varphi(\bar x;\bar c) \in \mathfrak{p} \quad &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar a; g(\bar c)) \\ &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar a; g(\bar c')) \quad \text{iff} \quad \varphi(\bar x;\bar c') \in \mathfrak{p}\,. \qquad \square \end{aligned}$$

Proposition 2.20. *Let $\bar a, U \subseteq \mathbb{M}$ and let $\mathfrak{M}$ be a model containing U that is $(|T| \oplus |U|)^+$-saturated and strongly $(|T| \oplus |U|)^+$-homogeneous. The following statements are equivalent:*

(1) $\bar a \sqrt[\mathrm{li}]{}_U M$.

(2) $\bar a \sqrt[\mathrm{ls}]{}_U M$.

(3) $\bar a \sqrt[\mathrm{q}]{}_U M$.

(4) $\bar b \equiv^{\mathrm{ls}}_U \bar b' \Rightarrow \bar b \equiv^{\mathrm{ls}}_{U\bar a} \bar b'$ *for all finite* $\bar b, \bar b' \subseteq M$.

(5) $\bar a \sqrt[\mathrm{s}]{}_N M$, *for all models $\mathfrak{N} \preceq \mathfrak{M}$ containing U.*

(6) *For all models $\mathfrak{N} \preceq \mathfrak{M}$ containing U, we have*

$$\bar b \equiv_N \bar b' \Rightarrow \bar b \equiv_{U\bar a} \bar b', \quad \textit{for all } \bar b, \bar b' \subseteq M\,.$$

(7) $\mathrm{tp}(\bar a/M)$ *is invariant under all automorphisms of $\mathfrak{M}$ that fix some model $\mathfrak{N} \preceq \mathfrak{M}$ containing U.*

(8) *Every indiscernible sequence $(\bar b_i)_{i<\omega}$ over U that is contained in M is also indiscernible over $U \cup \bar a$.*

(9) *For every indiscernible sequence $(\bar b_i)_{i<\omega}$ over U with $\bar b_0, \bar b_1 \subseteq M$, we can find some indiscernible sequence $(\bar b'_i)_{i<\omega}$ over $U \cup \bar a$ with $\bar b'_0 = \bar b_0$ and $\bar b'_1 = \bar b_1$.*

(10) $\bar b_0 \equiv_{U\bar a} \bar b_1$, *for every indiscernible sequence $(\bar b_i)_{i<\omega}$ over U with $\bar b_0, \bar b_1 \subseteq M$.*

Proof. Set $\kappa := |T| \oplus |U|$.

(3) $\Leftrightarrow$ (9) was already proved in Lemma 2.16.

(3) $\Rightarrow$ (4) Consider two finite tuples $\bar b, \bar b' \subseteq M$ with $\bar b \equiv^{\mathrm{ls}}_U \bar b'$. By definition of $\equiv^{\mathrm{ls}}$, there are tuples $\bar c_0, \dots, \bar c_n$ such that $\bar c_0 = \bar b$, $\bar c_n = \bar b'$ and $\bar c_i \approx^{\mathrm{ls}}_U \bar c_{i+1}$, for all $i < n$. As $\mathfrak{M}$ is κ^+-saturated, we may assume that $\bar c_0, \dots, \bar c_n$ are contained in M. By (3), it follows that $\bar c_i \approx^{\mathrm{ls}}_{U\bar a} \bar c_{i+1}$, for all $i < n$. This implies that $\bar b \equiv^{\mathrm{ls}}_{U\bar a} \bar b'$.

(4) $\Rightarrow$ (7) Let $\pi \in \mathrm{Aut}\,\mathfrak{M}_N$, for some model $\mathfrak{N} \preceq \mathfrak{M}$ containing U. For every finite $\bar b \subseteq M$, it follows by Proposition 2.5 that

$$\begin{aligned} \bar b \equiv_N \pi(\bar b) \quad &\Rightarrow \quad \bar b \equiv^{\mathrm{ls}}_N \pi(\bar b) \\ &\Rightarrow \quad \bar b \equiv^{\mathrm{ls}}_{N\bar a} \pi(\bar b) \quad \Rightarrow \quad \bar b \equiv_{\bar a} \pi(\bar b)\,. \end{aligned}$$

Consequently, for every formula $\varphi(\bar x;\bar y)$,

$$\varphi(\bar x;\bar b) \in \mathrm{tp}(\bar a/M) \quad \text{iff} \quad \varphi(\bar x;\pi(\bar b)) \in \mathrm{tp}(\bar a/M)\,.$$

(7) $\Rightarrow$ (2) Let $\bar b, \bar b' \subseteq M$ be tuples with $\bar b \equiv^{\mathrm{ls}}_U \bar b'$. First, we consider the case where $\bar b$ and $\bar b'$ are finite. By Proposition 2.10, there are tuples $\bar c_0, \dots, \bar c_n$ and models $N_0, \dots, N_{m-1} \supseteq U$ such that

$$\bar a = \bar c_0 \equiv_{N_0} \bar c_1 \equiv_{N_1} \cdots \equiv_{N_{n-2}} \bar c_{n-1} \equiv_{N_{n-1}} \bar c_n = \bar b\,.$$

Replacing each model $\mathfrak{N}_i$ by a suitable elementary substructure, we can ensure that $|N_i| = \kappa$. By κ^+-saturation of $\mathfrak{M}$, we may therefore assume that $N_i \subseteq M$. Hence, κ^+-homogeneity of $\mathfrak{M}$ implies that there are automorphisms $\pi_i \in \operatorname{Aut} \mathfrak{M}_{N_i}$ with $\pi_i(\bar{c}_i) = \bar{c}_{i+1}$. By (7) it follows that $\bar{c}_i \equiv_{N_i \bar{a}} \bar{c}_{i+1}$. Consequently, $\bar{b} \equiv_{U\bar{a}} \bar{b}'$.

For infinite tuples $\bar{b}, \bar{b}' \subseteq M$, it follows that

$$\begin{aligned} \bar{b} \equiv^{\mathrm{ls}}_U \bar{b}' \quad &\Rightarrow \quad \bar{b}|_I \equiv_{U\bar{a}} \bar{b}'|_I\,, \quad \text{for all finite sets of indices } I \\ &\Rightarrow \quad \bar{b} \equiv_{U\bar{a}} \bar{b}'\,. \end{aligned}$$

Consequently, $\bar{a} \sqrt[\mathrm{ls}]{}_U M$.

(2) ⇒ (5) Let $\mathfrak{N} \preceq \mathfrak{M}$ be a model containing U and consider two tuples $\bar{b}, \bar{b}' \subseteq M$ with $\bar{b} \equiv_N \bar{b}'$. Let $\bar{c}$ be an enumeration of N. By (2) and Proposition 2.5, it follows that

$$\begin{aligned} \bar{b} \equiv_N \bar{b}' \quad &\Rightarrow \quad \bar{b}\bar{c} \equiv_N \bar{b}'\bar{c} \\ &\Rightarrow \quad \bar{b}\bar{c} \equiv^{\mathrm{ls}}_N \bar{b}'\bar{c} \\ &\Rightarrow \quad \bar{b}\bar{c} \equiv^{\mathrm{ls}}_U \bar{b}'\bar{c} \\ &\Rightarrow \quad \bar{b}\bar{c} \equiv_{U\bar{a}} \bar{b}'\bar{c} \\ &\Rightarrow \quad \bar{b} \equiv_{U\bar{a}\bar{c}} \bar{b}' \quad \Rightarrow \quad \bar{b} \equiv_{N\bar{a}} \bar{b}'\,. \end{aligned}$$

(5) ⇒ (6) is trivial.

(6) ⇒ (10) Let $(\bar{b}_i)_{i<\omega}$ be an indiscernible sequence over U such that $\bar{b}_0, \bar{b}_1 \subseteq M$. We fix an arbitrary model $\mathfrak{N} \preceq \mathfrak{M}$ of size $|N| = \kappa$ containing U. By Lemma E5.3.11, there is some model $N' \equiv_U N$ such that $(\bar{b}_i)_{i<\omega}$ is indiscernible over N'. In particular, we have $\bar{b}_0 \equiv_{N'} \bar{b}_1$. By κ^+-saturation of $\mathfrak{M}$, we can find some set $N'' \subseteq M$ with $N'' \equiv_{U\bar{b}_0\bar{b}_1} N'$. Hence, $\bar{b}_0 \equiv_{N''} \bar{b}_1$ and (6) implies that $\bar{b}_0 \equiv_{U\bar{a}} \bar{b}_1$.

(10) ⇒ (8) Let $(\bar{b}_i)_{i<\omega}$ be an indiscernible sequence over U that is contained in M. To show that $(\bar{b}_i)_{i<\omega}$ is indiscernible over $U \cup \bar{a}$, we will prove that

$$\bar{b}[\bar{\imath}] \equiv_{U\bar{a}} \bar{b}[\bar{k}]\,, \quad \text{for all } \bar{\imath}, \bar{k} \in [\omega]^n,\ n < \omega\,.$$

It is sufficient to consider the case where $\bar{\imath} < \bar{k}$. Hence, let $\bar{\imath} < \bar{k}$ be elements of $[\omega]^n$. Fix some increasing sequence $\bar{l}_0 < \bar{l}_1 < \dots$ in $[\omega]^n$ with $\bar{l}_0 = \bar{\imath}$ and $\bar{l}_1 = \bar{k}$. We set $\bar{c}_j := \bar{b}[\bar{l}_j]$. Then $(\bar{c}_j)_{j<\omega}$ is indiscernible over U and it follows by (10) that $\bar{b}[\bar{\imath}] = \bar{c}_0 \equiv_{U\bar{a}} \bar{c}_1 = \bar{b}[\bar{k}]$.

(8) ⇒ (9) Let $(\bar{b}^n)_{n<\omega}$ be an indiscernible sequence over U such that $\bar{b}^0, \bar{b}^1 \subseteq M$. We first consider the special case where the tuples $\bar{b}^n$ are finite. Since $\mathfrak{M}$ is κ^+-saturated, it contains some sequence $(\bar{b}'_i)_{i<\omega}$ with $\bar{b}'[\omega] \equiv_{U\bar{b}_0\bar{b}_1} \bar{b}[\omega]$. Then $\bar{b}'_0 = \bar{b}_0$, $\bar{b}'_1 = \bar{b}_1$ and it follows by (8) that $(\bar{b}'_i)_{i<\omega}$ is indiscernible over $U \cup \bar{a}$.

For the general case, let $\Phi((\bar{x}^n)_{n<\omega})$ be a set of formulae stating that the sequence $(\bar{x}^n)_{n<\omega}$ is indiscernible over $U \cup \bar{a}$ and that $\bar{x}^0 = \bar{b}^0$ and $\bar{x}^1 = \bar{b}^1$. We have to show that Φ is satisfiable. Thus, consider a finite subset $\Phi_0 \subseteq \Phi$. Then there is a finite set I of indices such that the formulae in Φ_0 only contain variables x^n_i with $i \in I$. Applying the special case we have proved above to the sequence $(\bar{b}^n|_I)_{n<\omega}$, we obtain an indiscernible sequence $(\bar{b}'_n)_{n<\omega}$ over $U \cup \bar{a}$ with $\bar{b}'_0 = \bar{b}^0$ and $\bar{b}'_1 = \bar{b}^1$. This sequence satisfies Φ_0.

(1) ⇒ (2) follows since $\sqrt[\mathrm{li}]{} = {}^*(\sqrt[\mathrm{ls}]{}) \subseteq \sqrt[\mathrm{ls}]{}$.

(5) ⇒ (1) Fix some set $C \subseteq \mathbb{M}$. We have to show that there is some tuple $\bar{a}' \equiv_M \bar{a}$ with $\bar{a}' \sqrt[\mathrm{ls}]{}_U MC$. Let $\mathfrak{N} \preceq \mathfrak{M}$ be a model containing U of size $|N| = \kappa$. Then $\bar{a} \sqrt[\mathrm{s}]{}_N M$ and we can use Lemma 2.19 to find some tuple $\bar{a}_N \equiv_M \bar{a}$ such that $\bar{a}_N \sqrt[\mathrm{s}]{}_N MC$ and $\operatorname{tp}(\bar{a}_N/MC)$ is the unique $\sqrt[\mathrm{s}]{}$-free extension of $\operatorname{tp}(\bar{a}/M)$. Furthermore, if we are given two such models $\mathfrak{N}, \mathfrak{N}' \preceq \mathfrak{M}$, we can find some model $\mathfrak{N}^+ \preceq \mathfrak{M}$ containing $N \cup N'$ of size $|N^+| = \kappa$. Then

$$\bar{a}_N \sqrt[\mathrm{s}]{}_{N^+} MC\,, \quad \bar{a}_{N'} \sqrt[\mathrm{s}]{}_{N^+} MC\,, \quad \text{and} \quad \bar{a}_N \equiv_M \bar{a}_{N'}\,,$$

and it follows by uniqueness that $\bar{a}_N \equiv_{MC} \bar{a}_{N'}$. Consequently, choosing $\bar{a}' := \bar{a}_{N_0}$, for an arbitrary model $\mathfrak{N}_0$, we have

$$\bar{a}' \equiv_M \bar{a} \quad \text{and} \quad \bar{a}' \sqrt[\mathrm{s}]{}_N MC\,, \quad \begin{array}{l}\text{for all models } U \subseteq N \subseteq M \\ \text{of size } |N| = \kappa\,.\end{array}$$

We claim that $\bar{a}' \sqrt[\mathrm{ls}]{}_N MC$. Consider two tuples $\bar{b}, \bar{b}' \subseteq MC$ with $\bar{b} \approx^{\mathrm{ls}}_U \bar{b}'$. By Lemma 2.3, there is some model $N \supseteq U$ with $\bar{b} \equiv_N \bar{b}'$. We can choose N of size $|N| = \kappa$ and, by κ^+-saturation of $\mathfrak{M}$, we may assume that $N \subseteq M$. Consequently,

$$\bar a' \overset{\mathrm{s}}{\surd}_N MC \quad \text{implies} \quad \bar b \equiv_{N\bar a'} \bar b' ,$$

as desired. □

Corollary 2.21. $\overset{\mathrm{li}}{\surd} = {}^*(\overset{\mathrm{q}}{\surd})$ *is a forking relation.*

Proof. We have seen in Lemma 2.15 that $\overset{\mathrm{q}}{\surd}$ is a preforking relation. Consequently, ${}^*(\overset{\mathrm{q}}{\surd})$ is a forking relation and it remains to prove that it coincides with $\overset{\mathrm{li}}{\surd}$. The inclusion $\overset{\mathrm{ls}}{\surd} \subseteq \overset{\mathrm{q}}{\surd}$ follows immediately from the respective definitions. Consequently, $\overset{\mathrm{li}}{\surd} = {}^*(\overset{\mathrm{ls}}{\surd}) \subseteq {}^*(\overset{\mathrm{q}}{\surd})$. Conversely, by the implication (3) $\Rightarrow$ (1) of Proposition 2.20, we have

$$A \,{}^*(\overset{\mathrm{q}}{\surd})_U\, M \quad \text{implies} \quad A \overset{\mathrm{li}}{\surd}_U M ,$$

for sufficiently saturated models $\mathfrak{M}$. According to Lemma F2.4.7, this implies that ${}^*(\overset{\mathrm{q}}{\surd}) \subseteq \overset{\mathrm{li}}{\surd}$. □

Corollary 2.22. $\overset{\mathrm{s}}{\surd} \subseteq \overset{\mathrm{ls}}{\surd} \subseteq \overset{\mathrm{q}}{\surd}$ *and* $\overset{\mathrm{i}}{\surd} \subseteq \overset{\mathrm{li}}{\surd} \subseteq \overset{\mathrm{f}}{\surd}$

Proof. The first two inclusions follow immediately from the respective definitions. For the thrid one, it follows that

$$\overset{\mathrm{i}}{\surd} = {}^*(\overset{\mathrm{s}}{\surd}) \subseteq {}^*(\overset{\mathrm{ls}}{\surd}) = \overset{\mathrm{li}}{\surd} .$$

For the last inclusion, it is sufficient to prove that

$$A \overset{\mathrm{li}}{\surd}_U M \quad \text{implies} \quad A \overset{\mathrm{d}}{\surd}_U M ,$$

for every sufficiently saturated model $\mathfrak{M}$, since Lemma F2.4.7 then implies that $\overset{\mathrm{li}}{\surd} = {}^*(\overset{\mathrm{li}}{\surd}) \subseteq {}^*(\overset{\mathrm{d}}{\surd}) = \overset{\mathrm{f}}{\surd}$.

Hence, suppose that $A \overset{\mathrm{li}}{\surd}_U M$ where $\mathfrak{M}$ is a $(|T| \oplus |U|)^+$-saturated and strongly $(|T| \oplus |U|)^+$-homogeneous model containing U. By finite character it is sufficient to show that $A \overset{\mathrm{d}}{\surd}_U B$, for every finite subset $B \subseteq M$.

Hence, let $B \subseteq M$ be finite, and consider an indiscernible sequence $(\bar b'_i)_{i<\omega}$ over U where $\bar b'_0$ is an enumeration of B. By $(|T| \oplus |U|)^+$-saturation of $\mathfrak{M}$, we can find an indiscernible sequence $(\bar b_i)_{i<\omega}$ over U such that $\bar b[\omega] \subseteq M$ and $\bar b[\omega] \equiv_{U\bar b'_0} \bar b'[\omega]$. By Proposition 2.20 (8), this sequence is indiscernible over $U \cup A$. Let A' be some set such that

$$A\bar b[\omega] \equiv_{U\bar b'_0} A'\bar b'[\omega] .$$

Then $(\bar b'_i)_{i<\omega}$ is indiscernible over $U \cup A'$ and it follows by Lemma F3.1.3 that $A \overset{\mathrm{d}}{\surd}_U \bar b'_0$. □

It the remainder of this section we compare the relations $\overset{\mathrm{li}}{\surd}$ and $\overset{\mathrm{f}}{\surd}$.

Definition 2.23. We call an independence relation $\surd$ *weakly bounded* if, there exists a function $f : \mathrm{Cn} \to \mathrm{Cn}$ such that

$$\mathrm{mult}_{\surd}(\mathfrak{p}) \leq f\big(|T| \oplus |U|\big) , \quad \text{for all } \mathfrak{p} \in S^{<\omega}(U) .$$

In this case we also say that $\surd$ is weakly bounded *by* f.

We can characterise $\overset{\mathrm{li}}{\surd}$ as the coarsest weakly bounded forking relation.

Proposition 2.24.

(a) $\overset{\mathrm{li}}{\surd}$ *is weakly bounded by* $f(\kappa) = 2^{2^\kappa}$.

(b) $\surd \subseteq \overset{\mathrm{li}}{\surd}$, *for every weakly bounded forking relation* $\surd$.

Proof. (a) Fix a type $\mathfrak{p} \in S^{<\omega}(U)$ and some set $C \supseteq U$. We have to show that $\mathfrak{p}$ has at most $\kappa := 2^{2^{|T|\oplus|U|}}$ $\overset{\mathrm{li}}{\surd}$-free extensions over C. For $\mathfrak{q} \in S^{<\omega}(C)$, let $g_{\mathfrak{q}}$ be the function mapping a formula $\varphi(\bar x; \bar y)$ over U to the set

$$g_{\mathfrak{q}}(\varphi) := \big\{ [\bar b]_{\equiv^{\mathrm{ls}}_U} \bigm| \varphi(\bar x; \bar b) \in \mathfrak{q} \big\} .$$

We claim that $g_{\mathfrak{q}} = g_{\mathfrak{q}'}$ implies $\mathfrak{q} = \mathfrak{q}'$.

For the proof, suppose that $g_{\mathfrak{q}} = g_{\mathfrak{q}'}$ and let $\varphi(\bar x; \bar b) \in \mathfrak{q}$. Then $[\bar b]_{\equiv^{\mathrm{ls}}_U} \in g_{\mathfrak{q}} = g_{\mathfrak{q}'}$ implies that there is some tuple $\bar b' \equiv^{\mathrm{ls}}_U \bar b$ with $\varphi(\bar x; \bar b') \in \mathfrak{q}'$. Fix a tuple $\bar a'$ realising $\mathfrak{q}'$. Then $\bar a' \mathrel{\overset{\mathrm{ls}}{\surd}}_U C$ and

$$\bar b \equiv^{\mathrm{ls}}_U \bar b' \quad \text{implies} \quad \mathbb{M} \vDash \varphi(\bar a'; \bar b) \leftrightarrow \varphi(\bar a'; \bar b')\,.$$

Consequently, $\varphi(\bar x; \bar b) \in \mathfrak{q}'$, as desired.

To conclude the proof, let $N \supseteq U$ be a model of size $|T| \oplus |U|$. Note that the number of $\equiv_N$-classes of finite tuples is at most $|S^{<\omega}(N)| = 2^{|N|}$. By Proposition 2.5, it follows that there are also at most that many $\equiv^{\mathrm{ls}}_U$-equivalence classes of finite tuples. Hence, there are at most $2^{2^{|N|}} = \kappa$ functions of the form $g_{\mathfrak{q}}$. It follows that there are at most κ $\overset{\mathrm{li}}{\surd}$-free extensions of $\mathfrak{p}$ over C.

(b) For a contradiction, suppose that there is a weakly bounded forking relation $\surd$ with $\surd \nsubseteq \overset{\mathrm{li}}{\surd}$. Then there are $\bar a, B, U \subseteq \mathbb{M}$ such that

$$\bar a \surd_U B \quad \text{and} \quad \bar a \mathrel{\overset{\mathrm{li}}{\not\surd}}_U B\,.$$

Let $f : \mathrm{Cn} \to \mathrm{Cn}$ be the function bounding $\surd$ and let $M \supseteq U \cup B$ be a model that is $(|T| \oplus |U|)^+$-saturated and strongly $(|T| \oplus |U|)^+$-homogeneous. By (EXT), we can find some tuple $\bar a' \equiv_{UB} \bar a$ with $\bar a' \surd_U M$. By (MON), we have $\bar a' \mathrel{\overset{\mathrm{li}}{\not\surd}}_U M$. Hence, we can use Proposition 2.20 (10) to find an indiscernible sequence $(\bar b_i)_{i<\omega}$ over U with $\bar b_0, \bar b_1 \subseteq M$ such that $\bar b_0 \not\equiv_{U\bar a'} \bar b_1$. Fix some formula $\varphi(\bar x; \bar y)$ such that

$$\mathbb{M} \vDash \neg\varphi(\bar a'; \bar b_0) \wedge \varphi(\bar a'; \bar b_1)\,.$$

Let $I \subseteq \omega$ be an infinite set of indices such that

$$\mathbb{M} \vDash \varphi(\bar a'; \bar b_i) \leftrightarrow \varphi(\bar a'; \bar b_k) \quad \text{for all } i, k \in I\,,$$

and let $(\bar c_j)_{j\in J}$ be an extension of $(\bar b_i)_{i\in I\cup\{0,1\}}$ of size $|J| > f(|T| \oplus |U|)$ that is indiscernible over U and such that the order J is strongly $\aleph_0$-homogeneous. Fix a tuple $\bar a'' \equiv_{UM} \bar a'$ with $\bar a'' \surd_U M\bar c[J]$. For every $j \in J$, fix an order automorphism $\sigma_j : J \to J$ such that $\sigma_j(0) = j$ and let $\pi_j \in \mathrm{Aut}\,\mathbb{M}_U$ be an automorphism with

$$\pi_j(\bar c_k) = \bar c_{\sigma_j(k)}\,, \quad \text{for all } k \in J\,.$$

Setting $\bar a_j := \pi_j(\bar a'')$ it follows by invariance that

$$\bar a_j \surd_U \bar c[J] \quad \text{and} \quad \bar a_j \not\equiv_{U\bar c[J]} \bar a_k\,, \quad \text{for } j \neq k\,.$$

Hence, $\mathrm{mult}_{\surd}(\mathrm{tp}(\bar a/U)) \geq |J| > f(|T| \oplus |U|)$. A contradiction. ◻

Corollary 2.25. *Let T be a complete first-order theory. The following statements are equivalent.*

(1) $\overset{\mathrm{f}}{\surd} = \overset{\mathrm{li}}{\surd}$.

(2) $\overset{\mathrm{f}}{\surd}$ *is weakly bounded.*

(3) *If β is an indiscernible sequence over some set U and $A \mathrel{\overset{\mathrm{f}}{\surd}}_U \beta$, then β is indiscernible over $U \cup A$.*

Proof. (1) ⇒ (2) follows by Proposition 2.24 (a).

(2) ⇒ (1) The inclusion $\overset{\mathrm{li}}{\surd} \subseteq \overset{\mathrm{f}}{\surd}$ follows by Corollary 2.22, while $\overset{\mathrm{f}}{\surd} \subseteq \overset{\mathrm{li}}{\surd}$ follows by Proposition 2.24 (b).

(1) ⇒ (3) follows by Proposition 2.18.

(3) ⇒ (1) The inclusion $\overset{\mathrm{li}}{\surd} \subseteq \overset{\mathrm{f}}{\surd}$ follows by Corollary 2.22, while $\overset{\mathrm{f}}{\surd} \subseteq \overset{\mathrm{li}}{\surd}$ follows by Proposition 2.18. ◻

Theorem 2.26. *If a theory T does not have the independence property, then $\overset{\mathrm{li}}{\surd} = \overset{\mathrm{f}}{\surd}$.*

Proof. The inclusion $\overset{\mathrm{li}}{\surd} \subseteq \overset{\mathrm{f}}{\surd}$ was proved in Corollary 2.22. For the converse, it is sufficient, by Lemma F2.4.7, to prove that

$$\bar a \mathrel{\overset{\mathrm{f}}{\surd}}_U M \quad \text{implies} \quad \bar a \mathrel{\overset{\mathrm{li}}{\surd}}_U M\,,$$

for all models $\mathfrak{M}$ that are $(|T| \oplus |U|)^+$-saturated and strongly $(|T| \oplus |U|)^+$-homogeneous.

Hence, let $\bar{a} \overset{\mathrm{f}}{\surd}_U M$. We check condition (10) of Proposition 2.20. Let $(\bar{b}_i)_{i<\omega}$ be an indiscernible sequence over U with $\bar{b}_0, \bar{b}_1 \subseteq M$. Then $\bar{a} \overset{\mathrm{f}}{\surd}_U M$ implies that $\bar{a} \overset{\mathrm{d}}{\surd}_U \bar{b}_0\bar{b}_1$. By Lemma F3.1.3, there exists a tuple $\bar{a}' \equiv_{U\bar{b}_0\bar{b}_1} \bar{a}$ such that the sequence $(\bar{b}_{2i}\bar{b}_{2i+1})_{i<\omega}$ is indiscernible over $U \cup \bar{a}'$. For a contradiction, suppose that $\bar{b}_0 \not\equiv_{U\bar{a}} \bar{b}_1$. Then $\bar{b}_0 \not\equiv_{U\bar{a}'} \bar{b}_1$ and there is some formula $\varphi(\bar{x})$ over $U \cup \bar{a}'$ such that

$$\mathbb{M} \vDash \varphi(\bar{b}_0) \wedge \neg\varphi(\bar{b}_1)\,.$$

By indiscernibility of $(\bar{b}_{2i}\bar{b}_{2i+1})_{i<\omega}$ over $U \cup \bar{a}'$, it follows that

$$\mathbb{M} \vDash \varphi(\bar{b}_i) \quad \text{iff} \quad i \text{ is even.}$$

Hence, Proposition E5.4.2 implies that T has the independence property. A contradiction. □

Proposition 2.27. *A simple theory T does not have the independence property if, and only if, $\overset{\mathrm{li}}{\surd} = \overset{\mathrm{f}}{\surd}$.*

Proof. $(\Rightarrow)$ follows by Theorem 2.26.

$(\Leftarrow)$ Suppose that T is a simple theory with the independence property. We have to show that $\overset{\mathrm{li}}{\surd} \neq \overset{\mathrm{f}}{\surd}$. We can use Proposition E5.4.2 to find an indiscernible sequence $(\bar{a}_n)_{n<\omega}$ and a formula $\varphi(\bar{x};\bar{b})$ with parameters $\bar{b} \subseteq \mathbb{M}$ such that

$$\mathbb{M} \vDash \varphi(\bar{a}_n;\bar{b}) \quad \text{iff} \quad n \text{ is even.}$$

Using Proposition E5.3.6 we fix an indiscernible sequence $(\bar{a}'_n\bar{a}''_n)_{n<\omega+\omega}$ over $\bar{b}$ with

$$\mathrm{Av}((\bar{a}'_n\bar{a}''_n)_{n<\omega+\omega}/\bar{b}) \supseteq \mathrm{Av}((\bar{a}_{2n}\bar{a}_{2n+1})_{n<\omega}/\bar{b}).$$

Note that this implies that the interleaved sequence $\bar{a}'_0, \bar{a}''_0, \bar{a}'_1, \bar{a}''_1, \dots$ is indiscernible. In particular, we have

$$\bar{a}'_\omega \approx^{\mathrm{ls}}_U \bar{a}''_\omega \quad \text{where} \quad U := \bar{a}'[<\omega]\bar{a}''[<\omega]\,.$$

Let $A := \bar{a}'[<\omega+\omega]\bar{a}''[<\omega+\omega]$. Indiscernibility implies that $A \overset{\mathrm{u}}{\surd}_U \bar{b}$. Since $\overset{\mathrm{u}}{\surd} \subseteq \overset{\mathrm{f}}{\surd}$, it follows that $A \overset{\mathrm{f}}{\surd}_U \bar{b}$ and, by symmetry, $\bar{b} \overset{\mathrm{f}}{\surd}_U A$. But

$$\bar{a}'_\omega \not\equiv_{\bar{b}} \bar{a}''_\omega \quad \text{implies} \quad \bar{a}'_\omega \not\approx^{\mathrm{ls}}_{U\bar{b}} \bar{a}''_\omega\,.$$

Hence, $\bar{b} \overset{\mathrm{q}}{\not\surd}_U A$, which implies that $\bar{b} \overset{\mathrm{li}}{\not\surd}_U A$. Consequently, $\overset{\mathrm{f}}{\surd} \neq \overset{\mathrm{li}}{\surd}$. □

Theorem 2.28. *Let T be a complete first-order theory. The following statements are equivalent:*

(1) *T is stable.*

(2) *T is simple and it does not have the independence property.*

(3) *T is simple and $\overset{\mathrm{li}}{\surd} = \overset{\mathrm{f}}{\surd}$.*

(4) *$\overset{\mathrm{li}}{\surd}$ is symmetric.*

(5) *$\overset{\mathrm{li}}{\surd}$ is right local.*

Proof. (2) ⇔ (3) was already proved in Proposition 2.27.

(1) ⇒ (2) If T is stable, it is simple by Corollary F3.2.19 and it does not have the independence property by Proposition E5.4.11.

(2) ⇒ (1) Let T be a simple theory without the independence property. We have shown in Proposition F3.2.21 that T also does not have the strict order property. Consequently, it follows by Proposition E5.4.11 that T is stable.

(3) ⇒ (4) If T is simple, $\overset{\mathrm{f}}{\surd}$ is symmetric. Hence, so is $\overset{\mathrm{li}}{\surd} = \overset{\mathrm{f}}{\surd}$.

(4) ⇒ (5) Since $\overset{\mathrm{li}}{\surd}$ is a forking relation, this implication follows by Theorem F2.4.17.

(5) ⇒ (3) If $\overset{\mathrm{li}}{\surd}$ is right local, so is $\overset{\mathrm{f}}{\surd} \supseteq \overset{\mathrm{li}}{\surd}$. Consequently, T is simple. Furthermore, Theorem F2.4.17 implies that $\overset{\mathrm{li}}{\surd}$ is symmetric. Therefore, it follows by Theorem F3.1.9 that $\overset{\mathrm{f}}{\surd} \subseteq \overset{\mathrm{d}}{\surd} \subseteq \overset{\mathrm{li}}{\surd}$. □

3. $\sqrt[\mathrm{i}]{}$-*Morley sequences*

In this section we study $\sqrt[\mathrm{i}]{}$-Morley sequences in theories without the independence property.

Cofinal types

We start by noting that finiteness of the alternation number can be used to define a kind of 'limit type' of a sequences.

Definition 3.1. The *cofinal type* of a sequence $\alpha = (\bar a_i)_{i \in I}$ is the set

$$\mathrm{CF}(\alpha) := \big\{ \varphi(\bar x) \bigm| \varphi \text{ a formula over } \mathbb{M} \text{ such that } [\![\varphi(\bar a_i)]\!]_{i \in I} \text{ is cofinal in } I \big\}.$$

Lemma 3.2. *Let T be a theory without the independence property and let α be an indiscernible sequence. Then $\mathrm{CF}(\alpha)$ is a complete type over $\mathbb{M}$ which is finitely satisfiable in α.*

Proof. Suppose that $\alpha = (\bar a_i)_{i \in I}$. For completeness, consider a formula $\varphi(\bar x)$ over $\mathbb{M}$. Since $\mathrm{alt}_\varphi(\alpha) < \infty$, there exists some index $k \in I$ such that

$$\mathbb{M} \vDash \varphi(\bar a_i) \leftrightarrow \varphi(\bar a_j), \quad \text{for all } i, j \geq k.$$

Consequently,

$$\varphi \in \mathrm{CF}(\alpha) \quad \text{iff} \quad \mathbb{M} \vDash \varphi(\bar a_k) \quad \text{iff} \quad \neg\varphi \notin \mathrm{CF}(\alpha).$$

To show that $\mathrm{CF}(\alpha)$ is consistent, consider finitely many formulae $\varphi_0, \dots, \varphi_n \in \mathrm{CF}(\alpha)$. There exists some index $k \in I$ such that

$$\mathbb{M} \vDash \varphi_j(\bar a_i), \quad \text{for all } i \geq k \text{ and all } j \leq n.$$

In particular,

$$\mathbb{M} \vDash \varphi_0(\bar a_k) \wedge \dots \wedge \varphi_n(\bar a_k).$$

Hence, $\{\varphi_0, \dots, \varphi_n\}$ is satisfiable. As the tuple satisfying this set belongs to α, it further follows that $\mathrm{CF}(\alpha)$ is finitely satisfiable in α. $\square$

Cofinal types can be used to construct $\sqrt[\mathrm{i}]{}$-Morley sequences as follows.

Lemma 3.3. *Let T be a theory without the independence property and $\alpha = (\bar a_i)_{i \in I}$ an indiscernible sequence over U where the order I has no first element. Let $\alpha^{\mathrm{op}} := (\bar a_i)_{i \in I^{\mathrm{op}}}$ be the sequence with reverse ordering and let $\beta = (\bar b_j)_{j \in J}$ be generated by $\mathrm{CF}(\alpha^{\mathrm{op}})$ over $UC\alpha$.*

(a) *β is a $\sqrt[\mathrm{i}]{}$-Morley sequence over $UC\alpha$.*

(b) *$\beta\alpha$ is indiscernible over U.*

Proof. We start by proving that, for every formula φ over $UC\alpha$ and every tuple $\bar\jmath \in [J]^n$, there are arbitrarily small indices $\bar\imath \in [I]^n$ such that

$$\mathbb{M} \vDash \varphi(\bar b[\bar\jmath]) \leftrightarrow \varphi(\bar a[\bar\imath]).$$

We proceed by induction on n. For $n = 0$ there is nothing to do. Hence, suppose that we have proved the claim already for $n < \omega$ and that

$$\mathbb{M} \vDash \varphi(\bar b[\bar\jmath], \bar b_l),$$

where $\bar\jmath \in [J]^n$ and $l \in J$ are indices with $\bar\jmath < l$. Since $\bar b_l$ realises the type $\mathrm{CF}(\alpha^{\mathrm{op}}) \restriction UC\alpha\bar b[<l]$, we have $\varphi(\bar b[\bar\jmath], \bar x) \in \mathrm{CF}(\alpha^{\mathrm{op}})$. Consequently, there are arbitrarily small $k \in I$ such that

$$\mathbb{M} \vDash \varphi(\bar b[\bar\jmath], \bar a_k).$$

By inductive hypothesis, we can find arbitrarily small $\bar\imath < k$ such that

$$\mathbb{M} \vDash \varphi(\bar a[\bar\imath], \bar a_k).$$

Having proved the claim, it follows by Corollary E5.4.3 that

$$\mathbb{M} \vDash \varphi(\bar b[\bar\jmath]) \leftrightarrow \varphi(\bar b[\bar\jmath']), \quad \text{for all formulae } \varphi \text{ over } UC\alpha \text{ and all indices } \bar\jmath, \bar\jmath' \in [J]^n.$$

Hence, β is indiscernible over $UC\alpha$. As α is indiscernible over U, it further follows that

$$\mathbb{M} \vDash \varphi(\bar b[\bar\jmath], \bar a[\bar k]) \leftrightarrow \varphi(\bar a[\bar\imath], \bar a[\bar k])\,,$$

for all formulae φ over U and all indices $\bar\imath \in [I]^n$, $\bar k \in [I]^m$, $\bar\jmath \in [J]^n$ with $\bar\imath < \bar k$. This implies that $\beta\alpha$ is indiscernible over U.

To show that β is a $\sqrt[\mathrm{i}]{}$-Morley sequence, it remains to prove that

$$\bar b_j \sqrt[\mathrm{i}]{}_{UC\alpha}\ \bar b[<j]\,, \quad \text{for all } j \in J\,.$$

We have shown in Lemma 3.2 that $\mathrm{CF}(\alpha^{\mathrm{op}})$ is a global type that is finitely satisfiable in α. In particular, it is invariant over $UC\alpha$. Hence, the type $\mathrm{CF}(\alpha^{\mathrm{op}}) \restriction UC\alpha\bar b[<j]$ realised by $\bar b_j$ has a global extension $\mathrm{CF}(\alpha^{\mathrm{op}})$ that is invariant over $UC\alpha$. $\square$

As a concluding remark let us note that being generated by a type $\mathfrak{p}$ only depends on the average type of the sequence.

Lemma 3.4. *Let $\alpha = (\bar a_i)_{i\in I}$ and $\beta = (\bar a_j)_{j\in J}$ be infinite indiscernible sequences over U and $\mathfrak{p} \in S^{\bar s}(U\alpha\beta)$ a type that is invariant over U.*

(a) *If α is generated by $\mathfrak{p}$ over U and $\mathrm{Av}(\alpha/U) = \mathrm{Av}(\beta/U)$, then β is also generated by $\mathfrak{p}$ over U.*

(b) *If α and β are generated by $\mathfrak{p}$ over U, then $\mathrm{Av}(\alpha/U) = \mathrm{Av}(\beta/U)$.*

Proof. (a) Let $\varphi(\bar x; \bar y)$ be a formula over U such that $\mathbb{M} \vDash \varphi(\bar b_j; \bar b[\bar k])$, for some $\bar k < j$ in J. Let $\bar l i$ be a tuple in I with the same order type as $\bar k j$. Then $\mathrm{Av}(\alpha/U) = \mathrm{Av}(\beta/U)$ implies that $\mathbb{M} \vDash \varphi(\bar a_i; \bar a[\bar l])$. Consequently, $\varphi(\bar x; \bar a[\bar l]) \in \mathfrak{p} \restriction U\bar a[<i]$. Since $\bar a[\bar l] \equiv_U \bar b[\bar k]$, it follows by invariance of $\mathfrak{p}$ that $\varphi(\bar x; \bar b[\bar k]) \in \mathfrak{p}$.

(b) We prove by induction on n that

$$\bar a[\bar\imath] \equiv_U \bar b[\bar\jmath]\,, \quad \text{for all } \bar\imath \in [I]^n \text{ and } \bar\jmath \in [J]^n.$$

For $n = 0$, there is nothing to do. Hence, suppose that we have proved the claim already for tuples of length n and consider tuples $\bar\imath \in [I]^{n+1}$ and $\bar\jmath \in [J]^{n+1}$. Set $\bar\imath' := i_0 \dots i_{n-1}$ and $\bar\jmath' := j_0 \dots j_{n-1}$ and let $\varphi(\bar x_0, \dots, \bar x_n)$ be a formula over U. By inductive hypothesis and invariance of $\mathfrak{p}$, it follows that

$$\begin{aligned} \mathbb{M} \vDash \varphi(\bar a[\bar\imath'], \bar a_{i_n}) \quad &\text{iff} \quad \varphi(\bar a[\bar\imath'], \bar x) \in \mathfrak{p} \\ &\text{iff} \quad \varphi(\bar b[\bar\jmath'], \bar x) \in \mathfrak{p} \\ &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar b[\bar\jmath'], \bar b_{j_n})\,. \end{aligned}$$

$\square$

The confluence property

Our next aim is to prove a combinatorial characterisation of $\sqrt[\mathrm{i}]{}$-Morley sequences in terms of the so-called *confluence property*.

Definition 3.5. Let U be a set of parameters.

(a) Let $\alpha = (\alpha_k)_{k\in K}$ be a family of indiscernible sequences $\alpha_k = (\bar a^k_i)_{i\in I_k}$ over U. We say that α is *confluent over U* if there exists some tuple $\bar c$ such that, for every $k \in K$, the extended sequence $\alpha_k\bar c$ is still indiscernible over U.

(b) A complete type $\Phi((\bar x_i)_{i<\omega})$ over U has the *confluence property* if every family $\alpha = (\alpha_k)_{k\in K}$ of indiscernible sequences $\alpha_k = (\bar a^k_i)_{i\in I_k}$ over U with

$$\mathrm{Av}(\alpha_k/U) = \Phi\,, \quad \text{for all } k \in K\,,$$

is confluent over U.

(c) We say that a sequence $\alpha = (\bar a_i)_{i\in I}$ has the *confluence property* over a set U if it is indiscernible over U and $\mathrm{Av}(\alpha/U)$ has the confluence property.

We start by showing how to find sequences with the confluence property.

Lemma 3.6. *Every infinite sequence $\alpha = (\bar a_i)_{i\in I}$ such that*

$$\bar a_j \equiv_{U\bar a[<i]} \bar a_i \quad \textit{and} \quad \bar a_i \sqrt[\mathrm{i}]{}_U\ \bar a[<i]\,, \quad \textit{for all } i \le j \textit{ in } I\,,$$

has the confluence property over U.

Proof. Indiscernibility follows by Lemma F2.4.14. For the confluence property of $\mathrm{Av}(\alpha/U)$, we choose a $(|T| \oplus |U|)^+$-saturated model $\mathfrak{M}$ of T containing U and we use Proposition E5.3.6 to find an indiscernible sequence $\alpha' = (\bar a'_n)_{n<\omega}$ over U of length ω with $\mathrm{Av}(\alpha'/U) = \mathrm{Av}(\alpha/U)$. By invariance of $\sqrt[\mathrm{i}]{}$, we have

$$\bar a'_n \sqrt[\mathrm{i}]{}_U\, \bar a'[<n]\,, \quad \text{for all } n < \omega\,.$$

Since $\sqrt[\mathrm{i}]{}$ is a forking relation, we can choose, by induction on $n < \omega$, tuples

$$\bar b_n \equiv_{U\bar a'[<n]} \bar a'_n \quad \text{such that} \quad \bar b_n \sqrt[\mathrm{i}]{}_U\, M\bar a'[<n]\bar b[<n]\,.$$

By Lemma F2.4.14, we have $(\bar b_n)_{n<\omega} \equiv_U (\bar a'_n)_{n<\omega}$. Hence, $\beta = (\bar b_n)_{n<\omega}$ is an indiscernible sequence over U with

$$\mathrm{Av}(\beta/U) = \mathrm{Av}(\alpha'/U) = \mathrm{Av}(\alpha/U)\,.$$

To show that this average type has the confluence property over U, consider a family of indiscernible sequences $\beta_k = (\bar b^k_i)_{i\in I_k}$, for $k \in K$, over U with $\mathrm{Av}(\beta_k/U) = \mathrm{Av}(\beta/U)$. Since $\bar b_0 \sqrt[\mathrm{s}]{}_U\, M$, it follows by Lemma 2.19 that there is some tuple $\bar c \equiv_M \bar b_0$ such that

$$\bar c \sqrt[\mathrm{s}]{}_U\, M\beta \cup \bigcup_{k\in K} \beta_k\,.$$

We claim that every sequence $\beta_k \bar c$ is indiscernible over U. Note that $\bar c \sqrt[\mathrm{s}]{}_U\, \beta_k$. By Lemma F2.4.14, it is therefore sufficient to prove that

$$\bar c \equiv_{U\bar b^k[<i]} \bar b^k_i\,, \quad \text{for all } i \in I_k\,.$$

According to Lemma 2.19, $\mathrm{tp}(\bar b^k_i/M)$ has a unique $\sqrt[\mathrm{s}]{}$-free extension over $M \cup \bar b^k[<i]$. Consequently,

$$\bar c \sqrt[\mathrm{s}]{}_M\, \bar b^k[<i]\,, \quad \bar b^k_i \sqrt[\mathrm{s}]{}_M\, \bar b^k[<i]\,, \quad \text{and} \quad \bar c \equiv_M \bar b_0 \equiv_M \bar b^k_i$$

implies that $\bar c \equiv_{M\bar b^k[<i]} \bar b^k_i$. □

In particular, every $\sqrt[\mathrm{i}]{}$-Morley sequence has the confluence property. The converse statement also holds. The proof is split into several steps. We start by showing that every sequence α with the confluence property is generated by some invariant type. This type is the so-called *eventual type* of α.

Definition 3.7. The *eventual type* of a sequence $\alpha = (\bar a_i)_{i\in I}$ is the set

$$\mathrm{Ev}(\alpha/U) := \big\{ \varphi(\bar x) \bigm| \varphi(\bar x) \in \mathrm{CF}(\alpha\beta) \text{ for some maximally } \varphi\text{-alternating extension } \alpha\beta \text{ of } \alpha \text{ over } U \big\}\,.$$

Example. We consider the theory of open dense linear orders. By quantifier-elimination, every strictly increasing sequence $\alpha = (a_i)_{i\in I}$ in $\mathbb{M}$ is indiscernible. Furthermore, such a sequence α is maximally $(x > c)$-alternating, for $c \in \mathbb{M}$, if $a_i > c$, for some $i \in I$. It follows that the eventual type $\mathrm{Ev}(\alpha/\varnothing)$ contains all formulae of the form $x > c$ with $c \in \mathbb{M}$.

Lemma 3.8. *Let $\varphi(\bar x)$ be a formula over $\mathbb{M}$ and $\alpha = (\bar a_i)_{i\in I}$ an infinite indiscernible sequence over U.*

(a) *If α is maximally φ-alternating over U, then*

$$\varphi(\bar x) \in \mathrm{CF}(\alpha) \quad \text{iff} \quad \varphi(\bar x) \in \mathrm{CF}(\alpha\beta)\,,$$

for every extension $\alpha\beta$ of α that is indiscernible over U.

(b) *If α has the confluence property over U, then*

$$\varphi(\bar x) \in \mathrm{CF}(\alpha\beta) \quad \text{iff} \quad \varphi(\bar x) \in \mathrm{CF}(\alpha\gamma)\,.$$

for all maximally φ-alternating extensions $\alpha\beta$ and $\alpha\gamma$ of α.

Proof. (a) Set $n := \mathrm{alt}_\varphi(\alpha)$ and let $\bar k \in [I]^{n+1}$ be a sequence of indices such that

$$\mathbb{M} \vDash \varphi(\bar a_{k_i}) \leftrightarrow \neg\varphi(\bar a_{k_{i+1}})\,, \quad \text{for all } i < n\,.$$

Then

$$\varphi(\bar{x}) \in \mathrm{CF}(\alpha) \quad \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}_{k_n})\,.$$

For a contradiction, suppose that there is an extension $\alpha\beta = (\bar{a}_i)_{i \in I+J}$ that is indiscernible over U such that

$$\varphi(\bar{x}) \in \mathrm{CF}(\alpha/\mathbb{M}) \quad \text{iff} \quad \varphi(\bar{x}) \notin \mathrm{CF}(\alpha\beta/\mathbb{M})\,.$$

Then there is some index $j \in J$ such that

$$\mathbb{M} \vDash \varphi(\bar{a}_j) \leftrightarrow \neg\varphi(\bar{a}_{k_n})\,.$$

Consequently, the tuple $\bar{k}j \in [I+J]^{n+2}$ witnesses that $\mathrm{alt}_\varphi(\alpha\beta) > n$. Hence, α is not maximally φ-alternating. A contradiction.

(b) As $\alpha\beta$ and $\alpha\gamma$ have the same average type over U as α and this type has the confluence property, we can find some tuple $\bar{c}$ such that $\alpha\beta\bar{c}$ and $\alpha\gamma\bar{c}$ are indiscernible over U. Since $\alpha\beta$ and $\alpha\gamma$ are maximally φ-alternating, it follows by (a) that

$$\begin{aligned} \varphi(\bar{x}) \in \mathrm{CF}(\alpha\beta) \quad &\text{iff} \quad \varphi(\bar{x}) \in \mathrm{CF}(\alpha\beta\bar{c}) \\ &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{c}) \\ &\text{iff} \quad \varphi(\bar{x}) \in \mathrm{CF}(\alpha\gamma\bar{c}) \\ &\text{iff} \quad \varphi(\bar{x}) \in \mathrm{CF}(\alpha\gamma)\,. \end{aligned}$$

□

Lemma 3.9. *Let T be a theory without the independence property and let $\alpha = (\bar{a}_i)_{i \in I}$ be an infinite sequence with the confluence property over U.*

(a) $\mathfrak{p} := \mathrm{Ev}(\alpha/U)$ *is a complete type over* $\mathbb{M}$.

(b) $\mathfrak{p}$ *is invariant over* U.

(c) α *is generated by* $\mathfrak{p}$ *over* U.

Proof. (a) Let $\varphi(\bar{x})$ be a formula over $\mathbb{M}$. By Corollary 1.3 there exists a maximally φ-alternating extension $\alpha\beta$ of α. Then $\alpha\beta$ is also maximally

$\neg\varphi$-alternating and it follows by Lemma 3.8 (b) that

$$\begin{aligned} \varphi(\bar{x}) \in \mathrm{Ev}(\alpha/U) \quad &\text{iff} \quad \varphi(\bar{x}) \in \mathrm{CF}(\alpha\beta) \\ &\text{iff} \quad \neg\varphi(\bar{x}) \notin \mathrm{CF}(\alpha\beta) \\ &\text{iff} \quad \neg\varphi(\bar{x}) \notin \mathrm{Ev}(\alpha/U)\,. \end{aligned}$$

Hence, it remains to prove that $\mathrm{Ev}(\alpha/U)$ is satisfiable. Consider finitely many formulae $\varphi_0(\bar{x}), \dots, \varphi_n(\bar{x}) \in \mathrm{Ev}(\alpha/U)$. By Corollary 1.3 there exists an extension $\alpha\beta$ of α that is maximally φ_i-alternating over U, for all $i \leq n$. Suppose that $\beta = (\bar{b}_j)_{j \in J}$. Then

$$\varphi_i(\bar{x}) \in \mathrm{Ev}(\alpha/U) \quad \text{implies} \quad \varphi_i(\bar{x}) \in \mathrm{CF}(\alpha\beta)\,, \quad \text{for all } i \leq n\,,$$

and there exists some index $k \in J$ such that

$$\mathbb{M} \vDash \varphi_i(\bar{b}_j)\,, \quad \text{for all } j \geq k \text{ and } i \leq n\,.$$

This implies that $\mathbb{M} \vDash \varphi_0(\bar{b}_k) \wedge \dots \wedge \varphi_n(\bar{b}_k)$. Hence, $\{\varphi_0, \dots, \varphi_n\}$ is satisfiable.

(b) Consider tuples $\bar{b} \equiv_U \bar{b}'$ and a formula $\varphi(\bar{x}; \bar{y})$ over U. To show that

$$\varphi(\bar{x}; \bar{b}) \in \mathrm{Ev}(\alpha/U) \quad \text{iff} \quad \varphi(\bar{x}; \bar{b}') \in \mathrm{Ev}(\alpha/U)$$

we use Corollary 1.3 to find an extension $\alpha\beta$ of α that is maximally $\varphi(\bar{x}; \bar{b})$-alternating and maximally $\varphi(\bar{x}; \bar{b}')$-alternating over U. Choose a sequence $\alpha'\beta'$ such that

$$\alpha\beta\bar{b} \equiv_U \alpha'\beta'\bar{b}'\,.$$

Then $\alpha'\beta'$ is maximally $\varphi(\bar{x}; \bar{b}')$-alternating. As the type $\mathrm{Av}(\alpha\beta/U) = \mathrm{Av}(\alpha'\beta'/U)$ has the confluence property over U, there is some tuple $\bar{c}$ such that $\alpha\beta\bar{c}$ and $\alpha'\beta'\bar{c}$ are both indiscernible over U. It follows by

Lemma 3.8 (a) that

$$\begin{aligned}
\varphi(\bar{x};\bar{b}) \in \mathrm{Ev}(\alpha/U) \quad &\text{iff} \quad \varphi(\bar{x};\bar{b}) \in \mathrm{CF}(\alpha\beta) \\
&\text{iff} \quad \varphi(\bar{x};\bar{b}') \in \mathrm{CF}(\alpha'\beta') \\
&\text{iff} \quad \varphi(\bar{x};\bar{b}') \in \mathrm{CF}(\alpha'\beta'\bar{c}) \\
&\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{c};\bar{b}') \\
&\text{iff} \quad \varphi(\bar{x};\bar{b}') \in \mathrm{CF}(\alpha\beta\bar{c}) \\
&\text{iff} \quad \varphi(\bar{x};\bar{b}') \in \mathrm{CF}(\alpha\beta) \\
&\text{iff} \quad \varphi(\bar{x};\bar{b}') \in \mathrm{Ev}(\alpha/U)\,.
\end{aligned}$$

(c) To show that $\bar{a}_k$ realises the type $\mathfrak{p} \upharpoonright U\bar{a}[<k]$, we consider a formula $\varphi(\bar{x};\bar{y}_0,\dots,\bar{y}_{n-1})$ over U and a tuple $\bar{\imath} \in [I]^n$ of indices with $\bar{\imath} < k$. Fix a maximally $\varphi(\bar{x};\bar{a}[\bar{\imath}])$-alternating extension $\alpha\beta$ of α over U and let $\bar{c}$ be a tuple such that $\alpha\beta\bar{c}$ is indiscernible over U. Then Lemma 3.8 implies that

$$\begin{aligned}
\varphi(\bar{x};\bar{a}[\bar{\imath}]) \in \mathfrak{p} \upharpoonright U\bar{a}[<k] \quad &\text{iff} \quad \varphi(\bar{x};\bar{a}[\bar{\imath}]) \in \mathrm{CF}(\alpha\beta) \\
&\text{iff} \quad \varphi(\bar{x};\bar{a}[\bar{\imath}]) \in \mathrm{CF}(\alpha\beta\bar{c}) \\
&\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{c};\bar{a}[\bar{\imath}]) \\
&\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}_k;\bar{a}[\bar{\imath}])\,,
\end{aligned}$$

where the last step follows by indiscernibility. □

Combining the above results, we obtain the following characterisation of $\sqrt[\mathrm{i}]{\;}$-Morley sequences in theories without the independence property.

Theorem 3.10. *Let T be a theory without the independence property, $\alpha = (\bar{a}_i)_{i \in I}$ an infinite sequence, and $\mathfrak{p}$ a type. The following statements are equivalent:*

(1) *α is a $\sqrt[\mathrm{i}]{\;}$-Morley sequence for $\mathfrak{p} \upharpoonright U$ over U and $\mathfrak{p} = \mathrm{Ev}(\alpha/U)$.*

(2) *α has the confluence property over U and $\mathfrak{p} = \mathrm{Ev}(\alpha/U)$.*

(3) *$\mathfrak{p}$ is a global type that is invariant over U and α is generated by $\mathfrak{p}$ over U.*

Proof. The implication (1) ⇒ (2) follows by Lemma 3.6, and (2) ⇒ (3) was already proved in Lemma 3.9.

(3) ⇒ (1) For $i \leq j$ in I, we have

$$\mathrm{tp}(\bar{a}_j/U\bar{a}[<i]) = \mathfrak{p} \upharpoonright U\bar{a}[<i] = \mathrm{tp}(\bar{a}_i/U\bar{a}[<i])\,.$$

Furthermore, $\mathrm{tp}(\bar{a}_i/U\bar{a}[<i])$ extends to $\mathfrak{p}$, a complete type over $\mathbb{M}$ that is invariant over U. Consequently, we have $\bar{a}_i \sqrt[\mathrm{i}]{\;}_U \bar{a}[<i]$ and it follows by Lemma F2.4.14 that α is indiscernible over U.

We have shown that α is a $\sqrt[\mathrm{i}]{\;}$-Morley sequence for $\mathfrak{p} \upharpoonright U$ over U. It therefore remains to prove that $\mathfrak{p} = \mathrm{Ev}(\alpha/U)$. Let $\varphi(\bar{x};\bar{c}) \in \mathrm{Ev}(\alpha/U)$ be a formula with parameters $\bar{c} \subseteq \mathbb{M}$. Then $\varphi(\bar{x};\bar{c}) \in \mathrm{CF}(\alpha\beta)$, for some maximally $\varphi(\bar{x};\bar{c})$-alternating extension $\alpha\beta$ of α over U. Let $\bar{b}$ be a tuple realising $\mathfrak{p} \upharpoonright U\alpha\beta\bar{c}$. Applying Lemma 3.4 to the sequences α and $\alpha\beta$, it follows that $\alpha\beta$ is generated by $\mathfrak{p}$ over U. By choice of $\bar{b}$, so is $\alpha\beta\bar{b}$. Consequently, Lemma F2.4.14 implies that the sequence $\alpha\beta\bar{b}$ is indiscernible over U. As $\alpha\beta$ is maximally $\varphi(\bar{x};\bar{c})$-alternating, we therefore have $\varphi(\bar{x};\bar{c}) \in \mathrm{CF}(\alpha\beta\bar{b})$, which implies that $\mathbb{M} \vDash \varphi(\bar{b};\bar{c})$. By choice of $\bar{b}$, it follows that $\varphi(\bar{x};\bar{c}) \in \mathfrak{p} \upharpoonright U\alpha\beta\bar{c} \subseteq \mathfrak{p}$. □

Corollary 3.11. *Let α and β be infinite $\sqrt[\mathrm{i}]{\;}$-Morley sequences over U. The following statements are equivalent:*

(1) $\mathrm{Av}(\alpha/U) = \mathrm{Av}(\beta/U)$

(2) $\mathrm{Ev}(\alpha/U) = \mathrm{Ev}(\beta/U)$

(3) *There is some complete type $\mathfrak{p}$ over $\mathbb{M}$ that is invariant over U such that α and β are both generated by $\mathfrak{p}$.*

Proof. (2) ⇒ (3) By Theorem 3.10, both sequences are generated by the type $\mathrm{Ev}(\alpha/U) = \mathrm{Ev}(\beta/U)$, which is complete and invariant over U.

(3) ⇒ (2) If α and β are both generated by $\mathfrak{p}$, it follows by Theorem 3.10 that $\mathrm{Ev}(\alpha/U) = \mathfrak{p} = \mathrm{Ev}(\beta/U)$.

(1) ⇒ (3) By Theorem 3.10, α is generated by $\mathfrak{p} := \mathrm{Ev}(\alpha/U)$. Hence, Lemma 3.4 implies that so is β.

(3) ⇒ (1) follows by Lemma 3.4. □

As a consequence we can derive the following bound on the number of invariant global types.

Proposition 3.12. *Let T be a theory without the independence property and let $\mathfrak{M}$ be a model of T. There exists a bijection between types $\mathfrak{p} \in S^{<\omega}(\mathbb{M})$ that are invariant over M and average types $\mathrm{Av}(\alpha/M)$ of infinite $\sqrt[i]{}$-Morley sequences α over M.*

Proof. We map a type $\mathfrak{p} \in S^{<\omega}(\mathbb{M})$ that is invariant over M to the average type

$$\Phi_{\mathfrak{p}} := \mathrm{Av}(\alpha/M)\,,$$

where α is any infinite sequence generated by $\mathfrak{p}$ over M. According to Theorem 3.10, the resulting sequence is a $\sqrt[i]{}$-Morley sequence. Furthermore, if α and β are both generated by $\mathfrak{p}$ over M, it follows by Corollary 3.11 that $\mathrm{Av}(\alpha/M) = \mathrm{Av}(\beta/M)$. Consequently, $\Phi_{\mathfrak{p}}$ does not depend on the choice of α.

The inverse of the function $\mathfrak{p} \mapsto \Phi_{\mathfrak{p}}$ maps an average type Φ of an infinite $\sqrt[i]{}$-Morley sequence α over M to the type $\mathfrak{p}_\Phi := \mathrm{Ev}(\alpha/M)$. Again it follows by Corollary 3.11 that the type $\mathfrak{p}_\Phi$ does not depend on the choice of α.

It remains to prove that the functions $\mathfrak{p} \mapsto \Phi_{\mathfrak{p}}$ and $\Phi \mapsto \mathfrak{p}_\Phi$ are inverse to each other. Let $\mathfrak{p} \in S^{<\omega}(\mathbb{M})$ be a type that is invariant over M and let α be an infinite sequence that is generated by $\mathfrak{p}$ over M. Then it follows by Theorem 3.10 that $\mathfrak{p}_{\Phi_{\mathfrak{p}}} = \mathrm{Ev}(\alpha/M) = \mathfrak{p}$.

Conversely, consider an average type Φ of some infinite $\sqrt[i]{}$-Morley sequence α and let $\mathfrak{p}_\Phi := \mathrm{Ev}(\alpha/M)$. By Theorem 3.10, α is generated by $\mathfrak{p}_\Phi$, which implies that $\Phi_{\mathfrak{p}_\Phi} = \mathrm{Av}(\alpha/M) = \Phi$. □

As an application, we derive the following characterisation of theories without the independence property.

Theorem 3.13. *Let T be a complete first-order theory. The following statements are equivalent:*

1. *T does not have the independence property.*
2. *$\sqrt[f]{}$ is weakly bounded by $f(\kappa) = 2^\kappa$.*
3. *There is some cardinal $\kappa \geq |T|$ such that, for every type $\mathfrak{p} \in S^{<\omega}(M)$ where M is a model of size $|M| = \kappa$, there are less than 2^{2^κ} $\sqrt[u]{}$-free extensions of $\mathfrak{p}$ over any given set $C \supseteq M$.*
4. *For every $\kappa \geq |T|$, every set U of size $|U| = \kappa$, every type $\mathfrak{p} \in S^{<\omega}(U)$, and every set C, there are at most 2^κ $\sqrt[u]{}$-free extensions of $\mathfrak{p}$ over $U \cup C$.*

Proof. (4) ⇒ (3) is trivial.

(2) ⇒ (4) Let $\kappa \geq |T|$ and let U be a set of size $|U| = \kappa$. Consider a type $\mathfrak{p} \in S^{<\omega}(U)$ and some set $C \subseteq \mathbb{M}$. Let $(\mathfrak{q}_i)_{i<\lambda}$ be an enumeration of all $\sqrt[u]{}$-free extensions of $\mathfrak{p}$ over $U \cup C$. Since $\sqrt[u]{} \subseteq \sqrt[f]{}$, it follows that each $\mathfrak{q}_i$ is also a $\sqrt[f]{}$-free extension of $\mathfrak{p}$. By (2), there are at most $2^{|T| \oplus |U|}$ such extensions. Hence, $\lambda \leq 2^{|T| \oplus |U|} = 2^\kappa$.

(1) ⇒ (2) Let $U, C \subseteq \mathbb{M}$ be sets and let $(\mathfrak{p}_i)_{i<\lambda}$ be an enumeration without repetitions of all types over $U \cup C$ that do not fork over U. We have to show that $\lambda \leq 2^{|T| \oplus |U|}$. Let $\mathfrak{M}$ be a model of T containing U of size $|M| \leq |T| \oplus |U|$ and let $\mathfrak{N}$ be a model containing $M \cup C$ that is $(|T| \oplus |U|)^+$-saturated and strongly $(|T| \oplus |U|)^+$-homogeneous. By (EXT), we can fix, for every $i < \lambda$, some type $\mathfrak{q}_i \supseteq \mathfrak{p}_i$ over N that does not fork over U. Note that $\mathfrak{p}_i \neq \mathfrak{p}_k$ implies that $\mathfrak{q}_i \neq \mathfrak{q}_k$, for $i \neq k$. Since T does not have the independence property, it follows by Theorem 2.26 that $\sqrt[f]{} = \sqrt[li]{}$. Hence, each $\mathfrak{q}_i$ is $\sqrt[li]{}$-free over U and, thus, also over M. Consequently, we can use Proposition 2.20 to show that $\mathfrak{q}_i$ is $\sqrt[i]{}$-free over M. Note that there are at most $2^{|T| \oplus |M|} = 2^{|T| \oplus |U|}$ average types $\mathrm{Av}(\alpha/M)$ of $\sqrt[i]{}$-Morley sequences α over M. By Corollary 3.11, this means that there also are at most that many eventual type $\mathrm{Ev}(\alpha/M)$ of such sequences α. Therefore we can use Theorem 3.10 to show that there are at most that many types over N that are $\sqrt[i]{}$-free over M. This implies that $\lambda \leq 2^{|T| \oplus |U|}$.

(3) ⇒ (1) Suppose that there is some formula $\varphi(\bar{x};\bar{y})$ with the independence property. Then there are families $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_s)_{s\subseteq\omega}$ such that

$$\mathbb{M} \vDash \varphi(\bar{a}_i, \bar{b}_s) \quad \text{iff} \quad i \in s\,.$$

Let $\mathfrak{M}$ be a model of T of size $|M| = \kappa$ that contains α and β. We have seen in Theorem B2.4.13 that there are 2^{2^κ} ultrafilters over the set $A := \{\, \bar{a}_i \mid i < \kappa \,\}$. For every ultrafilter $\mathfrak{u}$ over A, set

$$\mathfrak{p}_{\mathfrak{u}} := \mathrm{Av}(\mathfrak{u}/MC)\,.$$

By Lemma F2.3.10, $\mathfrak{p}_{\mathfrak{u}}$ is a $\overset{\mathrm{u}}{\surd}$-free extension of $\mathfrak{p}_{\mathfrak{u}} \upharpoonright M$. Furthermore, if $\mathfrak{u} \neq \mathfrak{v}$ are distinct ultrafilters, we can fix some set $B \in \mathfrak{u} \setminus \mathfrak{v}$ and an index $s \subseteq \omega$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}_i; \bar{b}_s) \quad \text{iff} \quad \bar{a}_i \in B\,.$$

Consequently, $\varphi(\bar{x}; \bar{b}_s) \in \mathfrak{p}_{\mathfrak{u}} \setminus \mathfrak{p}_{\mathfrak{v}}$, which implies that $\mathfrak{p}_{\mathfrak{u}} \neq \mathfrak{p}_{\mathfrak{v}}$. It follows that there are at least 2^{2^κ} types over $M \cup C$ that are $\overset{\mathrm{u}}{\surd}$-free over M. □

4. Dp-rank

Mutually indiscernible sequences

We can characterise theories without the independence property also in terms of a rank that is based on mutually indiscernible sequences.

Definition 4.1. A family $(\alpha_k)_{k\in K}$ of sequences is *mutually indiscernible* over a set U if each sequence α_k is indiscernible over $U \cup \alpha[K \setminus \{k\}]$.

Before giving the definition of the dp-rank, we collect some technical properties of mutually indiscernible sequences. Let us start with ways to construct such families. The first observation is trivial.

Lemma 4.2. *Let $\alpha := (\bar{a}_i)_{i\in I}$ be an indiscernible sequence over U and let $\sim$ be a convex equivalence relation on I. The family $(\alpha|_E)_{E\in I/\sim}$ is mutually indiscernible over U.*

Lemma 4.3. *Let $(\alpha_k)_{k<\gamma}$ be a family of sequences and U a set of parameters. If $(\beta_k)_{k<\gamma}$ is a family such that each β_k is an indiscernible sequence over $U\alpha[>k]\beta[<k]$ with*

$$\mathrm{Av}(\beta_k/U\alpha[>k]\beta[<k]) \supseteq \mathrm{Av}(\alpha_k/U\alpha[>k]\beta[<k])\,,$$

then $(\beta_k)_{k<\gamma}$ is mutually indiscernible over U.

Proof. Suppose that $\alpha_k = (\bar{a}^k_i)_{i\in I_k}$ and $\beta_k = (\bar{b}^k_i)_{i\in J_k}$, for $k<\gamma$. To show that $(\beta_k)_{k<\gamma}$ is mutually indiscernible over U, we fix some index $k<\gamma$ and we prove by induction on $k < l \leq \gamma$ that β_k is indiscernible over $U\alpha[\geq l]\beta[\downarrow l \setminus \{k\}]$. The result then follows for $l = \gamma$.

For $l = k+1$, the claim holds by choice of β_k. For the inductive step, suppose that we have already shown that β_k is indiscernible over the set $U\alpha[\geq l]\beta[\downarrow l \setminus \{k\}]$. To show that it is also indiscernible over

$$U\alpha[\geq(l+1)]\beta[\downarrow(l+1) \setminus \{k\}]\,,$$

consider a formula $\varphi(\bar{x}_0,\dots,\bar{x}_{n-1}; \bar{c}, \bar{d})$ with parameters

$$\bar{c} \subseteq \beta^l \quad \text{and} \quad \bar{d} \subseteq U\alpha[\geq(l+1)]\beta[\downarrow l \setminus \{k\}]\,.$$

We have to show that

$$\mathbb{M} \vDash \varphi(\bar{b}^k[\bar{\imath}]; \bar{c}, \bar{d}) \leftrightarrow \varphi(\bar{b}^k[\bar{\jmath}]; \bar{c}, \bar{d})\,, \quad \text{for all } \bar{\imath}, \bar{\jmath} \in [J_k]^n\,.$$

W.l.o.g. we may assume that $\bar{c} = \bar{b}^l[\bar{s}]$, for some $\bar{s} \in [J_l]^m$. Fix indices $\bar{\imath}, \bar{\jmath} \in [J_k]^n$. By inductive hypothesis, the sequence β_k is indiscernible over $U\alpha[\geq l]\beta[\downarrow l \setminus \{k\}]$. Therefore, we have

$$\mathbb{M} \vDash \varphi(\bar{b}^k[\bar{\imath}]; \bar{a}^l[\bar{t}], \bar{d}) \leftrightarrow \neg\varphi(\bar{b}^k[\bar{\jmath}]; \bar{a}^l[\bar{t}], \bar{d})\,,$$

for all and all $\bar{t} \in [I_l]^m$. This implies that the formula

$$\varphi(\bar{b}^k[\bar{\imath}];\bar{x},\bar{d}) \leftrightarrow \varphi(\bar{b}^k[\bar{\jmath}];\bar{x},\bar{d})$$

belongs to

$$\operatorname{Av}(\alpha_l/U\alpha[>l]\beta[<l]) \subseteq \operatorname{Av}(\beta_l/U\alpha[>l]\beta[<l])\,.$$

Consequently,

$$\mathbb{M} \vDash \varphi(\bar{b}^k[\bar{\imath}];\bar{b}^l[\bar{s}],\bar{d}) \leftrightarrow \varphi(\bar{b}^k[\bar{\jmath}];\bar{b}^l[\bar{s}],\bar{d})\,, \qquad \square$$

Let us note the following property of sequences 'diagonally crossing' a family of mutually indiscernible sequences.

Lemma 4.4. *Let $\alpha = (\alpha_k)_{k\in K}$ be a family of mutually indiscernible sequences $\alpha_k = (\bar{a}^k_i)_{i\in I_k}$ over U.*

(a) $(\bar{a}^k_{\eta(k)})_{k\in K} \equiv_U (\bar{a}^k_{\zeta(k)})_{k\in K}\,,\quad$ *for all $\eta,\zeta \in \prod_{k\in K} I_k$.*

(b) *If the index set K is ordered and the sequence $\alpha = (\alpha_k)_{k\in K}$ is indiscernible over U, then each sequence of the form $(\bar{a}^k_{\eta(k)})_{k\in K}$ with $\eta \in \prod_{k\in K} I_k$ is also indiscernible over U.*

Proof. (a) We prove by induction on $n < \omega$ that

$$\bar{a}^{k_0}_{\eta(k_0)} \dots \bar{a}^{k_{n-1}}_{\eta(k_{n-1})} \equiv_{U\alpha[K\smallsetminus\bar{k}]} \bar{a}^{k_0}_{\zeta(k_0)} \dots \bar{a}^{k_{n-1}}_{\zeta(k_{n-1})}\,, \quad \text{for all } \bar{k} \in [K]^n.$$

For $n = 0$, there is nothing to do. For the inductive step, suppose that we have proved the claim already for n and let $\bar{k} \in [K]^{n+1}$. By mutual indiscernibility, we have

$$\bar{a}^{k_n}_{\eta(k_n)} \equiv_{U\alpha[K\smallsetminus\{k_n\}]} \bar{a}^{k_n}_{\zeta(k_n)}\,.$$

Therefore, it follows by inductive hypothesis that

$$\begin{aligned}\bar{a}^{k_0}_{\eta(k_0)} \dots \bar{a}^{k_{n-1}}_{\eta(k_{n-1})}\bar{a}^{k_n}_{\eta(k_n)} &\equiv_{U\alpha[K\smallsetminus\bar{k}]} \bar{a}^{k_0}_{\zeta(k_0)} \dots \bar{a}^{k_{n-1}}_{\zeta(k_{n-1})}\bar{a}^{k_n}_{\eta(k_n)} \\ &\equiv_{U\alpha[K\smallsetminus\bar{k}]} \bar{a}^{k_0}_{\zeta(k_0)} \dots \bar{a}^{k_{n-1}}_{\zeta(k_{n-1})}\bar{a}^{k_n}_{\zeta(k_n)}\,.\end{aligned}$$

(b) Note that indiscernibility of α implies that all index orders I_k are isomorphic. Hence, we may w.l.o.g. assume that $I_k = I$, for some fixed order I. Fix an element $i \in I$. Indiscernibility of α over U implies that the restriction $(\bar{a}^k_i)_{k\in K}$ is also indiscernible over U. By (a) it follows that so is every sequence of the form $(\bar{a}^k_{\eta(k)})_{k\in K}$ with $\eta \in I^K$. $\square$

We obtain the following generalisation of Lemma E5.3.11.

Corollary 4.5. *Suppose that $(\alpha_k)_{k\in K}$ is a family of mutually indiscernible sequences over U. For every set C, there exists a set $C' \equiv_U C$ such that $(\alpha_k)_{k\in K}$ is mutually indiscernible over $U \cup C'$.*

Proof. Suppose that $K = \kappa$ is a cardinal and let $\alpha_k = (\bar{a}^k_i)_{i\in I_k}$. By induction on $k < \kappa$, we use Proposition E5.3.6 to choose an indiscernible sequence $\beta_k = (\bar{b}^k_i)_{i\in I_k}$ over $U \cup C \cup \alpha[>k]\beta[<k]$ such that

$$\operatorname{Av}(\beta_k/U\alpha[>k]\beta[<k]) \supseteq \operatorname{Av}(\alpha_k/U\alpha[>k]\beta[<k])\,.$$

Then it follows by Lemma 4.3 that the family $(\beta_k)_{k\in K}$ is mutually indiscernible over $U \cup C$. As each α_k is indiscernible over $U \cup \alpha[K \smallsetminus \{k\}]$, we have

$$\operatorname{Av}(\beta_k/U\alpha[K\smallsetminus\{k\}]) = \operatorname{Av}(\alpha_k/U\alpha[K\smallsetminus\{k\}])\,.$$

This implies that

$$(\beta_k)_{k\in K} \equiv_U (\alpha_k)_{k\in K}\,.$$

Therefore, there exists an automorphism $\pi \in \operatorname{Aut}\mathbb{M}_U$ mapping one family to the other one. Consequently, $(\alpha_k)_{k\in K}$ is mutually indiscernible over $U \cup \pi[C]$. $\square$

Corollary 4.6. *Let $\alpha = (\alpha_k)_{k\in K}$ be a family of mutually indiscernible sequences $\alpha_k = (\bar{a}^k_i)_{i\in I_k}$ over U. For every family of linear orders $J_k \supseteq I_k$, $k \in K$, there exist sequences $\alpha'_k = (\bar{a}^k_j)_{j\in J_k}$ extending α_k such that the family $(\alpha'_k)_{k\in K}$ is mutually indiscernible over U.*

Proof. As in the preceding corollary, we choose by induction on k an indiscernible sequence $\beta_k = (\bar{b}^k_i)_{i \in J_k}$ over $U \cup \alpha[>k]\beta[<k]$ such that

$$\mathrm{Av}(\beta_k/U\alpha[>k]\beta[<k]) \supseteq \mathrm{Av}(\alpha_k/U\alpha[>k]\beta[<k])\,.$$

Then it follows by Lemma 4.3 that the family $(\beta_k)_{k \in K}$ is mutually indiscernible over U. As each α_k is indiscernible over $U \cup \alpha[K \setminus \{k\}]$, we have

$$\mathrm{Av}(\beta_k|_{I_k}/U\alpha[K \setminus \{k\}]) = \mathrm{Av}(\alpha_k/U\alpha[K \setminus \{k\}])\,.$$

Consequently, there exists an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ mapping each $\beta_k|_{I_k}$ to α_k. The family $(\pi(\beta_k))_{k \in K}$ is the desired extension of α. □

Proposition 4.7. *Let T be a theory without the independence property and let $(\alpha_k)_{k \in K}$ be a family of mutually indiscernible sequences over U. For every set C, there exists a subset $K_o \subseteq K$ of size $|K_o| \leq |T| \oplus |C|$ such that $(\alpha_k)_{k \in K \setminus K_o}$ is mutually indiscernible over $U \cup C$.*

Proof. Suppose that $\alpha_k = (\bar{a}^k_i)_{i \in I_k}$ where each $\bar{a}^k_i = (a^k_{i,j})_{j < \gamma_k}$ is a γ_k-tuple. Let $\mathfrak{M}$ be a model containing U and all sequences α_k, and define

$$\begin{aligned}
P &:= U \cup \{\, a^k_{i,j} \mid k \in K,\ i \in I_k,\ j < \gamma_k \,\}\,,\\
E &:= \{\, \langle a^k_{i,j}, a^k_{i,j'} \rangle \mid k \in K,\ i \in I_k,\ j, j' < \gamma_k \,\}\,,\\
F &:= \{\, \langle a^k_{i,j}, a^k_{i',j'} \rangle \mid k \in K,\ i, i' \in I_k,\ j, j' < \gamma_k \,\}\,,\\
R &:= \{\, \langle a^k_{i,j}, a^k_{i',j} \rangle \mid k \in K,\ i < i' \text{ in } I_k,\ j < \gamma_k \,\}\,.
\end{aligned}$$

Fix an $|M|^+$-saturated elementary extension

$$\langle \mathfrak{M}_+, P_+, U_+, E_+, F_+, R_+ \rangle \succeq \langle \mathfrak{M}, P, U, E, F, R \rangle\,.$$

Using the relations E_+, F_+, and R_+ we see that there are a set $K_+ \supseteq K$, linear orders I^+_k, ordinals γ^+_k, and a family

$$\big(b^k_{i,j}\big)_{k \in K_+, i \in I^+_k,\ j < \gamma^+_k}$$

of elements such that, setting $\bar{b}^k_i := (b^k_{i,j})_{j < \gamma^+_k}$ and $\beta_k := (\bar{b}^k_i)_{i \in I^+_k}$, we have

- $P_+ = U_+ \cup \beta[K_+]$,
- $I^+_k \supseteq I_k$, $\gamma^+_k \geq \gamma_k$, and $b^k_{i,j} = a^k_{i,j}$, for $k \in K$, $i \in I_k$, $j < \gamma_k$,
- the family $(\beta_k)_{k \in K_+}$ is mutually indiscernible over U_+.

By Lemma 1.12, we can find a set $W \subseteq P_+$ of size $|W| \leq |T| \oplus |C|$ such that

$$\bar{a} \equiv_W \bar{a}' \quad \text{implies} \quad \bar{a} \equiv_C \bar{a}'\,, \quad \text{for all } \bar{a}, \bar{a}' \subseteq P\,.$$

We choose a set $K_o \subseteq K$ of size $|K_o| \leq |W| \leq |T| \oplus |C|$ such that $W \subseteq \beta[K_o]$. We claim that the family $(\alpha_k)_{k \in K \setminus K_o}$ is mutually indiscernible over $U \cup C$. Fix $k \in K' := K \setminus K_o$ and let $\bar{\imath}, \bar{\jmath} \in [I_k]^m$. We have to show that

$$\bar{a}^k[\bar{\imath}] \equiv_{UC\alpha[K' \setminus \{k\}]} \bar{a}^k[\bar{\jmath}]\,.$$

Let $\bar{d} \subseteq U \cup \alpha[K' \setminus \{k\}]$ be finite. Since the sequence β_k is indiscernible over $U \cup \beta[K \setminus \{k\}] \supseteq \bar{d}\beta[K_o]$, we have

$$\bar{b}^k[\bar{\imath}] \equiv_{\bar{d}\beta[K_o]} \bar{b}^k[\bar{\jmath}]\,, \quad \text{which implies that} \quad \bar{a}^k[\bar{\imath}]\bar{d} \equiv_W \bar{a}^k[\bar{\jmath}]\bar{d}\,.$$

By choice of W, it follows that $\bar{a}^k[\bar{\imath}]\bar{d} \equiv_C \bar{a}^k[\bar{\jmath}]\bar{d}$. We have shown that

$$\bar{a}^k[\bar{\imath}] \equiv_{C\bar{d}} \bar{a}^k[\bar{\jmath}]\,, \quad \text{for all finite } \bar{d} \subseteq U \cup \alpha[K' \setminus \{k\}]\,.$$

Consequently, $\bar{a}^k[\bar{\imath}] \equiv_{UC\alpha[K' \setminus \{k\}]} \bar{a}^k[\bar{\jmath}]$. □

Dp-rank

After these preparations we can introduce the dp-rank.

Definition 4.8. Let $\Phi(\bar{x})$ be a set of formulae over $\mathbb{M}$ and $U \subseteq \mathbb{M}$ a set of parameters.

(a) The *dp-rank* $\mathrm{rk}_{\mathrm{dp}}(\Phi/U)$ of Φ over U is the least cardinal κ such that, for every tuple $\bar{b}$ realising Φ and every family $(\alpha_i)_{i < \kappa}$ of infinite mutually indiscernible sequences over U, there is some index $i < \kappa$ such that α_i is indiscernible over $U\bar{b}$. If such a cardinal does not exist, we set $\mathrm{rk}_{\mathrm{dp}}(\Phi/U) := \infty$.

(b) For a tuple $\bar{a} \subseteq \mathbb{M}$, we set

$$\mathrm{rk}_{\mathrm{dp}}(\bar{a}/U) := \mathrm{rk}_{\mathrm{dp}}(\mathrm{tp}(\bar{a}/U)/U)\,.$$

Remark. Note that $\mathrm{rk}_{\mathrm{dp}}(\Phi/U) = 0$ if, and only if, Φ is inconsistent.

Example. Let us consider the theory of $\langle \mathbb{Q}, \leq \rangle$. By quantifier-elimination it follows that a family $\alpha = (\alpha_k)_{k \in K}$ of sequences is mutually indiscernible over a set U if, and only if, all tuples in α_k have the same order type over the set $U \cup \alpha[K \setminus \{k\}]$.

Consider a partial type $\Phi(\bar{x})$ with n free variables $\bar{x}$. We claim that

$$\mathrm{rk}_{\mathrm{dp}}(\Phi/\varnothing) \leq n + 1\,.$$

Let $\bar{b}$ be an n-tuple realising Φ and $\alpha = (\alpha_k)_{k \leq n+1}$ a family of infinite mutually indiscernible sequences. For simplicity, let us assume that each α_k is a sequence of singletons. For $i \neq j$, it follows that either $\alpha_i < \alpha_j$ or $\alpha_j < \alpha_i$. Furthermore, for every $i < n$, there is at most one index k such that α_k contains both elements below and above b_i. Therefore, we can find some index $k \leq n + 1$ such that

$$\alpha_k < b_i \quad \text{or} \quad b_i < \alpha_k\,, \quad \text{for all } i < n\,.$$

This implies that α_k is indiscernible over $\bar{b}$.

We start by stating some basic monotonicity properties of the dp-rank.

Lemma 4.9. *Let Φ be a partial type over U. Then*

$$\mathrm{rk}_{\mathrm{dp}}(\Phi/U) = \mathrm{rk}_{\mathrm{dp}}(\Phi/UC)\,, \quad \textit{for every set } C\,.$$

Proof. Let $\kappa := \mathrm{rk}_{\mathrm{dp}}(\Phi/U)$ and consider a tuple $\bar{b}$ realising Φ and a family $(\alpha_k)_{k<\kappa}$ of infinite mutually indiscernible sequences over $U \cup C$. Suppose that $\alpha_k = (\bar{a}^k_i)_{i \in I_k}$ and let $\bar{c}$ be an enumeration of C. Setting $\alpha'_k := (\bar{a}^k_i \bar{c})_{i \in I_k}$, we obtain a family $(\alpha'_k)_{k<\kappa}$ of infinite mutually indiscernible sequences over U. By choice of κ, there exists some index $k < \kappa$ such that α'_k is indiscernible over $U \cup \bar{b}$. Consequently, α_k is indiscernible over $U \cup \bar{b}\bar{c}$. Hence, $\mathrm{rk}_{\mathrm{dp}}(\Phi/UC) \leq \kappa$.

For the converse inequality, let $\lambda < \kappa$. Then there exists a tuple $\bar{b}$ realising Φ and a family $(\alpha_k)_{k<\lambda}$ of infinite mutually indiscernible sequences over U such that no α_k is indiscernible over $U \cup \bar{b}$. By Corollary 4.5, there exists an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ such that the family $(\pi(\alpha_k))_{k<\lambda}$ is mutually indiscernible over $U \cup C$. It follows that the tuple $\pi(\bar{b})$ realises Φ and no sequence $\pi(\alpha_k)$ is indiscernible over $U \cup C \cup \pi(\bar{b})$. Hence, $\mathrm{rk}_{\mathrm{dp}}(\Phi/UC) > \lambda$. □

Corollary 4.10.

(a) $\Phi \subseteq \Psi \quad$ *implies* $\quad \mathrm{rk}_{\mathrm{dp}}(\Phi/U) \geq \mathrm{rk}_{\mathrm{dp}}(\Psi/U)$.

(b) $U \subseteq V \quad$ *implies* $\quad \mathrm{rk}_{\mathrm{dp}}(\bar{a}/U) \geq \mathrm{rk}_{\mathrm{dp}}(\bar{a}/V)$.

Proof. (a) follows immediately from the definition. For (b), note that Lemma 4.9 and (a) implies that

$$\mathrm{rk}_{\mathrm{dp}}(\bar{a}/U) = \mathrm{rk}_{\mathrm{dp}}(\mathrm{tp}(\bar{a}/U)/V) \geq \mathrm{rk}_{\mathrm{dp}}(\bar{a}/V)\,.$$ □

The next proposition collects several alternative characterisations of the dp-rank.

Proposition 4.11. *Let $\Phi(\bar{x})$ be a partial type over U and $\kappa > 0$ a cardinal. The following statements are equivalent:*

(1) $\mathrm{rk}_{\mathrm{dp}}(\Phi/U) \leq \kappa$

(2) *For every tuple $\bar{b}$ realising Φ and every family $(\alpha_k)_{k \in K}$ of infinite mutually indiscernible sequences over U, there is a set $K_0 \subseteq K$ of size $|K_0| < \kappa$ such that, for every $k \in K \setminus K_0$, all elements of α_k have the same type over $U\bar{b}$.*

(3) *For every tuple $\bar b$ realising Φ and every family $(\alpha_k)_{k\in K}$ of infinite mutually indiscernible sequences over U, there is a set $K_o \subseteq K$ of size $|K_o| < \kappa$ such that the subfamily $(\alpha_k)_{k\in K\setminus K_o}$ is mutually indiscernible over $U\bar b$.*

Proof. (3) ⇒ (2) is trivial.

(2) ⇒ (1) Suppose that there exist a tuple $\bar b$ realising $\Phi(\bar x)$ and a family $(\alpha_k)_{k<\kappa}$ of infinite mutually indiscernible sequences $\alpha_k = (\bar a^k_i)_{i\in I_k}$ over U such that no α_k is indiscernible over $U\bar b$. By Corollary 4.6, we may assume that every index order I_k is dense. For each $k < \kappa$, there are indices $\bar\imath, \bar\jmath \in [I_k]^{<\omega}$ such that

$$\bar a^k[\bar\imath] \not\equiv_{U\bar b} \bar a^k[\bar\jmath]\,.$$

Using Lemma E5.3.12 we obtain indices $\bar u^k < s^k < t^k < \bar v^k$ in I_k such that

$$\bar a^k[\bar u^k s^k \bar v^k] \not\equiv_{U\bar b} \bar a^k[\bar u^k t^k \bar v^k]\,.$$

It follows that the family $(\alpha'_k)_{k<\kappa}$ with $\alpha'_k := (\bar a^k[l\bar u^k\bar v^k])_{\bar u^k<l<\bar v^k}$ violates (2).

(1) ⇒ (3) First, we consider the case where κ is infinite. Suppose that there exist a tuple $\bar b$ realising Φ and a family $(\alpha_k)_{k\in K}$ of infinite mutually indiscernible sequences over U such that, for every $K_o \subseteq K$ of size $|K_o| < \kappa$, the subfamily $(\alpha_k)_{k\in K\setminus K_o}$ is not mutually indiscernible over $U \cup \bar b$. By induction on $i < \kappa$, we choose an index $k_i \in K$ and a finite subset $s_i \subseteq K$ as follows. Suppose that we have already defined k_j and s_j, for all $j < i$. Set $S := k[<i] \cup s[<i]$. Then $|S| < \kappa$ and, by assumption, we can find an index $k_i \in K \setminus S$ such that the sequence α_{k_i} is not indiscernible over $U \cup \bar b \cup \alpha[K \setminus (S \cup \{k_i\})]$. Therefore, we can find a finite subset $s_i \subseteq K \setminus (S \cup \{k_i\})$ such that α_{k_i} is not indiscernible over $U \cup \bar b \cup \alpha[s_i]$.

Having defined $(k_i)_{i<\kappa}$ and $(s_i)_{i<\kappa}$, we set

$$C := \bigcup_{i<\kappa} \alpha[s_i]\,.$$

Then the family $(\alpha_{k_i})_{i<\kappa}$ is mutually indiscernible over $U \cup C$, but no sequence α_{k_i} is indiscernible over $U \cup C \cup \bar b$. Consequently, it follows by Lemma 4.9 that $\mathrm{rk}_{\mathrm{dp}}(\Phi/U) = \mathrm{rk}_{\mathrm{dp}}(\Phi/UC) > \kappa$.

It remains to consider the case where $\kappa = n+1$ is finite. Let $(\alpha_k)_{k<\lambda}$ be a family of infinite mutually indiscernible sequences over U and let $\bar b$ be a tuple realising Φ. We construct the desired subset $K_o \subseteq \lambda$ by induction on λ.

If $\lambda \leq n$, we can take $K_o := \lambda$. Hence, suppose that $\lambda = n + m + 1 < \omega$ and that we have already proved the claim for families of size $n + m$. Extending the sequences α_k if necessary, we may assume that they do not have a last element. By induction on $k < \lambda$, we choose a sequence β_k indexed by $\mathbb{Z}$ such that the sequence β^{op} with the reversed ordering is generated by the type $\mathfrak{p}_k := \mathrm{CF}(\alpha_k)$ over $U\bar b\alpha[<\lambda]\beta[<k]$. By Lemma 3.3, the family $(\alpha^+_k)_{k<\lambda}$ with $\alpha^+_k := \alpha_k\beta_k$ is mutually indiscernible over U. As $(\alpha_k)_{k<\lambda}$ is mutually indiscernible over $U\beta[<\lambda]$ and

$$\mathrm{rk}_{\mathrm{dp}}(\Phi/U\beta[<\lambda]) = \mathrm{rk}_{\mathrm{dp}}(\Phi/U) \leq n+1 \leq \lambda\,,$$

we can find an index $k_o < \lambda$ such that α_{k_o} is indiscernible over $U\beta[<\lambda]\bar b$. Furthermore, since $(\alpha^+_k)_{k\in\lambda\setminus\{k_o\}}$ is mutually indiscernible over $U\alpha_{k_o}$, we can use the inductive hypothesis to find a set $H \subseteq \lambda \setminus \{k_o\}$ of size $|H| \leq n$ such that $(\alpha^+_k)_{k\in\lambda\setminus(H\cup\{k_o\})}$ is mutually indiscernible over $U\alpha_{k_o}\bar b$. If the sequence α_{k_o} is indiscernible over $U\bar b\alpha[\lambda \setminus (H \cup \{k_o\})]$, then $(\alpha_k)_{k\in\lambda\setminus H}$ is mutually indiscernible over $U\bar b$ and we are done.

For a contradiction, suppose otherwise. Then there is some finite set $C \subseteq U\bar b\alpha[\lambda \setminus (H \cup \{k_o\})]$ such that α_{k_o} is not indiscernible over C. Let $\bar c_k$ be an enumeration of $C \cap \alpha_k$ and set $C_o := C \cap (U \cup \bar b)$. Since $(\alpha^+_k)_{k\in\lambda\setminus(H\cup\{k_o\})}$ is mutually indiscernible over $U\bar b\alpha_{k_o}$, we can find, for every $k \in \lambda \setminus (H \cup \{k_o\})$, a tuple $\bar d_k \subseteq \beta_k$ such that

$$\bar d_k \equiv_{U\bar b\alpha_{k_o}\alpha^+[\lambda\setminus(H\cup\{k,k_o\})]} \bar c_k\,.$$

It follows that α_{k_o} is not indiscernible over $C_o \cup \bigcup_k \bar d_k \subseteq U\bar b\beta[<\lambda]$. This contradicts our choice of k_o.

It remains to consider the case where λ is an infinite cardinal. For every ordinal $\gamma < \lambda$, we can use the inductive hypothesis to find a set $H_\gamma \subseteq \gamma$ of size $|H_\gamma| \leq n$ such that the family $(\alpha_k)_{k \in \gamma \setminus H_\gamma}$ is mutually indiscernible over $U\bar{b}$. We will construct finite sets $K_0, \dots, K_{n-1} \subseteq \lambda$ and indices $s_0, \dots, s_{n-1} < \lambda$ as follows. Suppose that we have already chosen $K_0, \dots, K_{i-1}$ and $s_0, \dots, s_{i-1}$ such that

$$\{s_0, \dots, s_{i-1}\} \subseteq H_\gamma\,, \quad \text{for arbitrarily large } \gamma\,.$$

If the family $(\alpha_k)_{k \in \lambda \setminus \{s_0, \dots, s_{i-1}\}}$ is mutually indiscernible over $U\bar{b}$, we are done. Otherwise, there exists a finite set $K_i \subseteq \lambda \setminus \{s_0, \dots, s_{i-1}\}$ such that $(\alpha_k)_{k \in K_i}$ is not mutually indiscernible over $U\bar{b}$. By choice of the sets H_γ, we have $K_i \cap H_\gamma \neq \emptyset$, for all $\gamma < \lambda$. As the set K_i is finite, there is therefore some index $s_i \in K_i$ such that

$$\{s_0, \dots, s_{i-1}, s_i\} \subseteq H_\gamma\,, \quad \text{for arbitrarily large } \gamma\,.$$

Having constructed $s_0, \dots, s_{n-1}$ as above, it follows that there are arbitrarily large γ such that $H_\gamma = \{s_0, \dots, s_{n-1}\}$. Hence, there are arbitrarily large $\gamma < \lambda$ such that the family $(\alpha_k)_{k \in \gamma \setminus \{s_0, \dots, s_{n-1}\}}$ is mutually indiscernible over $U\bar{b}$. This implies that $(\alpha_k)_{k \in \lambda \setminus \{s_0, \dots, s_{n-1}\}}$ is also mutually indiscernible over $U\bar{b}$. □

We can use this characterisation to give a straightforward proof that the dp-rank is sub-additive.

Proposition 4.12. $\mathrm{rk}_{\mathrm{dp}}(\bar{a}\bar{b}/U) \oplus 1 \leq \mathrm{rk}_{\mathrm{dp}}(\bar{a}/U) \oplus \mathrm{rk}_{\mathrm{dp}}(\bar{b}/U\bar{a})$.

Proof. Let $\kappa := \mathrm{rk}_{\mathrm{dp}}(\bar{a}/U)$ and $\lambda := \mathrm{rk}_{\mathrm{dp}}(\bar{b}/U\bar{a})$. To show that

$$\mathrm{rk}_{\mathrm{dp}}(\bar{a}\bar{b}/U) \oplus 1 \leq \kappa \oplus \lambda\,,$$

consider a tuple $\bar{a}'\bar{b}' \equiv_U \bar{a}\bar{b}$ and a family $(\alpha_k)_{k \in K}$ of infinite mutually indiscernible sequences over U. According to Proposition 4.11 (3), it is sufficient to find a subset $K' \subseteq K$ of size $|K'| \oplus 1 < \kappa \oplus \lambda$ such that $(\alpha_k)_{k \in K \setminus K'}$ is mutually indiscernible over $U\bar{a}'\bar{b}'$.

Note that invariance implies that $\mathrm{rk}_{\mathrm{dp}}(\bar{b}'/U\bar{a}') = \mathrm{rk}_{\mathrm{dp}}(\bar{b}/U\bar{a})$. We use the characterisation in Proposition 4.11 (3) two times: first, to find a subset $K_0 \subseteq K$ of size $|K_0| < \kappa$ such that $(\alpha_k)_{k \in K \setminus K_0}$ is mutually indiscernible over $U \cup \bar{a}'$; and then, to find a subset $K_1 \subseteq K \setminus K_0$ of size $|K_1| < \lambda$ such that $(\alpha_k)_{k \in K \setminus (K_0 \cup K_1)}$ is mutually indiscernible over $U \cup \bar{a}'\bar{b}'$. Since $|K_0 \cup K_1| \oplus 1 < \kappa \oplus \lambda$, the claim follows. □

The dp-rank is well-behaved in theories without the independence properties. In particular, it always exists.

Theorem 4.13. *Let T be a complete first-order theory. The following statements are equivalent:*

(1) *T does not have the independence property.*

(2) $\mathrm{rk}_{\mathrm{dp}}(\Phi/U) \leq |T|^+ \oplus |\bar{x}|^+$*, for every partial type $\Phi(\bar{x})$ with variables $\bar{x}$ and every set U.*

(3) $\mathrm{rk}_{\mathrm{dp}}(\Phi/U) < \infty$*, for every partial type $\Phi(\bar{x})$ and every set U.*

Proof. (1) ⇒ (2) Let $\bar{b}$ be a tuple realising Φ and $(\alpha_k)_{k<\kappa}$ a family of infinite mutually indiscernible sequences over U of size $\kappa := |T|^+ \oplus |\bar{x}|^+$. By Proposition 4.7, there exists a set $K_0 \subseteq \kappa$ of size $|K_0| \leq |T| \oplus |\bar{b}| < \kappa$ such that the family $(\alpha_k)_{k \in \kappa \setminus K_0}$ is mutually indiscernible over $U \cup \bar{b}$. Fix $k \in \kappa \setminus K_0 \neq \emptyset$. Then α_k is indiscernible over $U \cup \bar{b}$.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) Let κ be an infinite cardinal and let $I := \omega \times \kappa$, ordered lexicographically. Suppose that there exists a formula $\varphi(\bar{x}; \bar{y})$ with the independence property. By compactness, there exists a tuple $\bar{b}$ and an indiscernible sequence $(\bar{a}_i)_{i \in I}$ such that

$$\mathbb{M} \vDash \varphi(\bar{a}_i; \bar{b}) \quad \text{iff} \quad i \in \{0\} \times \kappa\,.$$

By Lemma 4.2, the sequences $\alpha_i := (\bar{a}_{\langle i,k \rangle})_{k<\kappa}$ are mutually indiscernible over $\emptyset$, but none of them is indiscernible over $\bar{b}$. This implies that $\mathrm{rk}_{\mathrm{dp}}(\bar{b}/\emptyset) > \kappa$. □

F5. Theories without the array property

1. The array property

In this chapter we consider a property of formulae that generalises both the tree property and the independence property. It is based on families of tuples with a two-dimensional index set.

Definition 1.1. Let γ, δ be ordinals and $\alpha = (\bar{a}_{ij})_{i<\gamma, j<\delta}$ a family of tuples.

(a) The *i-th row* of α is the sequence $\alpha^i := (\bar{a}_{ij})_{j<\delta}$, its *j-th column* is $\alpha_j := (\bar{a}_{ij})_{i<\gamma}$, and its *diagonal* is $(\bar{a}_{ii})_{i<\min\{\gamma,\delta\}}$.

(b) For $I \subseteq \gamma$ and $J \subseteq \delta$, we set

$$\bar{a}[I;J] := \bigcup_{i \in I, j \in J} \bar{a}_{ij}\,.$$

(c) α is *biindiscernible* over a set U if the sequence $(\alpha^i)_{i<\gamma}$ of rows and the sequence $(\alpha_j)_{j<\delta}$ of columns are both indiscernible over U. We call α *strongly indiscernible* over U if, in addition, the sequence $(\alpha^i)_{i<\gamma}$ of rows is mutually indiscernible over U.

We start with presenting two methods to construct strongly indiscernible families.

Lemma 1.2. *Let $\alpha = (\bar{a}_{ij})_{i<\gamma, j<\delta}$ be a family such that the sequence of rows $(\alpha^i)_{i<\gamma}$ is both mutually indiscernible over U and indiscernible over U. Then α is strongly indiscernible.*

Proof. It remains to prove that the sequence of columns $(\alpha_j)_{j<\delta}$ is indiscernible over U. Fix indices $\bar{l} \in [\gamma]^m$ and $\bar{\imath}, \bar{\jmath} \in [\delta]^n$. We claim that

$$\bar{a}[\bar{l}; \bar{\imath}] \equiv_U \bar{a}[\bar{l}; \bar{\jmath}] .$$

Let $s < m$. Since α^{l_s} is indiscernible over $U \cup \bar{a}[\gamma \smallsetminus \{l_s\}; \delta]$, we have

$$\bar{a}[l_s; \bar{\imath}] \equiv_{U\bar{a}[\gamma\smallsetminus\{l_s\};\delta]} \bar{a}[l_s; \bar{\jmath}] ,$$

which implies that

$$\begin{aligned} &\bar{a}[l_0 \dots l_{s-1}; \bar{\imath}]\bar{a}[l_s; \bar{\imath}]\bar{a}[l_{s+1} \dots l_{m-1}; \bar{\jmath}] \\ \equiv_U\ &\bar{a}[l_0 \dots l_{s-1}; \bar{\imath}]\bar{a}[l_s; \bar{\jmath}]\bar{a}[l_{s+1} \dots l_{m-1}; \bar{\jmath}] . \end{aligned}$$

By transitivity, it follows that $\bar{a}[\bar{l}; \bar{\imath}] \equiv_U \bar{a}[\bar{l}; \bar{\jmath}]$. □

The next remark generalises Lemma F4.4.2.

Lemma 1.3. *Let $\beta = (\bar{b}_i)_{i<\delta\gamma}$ be an indiscernible sequence over U and define*

$$\alpha = (\bar{a}_{ij})_{i<\gamma, j<\delta} \quad \text{by} \quad \bar{a}_{ij} := \bar{b}_{\delta i+j} .$$

Then α is strongly indiscernible over U.

Proof. Note that the i-th row

$$\alpha^i = (\bar{a}_{ij})_{j<\delta} = (\bar{b}_{\delta i+j})_{j<\delta}$$

is indiscernible over

$$U \cup \bar{b}[<\delta i] \cup \bar{b}[\geq\delta(i+1)] = U \cup \bigcup_{l \neq i} \alpha^l .$$

By Lemma 1.2, it is therefore sufficient to show that the sequence of rows $(\alpha^i)_{i<\gamma}$ is indiscernible over U. Fix indices $\bar{\imath}, \bar{\jmath} \in [\gamma]^m$ and $\bar{l} \in [\delta]^n$. Then

$$(\bar{b}_{\delta i_s + l_t})_{s<m, t<n} \equiv_U (\bar{b}_{\delta j_s + l_t})_{s<m, t<n}$$

implies that $\bar{a}[\bar{\imath}; \bar{l}] \equiv_U \bar{a}[\bar{\jmath}; \bar{l}]$. □

Using two-dimensional families we can introduce the array property, which generalises the independence property and the tree property.

Definition 1.4. Let $\varphi(\bar{x}; \bar{y})$ be a formula and $k < \omega$.

(a) We say that $\varphi(\bar{x}; \bar{y})$ is *consistent over* a family $\beta = (\bar{b}_i)_{i\in I}$ of tuples if the set $\{ \varphi(\bar{x}; \bar{b}_i) \mid i \in I \}$ is consistent. Similarly, we say that φ is *inconsistent* or *k-inconsistent* over β, it the above set is, respectively, inconsistent or k-inconsistent.

(b) A *k-array* for φ is a family $\alpha = (\bar{a}_{ij})_{i,j<\omega}$ of tuples such that

- φ is k-inconsistent over each row $\alpha^i = (\bar{a}_{ij})_{j<\omega}$, $i < \omega$, and
- for every function $\eta : \omega \to \omega$, φ is consistent over the sequence $(\bar{a}_{i\eta(i)})_{i<\omega}$.

(c) We say that φ has the *array property,* or the *tree property of the second kind,* if, for some $k < \omega$, there exists a k-array for φ. A theory T has the *array property* if some formula does.

Let us first note that we can choose a k-array always to be strongly indiscernible.

Lemma 1.5. *A formula $\varphi(\bar{x}; \bar{y})$ has a k-array if, and only if, it has a strongly indiscernible k-array.*

Proof. ($\Leftarrow$) is trivial. For ($\Rightarrow$), suppose that the formula φ has a k-array $\alpha = (\bar{a}_{ij})_{i,j<\omega}$ with rows $(\alpha^i)_{i<\omega}$. By induction on i, we use Proposition E5.3.6 to choose an indiscernible sequence $\beta^i = (\bar{b}_{ij})_{j<\omega}$ over $\alpha[>i]\beta[<i]$ such that

$$\mathrm{Av}(\beta^i/\alpha[>i]\beta[<i]) \supseteq \mathrm{Av}(\alpha^i/\alpha[>i]\beta[<i]) .$$

According to Lemma F4.4.3, the family $(\beta^i)_{i<\omega}$ is mutually indiscernible. Furthermore, the k-inconsistency of $\{ \varphi(\bar{x}; \bar{a}_{ij}) \mid j < \omega \}$ implies the k-inconsistency of $\{ \varphi(\bar{x}; \bar{b}_{ij}) \mid j < \omega \}$.

To show that all sets of the form $\{ \varphi(\bar{x}; \bar{b}_{i\eta(i)}) \mid i < \omega \}$ are consistent, it is sufficient by compactness to prove that, for every $n < \omega$ and every

$\eta : [n] \to \omega$, there exists some tuple $\bar{c}$ with

$$\mathbb{M} \vDash \bigwedge_{i<n} \varphi(\bar{c}; \bar{b}_{i\eta(i)}) .$$

To do so, we prove by induction on $m \leq n$, that, for every function $\eta : [n] \to \omega$, there is some tuple $\bar{c}$ with

$$\mathbb{M} \vDash \bigwedge_{i<m} \varphi(\bar{c}; \bar{b}_{i\eta(i)}) \wedge \bigwedge_{m\leq i<n} \varphi(\bar{c}; \bar{a}_{i\eta(i)}) .$$

For $m = 0$, the existence of $\bar{c}$ follows by choice of the $\bar{a}_{ij}$. For the inductive step, suppose that, for every $\eta : [n] \to \omega$, we have already found a tuple $\bar{c}$ such that

$$\mathbb{M} \vDash \psi_\eta(\bar{c}; \bar{a}_{m\eta(m)}) ,$$

where

$$\psi_\eta(\bar{x}; \bar{y}) := \bigwedge_{i<m} \varphi(\bar{x}; \bar{b}_{i\eta(i)}) \wedge \varphi(\bar{x}; \bar{y}) \wedge \bigwedge_{m<i<n} \varphi(\bar{c}; \bar{a}_{i\eta(i)}) .$$

For a given $j < \omega$, we consider the function $\eta' : [n] \to \omega$ with $\eta'(m) := j$ and $\eta'(i) := \eta(i)$, for $i \neq m$. Then $\psi_{\eta'} = \psi_\eta$ and the inductive hypothesis implies that

$$\mathbb{M} \vDash \exists \bar{x} \psi_\eta(\bar{x}; \bar{a}_{mj}) , \quad \text{for every } j < \omega .$$

Hence,

$$\exists \bar{x} \psi_\eta(\bar{x}; \bar{y}) \in \mathrm{Av}(\alpha_m / \alpha[>m]\beta[<m]) \subseteq \mathrm{Av}(\beta_m / \alpha[>m]\beta[<m]) .$$

Consequently, there is some tuple $\bar{c}$ such that

$$\mathbb{M} \vDash \psi_\eta(\bar{c}; \bar{b}_{m\eta(m)}) .$$

We have shown that the family $\beta = (\beta^i)_{i<\omega}$ has all of the desired properties except possibly for biindiscernibility. To conclude the proof, we can use Proposition E5.3.6 to choose an indiscernible sequence $\beta' = (\beta'^i)_{i<\omega}$ such that

$$\mathrm{Av}(\beta'/\varnothing) \supseteq \mathrm{Av}(\beta/\varnothing) .$$

By Lemma 1.2, it follows that β' is strongly indiscernible. □

Next we show that the class of theories without the array property generalises both the simple theories and those without the independence property. We start by proving this implication for formulae.

Proposition 1.6. *Every formula with the array property has the tree property and the independence property.*

Proof. Suppose that φ has a k-array $(\bar{a}_{ij})_{i,j<\omega}$. We start by showing that φ has the tree property. We set

$$\bar{c}_{\langle\rangle} := \bar{a}_{00} \quad \text{and} \quad \bar{c}_w := \bar{a}_{n w_{n-1}} , \quad \text{for } w \in \omega^n,\ n > 0 .$$

Then the family $(\bar{c}_w)_{w\in\omega^{<\omega}}$ is a witness for the tree property of φ since

- for every $\eta \in \omega^\omega$, the set
$$\{ \varphi(\bar{x}; \bar{c}_w) \mid w \prec \eta \} = \{\varphi(\bar{x}; \bar{a}_{00})\} \cup \{ \varphi(\bar{x}; \bar{a}_{(n+1)\eta(n)}) \mid n < \omega \}$$
is consistent and
- for every $w \in \omega^{<\omega}$ of length $n := |w|$, the set
$$\{ \varphi(\bar{x}; \bar{c}_{wi}) \mid i < \omega \} = \{ \varphi(\bar{x}; \bar{a}_{(n+1)i}) \mid i < \omega \}$$
is k-inconsistent.

It remains to check the independence property. By Lemma 1.5, we may assume that α is strongly indiscernible. Let m be the maximal number such that, for some infinite subset $I \subseteq \omega$, there exists a tuple $\bar{c}$ with

$$\mathbb{M} \vDash \varphi(\bar{c}; \bar{a}_{ij}) , \quad \text{for all } i \in I \text{ and } j < m .$$

As φ is k-inconsistent over every column, we have $m < k$. Furthermore, it follows by maximality of m that there exists an infinite subset $J \subseteq I$ such that

$$\mathbb{M} \vDash \neg\varphi(\bar{c}; \bar{a}_{im}), \quad \text{for all } i \in J.$$

Choose a strictly increasing function $g : \omega \to J$ and define $\eta : \omega \to \omega$ by

$$\eta(i) := \begin{cases} 0 & \text{if } i \text{ is even,} \\ m & \text{if } i \text{ is odd.} \end{cases}$$

It follows that

$$\mathbb{M} \vDash \varphi(\bar{c}; \bar{a}_{g(i)\eta(i)}) \quad \text{iff} \quad i \text{ is even.}$$

Since, according to Lemma F4.4.4, the sequence $(\bar{a}_{g(i)\eta(i)})_{i<\omega}$ is indiscernible, it follows by Proposition E5.4.2 that φ has the independence property. □

Thus, theories without the array property generalise both simple theories and theories without the independence property.

Corollary 1.7. *Let T be a complete first-order theory with the array property. Then T is not simple and it has the independence property.*

Our next goal is an alternative characterisation of the array property.

Definition 1.8. Let $\alpha = (\bar{a}_{ij})_{i<\gamma, j<\delta}$ be a family of tuples.

(a) The *transpose* of α is $\alpha^T := (\bar{a}_{ji})_{i<\delta, j<\gamma}$.

(b) The *column k-condensation* of α is the family $\alpha^{(k)} := (\bar{a}'_{ij})_{i<\gamma, j<\delta}$ with

$$\bar{a}'_{ij} := \bar{a}[k * i; j] \quad \text{where} \quad k * i := \langle ki, ki+1, \dots ki+k-1 \rangle.$$

For $\bar{\imath} \in [\gamma]^n$, we similarly set

$$k * \bar{\imath} := (k * i_0) \dots (k * i_{n-1}).$$

(c) For a formula $\varphi(\bar{x}; \bar{y})$, we set

$$\varphi^{(k)}(\bar{x}; \bar{y}_0 \dots \bar{y}_{k-1}) := \bigwedge_{i<k} \varphi(\bar{x}; \bar{y}_i).$$

Remark. Note that a formula φ is consistent over a column α_j if, and only if, $\varphi^{(k)}$ is consistent over the condensed column $\alpha_j^{(k)}$.

Lemma 1.9. *Let $\alpha = (\bar{a}_{ij})_{i<\gamma, j<\delta}$ be a family of tuples and $k < \omega$.*

(a) *If α is biindiscernible over U, then so are α^T and $\alpha^{(k)}$.*

(b) *If α is strongly indiscernible over U, then so is $\alpha^{(k)}$.*

Proof. (a) Clearly, if α is biindiscernible over U, so is α^T. To see that the column k-condensation $\alpha^{(k)} = (\bar{b}_{ij})_{i<\gamma, j<\delta}$ is also biindiscernible over U, note that, for all tuples of indices $\bar{\imath}, \bar{\jmath}, \bar{l}$,

$$\bar{a}[k * \bar{l}; \bar{\imath}] \equiv_U \bar{a}[k * \bar{l}; \bar{\jmath}] \quad \text{implies} \quad \bar{b}[\bar{l}; \bar{\imath}] \equiv_U \bar{b}[\bar{l}; \bar{\jmath}],$$

$$\text{and} \quad \bar{a}[k * \bar{\imath}; \bar{l}] \equiv_U \bar{a}[k * \bar{\jmath}; \bar{l}] \quad \text{implies} \quad \bar{b}[\bar{\imath}; \bar{l}] \equiv_U \bar{b}[\bar{\jmath}; \bar{l}].$$

(b) Suppose that α is strongly indiscernible over U. It follows by (a) that the column k-condensation $\beta := \alpha^{(k)} = (\bar{b}_{ij})_{i<\gamma, j<\delta}$ is biindiscernible over U. To prove that the family $(\beta^i)_{i<\gamma}$ of rows is mutually indiscernible over U, consider indices $\bar{\imath}, \bar{\jmath} \in [\delta]^n$ and set

$$B_l := U \cup \bar{b}[\gamma \smallsetminus \{l\}; \delta].$$

Then $B_l = U \cup \bar{a}[\gamma \smallsetminus k * l; \delta]$ and

$$\bar{a}[k * l; \bar{\imath}] \equiv_{UB_l} \bar{a}[k * l; \bar{\jmath}] \quad \text{implies} \quad \bar{b}[l; \bar{\imath}] \equiv_{UB_l} \bar{b}[l; \bar{\jmath}].$$

Hence, β^l is indiscernible over $U \cup B_l$. □

Lemma 1.10. *Let T be a theory without the array property, $\varphi(\bar{x}; \bar{y})$ a formula, and $\alpha = (\bar{a}_{ij})_{i,j<\omega}$ a biindiscernible family.*

(a) *Suppose that α is strongly indiscernible. If φ is consistent over the 0-th column $\alpha_0 = (\bar{a}_{i0})_{i<\omega}$, it is consistent over all of α.*

(b) *If φ is consistent over the diagonal $(\bar{a}_{ii})_{i<\omega}$ of α, the formula $\varphi^{(k)}$ is consistent over the diagonal $(\bar{b}_{ii})_{i<\omega}$ of the column k-condensation $\alpha^{(k)} = (\bar{b}_{ij})_{i,j<\omega}$.*

Proof. (a) By compactness, it is sufficient to prove that, for every $k < \omega$, φ is consistent over $(\bar{a}_{ij})_{i<k,j<\omega}$. Fix $k < \omega$. By Lemma 1.9, the column k-condensation $\alpha^{(k)} = (\bar{b}_{ij})_{i,j<\omega}$ is also strongly indiscernible. Furthermore, as φ is consistent over $(\bar{a}_{i0})_{i<\omega}$ and $\bar{a}[\omega;0] = \bar{b}[\omega;0]$, it follows that $\varphi^{(k)}$ is consistent over $(\bar{b}_{i0})_{i<\omega}$. By Lemma F4.4.4, this implies that $\varphi^{(k)}$ is consistent over $(\bar{b}_{i,\eta(i)})_{i<\omega}$, for every $\eta : \omega \to \omega$. As $\varphi^{(k)}$ does not have the array property, there therefore exists some $i < \omega$ such that $\varphi^{(k)}$ is consistent over $(\bar{b}_{ij})_{j<\omega}$. By indiscernibility, it follows that it is also consistent over $(\bar{b}_{0j})_{j<\omega}$. This implies that φ is consistent over $(\bar{a}_{ij})_{i<k,j<\omega}$.

(b) We can use Corollary E5.3.10 to extend the sequence $(\alpha^i)_{i<\omega}$ of rows to an indiscernible sequence $(\alpha^i)_{i<\omega^2}$ of length ω^2. Suppose that $\alpha^i = (\bar{a}_{ij})_{j<\omega}$ and set $\bar{c}_{ij} := \bar{a}_{\omega i+j,i}$. By mutual indiscernibility of $(\alpha^i)_i$, we have

$$(\bar{c}_{ij})_{i,j<\omega} = (\bar{a}_{\omega i+j,i})_{i,j<\omega} \equiv (\bar{a}_{\omega i+j,0})_{i,j<\omega} .$$

Furthermore, according to Lemma 1.3, the latter family is strongly indiscernible. Hence, so is $(\bar{c}_{ij})_{i,j<\omega}$. Furthermore, by biindiscernibility of α, we have

$$(\bar{c}_{i0})_{i<\omega} = (\bar{a}_{\omega i,i})_{i<\omega} \equiv (\bar{a}_{ii})_{i<\omega} .$$

Consequently, the consistency of φ over $(\bar{a}_{ii})_{i<\omega}$ implies the consistency of φ over $(\bar{c}_{i0})_{i<\omega}$. It therefore follows by (a) that φ is consistent over $(\bar{c}_{ij})_{i,j<\omega}$. Finally, by biindiscernibility of α, we have

$$(\bar{c}_{ij})_{i<\omega,j<k} = (\bar{a}_{\omega i+j,i})_{i<\omega,j<k} \equiv (\bar{a}_{ki+j,i})_{i<\omega,j<k} .$$

Consequently, φ is consistent over $(\bar{a}_{ki+j,i})_{i<\omega,j<k}$, which implies that $\varphi^{(k)}$ is consistent over $(\bar{b}_{ii})_{i<\omega}$. □

Proposition 1.11. *A theory T does not have the array property if, and only if, for every biindiscernible family $\alpha = (\bar{a}_{ij})_{i,j<\omega}$, the consistency of a formula $\varphi(\bar{x};\bar{y})$ over the diagonal $(\bar{a}_{ii})_{i<\omega}$ implies the consistency of φ over α.*

Proof. ($\Leftarrow$) Suppose that some formula φ has a k-array. By Lemma 1.5, we can choose this k-array to be biindiscernible. It follows that φ is consistent over the diagonal of α, but not over α itself.

($\Rightarrow$) Suppose that T does not have the array property and let α be a biindiscernible family such that φ is consistent over the diagonal of α. By compactness, it is sufficient to prove that, for every $k < \omega$, φ is consistent over $(\bar{a}_{ij})_{i,j<k}$. By Lemma 1.10, $\varphi^{(k)}$ is consistent over the diagonal of $\alpha^{(k)}$. Since $\beta := (\alpha^{(k)})^T$ has the same diagonal, it follows by another application of Lemma 1.10 that $(\varphi^{(k)})^{(k)}$ is consistent over the diagonal of $\beta^{(k)} = (\bar{b}_{ij})_{i,j<\omega}$. In particular, $(\varphi^{(k)})^{(k)}(\bar{x};\bar{b}_{00})$ is consistent. Since $\bar{b}_{00} = (\bar{a}_{ij})_{i,j<k}$ the claim follows. □

As an application, let us show that, in theories without the array property, we can characterise dividing in the following way.

Definition 1.12. A formula $\varphi(\bar{x};\bar{b})$ *array-divides* over a set U if there exists a biindiscernible family $\beta = (\bar{b}_{ij})_{i,j<\omega}$ over U such that $\bar{b}_{00} = \bar{b}$ and φ is inconsistent over β.

Lemma 1.13. *Every formula that divides over U also array-divides over U.*

Proof. Suppose that $\varphi(\bar{x};\bar{b})$ divides over U. Then there exists an indiscernible sequence $\beta = (\bar{b}_i)_{i<\omega}$ over U such that $\bar{b}_0 = \bar{b}$ and φ is k-inconsistent over β. By Corollary E5.3.10, we can extend β to an indiscernible sequence $\beta = (\bar{b}_i)_{i<\omega^2}$ over U of length ω^2. Set $\alpha := (\bar{a}_{ij})_{i,j<\omega}$ with $\bar{a}_{ij} := \bar{b}_{\omega i+j}$. By Lemma 1.3, it follows that α is biindiscernible over U. Furthermore, $\bar{a}_{00} = \bar{b}$ and φ is inconsistent over α. Hence, $\varphi(\bar{x};\bar{b})$ array-divides over U. □

Corollary 1.14. *Let T be a theory without the array property. A formula $\varphi(\bar{x};\bar{b})$ divides over U if, and only if, it array-divides over U.*

Proof. We have proved the implication $(\Rightarrow)$ already in Lemma 1.13. For $(\Leftarrow)$, suppose that $\varphi(\bar{x};\bar{b})$ does not divide over U. To show that it does not array-divide over U, consider a biindiscernible family $\beta = (\bar{b}_{ij})_{i,j<\omega}$ over U such that $\bar{b}_{00} = \bar{b}$. Since the diagonal $(\bar{b}_{ii})_{i<\omega}$ is indiscernible over U, the fact that $\varphi(\bar{x};\bar{b}_{00})$ does not divide over U implies that φ is consistent over $(\bar{b}_{ii})_{i<\omega}$. By Proposition 1.11, it follows that φ is consistent over β. □

2. Forking and dividing

Extension bases

Our first question regarding theories without the array property is over which base sets forking and dividing coincide. For this to be the case, the forking relation should have all the properties of the dividing relation. Therefore, we start by collecting some of them.

Definition 2.1. Let $\overset{0}{\surd}$ and $\overset{1}{\surd}$ be preforking relations and $U \subseteq \mathbb{M}$. We say that *$\overset{0}{\surd}$-forking implies $\overset{1}{\surd}$-forking over U* if every formula that $\overset{0}{\surd}$-forks over U also $\overset{1}{\surd}$-forks over U. Similarly, we say that $\overset{0}{\surd}$ and $\overset{1}{\surd}$ *coincide over U* if we have implications in both directions.

Definition 2.2. Let $\surd$ be an independence relation and $U \subseteq \mathbb{M}$ a set.

(a) We say that $\surd$ has *left extension over a set U* if it satisfies the following axiom:

(LEXT) *Left Extension.* If $A_0 \surd_U B$ and $A_0 \subseteq A_1$ then there is some B' with

$$B' \equiv_{UA_0} B \quad \text{and} \quad A_1 \surd_U B' .$$

(b) U is a $\surd$*-base* if $A \surd_U U$, for all $A \subseteq \mathbb{M}$.

(c) U is a $\surd$*-extension base* if U is a $\surd$-base and $\surd$ has left extension over U.

Let us first note that $\surd$-bases do exist.

Lemma 2.3. (a) *Every set is a $\surd$-base if $\surd$ is one of the relations $\overset{\mathrm{ls}}{\surd}$, $\overset{\mathrm{s}}{\surd}$, or $\overset{\mathrm{d}}{\surd}$.*

(b) *$\overset{\mathrm{u}}{\surd}$ has left extension over every set.*

(c) *Every model is a $\overset{\mathrm{u}}{\surd}$-extension base.*

(d) *Every model is a $\surd$-base for all preforking relations $\surd$.*

Proof. (a) It follows immediately form the definition that $A \overset{\mathrm{s}}{\surd}_U U$, for all sets A and U. As we have seen in Corollary F4.2.22 that $\overset{\mathrm{s}}{\surd} \subseteq \overset{\mathrm{ls}}{\surd}$ it follows that $A \overset{\mathrm{ls}}{\surd}_U U$ as well. For $\overset{\mathrm{d}}{\surd}$, the claim follows immediately from the characterisation in Lemma F3.1.3.

(b) Suppose that $A \overset{\mathrm{u}}{\surd}_U \bar{b}$ and let $C \subseteq \mathbb{M}$. We have to show that there is some tuple $\bar{b}' \equiv_{UA} \bar{b}$ with $AC \overset{\mathrm{u}}{\surd}_U \bar{b}'$. In other words, we have to show that the set

$$\begin{aligned}\Phi(\bar{x}) := {} & \operatorname{tp}(\bar{b}/UA) \\ & \cup \{\, \varphi(\bar{x};\bar{c}) \mid \bar{c} \subseteq UAC \text{ and } \varphi(\bar{x};\bar{y}) \text{ a formula over } U \\ & \qquad \text{such that } \mathbb{M} \vDash \varphi(\bar{b};\bar{d}) \text{ for all } \bar{d} \subseteq U \,\}\end{aligned}$$

is satisfiable. For a contradiction, suppose that Φ is inconsistent. Then we can find a formula $\psi(\bar{x};\bar{a}) \in \operatorname{tp}(\bar{b}/UA)$, finitely many formulae $\varphi_i(\bar{x};\bar{y}_i)$ over U, and parameters $\bar{c}_i \subseteq UAC$ such that

$$\psi(\bar{x};\bar{a}) \vDash \bigvee_{i<n} \neg\varphi_i(\bar{x};\bar{c}_i) \quad \text{and} \quad \mathbb{M} \vDash \varphi_i(\bar{b};\bar{d}) \text{ for all } \bar{d} \subseteq U .$$

W.l.o.g. we may assume that the parameters $\bar{c}_i$ are all of the form $\bar{c}_i = \bar{a}\bar{c}$, for some tuple $\bar{c} \subseteq UAC$ that is disjoint from $\bar{a}$. Hence,

$$\psi(\bar{x};\bar{a}) \vDash \bigvee_{i<n} \neg\varphi_i(\bar{x};\bar{a},\bar{c})$$

and it follows by the Coincidence Lemma that

$$\psi(\bar{x};\bar{y}) \vDash \forall\bar{z} \bigvee_{i<n} \neg\varphi_i(\bar{x};\bar{y},\bar{z})\,.$$

Since $A \overset{\mathrm{u}}{\surd}_U \bar{b}$, there is some tuple $\bar{a}' \subseteq U$ such that $\mathbb{M} \vDash \psi(\bar{b};\bar{a}')$. Fix some tuple $\bar{d} \subseteq U$. Then it follows by the above implication that

$$\mathbb{M} \vDash \bigvee_{i<n} \neg\varphi_i(\bar{b};\bar{a}',\bar{d})\,.$$

Hence, there is some index i with $\mathbb{M} \vDash \neg\varphi_i(\bar{b};\bar{a}',\bar{d})$. As $\bar{a}'\bar{d} \subseteq U$, this contradicts our choice of φ_i.

(c) We have already seen in Lemma F2.3.15 that each model is a $\overset{\mathrm{u}}{\surd}$-base. Hence, the claim follows by (b).

(d) It follows by (c) that every model $\mathfrak{M}$ is a $\overset{\mathrm{u}}{\surd}$-base. Furthermore, we have shown in Theorem F2.3.13 that $\overset{\mathrm{u}}{\surd} \subseteq \surd$. Hence, $\mathfrak{M}$ is also a $\surd$-base. □

The reason we are interested in extension bases is the following result.

Lemma 2.4. *If forking equals dividing over U, then U is a $\overset{\mathrm{f}}{\surd}$-extension base.*

Proof. As forking equals dividing over U, it is sufficient to show that U is a $\overset{\mathrm{d}}{\surd}$-extension base. We have already shown in Lemma 2.3 that U is a $\overset{\mathrm{d}}{\surd}$-base. It therefore remains to show that $\overset{\mathrm{d}}{\surd}$ has left extension over U.

Suppose that $\bar{a} \overset{\mathrm{d}}{\surd}_U \bar{b}$ and let $\bar{c} \subseteq \mathbb{M}$. To find some tuple $\bar{b}' \equiv_{U\bar{a}} \bar{b}$ with $\bar{a}\bar{c} \overset{\mathrm{d}}{\surd}_U \bar{b}'$, we set $\mathfrak{p} := \mathrm{tp}(\bar{b}/U\bar{a})$ and

$$\Phi(\bar{x}) := \mathfrak{p}(\bar{x}) \cup \{\, \neg\varphi(\bar{x},\bar{a},\bar{c}) \mid \varphi(\bar{b},\bar{y},\bar{z}) \text{ divides over } U \,\}\,.$$

Clearly, every tuple $\bar{b}'$ realising $\Phi(\bar{x})$ has the desired properties. Hence, it remains to prove that Φ is consistent.

For a contradiction, suppose otherwise. Then

$$\mathfrak{p} \vDash \bigvee_{i<n} \varphi_i(\bar{x},\bar{a},\bar{c})\,,$$

where each formula $\varphi_i(\bar{b},\bar{y},\bar{z})$ divides over U. In particular, the disjunction

$$\psi(\bar{b},\bar{y},\bar{z}) := \bigvee_{i<n} \varphi_i(\bar{b},\bar{y},\bar{z})$$

forks over U. By assumption, this implies that ψ also divides over U. Thus, there exists an indiscernible sequence $\beta = (\bar{b}_i)_{i<\omega}$ over U such that $\bar{b}_0 = \bar{b}$ and $\{\, \psi(\bar{b}_i,\bar{y},\bar{z}) \mid i < \omega \,\}$ is k-inconsistent, for some $k < \omega$. By Lemma E5.3.11, we can find a sequence $\beta' \equiv_{U\bar{b}} \beta$ such that $\beta' = (\bar{b}'_i)_{i<\omega}$ is indiscernible over $U\bar{a}$. As $\mathfrak{p}$ is a type over $U\bar{a}$, it follows that

$$\mathrm{tp}(\bar{b}'_i/U\bar{a}) = \mathrm{tp}(\bar{b}'_0/U\bar{a}) = \mathrm{tp}(\bar{b}/U\bar{a}) = \mathfrak{p}\,, \quad \text{for all } i < \omega\,.$$

This implies that $\mathbb{M} \vDash \psi(\bar{b}'_i,\bar{a},\bar{c})$, for all i. Thus, the tuple $\bar{c}\bar{a}$ satisfies the set $\{\, \psi(\bar{b}'_i,\bar{y},\bar{z}) \mid i < \omega \,\}$, which is k-inconsistent by choice of β'. A contradiction. □

Quasi-dividing and the Broom Lemma

Before attacking the questions of when forking and dividing coincide, we take a look at a weakening of dividing called *quasi-dividing.*

Definition 2.5. A formula $\varphi(\bar{x};\bar{b})$ *quasi-divides* over a set U if there are tuples $\bar{b}_0,\dots,\bar{b}_{n-1}$, for some $n < \omega$, such that

$$\bar{b}_i \equiv_U \bar{b} \quad \text{and} \quad \{\, \varphi(\bar{x};\bar{b}_i) \mid i < n \,\} \text{ is inconsistent.}$$

Lemma 2.6. *Dividing implies quasi-dividing.*

Proof. Suppose that $\varphi(\bar{x};\bar{b})$ divides over U. Then there is a sequence $(\bar{b}_i)_{i<\omega}$ such that $\bar{b}_i \equiv_U \bar{b}$ and $\{\, \varphi(\bar{x};\bar{b}_i) \mid i < \omega \,\}$ is k-inconsistent, for some $k < \omega$. Consequently, the tuples $\bar{b}_0,\dots,\bar{b}_{k-1}$ show that $\varphi(\bar{x};\bar{b})$ quasi-divides over U. □

We start with a technical lemma that, given a forking relation with left extension, constructs something like a Morley sequence for the inverse relation.

Lemma 2.7. *Let $\surd$ be a forking relation with left extension over a set U, $\beta = (\bar b_n)_{n<\omega}$ an indiscernible sequence over $U \cup C$, and $\bar a$ a tuple such that*

$$C \surd_U \bar a\beta \quad \textit{and} \quad \bar b_n \surd_U \bar a\bar b[<n]\,, \quad \textit{for all } 0 < n < \omega\,.$$

For every number $k < \omega$, there exists a sequence $\alpha = (\bar a_i)_{i<k}$ such that $\bar a_0 = \bar a$ and, for all $i < k$,

$$\bar a_i\bar b_i \equiv_{UC} \bar a\bar b_0 \quad \textit{and} \quad C\bar a_{k-1}\bar b_{k-1}\dots\bar a_{i+1}\bar b_{i+1} \surd_U \bar a_i\bar b_i\,.$$

Proof. We prove the claim by induction on k. For $k = 0$, there is nothing to do. For the inductive step, suppose that we have already found a sequence $\alpha' = (\bar a'_i)_{i<k}$ of length k. We will construct one of length $k+1$. Let $\sigma \in \operatorname{Aut}\mathbb{M}_{UC}$ be an automorphism such that $\sigma(\bar b_n) = \bar b_{n+1}$, for all $n < \omega$. Note that $C \surd_U \bar a\bar b_0\dots\bar b_k$ and $\bar b_i \surd_U \bar a\bar b_0\dots\bar b_{i-1}$ implies, by Lemma F2.2.4 and induction on $i < k$, that

$$C\bar b_k\dots\bar b_{k-i+1} \surd_U \bar a\bar b_0\dots\bar b_{k-i}\,.$$

For $i = k$, we obtain

$$C\bar b_k\dots\bar b_1 \surd_U \bar a\bar b_0\,.$$

By (LEXT), we can therefore find tuples $\bar a'\bar b' \equiv_{UC\bar b_k\dots\bar b_1} \bar a\bar b_0$ such that

$$C\bar b_k\dots\bar b_1\sigma(\bar a'_{k-1})\dots\sigma(\bar a'_0) \surd_U \bar a'\bar b'\,.$$

Let $\pi \in \operatorname{Aut}\mathbb{M}_{UC\bar b_k\dots\bar b_1}$ be an automorphism with $\pi(\bar a'\bar b') = \bar a\bar b_0$ and set

$$\bar a_0 := \bar a \quad \text{and} \quad \bar a_{i+1} := \pi(\sigma(\bar a'_i))\,, \quad \text{for } i < k\,.$$

Then invariance implies that

$$C\bar b_k\dots\bar b_1\bar a_k\dots\bar a_1 \surd_U \bar a\bar b_0\,.$$

We claim that the sequence $\alpha := (\bar a_i)_{i<k+1}$ obtained in this way has the desired properties.

Clearly, we have $\bar a_0 = \bar a$. Furthermore, since $\pi(\bar b_i) = \bar b_i$ for $0 < i \le k$, we have

$$\bar a_{i+1}\bar b_{i+1} = \pi(\sigma(\bar a'_i))\bar b_{i+1} \equiv_{UC} \sigma(\bar a'_i)\sigma(\bar b_i) \equiv_{UC} \bar a'_i\bar b_i \equiv_{UC} \bar a\bar b_0\,.$$

For the last condition, note that, for $i < k$,

$$\begin{aligned} &C\bar a'_{k-1}\bar b_{k-1}\dots\bar a'_{i+1}\bar b_{i+1} \surd_U \bar a'_i\bar b_i \\ \Rightarrow\ &C\pi(\sigma(\bar a'_{k-1}\bar b_{k-1}\dots\bar a'_{i+1}\bar b_{i+1})) \surd_U \pi(\sigma(\bar a'_i\bar b_i)) \\ \Rightarrow\ &C\bar a_k\bar b_k\dots\bar a_{i+2}\bar b_{i+2} \surd_U \bar a_{i+1}\bar b_{i+1}\,. \end{aligned}$$

Furthermore, we have already seen above that

$$C\bar a_k\bar b_k\dots\bar a_1\bar b_1 \surd_U \bar a_0\bar b_0\,. \qquad \square$$

The following result is our main technical lemma. Note that, in the case where ψ = false, it states that a formula that forks in a particular way also quasi-divides.

Lemma 2.8 (Broom Lemma). *Let $\surd \subseteq \overset{\mathrm{li}}{\surd}$ be a forking relation with left extension over some set U. Suppose that*

$$\vartheta(\bar x;\bar a) \vDash \psi(\bar x;\bar c) \vee \bigvee_{i<n} \varphi_i(\bar x;\bar b^i)$$

and there are indiscernible sequences $\beta_i = (\bar b^i_j)_{j<\omega}$ over U such that

- $\bar b^i_0 = \bar b^i$ *and $\{\,\varphi_i(\bar x;\bar b^i_j) \mid j < \omega\,\}$ is k-inconsistent, for every $i < n$,*
- $\bar b^i_j \surd_U \beta[<i]\bar b^i[<j]\,, \quad$ *for all $i < n$ and $0 < j < \omega$,*

- $\bar{c} \sqrt{}_U \beta[<n]$.

Then there exist a number $m < \omega$ and tuples $\bar{a}_0, \dots, \bar{a}_{m-1} \subseteq \mathbb{M}$ such that

$$\bigwedge_{i<m} \vartheta(\bar{x}; \bar{a}_i) \vDash \psi(\bar{x}; \bar{c}) \quad \text{and} \quad \bar{a}_i \equiv_U \bar{a}\,, \quad \text{for all } i < m\,.$$

Proof. We prove the claim by induction on n. For $n = 0$, there is nothing to do. For the inductive step, suppose that we have already shown the claim for n. We aim to prove it for $n+1$. According to Proposition F4.2.18, $\bar{c} \sqrt{}_U \beta_0 \dots \beta_n$ implies that each sequence β_i is indiscernible over $U \cup \bar{c}$. Consequently, we can use Lemma 2.7 with $\bar{a} := \beta_0 \dots \beta_{n-1}$ and $\beta := \beta_n$ to construct a sequence $\alpha = (\alpha_i)_{i<k}$ such that

- $\alpha_0 = \beta_0 \dots \beta_{n-1}$,
- $\alpha_i \bar{b}^n_i \equiv_{U\bar{c}} \alpha_0 \bar{b}^n_0$, for all $i < k$,
- $\bar{c}\alpha_{k-1}\bar{b}^n_{k-1} \dots \alpha_{i+1}\bar{b}^n_{i+1} \sqrt{}_U \alpha_i \bar{b}^n_i$, for all $i < k$.

For each $j < k$, we choose an automorphism $\pi_j \in \operatorname{Aut} \mathbb{M}_{U\bar{c}}$ such that $\pi_j(\alpha_0 \bar{b}^n_0) = \alpha_j \bar{b}^n_j$. Then

$$\vartheta(\bar{x}; \pi_j(\bar{a})) \vDash \psi(\bar{x}; \bar{c}) \vee \bigvee_{i<n+1} \varphi_i(\bar{x}; \pi_j(\bar{b}^i))\,.$$

Consequently,

$$\bigwedge_{j<k} \vartheta(\bar{x}; \pi_j(\bar{a})) \vDash \bigwedge_{j<k} \Big[\psi(\bar{x}; \bar{c}) \vee \bigvee_{i<n} \varphi_i(\bar{x}; \pi_j(\bar{b}^i)) \vee \varphi_n(\bar{x}; \pi_j(\bar{b}^n))\Big]\,.$$

This implies that

$$\bigwedge_{j<k} \vartheta(\bar{x}; \pi_j(\bar{a})) \wedge \neg\Big[\psi(\bar{x}; \bar{c}) \vee \bigvee_{i<n} \bigvee_{j<k} \varphi_i(\bar{x}; \pi_j(\bar{b}^i))\Big] \vDash \bigwedge_{j<k} \varphi_n(\bar{x}; \pi_j(\bar{b}^n))\,.$$

Since $\{\, \varphi_n(\bar{x}; \bar{b}^n_j) \mid j < \omega \,\}$ is k-inconsistent and $\pi_j(\bar{b}^n) = \bar{b}^n_j$, it follows that the formula

$$\bigwedge_{j<k} \vartheta(\bar{x}; \pi_j(\bar{a})) \wedge \neg\Big[\psi(\bar{x}; \bar{c}) \vee \bigvee_{i<n} \bigvee_{j<k} \varphi_i(\bar{x}; \pi_j(\bar{b}^i))\Big]$$

is inconsistent. Hence,

$$\bigwedge_{j<k} \vartheta(\bar{x}; \pi_j(\bar{a})) \vDash \psi(\bar{x}; \bar{c}) \vee \bigvee_{i<n} \bigvee_{j<k} \varphi_i(\bar{x}; \pi_j(\bar{b}^i))\,.$$

For $s \leq k$, set

$$\psi_s(\bar{x}; \bar{c}^s) := \psi(\bar{x}; \bar{c}) \vee \bigvee_{i<n} \bigvee_{s\leq j<k} \varphi_i(\bar{x}; \pi_j(\bar{b}^i))\,.$$

By induction on s, we will find tuples $\bar{a}_0, \dots, \bar{a}_{m-1}$ such that

$$\bigwedge_{i<m} \vartheta(\bar{x}; \bar{a}_i) \vDash \psi_s(\bar{x}; \bar{c}^s) \quad \text{and} \quad \bar{a}_i \equiv_U \bar{a}\,, \quad \text{for all } i < m\,.$$

Then the statement of the lemma will follow for $s = k$. For $s = 0$, we can take the tuples $\pi_i(\bar{a})$ from above. For the inductive step, suppose that

$$\bigwedge_{i<m} \vartheta(\bar{x}; \bar{a}_i) \vDash \psi_s(\bar{x}; \bar{c}^s) \quad \text{where} \quad \bar{a}_i \equiv_U \bar{a}\,.$$

Note that

$$\psi_s(\bar{x}; \bar{c}^s) \equiv \psi_{s+1}(\bar{x}; \bar{c}^{s+1}) \vee \bigvee_{i<n} \varphi_i(\bar{x}; \pi_s(\bar{b}^i))$$

and the sequences $\pi_s(\beta_i)$ satisfy

- $\pi_s(\bar{b}^i_0) = \pi_s(\bar{b}^i)$ and $\{\, \varphi_i(\bar{x}; \pi_s(\bar{b}^i_j)) \mid j < \omega \,\}$ is k-inconsistent, for every $i < n$,
- $\pi_s(\bar{b}^i_j) \sqrt{}_U \pi_s(\beta[<i])\pi_s(\bar{b}^i[<j])$, for all $i < n$ and $j < \omega$.

Furthermore, $\bar b^{0} \dots \bar b^{n-1} \subseteq \beta_0 \dots \beta_{n-1} = \alpha_0$ implies

$$\pi_j(\bar b^{0}) \dots \pi_j(\bar b^{n-1}) \subseteq \pi_j(\alpha_0) = \alpha_j\,.$$

Consequently, we have $\bar c^{s+1} \subseteq \bar c\alpha_{k-1} \dots \alpha_{s+1}$ and

$$\bar c\alpha_{k-1} \dots \alpha_{s+1} \sqrt{}_U\, \alpha_s \quad \text{implies} \quad \bar c^{s+1} \sqrt{}_U\, \pi_s(\beta[<n])\,.$$

Therefore, we can use the inductive hypothesis on n to obtain a number $m' < \omega$ and tuples $\bar a_{ij}$, for $i < m$ and $j < m'$, such that $\bar a_{ij} \equiv_U \bar a_i \equiv_U \bar a$ and

$$\bigwedge_{j<m'} \bigwedge_{i<m} \vartheta(\bar x; \bar a_{ij}) \vDash \psi_{s+1}(\bar x; \bar c^{s+1})\,. \qquad \square$$

Remark. Note that we do *not* require that $\bar b^i_0 \sqrt{}_U\, \beta[<i]$. This will be essential in the applications below.

Recall that the Lemma of Kim states that, in a simple theory, every $\sqrt[\mathrm{f}]{}$-Morley sequence is a witness for dividing. The next result contains a similar statement for certain $\sqrt{}$-Morley sequences.

Lemma 2.9. *Let $\sqrt{} \subseteq \sqrt[\mathrm{li}]{}$ be a forking relation, U a $\sqrt{}$-extension base, and $\varphi(\bar x; \bar y)$ a formula without the array property. For every tuple $\bar b$ such that $\varphi(\bar x; \bar b)$ divides over U, there exists a model $\mathfrak{M}$ containing U and a global type $\mathfrak{p}$ extending* $\mathrm{tp}(\bar b/M)$ *such that $\mathfrak{p}$ is $\sqrt{}$-free over U and every sequence generated by $\mathfrak{p}$ over M witnesses that $\varphi(\bar x; \bar b)$ divides over U.*

Proof. Since $\varphi(\bar x; \bar b)$ divides over U, there exists a number $k < \omega$ and an indiscernible sequence $\beta = (\bar b_i)_{i<\omega}$ over U such that $\bar b_0 = \bar b$ and the set $\{\, \varphi(\bar x; \bar b_i) \mid i < \omega \,\}$ is k-inconsistent. Let $\mathfrak{N}$ be a $(|T| \oplus |U|)^+$-saturated and strongly $(|T| \oplus |U|)^+$-homogeneous model containing U. We can use Lemma E5.3.9 to extend β to an indiscernible sequence $\beta' = (\bar b_i)_{i<\lambda}$ over U of length $\lambda := (2^{|T| \oplus |N|})^+$. As $\beta' \sqrt{}_U\, U$, we find a sequence $\beta'' = (\bar b''_i)_{i<\lambda}$ such that $\beta'' \equiv_U \beta'$ and $\beta'' \sqrt{}_U\, N$.

As there are at most $2^{|T| \oplus |N|} < \lambda$ types over N, there exists an infinite subset $I \subseteq \lambda$ such that every tuple $\bar b''_i$ with $i \in I$ has the same type over N.

Let $\mathfrak{q}_0$ be this type and let $\mathfrak{M} \leq \mathfrak{N}$ be some model containing U of size $|M| \leq |T| \oplus |U|$. Choose a strictly increasing function $g : \omega \to I$ and set $\alpha := (\bar b''_{g(i)})_{i<\omega}$.

Let $\mathfrak{q}$ be the type of α over N. Since $\beta'' \sqrt{}_U\, N$ and $\sqrt{} \subseteq \sqrt[\mathrm{li}]{}$, it follows that $\mathfrak{q}_0$ and $\mathfrak{q}$ are $\sqrt[\mathrm{li}]{}$-free over U. By Proposition F4.2.20 (5), this implies that they are $\sqrt[\mathrm{s}]{}$-free over M. By saturation of $\mathfrak{N}$, there exists a sequence $(\alpha_i)_{i<\omega}$ in N that is generated by $\mathfrak{q}$ over M. By Lemma F2.4.14, $(\alpha_i)_{i<\omega}$ is indiscernible over M. Suppose that $\alpha_i = (\bar a^i_n)_{n<\omega}$.

Let $i, j, k < \omega$. As $\mathfrak{q}$ is $\sqrt[\mathrm{s}]{}$-free over M it follows by transitivity that

$$\alpha[>k] \sqrt[\mathrm{s}]{}_M\, \alpha[\leq k]\,.$$

Since $\bar a^k_i$ and $\bar a^k_j$ both realise $\mathfrak{q}_0 \upharpoonright M\alpha[<k]$, we furthermore have

$$\bar a^k_i \equiv_{M\alpha[<k]} \bar a^k_j\,.$$

Consequently, $\alpha[>k] \sqrt[\mathrm{s}]{}_{M\alpha[<k]}\, \alpha_k$ implies that

$$\bar a^k_i \equiv_{M\alpha[<k]\alpha[>k]} \bar a^k_j\,.$$

As in Lemma F4.4.4, it follows that

$$(\bar a^k_{\eta(k)})_{k<\omega} \equiv_M (\bar a^k_0)_{k<\omega}\,, \qquad \text{for all } \eta : \omega \to \omega\,.$$

By Proposition F2.4.3, $\mathfrak{q}_0$ has some global extension $\mathfrak{q}_1$ that is $\sqrt{}$-free over U. Fix a tuple $\bar b'$ realising $\mathfrak{q}_1 \upharpoonright M$. Then $\bar b' \equiv_U \bar b$ and there exists an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ with $\pi(\bar b') = \bar b$. Applying π to $\mathfrak{q}_1$ we obtain a global type $\mathfrak{p}$ extending $\mathrm{tp}(\bar b/\pi[M])$ that is $\sqrt{}$-free over U. We claim that this type $\mathfrak{p}$ and the model $M' := \pi[M]$ have the desired properties.

As $\mathfrak{q}_1$ is $\sqrt{}$-free over U, so is $\mathfrak{p}$. By base monotony it follows that $\mathfrak{p}$ is $\sqrt{}$-free over M. Hence, consider a sequence $(\bar c_i)_{i<\omega}$ generated by $\mathfrak{p}$ over M. As each tuple $\bar c_i$ realises $\mathfrak{p} \upharpoonright U = \mathfrak{q}_1 \upharpoonright U$, we have $\bar c_i \equiv_U \bar b$. Set $\bar d_i := \pi^{-1}(\bar c_i)$. Then the sequence $(\bar d_i)_{i<\omega}$ is generated by $\mathfrak{q}_1$ over M. Since so is the sequence $(\bar a^i_0)_{i<\omega}$, it follows by Lemma F2.4.14 that

$$(\bar d_i)_{i<\omega} \equiv_M (\bar a^i_0)_{i<\omega}\,.$$

Note that $\alpha_i \equiv_M \alpha$ implies that $\{ \varphi(\bar{x};\bar{a}^i_n) \mid n < \omega \}$ is k-inconsistent. If the set $\{ \varphi(\bar{x};\bar{a}^i_0) \mid i < \omega \}$ were consistent, the family $(\bar{a}^i_j)_{i,j<\omega}$ would form a k-array. Since the formula φ does not have the array property, the set $\{ \varphi(\bar{x};\bar{a}^i_0) \mid i < \omega \}$ is therefore inconsistent. By indiscernibility, it follows that it is l-inconsistent, for some l. Hence, so is the set $\{ \varphi(\bar{x};\bar{d}_i) \mid i < \omega \}$ and, applying the automorphism π, also the set $\{ \varphi(\bar{x};\bar{c}_i) \mid i < \omega \}$. □

Using these lemmas we can derive the first step of our proof that forking equals dividing over certain sets.

Lemma 2.10. *Let T be a theory without the array property and $\surd \subseteq \overset{\mathrm{li}}{\surd}$ a forking relation. Then forking implies quasi-dividing over every $\surd$-extension base U.*

Proof. Consider a formula $\varphi(\bar{x};\bar{a})$ that forks over U. By Lemma F2.4.4, there are formulae $\psi_i(\bar{x};\bar{b}_i)$ that divide over U such that $\varphi(\bar{x};\bar{a}) \vDash \bigvee_{i<n} \psi_i(\bar{x};\bar{b}_i)$. By Lemma 2.9, there are models $\mathfrak{M}_i$ and global types $\mathfrak{p}_i$, for $i < n$, such that $\mathfrak{p}_i$ extends $\mathrm{tp}(\bar{b}_i/M)$, $\mathfrak{p}_i$ is $\surd$-free over U, and every sequence generated by $\mathfrak{p}_i$ over M witnesses that $\psi_i(\bar{x};\bar{b}_i)$ divides over U. For $i < n$, we choose a sequence $\beta_i = (\bar{b}^i_j)_{j<\omega}$ generated by $\mathfrak{p}_i$ as follows. We start with $\bar{b}^i_0 := \bar{b}_i$, which realises $\mathfrak{p}_i \upharpoonright M$. For $j > 0$, we choose a tuple $\bar{b}^i_j$ realising $\mathfrak{p}_i \upharpoonright M\beta[<i]\bar{b}^i[<j]$. It follows that

- $\bar{b}^i_0 = \bar{b}_i$ and the set $\{ \varphi_i(\bar{x};\bar{b}^i_j) \mid j < \omega \}$ is k_i-inconsistent, for every $i < n$,
- $\bar{b}^i_j \surd_U \beta[<i]\bar{b}^i[<j]$, for all $i < n$ and $0 < j < \omega$,
- $\varnothing \surd_U \beta[<n]$.

By Lemma 2.8, we can therefore find tuples $\bar{a}_i \equiv_U \bar{a}$, for $i < m$, such that

$$\varphi(\bar{x};\bar{a}) \vDash \mathrm{false} \vee \bigvee_{i<n} \psi_i(\bar{x};\bar{b}_i) \quad \text{implies} \quad \bigwedge_{i<m} \varphi(\bar{x};\bar{a}_i) \vDash \mathrm{false}\,.$$

Consequently, $\varphi(\bar{x};\bar{a})$ quasi-divides over U. □

Strict Lascar invariance

Above we have found a criterion for the fact that forking implies quasi-dividing over a given set. It remains to find conditions showing that quasi-dividing implies dividing. To do so, we introduce the following combination of the relations $\overset{\mathrm{li}}{\surd}$ and $\overset{\mathrm{f}}{\surd}$.

Definition 2.11. For sets $A, B, U \subseteq \mathbb{M}$, we define

$$\begin{aligned} A \overset{\mathrm{fli}}{\surd}_U B \quad &:\text{iff} \quad A \overset{\mathrm{li}}{\surd}_U B \quad \text{and} \quad B \overset{\mathrm{f}}{\surd}_U A\,, \\ A \overset{\mathrm{sli}}{\surd}_U B \quad &:\text{iff} \quad A \;{}^*(\overset{\mathrm{fli}}{\surd})_U\; B\,. \end{aligned}$$

Lemma 2.12. $\bar{a} \overset{\mathrm{sli}}{\surd}_U B$ *if, and only if,* $\mathrm{tp}(\bar{a}/UB)$ *has a global extension* $\mathfrak{p}$ *that is Lascar-invariant over U and such that*

$$BC \overset{\mathrm{f}}{\surd}_U \bar{a}'\,, \quad \text{for all } C \subseteq \mathbb{M} \text{ and all } \bar{a}' \text{ realising } \mathfrak{p} \upharpoonright UBC\,.$$

Proof. ($\Leftarrow$) Let $\mathfrak{p}$ be an extension of $\mathrm{tp}(\bar{a}/UB)$ as above. To show that $\bar{a} \;{}^*(\overset{\mathrm{fli}}{\surd})_U\; B$, we fix some set $C \subseteq \mathbb{M}$. Let $\bar{a}'$ be a tuple realising $\mathfrak{p} \upharpoonright UBC$. Then $\bar{a}' \equiv_{UB} \bar{a}$ and, by choice of $\mathfrak{p}$, we have $\bar{a} \overset{\mathrm{li}}{\surd}_U BC$ and $BC \overset{\mathrm{f}}{\surd}_U \bar{a}$. This implies that $\bar{a} \overset{\mathrm{fli}}{\surd}_U BC$.

($\Rightarrow$) Let $\bar{a} \;{}^*(\overset{\mathrm{fli}}{\surd})_U\; B$. By Proposition F2.4.3, $\mathrm{tp}(\bar{a}/UB)$ has a global extension $\mathfrak{p}$ that is $\overset{\mathrm{fli}}{\surd}$-free over U. As $\overset{\mathrm{fli}}{\surd} \subseteq \overset{\mathrm{li}}{\surd}$, it is also Lascar invariant over U. For the second condition, suppose that $C \subseteq \mathbb{M}$ and let $\bar{a}'$ be a realisation of $\mathfrak{p} \upharpoonright UBC$. Then $\bar{a}' \overset{\mathrm{fli}}{\surd}_U BC$ implies $BC \overset{\mathrm{f}}{\surd}_U \bar{a}'$. □

Lemma 2.13. *The relation* $\overset{\mathrm{fli}}{\surd}$ *satisfies* (INV), (MON), (NOR), *and* (FIN).

Proof. (INV) follows from invariance of $\overset{\mathrm{li}}{\surd}$ and $\overset{\mathrm{f}}{\surd}$.

(MON) Suppose that $A \overset{\mathrm{fli}}{\surd}_U B$ and let $A_0 \subseteq A$ and $B_0 \subseteq B$. Then $A \overset{\mathrm{li}}{\surd}_U B$ and $B \overset{\mathrm{f}}{\surd}_U A$ and it follows that $A_0 \overset{\mathrm{li}}{\surd}_U B_0$ and $B_0 \overset{\mathrm{f}}{\surd}_U A_0$. Hence, $A_0 \overset{\mathrm{fli}}{\surd}_U B_0$.

(NOR) Suppose that $A \overset{\mathrm{fli}}{\surd}_U B$. Then $A \overset{\mathrm{li}}{\surd}_U B$ and $B \overset{\mathrm{f}}{\surd}_U A$ and it follows that $AU \overset{\mathrm{li}}{\surd}_U BU$ and $BU \overset{\mathrm{f}}{\surd}_U AU$. Hence, $AU \overset{\mathrm{fli}}{\surd}_U BU$.

(FIN) Suppose that $A_0 \overset{\text{fli}}{\surd}_U B$, for all finite $A_0 \subseteq A$. Then $A_0 \overset{\text{li}}{\surd}_U B$ and $B \overset{\text{f}}{\surd}_U A_0$, for all finite $A_0 \subseteq A$. This implies that $A \overset{\text{li}}{\surd}_U B$ and $B \overset{\text{f}}{\surd}_U A$. Hence, $A \overset{\text{fli}}{\surd}_U B$. □

Corollary 2.14. *The relation* $\overset{\text{sli}}{\surd}$ *satisfies* (INV), (MON), (NOR), (FIN), *and* (EXT).

Proof. A closer look at the proof of Proposition F2.4.5 reveals that, to establish the axioms (INV), (MON), (NOR), (FIN), and (EXT) for the relation $\overset{*}{\surd}$, we only need to assume that $\surd$ satisfies (INV), (MON), (NOR), and (FIN). □

The reason we are interested in the relation $\overset{\text{sli}}{\surd}$ is the following variant of the Lemma of Kim for theories with the array property.

Lemma 2.15. *Let T be a theory without the array property, $\varphi(\bar x;\bar b)$ a formula that divides over U, and $(\bar b_n)_{n<\omega}$ a sequence such that*

$$\bar b_n \equiv_U \bar b \quad \text{and} \quad \bar b_n \overset{\text{sli}}{\surd}_U \bar b[<n]\,, \quad \text{for all } n<\omega\,.$$

Then $\{\varphi(\bar x;\bar b_n) \mid n<\omega\}$ is inconsistent.

Proof. Applying a suitable automorphism, we may assume that $\bar b_0 = \bar b$. Since the formula $\varphi(\bar x;\bar b)$ divides over U, there exists an indiscernible sequence $\alpha = (\bar a_i)_{i<\omega}$ such that $\bar a_0 = \bar b$ and $\{\varphi(\bar x;\bar a_i) \mid i<\omega\}$ is k-inconsistent, for some $k<\omega$. By induction on $n<\omega$, we construct a family $(\alpha_j)_{j<n}$ of sequences $\alpha_j = (\bar a^j_i)_{i<\omega}$ such that

- each α_j is indiscernible over $U\alpha[<j]\bar b_{j+1}\dots\bar b_{n-1}$,
- $\alpha_j \equiv_U \alpha$, and $\bar a^j_0 = \bar b_j$.

For $n=1$, we can take the sequence $\alpha_0 := \alpha$. For the inductive step, suppose we have already constructed a family $(\alpha'_j)_{j<n}$ of size n. Since

$$\bar b_n \overset{\text{sli}}{\surd}_U \bar b[<n]\,,$$

we can use (EXT) to find a family $(\alpha''_i)_{i<n}$ such that

$$\alpha''[<n] \equiv_{U\bar b[<n]} \alpha'[<n] \quad \text{and} \quad \bar b_n \overset{\text{sli}}{\surd}_U \alpha''[<n]\,.$$

Since $\bar b_n \equiv_U \bar b$, there is some indiscernible sequence $\alpha'_n \equiv_U \alpha$ starting with $\bar b_n$. Note that $\bar b_n \overset{\text{sli}}{\surd}_U \alpha''[<n]$ implies that $\alpha''[<n] \overset{\text{d}}{\surd}_U \bar b_n$. By Lemma F3.1.3, we can therefore find a sequence $\alpha''_n \equiv_{U\bar b_n} \alpha'_n$ such that α''_n is indiscernible over $\alpha''[<n]$. We claim that the family $(\alpha''_i)_{i<n+1}$ has the desired properties.

Let $i<n$. By construction the sequence α''_i is indiscernible over $U\alpha''[<i]\bar b_{i+1}\dots\bar b_{n-1}$. Furthermore, we have $\bar b_n \overset{\text{li}}{\surd}_U \alpha''[<n]$, which implies that

$$\bar b_n \overset{\text{li}}{\surd}_{U\alpha''[<i]\bar b_{i+1}\dots\bar b_{n-1}} \alpha''_i\,.$$

By Proposition F4.2.18, it therefore follows that α''_i is also indiscernible over $U\alpha''[<i]\bar b_{i+1}\dots\bar b_{n-1}\bar b_n$. Finally, the sequence α''_n is indiscernible over $U\alpha''[<n]$ by construction.

Having constructed sequences $(\alpha_j)_{j<n}$ of length n, for every $n<\omega$, it follows by compactness that there also exists an infinite family $(\alpha_j)_{j<\omega}$ with the same properties.

To conclude the proof suppose, towards a contradiction, that the set $\{\varphi(\bar x;\bar b_n) \mid n<\omega\}$ is consistent. For $\eta:\omega\to\omega$ and $n<\omega$, a straightforward induction on i shows that

$$\bar a^0_{\eta(0)}\dots\bar a^{n-1}_{\eta(n-1)} \equiv_U \bar a^0_{\eta(0)}\dots\bar a^{n-i-1}_{\eta(n-i-1)}\bar a^{n-i}_0\dots\bar a^{n-1}_0\,.$$

This implies that

$$(\bar a^i_{\eta(i)})_{i<\omega} \equiv_U (\bar a^i_0)_{i<\omega} = (\bar b_i)_{i<\omega}\,.$$

Consequently, $\{\varphi(\bar x;\bar a^i_{\eta(i)}) \mid i<\omega\}$ is consistent, for every $\eta:\omega\to\omega$. Furthermore, $\alpha_j \equiv_U \alpha$ implies that $\{\varphi(\bar x;\bar a^i_n) \mid n<\omega\}$ is k-inconsistent, for some k. Consequently, the family $(\bar a^j_i)_{i,j<\omega}$ forms a k-array for φ. A contradiction. □

We obtain our first result for forking equalling dividing over $\overset{\text{sli}}{\surd}$-bases.

Proposition 2.16. *Let T be a theory without the array property and U a $\overset{\text{sli}}{\surd}$-base. Then forking equals dividing over U.*

Proof. Suppose that $\varphi(\bar{x};\bar{a})$ forks over U. Then there exist formulae $\psi_i(\bar{x};\bar{b}_i)$ that divide over U such that $\varphi(\bar{x};\bar{a}) \vDash \bigvee_{i<n} \psi_i(\bar{x};\bar{b}_i)$. Set $\bar{c} := \bar{a}\bar{b}_0 \dots \bar{b}_{n-1}$ and let $\mathfrak{p} := \mathrm{tp}(\bar{c}/U)$. Since $\bar{c} \overset{\text{sli}}{\surd}_U U$ there exists a global type $\mathfrak{q}$ extending $\mathfrak{p}$ that is $\overset{\text{fli}}{\surd}$-free over U. Let $\mathfrak{M}$ be a model containing U and let $\gamma = (\bar{c}_i)_{i<\omega}$ be a sequence generated by $\mathfrak{q}$ over M. Note that, by Proposition F4.2.20 (5), $\mathfrak{q}$ is $\overset{\text{s}}{\surd}$-free over M. Hence, it follows by Lemma F2.4.14, that γ is a $\overset{\text{sli}}{\surd}$-Morley sequence. Suppose that $\bar{c}_i = \bar{a}^i\bar{b}^i_0 \dots \bar{b}^i_{n-1}$. We claim that the set $\{\, \varphi(\bar{x};\bar{a}^i) \mid i < \omega \,\}$ is inconsistent. Since γ is indiscernible and $\bar{a}^i \equiv_U \bar{a}$, this implies that $\varphi(\bar{x};\bar{a})$ divides over U.

For a contradiction, suppose that there exists a tuple $\bar{d}$ realising the above set. Then there exists a function $g : \omega \to [n]$ such that

$$\mathbb{M} \vDash \psi_{g(i)}(\bar{d};\bar{b}^i_{g(i)}), \quad \text{for all } i < \omega .$$

Choose an infinite subset $I \subseteq \omega$ and an index $k < n$ such that $g(i) = k$, for all $i \in I$. It follows that $\{\, \psi_k(\bar{x};\bar{b}^i_k) \mid i < \omega \,\}$ is consistent. This contradicts Lemma 2.15 □

It remains to prove that $\surd$-extension bases are also $\overset{\text{sli}}{\surd}$-bases. We start with a technical lemma.

Lemma 2.17. *Let $\surd$ be a forking relation and U a $\surd$-base such that forking implies quasi-dividing over U.*

(a) *Every type $\mathfrak{p}$ over U has a global extension $\mathfrak{q}$ that is $\surd$-free over U and such that*

$$C \overset{\text{f}}{\surd}_U \bar{a}, \quad \textit{for all } C \subseteq \mathbb{M} \textit{ and all } \bar{a} \textit{ realising } \mathfrak{q} \restriction UC .$$

(b) *Every type $\mathfrak{p}$ over U has a global extension $\mathfrak{q}$ that is $\overset{\text{f}}{\surd}$-free over U and such that*

$$C \surd_U \bar{a}, \quad \textit{for all } C \subseteq \mathbb{M} \textit{ and all } \bar{a} \textit{ realising } \mathfrak{q} \restriction UC .$$

Proof. (a) Fix a tuple $\bar{a}$ realising $\mathfrak{p}$ and set

$$\begin{aligned} \Phi(\bar{x}) := \mathfrak{p}(\bar{x}) &\cup \{\, \neg\varphi(\bar{x};\bar{b}) \mid \bar{b} \subseteq \mathbb{M},\ \varphi(\bar{a};\bar{y}) \ \overset{\text{f}}{\surd}\text{-forks over } U \,\} \\ &\cup \{\, \neg\psi(\bar{x};\bar{b}) \mid \bar{b} \subseteq \mathbb{M},\ \psi(\bar{x};\bar{b}) \ \surd\text{-forks over } U \,\} . \end{aligned}$$

By (DEF), every global type containing Φ has the desired properties. Hence, it remains to show that Φ is satisfiable.

For a contradiction, suppose otherwise. Then there exist formulae $\varphi_i(\bar{x};\bar{y}_i)$, $i < m$, and $\psi_i(\bar{x};\bar{z}_i)$, $i < n$, and corresponding parameters $\bar{b}_0, \dots, \bar{b}_{m-1}, \bar{b}'_0, \dots, \bar{b}'_{n-1}$ such that

$$\mathfrak{p} \vDash \bigvee_{i<m} \varphi_i(\bar{x};\bar{b}_i) \vee \bigvee_{i<n} \psi_i(\bar{x};\bar{b}'_i) ,$$

each $\varphi_i(\bar{a};\bar{y})$ $\overset{\text{f}}{\surd}$-forks over U, and each $\psi_i(\bar{x};\bar{b}'_i)$ $\surd$-forks over U. As the disjunction $\bigvee_{i<m} \varphi_i(\bar{a};\bar{y}_i)$ also $\overset{\text{f}}{\surd}$-forks over U, we may assume that $m = 1$.

Since forking implies quasi-dividing over U, there are parameters $\bar{a}_0, \dots, \bar{a}_{k-1}$ such that $\bar{a}_i \equiv_U \bar{a}$ and the set $\{\, \varphi_0(\bar{a}_i;\bar{y}) \mid i < k \,\}$ is inconsistent. Set $\bar{c} := \bar{a}_0 \dots \bar{a}_{k-1}$ and $\mathfrak{r}(\bar{x}_0, \dots, \bar{x}_{k-1}) := \mathrm{tp}(\bar{c}/U)$. Then

$$\mathfrak{r} \restriction \bar{x}_j \vDash \varphi_0(\bar{x}_j;\bar{b}_0) \vee \bigvee_{i<n} \psi_i(\bar{x}_j;\bar{b}'_i) .$$

Hence,

$$\mathfrak{r} \vDash \bigwedge_{j<k} \Big[\varphi_0(\bar{x}_j;\bar{b}_0) \vee \bigvee_{i<n} \psi_i(\bar{x}_j;\bar{b}'_i) \Big] .$$

Consequently,

$$\mathfrak{r} \vDash \neg \bigwedge_{j<k} \varphi_0(\bar{x}_j;\bar{b}_0) \quad \text{implies that} \quad \mathfrak{r} \vDash \bigvee_{j<k} \bigvee_{i<n} \psi_i(\bar{x}_j;\bar{b}'_i) .$$

Since U is a $\sqrt{}$-base, we have $\bar c \sqrt{}_U U$. Hence, there is some tuple $\bar c' \equiv_U \bar c$ such that $\bar c' \sqrt{}_U \bar b'_0 \dots \bar b'_{n-1}$. As $\bar c' = \bar c'_0 \dots \bar c'_{k-1}$ realises $\mathfrak{r}$, there are indices $j < k$ and $i < n$ such that $\mathbb{M} \vDash \psi_i(\bar c'_j; \bar b'_i)$. But this implies that $\bar c'_j \not\sqrt{}_U \bar b'_i$. A contradiction.

(b) The proof is similar to the one above. Fix a tuple $\bar a$ realising $\mathfrak{p}$ and set

$$\begin{aligned}\Phi(\bar x) := \mathfrak{p}(\bar x) \cup \{ \neg\varphi(\bar x; \bar b) \mid \bar b \subseteq \mathbb{M},\ \varphi(\bar x; \bar b)\ \overset{\mathrm{f}}{\sqrt{}}\text{-forks over } U \} \\ \cup \{ \neg\psi(\bar x; \bar b) \mid \bar b \subseteq \mathbb{M},\ \psi(\bar a; \bar y) \sqrt{}\text{-forks over } U \}.\end{aligned}$$

Suppose that Φ is inconsistent. Then we can find formulae $\varphi_i(\bar x; \bar y_i)$, $i < m$, and $\psi_i(\bar x; \bar z_i)$, $i < n$, and parameters $\bar b_0, \dots, \bar b_{m-1}, \bar b'_0, \dots, \bar b'_{n-1}$ such that

$$\mathfrak{p} \vDash \bigvee_{i<m} \varphi_i(\bar x; \bar b_i) \vee \bigvee_{i<n} \psi_i(\bar x; \bar b'_i),$$

each $\varphi_i(\bar x; \bar b_i)$ $\overset{\mathrm{f}}{\sqrt{}}$-forks over U, and each $\psi_i(\bar a; \bar z_i)$ $\sqrt{}$-forks over U. As above, we may assume that $m = 1$.

Since forking implies quasi-dividing over U, there are parameters $\bar c_0, \dots, \bar c_{k-1}$ such that $\bar c_j \equiv_U \bar b_0$ and the set $\{ \varphi_0(\bar x; \bar c_j) \mid j < k \}$ is inconsistent. Choose tuples $\bar d_{ji}$ such that

$$\bar c_j \bar d_{j0} \dots \bar d_{j(n-1)} \equiv_U \bar b_0 \bar b'_0 \dots \bar b'_{n-1}, \quad \text{for } j < k.$$

Since the type $\mathfrak{p}$ is over U, it follows by invariance that

$$\mathfrak{p} \vDash \varphi_0(\bar x; \bar c_j) \vee \bigvee_{i<n} \psi_i(\bar x; \bar d_{ji}), \quad \text{for all } j < k.$$

As above, this implies that

$$\mathfrak{p} \vDash \bigvee_{j<k} \bigvee_{i<n} \psi_i(\bar x; \bar d_{ji}).$$

Set $\bar d := (\bar d_{ji})_{j<k, i<n}$. As U is a $\sqrt{}$-base, we have $\bar d \sqrt{}_U U$. Consequently, there is some tuple $\bar d' \equiv_U \bar d$ such that

$$\bar d' \sqrt{}_U \bar a.$$

Since $\bar a$ realises $\mathfrak{p}$, there are indices $j < k$ and $i < n$ such that

$$\mathbb{M} \vDash \psi_i(\bar a; \bar d'_{ji}).$$

But this implies that $\bar d'_{ji} \not\sqrt{}_U \bar a$. A contradiction. □

Corollary 2.18. *Let T be a theory without the array property and U a $\overset{\mathrm{li}}{\sqrt{}}$-base such that forking implies quasi-dividing over U. Then U is a $\overset{\mathrm{sli}}{\sqrt{}}$-base.*

Proof. Fix a tuple $\bar a \subseteq \mathbb{M}$. We can use Lemma 2.17 to find a global extension $\mathfrak{q}$ of $\operatorname{tp}(\bar a/U)$ that is $\overset{\mathrm{li}}{\sqrt{}}$-free over U and such that $C \overset{\mathrm{f}}{\sqrt{}}_U \bar a'$, for all sets $C \subseteq \mathbb{M}$ and all tuples $\bar a'$ realising $\mathfrak{q} \upharpoonright UC$. By Lemma 2.12, this implies that $\bar a \overset{\mathrm{sli}}{\sqrt{}}_U U$. □

Corollary 2.19. *Let T be a theory without the array property and $\sqrt{} \subseteq \overset{\mathrm{li}}{\sqrt{}}$ a forking relation. Every $\sqrt{}$-extension base is a $\overset{\mathrm{sli}}{\sqrt{}}$-base.*

Proof. Let U be a $\sqrt{}$-extension base. We have proved in Lemma 2.10 that forking implies quasi-dividing over U. Furthermore, since $\sqrt{} \subseteq \overset{\mathrm{li}}{\sqrt{}}$ and U is a $\sqrt{}$-base, it is also a $\overset{\mathrm{li}}{\sqrt{}}$-base. Consequently, the claim follows by Corollary 2.18. □

Proposition 2.20. *Let T be a theory without the array property. Then forking equals dividing over every set that is a $\sqrt{}$-extension base, for some forking relation $\sqrt{} \subseteq \overset{\mathrm{li}}{\sqrt{}}$.*

Proof. By Corollary 2.19, every $\sqrt{}$-extension base is a $\overset{\mathrm{sli}}{\sqrt{}}$-base. Hence, the claim follows by Proposition 2.16. □

Corollary 2.21. *Let T be a theory without the array property. Then forking equals dividing over every model M.*

Proof. We have seen in Lemma 2.3 (c) that every model is a $\overset{\mathrm{u}}{\sqrt{}}$-extension base. Consequently, the claim follows by Proposition 2.20. □

Combining the above results we obtain the following characterisation of those sets over which forking equals dividing.

Theorem 2.22 (Chernikov, Kaplan). *Let T be a theory without the array property and $U \subseteq \mathbb{M}$ be a set. The following statements are equivalent.*

(1) *Forking equals dividing over U.*

(2) *U is a $\overset{\mathrm{f}}{\surd}$-base.*

(3) *$\overset{\mathrm{f}}{\surd}$ has left extension over U.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) follow by Lemma 2.4. Conversely, suppose that (2) or (3) holds. Let $\varphi(\bar{x};\bar{b})$ be a formula that forks over U. To show that $\varphi(\bar{x};\bar{b})$ also divides over U, we fix a model $\mathfrak{M}$ containing U.

If (2) holds, we have $M \overset{\mathrm{f}}{\surd}_U U$ which, by (EXT), implies that there is some model $M' \equiv_U M$ with $M' \overset{\mathrm{f}}{\surd}_U \bar{b}$.

If (3) holds, we have $U \overset{\mathrm{f}}{\surd}_U \bar{b}$ which, by (LEXT), implies that there is some model $M' \equiv_U M$ with $M' \overset{\mathrm{f}}{\surd}_U \bar{b}$.

Thus, in both cases we have found a model M' such that $M' \overset{\mathrm{f}}{\surd}_U \bar{b}$. We claim that $\varphi(\bar{x};\bar{b})$ also forks over M'. Since forking equals dividing over models, it then follows that $\varphi(\bar{x};\bar{b})$ divides over M'. In particular, it divides over U.

To prove the claim suppose, for a contradiction, that $\varphi(\bar{x};\bar{b})$ does not fork over M'. Then we have $\bar{a} \overset{\mathrm{f}}{\surd}_{M'} \bar{b}$, for every tuple $\bar{a}$ satisfying $\varphi(\bar{x};\bar{b})$. By (LTR), this implies that $\bar{a}M' \overset{\mathrm{f}}{\surd}_U \bar{b}$, which contradicts the fact that $\varphi(\bar{x};\bar{b})$ forks over U. □

Corollary 2.23. *Let T be a theory without the array property.*

(a) *A set U is a $\overset{\mathrm{sli}}{\surd}$-base if, and only if, it is a $\overset{\mathrm{li}}{\surd}$-base.*

(b) *Forking equals dividing over every $\overset{\mathrm{li}}{\surd}$-base.*

Proof. (b) Let U be a $\overset{\mathrm{li}}{\surd}$-base. Since $\overset{\mathrm{li}}{\surd} \subseteq \overset{\mathrm{f}}{\surd}$, it is also a $\overset{\mathrm{f}}{\surd}$-base. By Theorem 2.22, it follows that forking equals dividing over U.

(a) The implication (⇒) follows by the inclusion $\overset{\mathrm{sli}}{\surd} \subseteq \overset{\mathrm{li}}{\surd}$. For (⇐), let U be a $\overset{\mathrm{li}}{\surd}$-base. By (b), forking equals dividing over U. Since dividing implies quasi-dividing, it follows that forking implies quasi-dividing over U. By Corollary 2.18, it follows that U is a $\overset{\mathrm{sli}}{\surd}$-base. □

3. *The Independence Theorem*

The Independence Theorem contains a characterisation of simple theories in terms of a certain property of the forking relation. A weaker version of this property also holds for theories without the array property. In this section we will present the weak version, use it to derive the strong one, and show that the latter characterises simple theories.

The chain condition

Before turning to the Independence Theorem itself, we first consider a closely related property called the *chain condition.*

Definition 3.1. A preforking relation $\surd$ satisfies the *chain condition* over a set $U \subseteq \mathbb{M}$ if, for every indiscernible sequence $(\bar{b}_i)_{i\in I}$ over U and every set of formulae $\Phi(\bar{x};\bar{y})$ such that, for some $i_0 \in I$, the set $\Phi(\bar{x};\bar{b}_{i_0})$ does not $\surd$-fork over U, the union $\bigcup_{i\in I} \Phi(\bar{x};\bar{b}_i)$ also does not $\surd$-fork over U.

The chain condition can be characterised is several equivalent ways. The following list is somewhat parallel to the characterisation of dividing in Lemma F3.1.3.

Proposition 3.2. *Let $\surd$ be a forking relation and $U \subseteq \mathbb{M}$ a set of parameters. The following statements are equivalent.*

(1) *$\surd$ satisfies the chain condition over U.*

(2) *If a formula $\varphi(\bar{x};\bar{b})$ does not $\surd$-fork over U and $\bar{b} \approx^{\mathrm{ls}}_U \bar{b}'$, then $\varphi(\bar{x};\bar{b}) \wedge \varphi(\bar{x};\bar{b}')$ also does not $\surd$-fork over U.*

(3) *For every cardinal λ, there exists a cardinal κ such that, for every partial type $\mathfrak{p}$ over U and every family $(\mathfrak{q}_i)_{i<\kappa}$ of partial types of size $|\mathfrak{q}_i| < \lambda$ such that no $\mathfrak{p} \cup \mathfrak{q}_i$ $\surd$-forks over U, there are indices $i < j$ such that $\mathfrak{p} \cup \mathfrak{q}_i \cup \mathfrak{q}_j$ does not $\surd$-fork over U.*

(4) *For every indiscernible sequence $\beta = (\bar b_i)_{i<\omega}$ over U and every tuple $\bar a \surd_U \bar b_0$, there exists a sequence $\beta' \equiv_{U\bar b_0} \beta$ such that β' is indiscernible over $U\bar a$ and $\bar a \surd_U \beta'$.*

Proof. (2) ⇒ (3) By Corollary F4.2.9, there exists a cardinal κ such that, for every sequence $(\bar b_i)_{i<\kappa}$ of tuples of size $|\bar b_i| < \lambda$, there are indices $i < j$ such that $\bar b_i \approx^{\mathrm{ls}}_U \bar b_j$. Increasing κ, if necessary, we may ensure that κ is larger than the number of sets of formulae of size less than λ. We claim that this cardinal κ has the desired properties.

Let $\mathfrak{p}$ and $(\mathfrak{q}_i)_{i<\kappa}$ be types as above. Then there exists a subset $I \subseteq \kappa$ of size $|I| = \kappa$, a set $\Phi(\bar x; \bar y)$ of formulae (without parameters), and tuples $\bar b_i \in \mathbb{M}^{<\lambda}$ such that

$$\mathfrak{q}_i(\bar x) = \Phi(\bar x; \bar b_i)\,, \quad \text{for all } i \in I\,.$$

By choice of κ, we can find indices $i < j$ in I such that $\bar b_i \approx^{\mathrm{ls}}_U \bar b_j$. We claim that the type

$$\mathfrak{p} \cup \mathfrak{q}_i \cup \mathfrak{q}_j = \mathfrak{p}(\bar x) \cup \Phi(\bar x; \bar b_i) \cup \Phi(\bar x; \bar b_j)$$

does not $\surd$-fork over U.

For a contradiction, suppose otherwise. By compactness, we can then find finite sets $\Psi_0 \subseteq \mathfrak{p}$ and $\Phi_0 \subseteq \Phi$ such that

$$\Psi_0(\bar x) \cup \Phi_0(\bar x; \bar b_i) \cup \Phi_0(\bar x; \bar b_j) \quad \surd\text{-forks over } U\,.$$

Setting

$$\varphi(\bar x; \bar y) := \bigwedge \Psi_0(\bar x) \wedge \bigwedge \Phi_0(\bar x; \bar y)\,,$$

it follows that the formula $\varphi(\bar x; \bar b_i) \wedge \varphi(\bar x; \bar b_j)$ $\surd$-forks over U. On the other hand, $\mathfrak{p} \cup \mathfrak{q}_i \vDash \varphi(\bar x; \bar b_i)$ implies that $\varphi(\bar x; \bar b_i)$ does not $\surd$-fork over U. As $\bar b_i \approx^{\mathrm{ls}}_U \bar b_j$, this contradicts (2).

(3) ⇒ (1) Let κ be the cardinal from (3) associated with $\lambda := |\Phi|^+$. Extending the sequence $(\bar b_i)_{i\in I}$ we may assume that $|I| \geq \kappa$. For $w \subseteq I$, set

$$\Phi_w := \bigcup_{i\in w} \Phi(\bar x; \bar b_i)\,.$$

By compactness, it is sufficient to show that there is no finite subset $w \subseteq I$ such that Φ_w $\surd$-forks over U. We proceed by induction on $|w|$. For $w = \{i\}$, the claim holds since $\bar b_i \equiv_U \bar b_{i_0}$ and $\Phi(\bar x; \bar b_{i_0})$ does not $\surd$-fork over U. Hence, suppose that $n := |w| > 1$. Let $F := [I]^{n-1}$. By inductive hypothesis, no set Φ_s with $s \in F$ $\surd$-forks over U. Hence, we can use (3) to find indices $s \neq t \in F$ such that $\Phi_s \cup \Phi_t$ does not $\surd$-fork over U. Choosing sets $u, v \in F$ such that $\mathrm{ord}(uv) = \mathrm{ord}(st)$ and $w \subseteq u \cup v$, it follows by indiscernibility that $\Phi_w \subseteq \Phi_u \cup \Phi_v$ does not $\surd$-fork over U.

(1) ⇒ (4) Set $\mathfrak{p}(\bar x, \bar x') := \mathrm{tp}(\bar a \bar b_0/U)$. We extend β to an indiscernible sequence $\beta = (\bar b_i)_{i<\gamma}$ over U of length $\gamma \geq \beth_{\lambda^+}$ where $\lambda := 2^{|T|\oplus|U|\oplus|\bar b_0|}$. By the chain condition, the union $\bigcup_{i<\gamma} \mathfrak{p}(\bar x, \bar b_i)$ does not $\surd$-fork over U. Hence, there exists a tuple $\bar a'$ realising $\bigcup_{i<\gamma} \mathfrak{p}(\bar x, \bar b_i)$ such that $\bar a' \surd_U \beta$. Then $\bar a' \equiv_{U\bar b_0} \bar a$ and we can find a sequence $\beta' = (\bar b'_i)_{i<\gamma}$ such that $\bar a'\beta \equiv_{U\bar b_0} \bar a\beta'$. By Theorem E5.3.7 and choice of γ, there exists an indiscernible sequence $\beta'' = (\bar b''_n)_{n<\omega}$ over $U\bar a\bar b_0$ such that, for every $\bar\imath \in [\omega]^{<\omega}$, there is some $\bar\jmath \in [\gamma]^{<\omega}$ with

$$\bar b''[\bar\imath] \equiv_{U\bar a\bar b_0} \bar b'[\bar\jmath]\,.$$

By finite character, $\bar a \surd_U \bar b_0\beta'$ implies that $\bar a \surd_U \bar b_0\beta''$. By choice of β'' we can find, for every $n < \omega$, some tuple $\bar\jmath \in [\gamma]^n$ such that

$$\bar b_0 \bar b''_0 \dots \bar b''_{n-1} \equiv_{U\bar a\bar b_0} \bar b_0 \bar b'[\bar\jmath] \equiv_{U\bar b_0} \bar b_0 \bar b[\bar\jmath] \equiv_{U\bar b_0} \bar b_0 \bar b_1 \dots \bar b_n\,.$$

This implies that $\bar b_0\beta'' \equiv_{U\bar b_0} \beta$. Hence, the sequence $\beta''' := \bar b_0\beta''$ has the desired properties.

(4) ⇒ (2) Suppose that (2) does not hold. Then we can find a formula $\varphi(\bar x; \bar y)$ and an indiscernible sequence $\beta = (\bar b_i)_{i<\omega}$ over U such

that the formula $\varphi(\bar x;\bar b_0)$ does not $\sqrt{}$-fork over U, but the conjunction $\varphi(\bar x;\bar b_0) \wedge \varphi(\bar x;\bar b_1)$ does. We choose a tuple $\bar a \in \varphi(\bar x;\bar b_0)^{\mathbb{M}}$ with $\bar a \sqrt{}_U \bar b_0$. For every sequence $\beta' = (\bar b'_i)_{i<\omega} \equiv_{U\bar b_0} \beta$ that is indiscernible over $U\bar a$, we then have $\mathbb{M} \vDash \varphi(\bar a;\bar b'_i)$, for all i. As the conjunction $\varphi(\bar x;\bar b'_0) \wedge \varphi(\bar x;\bar b'_1)$ $\sqrt{}$-forks over U, it follows that $\bar a \not\sqrt{}_U \beta'$, for each such sequence β'. Therefore, (4) fails as well. □

As several of the characterisations of the chain condition are similar to characterisations of the dividing relation, we obtain the following implication.

Lemma 3.3. *If a preforking relation $\sqrt{}$ satisfies the chain condition over a set U then*

$$\bar a \sqrt{}_U \bar b \quad \textit{implies} \quad \bar a \overset{\mathrm{d}}{\sqrt{}}_U \bar b\,.$$

Proof. Suppose that $\bar a \sqrt{}_U \bar b$. To show that $\bar a \overset{\mathrm{d}}{\sqrt{}}_U \bar b$, we use condition (3) from Lemma F3.1.3. Hence, let $(\bar b_n)_{n<\omega}$ be an indiscernible sequence over U with $\bar b_0 = \bar b$. Setting $\Phi(\bar x,\bar x') := \mathrm{tp}(\bar a\bar b/U)$, it follows by the chain condition that there exists a tuple $\bar a'$ realising $\bigcup_{n<\omega}\Phi(\bar x,\bar b_n)$ with $\bar a' \sqrt{}_U \bar b$. In particular, we have

$$\bar a' \equiv_{U\bar b} \bar a \quad \text{and} \quad \bar b_i \equiv_{U\bar a'} \bar b_k\,, \quad \text{for all } i,k<\omega\,. \qquad \square$$

As a first application of the chain condition, let us show that array-dividing equals dividing. Once we have shown that in theories without the array property $\overset{\mathrm{d}}{\sqrt{}}$ satisfies the chain condition, the following result will generalise Corollary 1.14.

Proposition 3.4. *Suppose that $\overset{\mathrm{d}}{\sqrt{}}$ satisfies the chain condition over a set U. A formula divides over U if, and only if, it array-divides over U.*

Proof. ($\Rightarrow$) was already proved in Lemma 1.13. For ($\Leftarrow$), suppose that $\varphi(\bar x;\bar b)$ does not divide over U. To show that it also does not array-divide over U, we consider a family $\beta = (\bar b_{ij})_{i,j<\omega}$ that is biindiscernible over U with $\bar b_{00} = \bar b$. We apply the chain condition to the sequence $\beta^0 = (\bar b_{i0})_{i<\omega}$ to show that the set $\{\,\varphi(\bar x;\bar b_{i0}) \mid i<\omega\,\}$ does not divide over U. Applying the chain condition again, this time to the sequence $(\beta^i)_{i<\omega}$ of rows, it follows that the set $\{\,\varphi(\bar x;\bar b_{ij}) \mid i,j<\omega\,\}$ does not divide over U. In particular, this set is consistent. □

Finally, we show that, in theories without the array property, $\overset{\mathrm{f}}{\sqrt{}}$ satisfies the chain condition. We start by proving this implication over models before generalising it to arbitrary $\overset{\mathrm{f}}{\sqrt{}}$-bases.

Lemma 3.5. *Let T be a theory without the array property and let $\mathfrak{M}$ be a model of T. Then $\overset{\mathrm{f}}{\sqrt{}}$ satisfies the chain condition over M.*

Proof. We check condition (2) of Proposition 3.2. Let $\bar b \approx^{\mathrm{ls}}_M \bar b'$ be tuples and $\varphi(\bar x;\bar y)$ a formula such that the conjunction $\varphi(\bar x;\bar b)\wedge\varphi(\bar x;\bar b')$ forks over M. We have to show that $\varphi(\bar x;\bar b)$ also forks over M. Set $\kappa := \beth_{\lambda^+}$ where $\lambda := 2^{|T|\oplus|M|}$. Since $\bar b \approx^{\mathrm{ls}}_U \bar b'$, there exists an indiscernible sequence $\beta' = (\bar b'_i)_{i<\kappa}$ over M of length κ such that $\bar b'_0 = \bar b$ and $\bar b'_1 = \bar b'$. We have seen in Lemma 2.3 that M is a $\overset{\mathrm{u}}{\sqrt{}}$-extension base. By Corollary 2.19 this implies that M is a $\overset{\mathrm{sli}}{\sqrt{}}$-base. Furthermore, we have shown in Corollary 2.14 that $\overset{\mathrm{sli}}{\sqrt{}}$ satisfies the extension axiom. Hence, we have $\beta' \overset{\mathrm{sli}}{\sqrt{}}_M M$ and there exists a global type $\mathfrak{p} \supseteq \mathrm{tp}(\beta'/M)$ that is $\overset{\mathrm{sli}}{\sqrt{}}$-free over M. Let $\beta = (\beta^i)_{i<\omega}$ be a sequence generated by $\mathfrak{p}$ over M where $\beta^i = (\bar b_{ij})_{j<\omega}$. By indiscernibility of β^0 and the fact that forking equals dividing over M, it follows for all pairs $j \neq j'$ of indices that the formula $\varphi(\bar x;\bar b_{0j}) \wedge \varphi(\bar x;\bar b_{0j'})$ divides over M. By choice of β and Lemma 2.15, this implies that the set

$$\{\,\varphi(\bar x;\bar b_{ij}) \wedge \varphi(\bar x;\bar b_{ij'}) \mid i<\omega\,\}$$

is inconsistent. We can use Theorem E5.3.7 to find an indiscernible sequence $\alpha = (\alpha^i)_{i<\omega}$ over M such that, for every $\bar\imath \in [\omega]^{<\omega}$, there is some $\bar\jmath \in [\kappa]^{<\omega}$ with $\alpha[\bar\imath] \equiv_M \beta[\bar\jmath]$. It follows that the family α is biindiscernible over M and the formula φ is inconsistent over α. Consequently,

$\varphi(\bar{x};\bar{b}_{00})$ array-divides over M. According to Corollary 1.14 and Theorem 2.22, this implies that $\varphi(\bar{x};\bar{b}_{00})$ also divides and forks over M. $\square$

Theorem 3.6. *In a theory without the array property, $\sqrt[\mathrm{f}]{}$ satisfies the chain condition over every $\sqrt[\mathrm{f}]{}$-base.*

Proof. Let U be a $\sqrt[\mathrm{f}]{}$-base, $\varphi(\bar{x};\bar{y})$ a formula, and $\beta = (\bar{b}_i)_{i<\omega}$ an indiscernible sequence over U such that $\varphi(\bar{x};\bar{b}_0)$ does not fork over U. Fix a model $\mathfrak{M}$ containing U. Then $M \sqrt[\mathrm{f}]{}_U U$ and it follows by (EXT) that there exists a model $M' \equiv_U M$ such that $M' \sqrt[\mathrm{f}]{}_U \beta$. According to Theorem 2.22, we have $M' \sqrt[\mathrm{d}]{}_U \beta$. By Lemma F2.2.4, it therefore follows that a formula over β divides over U if, and only if, it divides over M'. In particular, $\varphi(\bar{x};\bar{b}_0)$ does not divide over M'. By Lemma 3.5, the formula $\varphi(\bar{x};\bar{b}_0) \wedge \varphi(\bar{x};\bar{b}_1)$ does not divide over M'. Hence, it also does not divide over U. The claim follows since forking equals dividing over U. $\square$

Corollary 3.7. *In a theory without the array property, $\sqrt[\mathrm{d}]{}$ satisfies the chain condition over every $\sqrt[\mathrm{f}]{}$-base.*

Proof. Let U be a $\sqrt[\mathrm{f}]{}$-base. According to Theorem 2.22, forking equals dividing over U. Consequently, $\sqrt[\mathrm{d}]{}$ has the chain condition over U if, and only if, $\sqrt[\mathrm{f}]{}$ does. Hence, the claim follows by the preceding theorem. $\square$

The Independence Theorem

There are two versions of the Independence Theorem: a weak one that holds in all theories without the array property, and a strong one that characterises simple theories.

Definition 3.8. (a) A preforking relation $\sqrt{}$ satisfies the *Weak Independence Theorem* over a set $U \subseteq \mathbb{M}$ if it has the following property:

(WIND) If $\bar{a}, \bar{b}, \bar{b}', \bar{c} \subseteq \mathbb{M}$ are tuples satisfying

$$\bar{c} \sqrt{}_U \bar{a}\bar{b}\,, \quad \bar{a} \sqrt{}_U \bar{b}\bar{b}'\,, \quad \text{and} \quad \bar{b} \equiv^{\mathrm{ls}}_U \bar{b}'\,,$$

then there exists a tuple $\bar{c}'$ such that

$$\bar{c}' \sqrt{}_U \bar{a}\bar{b}'\,, \quad \bar{c}' \equiv_{U\bar{a}} \bar{c}\,, \quad \text{and} \quad \bar{b}'\bar{c}' \equiv_U \bar{b}\bar{c}\,.$$

(b) A preforking relation $\sqrt{}$ satisfies the *Independence Theorem* over a set $U \subseteq \mathbb{M}$ if it has the following property:

(IND) If $\bar{a}, \bar{b}, A, B \subseteq \mathbb{M}$ are tuples such that

$$\bar{a} \equiv_U \bar{b}\,, \quad \bar{a} \sqrt{}_U A\,, \quad \bar{b} \sqrt{}_U B\,, \quad \text{and} \quad A \sqrt{}_U B\,,$$

then there exists a tuple $\bar{c}$ such that

$$\bar{c} \equiv_{UA} \bar{a}\,, \quad \bar{c} \equiv_{UB} \bar{b}\,, \quad \text{and} \quad \bar{c} \sqrt{}_U AB\,.$$

We say that $\sqrt{}$ satisfies the Independence Theorem *for a class* $\mathcal{C} \subseteq \wp(\mathbb{M})$, if it satisfies the theorem over every $U \in \mathcal{C}$.

Remark. The statement of the second axiom becomes clearer when we rephrase it in terms of types. Then it reads:

> Let $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ be types over, respectively, U, $U \cup A$, and $U \cup B$. If $\mathfrak{q}$ and $\mathfrak{r}$ are $\sqrt{}$-free extensions of $\mathfrak{p}$ and $A \sqrt{}_U B$, then $\mathfrak{q} \cup \mathfrak{r}$ is also a $\sqrt{}$-free extension of $\mathfrak{p}$.

We start by proving that the weak version holds in all theories without the array property.

Theorem 3.9. *For a forking relation $\sqrt{}$, the chain condition over a set U implies the Weak Independence Theorem over U.*

Proof. Suppose that $\sqrt{}$ satisfies the chain condition over U and let

$$\bar{c} \sqrt{}_U \bar{a}\bar{b}\,, \quad \bar{a} \sqrt{}_U \bar{b}\bar{b}'\,, \quad \text{and} \quad \bar{b} \equiv^{\mathrm{ls}}_U \bar{b}'\,.$$

We first consider the case where $\bar b \approx^{\mathrm{ls}}_U \bar b'$. By Lemma 3.3, we have $\bar a \mathop{\sqrt[\mathrm{d}]{}}_U \bar b\bar b'$ and, hence, $\bar a \mathop{\sqrt[\mathrm{d}]{}}_{U\bar b} \bar b'$. Therefore, we can use Lemma F4.2.12 to find a tuple $\bar a'$ such that $\bar a\bar b \approx^{\mathrm{ls}}_U \bar a'\bar b'$. Thus, there exists an indiscernible sequence $(\bar a_i\bar b_i)_{i<\omega}$ over U with $\bar a_0\bar b_0\bar a_1\bar b_1 = \bar a\bar b\bar a'\bar b'$. Since we have $\bar c \mathop{\surd}_U \bar a_0\bar b_0$, it follows by Proposition 3.2 (4) that there is a tuple $\bar c' \equiv_{U\bar a_0\bar b_0} \bar c$ such that $\bar c' \mathop{\surd}_U \bar a[\omega]\bar b[\omega]$ and $(\bar a_i\bar b_i)_{i<\omega}$ is indiscernible over $U\bar c'$. This implies that

$$\bar c' \mathop{\surd}_U \bar a\bar b'\,,\quad \bar c' \equiv_{U\bar a} \bar c\,,\quad\text{and}\quad \bar b'\bar c' \equiv_U \bar b\bar c' \equiv_U \bar b\bar c\,.$$

It remains to prove the general case. Fix a sequence $\bar b_0 \approx^{\mathrm{ls}}_U \dots \approx^{\mathrm{ls}}_U \bar b_n$ such that $\bar b_0 = \bar b$ and $\bar b_n = \bar b'$. By (EXT), there is a tuple $\bar a' \equiv_{U\bar b\bar b'} \bar a$ such that $\bar a' \mathop{\surd}_U \bar b_0 \dots \bar b_n$. Choosing tuples $\bar b'_0,\dots,\bar b'_n$ with

$$\bar a\bar b'_0 \dots \bar b'_n \equiv_{U\bar b\bar b'} \bar a'\bar b_0 \dots \bar b_n$$

it follows that $\bar b'_0 = \bar b$, $\bar b'_n = \bar b'$,

$$\bar b'_0 \approx^{\mathrm{ls}}_U \dots \approx^{\mathrm{ls}}_U \bar b'_n \quad\text{and}\quad \bar a \mathop{\surd}_U \bar b'_0 \dots \bar b'_n\,.$$

By the special case we have proved above, we can inductively find tuples $\bar c_0,\dots,\bar c_n$ such that $\bar c_0 = \bar c$,

$$\bar c_{i+1} \mathop{\surd}_U \bar a\bar b'_{i+1}\,,\quad \bar c_{i+1} \equiv_{U\bar a} \bar c_i\,,\quad\text{and}\quad \bar b'_{i+1}\bar c_{i+1} \equiv_U \bar b'_i\bar c_i\,.$$

The tuple $\bar c' := \bar c_n$ has the desired properties. □

By Theorem 3.6, we can conclude that, in theories without the array property, $\sqrt[\mathrm{f}]{}$ satisfies the chain condition and, thus, the Weak Independence Theorem over $\sqrt[\mathrm{f}]{}$-bases.

Corollary 3.10 (Weak Independence Theorem; Ben Yaacov, Chernikov). *In a theory T without the array property, $\sqrt[\mathrm{f}]{}$ satisfies the Weak Independence Theorem over every $\sqrt[\mathrm{f}]{}$-base.*

Let us turn to the strong version of the Independence Theorem. Our goal is to show that it characterises $\mathop{\downarrow}^{\mathrm{f}}$ in simple theories: a symmetric forking relation $\mathop{\downarrow}$ satisfies the Independence Theorem if, and only if, $\mathop{\downarrow} = \mathop{\downarrow}^{\mathrm{f}}$ and the theory in question is simple. We start by proving that forking satisfies (IND) in simple theories.

Theorem 3.11 (Independence Theorem). *In a simple first-order theory $\mathop{\downarrow}^{\mathrm{f}}$ satisfies the Independence Theorem for the class of all models.*

Proof. Let $\mathfrak{M}$ be a model and suppose that

$$\bar a \equiv_M \bar b\,,\quad \bar a \mathop{\downarrow}^{\mathrm{f}}_M A\,,\quad \bar b \mathop{\downarrow}^{\mathrm{f}}_M B\,,\quad\text{and}\quad A \mathop{\downarrow}^{\mathrm{f}}_M B\,.$$

As in simple theories every set is a $\mathop{\downarrow}^{\mathrm{f}}$-base, we have $\bar a \mathop{\downarrow}^{\mathrm{f}}_{MA} MA$. Therefore, we can use Lemma F4.2.13 to find a tuple $\bar a' \equiv_{MA} \bar a$ such that $\bar a' \mathop{\downarrow}^{\mathrm{f}}_{MA} B\bar a_0\bar b$. Then it follows by transitivity that

$$\begin{aligned}
&\bar a' \mathop{\downarrow}^{\mathrm{f}}_{MA} B\bar b &&\text{and}&& \bar a' \mathop{\downarrow}^{\mathrm{f}}_M A &&\text{implies}&& \bar a' \mathop{\downarrow}^{\mathrm{f}}_M AB\bar b\,,\\
&\bar a' \mathop{\downarrow}^{\mathrm{f}}_M AB &&\text{and}&& B \mathop{\downarrow}^{\mathrm{f}}_M A &&\text{implies}&& B\bar a' \mathop{\downarrow}^{\mathrm{f}}_M A\,,\\
&\bar a' \mathop{\downarrow}^{\mathrm{f}}_M B\bar b &&\text{and}&& \bar b \mathop{\downarrow}^{\mathrm{f}}_M B &&\text{implies}&& \bar a'\bar b \mathop{\downarrow}^{\mathrm{f}}_M B\,.
\end{aligned}$$

Furthermore, $\bar a' \equiv_M \bar a \equiv_M \bar b$, which implies that $\bar a' \equiv^{\mathrm{ls}}_M \bar b$. Hence, we can apply Corollary 3.10 to the statement $A \mathop{\downarrow}^{\mathrm{f}}_M B\bar a'$ to find a set A' such that

$$A' \mathop{\downarrow}^{\mathrm{f}}_M B\bar b\,,\quad A' \equiv_{MB} A\,,\quad\text{and}\quad \bar bA' \equiv_M \bar a'A\,.$$

Let $\bar c$ be a tuple such that $A'B\bar b \equiv_M AB\bar c$. Then

$$A \mathop{\downarrow}^{\mathrm{f}}_M \bar cB \quad\text{and}\quad \bar c \mathop{\downarrow}^{\mathrm{f}}_M B \quad\text{implies}\quad AB \mathop{\downarrow}^{\mathrm{f}}_M \bar c\,.$$

Furthermore, we have

$$\bar cA \equiv_M \bar bA' \equiv_M \bar a'A \equiv_M \bar aA \quad\text{and}\quad \bar cB \equiv_M \bar bB\,.$$ □

It remains to prove that forking is the only symmetric forking relations satisfying the Independence Theorem.

Definition 3.12. A class $\mathcal{C} \subseteq \wp(\mathbb{M})$ of small sets is *invariant* if

$$C \equiv_\varnothing C' \quad \text{implies} \quad C \in \mathcal{C} \Leftrightarrow C' \in \mathcal{C}\,.$$

We call $\mathcal{C}$ *dense* if, for every set $A \subseteq \mathbb{M}$, there is some $C \in \mathcal{C}$ with $A \subseteq C$.

Example. Every class containing all models is dense. In particular, the class of all $\overset{\mathrm{u}}{\surd}$-extension bases and the class of all $\overset{\mathrm{f}}{\surd}$-bases are invariant and dense.

We start with a lemma constructing a Morley sequence. The proof follows the lines of the proofs of Lemmas F2.4.13 and F2.4.15.

Lemma 3.13. *Let $\downarrow$ be a right local forking relation, let $\mathcal{C} \subseteq \wp(\mathbb{M})$ be invariant and dense, and let $(\bar{a}_n)_{n<\omega}$ be an indiscernible sequence over U.*

There exists a set $C \in \mathcal{C}$ containing U and a type $\mathfrak{p} \in S^{\bar{s}}(C)$ extending $\mathrm{tp}(\bar{a}_0/U)$ such that $(\bar{a}_n)_{n<\omega}$ is a $\downarrow$-Morley sequence for $\mathfrak{p}$ over C.

Proof. Let $\kappa := \mathrm{loc}(\downarrow)^+ \oplus |\bar{a}_0|^+$. We can use Lemma E5.3.9 to extend $(\bar{a}_n)_{n<\omega}$ to an indiscernible sequence $(\bar{a}_\alpha)_{\alpha\leq\kappa}$ over U. We construct an increasing chain $(C_\alpha)_{\alpha<\kappa}$ of sets $C_\alpha \in \mathcal{C}$ such that, for every $\alpha < \kappa$,

$$U \cup \bar{a}[<\alpha] \subseteq C_\alpha \quad \text{and} \quad (\bar{a}_i)_{\alpha<i\leq\kappa} \text{ is indiscernible over } C_\alpha\,.$$

For the inductive step, suppose that C_i has already been defined for all $i < \alpha$. As $\mathcal{C}$ is dense, we can choose some set $C' \in \mathcal{C}$ containing $V_\alpha := U \cup \bar{a}[<\alpha] \cup \bigcup_{i<\alpha} C_i$. Since the sequence $(\bar{a}_i)_{\alpha<i<\kappa}$ is indiscernible over V_α, we can apply Lemma E5.3.11 to obtain a set $C_\alpha \equiv_{V_\alpha} C'$ such that $(\bar{a}_i)_{\alpha<i<\kappa}$ is indiscernible over $V_\alpha \cup C_\alpha$. By invariance, it follows that $C_\alpha \in \mathcal{C}$.

After having constructed the sequence $(C_\alpha)_{\alpha<\kappa}$, we can find a set $W \subseteq \bigcup_{\alpha<\kappa} C_\alpha$ of size $|W| < \mathrm{loc}(\downarrow) \oplus |\bar{a}_0|^+ \leq \kappa$ such that

$$\bar{a}_\kappa \downarrow_W \bigcup_{\alpha<\kappa} C_\alpha\,.$$

Since κ is regular, there exists an index $\gamma < \kappa$ such that $W \subseteq C_\gamma$. By (MON) and (BMON), it follows that

$$\bar{a}_\kappa \downarrow_{C_\gamma} \bigcup_{\gamma<i<\kappa} \bar{a}_i\,.$$

By (INV), we therefore have

$$\bar{a}_\alpha \downarrow_{C_\gamma} \bigcup_{\gamma<i<\alpha} \bar{a}_i\,, \quad \text{for all } \gamma < \alpha < \kappa\,.$$

Hence, $(\bar{a}_\alpha)_{\gamma<\alpha<\kappa}$ is a $\downarrow$-Morley sequence for $\mathrm{tp}(\bar{a}_\kappa/C_\gamma)$ over C_γ. Fix an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ such that $\pi[\bar{a}_{\gamma+n+1}] = \bar{a}_n$, for all $n < \omega$. By invariance, it follows that $(\bar{a}_n)_{n<\omega}$ is a $\downarrow$-Morley sequence for $\mathfrak{p} := \mathrm{tp}(\pi[\bar{a}_\kappa]/\pi[C_\gamma])$ over $C := \pi[C_\gamma]$. ☐

The main argument is contained in a technical lemma which states that the Independence Theorem implies the following weaker variant of the chain condition.

Definition 3.14. A preforking relation $\surd$ satisfies the *chain condition for Morley sequences* over a set $U \subseteq \mathbb{M}$ if, for every $\surd$-Morley sequence $(\bar{b}_i)_{i\in I}$ over U and every set of formulae $\Phi(\bar{x};\bar{y})$ such that, for some $i_0 \in I$, the set $\Phi(\bar{x};\bar{b}_{i_0})$ does not $\surd$-fork over U, the union $\bigcup_{i\in I} \Phi(\bar{x};\bar{b}_i)$ also does not $\surd$-fork over U.

Lemma 3.15. *Let $\surd$ be a forking relation satisfying the Independence Theorem over a set U. Then $\surd$ satisfies the chain condition for Morley sequence over U.*

Proof. Let $(\bar{b}_n)_{n<\omega}$ be a $\surd$-Morley sequence over U and let $\Phi(\bar{x};\bar{y})$ be a set such that $\Phi(\bar{x};\bar{b}_0)$ does not $\surd$-fork over U. We fix a tuple $\bar{a}$ with $\bar{a} \surd_U \bar{b}_0$ and we set $\mathfrak{p}(\bar{x},\bar{x}') := \mathrm{tp}(\bar{a}\bar{b}_0/U)$. We have to show that there exists a tuple $\bar{c}$ realising $\bigcup_{n<\omega} \mathfrak{p}(\bar{x},\bar{b}_n)$ such that $\bar{c} \surd_U \bar{b}[<\omega]$.

To do so, we construct a sequence $(\bar{c}_n)_{n<\omega}$ such that

$$\bar{c}_n \surd_U \bar{b}[\leq n] \quad \text{and} \quad \bar{c}_n \text{ realises } \bigcup_{i\leq n} \mathfrak{p}(\bar{x},\bar{b}_i)\,.$$

We start with $\bar{c}_0 := \bar{a}$. Then $\bar{c}_0$ realises $\mathfrak{p}(\bar{x}, \bar{b}_0)$ and $\bar{c}_0 \surd_U \bar{b}_0$. For the inductive step, suppose that $\bar{c}_n$ has already been defined. Let $\bar{a}'$ be a realisation of $\mathfrak{p}(\bar{x}, \bar{b}_{n+1})$. Then

$$\bar{a}' \equiv_U \bar{c}_n\,, \quad \bar{a}' \surd_U \bar{b}_{n+1}\,, \quad \bar{c}_n \surd_U \bar{b}[\leq n]\,,$$
$$\text{and} \quad \bar{b}_{n+1} \surd_U \bar{b}[\leq n]\,,$$

which, by the Independence Theorem, implies that there is a tuple $\bar{c}_{n+1}$ such that

$$\bar{c}_{n+1} \equiv_{U\bar{b}_{n+1}} \bar{a}'\,, \quad \bar{c}_{n+1} \equiv_{U\bar{b}[\leq n]} \bar{c}_n\,, \quad \text{and} \quad \bar{c}_{n+1} \surd_U \bar{b}[\leq n]\bar{b}_{n+1}\,.$$

It follows that $\bar{c}_{n+1}$ realises the types $\mathrm{tp}(\bar{a}'/U\bar{b}_{n+1}) = \mathfrak{p}(\bar{x}, \bar{b}_{n+1})$ and $\mathrm{tp}(\bar{c}_n/U\bar{b}[\leq n]) \supseteq \bigcup_{i \leq n} \mathfrak{p}(\bar{x}, \bar{b}_i)$.

In particular, note that $\bar{c}_{n+1} \equiv_{U\bar{b}[\leq n]} \bar{c}_n$. Hence, having constructed the sequence $(\bar{c}_n)_{n<\omega}$, we can use the Compactness Theorem to find a tuple $\bar{c}$ such that

$$\bar{c} \equiv_{U\bar{b}[\leq n]} \bar{c}_n\,, \quad \text{for all } n < \omega\,.$$

Consequently, $\bar{c}$ realises $\bigcup_{n<\omega} \mathfrak{p}(\bar{x}, \bar{b}_n)$. Furthermore, (INV) and (DEF) implies that $\bar{c} \surd_U \bar{b}[<\omega]$. ☐

For symmetric forking relations, we can strengthen Lemma 3.3 as follows.

Theorem 3.16. *If a symmetric forking relation $\downarrow$ satisfies the chain condition for Morley sequences for a class $\mathcal{C}$ that is invariant and dense, then $\downarrow = \overset{\mathrm{d}}{\surd}$.*

Proof. We have shown in Theorem F3.1.9 that $\overset{\mathrm{d}}{\surd} \subseteq \downarrow$, for every symmetric forking relation. Conversely, suppose that $\bar{a} \downarrow_U \bar{b}$. To show that $\bar{a} \overset{\mathrm{d}}{\surd}_U \bar{b}$, set $\mathfrak{p}(\bar{x}, \bar{x}') := \mathrm{tp}(\bar{a}\bar{b}/U)$ and let $(\bar{b}_n)_{n<\omega}$ by an indiscernible sequence over U with $\bar{b}_0 = \bar{b}$. By Lemma F3.1.3 (3), it is sufficient to show that there is a tuple realising $\bigcup_{n<\omega} \mathfrak{p}(\bar{x}, \bar{b}_n)$. As $\downarrow$ is right local, we can use Lemma 3.13 to find a set $C \in \mathcal{C}$ containing U such that $(\bar{b}_n)_{n<\omega}$ is a $\downarrow$-Morley sequence over C. Since $\bar{a} \downarrow_U \bar{b}_0$, there is some $\bar{a}' \equiv_{U\bar{b}_0} \bar{a}$ such that $\bar{a}' \downarrow_U C\bar{b}_0$. Set $\mathfrak{p}'(\bar{x}, \bar{x}') := \mathrm{tp}(\bar{a}'\bar{b}_0/C)$. By the chain condition for Morley sequences, the union $\bigcup_{n<\omega} \mathfrak{p}'(\bar{x}, \bar{b}_n)$ does not $\downarrow$-fork over C. In particular, it is consistent. Hence, it follows that there is a tuple realising

$$\bigcup_{n<\omega} \mathfrak{p}(\bar{x}, \bar{b}_n) \subseteq \bigcup_{n<\omega} \mathfrak{p}'(\bar{x}, \bar{b}_n)\,.$$ ☐

We obtain the following characterisation of simple theories.

Theorem 3.17. *Let T be a complete first-order theory. The following statements are equivalent.*

(1) *T is simple.*

(2) *There exists a symmetric forking relation $\downarrow$ satisfying the Independence Theorem for the class of all models.*

(3) *There exists a symmetric forking relation $\downarrow$ satisfying the chain condition for Morley sequences for the class of all models.*

(4) *There exists a symmetric forking relation $\downarrow$ satisfying the chain condition for the class of all models.*

Proof. (4) ⇒ (3) is trivial; (3) ⇒ (1) follows by Theorem 3.16; (1) ⇒ (4) by Lemma 3.5; (1) ⇒ (2) was already proved in Theorem 3.11; and (2) ⇒ (3) follows by Lemma 3.15. ☐

As an application we consider the theory of the random graph.

Proposition 3.18. *The theory of the random graph is simple.*

Proof. By Theorem 3.16, it is sufficient to prove that the relation

$$A \downarrow^{\circ}_U B \quad :\text{iff} \quad A \cap B \subseteq U$$

is a symmetric forking relation satisfying the Independence Theorem. $\downarrow^{\circ}$ obviously satisfies the axioms (MON), (NOR), (LRF), (BMON), and (SYM).

(LTR) Suppose that $A_2 \downarrow^{\circ}_{A_1} B$ and $A_1 \downarrow^{\circ}_{A_0} B$ where $A_0 \subseteq A_1 \subseteq A_2$. Then $A_2 \cap B \subseteq A_1$ and $A_1 \cap B \subseteq A_0$. Hence,

$$A_2 \cap B \subseteq A_1 \cap B \subseteq A_0\,,$$

which implies that $A_2 \downarrow^{\circ}_{A_0} B$.

(DEF) Suppose that $A \not\downarrow^{\circ}_{U} B$. Then there is some element $b \in A \cap B \setminus U$. For every element $a \in (x = b)^{\mathbb{M}}$ it follows that $a \not\downarrow^{\circ}_{U} b$.

(EXT) Suppose that $\bar a \downarrow^{\circ}_{U} B_0$ and let $B_0 \subseteq B_1$. Using the extension axioms, we can find a tuple $\bar a'$ such that

$$\operatorname{atp}(\bar a'/UB_0) = \operatorname{atp}(\bar a/UB_0) \quad \text{and} \quad (\bar a' \setminus U) \cap B_1 = \varnothing\,.$$

By ultrahomogeneity, there exists an automorphism $\pi \in \operatorname{Aut} \mathbb{M}_{UB_0}$ mapping $\bar a$ to $\bar a'$. Hence, $\bar a' \equiv_{UB_0} \bar a$ and $\bar a' \downarrow^{\circ}_{U} B_1$.

(IND) We prove that $\downarrow^{\circ}$ satisfies the Independence Theorem for the class of all subsets of $\mathbb{M}$. Suppose that

$$\bar a \equiv_U \bar b\,, \quad \bar a \downarrow^{\circ}_{U} A\,, \quad \bar b \downarrow^{\circ}_{U} B\,, \quad \text{and} \quad A \downarrow^{\circ}_{U} B\,.$$

Replacing A and B by, respectively, $A \setminus U$ and $B \setminus U$, we may assume that $A \cap U = \varnothing$ and $B \cap U = \varnothing$. Let

$$\bar d := \bar a \cap U\,, \quad \bar a' := \bar a \setminus U\,, \quad \text{and} \quad \bar b' := \bar b \setminus U\,.$$

Note that $\bar a' \cap (U \cup A) = \varnothing$ and $\bar b' \cap (U \cup B) = \varnothing$. Since U, A, B are disjoint, we can use the extension axioms to find a tuple $\bar c'$ disjoint from $U \cup A \cup B$ such that

$$\operatorname{atp}(\bar c'/UA) = \operatorname{atp}(\bar a'/UA) \quad \text{and} \quad \operatorname{atp}(\bar c'/UB) = \operatorname{atp}(\bar b'/UB)\,.$$

It follows that

$$\bar c'\bar d \equiv_{UA} \bar a'\bar d\,, \quad \bar c'\bar d \equiv_{UB} \bar b'\bar d\,, \quad \text{and} \quad \bar c'\bar d \downarrow^{\circ}_{U} AB\,. \qquad \square$$

Part G.

Geometric Model Theory

The following two theorems summarise the results of this section.

Theorem 6.12 (Cohen, Shelah). *Let T be a complete first-order theory. The following conditions are equivalent:*

(1) *T is stable.*

(2) *T has $\mathrm{Un}(\kappa, \lambda)$-representations, for some cardinals κ and λ.*

(3) *T has $\mathrm{Wf}(0, |T|)$-representations.*

(4) *T has $\mathrm{Wf}(|T|, |T|)$-representations.*

Proof. (2) $\Rightarrow$ (1) has been shown in Proposition 6.8 (a), the implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) follow from Lemmas 6.5 and 6.2, and (1) $\Rightarrow$ (4) follows by Proposition 6.11. $\square$

Theorem 6.13 (Cohen, Shelah). *Let T be a complete first-order theory. The following conditions are equivalent:*

(1) *T is $\aleph_0$-stable.*

(2) *T has $\mathrm{Lf}(\aleph_0, \aleph_0)$-representations.*

Proof. (2) $\Rightarrow$ (1) follows by Proposition 6.8 (b) and (1) $\Rightarrow$ (2) follows by Proposition 6.11. $\square$

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The Roman and Fraktur alphabets							
A	a	$\mathfrak{A}$	$\mathfrak{a}$	N	n	$\mathfrak{N}$	$\mathfrak{n}$
B	b	$\mathfrak{B}$	$\mathfrak{b}$	O	o	$\mathfrak{O}$	$\mathfrak{o}$
C	c	$\mathfrak{C}$	$\mathfrak{c}$	P	p	$\mathfrak{P}$	$\mathfrak{p}$
D	d	$\mathfrak{D}$	$\mathfrak{d}$	Q	q	$\mathfrak{Q}$	$\mathfrak{q}$
E	e	$\mathfrak{E}$	$\mathfrak{e}$	R	r	$\mathfrak{R}$	$\mathfrak{r}$
F	f	$\mathfrak{F}$	$\mathfrak{f}$	S	s	$\mathfrak{S}$	ſ $\mathfrak{s}$
G	g	$\mathfrak{G}$	$\mathfrak{g}$	T	t	$\mathfrak{T}$	$\mathfrak{t}$
H	h	$\mathfrak{H}$	$\mathfrak{h}$	U	u	$\mathfrak{U}$	$\mathfrak{u}$
I	i	$\mathfrak{I}$	$\mathfrak{i}$	V	v	$\mathfrak{V}$	$\mathfrak{v}$
J	j	$\mathfrak{J}$	$\mathfrak{j}$	W	w	$\mathfrak{W}$	$\mathfrak{w}$
K	k	$\mathfrak{K}$	$\mathfrak{k}$	X	x	$\mathfrak{X}$	$\mathfrak{x}$
L	l	$\mathfrak{L}$	$\mathfrak{l}$	Y	y	$\mathfrak{Y}$	$\mathfrak{y}$
M	m	$\mathfrak{M}$	$\mathfrak{m}$	Z	z	$\mathfrak{Z}$	$\mathfrak{z}$

The Greek alphabet					
A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ε	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	ϑ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega