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1. Infinitary first-order logic

Logics are languages to talk about structures and their elements. They can be used to assert that a given structure has a certain property, to define classes of structures, or to define relations inside a given structure. Let us start with a simple, but typical example.

Example. Let $\mathfrak{K}$ be a field and X a set of variables. The *Zariski logic* over $\mathfrak{K}$ is the set $\mathrm{ZL}[\mathfrak{K}, X] := \mathfrak{K}[X]$ of all polynomials over $\mathfrak{K}$ with unknowns from X.

Let $\mathfrak{L} \supseteq \mathfrak{K}$ be a field extending $\mathfrak{K}$. For a polynomial $p \in \mathrm{ZL}[\mathfrak{K}, X]$ and a variable assignment $\beta : X \to L$, recall that $p^{\mathfrak{L}}[\beta]$ denotes the value of p when we assign to each variable $x \in X$ the value $\beta(x)$. A polynomial $p \in \mathrm{ZL}[\mathfrak{K}, X]$ *defines* in a given field $\mathfrak{L} \supseteq \mathfrak{K}$ the set

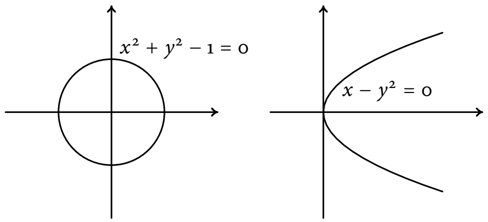
$$p^{\mathfrak{L}} := \{ \beta \in L^X \mid p^{\mathfrak{L}}[\beta] = 0 \}$$

of its roots. A set $A \subseteq L^n$ is *Zariski-definable over* $\mathfrak{K}$ if there exist finitely many polynomials $p_0, \dots, p_{k-1} \in \mathrm{ZL}[\mathfrak{K}, \{x_0, \dots, x_{n-1}\}]$ such that

$$A = \big\{ \langle \beta(x_0), \dots, \beta(x_{n-1}) \rangle \bigm| \beta \in p_0^{\mathfrak{L}} \cap \dots \cap p_{k-1}^{\mathfrak{L}} \big\} .$$

In case of algebraically closed fields the Zariski-definable relations are called *algebraic varieties.*

For instance, the polynomial $x^2 + y^2 - 1 = 0$ defines over $\mathbb{R}$ the unit circle S^1, while $x - y^2 = 0$ defines a rotated parabola.



Let us capture the above situation in a general definition.

Definition 1.1. A *logic* is a triple $\langle L, \mathcal{K}, \vDash \rangle$ consisting of a nonempty class L of *formulae,* a nonempty class $\mathcal{K}$ of *interpretations,* and a binary *satisfaction relation* $\vDash \subseteq \mathcal{K} \times L$.

Let $\mathfrak{J} \in \mathcal{K}$ be an interpretation and $\varphi \in L$ a formula. If $\mathfrak{J} \vDash \varphi$ then we say that φ *holds* in $\mathfrak{J}$, that $\mathfrak{J}$ *satisfies* φ, or that $\mathfrak{J}$ is a *model* of φ. For sets of formulae $\Phi \subseteq L$ we define

$$\mathfrak{J} \vDash \Phi \quad :\text{iff} \quad \mathfrak{J} \vDash \varphi \text{ for all } \varphi \in \Phi\,.$$

Example. (a) In the case of Zariski-logic $\mathrm{ZL}[\mathfrak{K}, X]$ the formulae are the polynomials $p \in \mathfrak{K}[X]$ and an interpretation is a pair $\langle \mathfrak{L}, \beta \rangle$ where $\mathfrak{L} \supseteq \mathfrak{K}$ is a field extension of $\mathfrak{K}$ and $\beta \in L^X$ is a variable assignment. We have

$$\langle \mathfrak{L}, \beta \rangle \vDash p \quad \text{iff} \quad p^{\mathfrak{L}}[\beta] = 0\,.$$

(b) For a boolean algebra $\mathfrak{B}$, we define *boolean logic*

$$\mathrm{BL}(\mathfrak{B}) := \langle B,\ \mathrm{spec}(\mathfrak{B}),\ \vDash \rangle\,,$$

where, for an element $b \in B$ and an ultrafilter $\mathfrak{u} \in \mathrm{spec}(\mathfrak{B})$,

$$\mathfrak{u} \vDash b \quad :\text{iff} \quad b \in \mathfrak{u}\,.$$

The main logic we will consider is *first-order logic,* also called *predicate logic.* We start by defining its *syntax,* that is, the set of first-order formulae. For convenience we simultaneously define two logics, basic first-order logic FO and a variant $\mathrm{FO}_{\kappa\aleph_0}$ where we allow infinite formulae.

Definition 1.2. Let Σ be a signature and κ an infinite cardinal. For each sort s of Σ, let X_s be a set of *variable symbols* of sort s, and set $X := \bigcup_s X_s$.

The set $\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ of *infinitary first-order formulae* is the smallest set of terms satisfying the following closure conditions:

- If $t_0, t_1 \in T[\Sigma, X]$ are of the same sort then $t_0 = t_1$ belongs to $\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$.
- If $R \in \Sigma$ is of type $s_0 \dots s_{n-1}$ and $t_i \in T_{s_i}[\Sigma, X]$, for $i < n$, then $Rt_0 \dots t_{n-1}$ is in $\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$.
- If $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ then $\neg\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$.
- If $\Phi \subseteq \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ and $|\Phi| < \kappa$ then $\bigwedge \Phi, \bigvee \Phi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$.
- If $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X \cup \{x\}]$ then $\exists x\varphi, \forall x\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$.

For $\kappa = \aleph_0$, we obtain (*finitary*) *first-order logic*

$$\mathrm{FO}[\Sigma, X] := \mathrm{FO}_{\aleph_0\aleph_0}[\Sigma, X]\,.$$

If we omit the cardinality restriction, we get

$$\mathrm{FO}_{\infty\aleph_0}[\Sigma, X] := \bigcup_\kappa \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]\,.$$

The operation $\neg$ is called *negation,* $\bigwedge$ and $\bigvee$ are *conjunction* and *disjunction,* and $\exists$ and $\forall$ are the *existential* and *universal quantifier.* An *atom* is a formula of the form

$$Rt_0 \dots t_{n-1} \quad \text{or} \quad t_0 = t_1\,.$$

A formula that is either an atom or the negation of an atom is called a *literal.*

Remark. Every formula $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ is a term $\varphi : T \to \Lambda$ where $T \subseteq \kappa^{<\omega}$ and

$$\Lambda := \Sigma \cup X' \cup \{=, \neg, \bigwedge, \bigvee\} \cup \{\, \exists x, \forall x \mid x \in X' \,\}\,,$$

for some $X' \supseteq X$. In particular, for $\kappa = \aleph_0$, we can regard $\mathrm{FO}[\Sigma, X]$ as a subset of $T[\Lambda, \varnothing]$.

It remains to define the meaning of these formulae, that is, the satisfaction relation. Before doing so, let us note that we can use induction on formulae.

Lemma 1.3. *If κ is a regular cardinal then we have* $\mathrm{frk}(\varphi) < \kappa$, *for all* $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$.

Proof. Let Λ be the same set of symbols as in the preceding remark. The set Γ of all terms $t : T \to \Lambda$ such that $\mathrm{frk}(t) < \kappa$ is closed under all operations of Definition 1.2. Since $\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ is the smallest such set we have $\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X] \subseteq \Gamma$, as desired. ☐

Corollary 1.4. $\mathrm{frk}(\varphi) < \infty$, *for all* $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$.

This result implies that the reversed ordering on the domain of a formula is well-founded. Therefore, we can give proofs and definitions by induction on this order. A proof or a construction *by induction on φ* takes the following form. We have to distinguish several cases:

- φ is an atom.
- $\varphi = \neg\psi$ and the inductive hypothesis holds for ψ.
- $\varphi = \bigwedge \Phi$ or $\varphi = \bigvee \Phi$ and the inductive hypothesis holds for every element of Φ.
- $\varphi = \exists x\psi$ or $\varphi = \forall x\psi$ and the inductive hypothesis holds for ψ.

We use induction to define the *semantics* of first-order logic, that is, the satisfaction relation.

Definition 1.5. Let $\mathfrak{A}$ be a Σ-structure and $\beta : X \to A$ a variable assignment. The pair $\langle \mathfrak{A}, \beta \rangle$ is called a (*first-order*) *interpretation*. For $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ we define the *satisfaction relation* $\mathfrak{A} \vDash \varphi[\beta]$ by induction on φ.

$$
\begin{array}{lll}
\mathfrak{A} \vDash t_0 = t_1[\beta] & :\text{iff} & t_0^{\mathfrak{A}}[\beta] = t_1^{\mathfrak{A}}[\beta]\,, \\
\mathfrak{A} \vDash Rt_0 \dots t_{n-1}[\beta] & :\text{iff} & \big\langle t_0^{\mathfrak{A}}[\beta], \dots, t_{n-1}^{\mathfrak{A}}[\beta] \big\rangle \in R^{\mathfrak{A}}\,, \\
\mathfrak{A} \vDash \neg\varphi[\beta] & :\text{iff} & \mathfrak{A} \nvDash \varphi[\beta]\,, \\
\mathfrak{A} \vDash \bigvee \Phi[\beta] & :\text{iff} & \text{there is some } \varphi \in \Phi \text{ such that} \\
 & & \mathfrak{A} \vDash \varphi[\beta]\,, \\
\mathfrak{A} \vDash \bigwedge \Phi[\beta] & :\text{iff} & \mathfrak{A} \vDash \varphi[\beta] \text{ for all } \varphi \in \Phi\,, \\
\mathfrak{A} \vDash \exists x\varphi[\beta] & :\text{iff} & \text{there is some } a \in A \text{ such that} \\
 & & \mathfrak{A} \vDash \varphi[\beta[x/a]]\,, \\
\mathfrak{A} \vDash \forall x\varphi[\beta] & :\text{iff} & \mathfrak{A} \vDash \varphi[\beta[x/a]] \text{ for all } a \in A\,.
\end{array}
$$

The set *defined* by a formula φ is $\varphi^{\mathfrak{A}} := \{\, \beta \in A^X \mid \mathfrak{A} \vDash \varphi[\beta] \,\}$.

Remark. For $X = \varnothing$, we simply write $\mathfrak{A} \vDash \varphi$ and we identify the pair $\langle \mathfrak{A}, \varnothing \rangle$ with the structure $\mathfrak{A}$. In this case $\varphi^{\mathfrak{A}}$ is either $\varnothing$ or $A^{\varnothing} = \{\varnothing\}$.

Exercise 1.1. Let $\mathfrak{N} := \langle \omega, +, 0, 1 \rangle$ be the natural numbers with addition and consider the formula

$$\varphi := \forall x \exists y [x = y + y \lor x = y + y + 1]\,.$$

Using the above definition, give a formal proof that $\mathfrak{N} \vDash \varphi$.

Definition 1.6. We will use the abbreviations

$$
\begin{array}{rlrl}
\text{true} := & \bigwedge \varnothing\,, & \text{false} := & \bigvee \varnothing\,, \\
\varphi \lor \psi := & \bigvee \{\varphi, \psi\}\,, & \varphi \to \psi := & \neg\varphi \lor \psi\,, \\
\varphi \land \psi := & \bigwedge \{\varphi, \psi\}\,, & \varphi \leftrightarrow \psi := & (\varphi \to \psi) \land (\psi \to \varphi)\,,
\end{array}
$$

and $\quad t_0 \neq t_1 := \neg(t_0 = t_1)\,.$

The operation $\to$ is called *implication.* We abbreviate $\exists x_0 \cdots \exists x_{n-1}$ as $\exists \bar{x}$ and $\forall x_0 \cdots \forall x_{n-1}$ as $\forall \bar{x}$. Furthermore, we set

$$(\exists \bar{x}.\gamma)\varphi := \exists \bar{x}(\gamma \land \varphi) \quad \text{and} \quad (\forall \bar{x}.\gamma)\varphi := \forall \bar{x}(\gamma \to \varphi)\,.$$

Quantifiers of the form $(\exists \bar{x}.\gamma)$ and $(\forall \bar{x}.\gamma)$ are called *relativised quantifiers,* the formula γ is the *guard* of the quantifier.

Remark. To avoid unnecessary parenthesis we employ the following precedence rules.

- Unary operators like quantifiers, negation, and the large conjunction and disjunction signs bind strongest. For instance, the formula

$$\exists x \neg \bigwedge_{i<5} P_i x \wedge \exists y P_0 y \quad \text{is read as} \quad \Big(\exists x \neg \bigwedge_{i<5} P_i x\Big) \wedge (\exists y P_0 y)\,.$$

- $\wedge$ binds stronger than $\vee$, $\rightarrow$, and $\leftrightarrow$.
- The precedence between $\vee$, $\rightarrow$, and $\leftrightarrow$ is left unspecified.

Example. (a) Let $x_0, \ldots, x_{n-1}$ be variables of sort s. The formula

$$\varphi_n := \exists x_0 \cdots \exists x_{n-1} \bigwedge_{i \neq k} x_i \neq x_k$$

expresses that the universe contains at least n different elements of sort s. Therefore, we can say that the domain of sort s is finite by the sentence

$$\varphi_{\text{fin}} := \bigvee \{\, \neg\varphi_n \mid n < \omega \,\}\,.$$

(b) Let $\Sigma = \{<\}$ be the signature of strict partial orders. We can express that an element y is the immediate successor of an element x by the formula

$$\varphi := x < y \wedge \neg\exists z (x < z \wedge z < y)\,.$$

(c) Let $\mathfrak{G} = \langle V, E\rangle$ be a graph. For every $n < \omega$, we can write down a first-order formula ψ_n saying that there exists a path of length at most n from the element x to y:

$$\psi_n := \exists z_0 \cdots \exists z_n \Big(z_0 = x \wedge z_n = y \wedge \bigwedge_{i<n} (z_i = z_{i+1} \vee E z_i z_{i+1})\Big)\,.$$

The $\mathrm{FO}_{\aleph_1 \aleph_0}$-formula

$$\varphi_{\text{sc}} := \forall x \forall y \bigvee_{n<\omega} \psi_n$$

expresses that the graph is strongly connected.

(d) Let $\langle \mathbb{R}, +, -, 0, <, f\rangle$ be the additive ordered group of the real numbers with one unary function symbol f. We can say that $|x - y| < z$ by the formula

$$x - y < z \;\wedge\; y - x < z\,.$$

Making heavy use of relativised quantifiers, we can express that the function f is continuous at x by the formula

$$\begin{aligned}&(\forall \varepsilon. \varepsilon > 0)(\exists \delta. \delta > 0)\\ &\quad (\forall y. x - y < \delta \wedge y - x < \delta)\\ &\qquad (fx - fy < \varepsilon \wedge fy - fx < \varepsilon)\,.\end{aligned}$$

Exercise 1.2. (a) Let $\langle A, \leq, P\rangle$ be a linear order with an additional unary predicate $P \subseteq A$. Write down a first-order formula $\varphi(x)$ which says that x is the supremum of P.

(b) Let $\langle V, E\rangle$ be a graph. Define a first-order formula φ which states that every vertex has exactly two outgoing edges.

Lemma 1.7. *For every ordinal $\alpha < \kappa$, there exists an $\mathrm{FO}_{\kappa\aleph_0}$-formula φ_α such that*

$$\mathfrak{A} \models \varphi_\alpha \quad \text{iff} \quad \mathfrak{A} \cong \langle \alpha, <\rangle\,.$$

Proof. We define a slightly more general formula $\psi_\alpha(x)$ such that

$$\mathfrak{A} \models \psi_\alpha(a) \quad \text{iff} \quad \langle {\downarrow} a, <\rangle \cong \langle \alpha, <\rangle\,.$$

The sentence

$$\vartheta := \forall x \neg (x < x) \wedge \forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$$

states that $<$ is a strict linear order. By induction on α, we set

$$\psi_0(x) := \neg\exists y (y < x) \wedge \vartheta\,,$$

and $\psi_\alpha(x) := \bigwedge_{\beta<\alpha} (\exists y.y < x)\psi_\beta(y) \wedge (\forall y.y < x) \bigvee_{\beta<\alpha} \psi_\beta(y)$.

Hence, we can define the desired formula φ_α by

$$\varphi_\alpha := \bigwedge_{\beta<\alpha} \exists y \psi_\beta(y) \wedge \forall y \bigvee_{\beta<\alpha} \psi_\beta(y). \qquad \square$$

We can define the notions of a free variable, a subformula, substitution, etc. for formulae in the same way as for terms. But note that, unlike terms, formulae can contain variables that are not free.

Definition 1.8. Let $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$.

(a) A subterm of φ is called a *subformula.*

(b) The set $\mathrm{free}(\varphi)$ of *free variables* of φ is the minimal set X_0 such that $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X_0]$. A formula without free variables is called a *sentence.*

(c) An occurrence of a variable x in a formula φ is *bound* if it lies in a subformula of the form $\exists x\psi$ or $\forall x\psi$. Otherwise, the occurrence of x is *free.*

(d) For a sequence $\bar{s} \in S^I$ of sorts, let $X_{\bar{s}} := \{ x_i \mid i \in I \}$ be a standard set of variables where x_i is of sort s_i. We set

$$\mathrm{FO}^{\bar{s}}_{\kappa\aleph_0}[\Sigma] := \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X_{\bar{s}}].$$

For ordinals α, we define

$$\mathrm{FO}^{\alpha}_{\kappa\aleph_0}[\Sigma] := \bigcup_{\bar{s}\in S^\alpha} \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X_{\bar{s}}]$$

and $$\mathrm{FO}^{<\alpha}_{\kappa\aleph_0}[\Sigma] := \bigcup_{\beta<\alpha} \mathrm{FO}^{\beta}_{\kappa\aleph_0}[\Sigma].$$

Remark. (a) Every $\mathrm{FO}_{\kappa\aleph_0}$-formula has less than κ free variables.

(b) Note that a variable x can occur both free and bound in the same formula φ.

Obviously, the truth value of a formula only depends on the symbols actually appearing in it. This triviality is recorded in the following lemma. Like the corresponding result for terms it can be proved by a straightforward induction on the structure of φ.

Lemma 1.9 (Coincidence Lemma). *Let $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Gamma, Y]$ be a formula and, for $i < 2$, let $\mathfrak{A}_i$ be a Σ_i-structure and $\beta_i : X_i \to A_i$ a variable assignment. If*

- $\Gamma \subseteq \Sigma_0 \cap \Sigma_1$ *and* $\mathrm{free}(\varphi) \subseteq X_0 \cap X_1$,
- $\mathfrak{A}_0|_\Gamma = \mathfrak{A}_1|_\Gamma$ *and* $\beta_0 \restriction \mathrm{free}(\varphi) = \beta_1 \restriction \mathrm{free}(\varphi)$

then we have $\mathfrak{A}_0 \vDash \varphi[\beta_0]$ *iff* $\mathfrak{A}_1 \vDash \varphi[\beta_1]$.

Remark. We will write $\varphi(x_0, \dots, x_{n-1})$ to indicate that

$$\mathrm{free}(\varphi) \subseteq \{x_0, \dots, x_{n-1}\}.$$

Furthermore, if $a_0, \dots, a_{n-1}$ are elements of the structure $\mathfrak{A}$, we write

$$\mathfrak{A} \vDash \varphi(a_0, \dots, a_{n-1})$$

instead of $\mathfrak{A} \vDash \varphi[\beta]$ for the assignment $\beta : x_i \mapsto a_i$. By the Coincidence Lemma, this notation is well-defined. Similarly, we write $\Phi(\bar{x})$ and $\mathfrak{A} \vDash \Phi(\bar{a})$, for sets $\Phi \subseteq \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$.

Let us compute the number of $\mathrm{FO}_{\kappa\aleph_0}$-formulae. Note that the number of finite formulae follows immediately from Lemma B3.1.5.

Lemma 1.10. *Let κ be a regular cardinal. Every formula $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ has less than κ subformulae.*

Proof. Using the same notation as in the remark after Definition 1.2, we see that φ is a Λ-term with $\mathrm{dom}\,\varphi \subseteq \kappa^{<\omega}$. If $\kappa = \aleph_0$ then φ is a finite term that has only finitely many subformulae. Suppose that $\kappa > \aleph_0$. Since κ is regular it follows by induction on φ that there exists a cardinal $\lambda < \kappa$ such that $\mathrm{dom}\,\varphi \subseteq \lambda^{<\omega}$. Hence, $|\mathrm{dom}\,\varphi| \leq \lambda^{<\omega} = \lambda \oplus \aleph_0 < \kappa$. $\square$

Lemma 1.11. *Let Σ be a signature, X a set of variables, and κ a regular cardinal.*

$$|\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]| \leq (|\Sigma| \oplus |X| \oplus \aleph_0)^{<\kappa}.$$

Proof. We have shown in the preceding lemma that every infinitary first-order formula $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ is a Λ-term with $|\mathrm{dom}\,\varphi| < \kappa$. Furthermore, we have

$$|\Lambda| \leq |\Sigma| \oplus |X'| \oplus \aleph_0 \leq |\Sigma| \oplus |X| \oplus |\mathrm{dom}\,\varphi| \oplus \aleph_0 \,.$$

Consequently, it follows that

$$\begin{aligned}|\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]| &\leq \sup_{\lambda<\kappa} \big(|\Sigma| \oplus |X| \oplus \lambda \oplus \aleph_0\big)^{\lambda} \\ &= \sup_{\lambda<\kappa} \big(|\Sigma| \oplus |X| \oplus \aleph_0\big)^{\lambda} = \big(|\Sigma| \oplus |X| \oplus \aleph_0\big)^{<\kappa}. \qquad \square\end{aligned}$$

Remark. In the preceding lemma, we have tacitly identified formulae φ and ψ that differ only in the names of bound variables, i.e., variables in $X' \smallsetminus X$. Hence, the above bound holds only up to this equivalence relation. Clearly, if we distinguish the formulae $\exists x Px$, $\exists y Py$, $\exists z Pz, \dots$ then we can construct arbitrarily many formulae by using that many different variable names.

Exercise 1.3. Prove that every formula $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$ can be rewritten to use only countably many different bound variables. That is, for every sort s, there exists a countable set Y_s such that φ can be written as Λ-term with

$$\Lambda := \Sigma \cup X \cup Y \cup \{=, \neg, \bigwedge, \bigvee\} \cup \{\, \exists x, \forall x \mid x \in X \cup Y \,\}\,,$$

where $Y = \bigcup_s Y_s$. *Hint.* If ψ is a subformula of φ then $\mathrm{free}(\psi) \smallsetminus X$ is finite.

We have seen that each $\mathrm{FO}_{\infty\aleph_0}$-formula has a foundation rank. Hence, we could measure the complexity of a formula by its foundation rank. But this measure is not very meaningful. There exists another rank for formulae that better reflects the semantics of first-order logic.

Definition 1.12. The *quantifier rank* $\mathrm{qr}(\varphi) \in \mathrm{On}$ of a formula $\varphi \in \mathrm{FO}_{\infty\aleph_0}$ is defined inductively by:

- $\mathrm{qr}(R\bar{t}) := 0$ and $\mathrm{qr}(t = t') := 0$.
- $\mathrm{qr}(\neg\varphi) := \mathrm{qr}(\varphi)$.
- $\mathrm{qr}(\exists x\varphi) := \mathrm{qr}(\forall x\varphi) := \mathrm{qr}(\varphi) + 1$.
- $\mathrm{qr}(\bigwedge \Phi) := \mathrm{qr}(\bigvee \Phi) := \sup\{\, \mathrm{qr}(\varphi) \mid \varphi \in \Phi \,\}$.

A formula φ is *quantifier-free* if $\mathrm{qr}(\varphi) = 0$.

Example. For the formulae φ_{fin} and ψ_{sc} from the example on page 448, we have

$$\begin{aligned} &\mathrm{qr}(\varphi_{\mathrm{fin}}) = \sup\{\, \mathrm{qr}(\varphi_n) \mid n < \omega \,\} = \omega\,,\\ \text{and}\quad &\mathrm{qr}(\psi_{\mathrm{sc}}) = \sup\{\, \mathrm{qr}(\psi_n) \mid n < \omega \,\} + 2 = \omega + 2\,.\end{aligned}$$

Immediately from the respective definitions it follows that the foundation rank bounds the quantifier rank of a formula.

Lemma 1.13. $\mathrm{qr}(\varphi) \leq \mathrm{frk}(\varphi)$*, for all* $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$.

Corollary 1.14. *If* κ *is a regular cardinal then we have* $\mathrm{qr}(\varphi) < \kappa$*, for all* $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$.

If κ is singular then $\mathrm{FO}_{\kappa\aleph_0}$ can exhibit pathological behaviour. Fortunately, it is safe to ignore these logics and only consider $\mathrm{FO}_{\kappa\aleph_0}$ for regular cardinals κ.

Lemma 1.15. *For singular cardinals* κ*, the logics* $\mathrm{FO}_{\kappa\aleph_0}$ *and* $\mathrm{FO}_{\kappa^+\aleph_0}$ *have the same expressive power.*

Proof. Let κ be singular and fix a cofinal function $f : \mathrm{cf}\,\kappa \to \kappa$. Every conjunction of κ formulae can be written equivalently as nested conjunction of less then κ formulae:

$$\bigwedge_{i<\kappa} \varphi_i \quad \text{is equivalent to} \quad \bigwedge_{\alpha<\mathrm{cf}(\kappa)} \bigwedge_{i<f(\alpha)} \varphi_i \,.$$

Consequently, we can inductively transform every formula $\varphi \in \mathrm{FO}_{\kappa^+\aleph_0}$ into an equivalent $\mathrm{FO}_{\kappa\aleph_0}$-formula. $\square$

2. Axiomatisations

Let us begin a more systematic investigation of what can be expressed in first-order logic. In this section we give examples of classes of structures that can be defined in $\mathrm{FO}_{\infty\aleph_0}$.

Definition 2.1. Let $\langle L, \mathcal{K}, \vDash \rangle$ be a logic.
(a) A set of formulae $\Phi \subseteq L$ *axiomatises* the class

$$\mathrm{Mod}_L(\Phi) := \{ \mathfrak{I} \in \mathcal{K} \mid \mathfrak{I} \vDash \Phi \} .$$

For a single formula we simply write $\mathrm{Mod}_L(\varphi) := \mathrm{Mod}_L(\{\varphi\})$.
(b) A class $\mathcal{C} \subseteq \mathcal{K}$ of interpretations is *L-axiomatisable* if

$$\mathcal{C} = \mathrm{Mod}_L(\Phi) , \quad \text{for some } \Phi \subseteq L .$$

If $\mathcal{C} = \mathrm{Mod}_L(\Phi)$, for a finite set $\Phi \subseteq L$, we say that $\mathcal{C}$ is *finitely L-axiomatisable.* If $\mathcal{C}$ is axiomatised by Φ, we call the set Φ an *axiom system* for $\mathcal{C}$ and every $\varphi \in \Phi$ is an *axiom.*
(c) A set of formulae $\Phi \subseteq L$ is *consistent,* or *satisfiable,* if $\mathrm{Mod}_L(\Phi) \neq \varnothing$. Otherwise, Φ is called *inconsistent,* or *unsatisfiable.* If $\mathrm{Mod}_L(\Phi) = \mathcal{K}$, then Φ is called *valid* or a *tautology.* We use the same terminology for single formulae φ.

Example (Partial orders). The class of all partial orders is finitely first-order axiomatised by

$$\forall x \forall y \forall z (x \leq y \wedge y \leq z \rightarrow x \leq z) ,$$
$$\forall x \forall y (x \leq y \wedge y \leq x \leftrightarrow x = y) .$$

We get an axiom system for the class of linear orders if we add the formula

$$\forall x \forall y (x \leq y \vee y \leq x) .$$

A linear order is *dense* if between any two elements there exists a third one. The corresponding first-order axiom is

$$\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y)) ,$$

where $x < y$ abbreviates $x \leq y \wedge x \neq y$. A dense linear order is *open* if it does not have a least and a greatest element.

$$\forall x \exists y \exists z (y < x \wedge x < z) .$$

A *discrete* linear order is an order where every element, except for the first one, has an immediate predecessor and every element, except for the last one, has an immediate successor.

$$\forall x \big[\exists y (y < x) \rightarrow \exists y (y < x \wedge \neg \exists z (y < z \wedge z < x)) \big] ,$$
$$\forall x \big[\exists y (x < y) \rightarrow \exists y (x < y \wedge \neg \exists z (x < z \wedge z < y)) \big] .$$

Example (Equivalence relations). The class of all structures $\mathfrak{A} = \langle A, \sim \rangle$ where $\sim$ is an equivalence relation can be axiomatised by the first-order formulae

$$\forall x (x \sim x) ,$$
$$\forall x \forall y (x \sim y \leftrightarrow y \sim x) ,$$
$$\forall x \forall y \forall z (x \sim y \wedge y \sim z \rightarrow x \sim z) .$$

Example (Lattices). An axiom system for the class of lattices was given in Lemma B2.2.4.

$$\forall x \forall y (x \sqsubseteq y \leftrightarrow x \sqcap y = x)$$
$$\forall x (x \sqcap x = x \ \wedge \ x \sqcup x = x)$$
$$\forall x \forall y (x \sqcap y = y \sqcap x \ \wedge \ x \sqcup y = y \sqcup x)$$
$$\forall x \forall y (x \sqcap (x \sqcup y) = x \ \wedge \ x \sqcup (x \sqcap y) = x)$$
$$\forall x \forall y \forall z (x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z)$$
$$\forall x \forall y \forall z (x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z)$$

For boolean algebras we have to add the axioms

$$\bot \neq \top\,,$$
$$\forall x(\bot \sqcap x = \bot \wedge \bot \sqcup x = x)\,,$$
$$\forall x(\top \sqcap x = x \wedge \top \sqcup x = \top)\,,$$
$$\forall x(x \sqcap x^* = \bot \wedge x \sqcup x^* = \top)\,,$$
$$\forall x \forall y \forall z[x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)]\,,$$
$$\forall x \forall y \forall z[x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)]\,.$$

Example (Groups). The class of all groups (in the signature $\{\,\cdot\,, {}^{-1}, e\}$) can be finitely axiomatised in first-order logic by the sentences

$$\forall x \forall y \forall z[x \cdot (y \cdot z) = (x \cdot y) \cdot z]\,,$$
$$\forall x(x \cdot e = x)\,,$$
$$\forall x(x \cdot x^{-1} = e)\,.$$

If we only allow multiplication then these axioms become

$$\forall x \forall y \forall z[x \cdot (y \cdot z) = (x \cdot y) \cdot z]\,,$$
$$\exists e \forall x[x \cdot e = x \wedge \exists y(x \cdot y = e)]\,.$$

We can add the $\mathrm{FO}_{\aleph_1 \aleph_0}$-sentence φ_{fin} from page 448 to obtain an axiom system for the class of all finite groups. But note that this is an infinitary formula. We will prove in Theorem C2.4.12 that this class cannot be axiomatised in first-order logic.

The class of all infinite groups on the other hand is first-order axiomatisable. To the group axioms we can add, for all $n < \omega$, the sentence

$$\exists x_0 \cdots \exists x_{n-1} \bigwedge_{i<k<n} x_i \neq x_k\,.$$

This axiom system is necessarily infinite. If the class of infinite groups where axiomatisable by a single first-order sentence, its negation could be used to construct an axiom system of the class of all finite groups.

Example (Rings). The class of all rings $\langle R, +, -, \cdot, 0, 1\rangle$ is defined by

$$\forall x \forall y \forall z[x + (y + z) = (x + y) + z]\,,$$
$$\forall x(x + 0 = x)\,,$$
$$\forall x(x + (-x) = 0)\,,$$
$$\forall x \forall y(x + y = y + x)\,,$$
$$\forall x \forall y \forall z[x \cdot (y \cdot z) = (x \cdot y) \cdot z]\,,$$
$$\forall x(x \cdot 1 = x \wedge 1 \cdot x = x)\,,$$
$$\forall x \forall y \forall z[x \cdot (y + z) = x \cdot y + x \cdot z]\,,$$
$$\forall x \forall y \forall z[(y + z) \cdot x = y \cdot x + z \cdot x]\,.$$

Example (Fields). We obtain an axiom system for the class of all fields if we add to the ring axioms the formulae

$$0 \neq 1\,,$$
$$\forall x \exists y(x \neq 0 \rightarrow x \cdot y = 1)\,,$$
$$\forall x \forall y(x \cdot y = y \cdot x)\,.$$

To get axioms for the class of ordered fields, we further have to add the axioms for a linear order and the formulae

$$\forall x \forall y \forall z(x < y \rightarrow x + z < y + z)\,,$$
$$\forall x \forall y \forall z(x < y \wedge 0 < z \rightarrow x \cdot z < y \cdot z)\,.$$

Example (Set theory). The axioms of set theory can be expressed in first-order logic. The signature consists just of one binary relation symbol $\in$.

First, let us collect some auxiliary formulae. The subset relation $x \subseteq y$ can be defined by the formula

$$\forall z(z \in x \rightarrow z \in y)\,.$$

There are formulae $\mathrm{Stage}(x)$ and $\mathrm{WellOrder}(x, y)$ that express, respectively, that the set x is a stage and that y is a well-order on the set x (exercise).

The Axiom of Extensionality reads

$$\forall a \forall b [a = b \leftrightarrow \forall x (x \in a \leftrightarrow x \in b)] .$$

To express the Axiom of Separation we need infinitely many formulae. For every first-order formula $\varphi(x, \bar{z}) \in \mathrm{FO}$, we have the formula

$$\forall \bar{z} \forall a \exists b \forall x [x \in b \leftrightarrow x \in a \wedge \varphi(x, \bar{z})] .$$

(Since the signature $\{\in\}$ of set theory does not contain constant symbols, we need parameters $\bar{z}$ for those sets that φ might refer to.)

The Axioms of Creation and Infinity are

$$\forall a (\exists s.\mathrm{Stage}(s))(a \in s)$$

and $(\exists s.\mathrm{Stage}(s)) \forall x [x \in s \rightarrow \wp(x) \in s]$,

where $\wp(x) \in s$ is an abbreviation for the formula

$$\exists z [z \in s \wedge \forall y (y \in z \leftrightarrow y \subseteq x)] .$$

For the Axiom of Choice we have the formula

$$\forall a \exists r \mathrm{WellOrder}(a, r) .$$

Finally, the Axiom of Replacement again consists of several formulae, one for every formula $\varphi(x, y, \bar{z}) \in \mathrm{FO}$.

$$(\forall \bar{z}.\mathrm{fun}_\varphi(\bar{z}))[\exists u \,\mathrm{dom}_\varphi(\bar{z}, u) \rightarrow \exists u \,\mathrm{rng}_\varphi(\bar{z}, u)] ,$$

where

$$\mathrm{fun}_\varphi(\bar{z}) := \forall x \forall y \forall y' [\varphi(x, y, \bar{z}) \wedge \varphi(x, y', \bar{z}) \rightarrow y = y']$$

says that φ defines a function and the formulae

$$\mathrm{dom}_\varphi(\bar{z}, u) := \forall x \forall y (\varphi(x, y, \bar{z}) \rightarrow x \in u)$$

and $\mathrm{rng}_\varphi(\bar{z}, u) := \forall x \forall y (\varphi(x, y, \bar{z}) \rightarrow y \in u)$

express that u contains, respectively, the domain and the range of the function defined by φ.

Exercise 2.1. Define the following formulae over the signature $\{\in\}$.

(a) $\mathrm{Stage}(x)$ states that the set x is a stage.

(b) $\mathrm{Pair}(x, y, z)$ expresses that $z = \langle x, y \rangle$.

(c) $\mathrm{WellOrder}(x, y)$ says that y is a well-order on the set x.

Lemma 2.2. *If $\mathfrak{A}$ is a finite Σ-structure then the class $\{ \mathfrak{B} \mid \mathfrak{B} \cong \mathfrak{A} \}$ is first-order axiomatisable. If Σ is finite then it is finitely axiomatisable.*

Proof. First, we consider the case that Σ is finite. Let $a_0, \dots, a_{n-1}$ be an enumeration of A without repetitions. If $\mathfrak{A}$ has only one sort then we can axiomatise $\mathfrak{A}$ by the formula

$$\begin{aligned} \exists x_0 \cdots \exists x_{n-1} \Big(& \bigwedge_{0 \le i < k < n} x_i \neq x_k \wedge \forall y \bigvee_{i < n} y = x_i \\ & \wedge \bigwedge \{ R x_{i_0} \dots x_{i_k} \mid \langle a_{i_0}, \dots, a_{i_k} \rangle \in R^{\mathfrak{A}},\ R \in \Sigma \} \\ & \wedge \bigwedge \{ \neg R x_{i_0} \dots x_{i_k} \mid \langle a_{i_0}, \dots, a_{i_k} \rangle \notin R^{\mathfrak{A}},\ R \in \Sigma \} \\ & \wedge \bigwedge \{ f x_{i_0} \dots x_{i_k} = x_l \mid f^{\mathfrak{A}}(a_{i_0}, \dots, a_{i_k}) = a_l ,\ f \in \Sigma \} \Big) . \end{aligned}$$

The case of several sorts requires two modifications of this formula. We have to replace the subformula $\forall y \bigvee_i y = x_i$ by a conjunction of several such formulae where y is of the respective sort s and the disjunction ranges only over those i such that x_i has the same sort s. Furthermore, we have to remove from the conjunction $\bigwedge_{i<k} x_i \neq x_k$ all inequations $x_i \neq x_k$ where x_i and x_k have different sorts.

Suppose that Σ is infinite. For each finite subsignature $\Sigma_0 \subseteq \Sigma$, we can construct a formula φ_{Σ_0} axiomatising the Σ_0-reduct $\mathfrak{A}|_{\Sigma_0}$ of $\mathfrak{A}$. We claim that the set

$$\Phi := \{ \varphi_{\Sigma_0} \mid \Sigma_0 \subseteq \Sigma \text{ is finite} \}$$

is the desired axiom system. Clearly, $\mathfrak{A} \vDash \Phi$. Conversely, suppose that $\mathfrak{B} \vDash \Phi$. Then B has exactly $n := |A|$ elements. For every finite signature

$\Sigma_0 \subseteq \Sigma$, there exists a sequence $\bar{b}^{\Sigma_0} \in B^n$ such that we can satisfy the formula φ_{Σ_0} if we assign to the variable x_i the element $b_i^{\Sigma_0}$. Define

$$S := \{ \Sigma_0 \subseteq \Sigma \mid \Sigma_0 \text{ finite} \}$$
$$\text{and} \quad S(\bar{b}) := \{ \Sigma_0 \in S \mid \bar{b}^{\Sigma_0} = \bar{b} \}, \quad \text{for } \bar{b} \in B^n.$$

Then $\langle S, \subseteq \rangle$ is a directed partial order with a finite partition $S = \bigcup_{\bar{b}} S(\bar{b})$. By Proposition B3.3.4, there exists some $\bar{b}$ such that $S(\bar{b})$ is a dense subset of S. It follows that the mapping $b_i \mapsto a_i$ is an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. □

3. Theories

In the previous section we have studied sets of formulae and the classes they axiomatise. Now we turn to the dual question. Given a class of structures we try to determine which formulae hold.

Definition 3.1. Let $\langle L, \mathcal{K}, \models \rangle$ be a logic, $\mathfrak{J} \in \mathcal{K}$ an interpretation, $\varphi, \psi \in L$ formulae, and $\Phi \subseteq L$ a set of formulae.

(a) We write

$$\Phi \models \varphi \quad :\text{iff} \quad \mathrm{Mod}_L(\Phi) \subseteq \mathrm{Mod}_L(\varphi).$$

If $\Phi \models \varphi$ then φ is called a *consequence* of Φ. We also say that φ *follows* from Φ or that Φ *entails* φ.

If $\Phi = \{\psi\}$ we simply write $\psi \models \varphi$ and, for $\Phi = \varnothing$, we write $\models \varphi$. Note that we use the same symbol $\models$ both for the satisfaction relation and for the entailment relation. The object on the left-hand side can be used to resolve any ambiguities.

(b) If $\varphi \models \psi$ and $\psi \models \varphi$ then φ and ψ are called *equivalent* and we write $\varphi \equiv \psi$. Similarly, if $\Phi \cup \{\varphi\} \models \psi$ and $\Phi \cup \{\psi\} \models \varphi$, we say that φ and ψ are *equivalent modulo* Φ.

(c) The *closure of* Φ *under entailment* is the set

$$\Phi^{\models} := \{ \varphi \in L \mid \Phi \models \varphi \}.$$

Remark. Note that if L_0 and L_1 are logics with the same class of interpretations, we can generalise the above definitions of $\Phi \models \varphi$ and $\varphi \equiv \psi$ also to the case that $\Phi \subseteq L_0$, $\psi \in L_0$, and $\varphi \in L_1$.

Example. If $p, q \in \mathrm{ZL}[\mathfrak{K}, X]$ where $\mathfrak{K}$ is algebraically closed then we have

$$\begin{aligned} p \models q \quad &\text{iff} \quad \text{every zero of } p \text{ is a zero of } q \\ &\text{iff} \quad p \mid q^n, \quad \text{for some } n < \omega. \end{aligned}$$

Consequently, $p^{\models} = \{ q \in \mathfrak{K}[X] \mid p \mid q^n \text{ for some } n < \omega \} \trianglelefteq \mathfrak{K}[X]$ is the radical ideal generated by p and we have

$$p \equiv q \quad \text{iff} \quad p^m = aq^n \text{ for some } a \in \mathfrak{K} \text{ and } m, n < \omega.$$

Each non-constant polynomial is satisfiable. A constant polynomial p is satisfiable if and only if $p = 0$. The polynomial $p = 0$ is the only tautology.

The following properties of the entailment relation follow immediately from the definition.

Lemma 3.2. *Let $\langle L, \mathcal{K}, \models \rangle$ be a logic.*

(a) *$\models$ is a preorder on L.*

(b) *A set $\Phi \subseteq L$ is a final segment of $\langle L, \models \rangle$ if, and only if, $\Phi = \Phi^{\models}$.*

(c) *If $\Phi \subseteq L$ is inconsistent, then $\Phi^{\models} = L$.*

(d) *φ is a tautology if, and only if, $\varnothing \models \varphi$.*

Definition 3.3. Let $\langle L, \mathcal{K}, \models \rangle$ be a logic.

(a) An L-*theory* is a set of formulae $T \subseteq L$ with $T^{\models} = T$. The L-theory of a class $\mathcal{C} \in \mathcal{K}$ is the set

$$\mathrm{Th}_L(\mathcal{C}) := \{ \varphi \in L \mid \mathfrak{J} \models \varphi, \text{ for all } \mathfrak{J} \in \mathcal{C} \}.$$

The L-theory of a single interpretation $\mathfrak{J} \in \mathcal{K}$ is $\mathrm{Th}_L(\mathfrak{J}) := \mathrm{Th}_L(\{\mathfrak{J}\})$.

(b) An L-theory T is *complete* if it is of the form $T = \mathrm{Th}_L(\mathfrak{J})$, for some L-interpretation $\mathfrak{J}$.

(c) Two L-interpretations $\mathfrak{J}_0$ and $\mathfrak{J}_1$ are *L-equivalent* if

$$\mathrm{Th}_L(\mathfrak{J}_0) = \mathrm{Th}_L(\mathfrak{J}_1)\,.$$

We write $\mathfrak{J}_0 \equiv_L \mathfrak{J}_1$ to denote this fact. As usual we omit the index L if $L = \mathrm{FO}^0[\Sigma]$.

Example. Let $\mathfrak{B}$ be a boolean algebra, $a, b \in B$, and $\mathfrak{u} \in \mathrm{spec}(\mathfrak{B})$. For boolean logic $\mathrm{BL}(\mathfrak{B}) = \langle B,\ \mathrm{spec}(\mathfrak{B}),\ \vDash\rangle$, we have

$$\begin{aligned} a \vDash b \quad &\text{iff} \quad \text{every ultrafilter containing } a \text{ also contains } b \\ &\text{iff} \quad a \sqsubseteq b\,, \end{aligned}$$

and $\quad \mathrm{Th}_{\mathrm{BL}(\mathfrak{B})}(\mathfrak{u}) = \{\, b \in B \mid \mathfrak{u} \vDash b \,\} = \mathfrak{u}\,.$

Remark. (a) The function $\Phi \mapsto \Phi^{\vDash}$ is a closure operator on L whose closed sets are the theories. Consequently, the set of all L-theories forms a complete partial order where the least element is the set $\varnothing^{\vDash}$ of all tautologies and the greatest element is the set L of all formulae.

(b) For $\Phi \subseteq L$, we have

$$\Phi = \mathrm{Th}_L(\mathrm{Mod}_L(\Phi)) \quad \text{iff} \quad \Phi = \Phi^{\vDash} \quad \text{iff} \quad \Phi \text{ is a theory}\,.$$

Exercise 3.1. Let T be a satisfiable L-theory such that there is no satisfiable L-theory T' with $T \subset T'$. Prove that T is complete.

The following properties of the entailment relation follow immediately from the definition. We say that a logic L is *closed under negation* if, for every formula $\varphi \in L$, there is some formula $\neg\varphi \in L$ with

$$\mathfrak{J} \vDash \neg\varphi \quad \text{iff} \quad \mathfrak{J} \nvDash \varphi\,.$$

Similarly, L is *closed under implication* if there are formulae $\varphi \to \psi$ such that

$$\mathfrak{J} \vDash \varphi \to \psi \quad \text{iff} \quad \mathfrak{J} \nvDash \varphi \text{ or } \mathfrak{J} \vDash \psi \text{ or both.}$$

Lemma 3.4. *Let L be a logic, $\Phi \subseteq L$, and $\varphi, \psi \in L$.*

(a) *$\Phi \vDash \varphi$ implies $\Psi \vDash \varphi$, for every $\Psi \supseteq \Phi$.*

If L is closed under negation then we have

(b) *$\Phi \vDash \varphi$ if, and only if, $\Phi \cup \{\neg\varphi\}$ is inconsistent;*

(c) *φ is satisfiable if, and only if, $\neg\varphi$ is no tautology;*

(d) *Φ is a complete theory if, and only if, we have*

$$\Phi \nvDash \varphi \quad \text{iff} \quad \Phi \vDash \neg\varphi\,, \quad \text{for all } \varphi \in L\,.$$

If L is closed under implication then we have

(f) *$\Phi \cup \{\varphi\} \vDash \psi$ if, and only if, $\Phi \vDash \varphi \to \psi$;*

(g) *$\varphi \equiv \psi$ modulo Φ if, and only if, $\Phi \vDash \varphi \to \psi$ and $\Phi \vDash \psi \to \varphi$.*

We conclude this section with a collection of equivalences that can be used to simplify first-order formulae. We start with the boolean operations which, of course, satisfy the laws of a boolean algebra.

Lemma 3.5. *The following equivalences hold for $\varphi, \psi, \vartheta \in \mathrm{FO}_{\infty\aleph_0}[\Sigma]$:*

(a) $\neg\neg\varphi \equiv \varphi$ (*elimination of double negation*)

(b) $\varphi \wedge \psi \equiv \psi \wedge \varphi$ (*commutativity*)
$\varphi \vee \psi \equiv \psi \vee \varphi$

(c) $(\varphi \wedge \psi) \wedge \vartheta \equiv \varphi \wedge (\psi \wedge \vartheta)$ (*associativity*)
$(\varphi \vee \psi) \vee \vartheta \equiv \varphi \vee (\psi \vee \vartheta)$

(d) $\varphi \wedge \varphi \equiv \varphi$ (*idempotence*)
$\varphi \vee \varphi \equiv \varphi$

(e) $\neg \bigwedge_{i<\alpha} \varphi_i \equiv \bigvee_{i<\alpha} \neg\varphi_i$ (*de Morgan's laws*)
$\neg \bigvee_{i<\alpha} \varphi_i \equiv \bigwedge_{i<\alpha} \neg\varphi_i$

(f) $\varphi \to \psi \equiv \neg\psi \to \neg\varphi$ (*contraposition*)

(g) $\varphi \wedge (\varphi \vee \psi) \equiv \varphi$ (*absorption*)
$\varphi \vee (\varphi \wedge \psi) \equiv \varphi$

(h) $\varphi \wedge (\psi \vee \vartheta) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \vartheta)$ (*distributivity*)
$\varphi \vee (\psi \wedge \vartheta) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \vartheta)$

Lemma 3.6. *The following equivalences hold for* $\varphi, \psi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma]$.

(a) $\exists x\varphi \vee \exists x\psi \equiv \exists x(\varphi \vee \psi)$
$\forall x\varphi \wedge \forall x\psi \equiv \forall x(\varphi \wedge \psi)$

(b) $\neg\exists x\varphi \equiv \forall x\neg\varphi$
$\neg\forall x\varphi \equiv \exists x\neg\varphi$

(c) $\exists x\exists y\varphi \equiv \exists y\exists x\varphi$
$\forall x\forall y\varphi \equiv \forall y\forall x\varphi$

Furthermore, if $x \notin \mathrm{free}(\varphi)$ *then we also have*

(d) $\varphi \wedge \exists x\psi \equiv \exists x(\varphi \wedge \psi)$
$\varphi \vee \forall x\psi \equiv \forall x(\varphi \vee \psi)$

(e) $\varphi \vee \exists x\psi \equiv \exists x(\varphi \vee \psi)$ *modulo* $\exists x(x = x)$
$\varphi \wedge \forall x\psi \equiv \forall x(\varphi \wedge \psi)$ *modulo* $\exists x(x = x)$

(f) $\varphi \equiv \exists x\varphi$ *modulo* $\exists x(x = x)$
$\varphi \equiv \forall x\varphi$ *modulo* $\exists x(x = x)$

Remark. Note that the equivalences (e) and (f) only hold in structures that contain at least one element of the corresponding sort.

Exercise 3.2. Prove some of the above equivalences.

Example. In general we have

$$\exists x(\varphi \wedge \psi) \not\equiv \exists x\varphi \wedge \exists x\psi\,,$$
$$\forall x(\varphi \vee \psi) \not\equiv \forall x\varphi \vee \forall x\psi\,,$$
$$\exists x\forall y\varphi \not\equiv \forall y\exists x\varphi\,.$$

For a counterexample, consider the structure $\mathfrak{A} = \langle A, P\rangle$ with $A = \{0, 1\}$ and $P = \{1\}$. We have

$$\mathfrak{A} \vDash \exists xPx \wedge \exists x\neg Px \quad \text{but} \quad \mathfrak{A} \nvDash \exists x(Px \wedge \neg Px)\,,$$
$$\mathfrak{A} \vDash \forall x(Px \vee \neg Px) \quad \text{but} \quad \mathfrak{A} \nvDash \forall xPx \vee \forall x\neg Px\,,$$
$$\mathfrak{A} \vDash \forall y\exists x(x = y) \quad \text{but} \quad \mathfrak{A} \nvDash \exists x\forall y(x = y)\,.$$

4. *Normal forms*

In this section we study syntactic operations on first-order formulae. In particular, we will define several ways to simplify a given formula. We start by generalising the operation of substitution from terms to formulae.

Definition 4.1. Let $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$ be a formula, $t \in T[\Sigma, X]$ a term, and $x \in X$ a variable. The *substitution* of t for x in φ is the formula $\varphi[x/t]$ obtained from φ by

- renaming the bound variables of φ such that no variable in $\mathrm{free}(t)$ is bound in φ, and
- replacing every free occurrence of x in φ by the term t.

Example. (a) When substituting terms in formulae, we have to take care to avoid clashes with bound variables in order not to change the meaning of the formula. For instance, consider the formula $\exists y(y + y = x)$ which expresses that x is divisible by 2. If we substitute y for x, we expect the formula to say that y is divisible by 2. If we rename the bound variable to z, we obtain the formula $\exists z(z + z = y)$ which has the expected semantics.

But if we forget the renaming, we get $\exists y(y + y = y)$ which has an altogether different meaning.

(b) Renaming bound variables does not change the meaning of a formula. But note that renaming of free variables does. For instance, we have $\exists zRxz \not\equiv \exists zRyz$ since the interpretation $\langle \mathfrak{A}, \beta \rangle$ with

$$\mathfrak{A} := \big\langle [2], \{\langle 0, 1 \rangle\} \big\rangle \quad \text{and} \quad \beta(x) := 0\,,\ \beta(y) := 1$$

satisfies the first formula but not the second one.

Remark. Note that, if $\varphi \equiv \psi$ are equivalent formulae, we have

$$\neg\varphi \equiv \neg\psi\,, \quad \exists x\varphi \equiv \exists x\psi\,, \quad \text{and} \quad \forall x\varphi \equiv \forall x\psi\,.$$

Similarly, $\varphi_i \equiv \psi_i$, for all i, implies that

$$\textstyle\bigwedge_i \varphi_i \equiv \bigwedge_i \psi_i \quad \text{and} \quad \bigvee_i \varphi_i \equiv \bigvee_i \psi_i\,.$$

By induction it follows that, if is φ a subformula of ϑ and $\varphi \equiv \psi$, then $\vartheta \equiv \vartheta[\varphi/\psi]$ where $\vartheta[\varphi/\psi]$ denotes the formula obtained from ϑ by replacing the subformula φ by ψ.

In the following we give a quick summary of various normal forms for first-order logic. That is, we present subsets $\Phi \subseteq \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ defined by some syntactic criterion and we prove that every formula of $\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ is logically equivalent to an element of Φ. We start by simplifying the terms appearing in a formula.

Definition 4.2. A formula $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$ is *term-reduced* if every atomic subformula of φ is of the form

$$R\bar{x}\,, \quad f\bar{x} = y\,, \quad \text{or} \quad y = z\,,$$

where $\bar{x}$, y, and z are variables.

Lemma 4.3. *For each formula $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$, we can construct a term-reduced formula $\psi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ such that $\varphi \equiv \psi$.*

Proof. If φ is not term-reduced, it contains a subformula ϑ of the form $R\bar{t}$ or $f\bar{t} = s$ where not all elements of $\bar{t}$ and s are variables. Suppose that t_0 is not a variable. If z is a variable that does not appear in ϑ, we can replace $Rt_0 \dots t_{n-1}$ by the equivalent formula

$$\exists z(t_0 = z \wedge Rzt_1 \dots t_{n-1})\,.$$

Similarly, we can replace $f\bar{t} = s$ by

$$\exists z(t_0 = z \wedge fzt_1 \dots t_{n-1} = s)\,.$$

By induction, it follows that, for every atomic subformula ϑ of φ, there exists a term-reduced formula $\chi_\vartheta \equiv \vartheta$. We obtain the desired formula ψ by replacing every atom ϑ in φ by the corresponding term-reduced formula χ_ϑ. □

Definition 4.4. (a) A formula is in *disjunctive normal form* if it is of the form

$$\bigvee \{ \textstyle\bigwedge \Phi_i \mid i \in I \}$$

where each Φ_i is a set of literals.

(b) A formula is in *conjunctive normal form* if it is of the form

$$\bigwedge \{ \textstyle\bigvee \Phi_i \mid i \in I \}$$

where each Φ_i is a set of literals.

Lemma 4.5. *For every quantifier-free formula $\varphi \in \mathrm{FO}[\Sigma, X]$, there exist equivalent $\mathrm{FO}[\Sigma, X]$-formulae $\mathrm{DNF}(\varphi)$ and $\mathrm{CNF}(\varphi)$ that are in, respectively, disjunctive normal form and conjunctive normal form.*

Proof. We construct $\mathrm{DNF}(\varphi)$ and $\mathrm{CNF}(\varphi)$ by induction on φ. If φ is a literal, we can set

$$\mathrm{DNF}(\varphi) := \varphi \quad \text{and} \quad \mathrm{CNF}(\varphi) := \varphi\,.$$

Suppose that, by inductive hypothesis, we have

$$\mathrm{DNF}(\psi) = \bigvee_i \bigwedge_k \alpha_{ik} \quad \text{and} \quad \mathrm{CNF}(\psi) = \bigwedge_i \bigvee_k \beta_{ik}$$

$$\mathrm{DNF}(\vartheta) = \bigvee_i \bigwedge_k \gamma_{ik} \quad \text{and} \quad \mathrm{CNF}(\vartheta) = \bigwedge_i \bigvee_k \delta_{ik}$$

Then we can set

$$\mathrm{DNF}(\neg\psi) := \bigvee_i \bigwedge_k \neg\beta_{ik}$$

$$\mathrm{CNF}(\neg\psi) := \bigwedge_i \bigvee_k \neg\alpha_{ik}$$

$$\mathrm{DNF}(\psi \wedge \vartheta) := \bigvee_i \bigvee_j \Big(\bigwedge_k \alpha_{ik} \wedge \bigwedge_k \gamma_{jk}\Big)$$

$$\mathrm{CNF}(\psi \wedge \vartheta) := \mathrm{CNF}(\psi) \wedge \mathrm{CNF}(\vartheta)$$

$$\mathrm{DNF}(\psi \vee \vartheta) := \mathrm{DNF}(\psi) \vee \mathrm{DNF}(\vartheta)$$

$$\mathrm{CNF}(\psi \vee \vartheta) := \bigwedge_i \bigwedge_j \Big(\bigvee_k \beta_{ik} \wedge \bigvee_k \delta_{jk}\Big).$$

□

Exercise 4.1. Prove the corresponding statement for $\mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$.

When doing inductions on the structure of a formula, it is sometimes useful not to have to treat the case of negations. In such cases we can use de Morgan's laws to move all negation signs directly in front of atoms.

Definition 4.6. Given a formula $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$, we construct two formulae φ^+ and φ^- as follows. If φ is atomic, we set $\varphi^+ := \varphi$ and $\varphi^- := \neg\varphi$. For other formulae we define

$$\begin{aligned}
(\neg\psi)^+ &:= \psi^-, & (\neg\psi)^- &:= \psi^+,\\
(\textstyle\bigwedge \Phi)^+ &:= \textstyle\bigwedge\{\, \psi^+ \mid \psi \in \Phi \,\}, & (\textstyle\bigwedge \Phi)^- &:= \textstyle\bigvee\{\, \psi^- \mid \psi \in \Phi \,\},\\
(\textstyle\bigvee \Phi)^+ &:= \textstyle\bigvee\{\, \psi^+ \mid \psi \in \Phi \,\}, & (\textstyle\bigvee \Phi)^- &:= \textstyle\bigwedge\{\, \psi^- \mid \psi \in \Phi \,\},\\
(\exists x\psi)^+ &:= \exists x\psi^+, & (\exists x\psi)^- &:= \forall x\psi^-,\\
(\forall x\psi)^+ &:= \forall x\psi^+, & (\forall x\psi)^- &:= \exists x\psi^-.
\end{aligned}$$

The formula φ^+ is called *the negation normal form* of φ. It is denoted by $\mathrm{NNF}(\varphi)$. We say that φ is *in negation normal form* if $\mathrm{NNF}(\varphi) = \varphi$.

The following basic properties of the negation normal form of φ can be shown by a straightforward induction on the structure of φ.

Lemma 4.7. *Let* $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$.

(a) $\mathrm{NNF}(\varphi) \equiv \varphi$ *and* $\varphi^- \equiv \neg\varphi$.

(b) $\mathrm{NNF}(\varphi)$ *is in negation normal form.*

(c) φ *is in negation normal form if, and only if, the only subformulae of* φ *of the form* $\neg\psi$ *are literals.*

(d) $\mathrm{qr}(\mathrm{NNF}(\varphi)) = \mathrm{qr}(\varphi)$.

Definition 4.8. A formula $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ is in *prenex normal form* if it is of the form

$$\varphi = Q_0 x_0 \cdots Q_{n-1} x_{n-1} \psi$$

where $Q_0, \dots, Q_{n-1} \in \{\exists, \forall\}$ and ψ is quantifier-free.

We can transform formulae into prenex normal form only for structures with nonempty universe.

Definition 4.9. Let T_{ne} be the theory consisting, for every sort s, of one formula $\exists x_s (x_s = x_s)$ where x_s is of sort s.

T_{ne} expresses that all domains of a structure are nonempty. For models of T_{ne} we can construct prenex normal forms.

Lemma 4.10. *For every formula* $\varphi \in \mathrm{FO}[\Sigma, X]$*, there exists a formula* $\psi \in \mathrm{FO}[\Sigma, X]$ *in prenex normal form such that* $\varphi \equiv \psi$ *modulo* T_{ne}.

Proof. By induction on φ, we can move the quantifiers to the front using the equivalences of Lemma 3.6. Suppose that the prenex normal forms of ψ and ϑ are, respectively,

$$Q_0 x_0 \cdots Q_{m-1} x_{m-1} \psi_0 \quad \text{and} \quad Q'_0 y_0 \cdots Q'_{n-1} y_{n-1} \vartheta_0 ,$$

where all variables x_i and y_k are distinct. For $Q \in \{\exists, \forall\}$, define $\overline{Q}$ by $\overline{\exists} := \forall$ and $\overline{\forall} := \exists$. The prenex normal form of φ is

$$\begin{array}{ll} \varphi & \text{if } \varphi \text{ is atomic}, \\ \overline{Q}_0 x_0 \cdots \overline{Q}_{m-1} x_{m-1} \neg\psi_0 & \text{for } \varphi = \neg\psi, \\ Q_0 x_0 \cdots Q_{m-1} x_{m-1} Q'_0 y_0 \cdots Q'_{n-1} y_{n-1} (\psi_0 \wedge \vartheta_0) & \text{for } \varphi = \psi \wedge \vartheta, \\ Q_0 x_0 \cdots Q_{m-1} x_{m-1} Q'_0 y_0 \cdots Q'_{n-1} y_{n-1} (\psi_0 \vee \vartheta_0) & \text{for } \varphi = \psi \vee \vartheta, \\ \exists z Q_0 x_0 \cdots Q_{m-1} x_{m-1} \psi_0 & \text{for } \varphi = \exists z\psi, \\ \forall z Q_0 x_0 \cdots Q_{m-1} x_{m-1} \psi_0 & \text{for } \varphi = \forall z\psi. \quad \square \end{array}$$

In some cases we can get a prenex normal form that is fully equivalent instead of being only equivalent modulo T_{ne}.

Corollary 4.11. *Let Σ be an S-sorted signature satisfying either of the following conditions:*

- *For every $s \in S$, there is a constant symbol of sort s.*
- *$|S| = 1$ and S does not contain relations of arity 0.*

For every formula $\varphi \in \mathrm{FO}[\Sigma, X]$, there exists a formula $\psi \in \mathrm{FO}[\Sigma, X]$ in prenex normal form such that $\varphi \equiv \psi$.

Proof. In the first case, every Σ-structure is a model of T_{ne}. Hence, logical equivalence and equivalence modulo T_{ne} coincide.

In the second case, we can obtain ψ as follows. There exists a formula ψ' in prenex normal form such that $\varphi \equiv \psi'$ modulo T_{ne}. Note that, up to isomorphism, there exists exactly one Σ-structure $\mathfrak{A}_0$ with empty universe since we have no relations of arity 0. Let $x \notin \text{free}(\varphi)$ be a new variable. Note that $\mathfrak{A}_0 \vDash \forall x\psi'$ and $\mathfrak{A}_0 \nvDash \exists x\psi'$ regardless of what the formula ψ' looks like. Hence, we can set

$$\psi := \begin{cases} \forall x\psi' & \text{if } \mathfrak{A}_0 \vDash \varphi, \\ \exists x\psi' & \text{otherwise}. \end{cases}$$

For every nonempty structure $\mathfrak{A}$, we have $\mathfrak{A} \vDash \psi'$ iff $\mathfrak{A} \vDash \psi$. Consequently, $\varphi \equiv \psi$. $\square$

Remark. Infinitary formulae usually have no prenex normal form. For example, consider the sentence

$$\varphi := \bigwedge_{n<\omega} \exists x_0 \cdots \exists x_{n-1} \bigwedge_{i \neq k} x_i \neq x_k .$$

If we move all quantifiers to the front, we obtain a formula starting with an infinite string of quantifiers. This is forbidden by the definition of $\mathrm{FO}_{\infty\aleph_0}$.

When we are interested in whether some theory is satisfiable, we can also perform translations that, while preserving satisfiability, do not respect logical equivalence. For infinitary formulae the following reduction to first-order logic is useful. Another example is Skolemisation which transforms an arbitrary theory into a universal one (see Section C2.3).

Lemma 4.12 (Chang's Reduction). *For every $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$, there exists a signature $\Sigma_\varphi \supseteq \Sigma$ and a set $\Phi_\varphi \subseteq \mathrm{FO}_{\kappa\aleph_0}[\Sigma_\varphi, X]$ with the following properties:*

- *Every model of φ can be expanded in exactly one way to a model of Φ_φ.*
- *Every model of Φ_φ is a model of φ.*
- *Every subformula of φ is equivalent modulo Φ_φ to an atomic formula.*
- *Every formula in Φ_φ is either a first-order formula or a sentence of the form $\forall \bar{x} \bigvee_i \psi_i(\bar{x})$ where each ψ_i is atomic.*

Proof. For every subformula $\psi(\bar{x})$ of φ with n free variables, choose two new n-ary relation symbols $R_\psi, R_{\neg\psi} \notin \Sigma$. Let Σ_φ be the signature consisting of Σ and all the new symbols $R_\psi, R_{\neg\psi}$. The set Φ_φ consists of the following formulae.

$$\begin{array}{ll} \forall \bar{x}(R_\psi \bar{x} \leftrightarrow \psi(\bar{x})), & \text{if } \psi \text{ is atomic.} \\ \forall \bar{x}(R_{\neg\psi} \bar{x} \leftrightarrow \neg R_\psi \bar{x}), & \end{array}$$

$$\forall \bar{x}(R_{\exists y\psi}\bar{x} \leftrightarrow \exists y R_\psi \bar{x} y)\,,$$
$$\forall \bar{x}(R_{\forall y\psi}\bar{x} \leftrightarrow \forall y R_\psi \bar{x} y)\,,$$
$$\forall \bar{x}(R_{\bigwedge_{i<\lambda}\psi_i}\bar{x} \to R_{\psi_i}\bar{x})\,, \qquad \text{for all } i < \lambda\,,$$
$$\forall \bar{x}(R_{\psi_i}\bar{x} \to R_{\bigvee_{i<\lambda}\psi_i}\bar{x})\,, \qquad \text{for all } i < \lambda\,,$$
$$\forall \bar{x}\Big[R_{\bigwedge_{i<\lambda}\psi_i}\bar{x} \vee \bigvee_{i<\lambda} R_{\neg\psi_i}\bar{x}\Big]\,,$$
$$\forall \bar{x}\Big[R_{\neg\bigvee_{i<\lambda}\psi_i}\bar{x} \vee \bigvee_{i<\lambda} R_{\psi_i}\bar{x}\Big]\,. \qquad \square$$

5. Translations

In the last section we have considered transformations of formulae respecting logical equivalence. Now we turn to operations on structures and we investigate how to compute the theory of the resulting structure from the theories of the original ones. We start with a trivial example that illustrates the general situation.

Lemma 5.1. *Let $\Sigma \subseteq \Gamma$ be signatures. For every formula $\varphi(\bar{x}) \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma]$, there exists a formula $\psi(\bar{x}) \in \mathrm{FO}_{\kappa\aleph_0}[\Gamma]$ such that*

$$\mathfrak{A}|_\Sigma \vDash \varphi(\bar{a}) \quad \text{iff} \quad \mathfrak{A} \vDash \psi(\bar{a})\,,$$

for every Γ-structure $\mathfrak{A}$ and all $\bar{a} \subseteq A$.

Proof. We can set $\psi := \varphi$. $\square$

Corollary 5.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ Γ-structures.*

$$\mathfrak{A} \equiv_{\mathrm{FO}_{\kappa\aleph_0}[\Gamma]} \mathfrak{B} \quad \textit{implies} \quad \mathfrak{A}|_\Sigma \equiv_{\mathrm{FO}_{\kappa\aleph_0}[\Sigma]} \mathfrak{B}|_\Sigma\,, \quad \textit{for all } \Sigma \subseteq \Gamma\,.$$

The other results of this section are all of the above form. We consider an operation F on structures and logics L and L', and we prove that, for every formula $\varphi \in L$, one can construct a formula $\varphi' \in L'$ such that

$$F(\mathfrak{A}) \vDash \varphi \quad \text{iff} \quad \mathfrak{A} \vDash \varphi'\,, \quad \text{for every } \mathfrak{A}\,.$$

As a consequence we obtain the result that

$$\mathfrak{A} \equiv_{L'} \mathfrak{B} \quad \text{implies} \quad F(\mathfrak{A}) \equiv_L F(\mathfrak{B})\,.$$

In the case that $L' = L$ we call such operations *compatible* with L.

As a converse to the introductory example we consider expansions of a structure. Of course, there is no hope to reduce the theory of an arbitrary expansion to the original structure. But if we expand a structure by definable relations, such a reduction is possible.

Definition 5.3. Let $\mathfrak{A}$ be a Σ-structure and $\Gamma \supseteq \Sigma$. A Γ-structure $\mathfrak{B}$ is an *L-definable expansion* of $\mathfrak{A}$ if $\mathfrak{B}|_\Sigma = \mathfrak{A}$ and, for every symbol $\xi \in \Gamma \setminus \Sigma$, there is some L-formula φ_ξ such that

$$\bar{a} \in R^{\mathfrak{B}} \quad \text{iff} \quad \mathfrak{A} \vDash \varphi_R(\bar{a})\,, \quad \text{for all relations } R \in \Gamma \setminus \Sigma\,,$$
$$f^{\mathfrak{B}}(\bar{a}) = b \quad \text{iff} \quad \mathfrak{A} \vDash \varphi_f(\bar{a}, b)\,, \quad \text{for all functions } f \in \Gamma \setminus \Sigma\,.$$

In this case we also say that $(\varphi_\xi)_{\xi\in\Gamma\setminus\Sigma}$ *defines the expansion* $\mathfrak{B}$ of $\mathfrak{A}$.

Lemma 5.4. *Let $\Sigma \subseteq \Gamma$ be signatures and let $\varphi_\xi(\bar{x}) \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma]$, for $\xi \in \Gamma \setminus \Sigma$, be formulae. For every formula $\psi(\bar{x}) \in \mathrm{FO}_{\kappa\aleph_0}[\Gamma]$, there exists a formula $\psi^+(\bar{x}) \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma]$ such that*

$$\mathfrak{A}_+ \vDash \psi(\bar{a}) \quad \text{iff} \quad \mathfrak{A} \vDash \psi^+(\bar{a})\,,$$

whenever $\bar{a} \subseteq A$ and $\mathfrak{A}_+$ is the expansion of $\mathfrak{A}$ defined by $(\varphi_\xi)_\xi$.

Proof. Let ψ' be a term-reduced formula equivalent to ψ. We can obtain ψ^+ by replacing in ψ'

- every atom $R\bar{t}$ with $R \in \Gamma \setminus \Sigma$ by the formula $\varphi_R(\bar{t})$ and
- every atom $f\bar{t} = s$ with $f \in \Gamma \setminus \Sigma$ by $\varphi_f(\bar{t}, s)$. $\square$

Next we consider substructures. Again we have to restrict ourselves to those where the universe is definable.

Definition 5.5. Let $\mathfrak{A}$ be an S-sorted Σ-structure and $\delta_s(x) \in \mathrm{FO}^s_{\kappa\aleph_0}[\Sigma]$, for $s \in S$.

(a) If $\bigcup_{s\in S} \delta_s^{\mathfrak{A}}$ induces a substructure $\mathfrak{A}_0$ of $\mathfrak{A}$, we call $\mathfrak{A}_0$ the substructure *defined* by $(\delta_s)_{s\in S}$.

(b) The *relativisation* of a formula $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma]$ to $(\delta_s)_{s\in S}$ is the formula $\varphi^{(\bar\delta)} \in \mathrm{FO}_{\kappa\aleph_0}$ obtained from φ by replacing every subformula of the form $\exists y\psi$ and $\forall y\psi$ by, respectively,

$$(\exists y.\delta_s(y))\psi \quad \text{and} \quad (\forall y.\delta_s(y))\psi\,,$$

where s is the sort of y.

Lemma 5.6. *If a sequence $(\delta_s)_{s\in S}$ of $\mathrm{FO}_{\kappa\aleph_0}$-formulae defines a substructure $\mathfrak{A}_0$ of $\mathfrak{A}$, we have*

$$\mathfrak{A}_0 \vDash \varphi(\bar a) \quad \text{iff} \quad \mathfrak{A} \vDash \varphi^{(\bar\delta)}(\bar a)\,,$$

for every $\varphi \in \mathrm{FO}_{\kappa\aleph_0}$ and all $\bar a \subseteq \bigcup_{s\in S} \delta_s^{\mathfrak{A}}$.

Exercise 5.1. Prove Lemma 5.6.

Factorisation by definable congruences is also compatible with first-order logic.

Lemma 5.7. *Let Σ be an S-sorted signature and $\varepsilon_s(x,y) \in \mathrm{FO}^{ss}_{\kappa\aleph_0}[\Sigma]$, for $s \in S$. For every formula $\varphi(\bar x) \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$, there exists a formula $\varphi'(\bar x) \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ such that, if $\approx\, := \bigcup_{s\in S} \varepsilon_s^{\mathfrak{A}}$ is a congruence relation on $\mathfrak{A}$, then*

$$\mathfrak{A}/{\approx} \vDash \varphi([\bar a]_\approx) \quad \text{iff} \quad \mathfrak{A} \vDash \varphi'(\bar a)\,, \quad \textit{for all } \bar a \subseteq A\,.$$

Proof. We can obtain φ' from φ by replacing every atom of the form $t = u$ by the formula $\varepsilon_s(t,u)$, where s is the sort of t and u. □

If we combine all of the above operations, we obtain the notion of a first-order interpretation.

Definition 5.8. Let Σ be an S-sorted signatures and Γ a T-sorted one.

(a) An $\mathrm{FO}_{\kappa\aleph_0}$-*interpretation* from Σ to Γ is a sequence

$$\mathcal{I} = \langle \alpha, (\delta_t)_{t\in T}, (\varepsilon_t)_{t\in T}, (\varphi_\xi)_{\xi\in\Gamma} \rangle$$

of formulae where, for some function $\sigma : T \to S^{<\omega}$,

$$\alpha \in \mathrm{FO}^{\circ}_{\kappa\aleph_0}[\Sigma]\,, \quad \delta_t \in \mathrm{FO}^{\sigma(t)}_{\kappa\aleph_0}[\Sigma]\,, \quad \varepsilon_t \in \mathrm{FO}^{\sigma(t)\sigma(t)}_{\kappa\aleph_0}[\Sigma]\,,$$

for every relation symbol $R \in \Gamma$ of type $t_0 \dots t_{n-1}$,

$$\varphi_R \in \mathrm{FO}^{\sigma(t_0)\dots\sigma(t_{n-1})}_{\kappa\aleph_0}[\Sigma]\,,$$

and for every function symbol $f \in \Gamma$ of type $t_0 \dots t_{n-1} \to t'$,

$$\varphi_f \in \mathrm{FO}^{\sigma(t_0)\dots\sigma(t_{n-1})\sigma(t')}_{\kappa\aleph_0}[\Sigma]\,.$$

(b) Each $\mathrm{FO}_{\kappa\aleph_0}$-interpretation $\mathcal{I}$ defines an operation on structures as follows. Intuitively, given a Σ-structure $\mathfrak{A}$ the interpretation $\mathcal{I}$ constructs a Γ-structure $\mathcal{I}(\mathfrak{A})$ every element of which is a tuple of elements of $\mathfrak{A}$ and where the relations and functions are defined by the formulae φ_ξ. The formulae δ_t define those tuples that encode elements of sort t and the formula ε_t is used to check whether two such tuples encode the same element. Finally, the *admissibility condition* α says when $\mathcal{I}(\mathfrak{A})$ is defined.

Formally, if $\mathfrak{A}$ is a Σ-structure with $\mathfrak{A} \vDash \alpha$, we define the Γ-structure

$$\mathcal{I}(\mathfrak{A}) := \big\langle (\delta_t^{\mathfrak{A}})_{t\in T},\ (\varphi_\xi^{\mathfrak{A}})_{\xi\in\Gamma} \big\rangle / {\approx}\,,$$

which is obtained from the structure $\big\langle (\delta_t^{\mathfrak{A}})_{t\in T},\ (\varphi_\xi^{\mathfrak{A}})_{\xi\in\Gamma} \big\rangle$, where the domain of sort t is $\delta_t^{\mathfrak{A}} \subseteq A^{\sigma(t)}$ and every symbol $\xi \in \Gamma$ is interpreted as the relation or function $\varphi_\xi^{\mathfrak{A}}$, by factorising by the congruence relation $\approx$ defined by the $\varepsilon_t^{\mathfrak{A}}$. We regard $\mathcal{I}(\mathfrak{A})$ as undefined if

- $\mathfrak{A} \nvDash \alpha$, or

- $\varepsilon_t^{\mathfrak{A}}$ is not a congruence relation of $\langle(\delta_t^{\mathfrak{A}})_t, (\varphi_\xi^{\mathfrak{A}})_\xi\rangle$, or
- there is some function symbol $f \in \Gamma$ such that $\varphi_f^{\mathfrak{A}}$ is not a function.

Example. We construct an interpretation

$$\mathcal{I} = \langle \delta(\bar{x}), \varepsilon(\bar{x},\bar{y}), \varphi_+(\bar{x},\bar{y},\bar{z}), \varphi_\cdot(\bar{x},\bar{y},\bar{z})\rangle$$

such that

$$\mathcal{I}\langle\mathbb{Z},+,\cdot,0,<\rangle \cong \langle\mathbb{Q},+,\cdot\rangle\,.$$

We encode a rational number p/q by the pair $\langle p,q\rangle$.

$$\begin{aligned}\delta(x,x') &:= x' > 0\,,\\ \varepsilon(x,x',y,y') &:= x\cdot y' = y\cdot x'\,,\\ \varphi_+(x,x',y,y',z,z') &:= \varepsilon(z,z',\ x\cdot y' + y\cdot x',\ x'\cdot y')\,,\\ \varphi_\cdot(x,x',y,y',z,z') &:= \varepsilon(z,z',\ x\cdot y,\ x'\cdot y')\,.\end{aligned}$$

Exercise 5.2. Consider the structures $\mathfrak{N} := \langle\mathbb{N},+,\cdot\rangle$ of arithmetic, $\mathfrak{S} := \langle \mathrm{HF}, \in\rangle$ of hereditary finite sets, and $\mathfrak{M} := \langle 2^{<\omega},\cdot\rangle$ of finite sequences over $[2]$ with concatenation. Define interpretations $\mathcal{I}_0$, $\mathcal{I}_1$, and $\mathcal{I}_2$ such that

$$\mathfrak{N} = \mathcal{I}_0(\mathfrak{S})\,,\quad \mathfrak{S} = \mathcal{I}_1(\mathfrak{M})\,,\quad \mathfrak{M} = \mathcal{I}_2(\mathfrak{N})\,.$$

For the next lemma, we denote by $\iota_s : \delta_s^{\mathfrak{A}} \to \mathcal{I}(\mathfrak{A})$ the canonical function mapping a tuple to the element it encodes.

Lemma 5.9 (Interpretation Lemma). *Let $\mathcal{I} = \langle \alpha, (\delta_s)_s, (\varepsilon_s)_s, (\varphi_\xi)_\xi\rangle$ be an $\mathrm{FO}_{\kappa\aleph_0}$-interpretation from Σ to Γ.*

(a) *For every formula $\psi(x_0,\dots,x_{m-1}) \in \mathrm{FO}^{\bar{s}}_{\kappa\aleph_0}[\Gamma]$, we can construct an formula $\psi^{\mathcal{I}}(\bar{x}_0,\dots,\bar{x}_{m-1}) \in \mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma]$ such that*

$$\mathcal{I}(\mathfrak{A}) \models \psi(\iota_{s_0}\bar{a}_0,\dots,\iota_{s_{m-1}}\bar{a}_{m-1}) \quad\text{iff}\quad \mathfrak{A} \models \psi^{\mathcal{I}}(\bar{a}_0,\dots,\bar{a}_{m-1})\,,$$

for all structures $\mathfrak{A}$ such that $\mathcal{I}(\mathfrak{A})$ is defined and all $\bar{a}_i \subseteq \delta_{s_i}^{\mathfrak{A}}$.

(b) *There exists a formula $\chi \in \mathrm{FO}^0_{\kappa\aleph_0}[\Sigma]$ such that, for every Σ-structure $\mathfrak{A}$,*

$$\mathfrak{A} \models \chi \quad\text{iff}\quad \mathcal{I}(\mathfrak{A}) \text{ is defined.}$$

Proof. (a) W.l.o.g. we may assume that ψ is term-reduced. We define $\psi^{\mathcal{I}}$ by induction on ψ. For atomic formulae, we have

$$\begin{aligned}(fx_0\dots x_{m-1} = y)^{\mathcal{I}} &:= \varphi_f(\bar{x}_0,\dots,\bar{x}_{m-1},\bar{y})\,,\\ (Rx_0\dots x_{m-1})^{\mathcal{I}} &:= \varphi_R(\bar{x}_0,\dots,\bar{x}_{m-1})\,,\end{aligned}$$

and, if x and y are of sort s then

$$(x = y)^{\mathcal{I}} := \varepsilon_s(\bar{x},\bar{y})\,.$$

(Note that we assume that every tuple satisfying φ_ξ also satisfies the corresponding δ_s. Otherwise, we have to add the conjunction of all $\delta_{s_i}(\bar{x}_i)$ to the above formulae.) Boolean combinations are left unchanged.

$$\begin{aligned}(\neg\vartheta)^{\mathcal{I}} &:= \neg\vartheta^{\mathcal{I}}\,,\\ (\bigwedge\Phi)^{\mathcal{I}} &:= \bigwedge\{\,\vartheta^{\mathcal{I}} \mid \vartheta\in\Phi\,\}\,,\\ (\bigvee\Phi)^{\mathcal{I}} &:= \bigvee\{\,\vartheta^{\mathcal{I}} \mid \vartheta\in\Phi\,\}\,.\end{aligned}$$

And if y is a variable of sort s, we have to restrict quantifiers over y to δ_s.

$$\begin{aligned}(\exists y\vartheta)^{\mathcal{I}} &:= (\exists\bar{y}.\delta_s(\bar{y}))\vartheta^{\mathcal{I}}\,,\\ (\forall y\vartheta)^{\mathcal{I}} &:= (\forall\bar{y}.\delta_s(\bar{y}))\vartheta^{\mathcal{I}}\,.\end{aligned}$$

(b) We can set

$$\chi := \alpha \wedge \bigwedge_{\xi\in\Gamma} \vartheta_\xi$$

where, for each relation symbol $R \in \Gamma$ of type $s_0 \dots s_{n-1}$, the formula

$$\vartheta_R := \forall \bar{x}_0 \cdots \bar{x}_{n-1} \bar{y}_0 \cdots \bar{y}_{n-1} \Big(\bigwedge_{i<n} \varepsilon_{s_i}(\bar{x}_i, \bar{y}_i) \rightarrow \big(\varphi_R(\bar{x}_0, \dots, \bar{x}_{n-1}) \leftrightarrow \varphi_R(\bar{y}_0, \dots, \bar{y}_{n-1})\big)\Big)$$

expresses that the ε_s define a congruence with respect to the relation defined by φ_R and, for each function symbol $f \in \Gamma$ of type $s_0 \dots s_{n-1} \rightarrow t$, the formula

$$\begin{aligned}\vartheta_f := {} & \forall \bar{x}_0 \cdots \bar{x}_{n-1} \exists \bar{y} \varphi_f(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{y}) \\ & \wedge \forall \bar{x}_0 \cdots \bar{x}_{n-1} \bar{y}_0 \cdots \bar{y}_{n-1} \bar{u} \bar{v} \\ & \Big(\Big(\bigwedge_{i<n} \varepsilon_{s_i}(\bar{x}_i, \bar{y}_i) \wedge \varphi_f(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{u}) \\ & \qquad \wedge \varphi_f(\bar{y}_0, \dots, \bar{y}_{n-1}, \bar{v})\Big) \rightarrow \varepsilon_t(\bar{u}, \bar{v})\Big)\end{aligned}$$

says that φ_f defines a function and the ε_s define a congruence with respect to this function. □

The general scheme of these constructions is summarised in the following definition.

Definition 5.10. Let $\langle L_0, \mathcal{K}_0, \models\rangle$ and $\langle L_1, \mathcal{K}_1, \models\rangle$ be logics.

(a) A *morphism* from L_0 to L_1 is a pair $\langle \alpha, \beta\rangle$ of functions $\alpha : L_0 \rightarrow L_1$ and $\beta : \mathcal{K}_1 \rightarrow \mathcal{K}_0$ such that

$$\mathfrak{J} \models \alpha(\varphi) \quad \text{iff} \quad \beta(\mathfrak{J}) \models \varphi\,, \qquad \text{for all } \varphi \in L_0 \text{ and } \mathfrak{J} \in \mathcal{K}_1\,.$$

The category consisting of all logics and these morphisms is called $\mathfrak{Logic}$.

(b) An *embedding* is a morphism $\langle \alpha, \beta\rangle : L_0 \rightarrow L_1$ where β is surjective.

(c) A *comorphism* from L_0 to L_1 is a morphism $\langle \alpha, \beta\rangle : L_1 \rightarrow L_0$.

(d) By abuse of terminology we call a function $\alpha : L_0 \rightarrow L_1$ a morphism if there exists a function $\beta : \mathcal{K}_1 \rightarrow \mathcal{K}_0$ such that the pair $\langle \alpha, \beta\rangle$ forms a morphism $L_0 \rightarrow L_1$. Similarly, we call $\beta : \mathcal{K}_0 \rightarrow \mathcal{K}_1$ a comorphism if there is some $\alpha : L_1 \rightarrow L_0$ such that $\langle \alpha, \beta\rangle$ is a comorphism $L_0 \rightarrow L_1$.

Remark. The only difference between a morphism and a comorphism is the direction of the arrow. We will use the former term if we want to stress the translation of formulae, while the latter term is used when we are mainly interested in the operation on structures.

Example. Each of the operations introduced in this section induces a comorphism. For instance, we have seen in Lemma 5.1 that the reduct operation $r : \mathfrak{A} \mapsto \mathfrak{A}|_\Sigma$ induces the comorphism

$$\langle i, r\rangle : \mathrm{FO}_{\kappa\aleph_0}[\Gamma, X] \rightarrow \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]\,,$$

where $i : \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X] \rightarrow \mathrm{FO}_{\kappa\aleph_0}[\Gamma, X]$ is the inclusion map.

In the case of interpretations we face a minor technical difficulty since these are partial operations. An $\mathrm{FO}_{\kappa\aleph_0}$-interpretation $\mathcal{I}$ from Σ to Γ induces a comorphism $L \rightarrow \mathrm{FO}_{\kappa\aleph_0}[\Gamma, X]$ where L is not $\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ but the sublogic $\langle \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X], \mathcal{C}, \models\rangle$, where the class $\mathcal{C} \subseteq \mathrm{Str}[\Sigma]$ of interpretations consists of those Σ-structures $\mathfrak{A}$ such that $\mathcal{I}(\mathfrak{A})$ is defined.

Exercise 5.3. Prove that a morphism $\langle \alpha, \beta\rangle : L_0 \rightarrow L_1$ is a monomorphism if, and only if, α is injective and β is surjective. Show that it is an epimorphism if, and only if, α is surjective and β is injective.

Lemma 5.11. *Let $\langle \alpha, \beta\rangle : L_0 \rightarrow L_1$ be a morphism of logics.*

(a) *$\langle \alpha, \beta\rangle$ is a monomorphism if, and only if, it has a left inverse.*

(b) *$\langle \alpha, \beta\rangle$ is an epimorphism if, and only if, it has a right inverse.*

Lemma 5.12. *Let $\langle \alpha, \beta\rangle : L_0 \rightarrow L_1$ be a morphism of logics, $\Phi \subseteq L_0$, $\varphi, \psi \in L_0$, and $\mathfrak{J}$ an L_1-interpretation.*

(a) *$\varphi \models \psi$ implies $\alpha(\varphi) \models \alpha(\psi)$.*

(b) *If Φ is inconsistent then so is $\alpha[\Phi]$.*

(c) $\mathrm{Th}_{L_0}(\beta(\mathfrak{J})) = \alpha^{-1}[\mathrm{Th}_{L_1}(\mathfrak{J})]$.

(d) $\mathrm{Mod}_{L_1}(\alpha[\Phi]) = \beta^{-1}[\mathrm{Mod}_{L_0}(\Phi)]$.

Proof. (a) For every L_1-interpretation $\mathfrak{J}$, we have the following chain of implications:

$$\begin{aligned}\mathfrak{J} \vDash \alpha(\varphi) \quad &\Rightarrow \quad \beta(\mathfrak{J}) \vDash \varphi \\ &\Rightarrow \quad \beta(\mathfrak{J}) \vDash \psi \quad \Rightarrow \quad \mathfrak{J} \vDash \alpha(\psi)\,.\end{aligned}$$

It follows that $\alpha(\varphi) \vDash \alpha(\psi)$.

(b) Suppose that $\alpha[\Phi]$ has a model $\mathfrak{J}$. Then $\mathfrak{J} \vDash \alpha[\Phi]$ implies that $\beta(\mathfrak{J}) \vDash \Phi$. Hence, Φ is satisfiable.

(c) For a formula $\varphi \in L_0$ and an L_1-interpretation $\mathfrak{J}$, we have

$$\beta(\mathfrak{J}) \vDash \varphi \quad \text{iff} \quad \mathfrak{J} \vDash \alpha(\varphi) \quad \text{iff} \quad \varphi \in \alpha^{-1}[\mathrm{Th}_{L_1}(\mathfrak{J})]\,.$$

(d) By definition of a morphism, we have

$$\mathfrak{J} \vDash \alpha[\Phi] \quad \text{iff} \quad \beta(\mathfrak{J}) \vDash \Phi \quad \text{iff} \quad \mathfrak{J} \in \beta^{-1}[\mathrm{Mod}_{L_1}(\Phi)]\,. \qquad \square$$

Corollary 5.13. *Let $\langle\alpha,\beta\rangle : L_0 \to L_1$ be a comorphism of logics and suppose that $\mathfrak{J}_0, \mathfrak{J}_1$ are L_0-interpretations.*

$$\mathfrak{J}_0 \equiv_{L_0} \mathfrak{J}_1 \quad \textit{implies} \quad \beta(\mathfrak{J}_0) \equiv_{L_1} \beta(\mathfrak{J}_1)\,.$$

Proof. The claim follows immediately from Lemma 5.12 (c). $\square$

Every monomorphism of logics is an embedding. Statement (a) of the following lemma states that, conversely, every embedding is a monomorphism 'up to logical equivalence'.

Lemma 5.14. *Let $\langle\alpha,\beta\rangle : L_0 \to L_1$ be an embedding of logics, $\Phi \subseteq L_0$, and $\varphi, \psi \in L_0$ formulae.*

(a) $\varphi \vDash \psi$ iff $\alpha(\varphi) \vDash \alpha(\psi)$.

(b) $\mathrm{Mod}_{L_0}(\Phi) = \beta[\mathrm{Mod}_{L_1}(\alpha[\Phi])]$.

Proof. (a) We have already seen in Lemma 5.12 (a) that $\varphi \vDash \psi$ implies $\alpha(\varphi) \vDash \alpha(\psi)$. Conversely, suppose that $\alpha(\varphi) \vDash \alpha(\psi)$ and let $\mathfrak{J}_0$ be an L_0-interpretation. By assumption, there is some L_1-interpretation $\mathfrak{J}_1$ with $\beta(\mathfrak{J}_1) = \mathfrak{J}_0$. Hence, we have

$$\mathfrak{J}_0 \vDash \varphi \quad \Rightarrow \quad \mathfrak{J}_1 \vDash \alpha(\varphi) \quad \Rightarrow \quad \mathfrak{J}_1 \vDash \alpha(\psi) \quad \Rightarrow \quad \mathfrak{J}_0 \vDash \psi\,.$$

It follows that $\varphi \vDash \psi$.

(b) By Lemmas A2.1.10 and 5.12 (d), it follows that

$$\beta[\mathrm{Mod}_{L_1}(\alpha[\Phi])] = \beta[\beta^{-1}[\mathrm{Mod}_{L_0}(\Phi)]] = \mathrm{Mod}_{L_0}(\Phi)\,. \qquad \square$$

6. *Extensions of first-order logic*

Lindström quantifiers

First-order logic seems to be ill-suited to talk about cardinalities. To express that there are infinitely many elements we had to use an infinite set of formulae, and we will see in Lemma C2.4.9 that, even if we allow infinitely many formulae, we cannot express that something is finite.

To obtain a logic where these things can be expressed, we add to ordinary first-order logic a *cardinality quantifier* $\exists^\lambda$ with the meaning of 'there are at least λ many'.

Definition 6.1. By $\mathrm{FO}_{\kappa\aleph_0}(\exists^\lambda)[\Sigma, X]$ we denote the logic obtained from $\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ by adding the syntax rule:

- if $\varphi \in \mathrm{FO}_{\kappa\aleph_0}(\exists^\lambda)[\Sigma, X \cup \{x\}]$ then $\exists^\lambda x\varphi \in \mathrm{FO}_{\kappa\aleph_0}(\exists^\lambda)[\Sigma, X]$.

We define the semantics of this quantifier by

$$\mathfrak{A} \vDash \exists^\lambda x\varphi[\beta] \quad :\text{iff} \quad \big|\big\{\, a \in A \;\big|\; \mathfrak{A} \vDash \varphi[\beta[x/a]] \,\big\}\big| \geq \lambda\,.$$

Example. We can axiomatise the order $\langle \omega, < \rangle$ up to isomorphism by the formula

$$\begin{aligned} &\forall x \neg (x < x) \wedge \forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z) \\ &\quad \wedge \forall x \forall y (x < y \vee x = y \vee y < x) \\ &\quad \wedge \forall x \exists y (x < y) \\ &\quad \wedge \forall x \neg \exists^{\aleph_0} y (y < x) \,. \end{aligned}$$

Another property that infinitary first-order logic is unable to express is well-foundedness. As above, we can introduce a new quantifier expressing that a definable relation is a well-order. This logic will play an important role in Section c5.6.

Definition 6.2. Let $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{wo})$ be the extension of $\mathrm{FO}_{\kappa\aleph_0}$ by the *well-ordering quantifier* W whose semantics is given by

$$\mathfrak{A} \vDash \mathsf{W}\bar{x}\bar{y}\varphi(\bar{x},\bar{y},\bar{c}) \quad \text{iff} \quad \text{the relation } \varphi^{\mathfrak{A}}(\bar{x},\bar{y},\bar{c}) \text{ is a well-ordering of its field.}$$

Note that the quantifier W cannot be used to express 'there exists a well-order'. We can only check whether some definable relation is a well-order.

Generalising the above examples we can define extensions of (infinitary) first-order logic by quantifiers for any given property.

Definition 6.3. Let $\Gamma = \{R_0, \dots, R_n\}$ be a finite relational signature and $\mathcal{K}$ a class of Γ-structures. The *Lindström quantifier* for $\mathcal{K}$ is of the form $\mathsf{Q}_{\mathcal{K}}\bar{x}_0 \dots \bar{x}_n \varphi_0(\bar{x}_0, \bar{z}) \dots \varphi_n(\bar{x}_n, \bar{z})$. The semantics of such a formula is defined by

$$\begin{aligned} &\mathfrak{A} \vDash \mathsf{Q}_{\mathcal{K}}\bar{x}_0 \dots \bar{x}_n \varphi_0(\bar{x}_0, \bar{c}) \dots \varphi_n(\bar{x}_n, \bar{c}) \\ &:\text{iff } \langle A, \varphi_0(\bar{x}_0,\bar{c})^{\mathfrak{A}}, \dots, \varphi_n(\bar{x}_n,\bar{c})^{\mathfrak{A}} \rangle \in \mathcal{K} \,. \end{aligned}$$

Example. (a) The cardinality quantifier $\exists^{\lambda}$ is the quantifier $\mathsf{Q}_{\mathcal{K}}$ where

$$\mathcal{K} := \{ \langle A, P \rangle \mid A \text{ a set}, P \subseteq A, |P| \geq \lambda \} \,.$$

(b) The *cardinality comparison quantifier* is defined by the class

$$\mathcal{K} := \{ \langle A, P, Q \rangle \mid |P| = |Q| \} \,.$$

(c) The well-ordering quantifier W is defined by the class

$$\mathcal{K} := \{ \langle A, R \rangle \mid R \text{ is a well-order on its field} \} \,.$$

Second-order logic

In second-order logic we extend first-order logic by variables for relations and functions and we allow quantification over such variables. When we equip each variable with a type, the set of variables becomes a signature where the constant symbols play the role of the first-order variables. This particular point of view makes the definition of syntax and semantics much more streamlined. We could also have adopted this convention for the definition of first-order logic. But for expositionary reasons we have refrained from doing so.

Definition 6.4. Let Σ and Ξ be S-sorted signatures. The set $\mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$ of *infinitary second-order formulae* is the smallest set of terms satisfying the following closure conditions:

- If $t_0, t_1 \in T[\Sigma \cup \Xi, \varnothing]$ are terms of the same sort, we have $t_0 = t_1 \in \mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$.
- If $R \in \Sigma \cup \Xi$ is of type $s_0 \dots s_{n-1}$ and $t_i \in T_{s_i}[\Sigma \cup \Xi, \varnothing]$, for $i < n$, then $Rt_0 \dots t_{n-1} \in \mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$.
- If $\varphi \in \mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$, then $\neg\varphi \in \mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$.
- If $\Phi \subseteq \mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$ and $|\Phi| < \kappa$, then $\bigwedge \Phi$, $\bigvee \Phi \in \mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$.
- If $\varphi \in \mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi \cup \{\xi\}]$, then $\exists\xi\varphi$, $\forall\xi\varphi \in \mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$.

We define *monadic second-order logic* $\mathrm{MSO}_{\kappa\aleph_0}[\Sigma, \Xi]$ as the restriction of $\mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$ were we allow only constant symbols and *unary* relation symbols in the variable signature Ξ.

For a $(\Sigma \cup \Xi)$-structure $\mathfrak{A}$ and an $\mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$-formula φ, we define the satisfaction relation $\mathfrak{A} \vDash \varphi$ by induction on φ.

$$\begin{aligned}
&\mathfrak{A} \vDash t_0 = t_1 &&:\text{iff} \quad t_0^{\mathfrak{A}} = t_1^{\mathfrak{A}}\,,\\
&\mathfrak{A} \vDash R t_0 \dots t_{n-1} &&:\text{iff} \quad \langle t_0^{\mathfrak{A}}, \dots, t_{n-1}^{\mathfrak{A}} \rangle \in R^{\mathfrak{A}}\,,\\
&\mathfrak{A} \vDash \neg\varphi &&:\text{iff} \quad \mathfrak{A} \nvDash \varphi\,,\\
&\mathfrak{A} \vDash \bigvee \Phi &&:\text{iff} \quad \text{there is some } \varphi \in \Phi \text{ such that } \mathfrak{A} \vDash \varphi\,,\\
&\mathfrak{A} \vDash \bigwedge \Phi &&:\text{iff} \quad \mathfrak{A} \vDash \varphi \text{ for all } \varphi \in \Phi\,,\\
&\mathfrak{A} \vDash \exists \xi \varphi &&:\text{iff} \quad \text{there is some relation or function } \xi^{\mathfrak{A}}\\
&&&\qquad\quad \text{such that } \langle \mathfrak{A}, \xi^{\mathfrak{A}} \rangle \vDash \varphi\,,\\
&\mathfrak{A} \vDash \forall \xi \varphi &&:\text{iff} \quad \langle \mathfrak{A}, \xi^{\mathfrak{A}} \rangle \vDash \varphi \text{ for all suitable relations or}\\
&&&\qquad\quad \text{functions } \xi^{\mathfrak{A}}.
\end{aligned}$$

Example (Peano Axioms). The structure $\langle \omega, s, 0 \rangle$, where $s : n \mapsto n + 1$ is the successor function, can be axiomatised in monadic second-order logic up to isomorphism by the *Peano Axioms.*

$$\begin{aligned}
&\forall x (sx \neq 0)\,,\\
&\forall x \forall y (sx = sy \to x = y)\,,\\
&\forall Z [Z0 \land \forall x (Zx \to Zsx) \to \forall x Zx]\,.
\end{aligned}$$

The third axiom which states the induction principle is not first-order.

Example. (a) The class of all well-orders can be axiomatised by the MSO-formulae

$$\begin{aligned}
&\forall x \forall y (x \leq y \land y \leq x \leftrightarrow x = y)\,,\\
&\forall x \forall y \forall z (x \leq y \land y \leq z \to x \leq z)\,,\\
&\forall x \forall y (x \leq y \lor y \leq x)\,,\\
&\forall Z [\exists x Zx \to (\exists x.Zx)(\forall y.Zy)(x \leq y)]\,,
\end{aligned}$$

which express that $\leq$ is a linear order such that every nonempty set Z has a minimal element.

(b) Let $\langle V, E \rangle$ be a graph. The transitive closure of the relation E can be defined by the monadic second-order formula

$$\varphi(x, y) := \forall Z [Zx \land \forall u \forall v (Zu \land Euv \to Zv) \to Zy]\,.$$

Consequently, we can express that a graph is strongly connected by

$$\psi := \forall x \forall y \varphi(x, y)\,.$$

(c) Let $\varphi(x)$ and $\psi(x)$ be second-order formulae. We can express that the sets defined by φ and ψ have the same cardinality by the second-order formula

$$\begin{aligned}
\exists f \big[&(\forall x.\varphi(x))(\forall y.\varphi(y))(x \neq y \to fx \neq fy)\\
&\land (\forall x.\varphi(x))(\exists y.\psi(y))(fx = y)\\
&\land (\forall x.\psi(x))(\exists y.\varphi(y))(fy = x)\big]
\end{aligned}$$

which states that there exists a bijection between these sets.

Logical systems

We have already introduced several logics and we will define some more below. To facilitate a uniform treatment let us define a general framework for the kind of logic we are interested in. We have two basic requirements. Firstly, the logic should talk about structures and, secondly, it should be well-behaved with respect to reducts and expansions of signatures. Like in the first-order case we will therefore consider not a single logic but a family of them, one logic for each signature. We start by giving a general definition of a family of logics.

Definition 6.5. Let $\mathfrak{S}$ be a category. A *logical system* parametrised by $\mathfrak{S}$ is a functor $\mathcal{L} : \mathfrak{S} \to \mathfrak{Logic}$. To each logical system $\mathcal{L}$ we associate a covariant functor L and a contravariant functor $\mathcal{C}$ such that

$$\begin{aligned}
\mathcal{L}[s] &= \langle L[s], \mathcal{C}[s], \vDash_s \rangle\,, &&\text{for } s \in \mathfrak{S}\,,\\
\mathcal{L}[f] &= \langle L[f], \mathcal{C}[f] \rangle\,, &&\text{for } f \in \mathfrak{S}(s, s')\,.
\end{aligned}$$

L is called the *syntax functor* of $\mathcal{L}$ and $\mathcal{C}$ is the *semantics functor.*

Remark. (a) An alternative, more concrete definition of a logical system would be as follows. A logical system consists of a covariant functor $L : \mathfrak{S} \to \mathfrak{L}$, a contravariant functor $\mathcal{C} : \mathfrak{S} \to \mathfrak{Int}$, and a family $(\vDash_s)_{s \in \mathfrak{S}}$ of binary relations $\vDash_s \subseteq \mathcal{C}[s] \times L[s]$ that satisfy the following conditions:

- $\langle L[s], \mathcal{C}[s], \vDash_s \rangle$ is a logic, for all $s \in \mathfrak{S}$.
- For every morphism $f : s \to t$ of $\mathfrak{S}$, all formulae $\varphi \in L[s]$, and each interpretation $\mathfrak{J} \in \mathcal{C}[t]$, we have

$$\mathcal{C}[f](\mathfrak{J}) \vDash_s \varphi \quad \text{iff} \quad \mathfrak{J} \vDash_t L[f](\varphi) .$$

Note that the second condition is a generalisation of the property of terms stated in Lemma b3.1.16.

(b) Usually the category $\mathfrak{S}$ specifies a signature Σ and a set of variables X, and $\mathcal{C}[\Sigma, X]$ is the class of all pairs $\langle \mathfrak{A}, \beta \rangle$ where $\mathfrak{A}$ is a Σ-structure and β a variable assignment for the variables in X. In fact, we will mostly deal with logics without free variables where the interpretations consists of only a structure (see Definition 6.7 below).

Example. We define a logical system based on Zariski logic. The category $\mathfrak{S}$ of parameters consists of all pairs $\langle \mathfrak{K}, X \rangle$ where $\mathfrak{K}$ is a field and X a set of variables. If $\mathfrak{L}$ is an extension of $\mathfrak{K}$ then $\mathfrak{S}(\langle \mathfrak{K}, X \rangle, \langle \mathfrak{L}, Y \rangle)$ consists of all functions $f : X \to Y$. If $\mathfrak{L}$ is not an extension of $\mathfrak{K}$ then there are no morphisms $\langle \mathfrak{K}, X \rangle \to \langle \mathfrak{L}, Y \rangle$.

The logical system maps a parameter $\langle \mathfrak{K}, X \rangle \in \mathfrak{S}$ to the Zariski logic $\text{ZL}[\mathfrak{K}, X]$. Each morphism $f : \langle \mathfrak{K}, X \rangle \to \langle \mathfrak{L}, Y \rangle$ of $\mathfrak{S}$ is mapped to the morphism $\langle \alpha, \beta \rangle : \text{ZL}[\mathfrak{K}, X] \to \text{ZL}[\mathfrak{L}, Y]$ where

- α maps a polynomial $p(\bar{x}) \in \mathfrak{K}[X]$ to $p(f(\bar{x})) \in \mathfrak{L}[Y]$ and
- β maps a variable assignment $\gamma \in \mathfrak{M}^Y$ to $\gamma \circ f \in \mathfrak{M}^X$.

Note that $\langle \alpha, \beta \rangle$ is indeed a morphism since

$$\gamma \circ f \vDash p(x_0, \dots, x_{n-1}) \quad \text{iff} \quad \gamma \vDash p(f(x_0), \dots, f(x_{n-1})) .$$

Recall the categories $\mathfrak{Sig}$, $\mathfrak{SigVar}$, and $\mathfrak{Str}$ introduced in Section b3.1.

Definition 6.6. By $\text{FO}_{\kappa\aleph_0}$ we denote the logical system $\mathfrak{SigVar} \to \mathfrak{Logic}$ with

$$\langle \Sigma, X \rangle \mapsto \langle \text{FO}_{\kappa\aleph_0}[\Sigma, X], \text{Str}[\Sigma, X], \vDash \rangle ,$$

and $\text{FO}^{\bar{s}}_{\kappa\aleph_0} : \mathfrak{Sig} \to \mathfrak{Logic}$ is the subsystem with

$$\Sigma \mapsto \langle \text{FO}^{\bar{s}}_{\kappa\aleph_0}[\Sigma], \text{Str}, \vDash \rangle .$$

Exercise 6.1. Prove that $\text{FO}_{\kappa\aleph_0}$ and $\text{FO}^{\bar{s}}_{\kappa\aleph_0}$ are indeed logical systems.

Exercise 6.2. Let $\mathcal{L} : \mathfrak{S} \to \mathfrak{Logic}$ be a logical system with

$$\mathcal{L}(s) = \langle L[s], \mathcal{C}[s], \vDash_s \rangle , \qquad \text{for } s \in \mathfrak{S} ,$$
$$\mathcal{L}(f) = \langle \alpha_f, \beta_f \rangle , \qquad \text{for } f \in \mathfrak{S}(s, t) .$$

Show that the function $\mathcal{L}^{\text{op}} : \mathfrak{S}^{\text{op}} \to \mathfrak{Logic}$ defined by

$$\mathcal{L}^{\text{op}}(s) := \langle \mathcal{C}[s], L[s], (\vDash_s)^{-1} \rangle , \qquad \text{for } s \in \mathfrak{S} ,$$
$$\mathcal{L}^{\text{op}}(f) := \langle \beta_f, \alpha_f \rangle , \qquad \text{for } f \in \mathfrak{S}(s, t)$$

is a logical system.

We are mainly interested in logical systems that, like first-order logic, talk about structures.

Definition 6.7. An *algebraic logic* is a logical system $\mathcal{L} : \mathfrak{Sig} \to \mathfrak{Logic}$ parametrised by $\mathfrak{Sig}$ such that

- the semantics functor is the canonical functor $\text{Str} : \mathfrak{Sig} \to \mathfrak{Str}$ and
- every logic $L[\Sigma]$ is invariant under isomorphisms, that is,

$$\mathfrak{A} \cong \mathfrak{B} \quad \text{implies} \quad \mathfrak{A} \equiv_{L[\Sigma]} \mathfrak{B} , \qquad \text{for all } \mathfrak{A}, \mathfrak{B} \in \text{Str}[\Sigma] .$$

Example. We will prove in Lemma c2.1.3 (c) that first-order logic is invariant under isomorphisms. Consequently, $\text{FO}^{0}_{\kappa\aleph_0}$ is algebraic. Clearly, $\text{FO}^{\alpha}_{\kappa\aleph_0}$ is not, for $\alpha > 0$, since the interpretations are not structures.

Remark. Note that it follows immediately from the definition of an algebraic logic that the reduct operation $\mathfrak{A} \mapsto \mathfrak{A}|_\Sigma$ is a comorphism $L[\Gamma] \to L[\Sigma]$, for every algebraic logic L.

When defining the semantics of second-order logic we have treated the variables as symbols of a signature. This trick can be used to simulate free variables in every algebraic logic.

Definition 6.8. Let L be an algebraic logic, Σ a signature, and X a set of variables disjoint from Σ. We set

$$L[\Sigma, X] := L[\Sigma \cup X],$$

where we regard the elements of X as constant symbols. If $\mathfrak{A}$ is a Σ-structure and $\beta : X \to A$ a variable assignment, we define

$$\mathfrak{A} \vDash \varphi[\beta] \quad :\text{iff} \quad \mathfrak{A}_\beta \vDash \varphi,$$

where $\mathfrak{A}_\beta$ is the $(\Sigma \cup X)$-expansion of $\mathfrak{A}$ where we assign to the additional constants $x \in X$ the value $x^{\mathfrak{A}_\beta} := \beta(x)$.

We define $\varphi^{\mathfrak{A}}$, $\text{free}(\varphi)$, $\mathfrak{A} \vDash \varphi(\bar{a})$, and $L^{\bar{s}}[\Sigma]$ in the same way as for first-order logic.

Lindenbaum algebras

Usually we are only interested in the expressive power of a logic and, hence, we will not distinguish between equivalent formulae. To this end we associate with every logic L a partial order that reflects the structural properties of L while abstracting away from the concrete syntax. We have seen in Lemma 3.2 that the entailment relation $\vDash$ is a preorder. If we identify equivalent formulae, we obtain the partial order $\langle L, \vDash \rangle / \equiv$. In this way we can define a functor $\mathfrak{Logic} \to \mathfrak{PO}$ where $\mathfrak{PO}$ is the category of all partial orders with homomorphisms.

Definition 6.9. The *Lindenbaum functor* $\mathfrak{Lb} : \mathfrak{Logic} \to \mathfrak{PO}$ is defined by

$$\mathfrak{Lb}(L) := \langle L, \vDash \rangle / \equiv, \qquad \text{for } L \in \mathfrak{Logic},$$
$$\mathfrak{Lb}(\mu)([\varphi]_\equiv) := [\alpha(\varphi)]_\equiv, \qquad \text{for } \mu = \langle \alpha, \beta \rangle \in \mathfrak{Logic}(L_0, L_1).$$

The partial order $\mathfrak{Lb}(L)$ is called the *Lindenbaum algebra* of L.

Remark. Note that it follows by Lemma 5.12 (a) that the image $\mathfrak{Lb}(\mu)$ of a morphism $\mu : L_0 \to L_1$ is well-defined and that it is indeed a homomorphism of partial orders.

Example. (a) Let $\mathfrak{K}$ be an algebraically closed field. For Zariski logic $\text{ZL}[\mathfrak{K}, X]$, we have shown that

$$p \equiv q \quad \text{iff} \quad p^m = aq^n \quad \text{for some } a \in K \text{ and } m, n < \omega.$$

The Lindenbaum algebra $\mathfrak{Lb}(\text{ZL}[\mathfrak{K}, X])$ is an upper semilattice where

$$\top = [0]_\equiv, \quad \bot = [1]_\equiv, \quad \text{and} \quad [p]_\equiv \sqcup [q]_\equiv = [pq]_\equiv.$$

(b) Let $\mathfrak{B}$ be a boolean algebra. The Lindenbaum algebra $\mathfrak{Lb}(\text{BL}(\mathfrak{B}))$ is isomorphic to $\mathfrak{B}$ since, for $a, b \in B$,

$$a \equiv b \quad \text{implies} \quad a = b.$$

Lemma 6.10. *Let $\mu : L_0 \to L_1$ be a morphism of logics.*

(a) *If μ is an epimorphism then so is $\mathfrak{Lb}(\mu)$.*

(b) *If μ is an embedding then so is $\mathfrak{Lb}(\mu)$.*

Proof. Suppose that $\mu = \langle \alpha, \beta \rangle$.

(a) Let $[\varphi]_\equiv \in \mathfrak{Lb}(L_1)$. The map α is surjective since μ is an epimorphism. Consequently, there is some $\psi \in L_0$ with $\alpha(\psi) = \varphi$. Hence, $\mathfrak{Lb}(\mu)([\psi]_\equiv) = [\varphi]_\equiv$, as desired.

(b) follows immediately from Lemma 5.14 (a). □

Definition 6.11. Let L be a logic and $\varphi, \psi \in L$ formulae.

(a) A *negation* of φ is a formula $\vartheta \in L$ such that, for all L-interpretations $\mathfrak{I}$, we have

$$\mathfrak{I} \vDash \vartheta \quad \text{iff} \quad \mathfrak{I} \nvDash \varphi.$$

If φ has negations, we fix one and denote it by $\neg\varphi$.

(b) A *disjunction* of φ and ψ is a formula $\vartheta \in L$ such that, for all L-interpretations $\mathfrak{J}$, we have

$$\mathfrak{J} \vDash \vartheta \quad \text{iff} \quad \mathfrak{J} \vDash \varphi \text{ or } \mathfrak{J} \vDash \psi \text{ or both.}$$

If disjunctions of φ and ψ exist, we fix one and denote it by $\varphi \vee \psi$.

(c) A *conjunction* of φ and ψ is a formula $\vartheta \in L$ such that, for all L-interpretations $\mathfrak{J}$, we have

$$\mathfrak{J} \vDash \vartheta \quad \text{iff} \quad \mathfrak{J} \vDash \varphi \text{ and } \mathfrak{J} \vDash \psi\,.$$

If conjunctions of φ and ψ exist, we fix one and denote it by $\varphi \wedge \psi$.

(d) We say that L is *closed under negation, disjunction,* or *conjunction* if all L-formulae have, respectively, negations, disjunctions, or conjunctions. We call L *boolean closed* if L is closed under all three operations.

Remark. (a) Note that $\neg\varphi$, $\varphi \vee \psi$, and $\varphi \wedge \psi$ are only determined up to logical equivalence, but they are unique when regarded as elements of $\mathfrak{Lb}(L)$.

(b) If L is closed under conjunction and disjunction, the Lindenbaum algebra $\mathfrak{Lb}(L)$ is a lattice where

$$[\varphi]_\equiv \sqcap [\psi]_\equiv = [\varphi \wedge \psi]_\equiv \quad \text{and} \quad [\varphi]_\equiv \sqcup [\psi]_\equiv = [\varphi \vee \psi]_\equiv\,.$$

Exercise 6.3. Define a logic L such that $\mathfrak{Lb}(L)$ is a boolean algebra but L is closed under neither negation, nor disjunction, nor conjunction.

Lemma 6.12. *Let $\langle L, \mathcal{K}, \vDash\rangle$ be a logic.*

(a) *If L is closed under conjunction and disjunction then $\mathfrak{Lb}(L)$ is a distributive lattice.*

(b) *If L is boolean closed then $\mathfrak{Lb}(L)$ is a boolean algebra.*

Proof. (a) $\mathfrak{Lb}(L)$ is clearly a lattice if it has the above closure properties. To show that it is distributive note that the function

$$f : \mathfrak{Lb}(L) \to \mathscr{P}(\mathcal{K}) : [\varphi]_\equiv \mapsto \mathrm{Mod}_L(\varphi)$$

is an embedding of $\mathfrak{Lb}(L)$ into a power-set lattice and such lattices are always distributive.

(b) If L is boolean closed, it contains tautologies $\varphi \vee \neg\varphi$ and unsatisfiable formulae $\varphi \wedge \neg\varphi$. Hence, $\mathfrak{Lb}(L)$ forms a boolean algebra. □

When investigating a logical theory T we usually are only interested in the class of models of T. In these cases we can restrict the logic by removing all interpretations that do not satisfy T.

Definition 6.13. Let $\langle L, \mathcal{K}, \vDash\rangle$ be a logic, $\Phi \subseteq L$ a set of formulae, and let $i : \Phi \to L$ and $j : \mathrm{Mod}_L(\Phi) \to \mathcal{K}$ be the corresponding inclusion maps.

(a) The *restriction* of L to Φ is the logic

$$L|_\Phi := \langle \Phi, \mathcal{K}, \vDash\rangle\,,$$

where the set of formulae is restricted to Φ. The morphism

$$\langle i, \mathrm{id}_\mathcal{K}\rangle : L|_\Phi \to L$$

is the *inclusion morphism* associated with Φ and L.

(b) The *localisation* of L to Φ is the logic

$$L/\Phi := \langle L,\ \mathrm{Mod}_L(\Phi),\ \vDash\rangle\,,$$

where the class of interpretations is restricted to those satisfying Φ. The morphism

$$\langle \mathrm{id}_L, j\rangle : L \to L/\Phi$$

is the *localisation morphism* associated with Φ and L. We define the relations

$$\begin{aligned} \varphi \vDash_\Phi \psi \quad &:\text{iff} \quad \Phi \cup \{\varphi\} \vDash \psi\,, \\ \varphi \equiv_\Phi \psi \quad &:\text{iff} \quad \varphi \equiv \psi \text{ modulo } \Phi\,. \end{aligned}$$

(c) If L is an algebraic logic and $\Phi \subseteq L^\circ[\Sigma]$ then we set

$$L^{\bar{s}}/\Phi := L^{\bar{s}}[\Sigma]/\Phi\,.$$

The next lemma and its corollary state that the restriction and the localisation of a logic yield something like 'short exact sequences' of logics and Lindenbaum algebras

$$L|_\Phi \to L \to L/\Phi \quad \text{and} \quad \mathfrak{Lb}(L|_\Phi) \to \mathfrak{Lb}(L) \to \mathfrak{Lb}(L/\Phi)\,.$$

Lemma 6.14. *Let $\langle L, \mathcal{K}, \models\rangle$ be a logic and $\Phi \subseteq L$ a set of formulae.*

(a) *The inclusion morphism $i : L|_\Phi \to L$ is a monomorphism of logics.*

(b) *The localisation morphism $\lambda : L \to L/\Phi$ is an epimorphism of logics.*

Corollary 6.15. *Let L be a logic and $\Phi \subseteq L$.*

(a) *There exists an embedding $\mathfrak{Lb}(L|_\Phi) \to \mathfrak{Lb}(L)$.*

(b) *There exists a surjective homomorphism $\mathfrak{Lb}(L) \to \mathfrak{Lb}(L/\Phi)$.*

Proof. The claims follow from Lemmas 6.14 and 6.10. □

We can describe the entailment relation of a localisation as follows.

Lemma 6.16. *Let L be a logic and $T \subseteq L$.*

(a) $\varphi \models \psi$ *in* L/T iff $\varphi \models_T \psi$ *in* L.

(b) $\mathfrak{Lb}(L/T) = \langle L, \models_T\rangle/\equiv_T$.

Proof. (a) We have $\varphi \models \psi$ in L/T if, and only if, every model of T that satisfies φ also satisfies ψ. This is equivalent to $T \cup \{\varphi\} \models \psi$.

(b) follows immediately from (a). □

c2. Elementary substructures and embeddings

1. Homomorphisms and embeddings

We can compare structures by looking at the functions between them. In this section we investigate how such maps are related to the theories of the structures in question.

Definition 1.1. Let L be an algebraic logic and $f : \mathfrak{A} \to \mathfrak{B}$ a partial function between Σ-structures.

(a) We say that f *preserves* a formula $\varphi(\bar x) \in L[\Sigma, X]$ if

$$\mathfrak{A} \models \varphi(\bar a) \quad \text{implies} \quad \mathfrak{B} \models \varphi(f\bar a)\,, \qquad \text{for all } \bar a \subseteq \operatorname{dom} f\,.$$

(b) Let $\Delta \subseteq L[\Sigma, X]$ be a set of formulae. We call f a *Δ-map* if it preserves every formula in Δ. A *Δ-embedding* is a Δ-map that is an embedding. We say that f is *strict* if we have

$$\mathfrak{A} \models \varphi(\bar a) \quad \text{iff} \quad \mathfrak{B} \models \varphi(f\bar a)\,,$$

for all formulae $\varphi(\bar x) \in \Delta$ and every $\bar a \subseteq \operatorname{dom} f$.

If $C \subseteq A \subseteq B$ then we say that $f : A \to B$ is a Δ-map or a Δ-embedding *over* C if f additionally satisfies $f \restriction C = \mathrm{id}_C$. For historical reasons FO-maps and FO-embeddings are usually called *elementary.*

(c) We denote by $\operatorname{Emb}_L(\mathfrak{A}, \mathfrak{B})$ the set of all $L^{<\omega}[\Sigma]$-embeddings $h : \mathfrak{A} \to \mathfrak{B}$. We write $\mathfrak{Emb}_L(\Sigma)$ for the category of all $L^{<\omega}[\Sigma]$-embeddings between Σ-structures.

Remark. If Δ is closed under negation then every Δ-map is strict.

Example. Let $f : \mathfrak{A} \to \mathfrak{B}$. Immediately from the definition it follows that

(a) f is injective if and only if it preserves the formula $x \neq y$;

(b) f is a homomorphism if and only if it preserves every atomic formula;

(c) f is an embedding if and only if it preserves every literal.

Definition 1.2. (a) We write $\mathrm{QF}_{\kappa\aleph_0}[\Sigma, X]$ for the set of all quantifier-free $\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$-formulae.

(b) For $\Delta \subseteq \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ we denote by $\exists\Delta$ the closure of Δ under existential quantifiers and conjunctions and disjunctions of less than κ formulae. Similarly, $\forall\Delta$ denotes the closure of Δ under conjunctions, disjunctions, and universal quantifiers. The intended value of κ should always be clear from the context.

(c) The set of *existential formulae* is $\exists_{\kappa\aleph_0}[\Sigma, X] := \exists\mathrm{QF}_{\kappa\aleph_0}[\Sigma, X]$ and the set of *universal formulae* is $\forall_{\kappa\aleph_0}[\Sigma, X] := \forall\mathrm{QF}_{\kappa\aleph_0}[\Sigma, X]$. For $\kappa = \aleph_0$, we simply write $\exists[\Sigma, X]$ and $\forall[\Sigma, X]$.

(d) The set $\exists^+_{\kappa\aleph_0}[\Sigma, X]$ of *positive existential formulae* consists of all $\mathrm{FO}_{\kappa\aleph_0}$-formulae containing neither negations nor universal quantifiers.

Lemma 1.3. *Let $f : \mathfrak{A} \to \mathfrak{B}$.*

(a) *f is a homomorphism if, and only if, it preserves all $\exists^+_{\infty\aleph_0}$-formulae.*

(b) *f is an embedding if, and only if, it preserves all $\exists_{\infty\aleph_0}$-formulae.*

(c) *If f is an isomorphism, it preserves all $\mathrm{FO}_{\infty\aleph_0}$-formulae.*

Proof. One direction follows immediately from the definition (see the above example) since every function preserving all atomic formulae is a homomorphism and every function preserving all literals is an embedding.

For the other direction, we prove all three claims simultaneously by induction on the structure of φ. For claims (b) and (c), we may assume that φ is in negation normal form.

If $\varphi = Rt_0 \dots t_n$ then we have

$$\begin{aligned}\mathfrak{A} \models (Rt_0 \dots t_{n-1})(\bar a) &\Rightarrow \langle t_0^{\mathfrak{A}}(\bar a), \dots, t_{n-1}^{\mathfrak{A}}(\bar a)\rangle \in R^{\mathfrak{A}} \\ &\Rightarrow \langle f(t_0^{\mathfrak{A}}(\bar a)), \dots, f(t_{n-1}^{\mathfrak{A}}(\bar a))\rangle \in R^{\mathfrak{B}} \\ &\Rightarrow \langle t_0^{\mathfrak{B}}(f\bar a), \dots, t_{n-1}^{\mathfrak{B}}(f\bar a)\rangle \in R^{\mathfrak{B}} \\ &\Rightarrow \mathfrak{B} \models (Rt_0, \dots, t_{n-1})(f\bar a)\,.\end{aligned}$$

The proof for $\varphi = t_0 = t_1$ is similar.

For (b) and (c), we also have to consider the case that $\varphi = \neg Rt_0 \dots t_n$. Since in these cases f is a strict homomorphism we have

$$\begin{aligned}\mathfrak{A} \models \neg(Rt_0, \dots, t_{n-1})(\bar a) &\Rightarrow \langle t_0^{\mathfrak{A}}(\bar a), \dots, t_{n-1}^{\mathfrak{A}}(\bar a)\rangle \notin R^{\mathfrak{A}} \\ &\Rightarrow \langle f(t_0^{\mathfrak{A}}(\bar a)), \dots, f(t_{n-1}^{\mathfrak{A}}(\bar a))\rangle \notin R^{\mathfrak{B}} \\ &\Rightarrow \langle t_0^{\mathfrak{B}}(f\bar a), \dots, t_{n-1}^{\mathfrak{B}}(f\bar a)\rangle \notin R^{\mathfrak{B}} \\ &\Rightarrow \mathfrak{B} \models \neg(Rt_0, \dots, t_{n-1})(f\bar a)\,.\end{aligned}$$

The proof for $\varphi = t_0 \neq t_1$ is similar.

The cases that $\varphi = \bigwedge \Phi$ or $\varphi = \bigvee \Phi$ follow immediately from the inductive hypothesis. Therefore, it remains to consider quantifiers. Suppose that $\varphi = \exists y\psi(\bar x, y)$. We have

$$\begin{aligned}\mathfrak{A} \models \exists y\psi(\bar a, y) &\Rightarrow \mathfrak{A} \models \psi(\bar a, b) \text{ for some } b \in A \\ &\Rightarrow \mathfrak{B} \models \psi(f\bar a, fb) \text{ for some } b \in A \\ &\Rightarrow \mathfrak{B} \models \exists y\psi(f\bar a, y)\,.\end{aligned}$$

Finally, for claim (c) there is the case that $\varphi = \forall y\psi(\bar x, y)$. Then we have

$$\begin{aligned}\mathfrak{A} \models \forall y\psi(\bar a, y) &\Rightarrow \mathfrak{A} \models \psi(\bar a, b) \text{ for all } b \in A \\ &\Rightarrow \mathfrak{B} \models \psi(f\bar a, fb) \text{ for all } b \in A \\ &\Rightarrow \mathfrak{B} \models \forall y\psi(f\bar a, y)\,,\end{aligned}$$

since f is surjective. □

Corollary 1.4. *Let $\mathfrak{A}$ be a Σ-structure. For every relation $R := \varphi^{\mathfrak{A}}$ defined by some formula $\varphi(\bar{x}) \in \mathrm{FO}^{<\omega}_{\infty\aleph_0}[\Sigma]$, we have*

$$\bar{a} \in R \quad \text{iff} \quad \pi\bar{a} \in R\,, \quad \text{for each automorphism } \pi : \mathfrak{A} \to \mathfrak{A}\,.$$

Example. We can use the above characterisation to prove that certain relations are not definable. Let $\mathfrak{A}$ be a structure and R a relation. If we can find an automorphism of $\mathfrak{A}$ that is not an automorphism of the expansion $\langle \mathfrak{A}, R \rangle$ then we know that R is not definable in $\mathfrak{A}$.

(a) Addition is not definable in the structure $\langle \mathbb{N}, \cdot \rangle$. Define the function $\pi : \mathbb{N} \to \mathbb{N}$ that maps a number of the form $2^m 3^n k$, where k is not divisible by 2 or 3, to the number $2^n 3^m k$. Then π is an automorphism of $\langle \mathbb{N}, \cdot \rangle$, but it is not an automorphism of $\langle \mathbb{N}, \cdot, + \rangle$ since we have

$$4 + 3 = 7 \quad \text{and} \quad \pi(4) + \pi(3) = 9 + 2 \neq 7 = \pi(7)\,.$$

(b) Similarly, we can show that multiplication is not definable in the structure $\langle \mathbb{Z}, + \rangle$ since the mapping $\pi : x \mapsto -x$ is an automorphism of $\langle \mathbb{Z}, + \rangle$ but not of $\langle \mathbb{Z}, +, \cdot \rangle$.

Definition 1.5. A formula $\varphi(\bar{x})$ is *preserved in substructures* if

$$\mathfrak{A} \vDash \varphi(\bar{a}) \quad \text{implies} \quad \mathfrak{A}_0 \vDash \varphi(\bar{a})\,,$$

whenever $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is a substructure containing $\bar{a}$.

Lemma 1.6. *$\forall_{\infty\aleph_0}$-formulae are preserved in substructures.*

Proof. This is just the dual statement of Lemma 1.3 (b). Let $\varphi \in \forall_{\infty\aleph_0}$ and suppose there exist structures $\mathfrak{A}_0 \subseteq \mathfrak{A}$ and elements $\bar{a} \subseteq A_0$ such that

$$\mathfrak{A} \vDash \varphi(\bar{a}) \quad \text{but} \quad \mathfrak{A}_0 \nvDash \varphi(\bar{a})\,.$$

Let $\mathrm{id} : \mathfrak{A}_0 \to \mathfrak{A}$ be the embedding of $\mathfrak{A}_0$ into $\mathfrak{A}$. Since $\neg\varphi$ is equivalent to some existential formula $\psi \in \exists_{\infty\aleph_0}$ it follows from Lemma 1.3 (b) that

$$\mathfrak{A}_0 \vDash \neg\varphi(\bar{a}) \quad \text{implies} \quad \mathfrak{A} \vDash \neg\varphi(\bar{a})\,.$$

Contradiction. □

Example. Groups can be axiomatised by universal sentences:

$$\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$$
$$\forall x (x \cdot e = x)$$
$$\forall x (x \cdot x^{-1} = e)$$

It follows that every substructure of a group $\langle G, \cdot, ^{-1}, e \rangle$ is itself a group.

Note that, if we use the smaller signature consisting only of group multiplication $\cdot$, this property fails since the axioms are no longer universal:

$$\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$$
$$\exists e \forall x [x \cdot e = x \wedge \exists y (x \cdot y = e)]$$

For instance, the group $\langle \mathbb{Z}, + \rangle$ has the substructure $\langle \mathbb{N}, + \rangle$ which is not a group.

Definition 1.7. A formula $\varphi(\bar{x})$ is *preserved in unions of chains* if, for all chains $(\mathfrak{A}_i)_{i<\alpha}$ and every tuple $\bar{a} \subseteq A_0$,

$$\mathfrak{A}_i \vDash \varphi(\bar{a})\,, \text{ for all } i < \alpha\,, \quad \text{implies} \quad \bigcup_{i<\alpha} \mathfrak{A}_i \vDash \varphi(\bar{a})\,.$$

Lemma 1.8. *Every $\forall\exists_{\infty\aleph_0}$-formula φ is preserved in unions of chains.*

Proof. Let $(\mathfrak{A}_i)_{i<\alpha}$ be a chain with union $\mathfrak{B} := \bigcup_{i<\alpha} \mathfrak{A}_i$. Suppose that $\varphi \in \forall\exists_{\infty\aleph_0}$ is a formula such that $\mathfrak{A}_i \vDash \varphi(\bar{a})$, for all $i < \alpha$, where $\bar{a} \subseteq A_0$. We prove by induction on φ that $\mathfrak{B} \vDash \varphi(\bar{a})$.

If $\varphi \in \exists_{\infty\aleph_0}$ then $\mathfrak{A}_0 \vDash \varphi(\bar{a})$ and $\mathfrak{A}_0 \subseteq \mathfrak{B}$ implies that $\mathfrak{B} \vDash \varphi(\bar{a})$, by Lemma1.3 (b). If $\varphi = \bigwedge \Phi$ or $\varphi = \bigvee \Phi$, for $\Phi \subseteq \forall\exists_{\infty\aleph_0}$ then the claim follows immediately from the inductive hypothesis.

Hence, it remains to consider the case that $\varphi = \forall y \psi(\bar{x}, y)$, for some $\psi \in \forall\exists_{\infty\aleph_0}$. For every $b \in B$, there is some index k such that $b \in A_k$. By assumption, we have $\mathfrak{A}_i \vDash \psi(\bar{a}, b)$, for every $i \geq k$. By inductive hypothesis, it follows that $\bigcup_{i \geq k} \mathfrak{A}_i \vDash \psi(\bar{a}, b)$. Since $\bigcup_{i \geq k} \mathfrak{A}_i = \mathfrak{B}$ we have shown that $\mathfrak{B} \vDash \psi(\bar{a}, b)$, for all $b \in B$. This implies that $\mathfrak{B} \vDash \forall y \psi(\bar{a}, y)$. □

Remark. Similarly to Lemma **??**, we can show that $\forall\exists_{\infty\aleph_0}$-formulae are preserved in direct limits of diagrams of embeddings. Analogously it follows that $\forall\exists^+_{\infty\aleph_0}$-formulae are preserved in arbitrary direct limits.

Example. The class of all fields is $\forall\exists$-axiomatisable. It follows that the union of a chain of fields is again a field.

Exercise 1.1. Prove that every $\forall\exists^+_{\infty\aleph_0}$-formula is preserved in direct limits.

2. *Elementary embeddings*

Definition 2.1. Let L be an algebraic logic, $\Delta \subseteq L[\Sigma, X]$ a set of formulae, and $\mathfrak{A}$ and $\mathfrak{B}$ Σ-structures.

We say that $\mathfrak{B}$ is a Δ-*extension* of $\mathfrak{A}$, or that $\mathfrak{A}$ is a Δ-*substructure* of $\mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and the inclusion map $A \to B$ is a Δ-embedding. We write $\mathfrak{A} \preceq_\Delta \mathfrak{B}$ to indicate that $\mathfrak{A}$ is a Δ-substructure of $\mathfrak{B}$. In the case $\Delta = \mathrm{FO}[\Sigma]$ we also speak of *elementary* embeddings and extensions, and we write $\mathfrak{A} \preceq \mathfrak{B}$ instead of $\mathfrak{A} \preceq_{\mathrm{FO}} \mathfrak{B}$.

Example. (a) $\langle \mathbb{N}, \leq \rangle \subseteq \langle \mathbb{Q}, \leq \rangle$ is not elementary since

$$\langle \mathbb{N}, \leq \rangle \vDash \exists x \forall y (x \leq y) \quad \text{but} \quad \langle \mathbb{Q}, \leq \rangle \nvDash \exists x \forall y (x \leq y)\,.$$

(b) There are structures $\mathfrak{A} \subseteq \mathfrak{B}$ such that $\mathfrak{A} \equiv \mathfrak{B}$ but $\mathfrak{A} \npreceq \mathfrak{B}$. For instance, let $\mathfrak{A} := \langle 2\mathbb{Z}, \leq \rangle$ and $\mathfrak{B} := \langle \mathbb{Z}, \leq \rangle$. Then we even have $\mathfrak{A} \cong \mathfrak{B}$ but $\mathfrak{A} \npreceq \mathfrak{B}$ since

$$\langle 2\mathbb{Z}, \leq \rangle \nvDash \exists x (0 < x \wedge x < 2) \quad \text{but} \quad \langle \mathbb{Z}, \leq \rangle \vDash \exists x (0 < x \wedge x < 2)\,.$$

(c) $\langle \mathbb{Q}, \leq \rangle \preceq_{\mathrm{FO}} \langle \mathbb{R}, \leq \rangle$. (The easiest proof of this statement is based on so-called 'back-and-forth' arguments which will be introduced in Chapter c4. See Lemma c4.1.4).

Exercise 2.1. Find an elementary extension of $\langle \mathbb{Z}, s \rangle$ where $s : x \mapsto x + 1$ is the successor function.

Remark. If L is closed under negation then $\mathfrak{A} \preceq_L \mathfrak{B}$ implies $\mathfrak{A} \equiv_L \mathfrak{B}$.

Definition 2.2. Let L be an algebraic logic and $\mathfrak{A}$ a Σ-structure.

(a) For a set $U \subseteq A$, we denote by $\mathfrak{A}_U$ the expansion of $\mathfrak{A}$ by one constant c_a, for each element $a \in U$, with value $c_a^{\mathfrak{A}} := a$. By Σ_U we denote the corresponding expansion of the signature. In the following we will not distinguish between the element a and the symbol c_a denoting it, and we simply write a in both cases.

(b) If T is a complete theory and $\mathfrak{A}$ a model of T with $U \subseteq A$ then we define $T(U) := \mathrm{Th}_L(\mathfrak{A}_U)$. For $U = A$, we call $T(A)$ the L-*diagram* of $\mathfrak{A}$.

Let $\Delta_0 \subseteq \mathrm{FO}[\Sigma]$ be the set of all atomic first-order formulae and $\Delta_1 \subseteq \mathrm{FO}[\Sigma]$ the set of all literals. The Δ_0-diagram of $\mathfrak{A}$ is called the *atomic diagram*, and the Δ_1-diagram is the *algebraic diagram*. As usual, the FO-diagram is called *elementary*.

The next lemma states that in order to construct an L-extension of a structure $\mathfrak{A}$ we can take any model of its L-diagram.

Lemma 2.3 (Diagram Lemma). *Let L be an algebraic logic and $\mathfrak{A}$ and $\mathfrak{B}$ Σ-structures. There exists an L-map $g : \mathfrak{A} \to \mathfrak{B}$ if and only if we have*

$$\mathfrak{B}^+ \vDash \mathrm{Th}_L(\mathfrak{A}_A)\,, \quad \textit{for some } \Sigma_A\textit{-expansion } \mathfrak{B}^+ \textit{ of } \mathfrak{B}\,.$$

Proof. ($\Rightarrow$) By definition, $\mathfrak{B} \vDash \varphi(g\bar{a})$, for all $\varphi(\bar{a}) \in \mathrm{Th}(\mathfrak{A}_A)$. Hence, if $\bar{a}$ is an enumeration of A then we can define the desired expansion of $\mathfrak{B}$ by $\mathfrak{B}^+ := \langle \mathfrak{B}, g(\bar{a}) \rangle$.

($\Leftarrow$) We claim that the function $g : A \to B : a \mapsto c_a^{\mathfrak{B}^+}$ is the desired L-embedding. Since $\mathrm{Th}_L(\mathfrak{B}^+) = \mathrm{Th}_L(\mathfrak{A}_A)$ we have

$$\begin{aligned} \mathfrak{A} \vDash \varphi(a_0, \dots, a_{n-1}) \quad &\text{iff} \quad \varphi(c_{a_0}, \dots, c_{a_{n-1}}) \in \mathrm{Th}_L(\mathfrak{A}_A) \\ &\Rightarrow \quad \mathfrak{B}^+ \vDash \varphi(c_{a_0}, \dots, c_{a_{n-1}}) \\ &\text{iff} \quad \mathfrak{B} \vDash \varphi(g(a_0), \dots, g(a_{n-1}))\,. \end{aligned}$$ □

Corollary 2.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures. Let $\Delta_0(\mathfrak{A})$ be the atomic diagram of $\mathfrak{A}$ and $\Delta_1(\mathfrak{A})$ the algebraic diagram.*

(a) *There exists a homomorphism* $\mathfrak{A} \to \mathfrak{B}$ *if and only if*

$$\mathfrak{B}_A \vDash \Delta_0(\mathfrak{A}), \quad \text{for some expansion } \mathfrak{B}_A \text{ of } \mathfrak{B}.$$

(b) *There exists an embedding* $\mathfrak{A} \to \mathfrak{B}$ *if and only if*

$$\mathfrak{B}_A \vDash \Delta_1(\mathfrak{A}), \quad \text{for some expansion } \mathfrak{B}_A \text{ of } \mathfrak{B}.$$

For first-order logic there is a simple test to check whether some extension is elementary.

Theorem 2.5 (Tarski-Vaught Test). *Let* $\mathfrak{A} \subseteq \mathfrak{B}$ *be* Σ*-structures and suppose that* $\Delta \subseteq \mathrm{FO}_{\infty\aleph_0}[\Sigma]$ *is closed under negation, subformulae, and negation normal forms.*

We have $\mathfrak{A} \preceq_\Delta \mathfrak{B}$ *if and only if, for every formula* $\exists y\varphi(\bar{x}, y) \in \Delta$ *and all tuples* $\bar{a} \subseteq A$,

$$\mathfrak{B} \vDash \exists y\varphi(\bar{a}, y) \quad \text{implies} \quad \mathfrak{B} \vDash \varphi(\bar{a}, b), \text{ for some } b \in A.$$

Proof. ($\Rightarrow$) Since $\mathfrak{A} \preceq_\Delta \mathfrak{B}$ and Δ is closed under negation we have

$$\begin{aligned} \mathfrak{B} \vDash \exists y\varphi(\bar{a}, y) \quad &\text{iff} \quad \mathfrak{A} \vDash \exists y\varphi(\bar{a}, y) \\ &\text{iff} \quad \mathfrak{A} \vDash \varphi(\bar{a}, b) \qquad \text{for some } b \in A \\ &\text{iff} \quad \mathfrak{B} \vDash \varphi(\bar{a}, b) \qquad \text{for some } b \in A. \end{aligned}$$

($\Leftarrow$) Since Δ is closed under subformulae we can prove by induction on φ that

$$\mathfrak{A} \vDash \varphi(\bar{a}) \quad \text{implies} \quad \mathfrak{B} \vDash \varphi(\bar{a}), \qquad \text{for all } \varphi \in \Delta.$$

Moreover, it is sufficient to consider only formulae φ in negation normal form.

We will only give the inductive step for the universal quantifier. The other cases are handled in the same way as in the proof of Lemma 1.3. Suppose that

$$\mathfrak{A} \vDash \forall y\psi(\bar{a}, y) \quad \text{but} \quad \mathfrak{B} \nvDash \forall y\psi(\bar{a}, y).$$

Since $\forall y\psi \in \Delta$ we have $\mathrm{NNF}(\neg\forall y\psi) = \exists y(\mathrm{NNF}(\neg\psi)) \in \Delta$. Therefore, $\mathfrak{B} \vDash \exists y\neg\psi(\bar{a}, y)$ implies that $\mathfrak{B} \vDash \neg\psi(\bar{a}, b)$, for some $b \in A$. On the other hand, $\mathfrak{A} \vDash \forall y\psi(\bar{a}, y)$ implies that $\mathfrak{A} \vDash \psi(\bar{a}, b)$ and, by inductive hypothesis, it follows that $\mathfrak{B} \vDash \psi(\bar{a}, b)$. Contradiction. □

Proposition 2.6. *Let* $\mathcal{D} : \mathfrak{I} \to \mathrm{Hom}_s(\Sigma)$ *be a directed diagram of strict homomorphisms with cone* $h_i : \mathcal{D}(i) \to \varinjlim \mathcal{D}$, $i \in I$, *and suppose that* $\Delta \subseteq \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$ *is closed under subformulae and negation. If each map* $\mathcal{D}(i, j)$ *is a* Δ*-map then so is every* h_i.

Proof. By induction on $\varphi \in \Delta$ we prove that

$$\mathcal{D}(i) \vDash \varphi(\bar{a}) \quad \text{iff} \quad \varinjlim \mathcal{D} \vDash \varphi(h_i(\bar{a})).$$

Since $\forall y\psi(\bar{x}, y) \equiv \neg\exists y\neg\psi(\bar{x}, y)$ and Δ is closed under negation we may w.l.o.g. assume that φ does not contain universal quantifiers.

If φ is atomic then the claim follows from the fact that h_i is a strict homomorphism. The cases that $\varphi = \neg\psi$, $\varphi = \bigwedge \Phi$, or $\varphi = \bigvee \Phi$ follow immediately from inductive hypothesis.

Suppose that $\varphi = \exists y\psi(\bar{x}, y)$. If $\mathcal{D}(i) \vDash \exists y\psi(\bar{a}, y)$ then there is some $b \in \mathcal{D}(i)$ such that $\mathcal{D}(i) \vDash \psi(\bar{a}, b)$. By inductive hypothesis, it follows that $\varinjlim \mathcal{D} \vDash \psi(h_i(\bar{a}b))$. Hence, $\varinjlim \mathcal{D} \vDash \varphi(h_i(\bar{a}))$. Conversely, suppose that $\varinjlim \mathcal{D} \vDash \exists y\psi(h_i(\bar{a}), y)$. Then there is some element b such that $\varinjlim \mathcal{D} \vDash \psi(h_i(\bar{a}), b)$. By definition of a direct limit there is some index k with $b \in \mathrm{rng}\, h_k$. Let $l \in I$ be an index with $i, k \leq l$ and let $c \in \mathcal{D}(l)$ be an element with $h_l(c) = b$. By inductive hypothesis, it follows that $\mathcal{D}(l) \vDash \psi(\mathcal{D}(i, l)(\bar{a}), c)$. Hence, $\mathcal{D}(l) \vDash \varphi(\mathcal{D}(i, l)(\bar{a}))$. Since $\mathcal{D}(i, l)$ is a Δ-map and Δ is closed under negation we have $\mathcal{D}(i) \vDash \varphi(\bar{a})$, as desired. □

Definition 2.7. A chain $(\mathfrak{A}_i)_{i<\alpha}$ is an *L-chain* if $\mathfrak{A}_i \preceq_L \mathfrak{A}_k$, for all $i < k$. As usual, FO-chains are also called *elementary.*

Corollary 2.8. *If* $(\mathfrak{A}_i)_{i<\alpha}$ *is an* $\mathrm{FO}_{\kappa\aleph_0}$*-chain then* $\mathfrak{A}_k \preceq_{\mathrm{FO}_{\kappa\aleph_0}} \bigcup_{i<\alpha} \mathfrak{A}_i$, *for all* $k < \alpha$.

If $\Delta \subseteq \mathrm{FO}_{\kappa\aleph_0}$ is not closed under negation then obtain a similar result if we require the diagram to be κ-directed and Δ to not contain universal quantifiers.

Proposition 2.9. *Let $\mathcal{D} : \mathfrak{J} \to \mathrm{Hom}(\Sigma)$ be a κ-directed diagram with cone $h_i : \mathcal{D}(i) \to \varinjlim \mathcal{D}$, $i \in I$, and suppose that $\Delta \subseteq \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ is closed under subformulae and no formula in Δ contains universal quantifiers. If each map $\mathcal{D}(i,j)$ is a Δ-map then so is every h_i.*

Proof. By induction on $\varphi \in \Delta$ we prove that

$$\varinjlim \mathcal{D} \vDash \varphi(\bar{a}) \quad \text{iff} \quad \text{there is some } i \in I \text{ and a tuple } \bar{b} \text{ with } h_i(\bar{b}) = \bar{a} \text{ such that } \mathcal{D}(i) \vDash \varphi(\bar{b})\,.$$

(φ atomic) follows from the definition of $\varinjlim \mathcal{D}$.

($\varphi = \bigvee \Psi$) If $\mathcal{D}(i) \vDash \varphi(\bar{b})$ then there is a formula $\psi \in \Psi$ with $\mathcal{D}(i) \vDash \psi(\bar{b})$. By inductive hypothesis it follows that $\varinjlim \mathcal{D} \vDash \varphi(\bar{a})$. Conversely, if $\varinjlim \mathcal{D} \vDash \psi(\bar{a})$, for some $\psi \in \Psi$, then we have $\mathcal{D}(i) \vDash \psi(\bar{b})$ and $h_i(\bar{b}) = \bar{a}$, for suitable i and $\bar{b}$.

($\varphi = \bigwedge \Psi$) If $\mathcal{D}(i) \vDash \varphi(\bar{b})$ then the inductive hypothesis implies that $\varinjlim \mathcal{D} \vDash \psi(\bar{a})$, for each $\psi \in \Psi$. Conversely, if $\varinjlim \mathcal{D} \vDash \varphi(\bar{a})$ then we can find, for every $\psi \in \Psi$, an index $i_\psi \in I$ and a tuple $\bar{b}_\psi$ with $h_{i_\psi}(\bar{b}_\psi) = \bar{a}$ and $\mathcal{D}(i_\psi) \vDash \psi(\bar{b}_\psi)$. Since $h_{i_\psi}(\bar{b}_\psi) = h_{i_\vartheta}(\bar{b}_\vartheta)$, for $\psi, \theta \in \Psi$, there exists, by definition of $\varinjlim \mathcal{D}$, an index $l_{\psi\vartheta} \geq i_\psi, i_\theta$ with

$$\mathcal{D}(i_\psi, l_{\psi\vartheta})(\bar{b}_\psi) = \mathcal{D}(i_\vartheta, l_{\psi\vartheta})(\bar{b}_\vartheta)\,.$$

Since $\mathfrak{J}$ is κ-directed we can find index $k \in I$ with $l_{\psi\vartheta} \leq k$, for all ψ, ϑ. Let $\bar{c} := \mathcal{D}(i_\psi, k)(\bar{b}_\psi)$, for some/all ψ. It follows that $h_k(\bar{c}) = \bar{a}$ and $\mathcal{D}(k) \vDash \psi(\bar{c})$, for every $\psi \in \Psi$.

($\varphi = \neg\psi$) Since all homomorphisms $\mathcal{D}(i,k)$ are Δ-maps and $\neg\psi \in \Delta$ we have

$$\mathcal{D}(i) \vDash \psi(\bar{b}) \quad \text{iff} \quad \mathcal{D}(k) \vDash \psi(\mathcal{D}(i,k)(\bar{b}))\,, \quad \text{for all } i \leq k\,.$$

Consequently, $h_i(\bar{b}) = h_j(\bar{c})$, for arbitrary $i, j \in I$, implies

$$\mathcal{D}(i) \vDash \psi(\bar{b}) \quad \text{iff} \quad \mathcal{D}(j) \vDash \psi(\bar{c})\,.$$

Therefore, we have

$$\begin{aligned}
&\varinjlim \mathcal{D} \nvDash \psi(\bar{a}) \\
\text{iff}\quad &\mathcal{D}(i) \nvDash \psi(\bar{b}) &&\text{for all } i \text{ and } \bar{b} \in h_i^{-1}(\bar{a})\,,\\
\text{iff}\quad &\mathcal{D}(i) \vDash \neg\psi(\bar{b}) &&\text{for all } i \text{ and } \bar{b} \in h_i^{-1}(\bar{a})\,,\\
\text{iff}\quad &\mathcal{D}(i) \vDash \neg\psi(\bar{b}) &&\text{for some } i \text{ and } \bar{b} \in h_i^{-1}(\bar{a})\,.
\end{aligned}$$

($\varphi = \exists y \psi(\bar{x}, y)$) If $\mathcal{D}(i) \vDash \exists y \psi(\bar{b}, y)$ then there is some $c \in \mathcal{D}(i)$ such that $\mathcal{D}(i) \vDash \psi(\bar{b}, c)$. By inductive hypothesis, it follows that

$$\varinjlim \mathcal{D} \vDash \psi(h_i(\bar{b}c))\,.$$

Hence, $\varinjlim \mathcal{D} \vDash \varphi(h_i(\bar{b}))$. Conversely, suppose that $\varinjlim \mathcal{D} \vDash \exists y \psi(\bar{a}, y)$. Then there is some element c such that $\varinjlim \mathcal{D} \vDash \psi(\bar{a}, c)$. By inductive hypothesis, we can find an index i and elements $\bar{b}d \in h_i^{-1}(\bar{a}c)$ such that $\mathcal{D}(i) \vDash \psi(\bar{b}, d)$. Hence, $\mathcal{D}(i) \vDash \varphi(\bar{b})$. □

Exercise 2.2. Find an example showing that the above Proposition does not hold if Δ contains a formula with a universal quantifier.

We conclude this section with the observation that interpretations preserve elementary embeddings.

Lemma 2.10. *Let Σ and Γ be signatures. Every first-order interpretation $\mathcal{I}$ from Σ to Γ induces a functor $\mathcal{I} : \mathfrak{Emb}_\mathcal{I} \to \mathfrak{Emb}_{\mathrm{FO}}(\Gamma)$, where $\mathfrak{Emb}_\mathcal{I}$ denotes the subcategory of $\mathfrak{Emb}_{\mathrm{FO}}(\Sigma)$ consisting of all structures $\mathfrak{A}$ such that $\mathcal{I}(\mathfrak{A})$ is defined.*

Proof. Suppose that $\mathcal{I} = \langle \alpha, (\delta_s)_s, (\varepsilon_s)_s, (\varphi_\xi)_\xi \rangle$, let $h : \mathfrak{A} \to \mathfrak{B}$ be an elementary embedding such that $\mathcal{I}(\mathfrak{A})$ and $\mathcal{I}(\mathfrak{B})$ are defined, and let

$$\iota_s : \delta_s^{\mathfrak{A}} \to \mathcal{I}(\mathfrak{A}) \quad \text{and} \quad \kappa_s : \delta_s^{\mathfrak{B}} \to \mathcal{I}(\mathfrak{B})$$

be the canonical functions mapping a tuple to the element it encodes. We define $\mathcal{I}(h) : \mathcal{I}(\mathfrak{A}) \to \mathcal{I}(\mathfrak{B})$ as follows. For every element c of $\mathcal{I}(\mathfrak{A})$ of sort s, we set

$$\mathcal{I}(h)(c) := \kappa_s(h(\bar{a})), \quad \text{for any } \bar{a} \in \iota_s^{-1}(c).$$

We claim that $\mathcal{I}(h)$ is a well-defined elementary embedding.

To show that it is well-defined, suppose that $\bar{a}, \bar{a}' \in \iota_s^{-1}(c)$. Then

$$\mathfrak{A} \vDash \varepsilon_s(\bar{a}, \bar{a}') \quad \text{implies} \quad \mathfrak{B} \vDash \varepsilon_s(h(\bar{a}), h(\bar{a}')).$$

Consequently,

$$\kappa_s(h(\bar{a})) = \kappa_s(h(\bar{a}')),$$

as desired.

Hence, it remains to show that $\mathcal{I}(h)$ is an elementary embedding. Let $\bar{c}$ be an n-tuple in $\mathcal{I}(\mathfrak{A})$ with sorts $\bar{s}$ and let $\varphi(\bar{x})$ be a first-order formula. Choosing tuples $\bar{a}_i \in \iota_{s_i}^{-1}(c_i)$, it follows by Lemma c1.5.9 that

$$\begin{aligned} & \mathcal{I}(\mathfrak{A}) \vDash \varphi(\bar{c}) \\ \text{iff} \quad & \mathfrak{A} \vDash \varphi^{\mathcal{I}}(\bar{a}_0, \dots, \bar{a}_{n-1}) \\ \text{iff} \quad & \mathfrak{B} \vDash \varphi^{\mathcal{I}}(h(\bar{a}_0), \dots, h(\bar{a}_{n-1})) \\ \text{iff} \quad & \mathcal{I}(\mathfrak{B}) \vDash \varphi(\kappa_{s_0}(h(\bar{a}_0)), \dots, \kappa_{s_{n-1}}(h(\bar{a}_{n-1}))) \\ \text{iff} \quad & \mathcal{I}(\mathfrak{B}) \vDash \varphi(\mathcal{I}(h)(\bar{c})). \end{aligned}$$

□

3. The Theorem of Löwenheim and Skolem

A general method to eliminate existential quantifiers consists in replacing them by functions. Consider a formula $\psi = \exists y \varphi(\bar{x}, y)$ which states that, for a given value of $\bar{x}$, there exists some element y satisfying φ. If we define a function f that maps all suitable values of $\bar{x}$ to such an element y then we can write ψ equivalently as $\varphi(\bar{x}, f\bar{x})$. Informally we say that the function f we constructed yields a 'witness' that asserts the truth of $\exists y \varphi$.

Definition 3.1. Let $\Phi \subseteq \mathrm{FO}^{0}_{\infty\aleph_0}[\Sigma]$ and $\Delta \subseteq \mathrm{FO}^{<\omega}_{\infty\aleph_0}[\Sigma]$.

(a) A Σ-term $t(\bar{x})$ defines a *Skolem function* for a formula $\exists y \varphi(\bar{x}, y)$ (w.r.t. Φ) if

$$\Phi \vDash \forall \bar{x}\big[\exists y \varphi(\bar{x}, y) \to \varphi(\bar{x}, t(\bar{x}))\big].$$

A formula of the form $\forall \bar{x}[\exists y \varphi(\bar{x}, y) \to \varphi(\bar{x}, t)]$ is called a *Skolem axiom* for $\exists y \varphi$.

(b) A *Δ-Skolemisation* of Φ is a set $\Phi^+ \subseteq \mathrm{FO}^{0}_{\infty\aleph_0}[\Sigma^+]$, for some signature $\Sigma^+ \supseteq \Sigma$, such that

- $\Phi \subseteq \Phi^+$,
- every model $\mathfrak{M} \vDash \Phi$ has an Σ^+-expansion $\mathfrak{M}^+ \vDash \Phi^+$ and,
- for every formula $\exists y \varphi \in \Delta$, there exists a Σ^+-term defining a Skolem function for $\exists y \varphi$.

(c) We say that a theory $T \subseteq \mathrm{FO}^{0}_{\infty\aleph_0}[\Sigma]$ is a *Δ-Skolem theory* if T is a Δ-Skolemisation of itself. If $\Delta = \mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma]$ we simply speak of a *Skolemisation* and a *Skolem theory.* The intended value of κ and Σ should always be clear from the context.

Example. Consider the ordered additive group of the real numbers $\mathfrak{R} = \langle \mathbb{R}, +, <, f \rangle$ expanded by the (definable) function $f(x) := x/2$. The term $f(x_0 + x_1)$ defines a Skolem function for the formula

$$\varphi(x_0, x_1) := \exists y (x_0 < y < x_1).$$

The main reason why Skolem theories are interesting is the property of their models that *all* substructures are elementary.

Lemma 3.2. *Let $T \subseteq \mathrm{FO}^{0}_{\kappa\aleph_0}[\Sigma]$ be a Δ-Skolem theory where the set $\Delta \subseteq \mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma]$ is closed under negation, subformulae, and negation normal forms. If $\mathfrak{A} \vDash T$ and $\mathfrak{B} \subseteq \mathfrak{A}$ then $\mathfrak{B} \preceq_\Delta \mathfrak{A}$.*

Proof. We apply the Tarski-Vaught Test. Suppose that $\exists y \varphi(\bar{x}, y) \in \Delta$ is a formula and $\bar{a} \subseteq B$ a tuple such that

$$\mathfrak{A} \vDash \exists y \varphi(\bar{a}, y).$$

Let t be a term defining a Skolem function for $\exists y\varphi$. Then

$$\mathfrak{A} \vDash \varphi(\bar{a}, t(\bar{a}))\,.$$

Since $\bar{a} \subseteq B$ and B is closed under all functions of $\mathfrak{A}$ it follows that $t^{\mathfrak{A}}(\bar{a}) \in B$, as desired. □

Syntactically we can use Skolemisation to eliminate existential quantifiers.

Lemma 3.3. *Suppose that $T \subseteq \mathrm{FO}^{\circ}_{\kappa\aleph_0}[\Sigma]$ is a Skolem theory. For every formula $\varphi \in \mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma]$, we can construct a formula $\varphi^* \in \forall^{<\omega}_{\kappa\aleph_0}[\Sigma]$ such that*

$$\varphi^* \vDash \varphi \quad \textit{and} \quad T \vDash \varphi \to \varphi^*.$$

In particular, $\varphi \equiv \varphi^*$ modulo T.

Proof. We define φ^* by induction on φ. W.l.o.g. we may assume that φ is in negation normal form. For $\varphi \in \forall_{\kappa\aleph_0}$ we set $\varphi^* := \varphi$. For conjunctions, disjunctions, and universal quantifiers, we set

$$(\textstyle\bigwedge \Psi)^* := \bigwedge \{\, \psi^* \mid \psi \in \Psi \,\}\,,$$
$$(\textstyle\bigvee \Psi)^* := \bigvee \{\, \psi^* \mid \psi \in \Psi \,\}\,,$$
$$(\forall y\psi)^* := \forall y\psi^*.$$

Finally, for $\varphi = \exists y\psi(\bar{x}, y)$ we set $\varphi^* := \psi^*(\bar{x}, t_\varphi)$ where the term t_φ defines a Skolem function for φ. □

Corollary 3.4. *For every Skolem theory $T \subseteq \mathrm{FO}^{\circ}_{\kappa\aleph_0}[\Sigma]$ there exists a set $\Phi \subseteq \forall_{\kappa\aleph_0}[\Sigma]$ such that $T \equiv \Phi$.*

Proof. Let $\Phi := \{\, \varphi \in \forall^{\circ}_{\kappa\aleph_0}[\Sigma] \mid T \vDash \varphi \,\}$. Then we have $T \vDash \Phi$. Conversely, we can use the preceding lemma to assign to every formula $\varphi \in T$ a formula $\varphi^* \in \Phi$ with $\varphi^* \vDash \varphi$. This implies that $\Phi \vDash T$. □

Constructing Δ-Skolemisations is easy. We just have to add Skolem axioms for all formulae in Δ.

Lemma 3.5. *For all $\Phi \subseteq \mathrm{FO}^{\circ}_{\kappa\aleph_0}[\Sigma]$ and $\Delta \subseteq \mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma]$, there exists a Δ-Skolemisation $\Phi^+ \subseteq \mathrm{FO}^{\circ}_{\kappa\aleph_0}[\Sigma^+]$ of Φ with $|\Phi^+| \leq |\Phi| \oplus |\Delta|$ and $|\Sigma^+| \leq |\Sigma| \oplus |\Delta|$.*

Proof. Let Σ^+ be the signature obtained from Σ by adding new function symbols $f_{\exists y\varphi}$, for every formula $\exists y\varphi \in \Delta$. We construct Φ^+ by adding to Φ all Skolem axioms

$$\chi_{\exists y\varphi} := \forall\bar{x}\big[\exists y\varphi(\bar{x}, y) \to \varphi(\bar{x}, f_{\exists y\varphi}\bar{x})\big]$$

with $\exists y\varphi \in \Delta$. Clearly, $|\Phi^+| \leq |\Phi| \oplus |\Delta|$ and $|\Sigma^+| \leq |\Sigma| \oplus |\Delta|$.

We claim that Φ^+ is a Δ-Skolemisation of Φ. By construction, we have $\Phi \subseteq \Phi^+$ and every formula $\exists y\varphi \in \Delta$ has the Skolem function $f_{\exists y\varphi}$. Hence, it remains to prove that every model of Φ can be expanded to one of Φ^+.

Suppose that $\mathfrak{A} \vDash \Phi$. We construct an expansion $\mathfrak{A}^+ \vDash \Phi^+$ as follows. Let $\exists y\varphi \in \Delta$ and $\bar{a} \subseteq A$. If $\mathfrak{A} \vDash \exists y\varphi(\bar{a}, y)$ then we select some $b \in A$ such that $\mathfrak{A} \vDash \varphi(\bar{a}, b)$ and we set $f^{\mathfrak{A}^+}_{\exists y\varphi}(\bar{a}) := b$. Otherwise, we set $f^{\mathfrak{A}^+}_{\exists y\varphi}(\bar{a}) := b$, for an arbitrary element $b \in A$. This ensures that $\mathfrak{A}^+ \vDash \chi_{\exists y\varphi}$. Since $\mathfrak{A} \vDash \Phi$ and the function symbols $f_{\exists y\varphi}$ do not appear in Φ we further have $\mathfrak{A}^+ \vDash \Phi$. Consequently, $\mathfrak{A}^+ \vDash \Phi^+$. □

In order to obtain a Skolem theory we can iterate this construction.

Theorem 3.6. *Let κ be a regular cardinal. Every set $\Phi \subseteq \mathrm{FO}^{\circ}_{\kappa\aleph_0}[\Sigma]$ has a Skolemisation $\Phi^+ \subseteq \mathrm{FO}^{\circ}_{\kappa\aleph_0}[\Sigma^+]$ such that $|\Phi^+| \leq (|\Sigma| \oplus \aleph_0)^{<\kappa}$ and $(\Phi^+)^{\vDash}$ is a Skolem theory.*

Proof. We construct an increasing sequence of sets $(\Phi_\alpha)_{\alpha<\kappa}$ with $\Phi_\alpha \subseteq \mathrm{FO}^{\circ}_{\kappa\aleph_0}[\Sigma_\alpha]$. We set $\Phi_0 := \Phi$ and $\Phi_\delta := \bigcup_{\alpha<\delta} \Phi_\alpha$, for limit ordinals δ. For the successor step, we use Lemma 3.5 to obtain a $\mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma_\alpha]$-Skolemisation $\Phi_{\alpha+1}$ of Φ_α such that

$$|\Phi_{\alpha+1}| \leq |\Phi_\alpha| \oplus \big|\mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma_\alpha]\big| \quad \text{and} \quad |\Sigma_{\alpha+1}| \leq |\Sigma_\alpha| \oplus \big|\mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma_\alpha]\big|\,.$$

We claim that the union $\Phi^+ := \bigcup_{\alpha<\kappa} \Phi_\alpha$ is the desired Skolemisation. Let $\Sigma^+ := \bigcup_{\alpha<\kappa} \Sigma_\alpha$. First, we show by induction on α that $|\Sigma_\alpha| \leq (|\Sigma| \oplus \aleph_0)^{<\kappa}$. Clearly, this holds for $\Sigma_0 = \Sigma$. For the successor step, we have

$$\begin{aligned} |\Sigma_{\alpha+1}| &\leq |\Sigma_\alpha| \oplus \left|\mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma_\alpha]\right| \\ &\leq |\Sigma_\alpha| \oplus (|\Sigma_\alpha| \oplus \aleph_0)^{<\kappa} = (|\Sigma_\alpha| \oplus \aleph_0)^{<\kappa} \\ &\leq ((|\Sigma| \oplus \aleph_0)^{<\kappa} \oplus \aleph_0)^{<\kappa} = (|\Sigma| \oplus \aleph_0)^{<\kappa}. \end{aligned}$$

For limit ordinals $\delta < \kappa$, it follows that

$$|\Sigma_\delta| = \sup_{\alpha<\delta} |\Sigma_\alpha| \leq |\delta| \otimes (|\Sigma| \oplus \aleph_0)^{<\kappa} = (|\Sigma| \oplus \aleph_0)^{<\kappa}.$$

Consequently, we have

$$|\Sigma^+| = \sup_{\alpha<\kappa} |\Sigma_\alpha| \leq \kappa \otimes (|\Sigma| \oplus \aleph_0)^{<\kappa} = (|\Sigma| \oplus \aleph_0)^{<\kappa},$$

by Corollary A4.4.32. This implies that

$$|\Phi^+| \leq \left|\mathrm{FO}^0_{\kappa\aleph_0}[\Sigma^+]\right| \leq (|\Sigma^+| \oplus \aleph_0)^{<\kappa} \leq (|\Sigma| \oplus \aleph_0)^{<\kappa}.$$

Next, we prove that $(\Phi^+)^\vDash$ is a Skolem theory. Let $\exists y\varphi \in \mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma^+]$. Since κ is regular it follows by induction on φ that $\exists y\varphi \in \mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma_\alpha]$, for some $\alpha < \kappa$. Hence, there is a $\Sigma_{\alpha+1}$-term that defines a Skolem function for $\exists y\varphi$.

Finally, to show that Φ^+ is a Skolemisation of Φ it remains to prove that every model of Φ can be expanded to one of Φ^+. Let $\mathfrak{A} \vDash \Phi$ be a model of Φ. We construct a sequence $(\mathfrak{A}_\alpha)_{\alpha\leq\kappa}$ of models $\mathfrak{A}_\alpha \vDash \Phi_\alpha$ with $\mathfrak{A}_0 = \mathfrak{A}$ such that, for all $\alpha \leq \beta$, $\mathfrak{A}_\beta$ is an expansion of $\mathfrak{A}_\alpha$. $\mathfrak{A}_\kappa \vDash \Phi^+$ is the desired expansion of $\mathfrak{A}$.

We start with $\mathfrak{A}_0 := \mathfrak{A}$. For the successor step, suppose that $\mathfrak{A}_\alpha$ has already been defined. Since $\Phi_{\alpha+1}$ is a Skolemisation of Φ_α we can expand $\mathfrak{A}_\alpha$ to a $\Sigma_{\alpha+1}$-structure $\mathfrak{A}_{\alpha+1}$ such that $\mathfrak{A}_{\alpha+1} \vDash \Phi_{\alpha+1}$. For limit ordinals δ, we let $\mathfrak{A}_\delta$ be the 'union' of all the $\mathfrak{A}_\alpha$, $\alpha < \delta$, that is, its universe is A and, for each function $f \in \Sigma_\alpha$, we add the function $f^{\mathfrak{A}_\alpha}$ to $\mathfrak{A}_\delta$. (Note that this is well-defined since, if $f \in \Sigma_\alpha$ and $\alpha < \beta$ then $f^{\mathfrak{A}_\alpha} = f^{\mathfrak{A}_\beta}$.) □

An important application of the technique of Skolemisation is the following result.

Theorem 3.7 (Downward Löwenheim-Skolem Theorem).
Let $\Delta \subseteq \mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma]$, for a regular cardinal κ, and set $\mu := |\Sigma| \oplus |\Delta| \oplus \kappa^-$ where $\kappa^- := \sup\{\lambda \mid \lambda < \kappa\}$.

For each Σ-structure $\mathfrak{A}$, every subset $X \subseteq A$, and all cardinals λ with $|X| \oplus \mu \leq \lambda \leq |A|$, there exists a Δ-substructure $\mathfrak{B} \leq_\Delta \mathfrak{A}$ of size $|B| = \lambda$ with $X \subseteq B$.

Proof. Let Γ be the closure of Δ under subformulae, negation, and negation normal form. Since every formula $\varphi \in \mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma]$ has less that κ subformulae it follows that $|\Gamma| \leq |\Delta| \otimes \kappa^-$. By Lemma 3.5, we can choose a Γ-Skolemisation $T^+ \subseteq \mathrm{FO}^0_{\kappa^+\aleph_0}[\Sigma^+]$ of $\mathrm{Th}_\Gamma(\mathfrak{A})$ such that

$$|T^+| \leq |\mathrm{Th}_\Gamma(\mathfrak{A})| \oplus |\Gamma| \quad \text{and} \quad |\Sigma^+| \leq |\Sigma| \oplus |\Gamma| \leq \mu.$$

Let $\mathfrak{A}^+$ be a Σ^+-expansion of $\mathfrak{A}$ such that $\mathfrak{A}^+ \vDash T^+$, and choose some set $X \subseteq Z \subseteq A$ of size $|Z| = \lambda$. By Corollary B3.1.11, the substructure $\mathfrak{B}^+ := \langle\!\langle Z \rangle\!\rangle_{\mathfrak{A}^+}$ has cardinality

$$\lambda = |Z| \leq |B^+| \leq |Z| \oplus |\Sigma^+| \oplus \aleph_0 = \lambda.$$

By Lemma 3.2, we have $\mathfrak{B}^+ \leq_\Gamma \mathfrak{A}^+$. Let $\mathfrak{B}$ be the Σ-reduct of $\mathfrak{B}^+$. Then $\mathfrak{B} \leq_\Delta \mathfrak{A}$, as desired. □

Corollary 3.8. *Let $\mathfrak{A}$ be a Σ-structure. For each set $X \subseteq A$ and every cardinal $|X| \oplus |\Sigma| \oplus \aleph_0 \leq \kappa \leq |A|$, there exists an elementary substructure $\mathfrak{B} \preceq \mathfrak{A}$ of size $|B| = \kappa$ such that $X \subseteq B$.*

Example. The field $\mathfrak{R} = \langle \mathbb{R}, +, \cdot, 0, 1 \rangle$ of real numbers contains a countable elementary substructure $\mathfrak{R}_0 \prec \mathfrak{R}$.

We can generalise the technique of Skolemisation to $\mathrm{FO}_{\kappa\aleph_0}(\exists^\lambda)$ and $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{wo})$ in a straightforward way. As a result we obtain a variant of the Löwenheim-Skolem Theorem for these logics.

Theorem 3.9. *Let $\Delta \subseteq \mathrm{FO}^{<\omega}_{\kappa\aleph_0}(\exists^\lambda)[\Sigma]$, for a regular cardinal κ, and set $\mu := |\Sigma| \oplus |\Delta| \oplus \lambda \oplus \kappa^-$ where $\kappa^- := \sup\{\, \lambda \mid \lambda < \kappa \,\}$.*

For each Σ-structure $\mathfrak{A}$, every subset $X \subseteq A$, and all cardinals ν with $|X| \oplus \mu \leq \nu \leq |A|$, there exists a Δ-substructure $\mathfrak{B} \preceq_\Delta \mathfrak{A}$ of size $|B| = \nu$ with $X \subseteq B$.

Proof. The proof is analogous to that of Theorem 3.7. We adapt the notion of a Skolem function and a Δ-Skolemisation as follows. We say that a sequence $(t_i)_{i<\lambda}$ defines a Skolem function for a formula of the form $\exists^\lambda y\varphi(\bar{x}, y)$ if, for all $i, k < \lambda$ with $i \neq k$,

$$\Phi \vDash \forall\bar{x}(\exists^\lambda y\varphi(\bar{x}, y) \to \varphi(\bar{x}, t_i(\bar{x})))\,,$$
$$\Phi \vDash \forall\bar{x}(\exists^\lambda y\varphi(\bar{x}, y) \to t_i(\bar{x}) \neq t_k(\bar{x}))\,.$$

A Δ-Skolemisation of Φ is a set $\Phi^+ \supseteq \Phi$ such that

- every model of Φ can be extended to one of Φ^+,
- for every formula $\exists y\varphi \in \Delta$, there is a term defining a Skolem function for $\exists y\varphi$,
- for every formula $\exists^\lambda y\varphi \in \Delta$, there is a sequence of terms defining a Skolem function for $\exists^\lambda y\varphi$.

With these definitions it follows as above that if $\mathfrak{A} \vDash \Phi^+$ and $\mathfrak{B} \subseteq \mathfrak{A}$ then $\mathfrak{B} \preceq_\Delta \mathfrak{A}$. Furthermore, for every set Φ, we can find a Δ-Skolemisation of size $|\Phi| \oplus |\Delta| \oplus \lambda$. Consequently, we can repeat the construction in the proof of Theorem 3.7. □

Theorem 3.10. *Let $\Delta \subseteq \mathrm{FO}^{<\omega}_{\kappa\aleph_0}(\mathrm{wo})[\Sigma]$, for a regular cardinal κ, and set $\mu := |\Sigma| \oplus |\Delta| \oplus \kappa^-$ where $\kappa^- := \sup\{\, \lambda \mid \lambda < \kappa \,\}$.*

For each Σ-structure $\mathfrak{A}$, every subset $X \subseteq A$, and all cardinals λ with $|X| \oplus \mu \leq \lambda \leq |A|$, there exists a Δ-substructure $\mathfrak{B} \preceq_\Delta \mathfrak{A}$ of size $|B| = \lambda$ with $X \subseteq B$.

Proof. We adapt the notion of a Skolem function and a Δ-Skolemisation as follows. A sequence $(t_n)_{n<\omega}$ defines a Skolem function for the formula $\mathsf{W}\bar{x}\bar{y}\varphi(\bar{x}, \bar{y}, \bar{z})$ if, for all $n < \omega$,

$$\Phi \vDash \forall\bar{z}\big[\neg\mathsf{W}\bar{x}\bar{y}\varphi(\bar{x}, \bar{y}, \bar{z}) \to \varphi(t_{n+1}(\bar{z}), t_n(\bar{z}), \bar{z})\big]\,,$$

that is, the sequence $(t_n)_n$ yields witnesses for the fact that the relation defined by φ is not well-founded.

A Δ-Skolemisation of Φ is a set $\Phi^+ \supseteq \Phi$ such that

- every model of Φ can be extended to one of Φ^+,
- for every formula $\exists y\varphi \in \Delta$, there is a term defining a Skolem function for $\exists y\varphi$,
- for every formula $\mathsf{W}\bar{x}\bar{y}\varphi \in \Delta$, there is a sequence of terms defining a Skolem function for $\mathsf{W}\bar{x}\bar{y}\varphi$.

With these definitions it follows as above that if $\mathfrak{A} \vDash \Phi^+$ and $\mathfrak{B} \subseteq \mathfrak{A}$ then $\mathfrak{B} \preceq_\Delta \mathfrak{A}$. (Note that, if $\varphi(\bar{x}, \bar{y}, \bar{c})^{\mathfrak{A}}$, for $\bar{c} \subseteq B$, is a well-order of its field then so is $\varphi(\bar{x}, \bar{y}, \bar{c})^{\mathfrak{A}} \cap B^n = \varphi(\bar{x}, \bar{y}, \bar{c})^{\mathfrak{B}}$. Conversely, if $\varphi(\bar{x}, \bar{y}, \bar{c})^{\mathfrak{A}}$ is not a well-order then the Skolem function yields an infinite strictly decreasing sequence of elements of $\mathfrak{B}$. Hence, $\varphi(\bar{x}, \bar{y}, \bar{c})^{\mathfrak{A}} \cap B^n$ is also not a well-order.)

Furthermore, for every set Φ, we can find a Δ-Skolemisation of size $|\Phi| \oplus |\Delta| \oplus \lambda$. Consequently, we can repeat the construction in the proof of Theorem 3.7. □

Exercise 3.1. Work out the missing details in the above proofs.

4. *The Compactness Theorem*

In this section we introduce an important method to construct models from diagrams. These models $\mathfrak{M}$ will have the additional nice property that every element is denoted by some term, that is, $\mathfrak{M} = \langle\!\langle \varnothing \rangle\!\rangle_{\mathfrak{M}}$.

Definition 4.1. Let $\Phi \subseteq \mathrm{FO}^{0}_{\infty\aleph_0}[\Sigma]$. A structure $\mathfrak{H}$ is a *Herbrand model* of Φ if $\mathfrak{H} \vDash \Phi$ and, for every $a \in H$, there is some term $t \in T[\Sigma, \varnothing]$ with $t^{\mathfrak{H}} = a$.

We start by characterising those sets of formulae that contain sufficient information to extract a model.

Definition 4.2. A set $\Phi \subseteq \mathrm{FO}^{\mathrm{o}}_{\infty\aleph_0}[\Sigma]$ is *=-closed* if

- $t = t \in \Phi$, for all $t \in T[\Sigma, \varnothing]$, and
- if $\varphi(x)$ is an atomic formula and $s, t \in T[\Sigma, \varnothing]$ are terms with $s = t \in \Phi$ then we have $\varphi(s) \in \Phi$ iff $\varphi(t) \in \Phi$.

Lemma 4.3. *Let $\Phi \subseteq \mathrm{FO}^{\mathrm{o}}_{\infty\aleph_0}[\Sigma]$ be =-closed. The relation*

$$s \sim t \quad :\text{iff} \quad s = t \in \Phi$$

is a congruence relation of the term algebra $\mathfrak{T}[\Sigma, \varnothing]$.

Proof. ~ is reflexive since $t = t \in \Phi$, for all t. For symmetry, suppose that $s = t \in \Phi$ and set $\varphi(x) := x = s$. It follows that

$$\varphi(s) = s = s \in \Phi \quad \text{implies} \quad \varphi(t) = t = s \in \Phi\,.$$

Similarly, if $r = s \in \Phi$ and $s = t \in \Phi$ then setting $\varphi(x) := r = x$ we see that

$$\varphi(s) = r = s \in \Phi \quad \text{implies} \quad \varphi(t) = r = t \in \Phi\,.$$

Consequently, ~ is an equivalence relation.

Suppose that $s_i \sim t_i$, for $i < n$, and let $f \in \Sigma$ be an n-ary function symbol. In the same way as above we can show, by induction on i, that

$$f s_0 \dots s_i s_{i+1} \dots s_{n-1} = f t_0 \dots t_i s_{i+1} \dots s_{n-1} \in \Phi\,.$$

It follows that $f^{\mathfrak{T}[\Sigma,\varnothing]}(s_0, \dots, s_{n-1}) \sim f^{\mathfrak{T}[\Sigma,\varnothing]}(t_0, \dots, t_{n-1})$, as desired. □

Lemma 4.4. *Every =-closed set of atomic sentences $\Phi \subseteq \mathrm{FO}^{\mathrm{o}}_{\infty\aleph_0}[\Sigma]$ has a Herbrand model $\mathfrak{H}$ such that*

$$\Phi = \{\, \varphi \mid \varphi \text{ atomic and } \mathfrak{H} \vDash \varphi \,\}\,.$$

Proof. By Lemma 4.3, the relation

$$s \sim t \quad :\text{iff} \quad s = t \in \Phi$$

is a congruence relation of the term algebra $\mathfrak{T}[\Sigma, \varnothing]$. Hence, we can take the quotient $\mathfrak{H}_0 := \mathfrak{T}[\Sigma, \varnothing]/\!\sim$. Let $\mathfrak{H}$ be the expansion of $\mathfrak{H}_0$ by relations

$$R^{\mathfrak{H}} := \{\, \langle [t_0]_\sim, \dots, [t_{n-1}]_\sim \rangle \mid R t_0 \dots t_{n-1} \in \Phi \,\}\,,$$

for each n-ary relation $R \in \Sigma$. We claim that $\mathfrak{H}$ is the desired model.

Clearly, every element of $\mathfrak{H}$ is denoted by some term. Furthermore, by definition of $\mathfrak{H}$, we have $\mathfrak{H} \vDash \varphi$, for every $\varphi \in \Phi$. Conversely, suppose that $\mathfrak{H} \vDash \varphi$, for some atomic sentence φ. If $\varphi = s = t$ then we have $[s]_\sim = [t]_\sim$ which, by definition of $\mathfrak{H}$, implies that $s = t \in \Phi$. Similarly, if $\varphi = R t_0 \dots t_{n-1}$ then $\langle [t_0]_\sim, \dots, [t_{n-1}]_\sim \rangle \in R^{\mathfrak{H}}$. Hence, there are terms $s_i \sim t_i$ such that $R s_0 \dots s_{n-1} \in \Phi$. Since Φ is =-closed it follows that $R t_0 \dots t_{n-1} \in \Phi$. □

We have shown how to construct a model for a set of atomic formulae. Next we turn to the case of formulae with quantifiers.

Definition 4.5. A *Hintikka set* is a set $\Phi \subseteq \mathrm{FO}^{\mathrm{o}}_{\infty\aleph_0}[\Sigma]$ of sentences with the following closure properties:

(H1) Φ is =-closed.

(H2) If $\varphi \in \Phi$ then $\neg\varphi \notin \Phi$.

(H3) If $\neg\neg\varphi \in \Phi$ then $\varphi \in \Phi$.

(H4) If $\bigwedge \Psi \in \Phi$ then $\Psi \subseteq \Phi$.

(H5) If $\neg \bigwedge \Psi \in \Phi$ then there is some $\psi \in \Psi$ such that $\neg\psi \in \Phi$.

(H6) If $\bigvee \Psi \in \Phi$ then there is some $\psi \in \Psi$ such that $\psi \in \Phi$.

(H7) If $\neg \bigvee \Psi \in \Phi$ then $\neg\psi \in \Phi$, for all $\psi \in \Psi$.

(H8) If $\forall x \varphi(x) \in \Phi$ then $\varphi(t) \in \Phi$, for all $t \in T[\Sigma, \varnothing]$.

(H9) If $\neg\forall x \varphi(x) \in \Phi$ then there is some $t \in T[\Sigma, \varnothing]$ with $\neg\varphi(t) \in \Phi$.

(H10) If $\exists x\varphi(x) \in \Phi$ then there is some $t \in T[\Sigma, \varnothing]$ with $\varphi(t) \in \Phi$.

(H11) If $\neg\exists x\varphi(x) \in \Phi$ then $\neg\varphi(t) \in \Phi$, for all $t \in T[\Sigma, \varnothing]$.

Remark. Every elementary diagram is a Hintikka set.

Lemma 4.6. *Every Hintikka set $\Phi \subseteq \mathrm{FO}^0_{\infty\aleph_0}[\Sigma]$ has a Herbrand model.*

Proof. Let $\Phi_0 \subseteq \Phi$ consist of all atomic sentences in Φ. By the definition of a Hintikka set it follows that Φ_0 is =-closed. Hence, we can apply Lemma 4.4 to obtain a Herbrand model $\mathfrak{H}$ of Φ_0. We claim that $\mathfrak{H} \vDash \Phi$.

We prove by induction on the structure of a formula φ that

$$\varphi \in \Phi \text{ implies } \mathfrak{H} \vDash \varphi \quad \text{and} \quad \neg\varphi \in \Phi \text{ implies } \mathfrak{H} \vDash \neg\varphi .$$

If φ is atomic then the claim follows by Lemma 4.4.

Suppose that $\varphi = \neg\psi$. If $\varphi \in \Phi$ then we can apply the inductive hypothesis to ψ and it follows that $\mathfrak{H} \vDash \neg\psi$. Similarly, if $\neg\varphi \in \Phi$ then we have $\psi \in \Phi$, which implies that $\mathfrak{H} \vDash \psi$ and $\mathfrak{H} \vDash \neg\varphi$.

Consider the case that $\varphi = \bigwedge \Psi$. If $\bigwedge \Psi \in \Phi$ then $\Psi \subseteq \Phi$ implies that $\mathfrak{H} \vDash \psi$, for all $\psi \in \Psi$, and we have $\mathfrak{H} \vDash \bigwedge \Psi$. Analogously, if $\neg \bigwedge \Psi \in \Phi$ then there is some $\psi \in \Psi$ with $\neg\psi \in \Phi$. By inductive hypothesis it follows that $\mathfrak{H} \vDash \neg\psi$ which implies that $\mathfrak{H} \vDash \neg \bigwedge \Psi$.

Suppose that $\varphi = \forall x\psi(x)$. If $\varphi \in \Phi$ then $\psi(t) \in \Phi$, for all $t \in T[\Sigma, \varnothing]$. Hence, $\mathfrak{H} \vDash \psi(t)$, for all $t \in T[\Sigma, \varnothing]$. Since every element of H is denoted by a term it follows that $\mathfrak{H} \vDash \psi(a)$, for all $a \in H$, that is, $\mathfrak{H} \vDash \forall x\psi(x)$. Finally, if $\neg\forall x\psi(x) \in \Phi$ then there is some $t \in T[\Sigma, \varnothing]$ such that $\neg\psi(t) \in \Phi$. Therefore, we have $\mathfrak{H} \vDash \neg\psi(t)$ which implies that $\mathfrak{H} \vDash \neg\forall x\psi(x)$. The remaining cases are proved analogously. ☐

It is quite tedious to check that a set Φ satisfies conditions (H1)–(H11). The following lemma provides a simpler criterion for Φ being a Hintikka set.

Lemma 4.7. *Let $\Phi \subseteq \mathrm{FO}^0_{\infty\aleph_0}[\Sigma]$ be a set of sentences with the following properties:*

(1) *Every finite subset $\Phi_0 \subseteq \Phi$ is satisfiable.*

(2) *For every sentence $\varphi \in \mathrm{FO}^0_{\infty\aleph_0}[\Sigma]$ we have $\varphi \in \Phi$ or $\neg\varphi \in \Phi$.*

(3) *If $\exists x\varphi(x) \in \Phi$ then there exists some term $t \in T[\Sigma, \varnothing]$ such that $\varphi(t) \in \Phi$.*

(4) *If $\bigvee \Psi \in \Phi$ where $|\Psi| \geq \aleph_0$ then there is some $\psi \in \Psi$ with $\psi \in \Phi$.*

(5) *If $\neg \bigwedge \Psi \in \Phi$ where $|\Psi| \geq \aleph_0$ then there is some $\psi \in \Psi$ with $\neg\psi \in \Phi$.*

Then Φ is a Hintikka set.

Proof. First we show that

(∗) if $\Phi_0 \subseteq \Phi$ is finite and $\Phi_0 \vDash \varphi$ then $\varphi \in \Phi$.

Suppose otherwise. By (2), $\varphi \notin \Phi$ implies $\neg\varphi \in \Phi$. Hence, (1) implies that $\Phi_0 \cup \{\neg\varphi\}$ is satisfiable, and it follows that $\Phi_0 \nvDash \varphi$. A contradiction.

From (∗) we can conclude that Φ satisfies (H1), (H3), (H4), (H7), (H8), and (H11). Furthermore, (1) implies (H2), and (3) and (∗) imply that Φ satisfies (H9) and (H10).

It remains to prove (H5) and (H6). If $\Psi = \{\psi_0, \dots, \psi_{n-1}\}$ is finite then $\psi_0, \dots, \psi_{n-1} \in \Phi$ implies, by (∗), that $\bigwedge \Psi \in \Phi$. Hence, $\neg \bigwedge \Psi \notin \Phi$. Similarly, If $\neg\psi_0, \dots, \neg\psi_{n-1} \in \Phi$ then it follows that $\bigvee \Psi \notin \Phi$. If, on the other hand, Ψ is infinite then (H5) and (H6) follow immediately from (4) and (5). ☐

Hintikka sets can be used to prove the Compactness Theorem which is the most fundamental result in first-order model theory. Most results in the remainder of this book are based on this theorem. It is frequently used to construct structures with some given properties. To do so, one describes the desired structure by a set of first-order formulae and then uses the Compactness Theorem to prove that this set of axioms is satisfiable.

Theorem 4.8 (Compactness Theorem)**.** *Let $\Phi \subseteq \mathrm{FO}[\Sigma, X]$ be a set of first-order formulae and $\varphi \in \mathrm{FO}[\Sigma, X]$.*

(a) *Φ is satisfiable if and only if every finite subset $\Phi_0 \subseteq \Phi$ is satisfiable.*

(b) *$\Phi \vDash \varphi$ if and only if there exists a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \vDash \varphi$.*

Proof. Let us first prove that (a) implies (b). We have

$$\begin{aligned}\Phi \vDash \varphi \ \text{iff}\ & \Phi \cup \{\neg\varphi\} \text{ is inconsistent}\\ \text{iff}\ & \text{there exists a finite subset } \Phi_0 \subseteq \Phi \text{ such that}\\ & \quad \Phi_0 \cup \{\neg\varphi\} \text{ is inconsistent}\\ \text{iff}\ & \text{there exists a finite subset } \Phi_0 \subseteq \Phi \text{ such that } \Phi_0 \vDash \varphi\,.\end{aligned}$$

It remains to prove (a). For the nontrivial direction, suppose that every finite subset of Φ is satisfiable. By replacing every free variable in Φ by a constant symbol we may assume that every formula in Φ is a sentence. We have to construct a model of Φ. By Lemma 4.6, it is sufficient to find a Hintikka set $\Psi \supseteq \Phi$.

We construct Ψ in stages. Let $\kappa := |\mathrm{FO}^0[\Sigma]| = |\Sigma| \oplus \aleph_0$. Let C be a set containing κ^+ constant symbols of each sort and set $\Sigma_C := \Sigma \cup C$. We fix an enumeration $(\varphi_\alpha)_{\alpha<\kappa^+}$ of $\mathrm{FO}^0[\Sigma_C]$ such that, for every $\psi \in \mathrm{FO}^0[\Sigma_C]$, the set $\{\, \alpha < \kappa^+ \mid \varphi_\alpha = \psi \,\}$ is cofinal in κ^+.

We construct an increasing sequence $(\Psi_\alpha)_{\alpha<\kappa^+}$ of sets $\Phi \subseteq \Psi_\alpha \subseteq \mathrm{FO}^0[\Sigma_C]$ such that every finite subset of Ψ_α is satisfiable and such that the limit $\Psi := \bigcup_\alpha \Psi_\alpha$ is a Hintikka set. By Lemma 4.7 it is sufficient to ensure that

- $\varphi_\alpha \in \Psi_{\alpha+1}$ or $\neg\varphi_\alpha \in \Psi_{\alpha+1}$,
- If $\varphi_\alpha = \exists x\vartheta$ and $\varphi_\alpha \in \Psi_{\alpha+1}$ then $\vartheta(c) \in \Psi_{\alpha+1}$, for some constant $c \in C$.

Set $\Psi_0 := \Phi$. For limit ordinals δ, we set $\Psi_\delta := \bigcup_{\alpha<\delta} \Psi_\alpha$. For the successor step, suppose that Ψ_α has already been defined. If every finite subset of $\Psi_\alpha \cup \{\varphi_\alpha\}$ is satisfiable then set $\psi := \varphi_\alpha$ else set $\psi := \neg\varphi_\alpha$. We claim that every finite subset of $\Psi_\alpha \cup \{\psi\}$ is satisfiable. If $\psi = \varphi_\alpha$ then this holds by choice of ψ. Hence, suppose that $\psi = \neg\varphi_\alpha$ and there is a finite subset $\Gamma_0 \subseteq \Psi_\alpha \cup \{\neg\varphi_\alpha\}$ that is inconsistent. By construction there is also a finite subset $\Gamma_1 \subseteq \Psi_\alpha \cup \{\varphi_\alpha\}$ which is inconsistent. Hence, $\Gamma_0 \vDash \varphi_\alpha$ and $\Gamma_1 \vDash \neg\varphi_\alpha$. It follows that $\Gamma := \Gamma_0 \cup \Gamma_1$ is a finite subset of Ψ_α with $\Gamma \vDash \varphi_\alpha \wedge \neg\varphi_\alpha$. Thus, Γ is inconsistent in contradiction to our assumption on Ψ_α.

We have found a set $\Psi_\alpha \cup \{\psi\}$ that satisfies the first of our conditions. If ψ is not of the form $\exists x\vartheta$ then we can set $\Psi_{\alpha+1} := \Psi_\alpha \cup \{\psi\}$ and we are done. Hence, suppose that $\psi = \exists x\vartheta(x)$. Since $|\Psi_\alpha| \leq \kappa$ we can find a constant symbol $c \in C$ that does not appear in Ψ_α. We define $\Psi_{\alpha+1} := \Psi_\alpha \cup \{\psi, \vartheta(c)\}$. Note that, since every finite subset of $\Psi_\alpha \cup \{\exists x\vartheta\}$ is satisfiable so is every finite subset of $\Psi_{\alpha+1}$. □

Exercise 4.1. Let $\varphi \in \mathrm{FO}$ and $\Phi, T \subseteq \mathrm{FO}$. Prove that, if $\varphi \equiv \Phi$ modulo T then there exists a finite subset $\Phi_0 \subseteq \Phi$ such that $\varphi \equiv \bigwedge \Phi_0$ modulo T.

Exercise 4.2. Let $\mathcal{K}_i$, $i \in I$, be a family of first-order axiomatisable classes such that, $\bigcap_{i\in I_0} \mathcal{K}_i \neq \varnothing$, for every finite set $I_0 \subseteq I$. Show that $\bigcap_{i\in I} \mathcal{K}_i \neq \varnothing$.

Exercise 4.3. Let T be a first-order theory and $\mathfrak{A}$ a structure. Prove that $\mathfrak{A}$ can be embedded into some model of T if, and only if, every finitely generated substructure of $\mathfrak{A}$ can be embedded into some model of T.

We conclude this section with some simple applications of the Compactness Theorem. First, we show that first-order logic is not able to count.

Lemma 4.9. *Let Σ be an S-sorted signature and $s \in S$ a sort. There exists no set $\Phi \subseteq \mathrm{FO}^0[\Sigma]$ such that*

$$\mathfrak{A} \vDash \Phi \quad \text{iff} \quad |A_s| < \aleph_0\,, \qquad \textit{for all } \Sigma\textit{-structures } \mathfrak{A}\,.$$

Proof. For a contradiction, suppose that there is such a set Φ. Let

$$\psi_n := \exists x_0 \cdots \exists x_{n-1} \bigwedge_{i<k} x_i \neq x_k$$

be the sentence expressing that there are at least n elements of sort s. We claim that

$$\Gamma := \Phi \cup \{\, \psi_n \mid n < \omega \,\}$$

is satisfiable. This yields the desired contradiction.

By the Compactness Theorem, we only need to check that every finite subset of Γ is satisfiable. If $\Gamma_0 \subseteq \Gamma$ is finite then there exists a number $k < \omega$ such that

$$\Gamma_0 \subseteq \Phi \cup \{ \psi_n \mid n < k \} .$$

Choose any finite Σ-structure $\mathfrak{A}$ with $|A_s| \geq k$. Since $\mathfrak{A}$ is finite we have $\mathfrak{A} \vDash \Phi$. Furthermore, $\mathfrak{A} \vDash \psi_n$, for all $n < k$. Hence, $\mathfrak{A}$ is a model of Γ_0. □

Example. Let us show that there is no set of first-order formulae expressing that a graph is connected. Suppose that $\Phi \subseteq \mathrm{FO}[E]$ is a set of formulae such that

$$\mathfrak{G} \vDash \Phi \quad \text{iff} \quad \mathfrak{G} \text{ is a connected undirected graph} .$$

We define formulae $\varphi_n(x, y)$ saying that there exists a path of length at most n from x to y by

$$\varphi_0(x, y) := x = y$$
$$\text{and} \quad \varphi_{n+1}(x, y) := \varphi_n(x, y) \vee \exists z(Exz \wedge \varphi_n(z, y)) .$$

Let c, d be new constant symbols and set

$$\Psi := \Phi \cup \{ \neg\varphi_n(c, d) \mid n < \omega \} .$$

Then Ψ is inconsistent since any model would be a connected graph that does not contain a path from c to d. Let $\Psi_0 \subseteq \Psi$ be a finite subset. There is some number k such that

$$\Psi_0 \subseteq \Phi \cup \{ \neg\varphi_n(c, d) \mid n < k \} .$$

Let $\mathfrak{P}_k$ be the graph consisting of a single path with k edges where the endpoints are denoted by c and d.

$$c - \bullet - \cdots - \bullet - d$$

Then we have $\mathfrak{P}_k \vDash \Psi_0$. Hence, every finite subset of Ψ is satisfiable and, by the Compactness Theorem, it follows that Ψ has a model. A contradiction.

Exercise 4.4. (a) Show that the class of all undirected, acyclic graphs is first-order axiomatisable. (A graph is *acyclic* if it does not contain a path $v_0, v_1, \dots, v_{n-1}, v_n, v_0$ where all the v_i are distinct.)

(b) Show that the class of all undirected graph that are not acyclic is not first-order axiomatisable.

(c) Use (b) to prove that the class of all undirected acyclic graphs is not finitely first-order axiomatisable.

Lemma 4.10. *The Compactness Theorem fails for* $\mathrm{FO}_{\kappa\aleph_0}[\Sigma]$ *if* $\kappa > \aleph_0$.

Proof. Let $\varphi_n := \exists x_0 \cdots \exists x_{n-1} \bigwedge_{i \neq k} x_i \neq x_k$ and

$$\varphi_{\mathrm{fin}} := \bigvee \{ \neg\varphi_n \mid n < \omega \} .$$

The set $\Phi := \{\varphi_{\mathrm{fin}}\} \cup \{ \varphi_n \mid n < \omega \}$ is unsatisfiable but each of its finite subsets has a model. □

Lemma 4.11. *Let $\mathcal{K}$ be a class of Σ-structures. If both $\mathcal{K}$ and* $\mathrm{Str}[\Sigma] \setminus \mathcal{K}$ *are first-order axiomatisable then the class $\mathcal{K}$ is finitely axiomatisable.*

Proof. Let Φ^+ and Φ^- be sets such that

$$\mathcal{K} = \mathrm{Mod}(\Phi^+) \quad \text{and} \quad \mathrm{Str}[\Sigma] \setminus \mathcal{K} = \mathrm{Mod}(\Phi^-) .$$

Then $\Phi^+ \cup \Phi^-$ is inconsistent. Hence, there are finite subsets $\Phi_0^+ \subseteq \Phi^+$ and $\Phi_0^- \subseteq \Phi^-$ such that $\Phi_0^+ \cup \Phi_0^-$ is inconsistent. Setting $\varphi := \bigwedge \Phi_0^-$ it follows that $\Phi^+ \vDash \neg\varphi_-$. Hence,

$$\mathfrak{A} \vDash \neg\varphi_- , \quad \text{for all } \mathfrak{A} \in \mathcal{K} .$$

Conversely, we have

$$\mathfrak{A} \vDash \varphi_- , \quad \text{for all } \mathfrak{A} \notin \mathcal{K} .$$

Consequently, $\mathrm{Mod}(\neg\varphi_-) = \mathcal{K}$, as desired. □

Generalising the idea behind Lemma 4.9 we obtain a converse to the Downward Löwenheim-Skolem Theorem.

Theorem 4.12 (Upward Löwenheim-Skolem-Tarski Theorem).
Let $T \subseteq \mathrm{FO}^0[\Sigma]$. If there exists a sort s such that, for every $n < \aleph_0$, T has a model $\mathfrak{A}$ with $|A_s| \geq n$ then T has models $\mathfrak{A}$ where $|A_s|$ has arbitrarily large cardinality.

Proof. Suppose that T has, for every $n < \aleph_0$, a model whose domain of sort s has size at least n. Let κ be an arbitrary cardinal and fix a set $C := \{ c_\alpha \mid \alpha < \kappa \}$ of κ constant symbols of sort s such that Σ and C are disjoint. We claim that the set

$$\Phi := T \cup \{ c \neq d \mid c, d \in C\,,\ c \neq d \}$$

has a model. By the Compactness Theorem, it is sufficient to show that every finite subset $\Phi_0 \subseteq \Phi$ is satisfiable. Since Φ_0 is finite, there exists a finite set $C_0 \subseteq C$ such that

$$\Phi_0 \subseteq T \cup \{ c \neq d \mid c, d \in C_0\,,\ c \neq d \}\,.$$

By assumption, there exists a model $\mathfrak{A} \vDash \varphi$ with $|A_s| \geq |C_0|$. We can turn it into a model of Φ_0 by interpreting the constant symbols $c \in C_0$ by distinct elements of A_s. □

The next example shows that, again, the above theorem fails for $\mathrm{FO}_{\kappa\aleph_0}$ with $\kappa > \aleph_0$. (Another counterexample is given by Lemma C1.1.7.)

Example. Let $\varphi \in \mathrm{FO}$ be a sentence axiomatising the class of ordered fields. The $\mathrm{FO}_{\aleph_1\aleph_0}$-sentence

$$\psi := \varphi \wedge \forall x \bigvee_{n<\omega} x < 1 + \cdots + 1$$

axiomatises the class of all archimedian ordered fields. It follows that ψ has only models of cardinality κ with $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$.

As an immediate consequence of the Upward Löwenheim-Skolem-Tarski Theorem we obtain the result that infinite structures cannot be characterised up to isomorphism in first-order logic.

Corollary 4.13. *If $\mathfrak{A}$ is a structure with at least one infinite domain then there exists no set $\Phi \subseteq \mathrm{FO}$ such that*

$$\mathfrak{B} \vDash \Phi \quad \text{iff} \quad \mathfrak{B} \cong \mathfrak{A}\,.$$

5. *Amalgamation*

We can use the Upward Löwenheim-Skolem-Tarski Theorem to construct elementary extensions of a single structure. In this section we present a way to find a common elementary extension of several structures.

Definition 5.1. Let L be a logic.

(a) For sets $\Phi, \Delta \subseteq L$ of formulae, we define the set of all *Δ-consequences* of Φ by

$$\Phi^{\vDash}_{\Delta} := \Phi^{\vDash} \cap \Delta\,.$$

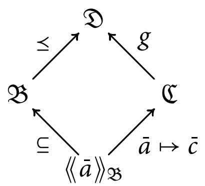
(b) Suppose that L is algebraic. For structures $\mathfrak{A}$ and $\mathfrak{B}$ and tuples $\bar{a} \subseteq A$ and $\bar{b} \subseteq B$, we write

$$\langle \mathfrak{A}, \bar{a} \rangle \leq_\Delta \langle \mathfrak{B}, \bar{b} \rangle \quad :\text{iff} \quad \mathfrak{A} \vDash \varphi(\bar{a}) \text{ implies } \mathfrak{B} \vDash \varphi(\bar{b})\,, \text{ for all } \varphi \in \Delta\,.$$

Theorem 5.2 (Amalgamation Theorem). *Let $\mathfrak{B}$ and $\mathfrak{C}$ be Σ-structures, $\Delta \subseteq \mathrm{FO}$, and $\bar{a} \subseteq B$, $\bar{c} \subseteq C$ sequences such that*

$$\langle \mathfrak{C}, \bar{c} \rangle \leq_{\exists\Delta} \langle \mathfrak{B}, \bar{a} \rangle\,.$$

There exists an elementary extensions $\mathfrak{D} \succeq \mathfrak{B}$ and a Δ-map $g : \mathfrak{C} \to \mathfrak{D}$ with $g(\bar{c}) = \bar{a}$.



Proof. By taking an isomorphic copy of $\mathfrak{C}$ we may assume that $\bar{a} = \bar{c}$ and $B \cap C = \bar{a}$. To find the desired structure $\mathfrak{D}$ we prove that

$$T := \mathrm{Th}(\mathfrak{B}_B) \cup \mathrm{Th}_\Delta(\mathfrak{C}_C)$$

is satisfiable. By the Compactness Theorem, it is sufficient to show that every finite subset $T_0 \subseteq T$ has a model. Given $T_0 \subseteq T$ set

$$\varphi(\bar{a}, \bar{d}) := \bigwedge (T_0 \cap \mathrm{Th}_\Delta(\mathfrak{C}_C))$$

where $\bar{d} \subseteq C \setminus \bar{a}$. Suppose, for a contradiction, that

$$\mathrm{Th}(\mathfrak{B}_B) \vDash \neg\exists\bar{y}\varphi(\bar{a}, \bar{y}) .$$

Then we have $\langle \mathfrak{B}, \bar{a} \rangle \vDash \neg\exists\bar{y}\varphi(\bar{a}, \bar{y})$ and, since $\exists\bar{y}\varphi \in \exists\Delta$, it follows that $\langle \mathfrak{C}, \bar{a} \rangle \vDash \neg\exists\bar{y}\varphi(\bar{a}, \bar{y})$. Consequently, we have $\mathrm{Th}_\Delta(\mathfrak{C}_C) \vDash \neg\varphi(\bar{a}, \bar{d})$. Contradiction.

Since $\mathrm{Th}(\mathfrak{B}_B)$ is complete it follows that $\mathrm{Th}(\mathfrak{B}_B) \vDash \exists\bar{y}\varphi(\bar{a}, \bar{y})$. Thus, there exists some tuple $\bar{b} \subseteq B$ such that $\mathfrak{B}_B \vDash \varphi(\bar{a}, \bar{b})$. The structure $\langle \mathfrak{B}_B, \bar{b} \rangle \vDash T_0$ is our desired model.

We have shown that there exists a model $\mathfrak{D} \vDash T$. Since $\mathfrak{D} \vDash \mathrm{Th}(\mathfrak{B}_B)$ there exists an elementary embedding $h : \mathfrak{B} \to \mathfrak{D}$ and, by taking isomorphic copies, we may assume that $\mathfrak{D} \succeq \mathfrak{B}$. We define a function $g : C \to D$ by setting $g(d) := d^{\mathfrak{D}}$, for $d \in C$. ($d^{\mathfrak{D}}$ is the value of the constant symbol d in $\mathfrak{D}$.) Since $\mathfrak{D} \vDash \mathrm{Th}_\Delta(\mathfrak{C}_C)$ it follows that $g : \mathfrak{C} \to \mathfrak{D}$ is a Δ-map. Furthermore, we have $g(\bar{c}) = \bar{c}^{\mathfrak{D}} = \bar{a}$. □

Corollary 5.3. *If $\mathfrak{A} \equiv \mathfrak{B}$ then there exists a structure $\mathfrak{C}$ such that $\mathfrak{A} \preceq \mathfrak{C}$ and $\mathfrak{B} \preceq \mathfrak{C}$.*

Let us record a special instance of the Amalgamation Theorem that will be used in the next section.

Corollary 5.4. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be Σ-structures and $\bar{a} \subseteq B$ a sequence of elements. If $f : \langle \bar{a} \rangle \to C$ is a homomorphism such that*

$$\langle \mathfrak{C}, f\bar{a} \rangle \leq_\exists \langle \mathfrak{B}, \bar{a} \rangle ,$$

then there exists an elementary extension $\mathfrak{D} \succeq \mathfrak{B}$ and an embedding $g : \mathfrak{C} \to \mathfrak{D}$ such that $g f(\bar{a}) = \bar{a}$.

Lemma 5.5. *Let $T, \Delta \subseteq \mathrm{FO}$ where Δ is closed under disjunctions. Then $\mathfrak{A} \vDash T^{\vDash}_{\Delta}$ if, and only if, there exists a model $\mathfrak{B} \vDash T$ such that $\mathfrak{B} \leq_\Delta \mathfrak{A}$.*

Proof. ($\Leftarrow$) Obviously, $\mathfrak{B} \vDash T^{\vDash}_{\Delta}$ and $\mathfrak{B} \leq_\Delta \mathfrak{A}$ implies that $\mathfrak{A} \vDash T^{\vDash}_{\Delta}$.

($\Rightarrow$) Set $\Phi := \mathrm{Th}_{\Delta^\neg}(\mathfrak{A})$ where $\Delta^\neg := \{ \neg\varphi \mid \varphi \in \Delta \}$. It is sufficient to find a model $\mathfrak{B}$ of $\Psi := \Phi \cup T$. If Ψ is unsatisfiable then there exists a finite subset $\{\varphi_0, \dots, \varphi_k\} \subseteq \Phi$ such that

$$T \vDash \neg\varphi_0 \vee \dots \vee \neg\varphi_k .$$

Suppose that $\varphi_i = \neg\psi_i$, for $\psi_i \in \Delta$. Then $T \vDash \psi_0 \vee \dots \vee \psi_k$ implies that $\psi_0 \vee \dots \vee \psi_k \in T^{\vDash}_{\Delta}$ and, hence, $\mathfrak{A} \vDash \psi_0 \vee \dots \vee \psi_k$ in contradiction to $\mathfrak{A} \vDash \varphi_i$, for all $i \leq k$. □

Corollary 5.6. *Let $T, \Delta \subseteq \mathrm{FO}$ where Δ is closed under disjunctions, and set $\Delta^\neg := \{ \neg\varphi \mid \varphi \in \Delta \}$. For every model $\mathfrak{A} \vDash T^{\vDash}_{\forall\Delta}$, there exists a model $\mathfrak{B} \vDash T$ and a $\Delta^\neg$-map $g : \mathfrak{A} \to \mathfrak{B}$.*

Proof. Suppose that $\mathfrak{A} \vDash T^{\vDash}_{\forall\Delta}$. By Lemma 5.5, we can find a model $\mathfrak{C} \vDash T$ such that $\mathfrak{A} \leq_{\exists\Delta^\neg} \mathfrak{C}$. By the Amalgamation Theorem, it follows that there exists some elementary extension $\mathfrak{B} \succeq \mathfrak{C}$ and a $\Delta^\neg$-map $g : \mathfrak{A} \to \mathfrak{B}$. □

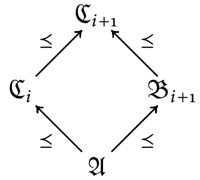
We can amalgamate several structures by iterating the Amalgamation Theorem.

Lemma 5.7. *Let $\mathfrak{B}_i$, $i < \alpha$, be a family of structures and suppose that $\mathfrak{A} \subseteq \mathfrak{B}_i$, for all $i < \alpha$, is a common substructure with universe $A = B_i \cap B_k$, for all $i \neq k$. There exists a structure $\mathfrak{C}$ such that $\mathfrak{B}_i \preceq \mathfrak{C}$, for all $i < \alpha$.*

Proof. We construct an elementary chain $(\mathfrak{C}_i)_{i<\alpha}$ such that $\mathfrak{B}_i \preceq \mathfrak{C}_i$, for $i < \alpha$. The structure $\mathfrak{C} := \bigcup_{i<\alpha} \mathfrak{C}_i$ has the desired properties since $\mathfrak{B}_i \preceq \mathfrak{C}_i \preceq \mathfrak{C}$.

We define $\mathfrak{C}_i$ by induction on i. We start with $\mathfrak{C}_0 := \mathfrak{B}_0$ and, for limit ordinals δ, we set $\mathfrak{C}_\delta := \bigcup_{i<\delta} \mathfrak{C}_i$. For the successor step, we can apply the

Amalgamation Theorem to obtain a common elementary extension $\mathfrak{C}_{i+1}$ of $\mathfrak{C}_i$ and $\mathfrak{B}_{i+1}$.



□

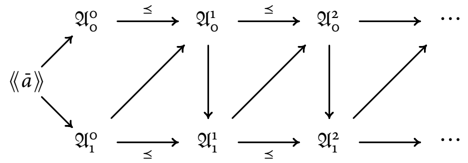
We conclude this section with an amalgamation theorem for expansions instead of extensions. We also record two applications.

Theorem 5.8. *Let Γ_0 and Γ_1 be signatures and set $\Sigma := \Gamma_0 \cap \Gamma_1$. Suppose that $\mathfrak{A}_i$ is a Γ_i-structure, for $i < 2$, and let $\bar{a} \subseteq A_0 \cap A_1$ be a sequence such that*

$$\langle \mathfrak{A}_0|_\Sigma, \bar{a} \rangle \equiv \langle \mathfrak{A}_1|_\Sigma, \bar{a} \rangle .$$

Then there exists a $(\Gamma_0 \cup \Gamma_1)$-structure $\mathfrak{B}$ with $\mathfrak{A}_0 \preceq \mathfrak{B}|_{\Gamma_0}$ and an elementary embedding $g : \mathfrak{A}_1 \to \mathfrak{B}|_{\Gamma_1}$ with $g(\bar{a}) = \bar{a}$.

Proof. We construct structures $\mathfrak{A}_i^n$ for $i < 2$ and $n < \omega$ as follows. We start with $\mathfrak{A}_i^0 := \mathfrak{A}_i$. If $\mathfrak{A}_0^n$ and $\mathfrak{A}_1^n$ are already defined then we apply the Amalgamation Theorem twice. First, we use it to obtain an elementary extension $\mathfrak{A}_0^{n+1} \succeq \mathfrak{A}_0^n$ and an elementary embedding $\mathfrak{A}_1^n|_\Sigma \to \mathfrak{A}_0^{n+1}|_\Sigma$. Then we construct an elementary extension $\mathfrak{A}_1^{n+1} \succeq \mathfrak{A}_1^n$ and an elementary embedding $\mathfrak{A}_0^{n+1}|_\Sigma \to \mathfrak{A}_1^{n+1}|_\Sigma$.



Let $\mathfrak{B}_i := \bigcup_n \mathfrak{A}_i^n$. The elementary embeddings induce an isomorphism $h : \mathfrak{B}_0|_\Sigma \to \mathfrak{B}_1|_\Sigma$. We use h to expand the Γ_0-structure $\mathfrak{B}_0$ to a $(\Gamma_0 \cup \Gamma_1)$-structure $\mathfrak{B}$ by setting

$$\xi^{\mathfrak{B}} := h^{-1}[\xi^{\mathfrak{B}_1}], \qquad \text{for } \xi \in \Gamma_1 \setminus \Sigma .$$

Since $\mathfrak{B}|_{\Gamma_0} \succeq \mathfrak{A}_0$ the claim follows. □

Corollary 5.9. *Let $T \subseteq \mathrm{FO}^0[\Sigma]$ and $\mathfrak{A}$ a Σ_0-structure where $\Sigma_0 \subseteq \Sigma$. We have $\mathfrak{A} \vDash T^{\vDash} \cap \mathrm{FO}^0[\Sigma_0]$ if and only if $\mathfrak{A} \leq \mathfrak{B}|_{\Sigma_0}$ for some model $\mathfrak{B}$ of T.*

Proof. $(\Leftarrow)$ is trivial. For $(\Rightarrow)$, we set $\Delta := \mathrm{FO}^0[\Sigma_0]$ and we assume that $\mathfrak{A} \vDash T_\Delta^{\vDash}$. We can use Lemma 5.5 to find a model $\mathfrak{C} \vDash T$ such that $\mathfrak{C} \leq_\Delta \mathfrak{A}$. By choice of Δ this implies that $\mathfrak{C}|_{\Sigma_0} \equiv \mathfrak{A}|_{\Sigma_0}$. Applying Theorem 5.8 we obtain an elementary extension $\mathfrak{B} \succeq \mathfrak{C}$ and the desired elementary map $g : \mathfrak{A} \to \mathfrak{B}|_{\Sigma_0}$. □

The second application is the Interpolation Theorem of Craig. We will prove a much more general version in Section C5.5.

Theorem 5.10 (Craig)**.** *Let Γ_0 and Γ_1 be signatures and set $\Sigma := \Gamma_0 \cap \Gamma_1$. Suppose that $\varphi_0 \vDash \varphi_1$ where $\varphi_0 \in \mathrm{FO}^0[\Gamma_0]$ and $\varphi_1 \in \mathrm{FO}^0[\Gamma_1]$. Then there exists a formula $\psi \in \mathrm{FO}^0[\Sigma]$ such that*

$$\varphi_0 \vDash \psi \quad \textit{and} \quad \psi \vDash \varphi_1 .$$

Proof. If φ_0 is inconsistent, we can set $\psi :=$ false. Hence, suppose that φ_0 has a model $\mathfrak{A}_0$ and set $\Psi := \mathrm{Th}(\mathfrak{A}_0|_\Sigma)$.

If $\Psi \vDash \varphi_1$, then we can use the Compactness Theorem to find a finite subset $\Psi_0 \subseteq \Psi$ with $\Psi_0 \vDash \varphi_1$. Hence, $\psi := \bigwedge \Psi_0$ is the desired formula.

Suppose that $\Psi \nvDash \varphi_1$. Then $\Psi \cup \{\neg\varphi_1\}$ has a model $\mathfrak{A}_1$. Since

$$\mathrm{Th}(\mathfrak{A}_1|_\Sigma) = \Psi = \mathrm{Th}(\mathfrak{A}_0|_\Sigma) ,$$

we can use Theorem 5.8, to find a $(\Gamma_0 \cup \Gamma_1)$-structure $\mathfrak{B}$ with

$$\mathrm{Th}(\mathfrak{B}|_{\Gamma_0}) = \mathrm{Th}(\mathfrak{A}_0) \quad \text{and} \quad \mathrm{Th}(\mathfrak{B}|_{\Gamma_1}) = \mathrm{Th}(\mathfrak{A}_1) .$$

In particular, we have $\mathfrak{B} \vDash \varphi_0$ and $\mathfrak{B} \vDash \neg\varphi_1$. Consequently, $\varphi_0 \nvDash \varphi_1$. A contradiction. □

C3. *Types and type spaces*

1. *Types*

In the same way that we can classify structures by their theory, we can distinguish elements of a structure by the formulae they satisfy. Such theories of elements are called *types*.

Definition 1.1. Let L be a logic.

(a) A (*partial*) L-*type* is a satisfiable set of L-formulae.

(b) An L-type $\mathfrak{p}$ is *complete* if it is a complete L-theory.

(c) We denote by $S(L)$ the set of all complete L-types.

(d) For $\Phi \subseteq L$, we define the set

$$\langle \Phi \rangle_L := \{ \mathfrak{p} \in S(L) \mid \Phi \subseteq \mathfrak{p} \}$$

of all types containing Φ. Usually we will omit the index L and just write $\langle \Phi \rangle$. Furthermore, for single formulae φ we write $\langle \varphi \rangle$ instead of $\langle \{ \varphi \} \rangle$.

Example. For boolean logic $\mathrm{BL}(\mathfrak{B})$ introduced in Section C1.1, interpretations are ultrafilters and the theory of an ultrafilter $\mathfrak{u}$ is $\mathfrak{u}$ itself. Hence,

$$\begin{aligned} S(\mathrm{BL}(\mathfrak{B})) &= \big\{ \mathrm{Th}(\mathfrak{u}) \bigm| \mathfrak{u} \in \mathrm{spec}(\mathfrak{B}) \big\} \\ &= \{ \mathfrak{u} \mid \mathfrak{u} \in \mathrm{spec}(\mathfrak{B}) \} = \mathrm{spec}(\mathfrak{B}) . \end{aligned}$$

Definition 1.2. Let L be an algebraic logic and $\bar{s}$ a sequence of sorts.

(a) Let $\mathfrak{M}$ be a Σ-structure. The *L-type* of a tuple $\bar{a} \in M^{\bar{s}}$ is the set

$$\operatorname{tp}_L(\bar{a}/\mathfrak{M}) := \{ \varphi(\bar{x}) \in L^{\bar{s}}[\Sigma] \mid \mathfrak{M} \vDash \varphi(\bar{a}) \}.$$

If the structure $\mathfrak{M}$ is known from the context we will omit it and simply write $\operatorname{tp}_L(\bar{a})$. Similarly, we omit the index L in case $L = \mathrm{FO}$.

(b) Let $T \subseteq L^{\circ}[\Sigma]$ be an L-theory. An *$\bar{s}$-type of T* is an L-type $\mathfrak{p} \subseteq L^{\bar{s}}[\Sigma]$ such that $\mathfrak{p} \cup T$ is consistent. The set of all complete $\bar{s}$-types of T is

$$S^{\bar{s}}_L(T) := \{ \mathfrak{p} \in S(L^{\bar{s}}[\Sigma]) \mid T \subseteq \mathfrak{p} \}.$$

An *α-type of T*, for an ordinal α, is an $\bar{s}$-type of T where $|\bar{s}| = \alpha$. The set of all complete α-types is

$$S^{\alpha}_L(T) := \bigcup \{ S^{\bar{s}}_L(T) \mid |\bar{s}| = \alpha \}.$$

(c) We also need types with parameters. If $\mathfrak{M}$ is a model of T and $U \subseteq M$ then we say that a type of $T(U)$ is a type *over U*. In particular, the set $\operatorname{tp}_L(\bar{a}/U) := \operatorname{tp}_L(\bar{a}/\mathfrak{M}_U)$ is the L-type of $\bar{a}$ over U. We set

$$S^{\bar{s}}_L(U) := S^{\bar{s}}_L(T(U)).$$

To simplify notation, we define $S^{<\omega}_L(U) := \bigcup_{n<\omega} S^n_L(U)$. Again, we usually omit the index if $L = \mathrm{FO}$.

(d) An $\bar{s}$-type $\mathfrak{p}$ over U is *realised* in $\mathfrak{M}$ if there is some tuple $\bar{a} \in M^{\bar{s}}$ such that $\mathfrak{p} \subseteq \operatorname{tp}_L(\bar{a}/U)$. Otherwise, we say that $\mathfrak{M}$ *omits* $\mathfrak{p}$.

Example. Let $\mathfrak{N} = \langle \omega, s, 0 \rangle$ where $s(n) := n + 1$ is the successor function. We have

$$S^1(\varnothing) = \{ \mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_\infty \}$$

where, for $n < \omega$,

$$\mathfrak{p}_n := \operatorname{tp}(n) \vDash x_0 = s^n(0),$$

and $\mathfrak{p}_\infty \vDash x_0 \neq s^n(0)$, for all n. Hence, $\mathfrak{p}_\infty$ is not realised in $\mathfrak{N}$.

Example. Consider $\langle \mathbb{Q}, < \rangle$. The elements of $S^1(\mathbb{Q})$ correspond to the set of cuts. For every cut $\langle A, B \rangle$ of $\mathbb{Q}$, i.e., every partition $A \cup B = \mathbb{Q}$ such that A is an initial segment and B is a final one, there exists a non-realised type $\mathfrak{p}$ such that

$$\begin{aligned} &\mathfrak{p} \vDash x > a &&\text{for all } a \in A, \\ \text{and}\quad &\mathfrak{p} \vDash x < b &&\text{for all } b \in B. \end{aligned}$$

It follows that $|S^1(\mathbb{Q})| = 2^{\aleph_0}$. Depending on whether A has a maximal element or B has a minimal one, we obtain the following classification.

(*realised*) For each $a \in \mathbb{Q}$, we have a type $\mathfrak{p} \vDash x = a$, i.e., $\mathfrak{p} = \operatorname{tp}(a/\mathbb{Q})$.

(a^+) For each $a \in \mathbb{Q}$, there exists a type $\mathfrak{p}$ of an element 'immediately above a'. That is,

$$\begin{aligned} &\mathfrak{p} \vDash x > b &&\text{for all } b \leq a, \\ \text{and}\quad &\mathfrak{p} \vDash x < b &&\text{for all } b > a. \end{aligned}$$

(a^-) Similarly, for each $a \in \mathbb{Q}$, we have the type of an element 'immediately below a'.

($+\infty$) We have one type $\mathfrak{p}$ of an infinite positive element. That is,

$$\mathfrak{p} \vDash x > a \qquad \text{for all } a \in \mathbb{Q}.$$

($-\infty$) Similarly, there is the type of an infinite negative element.

(*irrational*) Finally, for each cut $\langle A, B \rangle$ such that A has no maximal element and B has no minimal one, there is one type $\mathfrak{p}$ such that

$$\begin{aligned} &\mathfrak{p} \vDash x > a &&\text{for all } a \in A, \\ \text{and}\quad &\mathfrak{p} \vDash x < b &&\text{for all } b \in B. \end{aligned}$$

Exercise 1.1. Let $T := \operatorname{Th}(\mathfrak{Z})$ where $\mathfrak{Z} := \langle \mathbb{Z}, s \rangle$ and $s : x \mapsto x + 1$ is the successor function. Determine $S^n(T)$, for every $n < \omega$. In particular, compute $|S^n(T)|$. *Hint.* Note that, modulo T, every formula is equivalent to a quantifier-free one.

The set of types of $L|_\Phi$ and L/Φ can be computed as follows.

Lemma 1.3. *Let L be a logic and $\Phi \subseteq L$.*

(a) $S(L/\Phi) = \langle \Phi \rangle_L \subseteq S(L)$.

(b) $S(L|_\Phi) = \{ \mathfrak{p} \cap \Phi \mid \mathfrak{p} \in S(L) \}$.

Proof. (a) We have

$$\begin{aligned} \mathfrak{p} \in S(L/\Phi) \quad &\text{iff} \quad \mathfrak{p} = \mathrm{Th}_{L/\Phi}(\mathfrak{J}) \text{ for some } \mathfrak{J} \in \mathrm{Mod}_L(\Phi) \\ &\text{iff} \quad \mathfrak{p} = \mathrm{Th}_L(\mathfrak{J}) \text{ for some } \mathfrak{J} \vDash \Phi \\ &\text{iff} \quad \mathfrak{p} \in S(L) \text{ and } \Phi \subseteq \mathfrak{p}. \end{aligned}$$

(b) We have

$$\begin{aligned} S(L|_\Phi) &= \{ \mathrm{Th}_{L|_\Phi}(\mathfrak{J}) \mid \mathfrak{J} \text{ an } L\text{-interpretation} \} \\ &= \{ \mathrm{Th}_L(\mathfrak{J}) \cap \Phi \mid \mathfrak{J} \text{ an } L\text{-interpretation} \} \\ &= \{ \mathfrak{p} \cap \Phi \mid \mathfrak{p} \in S(L) \}. \end{aligned}$$ □

The relationship between a logic L and its set of types $S(L)$ is similar to that between a boolean algebra $\mathfrak{B}$ and its spectrum $\mathrm{spec}(\mathfrak{B})$. In fact, if L is boolean closed there exists an embedding $S(L) \to \mathrm{spec}(\mathfrak{Lb}(L))$.

Lemma 1.4. *Let L be a logic that is closed under disjunction and conjunction and that contains an unsatisfiable formula.*

(a) *If $\Phi \subseteq L$ then $\Phi^\vDash$ is a filter of $\langle L, \vDash \rangle$ and $\Phi^\vDash/\equiv$ is a filter of $\mathfrak{Lb}(L)$.*

(b) *Every complete L-theory T is an ultrafilter of $\langle L, \vDash \rangle$.*

Proof. Since (a) is obvious, we only need to prove (b). By (a), we know that $T = T^\vDash$ is a filter. Since there is an unsatisfiable formula, this filter is proper.

To prove that T is an ultrafilter consider a disjunction $\varphi \vee \psi \in T$. Since T is complete there exists an interpretation $\mathfrak{J}$ with $\mathrm{Th}_L(\mathfrak{J}) = T$. Hence, $\mathfrak{J} \vDash \varphi \vee \psi$ implies that $\mathfrak{J} \vDash \varphi$ or $\mathfrak{J} \vDash \psi$. In the former case we have $\varphi \in T$ and, otherwise, we have $\psi \in T$. □

Remark. If $\mathfrak{u}$ is a proper filter of $\mathfrak{Lb}(L)$ then the finite intersection property implies that every finite subset of $\mathfrak{u}$ is satisfiable.

In general the converse of statement (b) is not true, but there are some logics where every ultrafilter is a type. We have already seen in Section 1 that this is the case for boolean logic. A more important example of this phenomenon is first-order logic.

Lemma 1.5. *Every ultrafilter $\mathfrak{u}$ of $\langle \mathrm{FO}[\Sigma, X], \vDash \rangle$ is a complete type.*

Proof. If $\mathfrak{u}$ is an ultrafilter, it has the finite intersection property. Hence, every finite subset $\Phi \subseteq \mathfrak{u}$ is satisfiable. By the Compactness Theorem it follows that $\mathfrak{u}$ is satisfiable. Consequently, $\mathfrak{u}$ is a type. Since FO is boolean closed we can use Theorem B2.4.11 and Lemma C1.3.4 (d) to show that $\mathfrak{u}$ is complete. □

Corollary 1.6. *We have*

$$S(\mathrm{FO}[\Sigma, X]) = \mathrm{spec}(\langle \mathrm{FO}[\Sigma, X], \vDash \rangle).$$

Remark. In the next section we will see that the Stone topology on the spectrum induces a topology on the type space where the closed sets are precisely those of the form $\langle \Phi \rangle$, for $\Phi \subseteq \mathrm{FO}$. The name 'Compactness Theorem' stems from the fact that this theorem implies that the topology obtained in this way is compact.

For logics where the Compactness Theorem fails, there are ultrafilters that do not correspond to types. In fact, the Compactness Theorem is equivalent to the statement of Lemma 1.5.

Example. There are ultrafilters of $\mathfrak{Lb}(\mathrm{FO}_{\aleph_1 \aleph_0}[\Sigma])$ which are not types. Let $\psi := \bigwedge_{n<\omega} \varphi_n$ where φ_n is the formula stating that there are at least n elements. The formula $\neg\psi \wedge \varphi_n$ is satisfiable, for every n. Hence, the set $\{\neg\psi\} \cup \{ \varphi_n \mid n < \omega \}$ has the finite intersection property and there exists an ultrafilter

$$\mathfrak{u} \supseteq \{\neg\psi\} \cup \{ \varphi_n \mid n < \omega \}.$$

This ultrafilter is not a type since it is not satisfiable.

This example shows that, for $\kappa > \aleph_0$, the inclusion $S(\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]) \subset \operatorname{spec}(\langle \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X], \vDash\rangle)$ is proper. We can describe the subset of the spectrum corresponding to $S(\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X])$ as follows. Using Chang's reduction we can find a signature $\Sigma_+ \supseteq \Sigma$ and a first-order theory $T \subseteq \mathrm{FO}[\Sigma_+, X]$ such that

$$\operatorname{spec}(\langle \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X], \vDash\rangle) \cong S(T) .$$

Then we can characterise $S(\mathrm{FO}_{\kappa\aleph_0})$ as a subset of $S(T)$ by describing the types in $S(T) \smallsetminus S(\mathrm{FO}_{\kappa\aleph_0})$.

Lemma 1.7. *Let $\varphi \in \mathrm{FO}_{\kappa^+\aleph_0}[\Sigma, X]$ and $|\Sigma| \leq \kappa$. There exists a signature $\Sigma_+ \supseteq \Sigma$ and set C of (partial) $\mathrm{FO}[\Sigma_+, X]$-types such that*

$$\mathfrak{M} \vDash \varphi \quad \text{iff} \quad \text{there is some } \Sigma_+\text{-expansion of } \mathfrak{M} \text{ that omits every type in } C.$$

Furthermore, we can choose Σ_+ and C of size at most κ.

Proof. By Lemma c1.4.12, there exists an $\mathrm{FO}_{\kappa\aleph_0}$-theory T_φ such that $\mathfrak{M} \vDash \varphi$ if, and only if, some expansion of $\mathfrak{M}$ satisfies T_φ. We define a set C of types such that $\mathfrak{M}^+ \vDash T_\varphi$ iff $\mathfrak{M}^+$ omits all types in C.

For every first-order formula $\vartheta \in T_\varphi$, we define the type

$$\mathfrak{p}_\vartheta := \{\neg\vartheta\} .$$

Every formula $\vartheta \in T_\varphi \smallsetminus \mathrm{FO}$ is of the form $\vartheta = \forall\bar{x} \bigvee_{i<\lambda} \psi_i$. For these formulae, we set

$$\mathfrak{p}_\vartheta := \{ \neg\psi_i \mid i < \lambda \} .$$

By construction, a structure satisfies $\vartheta \in T_\varphi$ if, and only if, it omits $\mathfrak{p}_\vartheta$. Consequently, we can set $C := \{ \mathfrak{p}_\vartheta \mid \vartheta \in T_\varphi \}$. □

2. Type spaces

In this section we investigate the analogy between type spaces and spectra. We start by defining a topology on the set of type $S(L)$ that is analogous to the Stone topology of a spectrum.

Definition 2.1. The *type space* of a logic L is the topological space $\mathfrak{S}(L)$ with universe $S(L)$ where the basic closed sets are of the form

$$\langle\varphi_0\rangle_L \cup \dots \cup \langle\varphi_{n-1}\rangle_L , \quad \text{for } n < \omega \text{ and } \varphi_0, \dots, \varphi_{n-1} \in L .$$

If L is closed under disjunctions, the closed sets can be written in the simpler form $\langle\Phi\rangle_L$, for $\Phi \subseteq L$.

Lemma 2.2. *If L is closed under disjunctions, every nonempty closed set of $\mathfrak{S}(L)$ is of the form $\langle\Phi\rangle_L$, for $\Phi \subseteq L$.*

Proof. Let $\mathcal{C} := \{\varnothing\} \cup \{ \langle\Phi\rangle_L \mid \Phi \subseteq L \}$. Since $\langle\Phi\rangle_L = \bigcap_{\varphi\in\Phi}\langle\varphi\rangle_L$, every set of $\mathcal{C}$ is closed in $\mathfrak{S}(L)$. To prove the converse, it is sufficient to show that $\mathcal{C}$ forms a topology. Since $\bigcap_i\langle\Phi_i\rangle_L = \langle\bigcup_i \Phi_i\rangle_L$, $\mathcal{C}$ is closed under arbitrary intersections. Furthermore, note that $\varnothing \in \mathcal{C}$ and $S(L) = \langle\varnothing\rangle_L \in \mathcal{C}$.

Hence, it remains to show that $\mathcal{C}$ is closed under finite unions. We claim that

$$\langle\Phi\rangle_L \cup \langle\Psi\rangle_L = \langle\{ \varphi \vee \psi \mid \varphi \in \Phi,\ \psi \in \Psi \}\rangle_L .$$

For the non-trivial inclusion, let $\mathfrak{p} \in \langle\{ \varphi \vee \psi \mid \varphi \in \Phi,\ \psi \in \Psi \}\rangle_L$. We have to show that $\mathfrak{p} \in \langle\Phi\rangle_L \cup \langle\Psi\rangle_L$. If $\mathfrak{p} \in \langle\Psi\rangle_L$, we are done. Hence, suppose there is some formula $\psi \in \Psi \smallsetminus \mathfrak{p}$. For every $\varphi \in \Phi$, we have $\varphi \vee \psi \in \mathfrak{p}$. Since $\psi \notin \mathfrak{p}$ it follows as in the proof of Lemma 1.4 that $\varphi \in \mathfrak{p}$. Therefore, $\Phi \subseteq \mathfrak{p}$. □

For first-order logic, we have seen in Corollary 1.6 that types and ultrafilters coincide. Since the definitions of the respective topologies are also the same, it follows that the type space of a first-order logic is just its spectrum.

Theorem 2.3. *We have*

$$\mathfrak{S}(\mathrm{FO}[\Sigma, X]) = \mathrm{spec}(\langle \mathrm{FO}[\Sigma, X], \vDash \rangle) .$$

In particular, the type space $\mathfrak{S}(\mathrm{FO}[\Sigma, X])$ is a Stone space.

Proof. By Corollary 1.6 both spaces have the same universe and, according to Lemma 2.2, the closed sets are also the same. □

Example. Let $T := \mathrm{Th}(\mathfrak{C})$ where $\mathfrak{C} := \langle 2^\omega, (P_n)_{n<\omega} \rangle$ and

$$P_n := \{ \alpha \in 2^\omega \mid \alpha(n) = 1 \} .$$

Then $S^1(T) = \{ \mathfrak{p}_\alpha \mid \alpha \in 2^\omega \}$ where

$$\mathfrak{p}_\alpha \vDash P_n x \quad \text{for } n \in \alpha^{-1}(1) ,$$
$$\mathfrak{p}_\alpha \vDash \neg P_n x \quad \text{for } n \in \alpha^{-1}(0) .$$

The basic closed sets of $\mathfrak{S}^1(T)$ are of the form

$$\langle P_{i_0} x \wedge \dots \wedge P_{i_k} x \wedge \neg P_{j_0} x \wedge \dots \wedge \neg P_{j_m} x \rangle .$$

Since these sets are clopen it follows that the open sets are of the form

$$O_W := \{ \mathfrak{p}_\alpha \mid \text{there is some } w < \alpha \text{ with } w \in W \}$$

with $W \subseteq 2^{<\omega}$. Consequently, the type space $\mathfrak{S}^1(T)$ is homeomorphic to the Cantor discontinuum.

For logics different from first-order logic, the type spaces usually are not Stone spaces.

Definition 2.4. A topological space $\mathfrak{X}$ is a T_0-*space* if, for every pair $x, y \in X$ of distinct points, there exists a closed set C such that

$$x \in C \text{ and } y \in X \setminus C , \quad \text{or} \quad x \in X \setminus C \text{ and } y \in C .$$

Lemma 2.5. *Let L be a logic. The type space $\mathfrak{S}(L)$ is a T_0-space.*

Proof. If $\mathfrak{p}, \mathfrak{q} \in S(L)$ are distinct types, there exists a formula φ such that $\varphi \in \mathfrak{p} \setminus \mathfrak{q}$ or $\varphi \in \mathfrak{q} \setminus \mathfrak{p}$. Consequently, $\mathfrak{p} \in \langle \varphi \rangle$ and $\mathfrak{q} \in S(L) \setminus \langle \varphi \rangle$, or $\mathfrak{p} \in S(L) \setminus \langle \varphi \rangle$ and $\mathfrak{q} \in \langle \varphi \rangle$. □

As an application of the Stone topology of the type space, consider the question of whether a first-order theory T has a model that realises all types in a given set X but no other ones. This is not possible for every set of types. The next lemma provides a first, topological condition X has to satisfy.

Lemma 2.6. *Let T be a complete first-order theory, $\mathfrak{M}$ a model of T, $U \subseteq M$, $\bar{s}$ a sequence of sorts, and let X be the set of all $\bar{s}$-types over U that are realised in $\mathfrak{M}$. Then X is dense in $\mathfrak{S}^{\bar{s}}(U)$.*

Proof. For a contradiction, suppose that there exists a type $\mathfrak{p} \in S^{\bar{s}}(U)$ with $\mathfrak{p} \notin \mathrm{cl}(X)$. Then we can find some formula $\varphi(\bar{x})$ over U with $\mathfrak{p} \in \langle \varphi \rangle$ and $\langle \varphi \rangle \cap X = \varnothing$. It follows that $\mathfrak{M} \vDash \neg\varphi(\bar{a})$, for all $\bar{a} \in M^{\bar{s}}$. Hence, $\mathfrak{M} \vDash \forall \bar{x} \neg\varphi(\bar{x})$ which implies that $\forall \bar{x} \neg\varphi(\bar{x}) \in T \subseteq \mathfrak{p}$. Consequently, $\varphi(\bar{x}) \wedge \forall \bar{x} \neg\varphi(\bar{x}) \in \mathfrak{p}$ and $\mathfrak{p}$ is inconsistent. Contradiction. □

Example. Let $\mathfrak{N} := \langle \omega, s, 0 \rangle$ where $s : n \mapsto n + 1$ is the successor function. We have seen on page 528 that the types of $\mathrm{Th}(\mathfrak{N})$ are $\mathfrak{p}_n := \mathrm{tp}(n)$, for $n < \omega$, and the type $\mathfrak{p}_\infty$ of an infinite element. The set of realised types is $X := \{ \mathfrak{p}_n \mid n < \omega \}$, while $\mathfrak{p}_\infty$ is not realised. Note that a set $C \subseteq S(\varnothing)$ with $\mathfrak{p}_\infty \notin C$ is closed if, and only if, it is finite. Hence, $\mathfrak{p}_\infty$ is an accumulation point of X and X is dense in $\mathfrak{S}^1(\varnothing)$.

For most logics, the type space is not a spectrum. But, for a boolean closed logic L, we can at least prove the existence of an embedding $\mathfrak{S}(L) \to \mathrm{spec}(\mathfrak{Lb}(L))$. It turns out that this map is a homeomorphism if, and only if, the type space is compact.

Lemma 2.7. *Let L be a boolean closed logic. The type space $\mathfrak{S}(L)$ is compact if, and only if, every ultrafilter of $\langle L, \vDash \rangle$ is a complete type.*

Proof. ($\Leftarrow$) If every ultrafilter is a type, then $S(L) = \mathrm{spec}(\langle L, \vDash \rangle)$. Since the topologies of both spaces also coincide, they are homeomorphic.

Consequently, the compactness of $\mathrm{spec}(\langle L, \vDash\rangle)$ implies the compactness of $\mathfrak{S}(L)$.

($\Rightarrow$) Let $\mathfrak{u}$ be an ultrafilter of $\langle L, \vDash\rangle$. First, we show that $\mathfrak{u}$ is satisfiable. For a contradiction, suppose otherwise. Then

$$\varnothing = \langle \mathfrak{u}\rangle_L = \bigcap_{\varphi\in\mathfrak{u}} \langle\varphi\rangle_L .$$

Since $\mathfrak{S}(L)$ is compact, there is a finite subset $\Phi_0 \subseteq \mathfrak{u}$ such that

$$\varnothing = \bigcap_{\varphi\in\Phi_0} \langle\varphi\rangle_L .$$

Hence, $\bigwedge \Phi_0 \equiv \bot$ and $\Phi_0 \subseteq \mathfrak{u}$ implies $\bot \in \mathfrak{u}$. A contradiction.

Consequently, there is some model $\mathfrak{J} \vDash \mathfrak{u}$. Since L is boolean closed, it follows that $\mathrm{Th}_L(\mathfrak{J}) = \mathfrak{u}$. Therefore, $\mathfrak{u}$ is a complete type. ☐

Lemma 2.8. *Let L be a boolean closed logic.*

(a) *The function*

$$h : \mathfrak{S}(L) \to \mathrm{spec}(\mathfrak{Lb}(L)) : \mathfrak{p} \mapsto \mathfrak{p}/{\equiv}$$

is continuous and injective.

(b) *h is a homeomorphism if, and only if, $\mathfrak{S}(L)$ is compact.*

Proof. (a) First, note that, according to Lemma 1.4, for every $\mathfrak{p} \in S(L)$, $h(\mathfrak{p}) = \mathfrak{p}/{\equiv}$ is indeed an ultrafilter of $\mathfrak{Lb}(L)$.

For injectivity, consider types $\mathfrak{p} \neq \mathfrak{q}$. By symmetry, we may assume that there is some formula $\varphi \in \mathfrak{p} \setminus \mathfrak{q}$. If $h(\mathfrak{p}) = h(\mathfrak{q})$ then

$$[\varphi]_\equiv \in \mathfrak{p}/{\equiv} = h(\mathfrak{p}) = h(\mathfrak{q}) = \mathfrak{q}/{\equiv}$$

would imply that $\varphi \in \mathfrak{q}$. A contradiction.

To show that h is continuous, let $\Phi \subseteq \mathfrak{Lb}(L)$. Then

$$\begin{aligned} h^{-1}[\langle\Phi\rangle_{\mathfrak{Lb}(L)}] &= \big\{ \mathfrak{p} \in S(L) \;\big|\; \Phi \subseteq \mathfrak{p}/{\equiv} \big\} \\ &= \big\{ \mathfrak{p} \in S(L) \;\big|\; \bigcup\Phi \subseteq \mathfrak{p} \big\} = \langle \bigcup\Phi\rangle_L \end{aligned}$$

is closed.

(b) ($\Rightarrow$) If h is a homeomorphism, then $\mathfrak{S}(L) \cong \mathrm{spec}(\mathfrak{Lb}(L))$ is a Stone space and, hence, compact.

($\Leftarrow$) By (a), it remains to show that h is closed and surjective. For surjectivity, fix an ultrafilter $\mathfrak{u} \in \mathrm{spec}(\mathfrak{Lb}(L))$. Then $\bigcup\mathfrak{u}$ is an ultrafilter of $\langle L, \vDash\rangle$. Hence, Lemma 2.7 implies that $\bigcup\mathfrak{u} \in S(L)$. Consequently,

$$h(\bigcup\mathfrak{u}) = (\bigcup\mathfrak{u})/{\equiv} = \mathfrak{u},$$

as desired.

It remains to prove that h is closed. By Lemma b5.2.3, it is sufficient to show that $h[\langle\Phi\rangle_L]$ is closed, for every $\Phi \subseteq L$. For $\Phi \subseteq L$, it follows that

$$\begin{aligned} h[\langle\Phi\rangle_L] &= \{ \mathfrak{p}/{\equiv} \mid \mathfrak{p} \in S(L),\ \Phi \subseteq \mathfrak{p} \} \\ &= \big\{ \mathfrak{p}/{\equiv} \;\big|\; \mathfrak{p} \in S(L),\ \Phi/{\equiv} \subseteq \mathfrak{p}/{\equiv} \big\} \\ &= \langle \Phi/{\equiv}\rangle_{\mathfrak{Lb}(L)} \cap \mathrm{rng}\, h \\ &= \langle \Phi/{\equiv}\rangle_{\mathfrak{Lb}(L)} \end{aligned}$$

is closed. ☐

Corollary 2.9. *Let L be a boolean closed logic. The following conditions are equivalent:*

(1) *$\mathfrak{S}(L)$ is compact.*

(2) *$\mathfrak{S}(L) \cong \mathrm{spec}(\mathfrak{Lb}(L))$.*

(3) *Every ultrafilter of $\langle L, \vDash\rangle$ is a complete type.*

Many results of Section b5.6 on spectra generalise to type spaces. In particular, the type space operation $L \mapsto \mathfrak{S}(L)$ is a functor form the category of logics to the category of topological spaces.

Definition 2.10. Let $\mu := \langle\alpha, \beta\rangle : L_0 \to L_1$ be a morphism of logics. We define a function $\mathfrak{S}(\mu)$ by setting

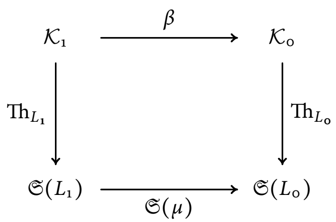
$$\mathfrak{S}(\mu)(\mathfrak{p}) := \alpha^{-1}[\mathfrak{p}], \quad \text{for } \mathfrak{p} \in S(L_1).$$

Example. For the inclusion morphism $i : L|_\Phi \to L$ and the localisation morphism $\lambda : L \to L/\Phi$ from Lemma C1.6.14, we obtain

$$\mathfrak{S}(i)(\mathfrak{p}) = \mathfrak{p} \cap \Phi \quad \text{and} \quad \mathfrak{S}(\lambda)(\mathfrak{p}) = \mathfrak{p}\,.$$

Proposition 2.11. *Let $\mu := \langle \alpha, \beta \rangle : \langle L_0, \mathcal{K}_0, \vDash \rangle \to \langle L_1, \mathcal{K}_1, \vDash \rangle$ be a morphism of logics.*

(a) *$\mathfrak{S}(\mu)$ is the unique function that makes the following diagram commute:*



(b) *$\mathfrak{S}(\mu) : \mathfrak{S}(L_1) \to \mathfrak{S}(L_0)$ is continuous.*

(c) *If μ is an embedding then $\mathfrak{S}(\mu)$ is surjective.*

(d) *If α is surjective then $\mathfrak{S}(\mu)$ is injective.*

(e) *If α is surjective and $\operatorname{rng}\beta = \mathrm{Mod}_{L_0}(\Phi)$, for some $\Phi \subseteq L_0$, then $\mathfrak{S}(\mu)$ is closed and injective.*

(f) *If $\mathfrak{S}(\mu)$ is surjective, then $\mathfrak{Lb}(\mu) : \mathfrak{Lb}(L_0) \to \mathfrak{Lb}(L_1)$ is injective.*

Proof. (a) We have seen in Lemma C1.5.12 (c) that $\mathfrak{S}(\mu) \circ \mathrm{Th}_{L_1} = \mathrm{Th}_{L_0} \circ \beta$. In particular, $\operatorname{rng}\mathfrak{S}(\mu) \subseteq \operatorname{rng}\mathrm{Th}_{L_0} = S(L_0)$ and the above diagram commutes. For uniqueness, note that $\mathrm{Th}_{L_1} : \mathcal{K}_1 \to S(L_1)$ is surjective. Hence, for every function f making the above diagram commute,

$$\mathfrak{S}(\mu) \circ \mathrm{Th}_{L_1} = \mathrm{Th}_{L_0} \circ \beta = f \circ \mathrm{Th}_{L_1} \quad \text{implies} \quad \mathfrak{S}(\mu) = f\,,$$

by Lemma A2.1.10.

(b) For every $\varphi \in L_0$, we have

$$\begin{aligned} \mathfrak{p} \in \mathfrak{S}(\mu)^{-1}[\langle \varphi \rangle_{L_0}] \quad &\text{iff} \quad \mathfrak{S}(\mu)(\mathfrak{p}) = \alpha^{-1}[\mathfrak{p}] \in \langle \varphi \rangle_{L_0} \\ &\text{iff} \quad \varphi \in \alpha^{-1}[\mathfrak{p}] \\ &\text{iff} \quad \alpha(\varphi) \in \mathfrak{p} \quad \text{iff} \quad \mathfrak{p} \in \langle \alpha(\varphi) \rangle_{L_1}\,. \end{aligned}$$

Hence, $\mathfrak{S}(\mu)^{-1}[\langle \varphi \rangle_{L_0}] = \langle \alpha(\varphi) \rangle_{L_1}$. The claim follows by Lemma B5.2.3 since the sets $\langle \varphi \rangle_{L_0}$, for $\varphi \in L_0$, form a closed subbase of the topology of $\mathfrak{S}(L_0)$.

(c) Since β and Th_{L_0} are surjective, so is $\mathrm{Th}_{L_0} \circ \beta = \mathfrak{S}(\mu) \circ \mathrm{Th}_{L_1}$. Consequently, $\mathfrak{S}(\mu)$ is also surjective.

(d) Suppose that α is surjective and let $\mathfrak{p}, \mathfrak{q} \in S(L_1)$ be types such that $\mathfrak{S}(\mu)(\mathfrak{p}) = \mathfrak{S}(\mu)(\mathfrak{q})$. Then $\alpha^{-1}[\mathfrak{p}] = \alpha^{-1}[\mathfrak{q}]$ implies, by Lemma A2.1.10, that

$$\mathfrak{p} = \alpha[\alpha^{-1}[\mathfrak{p}]] = \alpha[\alpha^{-1}[\mathfrak{q}]] = \mathfrak{q}\,.$$

(e) We have already seen in (d) that $\mathfrak{S}(\mu)$ is injective. To show that it is closed, it is sufficient, by Lemma B5.2.3, to prove that $\mathfrak{S}(\mu)[\langle \varphi \rangle_{L_1}]$ is closed, for every $\varphi \in L_1$. We claim that

$$\mathfrak{S}(\mu)[\langle \varphi \rangle_{L_1}] = \langle \Phi \cup \alpha^{-1}(\varphi) \rangle_{L_0}\,.$$

($\subseteq$) Let $\mathfrak{p} \in \langle \varphi \rangle_{L_1}$ and fix an L_1-interpretation $\mathfrak{J}$ with $\mathrm{Th}_{L_1}(\mathfrak{J}) = \mathfrak{p}$. Then $\beta(\mathfrak{J}) \in \operatorname{rng}\beta = \mathrm{Mod}_{L_0}(\Phi)$ implies $\Phi \subseteq \mathrm{Th}_{L_0}(\beta(\mathfrak{J})) = \mathfrak{S}(\mu)(\mathfrak{p})$. Furthermore, $\varphi \in \mathfrak{p}$ implies $\alpha^{-1}(\varphi) \subseteq \alpha^{-1}[\mathfrak{p}] = \mathfrak{S}(\mu)(\mathfrak{p})$. Consequently, $\mathfrak{S}(\mu)(\mathfrak{p}) \in \langle \Phi \cup \alpha^{-1}(\varphi) \rangle_{L_0}$.

($\supseteq$) Let $\mathfrak{p} \in \langle \Phi \cup \alpha^{-1}(\varphi) \rangle_{L_0}$ and let $\mathfrak{J}_0$ be an L_0-interpretation with $\mathrm{Th}_{L_0}(\mathfrak{J}_0) = \mathfrak{p}$. Then $\mathfrak{J}_0 \vDash \Phi$ and $\operatorname{rng}\beta = \mathrm{Mod}_{L_0}(\Phi)$ implies that there is some L_1-interpretation $\mathfrak{J}$ with $\beta(\mathfrak{J}) = \mathfrak{J}_0$. Set $\mathfrak{q} := \mathrm{Th}_{L_1}(\mathfrak{J})$. Since α is surjective, we have

$$\begin{aligned} \alpha^{-1}(\varphi) \subseteq \mathfrak{p} \quad &\Rightarrow \quad \beta(\mathfrak{J}) \vDash \alpha^{-1}(\varphi) \\ &\Rightarrow \quad \mathfrak{J} \vDash \alpha[\alpha^{-1}(\varphi)] = \{\varphi\} \quad \Rightarrow \quad \varphi \in \mathfrak{q}\,. \end{aligned}$$

Hence, $\mathfrak{q} \in \langle\varphi\rangle_{L_1}$ and $\mathfrak{S}(\mu)(\mathfrak{q}) = \mathfrak{p}$.

(f) Let $\varphi, \psi \in L_0$ be formulae with $\alpha(\varphi) \equiv_{L_1} \alpha(\psi)$. We claim that $\varphi \equiv_{L_0} \psi$. By symmetry, it is sufficient to show that $\varphi \vDash \psi$.

Let $\mathfrak{J}$ be an L_0-interpretation with $\mathfrak{J} \vDash \varphi$. Since $\mathfrak{S}(\mu)$ is surjective, there is some type $\mathfrak{p} \in S(L_1)$ with $\mathfrak{S}(\mu)(\mathfrak{p}) = \mathrm{Th}_{L_0}(\mathfrak{J})$. Consequently,

$$\varphi \in \mathrm{Th}_{L_0}(\mathfrak{J}) = \mathfrak{S}(\mu)(\mathfrak{p}) = \alpha^{-1}[\mathfrak{p}] \quad \text{implies} \quad \alpha(\varphi) \in \mathfrak{p}\,.$$

Since $\alpha(\psi) \equiv_{L_1} \alpha(\varphi)$, it follows that $\alpha(\psi) \in \mathfrak{p}$. Hence, $\psi \in \alpha^{-1}(\mathfrak{p}) = \mathrm{Th}_{L_0}(\mathfrak{J})$ and $\mathfrak{J} \vDash \psi$. ☐

Corollary 2.12. *$\mathfrak{S}$ is a contravariant functor from $\mathfrak{Logic}$ to $\mathfrak{Top}_0$, the category of all T_0-spaces.*

Corollary 2.13. *Let $\mu : L_0 \to L_1$ be a morphism of logics.*

(a) *If μ is an embedding then $\mathfrak{S}(\mu)$ is a continuous surjection.*

(b) *If μ is an epimorphism then $\mathfrak{S}(\mu)$ is a continuous injection.*

(c) *If μ is an isomorphism then $\mathfrak{S}(\mu)$ is a homeomorphism.*

We can strengthen statement (c) of this corollary as follows.

Corollary 2.14. *Let $\mu = \langle\alpha, \beta\rangle : L_0 \to L_1$ be a morphism of logics where α and β are surjective. Then $\mathfrak{S}(\mu) : \mathfrak{S}(L_1) \to \mathfrak{S}(L_0)$ is a homeomorphism.*

Proof. As μ is an embedding, Corollary 2.13 (a) implies that $\mathfrak{S}(\mu)$ is continuous and surjective. Furthermore, $\mathrm{rng}\,\beta = \mathrm{Mod}_{L_0}(\varnothing)$. Therefore, we can use Proposition 2.11 (e) to show that $\mathfrak{S}(\mu)$ is closed and injective. ☐

Corollary 2.15. *Let L be a logic, $\Phi \subseteq L$, and $\lambda : L \to L/\Phi$ the localisation morphism. The function*

$$\mathfrak{S}(\lambda) : \mathfrak{S}(L/\Phi) \to \mathfrak{S}(L) : \mathfrak{p} \mapsto \mathfrak{p}$$

is continuous, closed, and injective.

Proof. Note that $\lambda = \langle \mathrm{id}, j\rangle$ where $j : \mathrm{Mod}_L(\Phi) \to \mathrm{Mod}_L(\varnothing)$ is the inclusion map. Since $\mathrm{rng}\, j = \mathrm{Mod}_L(\Phi)$, the claim follows by Proposition 2.11 (e). ☐

Example. In analogy to boolean logic, we define the *Lindenbaum quotient* $\mathcal{Q}(L)$ of a logic L by

$$\mathcal{Q}(L) := \langle \mathfrak{Lb}(L), \mathfrak{S}(L), \vDash\rangle$$

where, for $\mathfrak{p} \in S(L)$ and $\varphi \in L$,

$$\mathfrak{p} \vDash [\varphi]_\equiv \quad :\text{iff} \quad \varphi \in \mathfrak{p}\,.$$

We can turn $\mathcal{Q}$ into a functor $\mathcal{Q} : \mathfrak{Logic} \to \mathfrak{Logic}$ by setting, for a morphism $\mu : L_0 \to L_1$,

$$\mathcal{Q}(\mu) := \langle \mathfrak{Lb}(\mu), \mathfrak{S}(\mu)\rangle : \mathcal{Q}(L_0) \to \mathcal{Q}(L_1)\,.$$

The functor $\mathcal{Q}$ is idempotent in the sense that there exists a natural isomorphism $\eta : \mathcal{Q} \circ \mathcal{Q} \to \mathcal{Q}$. This natural isomorphism is defined as follows. For $\mathfrak{p} \in S(L)$, we have

$$\mathrm{Th}_{\mathcal{Q}(L)}(\mathfrak{p}) = \{\, [\varphi]_\equiv \mid \varphi \in \mathfrak{p} \,\} = \mathfrak{p}/{\equiv}\,.$$

Hence,

$$\mathfrak{S}(\mathcal{Q}(L)) = \{\, \mathrm{Th}_{\mathcal{Q}(L)}(\mathfrak{p}) \mid \mathfrak{p} \in S(L) \,\} = \{\, \mathfrak{p}/{\equiv} \mid \mathfrak{p} \in S(L) \,\}\,.$$

Since $\mathfrak{p}/{\equiv} = \mathfrak{q}/{\equiv}$ implies $\mathfrak{p} = \mathfrak{q}$, it follows that the function

$$\beta : \mathfrak{S}(L) \to \mathfrak{S}(\mathcal{Q}(L)) : \mathfrak{p} \mapsto \mathfrak{p}/{\equiv}$$

is a homeomorphism. Furthermore, since $\big[[\varphi]_\equiv\big]_\equiv = \{[\varphi]_\equiv\}$, the map

$$\alpha : \mathfrak{Lb}(\mathcal{Q}(L)) \to \mathfrak{Lb}(L) : [[\varphi]_\equiv]_\equiv \mapsto [\varphi]_\equiv$$

is a well-defined isomorphism of partial orders. Consequently, we obtain an isomorphism of logics

$$\eta_L := \langle \alpha, \beta \rangle : \mathcal{Q}(\mathcal{Q}(L)) \to \mathcal{Q}(L)\,.$$

Since, for every morphism $\mu : L_0 \to L_1$, we have

$$\eta_{L_1} \circ \mathcal{Q}(\mathcal{Q}(\mu)) = \mathcal{Q}(\mu) \circ \eta_{L_0}\,,$$

it follows that $(\eta_L)_L$ is a natural isomorphism.

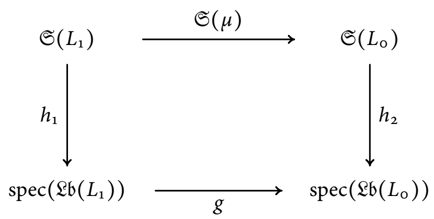
For boolean closed logics where the type space is compact and, hence, homeomorphic to the spectrum of the Lindenbaum algebra, we can strengthen Corollary 2.13 (a) as follows.

Lemma 2.16. *Let L_0 and L_1 be boolean closed logics where $\mathfrak{S}(L_1)$ is compact. If $\mu : L_0 \to L_1$ is an embedding, $\mathfrak{S}(\mu) : \mathfrak{S}(L_1) \to \mathfrak{S}(L_0)$ is continuous, closed, and surjective.*

Proof. We have already seen in Corollary 2.13 that $\mathfrak{S}(\mu)$ is continuous and surjective. Hence, it remains to prove that it is closed. Note that it follows by Lemma B5.3.10 that $\mathfrak{S}(L_0) = \mathfrak{S}(\mu)[S(L_1)]$ is also compact. By Lemma 2.8, there exist homeomorphisms

$$h_i : \mathfrak{S}(L_i) \to \operatorname{spec}(\mathfrak{Lb}(L_i)) : \mathfrak{p} \mapsto \mathfrak{p}/\equiv\,, \quad \text{for } i \in [2]\,.$$

Furthermore, we have seen in Lemma C1.6.10 that $\mathfrak{Lb}(\mu)$ is injective. Hence, Lemma B5.6.7 implies that the function $g := \operatorname{spec}(\mathfrak{Lb}(\mu))$ is continuous, closed, and surjective.



Since $\mathfrak{S}(\mu) = h_0^{-1} \circ g \circ h_1$, it follows that $\mathfrak{S}(\mu)$ is closed. □

Lemma 2.17. *Let L be a logic and $\Phi \subseteq L$. If $\mathfrak{S}(L)$ is compact, then so are $\mathfrak{S}(L|_\Phi)$ and $\mathfrak{S}(L/\Phi)$.*

Proof. Let $\lambda : L \to L/\Phi$ and $i : L|_\Phi \to L$ be the canonical morphisms. We have seen in Corollary 2.13 that $\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(L|_\Phi)$ is continuous and surjective. Since $\mathfrak{S}(L)$ is compact, it follows by Lemma B5.3.10 that $\mathfrak{S}(i)[S(L)] = \mathfrak{S}(L|_\Phi)$ is also compact.

By Corollary 2.15, $\mathfrak{S}(\lambda) : \mathfrak{S}(L/\Phi) \to \mathfrak{S}(L)$ is continuous, closed, and injective. Consequently, $\mathfrak{S}(L/\Phi)$ is homeomorphic to a closed subset $\operatorname{rng} \mathfrak{S}(\lambda) \subseteq \mathfrak{S}(L)$ of $\mathfrak{S}(L)$. By Lemma B5.3.9, it follows that $\mathfrak{S}(L/\Phi)$ is compact. □

As a consequence, we obtain the following generalisation of Theorem 2.3.

Theorem 2.18. *For all first-order theories $T \subseteq \mathrm{FO}^0[\Sigma]$,*

$$\mathfrak{S}^{\bar{s}}(T) \cong \operatorname{spec}(\mathfrak{Lb}(\mathrm{FO}^{\bar{s}}[\Sigma]/T))$$

is a Stone space.

Proof. By Lemma 2.17, $\mathfrak{S}^{\bar{s}}(T) = \mathfrak{S}(\mathrm{FO}^{\bar{s}}[\Sigma]/T)$ is compact. Hence, the claim follows by Lemma 2.8 (b). □

For algebraic logics L, every map $\mu : \Sigma \to \Gamma$ between signatures gives rise to a morphism $L[\mu] : L[\Sigma] \to L[\Gamma]$ and a corresponding continuous map $\mathfrak{S}(L[\mu]) : \mathfrak{S}(L[\Gamma]) \to \mathfrak{S}(L[\Sigma])$. In the lemma below, we take a closer look at such a map, where $\mu : \Sigma_U \to \Sigma_V$ corresponds to a renaming of parameters.

Definition 2.19. Let L be an algebraic logic. For a type $\mathfrak{p}$ over U and a function $f : U \to V$, we write

$$f(\mathfrak{p}) := \{\, \varphi(\bar{x}; f(\bar{a})) \mid \varphi(\bar{x}; \bar{a}) \in \mathfrak{p} \,\}\,.$$

Remark. Suppose that f is a strict L-map and let $\mathfrak{p} := \mathrm{tp}_L(\bar{a}/U)$ where $\bar{a}, U \subseteq \mathrm{dom}\, f$. Then

$$f(\mathfrak{p}) = \mathrm{tp}_L\big(f(\bar{a}) \,/\, f[U]\big)\,.$$

Lemma 2.20. *Suppose that L is an algebraic logic, let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures, and $U \subseteq A$. Every injective, strict L-map $h : U \to B$ induces a homeomorphism*

$$\mathfrak{S}^{\bar{s}}_L(U) \to \mathfrak{S}^{\bar{s}}_L(h[U]) : \mathfrak{p} \mapsto h(\mathfrak{p})\,.$$

Proof. Set $V := h[U]$. Let $\mu : \Sigma_U \to \Sigma_V$ be the morphism of signatures with $\mu \restriction \Sigma = \mathrm{id}_\Sigma$ and $\mu \restriction U = h$, and let

$$\langle \alpha, \beta \rangle := L^{\bar{s}}[\mu] : L^{\bar{s}}[\Sigma_U] \to L^{\bar{s}}[\Sigma_V]$$

be the corresponding morphism of logics. Since μ is bijective so are α and β and we have $L^{\bar{s}}[\mu^{-1}] = \langle \alpha^{-1}, \beta^{-1} \rangle$.

We claim that β induces a bijection $\mathrm{Mod}_L(T(V)) \to \mathrm{Mod}_L(T(U))$. Let $\mathfrak{M} \vDash T(V)$, $\varphi(\bar{x}) \in L[\Sigma, X]$, and $\bar{c} \subseteq U$. As h is a strict L-map, it follows that

$$\begin{aligned}
\beta(\mathfrak{M}) \vDash \varphi(\bar{c}) \quad &\text{iff} \quad \mathfrak{M} \vDash \alpha(\varphi(\bar{c})) = \varphi(h(\bar{c}))\\
&\text{iff} \quad \varphi(h(\bar{c})) \in T(V)\\
&\text{iff} \quad \mathfrak{B} \vDash \varphi(h(\bar{c}))\\
&\text{iff} \quad \mathfrak{A} \vDash \varphi(\bar{c})\\
&\text{iff} \quad \varphi(\bar{c}) \in T(U)\,.
\end{aligned}$$

Similarly, it follows that $\beta^{-1}(\mathfrak{M}) \in \mathrm{Mod}(T(V))$, for every model $\mathfrak{M}$ of $T(U)$. Therefore, $\langle \alpha, \beta \rangle$ induces a morphism

$$\langle \alpha, \beta_0 \rangle : L^{\bar{s}}/T(U) \to L^{\bar{s}}/T(V)$$

where $\beta_0 = \beta \restriction \mathrm{Mod}_L(T(V))$ is bijective. As α is also bijective, it follows by Corollary 2.13 that the induced map

$$\mathfrak{S}^{\bar{s}}_L\langle \alpha, \beta_0 \rangle : \mathfrak{S}^{\bar{s}}_L(V) \to \mathfrak{S}^{\bar{s}}_L(U) : h(\mathfrak{p}) \mapsto \mathfrak{p}$$

is a homeomorphism. □

For first-order type spaces, we can say more on the dependence of a type space on the signature.

Proposition 2.21. *Let $\Sigma_0 \subseteq \Sigma$ be signatures, $T \subseteq \mathrm{FO}^\circ[\Sigma]$ a theory, and set $T_0 := T \cap \mathrm{FO}^\circ[\Sigma_0]$.*

(a) *For every $\Delta \subseteq \mathrm{FO}^\circ[\Sigma_0]$, we have*

$$\mathfrak{S}((\mathrm{FO}^\circ[\Sigma_0]/T_0)|_\Delta) = \mathfrak{S}((\mathrm{FO}^\circ[\Sigma]/T)|_\Delta)\,.$$

(b) *The function*

$$h : \mathfrak{S}(\mathrm{FO}^\circ[\Sigma]/T) \to \mathfrak{S}(\mathrm{FO}^\circ[\Sigma_0]/T_0) : \mathfrak{p} \mapsto \mathfrak{p} \cap \mathrm{FO}^\circ[\Sigma_0]$$

is continuous, closed, and surjective.

Proof. To simplify notation, set $L := \mathrm{FO}^\circ[\Sigma]$ and $L_0 := \mathrm{FO}^\circ[\Sigma_0]$.

(a) We start by showing that both type spaces have the same universe.

Let $\mathfrak{p} \in S((L/T)|_\Delta)$. Then there is some $\mathfrak{M} \in \mathrm{Mod}_L(T)$ with $\mathfrak{p} = \mathrm{Th}_\Delta(\mathfrak{M})$. Setting $\mathfrak{M}_0 := \mathfrak{M}|_{\Sigma_0}$ we obtain a model $\mathfrak{M}_0 \in \mathrm{Mod}_{L_0}(T_0)$ with $\mathfrak{p} = \mathrm{Th}_\Delta(\mathfrak{M}_0)$. It follows that $\mathfrak{p} \in S((L_0/T_0)|_\Delta)$.

Conversely, let $\mathfrak{p} \in S((L_0/T_0)|_\Delta)$. Then there is some model $\mathfrak{M}_0 \in \mathrm{Mod}_{L_0}(T_0)$ with $\mathfrak{p} = \mathrm{Th}_\Delta(\mathfrak{M}_0)$. We can use Corollary c2.5.9 to find a model $\mathfrak{M} \in \mathrm{Mod}_L(T)$ such that $\mathfrak{M}_0 \preceq \mathfrak{M}|_\Sigma$. It follows that $\mathfrak{p} = \mathrm{Th}_\Delta(\mathfrak{M})$. Hence, $\mathfrak{p} \in S((L/T)|_\Delta)$.

It remains to show that the two topologies coincide. For $\Phi \subseteq \Delta$, it follows by definition that

$$\begin{aligned}
\langle \Phi \rangle_{(L_0/T_0)|_\Delta} &= \big\{ \mathfrak{p} \in S((L_0/T_0)|_\Delta) \bigm| \Phi \subseteq \mathfrak{p} \big\}\\
&= \big\{ \mathfrak{p} \in S((L/T)|_\Delta) \bigm| \Phi \subseteq \mathfrak{p} \big\} = \langle \Phi \rangle_{(L/T)|_\Delta}\,.
\end{aligned}$$

(b) Consider the inclusion map $i : (L/T)|_{L_0} \to L/T$. By Lemma 2.16, the map

$$\mathfrak{S}(i) : \mathfrak{S}(L/T) \to \mathfrak{S}((L/T)|_{L_0}) : \mathfrak{p} \mapsto \mathfrak{p} \cap L$$

is continuous, closed, and surjective. Furthermore, we have seen in (a) that the identity map

$$\mathrm{id} : \mathfrak{S}((L/T)|_{L_0}) \to \mathfrak{S}(L_0/T_0)$$

is a homeomorphism. It follows that the composition $h = \mathrm{id} \circ \mathfrak{S}(i)$ is continuous, closed, and surjective. □

A special case of Proposition 2.21 (b) is worth singling out.

Corollary 2.22. *Let $T \subseteq \mathrm{FO}^{\circ}[\Sigma]$ be a first-order theory, $U \subseteq V$ sets of parameters, and*

$$i : \mathrm{FO}^{\bar{s}}[\Sigma_U]/T(U) \to \mathrm{FO}^{\bar{s}}[\Sigma_V]/T(V)$$

the inclusion morphism. The induced map

$$\mathfrak{S}(i) : \mathfrak{S}^{\bar{s}}(V) \to \mathfrak{S}^{\bar{s}}(U)$$

is continuous, closed, and surjective.

3. Retracts

For every fragment Δ of a logic L, we have seen above that the inclusion morphism $i : \Delta \to L$ induces a surjective, continuous map

$$\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(\Delta) : \mathfrak{p} \mapsto \mathfrak{p} \cap \Delta\,.$$

It follows that the type space of Δ is a quotient of the type space of L. In this section we take a closer look at the relationship between these two type spaces.

Definition 3.1. Let L be a logic, $L_0 \subseteq L$ a fragment, and $i : L_0 \to L$ the inclusion morphism.

(a) A morphism $r : L \to L_0$ is a *retraction* if $r \circ i = \mathrm{id}$.

(b) L_0 is a *retract* of L if there exists a retraction $L \to L_0$.

The type space of a retract is homeomorphic to the type space of the full logic.

Lemma 3.2. *Let $r = \langle \alpha, \beta \rangle : L \to L_0$ be a retraction and $i : L_0 \to L$ the inclusion morphism.*

(a) $\beta = \mathrm{id}$.

(b) $\alpha(\varphi) \equiv_L \varphi$*, for every* $\varphi \in L$.

(c) $\mathfrak{S}(r) = \mathfrak{S}(i)^{-1}$.

Proof. (a) Note that $i = \langle \iota, \mathrm{id} \rangle$, where $\iota : L_0 \to L$ is the inclusion function. Hence, $r \circ i = \mathrm{id}$ implies that $\mathrm{id} \circ \beta = \mathrm{id}$.

(b) To show that $\alpha(\varphi) \equiv_L \varphi$, let $\mathfrak{I}$ be an L-interpretation. We have seen in (a) that $\beta(\mathfrak{I}) = \mathfrak{I}$. Since r is a morphism of logics, it follows that

$$\mathfrak{I} \vDash \varphi \quad \text{iff} \quad \mathfrak{I} \vDash \alpha(\varphi)\,.$$

(c) Note that $r \circ i = \mathrm{id}$ implies $\mathfrak{S}(i) \circ \mathfrak{S}(r) = \mathrm{id}$. Hence, it remains to show that $\mathfrak{S}(r) \circ \mathfrak{S}(i) = \mathrm{id}$. Consider $\mathfrak{p} \in S(L)$. By (b), it follows that

$$\varphi \in \mathfrak{p} \quad \text{iff} \quad \alpha(\varphi) \in \mathfrak{p}\,, \quad \text{for all } \varphi \in L\,.$$

Hence, $\mathfrak{p} = \alpha^{-1}[\mathfrak{p}] = \mathfrak{S}(i \circ r)(\mathfrak{p}) = (\mathfrak{S}(r) \circ \mathfrak{S}(i))(\mathfrak{p})$. □

Corollary 3.3. *Let $r : L \to L_0$ be a retraction and $i : L_0 \to L$ the inclusion morphism.*

(a) $\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(L_0)$ *is a homeomorphism.*

(b) $\mathfrak{S}(r) : \mathfrak{S}(L_0) \to \mathfrak{S}(L)$ *is a homeomorphism.*

Proof. Both statements follow from Lemma 3.2 (c). □

Lemma 3.4. *Let L be a logic, $L_0 \subseteq L$ a fragment, and $i = \langle \iota, \mathrm{id} \rangle : L_0 \to L$ the inclusion morphism. The following statements are equivalent:*

(1) L_0 *is a retract of* L.

(2) *For every formula* $\varphi \in L$*, there is a formula* $\varphi_0 \in L_0$ *such that* $\varphi \equiv_L \varphi_0$.

(3) *The function* $\mathfrak{Lb}(i) : \mathfrak{Lb}(L_0) \to \mathfrak{Lb}(L)$ *is an isomorphism.*

Proof. (1) ⇒ (2) follows immediately by Lemma 3.2 (b).

(2) ⇒ (3) We have seen in Lemma c1.6.10 that $\mathfrak{Lb}(i)$ is an embedding. Hence, it remains to show that it is surjective. Let $[\varphi]_\equiv \in \mathfrak{Lb}(L)$. By (2), there is some formula $\varphi_0 \in L_0$ with $\varphi_0 \equiv_L \varphi$. It follows that

$$\mathfrak{Lb}(i)([\varphi_0]_\equiv) = [\varphi_0]_\equiv = [\varphi]_\equiv .$$

(3) ⇒ (1) We define a function $\alpha : L \to L_0$ as follows. For $\varphi \in L_0$, we set $\alpha(\varphi) := \varphi$. For $\varphi \in L \setminus L_0$, we choose an arbitrary formula ψ such that $[\psi]_\equiv \in \mathfrak{Lb}(i)^{-1}([\varphi]_\equiv)$ and set $\alpha(\varphi) := \psi$. Note that, for every $\varphi \in L$,

$$\alpha(\varphi) \in \mathfrak{Lb}(i)^{-1}([\varphi]_\equiv)$$

implies that

$$[\varphi]_\equiv = \mathfrak{Lb}(i)([\alpha(\varphi)]_\equiv) = [\alpha(\varphi)]_\equiv .$$

Hence, $\alpha(\varphi) \equiv_L \varphi$, for all $\varphi \in L$. By definition of α, we further have

$$(\alpha \circ \iota)(\varphi) = \alpha(\varphi) = \varphi , \quad \text{for all } \varphi \in L_0 .$$

Hence, to show that $r := \langle \alpha, \mathrm{id} \rangle$ is a left inverse of i it remains to prove that r is a morphism of logics. Let $\varphi \in L$ be a formula and $\mathfrak{J}$ an L-interpretation. Since $\varphi \equiv_L \alpha(\varphi)$, we have

$$\mathfrak{J} \vDash \varphi \quad \text{iff} \quad \mathfrak{J} \vDash \alpha(\varphi) ,$$

as desired. □

Below we will present several results that assume Δ to be boolean closed. The following lemma can sometimes be used to replace this restriction by the requirement that Δ is closed under negation.

Lemma 3.5. *Let* L *be a boolean closed logic,* $\Delta \subseteq L$ *closed under negation, and let* $i : \Delta \to L$ *be the inclusion morphism. If every formula in* L *is equivalent to a finite boolean combination of formulae in* Δ*, then*

$$\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(\Delta)$$

is a homeomorphism.

Proof. By Corollary 2.13, $\mathfrak{S}(i)$ is continuous and surjective.

For injectivity, suppose that $\mathfrak{S}(i)(\mathfrak{p}) = \mathfrak{S}(i)(\mathfrak{q})$. Then $\mathfrak{p} \cap \Delta = \mathfrak{q} \cap \Delta$. Since every formula in L is equivalent to a boolean combination of formulae in Δ, it follows that $\mathfrak{p} = \mathfrak{q}$.

It remains to show that $\mathfrak{S}(i)$ is closed. By Lemma b5.2.3, it is sufficient to prove that $\mathfrak{S}(i)[\langle\varphi\rangle_L]$ is closed, for every $\varphi \in L$. Fix $\varphi \in L$. By assumption on Δ and L, there are sets $\Psi_0, \dots, \Psi_{n-1} \subseteq \Delta$ such that $\varphi \equiv_L \bigvee_{k<n} \bigwedge \Psi_k$. Since, trivially, $\Psi_k \equiv_L \Psi_k$, it follows by Lemma 3.9 that

$$\mathfrak{S}(i)[\langle\Psi_k\rangle_L] = \langle\Psi_k\rangle_\Delta .$$

Consequently,

$$\mathfrak{S}(i)[\langle\varphi\rangle_L] = \mathfrak{S}(i)\Big[\bigcup_{k<n}\langle\Psi_k\rangle_L\Big] = \bigcup_{k<n}\mathfrak{S}(i)\big[\langle\Psi_k\rangle_L\big] = \bigcup_{k<n}\langle\Psi_k\rangle_\Delta$$

is closed. □

Exercise 3.1. Show that the preceding lemma may fail if Δ is not closed under negation.

Corollary 3.6. *Let* L *be a boolean closed logic such that* $\mathfrak{S}(L)$ *is compact, let* $\Delta_0 \subseteq \Delta \subseteq L$ *be closed under negation, and let* $i : \Delta_0 \to \Delta$ *be the inclusion morphism. The induced map*

$$\mathfrak{S}(i) : \mathfrak{S}(\Delta) \to \mathfrak{S}(\Delta_0)$$

is continuous, closed, and surjective.

Proof. Let $\Delta_o^+ \subseteq \Delta^+ \subseteq L$ be the boolean closures of Δ_o and Δ, and let $j_o : \Delta_o \to \Delta_o^+$, $j : \Delta \to \Delta^+$, and $i_+ : \Delta_o^+ \to \Delta^+$ be the corresponding inclusion morphisms. By Lemma 3.5, $\mathfrak{S}(j_o)$ and $\mathfrak{S}(j)$ are homeomorphisms. Hence,

$$j \circ i = i_+ \circ j_o \quad \text{implies} \quad \mathfrak{S}(i) = \mathfrak{S}(j_o) \circ \mathfrak{S}(i_+) \circ \mathfrak{S}(j)^{-1}.$$

Since, by Lemma 2.16, the functions on the right-hand side are continuous, closed, and surjective, so is $\mathfrak{S}(i)$. □

In the remainder of this section we consider to which extend the reverse of Corollary 3.3 (a) holds: in which cases is $\mathfrak{S}(i)$ being a homeomorphism sufficient for Δ to have the same expressive power as L.

Lemma 3.7. *Let L_o and L_1 be logics and $\mu : \mathfrak{S}(L_o) \to \mathfrak{S}(L_1)$ a homeomorphism. Then*

$$\mathfrak{p} \subseteq \mathfrak{q} \quad \text{iff} \quad \mu(\mathfrak{p}) \subseteq \mu(\mathfrak{q}), \quad \textit{for all } \mathfrak{p}, \mathfrak{q} \in S(L_o).$$

Proof. It is sufficient to prove that $\mathfrak{p} \subseteq \mathfrak{q}$ implies $\mu(\mathfrak{p}) \subseteq \mu(\mathfrak{q})$. Then we can prove the converse implication, by considering the homeomorphism μ^{-1}. Note that we have

$$\begin{aligned} &\mathfrak{p} \subseteq \mathfrak{q} &&\text{iff} \quad \text{for all } \Phi, \quad \mathfrak{p} \in \langle\Phi\rangle_{L_o} \Rightarrow \mathfrak{q} \in \langle\Phi\rangle_{L_o}, \\ \text{and} \quad &\mu(\mathfrak{p}) \subseteq \mu(\mathfrak{q}) &&\text{iff} \quad \text{for all } \Psi, \quad \mu(\mathfrak{p}) \in \langle\Psi\rangle_{L_1} \Rightarrow \mu(\mathfrak{q}) \in \langle\Psi\rangle_{L_1}. \end{aligned}$$

Let us show that the condition

$$\mathfrak{p} \in \langle\Phi\rangle_{L_o} \Rightarrow \mathfrak{q} \in \langle\Phi\rangle_{L_o}, \quad \text{for all } \Phi \subseteq L_o$$

is equivalent to

$$\mathfrak{p} \in C \Rightarrow \mathfrak{q} \in C, \quad \text{for all closed } C \subseteq S(L_o).$$

Clearly, if the implication holds for all closed sets C, it in particular holds for closed sets of the form $\langle\Phi\rangle_{L_o}$. Hence, it is sufficient to prove the converse. Suppose that every set $\langle\Phi\rangle_{L_o}$ containing $\mathfrak{p}$ also contains $\mathfrak{q}$ and let C be a closed set with $\mathfrak{p} \in C$. By definition, there is a family $(\Psi_i)_{i\in I}$ of finite sets $\Psi_i \subseteq L_o$ such that

$$C = \bigcap_{i\in I} \bigcup_{\psi\in\Psi_i} \langle\psi\rangle_{L_o}.$$

Since $\mathfrak{p} \in C$, there are formulae $\psi_i \in \Psi_i$, for $i \in I$, such that $\mathfrak{p} \in \langle\psi_i\rangle_{L_o}$. By assumption, this implies that $\mathfrak{q} \in \langle\psi_i\rangle_{L_o}$. Hence,

$$\mathfrak{q} \in \bigcap_{i\in I} \langle\psi_i\rangle_{L_o} \subseteq \bigcap_{i\in I} \bigcup_{\psi\in\Psi_i} \langle\psi\rangle_{L_o} = C.$$

To prove the lemma, suppose that $\mathfrak{p} \subseteq \mathfrak{q}$. We have just seen that this implies that

$$\mathfrak{p} \in C \Rightarrow \mathfrak{q} \in C, \quad \text{for all closed } C \subseteq S(L_o).$$

Hence,

$$\mu(\mathfrak{p}) \in \mu[C] \Rightarrow \mu(\mathfrak{q}) \in \mu[C], \quad \text{for all closed } C \subseteq S(L_o).$$

Since μ is a homeomorphism, it follows that

$$\mu(\mathfrak{p}) \in D \Rightarrow \mu(\mathfrak{q}) \in D, \quad \text{for all closed } D \subseteq S(L_1).$$

As we have seen above, this implies that

$$\mathfrak{p} \in \langle\Psi\rangle_{L_1} \Rightarrow \mathfrak{q} \in \langle\Psi\rangle_{L_1}, \quad \text{for all } \Psi \subseteq L_1.$$

Consequently, we have $\mu(\mathfrak{p}) \subseteq \mu(\mathfrak{q})$. □

Corollary 3.8. *Let L be a logic, $\Delta \subseteq L$, and $i : \Delta \to L$ the inclusion morphism. If $\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(\Delta)$ is a homeomorphism, then*

$$\mathfrak{p} \cap \Delta \vDash \mathfrak{p}, \quad \textit{for all } \mathfrak{p} \in S(L).$$

Proof. Suppose that $\mathfrak{J} \vDash \mathfrak{p} \cap \Delta$. To show that $\mathfrak{J} \vDash \mathfrak{p}$, consider $\mathfrak{q} := \mathrm{Th}_L(\mathfrak{J})$. Since $\mathfrak{S}(i)$ is bijective we have

$$\begin{aligned} &\mathfrak{S}(i)^{-1}(\mathfrak{p} \cap \Delta) = \mathfrak{S}(i)^{-1}(\mathfrak{S}(i)(\mathfrak{p})) = \mathfrak{p} \\ \text{and} \quad &\mathfrak{S}(i)^{-1}(\mathfrak{q} \cap \Delta) = \mathfrak{S}(i)^{-1}(\mathfrak{S}(i)(\mathfrak{q})) = \mathfrak{q} \,. \end{aligned}$$

Hence, $\mathfrak{p} \cap \Delta \subseteq \mathrm{Th}_\Delta(\mathfrak{J}) = \mathfrak{q} \cap \Delta$ implies, by Lemma 3.7, that

$$\mathfrak{p} = \mathfrak{S}(i)^{-1}(\mathfrak{p} \cap \Delta) \subseteq \mathfrak{S}(i)^{-1}(\mathfrak{q} \cap \Delta) = \mathfrak{q} = \mathrm{Th}_L(\mathfrak{J}) \,.$$

Consequently, $\mathfrak{J} \vDash \mathfrak{p}$. □

Below we will provide several characterisations of when Δ has the same expressive power as L. We start with a technical lemma containing a condition on when two sets Φ, Ψ of formulae are equivalent.

Lemma 3.9. *Let L be a logic, $\Delta \subseteq L$, and $i : \Delta \to L$ the inclusion morphism. If $\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(\Delta)$ is bijective, we have*

$$\Phi \equiv_L \Psi \quad \text{iff} \quad \mathfrak{S}(i)[\langle \Phi \rangle_L] = \langle \Psi \rangle_\Delta \,,$$

for all sets $\Phi \subseteq L$ and $\Psi \subseteq \Delta$.

Proof. ($\Rightarrow$) Suppose that $\Phi \equiv_L \Psi$. For ($\subseteq$), let $\mathfrak{p} \in \langle \Phi \rangle_L$. Then $\Phi \subseteq \mathfrak{p}$ implies

$$\Psi \subseteq \Phi^{\vDash} \cap \Delta \subseteq \mathfrak{p} \cap \Delta = \mathfrak{S}(i)(\mathfrak{p}) \,.$$

Hence, $\mathfrak{S}(i)(\mathfrak{p}) \in \langle \Psi \rangle_\Delta$.

For ($\supseteq$), let $\mathfrak{p} \in \langle \Psi \rangle_\Delta$. Since $\mathfrak{S}(i)$ is surjective, there is some $\mathfrak{q} \in S(L)$ with $\mathfrak{S}(i)(\mathfrak{q}) = \mathfrak{p}$. Hence,

$$\Psi \subseteq \mathfrak{p} = \mathfrak{S}(i)(\mathfrak{q}) = \mathfrak{q} \cap \Delta \subseteq \mathfrak{q} \,.$$

Since $\Psi \vDash \Phi$, it follows that $\Phi \subseteq \mathfrak{q}$. Consequently, we have $\mathfrak{q} \in \langle \Phi \rangle_L$, which implies that $\mathfrak{p} = \mathfrak{S}(i)(\mathfrak{q}) \in \mathfrak{S}(i)[\langle \Phi \rangle_L]$.

($\Leftarrow$) We have to show that $\Phi \equiv_L \Psi$. First, suppose that $\mathfrak{J} \vDash \Phi$ and let $\mathfrak{p} := \mathrm{Th}_\Delta(\mathfrak{J})$. Then $\mathfrak{p} \in \langle \Phi \rangle_L$ implies that

$$\mathrm{Th}_\Delta(\mathfrak{J}) = \mathfrak{p} \cap \Delta = \mathfrak{S}(i)(\mathfrak{p}) \in \mathfrak{S}(i)[\langle \Phi \rangle_L] = \langle \Psi \rangle_\Delta \,.$$

Hence, $\mathfrak{J} \vDash \Psi$. Conversely, suppose that $\mathfrak{J} \vDash \Psi$ and let $\mathfrak{p} := \mathrm{Th}_L(\mathfrak{J})$. Then $\mathfrak{S}(i)(\mathfrak{p}) = \mathfrak{p} \cap \Delta \in \langle \Psi \rangle_\Delta$. Since $\mathfrak{S}(i)$ is injective, we have

$$\mathfrak{p} \in \mathfrak{S}(i)^{-1}(\mathfrak{p} \cap \Delta) \subseteq \mathfrak{S}(i)^{-1}[\langle \Psi \rangle_\Delta] = \langle \Phi \rangle_L$$

and, therefore, $\mathfrak{J} \vDash \Phi$. □

For fragments $\Delta \subseteq L$ that are closed under disjunctions, we obtain the following characterisation of when every L-formula is equivalent to a set of Δ-formulae.

Proposition 3.10. *Let L be a logic, $\Delta \subseteq L$, and let $i : \Delta \to L$ be the inclusion morphism. If Δ is closed under disjunctions, the following statements are equivalent.*

(1) *For every $\Phi \subseteq L$, there is some $\Psi \subseteq \Delta$ such that $\Phi \equiv_L \Psi$.*

(2) *$\Phi \equiv_L \Phi^{\vDash} \cap \Delta$, for all $\Phi \subseteq L$.*

(3) *$\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(\Delta)$ is a homeomorphism.*

Proof. (1) $\Rightarrow$ (2) Let $\Phi \subseteq L$. Clearly, $\Phi \vDash \Phi^{\vDash} \cap \Delta$. Hence, we only need to prove that $\Phi^{\vDash} \cap \Delta \vDash \Phi$. By (1), there is a set $\Psi \subseteq \Delta$ such that $\Psi \equiv_L \Phi$. Hence, $\Phi \vDash \Psi$ implies that $\Psi \subseteq \Phi^{\vDash} \cap \Delta$. Since $\Psi \vDash \Phi$, it therefore follows that $\Phi^{\vDash} \cap \Delta \vDash \Phi$.

(2) $\Rightarrow$ (3) Suppose that every $\Phi \subseteq L$ is equivalent to $\Phi^{\vDash} \cap \Delta$. We have to prove that $\mathfrak{S}(i)$ is continuous, closed, and bijective. Continuity and surjectivity follow from Corollary 2.13.

For injectivity, suppose that $\mathfrak{p}, \mathfrak{q} \in S(L)$ are two types with $\mathfrak{S}(i)(\mathfrak{p}) = \mathfrak{S}(i)(\mathfrak{q})$. By (2), $\mathfrak{p} \equiv_L \mathfrak{p} \cap \Delta$ and $\mathfrak{q} \equiv_L \mathfrak{q} \cap \Delta$. Consequently, we have

$$\mathfrak{p} \equiv_L \mathfrak{p} \cap \Delta = \mathfrak{S}(i)(\mathfrak{p}) = \mathfrak{S}(i)(\mathfrak{q}) = \mathfrak{q} \cap \Delta \equiv_L \mathfrak{q} \,.$$

It follows that $\mathfrak{p} = \mathfrak{q}$, as desired.

It remains to prove that $\mathfrak{S}(i)$ is closed. Since $\mathfrak{S}(i)$ is injective, it is sufficient, by Lemma B5.2.3, to prove that $\mathfrak{S}(i)[\langle \Phi \rangle_L]$ is closed, for every $\Phi \subseteq L$. By (2), $\Phi \equiv_L \Phi^{\vDash} \cap \Delta$. Hence, it follows by Lemma 3.9 that the set $\mathfrak{S}(i)[\langle \Phi \rangle_L] = \langle \Phi^{\vDash} \cap \Delta \rangle_\Delta$ is closed.

(3) ⇒ (1) Suppose that $\mathfrak{S}(i)$ is a homeomorphism. To show that every $\Phi \subseteq L$ is equivalent to some $\Psi \subseteq \Delta$, we fix $\Phi \subseteq L$. Since $\langle \Phi \rangle_L$ is closed in $\mathfrak{S}(L)$, it follows that $C := \mathfrak{S}(i)[\langle \Phi \rangle_L]$ is a closed subset of $\mathfrak{S}(\Delta)$. By Lemma 2.2, there exists a set $\Psi \subseteq \Delta$ such that $C = \langle \Psi \rangle_\Delta$. Hence, $\mathfrak{S}(i)[\langle \Phi \rangle_L] = \langle \Psi \rangle_\Delta$ implies, by Lemma 3.9, that $\Phi \equiv_L \Psi$. □

Exercise 3.2. Show that the preceding lemma may fail if Δ is not closed under disjunctions.

For logics with compact type space, we can strengthen this proposition as follows.

Proposition 3.11. *Let L be a boolean closed logic such that $\mathfrak{S}(L)$ is compact, let $\Delta \subseteq L$, and let $i : \Delta \to L$ be the inclusion morphism. The following statements are equivalent:*

(1) *For every $\varphi \in L$, there is some $\psi \in \Delta$ with $\psi \equiv_L \varphi$.*

(2) *Δ is a retract of L.*

(3) *Δ is boolean closed and*

$$\mathrm{spec}(\mathfrak{Lb}(i)) : \mathrm{spec}(\mathfrak{Lb}(L)) \to \mathrm{spec}(\mathfrak{Lb}(\Delta))$$

is a homeomorphism.

(4) *Δ is boolean closed and $\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(\Delta)$ is a homeomorphism.*

Proof. (1) ⇔ (2) was already proved in Lemma 3.4.

(3) ⇒ (2) According to Lemma B5.6.7, $\mathfrak{Lb}(i) : \mathfrak{Lb}(\Delta) \to \mathfrak{Lb}(L)$ is an isomorphism. Hence, the claim follows by Lemma 3.4.

(1) ⇒ (4) $\mathfrak{S}(i)$ is a homeomorphism by Corollary 3.3 (a). Therefore, we only need to show that Δ is boolean closed. Let $\varphi, \vartheta \in \Delta$. Then $\varphi \wedge \vartheta$, $\varphi \vee \vartheta$, and $\neg\varphi$ are L-formulae. By (1), there are formulae $\psi_0, \psi_1, \psi_2 \in \Delta$ with

$$\psi_0 \equiv_L \varphi \wedge \vartheta\,, \quad \psi_1 \equiv_L \varphi \vee \vartheta\,, \quad \text{and} \quad \psi_2 \equiv_L \neg\varphi\,.$$

Hence, Δ is boolean closed.

(4) ⇒ (3) According to Lemma 2.17, $\mathfrak{S}(L)$ and $\mathfrak{S}(\Delta)$ are both compact. Therefore, we can use Lemma 2.8 to obtain homeomorphisms

$$h : \mathfrak{S}(L) \to \mathrm{spec}(\mathfrak{Lb}(L)) \quad \text{and} \quad h_0 : \mathfrak{S}(\Delta) \to \mathrm{spec}(\mathfrak{Lb}(\Delta))\,.$$

If $\mathfrak{S}(i)$ is a homeomorphism, then so is $\mathrm{spec}(\mathfrak{Lb}(i)) = h_0 \circ \mathfrak{S}(i) \circ h^{-1}$. □

Corollary 3.12. *Let L be a boolean closed logic such that $\mathfrak{S}(L)$ is compact, and let $\Delta \subseteq \Phi \subseteq L$. The following statements are equivalent.*

(1) *Every formula in Φ is equivalent to a finite boolean combination of formulae in Δ.*

(2) *$\mathfrak{p} \cap \Delta = \mathfrak{q} \cap \Delta$ implies $\mathfrak{p} \cap \Phi = \mathfrak{q} \cap \Phi$, for all $\mathfrak{p}, \mathfrak{q} \in S(L)$.*

Proof. (1) ⇒ (2) is obvious. For (2) ⇒ (1), let Δ_+ and Φ_+ be the boolean closures of, respectively, Δ and Φ and let $i : \Delta_+ \to \Phi_+$ be the inclusion morphism. By Proposition 3.11, it is sufficient to show that $\mathfrak{S}(i) : \mathfrak{S}(\Phi_+) \to \mathfrak{S}(\Delta_+)$ is a homeomorphism.

According to Lemma 2.16, $\mathfrak{S}(i)$ is continuous, closed, and surjective. Hence, it remains to prove that it is injective. Suppose that $\mathfrak{S}(i)(\mathfrak{p}) = \mathfrak{S}(i)(\mathfrak{q})$. Fix models $\mathfrak{J}_0 \vDash \mathfrak{p}$ and $\mathfrak{J}_1 \vDash \mathfrak{q}$, and set $\mathfrak{p}_+ := \mathrm{Th}_L(\mathfrak{J}_0)$ and $\mathfrak{q}_+ := \mathrm{Th}_L(\mathfrak{J}_1)$. Then

$$\mathfrak{p}_+ \cap \Delta_+ = \mathfrak{p} \cap \Delta_+ = \mathfrak{S}(i)(\mathfrak{p}) = \mathfrak{S}(i)(\mathfrak{q}) = \mathfrak{q} \cap \Delta_+ = \mathfrak{q}_+ \cap \Delta_+\,.$$

In particular, we have $\mathfrak{p}_+ \cap \Delta = \mathfrak{q}_+ \cap \Delta$. By (2), we obtain $\mathfrak{p}_+ \cap \Phi = \mathfrak{q}_+ \cap \Phi$, which implies that

$$\mathfrak{p} = \mathfrak{p}_+ \cap \Phi_+ = \mathfrak{q}_+ \cap \Phi_+ = \mathfrak{q}\,.$$ □

As an application we prove the intuitively obvious fact that, if there are more formulae than types, many formulae have to be equivalent.

Proposition 3.13. *Let L be a boolean closed logic such that $\mathfrak{S}(L)$ is compact. There exists a retract L_0 of L of size $|L_0| \leq |S(L)| \oplus \aleph_0$.*

Proof. Let $(\mathfrak{p}_\alpha)_{\alpha<\kappa}$ be an enumeration of $S(L)$ without repetitions. For every pair of indices $\alpha, \beta < \kappa$, $\alpha \neq \beta$, fix a formula $\psi_{\alpha\beta} \in \mathfrak{p}_\alpha \setminus \mathfrak{p}_\beta$. Set $\Psi := \{ \psi_{\alpha\beta} \mid \alpha, \beta < \kappa \}$ and let L_0 be the set of all finite boolean combinations of formulae in Ψ. Then $|L_0| \leq \kappa \otimes \kappa \otimes \aleph_0 \leq \kappa \oplus \aleph_0$ and

$$\mathfrak{p}_\alpha \cap \Psi = \mathfrak{p}_\beta \cap \Psi \quad \text{implies} \quad \mathfrak{p}_\alpha = \mathfrak{p}_\beta \,.$$

Therefore, Corollary 3.12 implies that L_0 is a retract of L. ☐

Corollary 3.14. *Let $T \subseteq \mathrm{FO}^0[\Sigma]$ be a first-order theory. There exists a subset $\Sigma_0 \subseteq \Sigma$ of size $|\Sigma_0| \leq |S^{<\omega}(T)|$ and a family of formulae $\varphi_\xi(\bar{x})$, for $\xi \in \Sigma \setminus \Sigma_0$, such that, for every model $\mathfrak{M}$ of T,*

$$\xi^{\mathfrak{M}} = \varphi_\xi^{\mathfrak{M}|_{\Sigma_0}}\,, \quad \textit{for all } \xi \in \Sigma \setminus \Sigma_0 \,.$$

Proof. For each finite tuple $\bar{s}$ of sorts, we can use Proposition 3.13 to obtain a retract $\Delta_{\bar{s}}$ of $\mathrm{FO}^{\bar{s}}[\Sigma]/T$ such that $|\Delta_{\bar{s}}| \leq |S^{\bar{s}}(T)|$. Let Σ_0 be the set of all symbols from Σ that appear in some $\Delta_{\bar{s}}$. Note that $S^{\bar{s}}(T) \neq \varnothing$ implies that

$$|S^{<\omega}(T)| = \Big|\bigcup_{\bar{s}} S^{\bar{s}}(T)\Big| \geq \aleph_0 \,.$$

Hence,

$$|\Sigma_0| \leq \sum_{\bar{s}} |\Delta_{\bar{s}}| \oplus \aleph_0 = |S^{<\omega}(T)| \oplus \aleph_0 = |S^{<\omega}(T)| \,.$$

Furthermore, for every relation symbol $R \in \Sigma \setminus \Sigma_0$ of type $\bar{s}$, there exists a formula $\varphi_R(\bar{x}) \in \Delta_{\bar{s}} \subseteq \mathrm{FO}^{\bar{s}}[\Sigma_0]$ such that $R\bar{x} \equiv \varphi_R(\bar{x})$. Similarly, for every function symbol $f \in \Sigma \setminus \Sigma_0$ of type $\bar{s} \to t$, there exists a formula $\varphi_f(\bar{x}, y) \in \Delta_{\bar{s}t} \subseteq \mathrm{FO}^{\bar{s}t}[\Sigma_0]$ such that $f\bar{x} = y \equiv \varphi_f(\bar{x}, y)$. ☐

4. Local type spaces

For technical reasons we will consider in the next section certain quotients of first-order type spaces $\mathfrak{S}^{\bar{s}}(U)$. To define these quotients we consider a restriction $L|_\Delta$ of some logic L and we equip the corresponding set of types $S(L|_\Delta)$ with a topology that is finer than the usual one. Our aim is to show that, for first-order logic, this topology coincides with the usual one. For simplicity, we only consider logics L that are closed under disjunction.

Definition 4.1. Let L be a logic that is closed under disjunction, $\Delta \subseteq L$ a fragment, and let $i : \Delta \to L$ be the inclusion morphism. We denote by $\mathfrak{S}_\Delta(L)$ the topological space with universe $S(\Delta)$ where the topology consists of all sets

$$\langle \Phi \rangle_\Delta := \mathfrak{S}(i)[\langle \Phi \rangle_L]\,, \quad \text{for } \Phi \subseteq L \,.$$

Lemma 4.2. *Let L be a logic that is closed under disjunctions, $\Delta \subseteq L$, and let $i : \Delta \to L$ be the inclusion morphism.*

(a) *The restriction function*

$$\rho_\Delta : \mathfrak{S}(L) \to \mathfrak{S}_\Delta(L) : \mathfrak{p} \mapsto \mathfrak{p} \cap \Delta$$

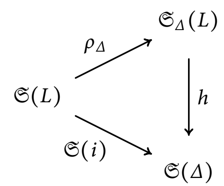
is closed and surjective.

(b) *The identity function*

$$h : \mathfrak{S}_\Delta(L) \to \mathfrak{S}(\Delta) : \mathfrak{p} \mapsto \mathfrak{p}$$

is continuous and bijective.

(c) *$\mathfrak{S}_\Delta(L) = \mathfrak{S}(\Delta)$ if, and only if, $\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(\Delta)$ is closed and ρ_Δ is continuous.*



Proof. First, note that $\mathfrak{S}(i) = h \circ \rho_\Delta$ since

$$h(\rho_\Delta(\mathfrak{p})) = h(\mathfrak{p} \cap \Delta) = \mathfrak{p} \cap \Delta = \mathfrak{S}(i)(\mathfrak{p}), \quad \text{for every } \mathfrak{p} \in S(L).$$

(a) Since L is closed under disjunctions, each closed set of $\mathfrak{S}(L)$ is of the form $\langle \Phi \rangle_L$, for some $\Phi \subseteq L$. The function ρ_Δ is closed since, for every $\Phi \subseteq L$,

$$\rho_\Delta[\langle \Phi \rangle_L] = \{ \mathfrak{p} \cap \Delta \mid \mathfrak{p} \in \langle \Phi \rangle_L \} = \big\{ \mathfrak{S}(i)(\mathfrak{p}) \bigm| \mathfrak{p} \in \langle \Phi \rangle_L \big\} = \langle \Phi \rangle_\Delta$$

is a closed set of $\mathfrak{S}(\Delta)$.

For surjectivity, note that h^{-1} and $\mathfrak{S}(i)$ are both surjective. Therefore, so is $\rho_\Delta = h^{-1} \circ \mathfrak{S}(i)$.

(b) h is clearly bijective. For continuity, note that $h \circ \rho_\Delta = \mathfrak{S}(i)$. Since $\mathfrak{S}(i)$ is surjective, it follows by Lemma A2.1.10 for a closed set $C \subseteq S(\Delta)$ that

$$\begin{aligned} h^{-1}[C] &= h^{-1}[\mathfrak{S}(i)[\mathfrak{S}(i)^{-1}[C]]] \\ &= h^{-1}[h[\rho_\Delta[\mathfrak{S}(i)^{-1}[C]]]] = \rho_\Delta[\mathfrak{S}(i)^{-1}[C]]. \end{aligned}$$

This set is closed, since $\mathfrak{S}(i)$ is continuous and ρ_Δ is closed.

(c) ($\Rightarrow$) If $\mathfrak{S}_\Delta(L) = \mathfrak{S}(\Delta)$, then h is a homeomorphism. Hence, $\rho_\Delta = h^{-1} \circ \mathfrak{S}(i)$ is a composition of continuous functions and, therefore, continuous. Similarly, $\mathfrak{S}(i) = h \circ \rho_\Delta$ is a composition of closed functions and, therefore, closed.

($\Leftarrow$) It is sufficient to show that the identity function

$$h : \mathfrak{S}_\Delta(L) \to \mathfrak{S}(\Delta) : \mathfrak{p} \mapsto \mathfrak{p}$$

is a homeomorphism. We have already seen in (b) that it is bijective and continuous. Hence, it remains to prove that h is closed.

By assumption, ρ_Δ is continuous and $\mathfrak{S}(i)$ is closed. It follows as in (b) that

$$h[C] = \mathfrak{S}(i)[\rho_\Delta^{-1}[C]]$$

is closed, for every closed set $C \subseteq S_\Delta(L)$. ◻

In the applications below we are interested in the case where $\mathfrak{S}(L)$ is compact and Δ closed under negation. In this situation the topologies of $\mathfrak{S}_\Delta(L)$ and $\mathfrak{S}(\Delta)$ coincide.

Theorem 4.3. *Let L be a boolean closed logic such that $\mathfrak{S}(L)$ is compact and let $\Delta \subseteq L$.*

(a) *The restriction function*

$$\rho_\Delta : \mathfrak{S}(L) \to \mathfrak{S}_\Delta(L) : \mathfrak{p} \mapsto \mathfrak{p} \cap \Delta$$

is continuous, closed, and surjective.

(b) *If Δ is closed under negation, then $\mathfrak{S}_\Delta(L) = \mathfrak{S}(\Delta)$.*

Proof. (a) We have already seen in Lemma 4.2 (a) that ρ_Δ is closed and surjective. Hence, it remains to prove that it is continuous.

Let Δ_+ be the set of all finite boolean combinations of formulae in Δ. We claim that

$$\rho_\Delta^{-1}[\langle \Phi \rangle_\Delta] = \langle \Phi^\vDash \cap \Delta_+ \rangle_L.$$

($\subseteq$) Let $\mathfrak{p} \in \rho_\Delta^{-1}[\langle \Phi \rangle_\Delta]$. Then $\mathfrak{p} \cap \Delta = \rho_\Delta(\mathfrak{p}) \in \langle \Phi \rangle_\Delta$ and there is some type $\mathfrak{q} \in \langle \Phi \rangle_L$ with $\mathfrak{q} \cap \Delta = \mathfrak{p} \cap \Delta$. Since every formula in $\mathfrak{q} \cap \Delta_+$ is a boolean combination of formulae in $\mathfrak{q} \cap \Delta$, it follows that $\mathfrak{q} \cap \Delta_+ = \mathfrak{p} \cap \Delta_+$. Hence,

$$\Phi^\vDash \subseteq \mathfrak{q} \quad \text{implies} \quad \Phi^\vDash \cap \Delta_+ \subseteq \mathfrak{q} \cap \Delta_+ = \mathfrak{p} \cap \Delta_+.$$

Consequently, $\mathfrak{p} \in \langle \Phi^\vDash \cap \Delta_+ \rangle_L$.

($\supseteq$) Let $\mathfrak{p} \in \langle \Phi^\vDash \cap \Delta_+ \rangle_L$ and set $\mathfrak{p}_o := \mathfrak{p} \cap \Delta_+$. If there is some $\mathfrak{q} \in S(L)$ with $\Phi \cup \mathfrak{p}_o \subseteq \mathfrak{q}$, then

$$\mathfrak{q} \cap \Delta_+ = \mathfrak{p}_o \quad \text{implies} \quad \rho_\Delta(\mathfrak{p}) = \mathfrak{p} \cap \Delta = \mathfrak{p}_o \cap \Delta = \mathfrak{q} \cap \Delta \in \langle \Phi \rangle_\Delta.$$

Hence, $\mathfrak{p} \in \rho_\Delta^{-1}[\langle \Phi \rangle_\Delta]$. Consequently, it remains to show that $\Phi \cup \mathfrak{p}_o$ is satisfiable.

For a contradiction, suppose otherwise. Then

$$\langle \Phi \rangle_L \cap \bigcap_{\psi \in \mathfrak{p}_0} \langle \psi \rangle_L = \langle \Phi \cup \mathfrak{p}_0 \rangle_L = \varnothing .$$

Since $\mathfrak{S}(L)$ is compact, we can find a finite subset $\Psi \subseteq \mathfrak{p}_0$ such that

$$\langle \Phi \rangle_L \cap \bigcap_{\psi \in \Psi} \langle \psi \rangle_L = \varnothing .$$

Hence, $\Phi \vDash \neg \bigwedge \Psi$. Note that $\Psi \subseteq \Delta_+$ implies $\neg \bigwedge \Psi \in \Delta_+$. Hence, $\neg \bigwedge \Psi \in \Phi^{\vDash} \cap \Delta_+ \subseteq \mathfrak{p}_0$ and $\mathfrak{p}_0$ is inconsistent. A contradiction.

(b) We have seen in (a) that ρ_Δ is continuous. By Lemma 4.2 (c), it is therefore sufficient to show that $\mathfrak{S}(i) : \mathfrak{S}(L) \to \mathfrak{S}(\Delta)$ is closed. Let $\Delta_+ \subseteq L$ be the set of all finite boolean combinations of formulae in Δ, and let $i_0 : \Delta \to \Delta_+$ and $i_+ : \Delta_+ \to L$ be the corresponding inclusion morphisms. Then $\mathfrak{S}(i_+)$ is closed by Lemma 2.16, and $\mathfrak{S}(i_0)$ is closed by Lemma 3.5. Hence, $\mathfrak{S}(i) = \mathfrak{S}(i_0) \circ \mathfrak{S}(i_+)$ is also closed. □

We will mainly use type spaces of the form $\mathfrak{S}_\Delta(L)$ in the case of first-order logic. In this case the definitions are as follows.

Definition 4.4. Let $T \subseteq \mathrm{FO}^0[\Sigma]$ be a theory and $\Delta \subseteq \mathrm{FO}[\Sigma, X \cup Y]$ a set of formulae where X and Y are disjoint sets of variables. For a set U of parameters, we set

$$\begin{aligned} \Delta_U^{\neg} := \{\, \varphi(\bar{x};\bar{c}) \mid \varphi(\bar{x};\bar{y}) \in \Delta,\ \bar{x} \subseteq X,\ \bar{y} \subseteq Y,\ \bar{c} \subseteq U \,\} \\ \cup \{\, \neg\varphi(\bar{x};\bar{c}) \mid \varphi(\bar{x};\bar{y}) \in \Delta,\ \bar{x} \subseteq X,\ \bar{y} \subseteq Y,\ \bar{c} \subseteq U \,\} . \end{aligned}$$

(a) A partial type $\mathfrak{p}$ over a set U is a *Δ-type* if $\mathfrak{p} \subseteq \Delta_U^{\neg}$. For $\Delta = \{\varphi\}$ we simply speak of a φ-type.

(b) The *restriction* of a partial type $\mathfrak{p}$ is the type

$$\mathfrak{p}|_\Delta := \mathfrak{p} \cap \Delta_U^{\neg} .$$

(c) Let $\mathfrak{M}$ be a structure. The *Δ-type* of a tuple $\bar{a} \subseteq M$ over a set $U \subseteq M$ is

$$\mathrm{tp}_\Delta(\bar{a}/U) := \mathrm{tp}(\bar{a}/U)|_\Delta .$$

(d) A Δ-type $\mathfrak{p}$ over U is *complete* if, for every formula $\varphi(\bar{x};\bar{y}) \in \Delta$ and each tuple $\bar{c} \subseteq U$, we have $\varphi(\bar{x};\bar{c}) \in \mathfrak{p}$ or $\neg\varphi(\bar{x};\bar{c}) \in \mathfrak{p}$.

(e) The space of all complete Δ-types over U is

$$\mathfrak{S}_\Delta(U) := \mathfrak{S}_{\Delta_U^{\neg}}\big(\mathrm{FO}[\Sigma_U, X]/T(U)\big) .$$

As usual we also write $\mathfrak{S}_\Delta(T)$ for $\mathfrak{S}_\Delta(\varnothing)$.

Since first-order type spaces are compact, it follows by the above results that $\mathfrak{S}_\Delta(U)$ is equal to $\mathfrak{S}((\mathrm{FO}[\Sigma_U, X]/T(U))|_{\Delta_U^{\neg}})$. Our aim is to show that this definition does not depend on the signature Σ.

Theorem 4.5. *Let $T \subseteq \mathrm{FO}^0[\Sigma]$ be a theory and $\Delta \subseteq \mathrm{FO}[\Sigma, X \cup Y]$ a set of formulae where X and Y are disjoint sets of variables. For a set U of parameters, set*

$$\begin{aligned} \Delta_U^{\neg} := \{\, \varphi(\bar{x};\bar{c}) \mid \varphi(\bar{x};\bar{y}) \in \Delta,\ \bar{c} \subseteq U \,\} \\ \cup \{\, \neg\varphi(\bar{x};\bar{c}) \mid \varphi(\bar{x};\bar{y}) \in \Delta,\ \bar{c} \subseteq U \,\} . \end{aligned}$$

(a) $\mathfrak{S}_\Delta(U) = \mathfrak{S}\big((\mathrm{FO}[\Sigma_U, X]/T(U))\,|_{\Delta_U^{\neg}}\big)$.

(b) *If $\Delta \subseteq \mathrm{FO}[\Sigma_0, X_0 \cup Y]$, for some $\Sigma_0 \subseteq \Sigma$ and $X_0 \subseteq X$, then*

$$\mathfrak{S}_\Delta(T) = \mathfrak{S}_\Delta(T \cap \mathrm{FO}^0[\Sigma_0]) ,$$

where the local type space on the left-hand side is with respect to the logic $\mathrm{FO}[\Sigma, X]$ and the one one the right-hand side with respect to $\mathrm{FO}[\Sigma_0, X_0]$.

Proof. (a) follows by Theorem 4.3 (b), while (b) follows from (a) and Proposition 2.21 (a) (treating the free variables from X and X_0 as constant symbols). □

Corollary 4.6. *Let $T \subseteq \mathrm{FO}^0[\Sigma]$ be a theory, $\Delta \subseteq \mathrm{FO}[\Sigma, X \cup Y]$ a set of formulae, U a set of parameters, and $\Delta_U^{\neg}$ the set from Definition 4.4. Then*

$$\mathfrak{S}_\Delta(U) \cong \mathrm{spec}(\mathfrak{Lb}(\Delta_U^{\neg})) ,$$

where $\mathfrak{Lb}(\Delta_U^{\neg})$ denotes the subalgebra of $\mathfrak{Lb}(\mathrm{FO}[\Sigma_U, X]/T(U))$ generated by $\Delta_U^{\neg}$.

Proof. Set $L := \mathrm{FO}[\Sigma_U, X]/T(U)$ and let Δ^+ be the boolean closure of $\Delta_U^{\neg}$. By Theorems 4.5 (a) and 2.18 and Lemma 3.5, it follows that

$$\mathfrak{S}_\Delta(U) = \mathfrak{S}(\Delta_U^{\neg}) \cong \mathfrak{S}(\Delta^+) \cong \operatorname{spec}(\mathfrak{Lb}(\Delta^+)) = \operatorname{spec}(\mathfrak{Lb}(\Delta_U^{\neg}))\,. \qquad \square$$

5. Stable theories

In this section we consider the size of first-order type spaces. First, let us state two trivial bounds.

Lemma 5.1. *Let T be a complete first-order theory and $\bar{s}$ a sequence of sorts. Then*

$$|U| \leq |S^{\bar{s}}(U)| \leq 2^{|T| \oplus |U| \oplus |\bar{s}|}, \quad \textit{for every set } U \textit{ of parameters.}$$

One situation where the size of a type space is important is when we want to construct a model realising all types. First note that we can use the Compactness Theorem to show that, for every structure $\mathfrak{A}$, we can add a tuple realising any given type $\mathfrak{p}$ over a subset $U \subseteq A$.

Lemma 5.2. *Let $\mathfrak{A}$ be a Σ-structure, $U \subseteq A$, and $\mathfrak{p} \in S^\alpha(U)$. There exists an elementary extension $\mathfrak{B} \succeq \mathfrak{A}$ of size $|B| \leq |A| \oplus |\Sigma| \oplus |\alpha| \oplus \aleph_0$ in which $\mathfrak{p}$ is realised.*

Proof. Let $\Phi := \mathfrak{p} \cup \mathrm{Th}(\mathfrak{A}_A)$. We regard the free variables x_i, $i < \alpha$, of $\mathfrak{p}$ as constant symbols. If Φ is satisfiable then, by the Theorem of Löwenheim and Skolem, there exists a model $\mathfrak{B} \models \Phi$ of size $|B| \leq |A| \oplus |\Sigma| \oplus |\alpha| \oplus \aleph_0$. Furthermore, we have $\mathfrak{B} \succeq \mathfrak{A}$, by Lemma c2.2.3, and there exists some $\bar{a} \in B^\alpha$ with $\operatorname{tp}(\bar{a}/U) = \mathfrak{p}$.

Hence, it is sufficient to show that Φ is satisfiable. Let $\Phi_0 \subseteq \Phi$ be finite. We write

$$\bigwedge \Phi_0 = \varphi(\bar{x}, \bar{a}) \wedge \psi(\bar{a}, \bar{b})$$

where $\bar{a} \subseteq U$, $\bar{b} \subseteq A \setminus U$, $\mathfrak{p} \models \varphi(\bar{x}, \bar{a})$, and $\mathfrak{A} \models \psi(\bar{a}, \bar{b})$. The last statement implies that $\exists \bar{y} \psi(\bar{a}, \bar{y}) \in \mathrm{Th}(\mathfrak{A}_U)$. By definition of a type, there exists a model $\mathfrak{C} \models \mathfrak{p} \cup \mathrm{Th}(\mathfrak{A}_U)$. In particular, we have

$$\mathfrak{C} \models \varphi(\bar{x}, \bar{a}) \wedge \exists \bar{y} \psi(\bar{a}, \bar{y})\,.$$

Choose a tuple $\bar{c} \subseteq C$ such that $\mathfrak{C} \models \psi(\bar{a}, \bar{c})$. We obtain a model of Φ_0 by interpreting the constant symbol b_i by the element c_i, for every i. $\square$

Corollary 5.3. *For every Σ-structure $\mathfrak{A}$, there exists an elementary extension $\mathfrak{B} \succeq \mathfrak{A}$ of size at most $|S^{<\omega}(A)|$ in which every type $\mathfrak{p} \in S^{<\omega}(A)$ is realised.*

Proof. According to Corollary 3.14, we can find a signature $\Sigma_0 \subseteq \Sigma_A$ of size $|\Sigma_0| \leq |S^{<\omega}(A)|$ such that there exists a retraction

$$\langle \alpha, \beta \rangle : \mathrm{FO}^{<\omega}[\Sigma_A]/T(A) \to \mathrm{FO}^{<\omega}[\Sigma_0]/T_0\,,$$

where $T_0 := T(A) \cap \mathrm{FO}^0[\Sigma_0]$. If we can show that there exists a model $\mathfrak{B}$ of T_0 realising every type in $S^{<\omega}(T_0)$, it follows that its expansion $\beta(\mathfrak{B})$ is a model of $T(A)$ realising every type in $S^{<\omega}(A)$. Therefore, we may assume without loss of generality that $|\Sigma| \leq |S^{<\omega}(A)|$.

Fix an enumeration $(\mathfrak{p}_\alpha)_{\alpha<\kappa}$ of $S^{<\omega}(A)$. We can use Lemma 5.2 to find, for every $\alpha < \kappa$, an elementary extension $\mathfrak{C}_\alpha \succeq \mathfrak{A}$ realising $\mathfrak{p}_\alpha$. By Lemma c2.5.7, there exists a common elementary extension $\mathfrak{C}$ of all $\mathfrak{C}_\alpha$. It follows that $\mathfrak{C}$ realises every type $\mathfrak{p}_\alpha$. By the Theorem of Löwenheim and Skolem, we can find an elementary substructure $\mathfrak{A} \preceq \mathfrak{B} \preceq \mathfrak{C}$ of size at most $|S^{<\omega}(A)| \oplus |\Sigma| \oplus \aleph_0$ such that every $\mathfrak{p}_\alpha$ is realised in $\mathfrak{B}$. Since $S^{<\omega}(A) = \bigcup_{n<\omega} S^n(A)$ is infinite, we have $|S^{<\omega}(A)| \oplus |\Sigma| \oplus \aleph_0 = |S^{<\omega}(A)|$ and the claim follows. $\square$

The number of different types a theory possesses also serves as a rough measure of its complexity. Intuitively, if there are only a few types the number of different configurations that can appear in a model is small. Before considering full type spaces $\mathfrak{S}^{\bar{s}}(U)$, we start by looking at those of the form $\mathfrak{S}_\varphi(U)$.

Definition 5.4. Let T be a complete first-order theory and κ an infinite cardinal. A formula $\varphi(\bar{x};\bar{y})$ is *κ-stable* (with respect to T) if we have $|S_\varphi(U)| \leq \kappa$, for all sets U of size $|U| \leq \kappa$. We call $\varphi(\bar{x};\bar{y})$ *stable* if it is κ-stable, for some infinite cardinal κ. Otherwise, $\varphi(\bar{x};\bar{y})$ is *unstable.*

Example. If $\sim$ is an equivalence relation with infinitely many classes, then the formula $x \sim y$ is κ-stable, for all infinite κ, since

$$|S_{x\sim y}(U)| = |U/{\sim}| \oplus 1 \leq |U| \oplus 1 .$$

The definition does not tell us much about stable formulae. We will therefore present three equivalent characterisations, two combinatorial ones that can be checked more easily, and one logical characterisation.

The equivalence proofs rest on two combinatorial results. The first one is a special case of the Theorem of Ramsey. We will prove the full version in Section E5.1 below.

Lemma 5.5. *Let $(a_n)_{n<\omega}$ be a sequence of elements and let $(B_n)_{n<\omega}$ be a sequence of sets. There exists an infinite set $I \subseteq \omega$ such that either*

$$a_i \in B_k\,, \quad \text{for all } i < k \text{ in } I\,,$$
or
$$a_i \notin B_k\,, \quad \text{for all } i < k \text{ in } I\,.$$

Proof. We construct an increasing sequence $n_0 < n_1 < \cdots$ of indices, a sequence $m_0, m_1, \ldots \in [2]$ of numbers, and a decreasing sequence $J_0 \supseteq J_1 \supseteq \cdots$ of infinite sets such that, for every $i < \omega$, we have $n_i \in J_i$ and either

$$m_i = 0 \quad \text{and} \quad a_{n_i} \notin B_k\,, \quad \text{for all } k \in J_{i+1}\,,$$
or
$$m_i = 1 \quad \text{and} \quad a_{n_i} \in B_k\,, \quad \text{for all } k \in J_{i+1}\,.$$

We start with $n_0 := 0$ and $J_0 := \omega$. By induction, suppose that we have already defined n_i and J_i. Set

$$L_0 := \{\, k \in J_i \mid a_{n_i} \notin B_k \,\} \quad \text{and} \quad L_1 := \{\, k \in J_i \mid a_{n_i} \in B_k \,\}\,.$$

Then $J_i = L_0 \cup L_1$. As J_i is infinite, at least one of L_0 and L_1 must also be infinite. Choose $m_i < 2$ such that L_{m_i} is infinite. We set

$$J_{i+1} := L_{m_i} \setminus [n_i + 1] \quad \text{and} \quad n_{i+1} := \min J_{i+1}\,.$$

Having defined $(n_i)_{i<\omega}$, $(m_i)_{i<\omega}$, and $(J_i)_{i<\omega}$, we consider the sets

$$M_0 := \{\, i < \omega \mid m_i = 0 \,\} \quad \text{and} \quad M_1 := \{\, i < \omega \mid m_i = 1 \,\}\,.$$

Note that $n_j \in J_j \subseteq J_i$ implies that

$$a_{n_i} \notin B_{n_j}\,, \quad \text{for all } i < j \text{ in } M_0\,,$$
and
$$a_{n_i} \in B_{n_j}\,, \quad \text{for all } i < j \text{ in } M_1\,.$$

Since $M_0 \cup M_1 = \omega$, at least one of M_0 and M_1 is infinite. If M_0 is infinite, we can therefore set $I := \{\, n_i \mid i \in M_0 \,\}$. Otherwise, we use $I := \{\, n_i \mid i \in M_1 \,\}$. □

Theorem 5.6 (Erdős, Makkai). *Let X be an infinite set and $S \subseteq \wp(X)$ a family of size $|S| > |X|$. Then there are sequences $(a_i)_{i<\omega}$ in X and $(B_i)_{i<\omega}$ in S such that either*

$$a_i \in B_k \quad \text{iff} \quad i \leq k\,, \quad \text{for all } i, k < \omega\,,$$
or
$$a_i \in B_k \quad \text{iff} \quad i \geq k\,, \quad \text{for all } i, k < \omega\,.$$

Proof. For every pair of disjoint finite subsets $Y, Z \subseteq X$, choose, if possible, a set $B \in S$ with $Y \subseteq B$ and $Z \subseteq X \setminus B$. Let $S_0 \subseteq S$ be the set of the chosen subsets B. As there are only $|X^{<\omega} \times X^{<\omega}| = |X|$ pairs of finite subsets, it follows that $|S_0| \leq |X| < |S|$. Consequently, there exists a set $A \in S$ that cannot be expressed as a finite boolean combination of sets from S_0. (We allow empty boolean combinations, so that A is different from $\varnothing$ and X.)

We inductively construct sequences $(c_n)_{n<\omega}$ in A, $(d_n)_{n<\omega}$ in $X \setminus A$, and $(B_n)_{n<\omega}$ in S_0 such that, for all n,

- $\{c_0, \ldots, c_n\} \subseteq B_n$,

- $\{d_0, \dots, d_n\} \subseteq X \setminus B_n$, and
- $c_i \in B_n \Leftrightarrow d_i \in B_n$, for all $i > n$.

For the inductive step, suppose that we have already defined elements $c_0, \dots, c_{n-1}, d_0, \dots, d_{n-1}$, and sets $B_0, \dots, B_{n-1}$. Since A is not a boolean combination of $B_0, \dots, B_{n-1}$, there are elements $c_n \in A$ and $d_n \in X \setminus A$ such that

$$c_n \in B_k \quad \text{iff} \quad d_n \in B_k\,, \quad \text{for all } k < n\,.$$

Then $\{c_0, \dots, c_n\} \subseteq A$ and $\{d_0, \dots, d_n\} \subseteq X \setminus A$. By choice of S_0, it follows that we can choose a set $B_n \in S_0$ with $\{c_0, \dots, c_n\} \subseteq B_n$ and $\{d_0, \dots, d_n\} \subseteq X \setminus B_n$. This concludes the inductive step.

We have constructed sequences such that

$$c_i \in B_k \quad \text{and} \quad d_i \notin B_k\,, \quad \text{for } i \leq k\,,$$

$$c_i \in B_k \Leftrightarrow d_i \in B_k\,, \qquad \text{for } i > k\,.$$

By Lemma 5.5, there exists an infinite subset $I \subseteq \omega$ such that either

- $c_i \notin B_k$, for all indices $i > k$ in I, or
- $c_i \in B_k$, for all indices $i > k$ in I.

In the first case, the sequences $(c_n)_{n \in I}$ and $(B_n)_{n \in I}$ satisfy

$$c_i \in B_k \quad \text{iff} \quad i \leq k\,, \quad \text{for all } i, k \in I\,.$$

In the second case, the sequences $(d_n)_{n \in I}$ and $(B_n)_{n \in I}$ satisfy

$$d_i \in B_k \quad \text{iff} \quad i > k\,, \quad \text{for all } i, k \in I\,.$$

Shifting the sequence $(d_i)_{i \in I}$ by one, we obtain the desired sequences $(a_i)_{i < \omega}$ and $(B_i)_{i < \omega}$. □

Using these two results, we can present our characterisations. We introduce each in turn, before proving that they are all equivalent to (un-)stability. The first combinatorial characterisation is based on the non-existence of a definable linear order.

Definition 5.7. Let T be a theory. A formula $\varphi(\bar{x}, \bar{y})$ has the *order property* (with respect to T) if there exists a model $\mathfrak{M} \vDash T$ containing two sequences $(\bar{a}^n)_{n < \omega}$ and $(\bar{b}^n)_{n < \omega}$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}^i, \bar{b}^k) \quad \text{iff} \quad i \leq k\,.$$

Using compactness we obtain several equivalent definitions of the order property.

Lemma 5.8. *Let T be a complete first-order theory and $\varphi(\bar{x}, \bar{y})$ a formula. The following statements are equivalent.*

(1) *φ has the order property with respect to T.*

(2) *For every linear order $\langle I, \leq \rangle$, there exists a model $\mathfrak{M}$ of T that contains sequences $(\bar{a}^i)_{i \in I}$ and $(\bar{b}^i)_{i \in I}$ such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^i, \bar{b}^k) \quad \text{iff} \quad i \leq k\,.$$

(3) *For every model $\mathfrak{M}$ of T and all finite linear orders $\langle I, \leq \rangle$, there are sequences $(\bar{a}^i)_{i \in I}$ and $(\bar{b}^i)_{i \in I}$ in M such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^i, \bar{b}^k) \quad \text{iff} \quad i \leq k\,.$$

Proof. (2) ⇒ (1) The claim follows from (2) if we set $I = \omega$.

(3) ⇒ (2) This is a direct application of the Compactness Theorem. Given I, choose new constant symbols $\bar{c}^i$ and $\bar{d}^i$, for $i \in I$, and define

$$\begin{aligned} \Phi := T &\cup \{\, \varphi(\bar{c}^i, \bar{d}^k) \mid i, k \in I,\ i \leq k \,\} \\ &\cup \{\, \neg\varphi(\bar{c}^i, \bar{d}^k) \mid i, k \in I,\ i > k \,\}\,. \end{aligned}$$

Clearly, every model of Φ contains two sequences with the desired properties. Hence, it remains to prove that Φ is satisfiable. By the Compactness Theorem, we only have to show that every finite subset of Φ has a model. Let $\Phi_0 \subseteq \Phi$ be finite. Then there exists a finite subset $I_0 \subseteq I$ such that

$$\begin{aligned} \Phi_0 \subseteq T &\cup \{\, \varphi(\bar{c}^i, \bar{d}^k) \mid i, k \in I_0,\ i \leq k \,\} \\ &\cup \{\, \neg\varphi(\bar{c}^i, \bar{d}^k) \mid i, k \in I_0,\ i > k \,\}\,. \end{aligned}$$

Let $\mathfrak{M}$ be an arbitrary model of T. By (3), we can find sequences $(\bar a^i)_{i \in I_0}$ and $(\bar b^i)_{i \in I_0}$ in M such that

$$\mathfrak{M} \vDash \varphi(\bar a^i, \bar b^k) \quad \text{iff} \quad i \leq k\,.$$

Consequently, we can satisfy Φ_0 in the structure $\mathfrak{M}$ if we interpret the constants $\bar c^i$ by $\bar a^i$ and the constants $\bar d^i$ by $\bar b^i$.

(1) ⇒ (3) Fix a model $\mathfrak{N}$ of T that contains sequences $(\bar a^n)_{n<\omega}$ and $(\bar b^n)_{n<\omega}$ such that

$$\mathfrak{N} \vDash \varphi(\bar a^i, \bar b^k) \quad \text{iff} \quad i \leq k\,.$$

Consider the formula

$$\psi_m := \exists \bar x_0 \cdots \exists \bar x_{m-1} \exists \bar y_0 \cdots \exists \bar y_{m-1} \bigwedge_{i<k} [\varphi(\bar x_i, \bar y_k) \wedge \varphi(\bar x_i, \bar x_i) \wedge \neg\varphi(\bar x_k, \bar y_i)]\,.$$

Suppose that $|I| = m < \omega$ and let $\mathfrak{M}$ be an arbitrary model of T. Since $\mathfrak{N} \vDash \varphi_m$ we have $T \vDash \varphi_m$ which, in turn, implies that $\mathfrak{M} \vDash \varphi_m$. Consequently, $\mathfrak{M}$ contains two finite sequences $(\bar a^n)_{n<m}$ and $(\bar b^n)_{n<m}$ with the desired properties. □

The second combinatorial characterisation is based on the non-existence of certain trees.

Definition 5.9. Let T be a complete first-order theory, $\varphi(\bar x; \bar y)$ a formula, U a set of parameters, and γ an ordinal. A *φ-tree of height γ over U* is a family $(\bar c_w)_{w \in 2^{<\gamma}}$ of parameters $\bar c_w \subseteq U$ such that, for every $\eta \in 2^\gamma$, the set

$$T(U) \cup \{\, \varphi^{\eta(\alpha)}(\bar x; \bar c_{\eta\restriction\alpha}) \mid \alpha < \gamma \,\}$$

is consistent, where

$$\varphi^0(\bar x; \bar y) := \varphi(\bar x; \bar y) \quad \text{and} \quad \varphi^1(\bar x; \bar y) := \neg\varphi(\bar x; \bar y)\,.$$

Lemma 5.10. *Let T be a first-order theory and $\varphi(\bar x; \bar y)$ a formula such that, for every $n < \omega$, there exists a model of T containing a φ-tree of height n. Then, for every ordinal γ, there exists a model of T containing a φ-tree of height γ.*

Proof. Given γ, set

$$\Phi_\gamma := T \cup \{\, \varphi^{\eta(\alpha)}(\bar x_\eta; \bar y_{\eta\restriction\alpha}) \mid \alpha < \gamma,\ \eta \in 2^\gamma \,\}\,.$$

If this set is satisfiable, there exists a model of T containing elements $\bar a_\eta$ and $\bar c_w$, for $\eta \in 2^\gamma$ and $w \in 2^{<\gamma}$, such that every $\bar a_\eta$ satisfies

$$T(\textstyle\bigcup_w \bar c_w) \cup \{\, \varphi^{\eta(\alpha)}(\bar x; \bar c_{\eta\restriction\alpha}) \mid \alpha < \gamma \,\}\,.$$

Hence, $(\bar c_w)_{w \in 2^{<\gamma}}$ is a φ-tree of height γ.

It therefore remains to show that Φ_γ is satisfiable. By the Compactness Theorem, it is sufficient to prove that every finite subset is satisfiable. Hence, consider a finite set $\Psi \subseteq \Phi_\gamma$. Let $\alpha_0 < \cdots < \alpha_{n-1}$ be an enumeration of all ordinals α such that Ψ contains a formula of the form $\varphi^{\eta(\alpha)}(\bar x_\eta; \bar y_{\eta\restriction\alpha})$ and let $\sigma : 2^{\leq\gamma} \to 2^{\leq n}$ be the function mapping a sequence $\eta \in 2^\beta$ of length $\beta \leq \gamma$ to its restriction $\langle \eta(\alpha_0), \dots, \eta(\alpha_k) \rangle$, where $k < n$ is the maximal index such that $\alpha_k < \beta$. By assumption, there exists a φ-tree $(\bar d_w)_{w \in 2^{<n}}$ of height n. For each branch $\zeta \in 2^n$, fix a tuple $\bar a_\zeta$ satisfying

$$T(\textstyle\bigcup_w d_w) \cup \{\, \varphi^{\zeta(i)}(\bar x; \bar d_{\zeta\restriction i}) \mid i < n \,\}\,.$$

Then Ψ is satisfied if we assign the value $\bar a_{\sigma(\eta)}$ to the variable $\bar x_\eta$ and the value $\bar d_{\sigma(w)}$ to the variable $\bar y_w$. □

The existence of large φ-trees implies that the local type spaces are also large. In particular, formulae with large φ-trees are unstable.

Lemma 5.11. *Let T be a complete theory and $\varphi(\bar x; \bar y)$ a formula such that there are φ-trees of height n, for all $n < \omega$. For every infinite cardinal κ there exists a set U of parameters such that $|S_\varphi(U)| > \kappa = |U|$.*

Proof. Let μ be the minimal cardinal such that $2^\mu > \kappa$. By Lemma 5.10, there exists a φ-tree $(\bar{c}_w)_{w \in 2^{<\mu}}$ of height μ. Since $2^{<\mu} \leq \kappa$, we can choose a set U of size $|U| = \kappa$ containing all parameters $\bar{c}_w$, for $w \in 2^{<\mu}$. For every branch $\eta \in 2^\mu$, fix a tuple $\bar{a}_\eta$ satisfying

$$\{ \varphi^{\eta(\alpha)}(\bar{x}; \bar{c}_{\eta\restriction\alpha}) \mid \alpha < \mu \}.$$

For $\eta \neq \zeta$, it follows that $\mathrm{tp}_\varphi(\bar{a}_\eta/U) \neq \mathrm{tp}_\varphi(\bar{a}_\zeta/U)$. Hence,

$$|S_\varphi(U)| \geq 2^\mu > \kappa = |U|. \qquad \square$$

Before proving the converse, let us present a third, logical characterisation of stability.

Definition 5.12. Let $\mathfrak{M}$ be a structure, $C, U \subseteq M$ sets of parameters, Δ a set of formulae, and $\varphi(\bar{x}; \bar{y})$ a formula.

(a) A *φ-definition* of a type $\mathfrak{p} \in S_\varphi(U)$ over C is a formula $\delta(\bar{y})$ over C such that

$$\varphi(\bar{x}; \bar{c}) \in \mathfrak{p} \quad \text{iff} \quad \mathfrak{M} \vDash \delta(\bar{c}), \quad \text{for all } \bar{c} \subseteq U.$$

(b) A complete type $\mathfrak{p} \in S_\Delta(U)$ is *definable* over C if, for every $\varphi \in \Delta$, the type $\mathfrak{p}|_\varphi$ has a φ-definition over C.

Example. Recall the example on page 529, where we described $S^1(\mathbb{Q})$ for the theory $T := \mathrm{Th}(\langle \mathbb{Q}, < \rangle)$. The definable types are those of the form (a^+), (a^-), $(+\infty)$, $(-\infty)$, and all realised types. The irrational types are not definable. For instance, for (a^+) and $\varphi(x; y) := x < y$, we can use the definition $\delta(y) := y > a$.

The number of definable types is always small.

Lemma 5.13. *Let $\varphi(\bar{x}; \bar{y}) \in \mathrm{FO}[\Sigma, X \cup Y]$. Then $S_\varphi(U)$ contains at most*

$$|\Sigma| \oplus |C| \oplus \aleph_0$$

types that are definable over C.

Proof. W.l.o.g. we may assume that X and Y are finite. Then there are $|\Sigma| \oplus |C| \oplus \aleph_0$ first-order formulae over C and, hence, at most that many φ-definitions. Furthermore, if $\mathfrak{p}, \mathfrak{q} \in S_\varphi(U)$ are types with the same φ-definition then $\mathfrak{p} = \mathfrak{q}$. $\square$

Lemma 5.14. *Let U be a set of parameters and let $\varphi(\bar{x}; \bar{y})$ be a first-order formula that has no φ-tree of height $N < \omega$. Then every φ-type in $S_\varphi(U)$ is definable over U.*

Proof. For a formula $\psi(\bar{x})$ over U, let $D_\varphi(\psi)$ be the maximal number n such that there exists a φ-tree $(\bar{c}_w)_{w \in 2^{<n}}$ of height n such that, for every $\eta \in 2^n$, the set

$$T(U) \cup \{\psi(\bar{x})\} \cup \{ \varphi^{\eta(i)}(\bar{x}; \bar{c}_{\eta\restriction i}) \mid i < n \}$$

is consistent. By assumption, $D_\varphi(\psi) < N$. In particular, the maximum is well-defined. Furthermore, D_φ is monotone in the sense that

$$\psi \vDash \vartheta \quad \text{implies} \quad D_\varphi(\psi) \leq D_\varphi(\vartheta).$$

Given $\mathfrak{p} \in S_\varphi(U)$, choose a finite subset $\Phi \subseteq \mathfrak{p}$ such that $D_\varphi(\bigwedge \Phi)$ is minimal. By choice of Φ and monotonicity of D_φ, it follows for every $\bar{c} \subseteq U$ that

$$\varphi(\bar{x}; \bar{c}) \in \mathfrak{p} \quad \text{iff} \quad D_\varphi\big(\bigwedge \Phi(\bar{x}) \wedge \varphi(\bar{x}; \bar{c})\big) = D_\varphi\big(\bigwedge \Phi(\bar{x})\big).$$

Since the non-existence of a φ-tree of height n with the above property is definable in first-order logic, it follows that, for every $n < \omega$ and every formula $\psi(\bar{x}; \bar{y})$ over U, there is a formula $\delta^n_\psi(\bar{y})$ over U such that

$$\mathfrak{M} \vDash \delta^n_\psi(\bar{c}) \quad \text{iff} \quad D_\varphi(\psi(\bar{x}; \bar{c})) < n.$$

Hence, we can use the formula $\delta^n_{\bigwedge \Phi \wedge \varphi}$ with $n := D_\varphi(\bigwedge \Phi) + 1$ to define $\mathfrak{p}$. $\square$

After having introduced three properties of formulae, we can show that they are all equivalent to (un-)stability.

Theorem 5.15. *Let T be a complete first-order theory and $\varphi(\bar{x};\bar{y})$ a formula. The following statements are equivalent:*

(1) *φ is stable.*

(2) *φ is κ-stable, for all infinite cardinals κ.*

(3) *φ does not have the order property.*

(4) *There exists some $n<\omega$ such that there is no φ-tree of height n.*

(5) *Every complete φ-type is definable over its domain.*

Proof. (2) ⇒ (1) is trivial and (1) ⇒ (4) ⇒ (5) ⇒ (1) were already proved in, respectively, Lemmas 5.11, 5.14, and 5.13.

(4) ⇒ (3) Suppose that φ has the order property. Let $\leq$ be the infix ordering on $I := 2^{\leq\omega}$, which is defined by

$$\begin{aligned} u < v \quad :\text{iff} \quad & v = u1x\,, && \text{for some } x \in 2^{\leq\omega},\\ \text{or } & u = w0x \text{ and } v = w1y\,, && \text{for some } w \in 2^{<\omega} \text{ and}\\ & && x,y \in 2^{\leq\omega}. \end{aligned}$$

By Lemma 5.8, we can find a model $\mathfrak{M}$ of T that contains sequences $(\bar{a}_w)_{w\in I}$ and $(\bar{b}_w)_{w\in I}$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}_u,\bar{b}_v) \quad \text{iff} \quad u \leq v\,.$$

For $\eta \in 2^\omega$ and $n<\omega$, it follows that

$$\mathfrak{M} \vDash \varphi(\bar{a}_\eta,\bar{b}_{\eta\restriction n}) \quad \text{iff} \quad \eta \leq \eta\restriction n \quad \text{iff} \quad \eta(n) = 0\,.$$

Consequently, for every $\eta \in 2^\omega$ and every $n<\omega$, the tuple $\bar{a}_\eta$ satisfies

$$T(U) \cup \{\, \varphi^{\eta(i)}(\bar{x};\bar{b}_{\eta\restriction i}) \mid i < n \,\}\,,$$

and $(\bar{b}_w)_{w\in 2^{<n}}$ is a φ-tree of height n.

(3) ⇒ (2) Suppose that there is an infinite set U with $|S_\varphi(U)| > |U|$. Fix a model $\mathfrak{M}$ containing realisations of every φ-type over U. Let $\bar{s}$ be the sorts of those variables in $\bar{x}$ that actually appear in φ and let $\bar{t}$ be those in $\bar{y}$. For $\bar{a} \in M^{\bar{s}}$, we set

$$S(\bar{a}) := \{\, \bar{c} \in U^{\bar{t}} \mid \mathfrak{M} \vDash \varphi(\bar{a};\bar{c}) \,\}\,.$$

Note that $\mathrm{tp}_\varphi(\bar{a}/U) \neq \mathrm{tp}_\varphi(\bar{b}/U)$ implies $S(\bar{a}) \neq S(\bar{b})$. Hence,

$$\mathcal{S} := \{\, S(\bar{a}) \mid \bar{a} \in M^{\bar{s}} \,\} \subseteq \wp(U^{\bar{t}})$$

is a family of size $|\mathcal{S}| = |S_\varphi(U)| > |U| = |U^{\bar{t}}|$. By Theorem 5.6, there exist sequences $(\bar{c}_i)_{i<\omega}$ in $U^{\bar{t}}$ and $(\bar{a}_i)_{i<\omega}$ in $M^{\bar{s}}$ such that either

$$\begin{aligned} & \bar{c}_i \in S(\bar{a}_k) && \text{iff} && i \leq k\,,\\ \text{or}\quad & \bar{c}_i \in S(\bar{a}_k) && \text{iff} && i \geq k\,. \end{aligned}$$

It follows that

$$\begin{aligned} & \mathfrak{M} \vDash \varphi(\bar{a}_i;\bar{c}_k) && \text{iff} && i \leq k\\ \text{or}\quad & \mathfrak{M} \vDash \varphi(\bar{a}_i;\bar{c}_k) && \text{iff} && i \geq k\,. \end{aligned}$$

In the first case, φ has the order property and we are done. In the second case, we can take, for every $n<\omega$, a prefix of length n of these two sequences and reverse their ordering to obtain sequences $(\bar{a}'_i)_{i<n}$ and $(\bar{c}'_i)_{i<n}$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}'_i;\bar{c}'_k) \quad \text{iff} \quad i \leq k\,.$$

Consequently, it follows by Lemma 5.8 (3) that φ has the order property. ☐

Having characterised stable formulae, we turn to theories and their type spaces.

Definition 5.16. (a) A complete first-order theory T is *κ-stable* if we have $|S^{\bar{s}}(U)| \leq \kappa$, for all finite tuples of sorts $\bar{s}$ and every set U of size

$|U| \leq \kappa$. We call T *stable* if it is κ-stable, for some infinite cardinal κ. Otherwise, T is *unstable*.

(b) A complete first-order theory T is *totally transcendental* if

$$\mathrm{rk}_{\mathrm{CB}}(\mathfrak{S}^{\bar{s}}(U)) < \infty \quad \text{for all sets } U \text{ and all finite tuples } \bar{s}\,.$$

We obtain equivalent characterisations to those of Theorem 5.15.

Theorem 5.17. *Let T be a complete first-order theory. The following statements are equivalent:*

(1) *T is stable.*

(2) *T is κ-stable, for every cardinal κ such that $\kappa^{|T|} = \kappa$.*

(3) *Every first-order formula is stable.*

(4) *Every complete type is definable over its domain.*

(5) $\mathrm{rk}_{\mathrm{CB}}(\mathfrak{S}_\Delta(U)) < \infty$, *for all sets U and all finite sets Δ.*

(6) $|S_\Delta(U)| \leq \kappa$, *for all infinite cardinals κ, all finite sets Δ, and all sets U of size $|U| \leq \kappa$.*

Proof. (2) ⇒ (1) is trivial.

(1) ⇒ (3) Suppose that some formula $\varphi(\bar{x}, \bar{y})$ is not stable. By Theorem 5.15, it follows that, for every infinite cardinal κ, there exists a set U of size $|U| \leq \kappa$ such that

$$\kappa < |S_\varphi(U)| \leq |S^{\bar{s}}(U)|\,,$$

where $\bar{s}$ are the sorts of $\bar{x}$. Consequently, T is not κ-stable, for any $\kappa \geq \aleph_0$.

(3) ⇒ (4) Every type $\mathfrak{p} \in S^{\bar{s}}(U)$ is definable over its domain since, by Theorem 5.15, all of its restrictions $\mathfrak{p}|_\varphi$ are definable.

(4) ⇒ (6) Let Δ be a finite set of formulae and U a set of size $|U| \leq \kappa$. There exists an injective function $S_\Delta(U) \to \prod_{\varphi \in \Delta} S_\varphi(U)$ mapping a Δ-type $\mathfrak{p}$ to the tuple of its restrictions $(\mathfrak{p}|_\varphi)_{\varphi \in \Delta}$. If every type is definable over its domain, it follows by Theorem 5.15 that

$$|S_\Delta(U)| \leq \prod_{\varphi \in \Delta} |S_\varphi(U)| \leq \kappa^{|\Delta|} = \kappa\,.$$

(6) ⇒ (2) Let κ be a cardinal with $\kappa^{|T|} = \kappa$ and let U be a set of size $|U| \leq \kappa$. Since there exists an injective function

$$S^{<\omega}(U) \to \prod_\varphi S_\varphi(U) : \mathfrak{p} \mapsto (\mathfrak{p}|_\varphi)_\varphi\,,$$

it follows that

$$|S^{<\omega}(U)| \leq \prod_\varphi |S_\varphi(U)| \leq \kappa^{|T|} = \kappa\,.$$

(6) ⇒ (5) Suppose that $\mathrm{rk}_{\mathrm{CB}}(\mathfrak{S}_\Delta(U)) = \infty$. We have seen in Corollary 4.6 that

$$\mathfrak{S}_\Delta(U) \cong \mathrm{spec}(\mathfrak{Lb}(\Delta^{\neg}_U))\,.$$

By Lemma B2.5.15, there exists an embedding $(\psi_w)_{w \in 2^{<\omega}}$ of $2^{<\omega}$ into $\mathfrak{Lb}(\Delta^{\neg}_U)$. Let $U_0 \subseteq U$ be the set of all parameters appearing in these formulae ψ_w. Then U_0 is countable and $(\psi_w)_{w \in 2^{<\omega}}$ is an embedding of $2^{<\omega}$ into $\mathfrak{Lb}(\Delta^{\neg}_{U_0})$. Consequently,

$$\mathrm{rk}_{\mathrm{CB}}(\mathfrak{S}_\Delta(U_0)) = \mathrm{rk}_{\mathrm{CB}}(\mathrm{spec}(\mathfrak{Lb}(\Delta^{\neg}_{U_0}))) = \infty\,.$$

It follows by Corollary B5.7.4 that $|S_\Delta(U_0)| \geq 2^{\aleph_0} > |U_0|$. This contradicts (6).

(5) ⇒ (6) Suppose that there is some infinite set U with $|S_\Delta(U)| > |U|$. We have seen in Corollary 4.6 that

$$\mathfrak{S}_\Delta(U) \cong \mathrm{spec}(\mathfrak{Lb}(\Delta^{\neg}_U))\,.$$

Consequently, $|\mathrm{spec}(\mathfrak{Lb}(\Delta^{\neg}_U))| > |\mathfrak{Lb}(\Delta^{\neg}_U)|$ implies, by Corollary B2.5.22, that

$$\mathrm{rk}_{\mathrm{CB}}(\mathfrak{S}_\Delta(U)) = \mathrm{rk}_{\mathrm{CB}}(\mathrm{spec}(\mathfrak{Lb}(\Delta^{\neg}_U))) = \infty\,. \qquad \square$$

$\aleph_0$-stable theories are particularly simple. They are κ-stable, for every cardinal κ, and not only the local type spaces $\mathfrak{S}_\Delta(U)$, but even the full type space $\mathfrak{S}^{<\omega}(U)$ has a Cantor-Bendixson rank.

Theorem 5.18. *Let T be a complete theory over a countable signature. The following statements are equivalent:*

(1) *T is $\aleph_0$-stable.*

(2) *T is κ-stable, for all infinite cardinals κ.*

(3) *T is totally transcendental.*

Proof. By Theorem 2.18, we have

$$\mathfrak{S}^{\bar{s}}(U) \cong \operatorname{spec}(\mathfrak{B}(U)) \quad \text{where} \quad \mathfrak{B}(U) := \mathfrak{Lb}(\mathrm{FO}^{\bar{s}}[\Sigma_U]/T(U)) .$$

(2) ⇒ (1) is trivial.

(3) ⇒ (2) Suppose that there is some infinite cardinal κ such that T is not κ-stable, that is, we have $|S^{\bar{s}}(U)| > |U|$, for some set U of size $|U| = \kappa$. By Corollary B2.5.22 there is some type $\mathfrak{p} \in S^{\bar{s}}(U)$ with $\mathrm{rk}_\mathrm{P}(\varphi) = \infty$. Hence, Theorem B5.7.8 implies that $\mathrm{rk}_{\mathrm{CB}}(\mathfrak{S}^{\bar{s}}(U)) = \infty$.

(1) ⇒ (3) Suppose that $\mathrm{rk}_{\mathrm{CB}}(\mathfrak{S}^{\bar{s}}(U)) = \infty$, for some set U and some finite tuple $\bar{s}$. By Theorem B5.7.8, there is some formula $\varphi \in \mathfrak{B}(U)$ with $\mathrm{rk}_\mathrm{P}(\varphi) = \infty$. Hence, we an use Lemma B2.5.15 to find an embedding $(\psi_w)_{w \in 2^{<\omega}}$ of $2^{<\omega}$ into $\mathfrak{B}(U)$. Let $U_0 \subseteq U$ be the set of all parameters appearing in these formulae ψ_w. Then U_0 is countable and $(\psi_w)_{w \in 2^{<\omega}}$ is an embedding of $2^{<\omega}$ into $\mathfrak{B}(U_0)$. By Lemma B5.7.3, it follows that

$$|S^{\bar{s}}(U_0)| = |\operatorname{spec}(\mathfrak{B}(U_0))| \geq 2^{\aleph_0} > |U| .$$

Hence, T is not $\aleph_0$-stable. □

C4. *Back-and-forth equivalence*

1. *Partial isomorphisms*

In many constructions and proofs we will have to find two sequences $\bar{a}$ and $\bar{b}$ that cannot be told apart by any formula of a given logic, i.e., we are interested in the relation $\langle \mathfrak{A}, \bar{a} \rangle \equiv_L \langle \mathfrak{B}, \bar{b} \rangle$. In the present chapter we take a closer look at such relations for $L = \mathrm{FO}_{\infty\aleph_0}$ and $L = \mathrm{FO}$.

Definition 1.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures, $\bar{a} \subseteq A$ and $\bar{b} \subseteq B$ sequences of the same length, and α an ordinal.

(a) We write $\langle \mathfrak{A}, \bar{a} \rangle \equiv_\alpha \langle \mathfrak{B}, \bar{b} \rangle$ iff

$$\mathfrak{A} \vDash \varphi(\bar{a}) \quad \text{iff} \quad \mathfrak{B} \vDash \varphi(\bar{b}) ,$$

for all formulae $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma]$ of quantifier rank $\mathrm{qr}(\varphi) \leq \alpha$. If $\mathfrak{A} \equiv_\alpha \mathfrak{B}$ we say that $\mathfrak{A}$ is *α-equivalent* to $\mathfrak{B}$.

(b) We write $\langle \mathfrak{A}, \bar{a} \rangle \equiv_\infty \langle \mathfrak{B}, \bar{b} \rangle$ iff $\langle \mathfrak{A}, \bar{a} \rangle \equiv_\alpha \langle \mathfrak{B}, \bar{b} \rangle$, for all ordinals α. Hence, we have

$$\langle \mathfrak{A}, \bar{a} \rangle \equiv_\infty \langle \mathfrak{B}, \bar{b} \rangle \quad \text{iff} \quad \langle \mathfrak{A}, \bar{a} \rangle \equiv_{\mathrm{FO}_{\infty\aleph_0}} \langle \mathfrak{B}, \bar{b} \rangle .$$

The relations $\equiv_\alpha$ can be computed by induction on α. Note that we have $\langle \mathfrak{A}, \bar{a} \rangle \equiv_0 \langle \mathfrak{B}, \bar{b} \rangle$ if and only if the function $a_i \mapsto b_i$ induces an isomorphism $\langle\!\langle \bar{a} \rangle\!\rangle_\mathfrak{A} \cong \langle\!\langle \bar{b} \rangle\!\rangle_\mathfrak{B}$.

Definition 1.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures. A *partial isomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a function p with $\operatorname{dom} p \subseteq A$ and $\operatorname{rng} p \subseteq B$ such that p can be extended to an isomorphism

$$\langle\!\langle \operatorname{dom} p \rangle\!\rangle_\mathfrak{A} \cong \langle\!\langle \operatorname{rng} p \rangle\!\rangle_\mathfrak{B} .$$

We denote the set of all partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ whose domains have cardinality less than κ by $\mathrm{pIso}_\kappa(\mathfrak{A},\mathfrak{B})$. The union for all cardinals κ is $\mathrm{pIso}(\mathfrak{A},\mathfrak{B}) := \bigcup_\kappa \mathrm{pIso}_\kappa(\mathfrak{A},\mathfrak{B})$.

For sequences $(a_i)_{i<\alpha}$ and $(b_i)_{i<\alpha}$ we simplify notation by writing $p : \bar{a} \mapsto \bar{b}$ for the function $p = \{\langle a_i, b_i\rangle \mid i < \alpha\}$. (Note that, if we reorder the sequences $\bar{a}$ and $\bar{b}$ then we obtain the same function p.)

Remark. (a) Note that, by Theorem B3.1.9, in the above definition the isomorphism

$$\pi : \langle\!\langle \mathrm{dom}\, p \rangle\!\rangle_{\mathfrak{A}} \to \langle\!\langle \mathrm{rng}\, p \rangle\!\rangle_{\mathfrak{B}}$$

extending p is unique, if it exists.

(b) If Σ is a relational signature then $\langle\!\langle X \rangle\!\rangle_{\mathfrak{A}} = X$ and a function p is a partial isomorphism iff $p : \mathrm{dom}\, p \cong \mathrm{rng}\, p$.

(c) Finally, note that $\langle\rangle \mapsto \langle\rangle = \varnothing$ is the unique function p with $\mathrm{dom}\, p = \varnothing$ and $\mathrm{rng}\, p = \varnothing$. It is a partial isomorphism iff $\langle\!\langle \varnothing \rangle\!\rangle_{\mathfrak{A}} \cong \langle\!\langle \varnothing \rangle\!\rangle_{\mathfrak{B}}$, that is, if the substructures generated by the constants of $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic and if the same relations of arity 0 hold in $\mathfrak{A}$ and $\mathfrak{B}$.

Definition 1.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures.

(a) A partial isomorphism $p \in \mathrm{pIso}(\mathfrak{A},\mathfrak{B})$ has the *back-and-forth property* with respect to a set $I \subseteq \mathrm{pIso}(\mathfrak{A},\mathfrak{B})$ of partial isomorphisms if the following conditions are satisfied:

Forth. For all $a \in A$, there is some $q \in I$ such that $p \subseteq q$ and $a \in \mathrm{dom}\, q$.

Back. For all $b \in B$, there is some $q \in I$ such that $p \subseteq q$ and $b \in \mathrm{rng}\, q$.

A set $J \subseteq \mathrm{pIso}(\mathfrak{A},\mathfrak{B})$ of partial isomorphisms has the back-and-forth property with respect to I if every element of J has the back-and-forth property.

(b) A *back-and-forth system* between $\mathfrak{A}$ and $\mathfrak{B}$ is a sequence $(I_\alpha)_\alpha$ of sets $I_\alpha \subseteq \mathrm{pIso}(\mathfrak{A},\mathfrak{B})$ such that

- for every α, $I_{\alpha+1}$ has the back-and-forth property with respect to I_α, and
- $I_\delta \subseteq \bigcap_{\alpha<\delta} I_\alpha$, for limit ordinals δ.

The *canonical back-and-forth system* $(I_\alpha(\mathfrak{A},\mathfrak{B}))_\alpha$ between $\mathfrak{A}$ and $\mathfrak{B}$ is defined inductively by

$$\begin{aligned} I_0(\mathfrak{A},\mathfrak{B}) &:= \mathrm{pIso}(\mathfrak{A},\mathfrak{B})\,, \\ I_{\alpha+1}(\mathfrak{A},\mathfrak{B}) &:= \{\, p \in I_\alpha(\mathfrak{A},\mathfrak{B}) \mid p \text{ has the back-and-forth} \\ &\qquad\qquad \text{property w.r.t. } I_\alpha(\mathfrak{A},\mathfrak{B}) \,\}\,, \\ \text{and} \quad I_\delta(\mathfrak{A},\mathfrak{B}) &:= \bigcap_{\alpha<\delta} I_\alpha(\mathfrak{A},\mathfrak{B})\,, \quad \text{for limit ordinals } \delta\,. \end{aligned}$$

We will also need the restrictions

$$I^\kappa_\alpha(\mathfrak{A},\mathfrak{B}) := I_\alpha(\mathfrak{A},\mathfrak{B}) \cap \mathrm{pIso}_\kappa(\mathfrak{A},\mathfrak{B})$$

to domains of size less than κ.

Example. Let $\mathfrak{A} = \langle \mathbb{Z}, < \rangle$ and $\mathfrak{B} = \langle \mathbb{Q}, < \rangle$. We have

$$\begin{aligned} I_0(\mathfrak{A},\mathfrak{B}) &= \{\, \bar{a} \mapsto \bar{b} \mid a_i < a_k \Leftrightarrow b_i < b_k \,\}\,, \\ I_1(\mathfrak{A},\mathfrak{B}) &= \{\, \bar{a} \mapsto \bar{b} \mid a_i < a_k \Leftrightarrow b_i < b_k \text{ and } |a_i - a_k| \neq 1 \,\} \\ I_2(\mathfrak{A},\mathfrak{B}) &= \{ \langle\rangle \mapsto \langle\rangle \}\,, \\ I_3(\mathfrak{A},\mathfrak{B}) &= \varnothing\,. \end{aligned}$$

Recall that an open dense linear order is a linear order without first and last element such that between any two elements there is a third one.

Lemma 1.4. *If $\mathfrak{A} = \langle A, < \rangle$ and $\mathfrak{B} = \langle B, < \rangle$ are open dense linear orders then* $\mathrm{pIso}_{\aleph_0}(\mathfrak{A},\mathfrak{B})$ *has the back-and-forth property with respect to itself.*

Proof. Suppose that $\bar{a} \mapsto \bar{b} \in \mathrm{pIso}_{\aleph_0}(\mathfrak{A},\mathfrak{B})$ where w.l.o.g. we may assume that $a_0 \leq \dots \leq a_{n-1}$. By symmetry it is sufficient to prove the forth property. Let $c \in A$. If $c = a_i$, for some i, then $\bar{a}c \mapsto \bar{b}b_i$ is a partial isomorphism and we are done. Suppose that there is some i such that $a_i < c < a_{i+1}$. Since $\mathfrak{B}$ is dense we can select an arbitrary element $b_i < d < b_{i+1}$ and the mapping $\bar{a}c \mapsto \bar{b}d$ is a partial isomorphism. Similarly, if $c < a_0$ or $c > a_{n-1}$ then we can take any element $d < b_0$ or $d > b_{n-1}$ to obtain a partial isomorphism $\bar{a}c \mapsto \bar{b}d$. ☐

Theorem 1.5 (Cantor). *Any two countable open dense linear orders are isomorphic.*

Proof. Let $\mathfrak{A} = \langle A, < \rangle$ and $\mathfrak{B} = \langle B, < \rangle$ be countable open dense linear orders and fix enumerations $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ of A and B, respectively. Let $I := \mathrm{pIso}_{\aleph_0}(\mathfrak{A}, \mathfrak{B})$. We construct an increasing chain $p_0 \subseteq p_1 \subseteq \dots$ of partial isomorphisms $p_i \in I$ such that $a_i \in \operatorname{dom} p_{2i+1}$ and $b_i \in \operatorname{dom} p_{2i+2}$. Their union $p := \bigcup_i p_i$ is a partial isomorphism with domain $\operatorname{dom} p = A$ and range $\operatorname{rng} p = B$, that is, it is the desired total isomorphism $p : \mathfrak{A} \cong \mathfrak{B}$.

We define p_i by induction on i. Let $p_0 := \varnothing$. Suppose that $p_i \in I$ has already been defined and that $i = 2n$ is even. Since I has the forth property with respect to itself we can find some $p_{i+1} \in I$ extending p_i such that $a_n \in \operatorname{dom} p_{i+1}$. Similarly, if $i = 2n + 1$ is odd then we use the back property to find a partial isomorphism $p_{i+1} \in I$ extending p_i with $b_n \in \operatorname{rng} p_{i+1}$. □

Exercise 1.1. Let $\mathfrak{R} = \langle \mathbb{R}, + \rangle$ be the additive group of real numbers. Show that $\mathrm{pIso}_{\aleph_0}(\mathfrak{R}, \mathfrak{R})$ has the back-and-forth property with respect to itself.

Exercise 1.2. Prove that any two countable atomless boolean algebras are isomorphic.

Remark. (a) The canonical back-and-forth system $(I_\alpha(\mathfrak{A}, \mathfrak{B}))_\alpha$ is maximal, that is, for any back-and-forth system $(I_\alpha)_\alpha$ we have $I_\alpha \subseteq I_\alpha(\mathfrak{A}, \mathfrak{B})$, for all α.

(b) Obviously, a back-and-forth system forms a descending chain

$$I_0 \supseteq I_1 \supseteq \dots \supseteq I_\alpha \supseteq \dots .$$

Furthermore, if there is some ordinal α such that

$$I_\alpha(\mathfrak{A}, \mathfrak{B}) = I_{\alpha+1}(\mathfrak{A}, \mathfrak{B})$$

then $I_\alpha(\mathfrak{A}, \mathfrak{B}) = I_\beta(\mathfrak{A}, \mathfrak{B})$, for all $\beta \geq \alpha$. Hence, there always exists an ordinal $\alpha < |I_0(\mathfrak{A}, \mathfrak{B})|^+$ such that

$$I_\alpha(\mathfrak{A}, \mathfrak{B}) = I_\beta(\mathfrak{A}, \mathfrak{B}), \quad \text{for all } \beta \geq \alpha .$$

Definition 1.6. Let α be the minimal ordinal such that

$$I_\alpha(\mathfrak{A}, \mathfrak{B}) = I_{\alpha+1}(\mathfrak{A}, \mathfrak{B}) .$$

We denote this limit by $I_\infty(\mathfrak{A}, \mathfrak{B}) := I_\alpha(\mathfrak{A}, \mathfrak{B})$ and the corresponding restrictions by $I^\kappa_\infty(\mathfrak{A}, \mathfrak{B}) := I^\kappa_\alpha(\mathfrak{A}, \mathfrak{B})$.

Remark. $I_\infty(\mathfrak{A}, \mathfrak{B})$ has the back-and-forth property with respect to itself.

Exercise 1.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be finite structures with $|A|, |B| \leq n$. Prove that $I_n(\mathfrak{A}, \mathfrak{B}) = I_\infty(\mathfrak{A}, \mathfrak{B})$.

Lemma 1.7. *If $p \in I_\alpha(\mathfrak{A}, \mathfrak{B})$ and $q \subseteq p$ then $q \in I_\alpha(\mathfrak{A}, \mathfrak{B})$.*

Proof. The claim follows by a straightforward induction on α. □

Corollary 1.8. $I_\alpha(\mathfrak{A}, \mathfrak{B}) \neq \varnothing$ iff $\langle\rangle \mapsto \langle\rangle \in I_\alpha(\mathfrak{A}, \mathfrak{B})$.

Lemma 1.9. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures and κ an infinite cardinal. The sequence $(I^\kappa_\alpha(\mathfrak{A}, \mathfrak{B}))_\alpha$ is a back-and-forth system.*

Proof. The claim follows by induction on α since, if

$$\bar{a} \mapsto \bar{b} \in I^\kappa_{\alpha+1}(\mathfrak{A}, \mathfrak{B}) \quad \text{and} \quad \bar{a}c \mapsto \bar{b}d \in I_\alpha(\mathfrak{A}, \mathfrak{B})$$

then the set $\bar{a}c$ has cardinality less than κ. Therefore,

$$\bar{a}c \mapsto \bar{b}d \in \mathrm{pIso}_\kappa(\mathfrak{A}, \mathfrak{B})$$

which implies that $\bar{a}c \mapsto \bar{b}d \in I^\kappa_\alpha(\mathfrak{A}, \mathfrak{B})$. □

Definition 1.10. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures, $\bar{a} \subseteq A$, $\bar{b} \subseteq B$, and α an ordinal. We define

$$\langle \mathfrak{A}, \bar{a} \rangle \cong_\alpha \langle \mathfrak{B}, \bar{b} \rangle \quad :\text{iff} \quad \bar{a} \mapsto \bar{b} \in I_\alpha(\mathfrak{A}, \mathfrak{B}) ,$$
$$\langle \mathfrak{A}, \bar{a} \rangle \cong_\infty \langle \mathfrak{B}, \bar{b} \rangle \quad :\text{iff} \quad \bar{a} \mapsto \bar{b} \in I_\infty(\mathfrak{A}, \mathfrak{B}) .$$

If $\mathfrak{A} \cong_\alpha \mathfrak{B}$ we say that $\mathfrak{A}$ is *α-isomorphic* to $\mathfrak{B}$. For an arbitrary back-and-forth system $(I_\beta)_\beta$ we write

$$(I_\beta)_\beta : \langle \mathfrak{A}, \bar{a} \rangle \cong_\alpha \langle \mathfrak{B}, \bar{b} \rangle \quad :\text{iff} \quad \bar{a} \mapsto \bar{b} \subseteq p \text{, for some } p \in I_\alpha .$$

Example. Let $\Sigma = \{ P_i \mid i < n \}$ be a signature consisting of n unary predicates. For a Σ-structure $\mathfrak{A} = \langle A, \bar{P} \rangle$ and a set $I \subseteq [n]$, we set

$$P_I^{\mathfrak{A}} := \{ a \in A \mid a \in P_i^{\mathfrak{A}} \text{ iff } i \in I \} .$$

For $k, l, m < \omega$, define

$$k =_m l \quad :\text{iff} \quad k = l \text{ or } k, l \geq m .$$

We claim that $\langle \mathfrak{A}, \bar{a} \rangle \cong_m \langle \mathfrak{B}, \bar{b} \rangle$ if and only if $\bar{a} \mapsto \bar{b} \in \mathrm{pIso}(\mathfrak{A}, \mathfrak{B})$ and

$$|P_I^{\mathfrak{A}} \smallsetminus \bar{a}| =_m |P_I^{\mathfrak{B}} \smallsetminus \bar{b}| , \quad \text{for all } I \subseteq [n] .$$

We prove the claim by induction on m. If $m = 0$ then $\bar{a} \mapsto \bar{b} \in I_0(\mathfrak{A}, \mathfrak{B})$ iff $\bar{a} \mapsto \bar{b}$ is a partial isomorphism. Suppose that $m > 0$.

For one direction, assume that there is some I such that

$$|P_I^{\mathfrak{A}} \smallsetminus \bar{a}| \neq_m |P_I^{\mathfrak{B}} \smallsetminus \bar{b}| .$$

By symmetry we may assume that $|P_I^{\mathfrak{A}} \smallsetminus \bar{a}| > |P_I^{\mathfrak{B}} \smallsetminus \bar{b}|$. If $c \in P_I^{\mathfrak{A}} \smallsetminus \bar{a}$ then we have

$$|P_I^{\mathfrak{A}} \smallsetminus \bar{a}c| \neq_{m-1} |P_I^{\mathfrak{B}} \smallsetminus \bar{b}d| , \qquad \text{for every } d \in P_I^{\mathfrak{B}} \smallsetminus \bar{b} .$$

By inductive hypothesis it follows that $\bar{a}c \mapsto \bar{b}d \notin I_{m-1}(\mathfrak{A}, \mathfrak{B})$, for all $d \in B$. Consequently, $\bar{a} \mapsto \bar{b} \notin I_m(\mathfrak{A}, \mathfrak{B})$.

For the other direction, let $\bar{a} \mapsto \bar{b}$ be a partial isomorphism such that

$$|P_I^{\mathfrak{A}} \smallsetminus \bar{a}| =_m |P_I^{\mathfrak{B}} \smallsetminus \bar{b}| , \quad \text{for all } I \subseteq [n] ,$$

and let $c \in A \smallsetminus \bar{a}$. Set $I := \{ i < n \mid c \in P_i^{\mathfrak{A}} \}$ and choose an arbitrary element $d \in P_I^{\mathfrak{B}} \smallsetminus \bar{b}$. It follows that

$$|P_I^{\mathfrak{A}} \smallsetminus \bar{a}c| =_{m-1} |P_I^{\mathfrak{B}} \smallsetminus \bar{b}d| .$$

By inductive hypothesis, this implies that $\bar{a}c \mapsto \bar{b}d \in I_{m-1}(\mathfrak{A}, \mathfrak{B})$, as desired. The back property follows by symmetry.

We will show below that the relations $\cong_\alpha$ and $\equiv_\alpha$ coincide. Hence, we can determine whether $\mathfrak{A} \equiv_\alpha \mathfrak{B}$ holds by defining a back-and-forth system $(I_\beta)_\beta : \mathfrak{A} \cong_\alpha \mathfrak{B}$ with $I_\alpha \neq \varnothing$.

Lemma 1.11. *We have $\mathfrak{A} \cong_\infty \mathfrak{B}$ if and only if there exists a nonempty set $I \subseteq \mathrm{pIso}_{\aleph_0}(\mathfrak{A}, \mathfrak{B})$ that has the back-and-forth property with respect to itself.*

Proof. ($\Rightarrow$) By Lemma 1.9, we can set $I := I_\infty^{\aleph_0}(\mathfrak{A}, \mathfrak{B})$.

($\Leftarrow$) We prove by induction on α that $I \subseteq I_\alpha(\mathfrak{A}, \mathfrak{B})$, for all α. Then we have $I \subseteq I_\infty(\mathfrak{A}, \mathfrak{B})$ which implies that $I_\infty(\mathfrak{A}, \mathfrak{B}) \neq \varnothing$.

Clearly, $I \subseteq \mathrm{pIso}(\mathfrak{A}, \mathfrak{B}) = I_0(\mathfrak{A}, \mathfrak{B})$. Suppose that $I \subseteq I_\alpha(\mathfrak{A}, \mathfrak{B})$. Each $p \in I$ has the back-and-forth property with respect to I and, therefore, also with respect to $I_\alpha(\mathfrak{A}, \mathfrak{B}) \supseteq I$. Hence, $p \in I_{\alpha+1}(\mathfrak{A}, \mathfrak{B})$. Finally, if δ is a limit ordinal and $I \subseteq I_\alpha(\mathfrak{A}, \mathfrak{B})$, for all $\alpha < \delta$, then

$$I \subseteq \bigcap_{\alpha < \delta} I_\alpha(\mathfrak{A}, \mathfrak{B}) = I_\delta(\mathfrak{A}, \mathfrak{B}) .$$

□

As an application we consider discrete linear orders.

Definition 1.12. Let $\mathfrak{A} = \langle A, \leq \rangle$ be a linear order.

(a) $\mathfrak{A}$ is *discrete* if every element of $\mathfrak{A}$ that is not the least one has an immediate predecessor, and every element that is not the greatest one has an immediate successor. We say that $\mathfrak{A}$ is *bounded* if it has a least and a greatest element.

(b) We define the *distance* $d(a, b)$ of two elements $a, b \in A$ by

$$d(a, b) := |\{ c \in A \mid a \leq c < b \text{ or } b \leq c < a \}| .$$

Furthermore, we set

$$d(-\infty, b) := |{\Downarrow} b| ,$$
$$d(a, \infty) := |{\Uparrow} a| ,$$
and $$d(-\infty, \infty) := |A| \oplus 1 .$$

(c) For numbers $m, n, k < \omega$, we define

$$m =_k n \quad :\text{iff} \quad m = n \quad \text{or} \quad m, n \geq k .$$

Lemma 1.13. *Let $\mathfrak{A} = \langle A, \leq \rangle$ and $\mathfrak{B} = \langle B, \leq \rangle$ be bounded discrete linear orders, $\bar{a} \in A^m$ and $\bar{b} \in B^m$, and $n < \omega$. We have*

$$\langle \mathfrak{A}, \bar{a} \rangle \cong_n \langle \mathfrak{B}, \bar{b} \rangle$$

if and only if $\bar{a} \mapsto \bar{b}$ is a partial isomorphism such that, for all i, k,

$$d(a_i, a_k) =_{2^n} d(b_i, b_k)\,, \qquad d(a_i, \infty) =_{2^n} d(b_i, \infty)\,,$$
$$d(-\infty, a_k) =_{2^n} d(-\infty, b_k)\,, \quad d(-\infty, \infty) =_{2^n} d(-\infty, \infty)\,.$$

Proof. ($\Rightarrow$) We prove the claim by induction on n. Let $m := |\bar{a}|$. To avoid case distinctions we add new least and greatest elements $-\infty$ and ∞ to $\mathfrak{A}$ and $\mathfrak{B}$ and we set $a_{-1} := -\infty$ and $a_m := \infty$, and similarly for b_{-1} and b_m.

For $n = 0$, we have

$$\langle \mathfrak{A}, \bar{a} \rangle \cong_0 \langle \mathfrak{B}, \bar{b} \rangle \quad \text{iff} \quad \bar{a} \mapsto \bar{b} \in \mathrm{pIso}(\mathfrak{A}, \mathfrak{B})\,.$$

Note that every partial automorphism trivially satisfies the condition $d(a_i, a_k) =_1 d(b_i, b_k)$.

Consider the case that $n > 0$ and suppose that $\langle \mathfrak{A}, \bar{a} \rangle \cong_n \langle \mathfrak{B}, \bar{b} \rangle$. Clearly, the first condition is satisfied since $\bar{a} \mapsto \bar{b}$ is a partial isomorphism. Therefore, it remains to show that

$$d(a_i, a_k) =_{2^n} d(b_i, b_k)\,, \quad \text{for all } -1 \leq i, k \leq m\,.$$

For a contradiction, suppose that there are i and k such that

$$d(a_i, a_k) \neq_{2^n} d(b_i, b_k)\,.$$

By symmetry we may assume that $a_i < a_k$ and $d(a_i, a_k) < d(b_i, b_k)$. In particular, we have $d(a_i, a_k) < 2^n$. Furthermore, by inductive hypothesis, we have

$$d(a_i, a_k) =_{2^{n-1}} d(b_i, b_k)\,,$$

which is only possible if $d(a_i, a_k) \geq 2^{n-1}$. Hence, there exists some element $b_i < d \leq b_k$ with $d(b_i, d) = 2^{n-1}$. By the back-and-forth property, we can find an element $c \in A$ such that

$$\langle \mathfrak{A}, \bar{a}c \rangle \cong_{n-1} \langle \mathfrak{B}, \bar{b}d \rangle\,.$$

By inductive hypothesis, we have $d(a_i, c) =_{2^{n-1}} d(b_i, d)$ which implies that $d(a_i, c) \geq 2^{n-1} = d(b_i, d)$. Consequently, we have

$$d(c, a_k) = d(a_i, a_k) - d(a_i, c) \leq 2^n - 1 - 2^{n-1} = 2^{n-1} - 1$$

which implies that $d(c, a_k) = d(d, b_k)$. Together, it follows that that

$$\begin{aligned} d(a_i, a_k) &= d(a_i, c) + d(c, a_k) \\ &\geq d(b_i, d) + d(d, b_k) = d(b_i, b_k)\,. \end{aligned}$$

A contradiction.

($\Leftarrow$) Let I_n be the set of all partial functions $\bar{a} \mapsto \bar{b}$ where the tuples $\bar{a}$ and $\bar{b}$ satisfy the above conditions. We claim that $(I_n)_{n<\omega}$ is a back-and-forth system. Clearly, every $\bar{a} \mapsto \bar{b} \in I_0$ is a partial isomorphism. It remains to check the back-and-forth property. By symmetry, we only need to prove one direction. Let $\bar{a} \mapsto \bar{b} \in I_n$ and $c \in A$. Fix indices i and k such that $a_i \leq c \leq a_k$ and there is no index l with $a_i < a_l < a_k$.

We distinguish three cases. If $d(a_i, c) < 2^{n-1}$ then let $d \in B$ be the element such that $b_i \leq d \leq b_k$ and $d(b_i, d) = d(a_i, c)$. If $d(a_i, a_k) = d(b_i, b_k)$ then we clearly have $d(c, a_k) = d(d, b_k)$. If, on the other hand, $d(a_i, a_k), d(b_i, b_k) \geq 2^n$ then $d(c, a_k) \geq 2^{n-1}$ and $d(d, b_k) \geq 2^{n-1}$. Hence, in both cases we have $d(d, b_k) =_{2^{n-1}} d(c, a_k)$.

Similarly, if $d(a_i, c) \geq 2^{n-1}$ but $d(c, a_k) < 2^{n-1}$ then we choose $d \in B$ such that $b_i \leq d \leq b_k$ and $d(d, b_k) = d(c, a_k)$. As above it follows that $d(a_i, c) =_{2^{n-1}} d(b_i, d)$.

Finally, suppose that $d(a_i, c), d(c, a_k) \geq 2^{n-1}$. Then we select an element $b_i < d < b_k$ such that $d(b_i, d) = 2^{n-1}$. Since $d(a_i, a_k), d(b_i, b_k) \geq 2^n$ it follows that $d(d, b_k) = d(b_i, b_k) - d(b_i, c) \geq 2^{n-1}$. ☐

Corollary 1.14. *For discrete linear orders $\mathfrak{A}$ and $\mathfrak{B}$ and $n < \omega$, we have*

$$\mathfrak{A} \cong_n \mathfrak{B} \quad \text{iff} \quad |A| =_{2^n-1} |B| .$$

Lemma 1.15. *Let $\mathfrak{A}_i = \langle A_i, <, \bar{P} \rangle$ and $\mathfrak{B}_i = \langle B_i, <, \bar{P} \rangle$, for $i \in [2]$, be linear orders expanded by unary predicates $\bar{P}$.*

$$\mathfrak{A}_0 \cong_\alpha \mathfrak{B}_0 \quad \text{and} \quad \mathfrak{A}_1 \cong_\alpha \mathfrak{B}_1 \quad \text{implies} \quad \mathfrak{A}_0 + \mathfrak{A}_1 \cong_\alpha \mathfrak{B}_0 + \mathfrak{B}_1 .$$

Proof. Fix back-and-forth systems $(I^i_\beta)_{\beta \leq \alpha} : \mathfrak{A}_i \cong_\alpha \mathfrak{B}_i$. We claim that

$$(J_\beta)_{\beta \leq \alpha} : \mathfrak{A}_0 + \mathfrak{A}_1 \cong_\alpha \mathfrak{B}_0 + \mathfrak{B}_1$$

where

$$J_\beta := \{ \bar{a}\bar{c} \mapsto \bar{b}\bar{d} \mid \bar{a} \mapsto \bar{b} \in I^0_\beta \text{ and } \bar{c} \mapsto \bar{d} \in I^1_\beta \} .$$

We have $J_\alpha \neq \varnothing$ since $I^i_\alpha \neq \varnothing$, for both i. Furthermore, $J_\delta = \bigcap_{\beta<\delta} J_\beta$, for limit ordinals δ. It remains to prove the back-and-forth property. Suppose that $\bar{a}\bar{c} \mapsto \bar{b}\bar{d} \in J_{\beta+1}$ and $e \in A$. If $e \in A_0$ then there is some $f \in B_0$ with $\bar{a}e \mapsto \bar{b}f \in I^0_\beta$. Hence, it follows that $\bar{a}e\bar{c} \mapsto \bar{b}f\bar{d} \in J_\beta$. If $e \in A_1$ then the same argument provides a suitable element $f \in B_1$. The back property follows analogously. □

2. *Hintikka formulae*

The relations $\cong_\alpha$ are definable in $\mathrm{FO}_{\infty\aleph_0}$ by a formula of quantifier rank α. Consequently, we have $\equiv_\alpha \subseteq \cong_\alpha$. The other inclusion will be shown in Section 3.

Lemma 2.1. *Let $\mathfrak{A}$ be a Σ-structure, $\bar{a} \subseteq A$, and α an ordinal. There exists a formula $\varphi^\alpha_{\mathfrak{A},\bar{a}}(\bar{x}) \in \mathrm{FO}_{\infty\aleph_0}[\Sigma]$ of quantifier rank $\mathrm{qr}(\varphi^\alpha_{\mathfrak{A},\bar{a}}) = \alpha$ such that*

$$\mathfrak{B} \vDash \varphi^\alpha_{\mathfrak{A},\bar{a}}(\bar{b}) \quad \text{iff} \quad \bar{a} \mapsto \bar{b} \in I_\alpha(\mathfrak{A}, \mathfrak{B}) ,$$

for all Σ-structures $\mathfrak{B}$ and every $\bar{b} \subseteq B$.

Proof. We construct $\varphi^\alpha_{\mathfrak{A},\bar{a}}$ by induction on α.

($\alpha = 0$) Let Φ be the set of all literals $\psi(\bar{x})$ such that $\mathfrak{A} \vDash \psi(\bar{a})$. We set $\varphi^0_{\mathfrak{A},\bar{a}} := \bigwedge \Phi$.

($\alpha = \beta + 1$) We have to express the back-and-forth property.

$$\varphi^{\beta+1}_{\mathfrak{A},\bar{a}}(\bar{x}) := \varphi^\beta_{\mathfrak{A},\bar{a}}(\bar{x}) \wedge \bigwedge_{c \in A} \exists y \varphi^\beta_{\mathfrak{A},\bar{a}c}(\bar{x}y) \wedge \forall y \bigvee_{c \in A} \varphi^\beta_{\mathfrak{A},\bar{a}c}(\bar{x}y) .$$

(α limit) We take the conjunction over all $\beta < \alpha$.

$$\varphi^\alpha_{\mathfrak{A},\bar{a}}(\bar{x}) := \bigwedge_{\beta<\alpha} \varphi^\beta_{\mathfrak{A},\bar{a}}(\bar{x}) .$$

□

Remark. Formulae of the form $\varphi^\alpha_{\mathfrak{A},\bar{a}}$ are called *Hintikka formulae*. Note that $\varphi^\alpha_{\mathfrak{A},\bar{a}} \in \mathrm{FO}_{\kappa^+\aleph_0}[\Sigma]$ where $\kappa := |A| \oplus |\Sigma| \oplus |\alpha| \oplus \aleph_0$. If Σ, $\bar{a}$, and α are finite then it follows by induction on α that there are only finitely many formulae of the form $\varphi^\alpha_{\mathfrak{A},\bar{a}}$ and that we can choose them to be in $\mathrm{FO}^{<\omega}[\Sigma]$.

Since $\cong_\infty = \cong_\alpha$, for some ordinal α, we can also define the relation $\cong_\infty$.

Definition 2.2. Let $\mathfrak{A}$ be a structure. The *Scott height* of $\mathfrak{A}$ is the least ordinal α such that $I^{\aleph_0}_\infty(\mathfrak{A}, \mathfrak{A}) = I^{\aleph_0}_\alpha(\mathfrak{A}, \mathfrak{A})$. The *Scott sentence* $\varphi^\infty_\mathfrak{A}$ of $\mathfrak{A}$ is defined by

$$\varphi^\infty_\mathfrak{A} := \varphi^\alpha_{\mathfrak{A},\langle\rangle} \wedge \bigwedge_{\bar{a} \in A^{<\omega}} \forall \bar{x} \big[\varphi^\alpha_{\mathfrak{A},\bar{a}}(\bar{x}) \rightarrow \varphi^{\alpha+1}_{\mathfrak{A},\bar{a}}(\bar{x}) \big] ,$$

where α is the Scott height of $\mathfrak{A}$.

Lemma 2.3. *The Scott height of $\mathfrak{A}$ is less than $|A|^+$.*

Proof. If A is finite then $I_{|A|}(\mathfrak{A}, \mathfrak{A}) = I_\infty(\mathfrak{A}, \mathfrak{A})$ and the Scott height is at most $|A| < \aleph_0$. Similarly, if A is infinite then there exists some ordinal

$$\alpha < |I^{\aleph_0}_0(\mathfrak{A}, \mathfrak{A})|^+ \leq (|A|^{<\aleph_0})^+ = |A|^+$$

such that $I^{\aleph_0}_\alpha(\mathfrak{A}, \mathfrak{A}) = I^{\aleph_0}_\infty(\mathfrak{A}, \mathfrak{A})$. □

Exercise 2.1. Compute the Scott height of $\langle \omega, \leq \rangle$.

Theorem 2.4. *For all structures $\mathfrak{A}$ and $\mathfrak{B}$, we have*

$$\mathfrak{B} \vDash \varphi^\infty_\mathfrak{A} \quad \text{iff} \quad \mathfrak{B} \cong_\infty \mathfrak{A} .$$

Proof. Let α be the Scott height of $\mathfrak{A}$.

($\Rightarrow$) If $\mathfrak{B} \vDash \varphi^\infty_\mathfrak{A}$ then $\bar a \mapsto \bar b \in I^{\aleph_0}_\alpha(\mathfrak{A}, \mathfrak{B})$ implies $\bar a \mapsto \bar b \in I^{\aleph_0}_{\alpha+1}(\mathfrak{A}, \mathfrak{B})$. Hence,

$$I^{\aleph_0}_{\alpha+1}(\mathfrak{A}, \mathfrak{B}) = I^{\aleph_0}_\alpha(\mathfrak{A}, \mathfrak{B}) = I^{\aleph_0}_\infty(\mathfrak{A}, \mathfrak{B}) .$$

Furthermore, $I^{\aleph_0}_\infty(\mathfrak{A}, \mathfrak{B})$ is not empty since $\mathfrak{B} \vDash \varphi^\alpha_{\mathfrak{A},\langle\rangle}$ implies $\langle\rangle \mapsto \langle\rangle \in I_\alpha(\mathfrak{A}, \mathfrak{B})$.

($\Leftarrow$) Suppose that $\mathfrak{B} \cong_\infty \mathfrak{A}$. Then we have $\mathfrak{B} \vDash \varphi^\alpha_{\mathfrak{A},\langle\rangle}$. To see that $\mathfrak{B}$ also satisfies the second part of the formula $\varphi^\infty_\mathfrak{A}$ we have to show that

$$\bar a \mapsto \bar b \in I^{\aleph_0}_\alpha(\mathfrak{A}, \mathfrak{B}) \quad \text{implies} \quad \bar a \mapsto \bar b \in I^{\aleph_0}_{\alpha+1}(\mathfrak{A}, \mathfrak{B}) .$$

Let $\bar a \mapsto \bar b \in I^{\aleph_0}_\alpha(\mathfrak{A}, \mathfrak{B})$. We claim that $\bar a \mapsto \bar b$ has the back-and-forth property with respect to $I^{\aleph_0}_\alpha(\mathfrak{A}, \mathfrak{B})$.

For the forth property let $c \in A$. Since $\mathfrak{A} \cong_\infty \mathfrak{B}$ there exist some tuple $\bar b' \subseteq A$ with $\langle \mathfrak{A}, \bar b' \rangle \cong_\infty \langle \mathfrak{B}, \bar b \rangle$. Hence,

$$\langle \mathfrak{A}, \bar b' \rangle \cong_\alpha \langle \mathfrak{B}, \bar b \rangle \cong_\alpha \langle \mathfrak{A}, \bar a \rangle .$$

Since α is the Scott height of $\mathfrak{A}$ is follows that

$$\langle \mathfrak{A}, \bar b' \rangle \cong_{\alpha+1} \langle \mathfrak{A}, \bar a \rangle .$$

Hence, we can find some $d' \in A$ with

$$\langle \mathfrak{A}, \bar b' d' \rangle \cong_\alpha \langle \mathfrak{A}, \bar a c \rangle .$$

Since $\langle \mathfrak{A}, \bar b' \rangle \cong_\infty \langle \mathfrak{B}, \bar b \rangle$ there is some $d \in B$ such that

$$\langle \mathfrak{A}, \bar b' d' \rangle \cong_\infty \langle \mathfrak{B}, \bar b d \rangle .$$

Consequently, we have

$$\langle \mathfrak{A}, \bar a c \rangle \cong_\alpha \langle \mathfrak{A}, \bar b' d' \rangle \cong_\alpha \langle \mathfrak{B}, \bar b d \rangle ,$$

and $\bar a c \mapsto \bar b d \in I_\alpha(\mathfrak{A}, \mathfrak{B})$. The back property follows analogously. □

Corollary 2.5. $\mathfrak{A} \equiv_{|A|^+} \mathfrak{B}$ *implies* $\mathfrak{A} \cong_\infty \mathfrak{B}$.

Proof. If α is the Scott height of $\mathfrak{A}$ then $\operatorname{qr}(\varphi^\infty_\mathfrak{A}) \leq \alpha + \omega < |A|^+$. □

3. *Ehrenfeucht-Fraïssé games*

Ehrenfeucht-Fraïssé games provide an intuitive way of describing back-and-forth systems.

Definition 3.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ Σ-structures, $\bar a_0 \subseteq A$, $\bar b_0 \subseteq B$, and let α be an ordinal.

(a) The *Ehrenfeucht-Fraïssé game* $\mathrm{EF}_\alpha(\mathfrak{A}, \bar a_0, \mathfrak{B}, \bar b_0)$ is played by two players (*spoiler* and *duplicator*) according to the following rules:

- A *position* in the game is a tuple $\langle \beta, \bar a, \bar b \rangle$ where $\beta \leq \alpha$, $\bar a \subseteq A$, $\bar b \subseteq B$, and $|\bar a| = |\bar b|$.
- The *initial position* is $\langle \alpha, \bar a_0, \bar b_0 \rangle$.
- In the position $\langle \beta, \bar a, \bar b \rangle$ spoiler chooses an ordinal $\gamma < \beta$ and either an element $c \in A$ or some $d \in B$. Duplicator responds by selecting an element of the other structure, i.e., either $d \in B$ or $c \in A$. The new position is $\langle \gamma, \bar a c, \bar b d \rangle$.
- Spoiler loses if he cannot choose γ because $\beta = 0$. Duplicator loses if a position $\langle \beta, \bar a, \bar b \rangle$ is reached where $\bar a \mapsto \bar b \notin \mathrm{pIso}(\mathfrak{A}, \mathfrak{B})$.

(b) The *infinite* version $\mathrm{EF}^\kappa_\infty(\mathfrak{A}, \bar a_0, \mathfrak{B}, \bar b_0)$ of the Ehrenfeucht-Fraïssé game is played in the same way as $\mathrm{EF}_\alpha(\mathfrak{A}, \bar a_0, \mathfrak{B}, \bar b_0)$ with the exception that the first component of all positions is omitted and every play has length κ. Hence, duplicator wins if she can continue the game for κ steps while, as before, spoiler wins if a position $\langle \bar a, \bar b\rangle$ is reached such that $\bar a \mapsto \bar b$ is not a partial isomorphism.

(c) A *winning strategy* of one of the players is a function mapping positions to moves such that, regardless of the moves of his opponent, the player wins if he always plays the moves given by the strategy. We say that a player *wins* the game $\mathrm{EF}_\alpha(\mathfrak A, \bar a, \mathfrak B, \bar b)$ if he has a winning strategy.

Example. Let $\mathfrak A = \langle \mathbb Z, <\rangle$ and $\mathfrak B = \langle \mathbb Q, <\rangle$. Spoiler wins the 3 round game $\mathrm{EF}_3(\mathfrak A, \mathfrak B)$. The game starts in position

$$\langle 3,\ \langle\rangle,\ \langle\rangle\rangle\,.$$

In the first round, spoiler chooses $2 < 3$ and $0 \in \mathbb Z$. Duplicator has to answer with some number $a \in \mathbb Q$. The new position is

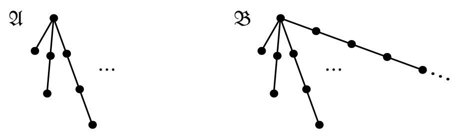
$$\langle 2,\ \langle 0\rangle,\ \langle a\rangle\rangle\,.$$

In the second round, spoiler chooses $1 < 2$ and $1 \in \mathbb Z$. Duplicator replies with some $b \in \mathbb Q$ such that $b > a$. The new position is

$$\langle 1,\ \langle 0,1\rangle,\ \langle a,b\rangle\rangle\,.$$

Finally, spoiler chooses $0 < 1$ and $(a+b)/2 \in \mathbb Q$. Duplicator has to respond with some element $z \in \mathbb Z$ such that $0 < z < 1$. Since there is no such element she loses.

Exercise 3.1. Let $\mathfrak A$ be the tree consisting of one path of length n, for every $n < \omega$, and let $\mathfrak B$ be the tree consisting of one path of length α, for every $\alpha \le \omega$.



Find the least ordinal α such that Spoiler wins $\mathrm{EF}_\alpha(\mathfrak A, \mathfrak B)$.

Immediately from the definition we obtain the following connection between Ehrenfeucht-Fraïssé games and the back-and-forth property.

Lemma 3.2. *Duplicator wins* $\mathrm{EF}_\alpha(\mathfrak A, \bar a, \mathfrak B, \bar b)$ *if and only if*

- *for all* $\beta < \alpha$ *and every* $c \in A$ *there is some* $d \in B$ *such that she wins* $\mathrm{EF}_\beta(\mathfrak A, \bar a c, \mathfrak B, \bar b d)$, *and*
- *for all* $\beta < \alpha$ *and every* $d \in B$ *there is some* $c \in A$ *such that she wins* $\mathrm{EF}_\beta(\mathfrak A, \bar a c, \mathfrak B, \bar b d)$.

By induction it follows that the winning positions in the game form a back-and-forth system.

Lemma 3.3. *Duplicator wins* $\mathrm{EF}_\alpha(\mathfrak A, \bar a, \mathfrak B, \bar b)$ *iff* $\bar a \mapsto \bar b \in I_\alpha(\mathfrak A, \mathfrak B)$.

Proof. We show the claim by induction on α.

($\alpha = 0$) By definition, duplicator wins $\mathrm{EF}_0(\mathfrak A, \bar a, \mathfrak B, \bar b)$ iff $\bar a \mapsto \bar b \in \mathrm{pIso}(\mathfrak A, \mathfrak B) = I_0(\mathfrak A, \mathfrak B)$.

($\alpha = \beta + 1$) Duplicator wins $\mathrm{EF}_{\beta+1}(\mathfrak A, \bar a, \mathfrak B, \bar b)$

iff for all $c \in A$ there is $d \in B$ such that she wins $\mathrm{EF}_\beta(\mathfrak A, \bar a c, \mathfrak B, \bar b d)$
and for all $d \in B$ there is $c \in A$ such that she wins $\mathrm{EF}_\beta(\mathfrak A, \bar a c, \mathfrak B, \bar b d)$

iff for all $c \in A$ there is $d \in B$ such that $\bar a c \mapsto \bar b d \in I_\beta(\mathfrak A, \mathfrak B)$
and for all $d \in B$ there is $c \in A$ such that $\bar a c \mapsto \bar b d \in I_\beta(\mathfrak A, \mathfrak B)$

iff $\bar a \mapsto \bar b$ has the back-and-forth property w.r.t $I_\beta(\mathfrak A, \mathfrak B)$

iff $\bar a \mapsto \bar b \in I_{\beta+1}(\mathfrak A, \mathfrak B)$.

(α limit) Duplicator wins $\mathrm{EF}_\alpha(\mathfrak A, \bar a, \mathfrak B, \bar b)$

iff she wins $\mathrm{EF}_\beta(\mathfrak A, \bar a, \mathfrak B, \bar b)$ for all $\beta < \alpha$

iff $\bar a \mapsto \bar b \in I_\beta(\mathfrak A, \mathfrak B)$ for all $\beta < \alpha$

iff $\bar a \mapsto \bar b \in I_\alpha(\mathfrak A, \mathfrak B)$. □

We have seen that the relation $\equiv_\alpha$ refines $\cong_\alpha$. The following lemma establishes the converse.

Lemma 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures with elements $\bar{a} \subseteq A$ and $\bar{b} \subseteq B$. If there exists a formula $\varphi(\bar{x}) \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$ of quantifier rank $\mathrm{qr}(\varphi) \leq \alpha$ such that*

$$\mathfrak{A} \vDash \varphi(\bar{a}) \quad \textit{and} \quad \mathfrak{B} \nvDash \varphi(\bar{b})$$

then spoiler wins $\mathrm{EF}_\alpha(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$.

Proof. W.l.o.g. we may assume that φ is in negation normal form. We prove the claim by induction on φ.

(φ literal) As $\bar{a}$ and $\bar{b}$ are distinguished by an atomic formula the mapping $\bar{a} \mapsto \bar{b}$ cannot be a partial isomorphism. Hence, spoiler wins the game $\mathrm{EF}_0(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$ immediately.

($\varphi = \bigwedge \Phi$) There is some formula $\psi \in \Phi$ such that

$$\mathfrak{A} \vDash \psi(\bar{a}) \quad \text{and} \quad \mathfrak{B} \nvDash \psi(\bar{b}) .$$

Since $\mathrm{qr}(\psi) \leq \alpha$ spoiler wins $\mathrm{EF}_\alpha(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$, by inductive hypothesis.

($\varphi = \bigvee \Phi$) follows in the same way.

($\varphi = \exists x \psi$) Let $\beta := \mathrm{qr}(\psi) < \alpha$. There is some element $c \in A$ such that $\mathfrak{A} \vDash \psi(\bar{a}, c)$, but $\mathfrak{B} \nvDash \psi(\bar{b}, d)$, for all $d \in B$. In the first move spoiler can choose β and the element $c \in A$. Duplicator responds with some element $d \in B$. By inductive hypothesis, spoiler can win the resulting game $\mathrm{EF}_\beta(\mathfrak{A}, \bar{a}c, \mathfrak{B}, \bar{b}d)$. Therefore, he also wins $\mathrm{EF}_\alpha(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$.

($\varphi = \forall x \psi$) analogously by choosing some $d \in B$. □

Theorem 3.5 (Karp)**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures and α an ordinal.*

(a) *The following statements are equivalent:*

  (1) $\langle \mathfrak{A}, \bar{a} \rangle \equiv_\alpha \langle \mathfrak{B}, \bar{b} \rangle$.
  (2) $\langle \mathfrak{A}, \bar{a} \rangle \cong_\alpha \langle \mathfrak{B}, \bar{b} \rangle$.
  (3) $\langle \mathfrak{B}, \bar{b} \rangle \vDash \varphi^\alpha_{\mathfrak{A}, \bar{a}}$.
  (4) *Duplicator wins* $\mathrm{EF}_\alpha(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$.

(b) *The following statements are equivalent:*

  (1) $\langle \mathfrak{A}, \bar{a} \rangle \equiv_\infty \langle \mathfrak{B}, \bar{b} \rangle$.
  (2) $\langle \mathfrak{A}, \bar{a} \rangle \cong_\infty \langle \mathfrak{B}, \bar{b} \rangle$.
  (3) $\langle \mathfrak{B}, \bar{b} \rangle \vDash \varphi^\infty_{\mathfrak{A}, \bar{a}}$.
  (4) *Duplicator wins* $\mathrm{EF}^{\aleph_0}_\infty(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$.

Proof. (a) We have already shown in Lemmas 2.1 and 3.3 that (2), (3), and (4) are equivalent.

(1) ⇒ (3) follows directly from the definition of $\equiv_\alpha$ since $\mathrm{qr}(\varphi^\alpha_{\mathfrak{A},\bar{a}}) \leq \alpha$.

(4) ⇒ (1) If $\langle \mathfrak{A}, \bar{a} \rangle \not\equiv_\alpha \langle \mathfrak{B}, \bar{b} \rangle$ then there is some formula $\varphi \in \mathrm{FO}_{\infty\aleph_0}$ of quantifier rank $\mathrm{qr}(\varphi) \leq \alpha$ such that $\mathfrak{A} \vDash \varphi(\bar{a})$ and $\mathfrak{B} \nvDash \varphi(\bar{b})$. By Lemma 3.4, spoiler wins $\mathrm{EF}_\alpha(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$.

(b) The equivalence (2) ⇔ (3) was proved in Theorem 2.4, and the implication (1) ⇒ (3) is trivial.

(2) ⇒ (4) Duplicator can win if she ensures that only positions $\langle \bar{c}, \bar{d} \rangle$ are reached where $\bar{c} \mapsto \bar{d} \in I_\infty(\mathfrak{A}, \mathfrak{B}) \subseteq \mathrm{pIso}(\mathfrak{A}, \mathfrak{B})$. But this is easily done since $I_\infty(\mathfrak{A}, \mathfrak{B})$ has the back-and-forth property with respect to itself. If spoiler chooses some element $c \in A$ then there exists an element $d \in B$ with $\bar{a}c \mapsto \bar{b}d \in I_\infty(\mathfrak{A}, \mathfrak{B})$. Similarly, if spoiler plays in $\mathfrak{B}$ then duplicator can respond in $\mathfrak{A}$.

(4) ⇒ (1) If $\langle \mathfrak{A}, \bar{a} \rangle \not\equiv_\infty \langle \mathfrak{B}, \bar{b} \rangle$ then there is some formula $\varphi \in \mathrm{FO}_{\infty\aleph_0}$ such that $\mathfrak{A} \vDash \varphi(\bar{a})$ and $\mathfrak{B} \nvDash \varphi(\bar{b})$. Let $\alpha := \mathrm{qr}(\varphi)$. By Lemma 3.4, spoiler wins $\mathrm{EF}_\alpha(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$. He can use the same strategy to win the infinite game $\mathrm{EF}^{\aleph_0}_\infty(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$. □

Corollary 3.6 (Ehrenfeucht, Fraïssé)**.** *Let Σ be a relational signature and $\mathfrak{A}$ and $\mathfrak{B}$ Σ-structures. For $m < \omega$, let $\Delta_m \subseteq \mathrm{FO}[\Sigma]$ be the set of all first-order formulae of quantifier rank at most m.*

(a) $\mathfrak{A} \equiv_{\Delta_m} \mathfrak{B}$ iff $\mathfrak{A}|_{\Sigma_0} \cong_m \mathfrak{B}|_{\Sigma_0}$ *for all finite* $\Sigma_0 \subseteq \Sigma$,

(b) $\mathfrak{A} \equiv_{\mathrm{FO}} \mathfrak{B}$ iff $\mathfrak{A}|_{\Sigma_0} \cong_\omega \mathfrak{B}|_{\Sigma_0}$ *for all finite* $\Sigma_0 \subseteq \Sigma$.

Exercise 3.2. Find structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \equiv_{\mathrm{FO}} \mathfrak{B}$ but $\mathfrak{A} \not\cong_\omega \mathfrak{B}$.

Corollary 3.7. *Every formula $\psi \in \mathrm{FO}_{\infty\aleph_0}[\Sigma, X]$ of quantifier rank α is equivalent to a disjunction of Hintikka formulae of quantifier rank α. For $\psi \in \mathrm{FO}[\Sigma, X]$ and relational Σ, we can choose this disjunction in $\mathrm{FO}[\Sigma, X]$.*

Proof. We have $\psi \equiv \bigvee \Phi$ where

$$\Phi := \{ \varphi^{\alpha}_{\mathfrak{A},\bar a} \mid \mathfrak{A} \vDash \psi(\bar a) \}$$

is the set of all Hintikka formulae corresponding to models of ψ.

If $\psi \in \mathrm{FO}[\Sigma, X]$ then $\alpha < \omega$ and there exist finite subsets $\Sigma_0 \subseteq \Sigma$ and $X_0 \subseteq X$ such that $\psi \in \mathrm{FO}[\Sigma_0, X_0]$. Hence, we have $\psi \equiv \bigvee \Phi_0$ where $\Phi_0 := \Phi \cap \mathrm{FO}[\Sigma_0, X_0]$ is finite. □

We conclude this section with several applications of Ehrenfeucht-Fraïssé games.

Lemma 3.8. *There exists no first-order formula φ such that, for every finite structure $\mathfrak{A}$, we have*

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad |A| \text{ is even.}$$

Proof. Suppose that such a formula φ exists and let $m := \mathrm{qr}(\varphi)$. By Corollary 1.14, we have

$$\langle [2^m], \leq \rangle \vDash \varphi \quad \text{iff} \quad \langle [2^m + 1], \leq \rangle \vDash \varphi .$$

A contradiction. □

Let us apply Ehrenfeucht-Fraïssé games to equivalence relations. Recall that we write $m =_k n$ iff $m = n$ or $m, n \geq k$. If E is an equivalence relation then we denote by $N^{=}_k(E)$ the number of E-classes $[a]_E$ of size $|[a]_E| = k$ and $N^{>}_k(E)$ denotes the number of classes of size $|[a]_E| > k$.

Lemma 3.9. *Let E and F be equivalence relations on the sets A and B, respectively. We have $\langle A, E\rangle \cong_m \langle B, F\rangle$ if and only if*

$$N^{=}_k(E) =_{m-k} N^{=}_k(F) \quad \text{and} \quad N^{>}_k(E) =_{m-k} N^{>}_k(F) ,$$

for all $k \leq m$.

Proof. (⇒) First, suppose that $N^{=}_k(E) > N^{=}_k(F) =: s$. We claim that spoiler wins $\mathrm{EF}_{s+k+1}(\mathfrak{A}, \mathfrak{B})$. Since $N^{=}_k(E) > s$ we can find $s + 1$ different E-classes $[a_0]_E, \dots, [a_s]_E$ of size $|[a_i]_E| = k$. In the first part of the game spoiler plays their representatives $a_0, \dots, a_s$. Duplicator has to answer with elements $b_0, \dots, b_s$ of different F-classes in B. Since we have $N^{=}_k(F) < s + 1$ there is an index i such that $l := |[b_i]_F| \neq k$. If $l < k$ then spoiler continues by playing $k - 1$ different elements

$$c_0, \dots, c_{k-2} \in [a_i]_E \smallsetminus \{a_i\} .$$

Since $|[b_i]_F \smallsetminus \{b_i\}| < k - 1$ duplicator cannot answer all of them. Consequently, spoiler wins after at most $s + 1 + k - 1 = s + k$ rounds. Similarly, if $l > k$ then spoiler plays k different elements

$$d_0, \dots, d_{k-1} \in [b_i]_F \smallsetminus \{b_i\} ,$$

and again duplicator cannot answer all of them. In this case spoiler wins after at most $s + 1 + k$ rounds.

It remains to consider the case that $N^{>}_k(E) > N^{>}_k(F) =: s$. By a similar argument as above we show that spoiler wins $\mathrm{EF}_{s+k+1}(\mathfrak{A}, \mathfrak{B})$. Since $N^{>}_k(E) > s$ we can find $s + 1$ different E-classes $[a_0]_E, \dots, [a_s]_E$ of size $|[a_i]_E| > k$. In the first part of the game spoiler plays their representatives $a_0, \dots, a_s$. Duplicator has to answer with elements $b_0, \dots, b_s$ of different F-classes in B. Since we have $N^{>}_k(F) < s + 1$ there is an index i such that $|[b_i]_F| \leq k$. In the second part of the game spoiler plays k different elements

$$c_0, \dots, c_{k-1} \in [a_i]_E \smallsetminus \{a_i\} .$$

Since $|[b_i]_F \smallsetminus \{b_i\}| < k$ duplicator cannot answer all of them. Consequently, spoiler wins after at most $s + 1 + k$ rounds.

(⇐) For $k \leq m$, let I_k be the set of all partial isomorphisms $\bar a \mapsto \bar b$ with $\bar a \in A^{m-k}$ and $\bar b \in B^{m-k}$ such that

$$\big|[a_i]_E \smallsetminus \bar a\big| =_k \big|[b_i]_F \smallsetminus \bar b\big| , \quad \text{for all } i < m - k .$$

We claim that $(I_k)_k : \langle A, E\rangle \cong_m \langle B, F\rangle$. Clearly, we have $\langle\rangle \mapsto \langle\rangle \in I_m$. By symmetry, it is therefore sufficient to prove the forth property.

Let $\bar{a} \mapsto \bar{b} \in I_{k+1}$, and $c \in A$. We have to find some $d \in B$ such that $\bar{a}c \mapsto \bar{b}d \in I_k$. We consider several cases. If $c = a_i$, for some i, then $\bar{a}c \mapsto \bar{b}b_i \in I_k$. If $c \in [a_i]_E \setminus \bar{a}$, for some i, then

$$\left|[a_i]_E \setminus \bar{a}\right| =_{k+1} \left|[b_i]_E \setminus \bar{b}\right|$$

implies that there is some $d \in [b_i]_E \setminus \bar{b}$. It follows that $\bar{a}c \mapsto \bar{b}d \in I_k$.

It remains to consider the case that $c \notin [a_i]_E$, for all i. Set $s := |[c]_E|$. We are looking for an element $d \in B$ with $s =_{k+1} |[d]_F|$ and $[d]_F \cap \bar{b} = \emptyset$. First, consider the case that $s \leq k$. Then we have

$$|[a_i]_E| = s \quad \text{iff} \quad |[b_i]_F| = s\,.$$

Let l be the number of indices i with $|[a_i]_E| = s$. Since

$$N_s^{=}(E) =_{m-s} N_s^{=}(F) \quad \text{and} \quad l + 1 \leq m - k - 1 + 1 \leq m - s\,,$$

it follows that $N_s^{=}(E) \geq l + 1$ implies $N_s^{=}(F) \geq l + 1$. Consequently, we can choose some element $d \in B$ such that $|[d]_F| = s$ and $[d]_F \cap \bar{b} = \emptyset$.

The proof for the case that $s > k$ is analogous. Then we have

$$|[a_i]_E| > k \quad \text{iff} \quad |[b_i]_F| > k\,,$$

and we denote by l the number of indices i with $|[a_i]_E| > k$. Since

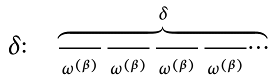
$$N_k^{>}(E) =_{m-k} N_k^{>}(F) \quad \text{and} \quad l + 1 \leq m - k\,,$$

it follows that $N_k^{>}(E) \geq l+1$ implies $N_k^{>}(F) \geq l+1$. Consequently, we can choose some element $d \in B$ such that $|[d]_F| > k$ and $[d]_F \cap \bar{b} = \emptyset$. □

We have seen in Lemma C1.1.7 that we can define every ordinal $\alpha < \kappa$ in $\mathrm{FO}_{\kappa\aleph_0}[<]$. Nevertheless there is no $\mathrm{FO}_{\infty\aleph_0}[<]$-formula that axiomatises the class of all well-orders.

Lemma 3.10. *For every ordinal α, there exists an ordinal $\delta > \alpha$ such that $\delta \equiv_\delta \delta + \delta \cdot \tau$, for each linear order τ.*

Proof. By Lemma A4.5.6, we can choose $\delta > \alpha$ such that $\omega^{(\delta)} = \delta$. Then δ is a limit ordinal such that $\delta = \omega^{(\beta)}\delta$, for all $\beta < \delta$. Hence, for each $\beta < \delta$, we can write δ as sum of δ copies of the order $\omega^{(\beta)}$. We call such a summand a $\omega^{(\beta)}$*-interval* of δ.



In the same way we can write linear orders of the form $\delta + \delta\tau$ as a sum of $\omega^{(\beta)}$-intervals.

For $\beta < \delta$, let I_β be the set of all finite partial isomorphisms $\bar{a} \mapsto \bar{b} \in \mathrm{pIso}_{\aleph_0}(\delta, \delta + \delta\tau)$ satisfying the following conditions. For notational simplicity we assume that $a_0 < \cdots < a_{n-1}$.

(1) a_i and a_{i+1} belong to the same $\omega^{(\beta)}$-interval iff b_i and b_{i+1} belong to the same $\omega^{(\beta)}$-interval.

(2) a_i is the α-th element of the $\omega^{(\beta)}$-interval containing a_i if and only if b_i is the α-th element of the $\omega^{(\beta)}$-interval containing b_i.

(3) a_0 is in the first $\omega^{(\beta)}$-interval if and only if b_0 is in the first $\omega^{(\beta)}$-interval.

Further, we set $I_\delta := \{\langle\rangle \mapsto \langle\rangle\}$. We claim that $(I_\beta)_{\beta<\delta} : \delta \cong_\delta \delta + \delta\tau$.

To prove the back property, suppose that $\bar{a} \mapsto \bar{b} \in I_{\beta+1}$ where $a_0 < \cdots < a_{n-1}$, and let $d \in \delta + \delta\tau$.

If d belongs to the $\omega^{(\beta)}$-interval of some b_i then let c be the corresponding element in the $\omega^{(\beta)}$-interval of a_i. It follows that $\bar{a}c \mapsto \bar{b}d \in I_\beta$. If d belongs to the first $\omega^{(\beta)}$-interval or if $d > b_{n-1}$ then we can easily find a suitable element $c \in \delta$ such that $\bar{a}c \mapsto \bar{b}d \in I_\beta$.

It remains to consider the case that the $\omega^{(\beta)}$-interval of d lies strictly between those of b_i and b_{i+1}. Since a_i and a_{i+1} do not belong to the same $\omega^{(\beta)}$-interval we can choose some $\omega^{(\beta)}$-interval between those containing a_i and a_{i+1}. Let c be the α-th element of this interval, where

α is the position of d in its $\omega^{(\beta)}$-interval. Again, it follows that $\bar{a}c \mapsto \bar{b}d \in I_\beta$.

In the same way, we can prove the forth property. Since $\langle\rangle \mapsto \langle\rangle \in I_\beta$, for all $\beta < \delta$, it follows that $(I_\beta)_{\beta<\delta} : \delta \cong_\delta \delta + \delta\tau$. □

Theorem 3.11. *There is no sentence* $\varphi \in \mathrm{FO}_{\infty\aleph_0}[\leq]$ *such that*

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad \mathfrak{A} \textit{ is a well-order.}$$

Proof. Suppose there is such a formula φ. Let $\alpha := \mathrm{qr}(\varphi)$. By the preceding lemma we can find an ordinal $\delta > \alpha$ such that $\delta \equiv_\delta \delta + \delta\zeta$ where $\zeta := \langle \mathbb{Z}, \leq \rangle$. Since δ is a well-order we have $\delta \vDash \varphi$. This implies that $\delta + \delta\zeta \vDash \varphi$. Contradiction. □

4. κ-complete back-and-forth systems

Sometimes the partial isomorphisms of a back-and-forth systems can be used to construct a total isomorphism between two structures.

Definition 4.1. Let κ be an infinite cardinal and $I \subseteq \mathrm{pIso}(\mathfrak{A}, \mathfrak{B})$.

(a) The set I is *κ-complete* if, for every increasing chain $(p_i)_{i<\alpha} \subseteq I$ and every subset $X \subseteq \bigcup_{i<\alpha} p_i$ of size $|X| < \kappa$, there is some $q \in I$ with $\bigcup_{i<\alpha} p_i \restriction X \subseteq q$.

(b) I is *κ-bounded* if, for every $p \in I$ and each subset $X \subseteq \mathrm{dom}\, p$, there is a partial isomorphism $q \in I$ of size $|q| < |X|^+ \oplus \kappa$ such that $p \restriction X \subseteq q \subseteq p$.

(c) We call I *κ-finitary* if, for every $p \in \mathrm{pIso}_\kappa(\mathfrak{A}, \mathfrak{B})$, we have

$$p \in I \quad \text{iff} \quad p \restriction X \in I \quad \text{for all } \textit{finite } X \subseteq \mathrm{dom}\, p\,.$$

Remark. Note that every κ-finitary set is κ-complete and $\aleph_0$-bounded.

Definition 4.2. For structures $\mathfrak{A}$ and $\mathfrak{B}$ and a cardinal κ, we set

$$I^\kappa_{\mathrm{FO}}(\mathfrak{A}, \mathfrak{B}) := \{\, \bar{a} \mapsto \bar{b} \in \mathrm{pIso}_\kappa(\mathfrak{A}, \mathfrak{B}) \mid \langle \mathfrak{A}, \bar{a} \rangle \equiv_{\mathrm{FO}} \langle \mathfrak{B}, \bar{b} \rangle \,\}\,.$$

Remark. Since every first-order formula refers only to finitely many constants it follows that the sets $I^\kappa_{\mathrm{FO}}(\mathfrak{A}, \mathfrak{B})$ are κ-finitary and, hence, κ-complete.

Definition 4.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures and κ an infinite cardinal.

(a) For a set $I \subseteq \mathrm{pIso}(\mathfrak{A}, \mathfrak{B})$, we write $I : \langle \mathfrak{A}, \bar{a} \rangle \sqsubseteq^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle$ if

- $\bar{a} \mapsto \bar{b} \subseteq p$, for some $p \in I$ with $|\mathrm{dom}\, p \setminus \bar{a}| < \kappa$,
- I is κ-complete and κ-bounded,
- I has the *forth property* with respect to itself. (We do not require the back property.)

Similarly, we define $I : \langle \mathfrak{A}, \bar{a} \rangle \cong^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle$ if

- $\bar{a} \mapsto \bar{b} \subseteq p$, for some $p \in I$ with $|\mathrm{dom}\, p \setminus \bar{a}| < \kappa$,
- I is κ-complete and κ-bounded,
- I has the back-and-forth property with respect to itself,

that is, if

$$I : \langle \mathfrak{A}, \bar{a} \rangle \sqsubseteq^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle \quad \text{and} \quad I : \langle \mathfrak{A}, \bar{a} \rangle \sqsupseteq^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle\,.$$

We write $\mathfrak{A} \cong^\kappa_{\mathrm{iso}} \mathfrak{B}$ if there exists some set I with $I : \mathfrak{A} \cong^\kappa_{\mathrm{iso}} \mathfrak{B}$, and similarly for $\sqsubseteq^\kappa_{\mathrm{iso}}$.

(b) Of particular interest are the following special cases.

$$\begin{aligned}
\langle \mathfrak{A}, \bar{a} \rangle \sqsubseteq^\kappa_0 \langle \mathfrak{B}, \bar{b} \rangle \quad &:\text{iff} \quad I^\kappa_0(\mathfrak{A}, \mathfrak{B}) : \langle \mathfrak{A}, \bar{a} \rangle \sqsubseteq^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle\,,\\
\langle \mathfrak{A}, \bar{a} \rangle \cong^\kappa_0 \langle \mathfrak{B}, \bar{b} \rangle \quad &:\text{iff} \quad I^\kappa_0(\mathfrak{A}, \mathfrak{B}) : \langle \mathfrak{A}, \bar{a} \rangle \cong^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle\,,\\
\langle \mathfrak{A}, \bar{a} \rangle \sqsubseteq^\kappa_{\mathrm{FO}} \langle \mathfrak{B}, \bar{b} \rangle \quad &:\text{iff} \quad I^\kappa_{\mathrm{FO}}(\mathfrak{A}, \mathfrak{B}) : \langle \mathfrak{A}, \bar{a} \rangle \sqsubseteq^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle\,,\\
\langle \mathfrak{A}, \bar{a} \rangle \cong^\kappa_{\mathrm{FO}} \langle \mathfrak{B}, \bar{b} \rangle \quad &:\text{iff} \quad I^\kappa_{\mathrm{FO}}(\mathfrak{A}, \mathfrak{B}) : \langle \mathfrak{A}, \bar{a} \rangle \cong^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle\,,\\
\langle \mathfrak{A}, \bar{a} \rangle \sqsubseteq^\kappa_\infty \langle \mathfrak{B}, \bar{b} \rangle \quad &:\text{iff} \quad I^\kappa_\infty(\mathfrak{A}, \mathfrak{B}) : \langle \mathfrak{A}, \bar{a} \rangle \sqsubseteq^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle\,,\\
\langle \mathfrak{A}, \bar{a} \rangle \cong^\kappa_\infty \langle \mathfrak{B}, \bar{b} \rangle \quad &:\text{iff} \quad I^\kappa_\infty(\mathfrak{A}, \mathfrak{B}) : \langle \mathfrak{A}, \bar{a} \rangle \cong^\kappa_{\mathrm{iso}} \langle \mathfrak{B}, \bar{b} \rangle\,.
\end{aligned}$$

Remark. (a) $I^{\aleph_0}_\infty(\mathfrak{A}, \mathfrak{B})$ is trivially $\aleph_0$-complete and $\aleph_0$-bounded. Hence, we have

$$\mathfrak{A} \cong^{\aleph_0}_\infty \mathfrak{B} \quad \text{iff} \quad \mathfrak{A} \cong_\infty \mathfrak{B}\,.$$

(b) The sets $I_0^\kappa(\mathfrak{A},\mathfrak{B})$ and $I_{\mathrm{FO}}^\kappa(\mathfrak{A},\mathfrak{B})$ are κ-finitary and, hence, κ-complete and κ-bounded. Consequently, we have

$$\mathfrak{A} \sqsubseteq_0^\kappa \mathfrak{B} \quad \text{iff} \quad I_0^\kappa(\mathfrak{A},\mathfrak{B}) \text{ is nonempty and it has the forth property with respect to itself.}$$

and similarly for the relations $\cong_0^\kappa$, $\sqsubseteq_{\mathrm{FO}}^\kappa$, and $\cong_{\mathrm{FO}}^\kappa$.

(c) Note that we have

$$I_\infty^\kappa(\mathfrak{A},\mathfrak{B}) \subseteq I_{\mathrm{FO}}^\kappa(\mathfrak{A},\mathfrak{B}) \subseteq I_0^\kappa(\mathfrak{A},\mathfrak{B})\,.$$

Furthermore, we have shown in Lemma 1.11 that

$$I : \mathfrak{A} \cong_{\mathrm{iso}}^\kappa \mathfrak{B} \quad \text{implies} \quad I \subseteq I_\infty(\mathfrak{A},\mathfrak{B})\,.$$

Let us summarise these remarks in the following lemma.

Lemma 4.4. *Let κ be a cardinal and $x \in \{0, \mathrm{FO}\}$.*

(a) *The following statements are equivalent:*
 (1) $\mathfrak{A} \sqsubseteq_x^\kappa \mathfrak{B}$.
 (2) *The set $I_x^\kappa(\mathfrak{A},\mathfrak{B})$ is nonempty and it has the forth property with respect to itself.*

(b) *The following statements are equivalent:*
 (1) $\mathfrak{A} \cong_x^\kappa \mathfrak{B}$.
 (2) $I_x^\kappa(\mathfrak{A},\mathfrak{B}) = I_\infty^\kappa(\mathfrak{A},\mathfrak{B}) \neq \varnothing$.
 (3) *The set $I_x^\kappa(\mathfrak{A},\mathfrak{B})$ is nonempty and it has the back-and-forth property with respect to itself.*

As an example we consider dense linear orders.

Definition 4.5. Let $\mathfrak{A} = \langle A, < \rangle$ be a linear order.

(a) For $C, D \subseteq A$, we write $C < D$ if $c < d$, for all $c \in C$ and $d \in D$.

(b) $\mathfrak{A}$ is *κ-dense* if, for all sets $C, D \subseteq A$ of size $|C|, |D| < \kappa$ with $C < D$, there exists an element $a \in A$ such that $C < a < D$. Note that we allow $C = \varnothing$ or $D = \varnothing$.

Lemma 4.6. *If $\mathfrak{B} = \langle B, < \rangle$ is a κ-dense linear order then we have*

$$\mathfrak{A} \sqsubseteq_0^\kappa \mathfrak{B}\,, \quad \text{for every linear order } \mathfrak{A}\,.$$

Proof. We have already noted that $\mathrm{pIso}_\kappa(\mathfrak{A},\mathfrak{B})$ is κ-complete. Furthermore, since linear orders are relational structures we have

$$\langle\rangle \mapsto \langle\rangle \in \mathrm{pIso}_\kappa(\mathfrak{A},\mathfrak{B}) \neq \varnothing\,.$$

Consequently, it remains to prove the forth property.

Let $p \in \mathrm{pIso}_\kappa(\mathfrak{A},\mathfrak{B})$ and $a \in A$. If $a \in \mathrm{dom}\, p$ then we are done. Otherwise, we can partition the domain of p into

$$C := \{\, c \in \mathrm{dom}\, p \mid c < a \,\} \quad \text{and} \quad D := \{\, d \in \mathrm{dom}\, p \mid a < d \,\}\,.$$

Then $C < D$ which implies that $p[C] < p[D]$. Since $\mathfrak{B}$ is κ-dense and $|C|, |D| \leq |\mathrm{dom}\, p| < \kappa$ we can find some element $b \in B$ with

$$p[C] < b < p[D]\,.$$

Hence, $p \cup \{\langle a, b\rangle\}$ is the desired partial isomorphism extending p. □

Corollary 4.7. *If $\mathfrak{A}$ and $\mathfrak{B}$ are κ-dense linear orders then $\mathfrak{A} \cong_0^\kappa \mathfrak{B}$.*

The relation $\cong_{\mathrm{iso}}^\kappa$ can also be characterised via Ehrenfeucht-Fraïssé games. The proof is completely analogous to that of Lemma 3.3.

Theorem 4.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures and κ a cardinal. The following statements are equivalent:*

(1) $\langle \mathfrak{A}, \bar{a} \rangle \cong_{\mathrm{iso}}^\kappa \langle \mathfrak{B}, \bar{b} \rangle$.
(2) *Duplicator wins* $\mathrm{EF}_\infty^\kappa(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$.

κ-complete sets with the back-and-forth property can be used to construct embeddings and isomorphisms.

Lemma 4.9. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures and κ an infinite cardinal.*

(a) *Suppose that* $I : \mathfrak{A} \cong^{\kappa}_{\mathrm{iso}} \mathfrak{B}$. *For all sequences* $\bar{a} \in A^\kappa$ *and* $\bar{b} \in B^\kappa$, *there exist sequences* $\bar{c} \in A^\kappa$ *and* $\bar{d} \in B^\kappa$ *such that, for all* $\alpha < \kappa$,

$$I : \langle \mathfrak{A}, (a_i)_{i<\alpha}, (c_i)_{i<\alpha} \rangle \cong^{\kappa}_{\mathrm{iso}} \langle \mathfrak{B}, (d_i)_{i<\alpha}, (b_i)_{i<\alpha} \rangle .$$

In particular, we have $\langle \mathfrak{A}, \bar{a}\bar{c} \rangle \cong_0 \langle \mathfrak{B}, \bar{d}\bar{b} \rangle$.

(b) *Suppose that* $I : \mathfrak{A} \sqsubseteq^{\kappa}_{\mathrm{iso}} \mathfrak{B}$. *For every sequence* $\bar{a} \in A^\kappa$, *there exist a sequence* $\bar{b} \in B^\kappa$ *such that*

$$I : \langle \mathfrak{A}, (a_i)_{i<\alpha} \rangle \sqsubseteq^{\kappa}_{\mathrm{iso}} \langle \mathfrak{B}, (b_i)_{i<\alpha} \rangle , \quad \textit{for all } \alpha < \kappa .$$

In particular, we have $\langle \mathfrak{A}, \bar{a} \rangle \cong_0 \langle \mathfrak{B}, \bar{b} \rangle$.

Proof. (a) We construct an increasing chain $(p_i)_{i<\kappa}$ of partial isomorphisms $p_i \in I$ with $|p_i| < \kappa$ such that $a_i \in \mathrm{dom}\, p_{i+1}$ and $b_i \in \mathrm{rng}\, p_{i+1}$, for all $i < \kappa$. Then we obtain the desired sequences $\bar{c}$ and $\bar{d}$ by setting

$$c_i := (p_{i+1})^{-1}(b_i) \quad \text{and} \quad d_i := p_{i+1}(a_i) .$$

Since $I : \mathfrak{A} \sqsubseteq^{\kappa}_{\mathrm{iso}} \mathfrak{B}$ there is some $p_0 \in I$ with $|p_0| < \kappa$. Suppose that we have already defined $p_i \in I$, for $i < \alpha$. If α is a limit ordinal then, I being κ-complete, there is some $p_\alpha \in I$ such that

$$\bigcup_{i<\alpha} p_i \restriction \big[\{ a_i \mid i < \alpha \} \cup \{ p^{-1}_{i+1}(b_i) \mid i < \alpha \} \big] \subseteq p_\alpha .$$

Finally, suppose that $\alpha = \gamma + 1$ is a successor. By the forth property we can find some $q \in I$ extending p_γ with $a_\gamma \in \mathrm{dom}\, q$. Analogously, there is some $p_\alpha \in I$ extending q with $b_\gamma \in \mathrm{rng}\, p_\alpha$.

(b) is proved in the same way as (a). We define an increasing chain $(p_i)_{i<\kappa}$ of partial isomorphisms such that $a_i \in \mathrm{dom}\, p_{i+1}$. For every a_i, we can use the forth property to find a suitable b_i. □

Lemma 4.10. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be structures generated by* $A_0 \subseteq A$ *and* $B_0 \subseteq B$, *respectively.*

(a) *If* $\kappa \geq |A_0| \oplus |B_0|$ *and* $I : \mathfrak{A} \cong^{\kappa}_{\mathrm{iso}} \mathfrak{B}$ *then* $\mathfrak{A} \cong \mathfrak{B}$.

(b) *If* $\kappa \geq |A_0|$ *and* $I : \mathfrak{A} \sqsubseteq^{\kappa}_{\mathrm{iso}} \mathfrak{B}$ *then there exists an embedding* $\mathfrak{A} \to \mathfrak{B}$.

Proof. (a) Let $\bar{a}$ be an enumeration of A_0 and $\bar{b}$ one of B_0. By the preceding lemma, there are sequences $\bar{c} \subseteq A$ and $\bar{d} \subseteq B$ such that

$$\langle \mathfrak{A}, \bar{a}\bar{c} \rangle \cong_0 \langle \mathfrak{B}, \bar{d}\bar{b} \rangle .$$

In particular, the map $p : \bar{a}\bar{c} \mapsto \bar{d}\bar{b}$ is a partial isomorphism. By definition, there exists an isomorphism

$$\pi : \langle\!\langle \mathrm{dom}\, p \rangle\!\rangle_{\mathfrak{A}} \cong \langle\!\langle \mathrm{rng}\, p \rangle\!\rangle_{\mathfrak{B}}$$

extending p. Since $\mathrm{dom}\, p \supseteq \bar{a} = A_0$ and $\mathrm{rng}\, p \supseteq \bar{b} = B_0$ it follows that π is a total isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

(b) Given an enumeration $\bar{a}$ of A_0 we can find a sequence $\bar{b} \subseteq B$ such that $\langle \mathfrak{A}, \bar{a} \rangle \cong_0 \langle \mathfrak{B}, \bar{b} \rangle$. Hence, $\bar{a} \mapsto \bar{b}$ is a partial isomorphism that can be extended to an isomorphism

$$\pi : \langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}} \cong \langle\!\langle \bar{b} \rangle\!\rangle_{\mathfrak{B}} .$$

Since $\langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}} = A$ it follows that π is the desired embedding. □

Corollary 4.11. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are countable structures with* $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$ *then* $\mathfrak{A} \cong \mathfrak{B}$.

Proof. Let α be the Scott height of $\mathfrak{A}$. Then $\alpha < |A|^+ \leq \aleph_1$. Hence, $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$ implies that $\mathfrak{B} \vDash \varphi^\infty_{\mathfrak{A}}$ where $\varphi^\infty_{\mathfrak{A}}$ is the Scott sentence of $\mathfrak{A}$. By Theorem 2.4, it follows that $\mathfrak{A} \cong_\infty \mathfrak{B}$. This is equivalent to $\mathfrak{A} \cong^{\aleph_0}_\infty \mathfrak{B}$ since $I^{\aleph_0}_\infty(\mathfrak{A}, \mathfrak{B})$ is always $\aleph_0$-complete. Hence, Lemma 4.10 (a) implies that $\mathfrak{A} \cong \mathfrak{B}$. □

Corollary 4.12. (a) *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* κ*-dense linear orders of size at most* κ *then* $\mathfrak{A} \cong \mathfrak{B}$.

(b) *If* $\mathfrak{B}$ *is a* κ*-dense linear order then every linear order* $\mathfrak{A}$ *of size at most* κ *can be embedded into* $\mathfrak{B}$.

Proof. Immediately from Lemma 4.6 and Corollary 4.7. □

We can show that the relation $\cong^\kappa_{\text{iso}}$ is reflexive and symmetric, but it is unknown whether it is also transitive. The relations $\cong^\kappa_0$, $\cong^\kappa_{\text{FO}}$, and $\cong^\kappa_\infty$, on the other hand, are transitive and symmetric but not reflexive.

Lemma 4.13. *If* $\mathfrak{A} \cong \mathfrak{B}$ *then* $\mathfrak{A} \cong^\kappa_{\text{iso}} \mathfrak{B}$, *for all* κ.

Proof. Fix an isomorphism $\pi : \mathfrak{A} \cong \mathfrak{B}$. The set

$$I := \{ p \in \text{pIso}_\kappa(\mathfrak{A}, \mathfrak{B}) \mid p \subseteq \pi \}$$

is nonempty, κ-finitary, and it has the back-and-forth property with respect to itself. Hence, we have $I : \mathfrak{A} \cong^\kappa_{\text{iso}} \mathfrak{B}$. □

Remark. The above lemma fails for the relations $\cong^\kappa_0$, $\cong^\kappa_{\text{FO}}$, and $\cong^\kappa_\infty$. In fact, we can even find structures $\mathfrak{A}$ such that $\mathfrak{A} \not\cong^{\aleph_0}_0 \mathfrak{A}$ or $\mathfrak{A} \not\cong^{\aleph_0}_{\text{FO}} \mathfrak{A}$. For instance, if we take $\mathfrak{A} := \langle \omega, \leq \rangle$ then $0 \mapsto 1 \in I^{\aleph_0}_0(\mathfrak{A}, \mathfrak{A})$ but there exists no element $a \in \omega$ with $\langle 0, a \rangle \mapsto \langle 1, 0 \rangle \in I^{\aleph_0}_0(\mathfrak{A}, \mathfrak{A})$. Structures such that $\mathfrak{A} \cong^\kappa_{\text{FO}} \mathfrak{A}$ are called *κ-homogeneous.* They are the subject of Section E1.1.

Lemma 4.14. *Let κ be a cardinal and $x \in \{0, \text{FO}, \infty\}$.*

$$\mathfrak{A} \sqsubseteq^\kappa_x \mathfrak{B} \sqsubseteq^\kappa_x \mathfrak{C} \quad \textit{implies} \quad \mathfrak{A} \sqsubseteq^\kappa_x \mathfrak{C}\,.$$

Proof. Let $L_x \subseteq \text{FO}_{\infty\aleph_0}$ be the logic such that

$$\bar{a} \mapsto \bar{b} \in I^\kappa_x(\mathfrak{A}, \mathfrak{B}) \quad \text{iff} \quad \langle \mathfrak{A}, \bar{a} \rangle \equiv_{L_x} \langle \mathfrak{B}, \bar{b} \rangle\,.$$

We start by showing that

$$I^\kappa_x(\mathfrak{A}, \mathfrak{C}) = \{ q \circ p \mid p \in I^\kappa_x(\mathfrak{A}, \mathfrak{B}),\ q \in I^\kappa_x(\mathfrak{B}, \mathfrak{C}) \}\,.$$

Clearly, if p and q preserve all L_x-formulae then so does $q \circ p$. Therefore, we only have to show that, for every $\bar{a} \mapsto \bar{c} \in I^\kappa_x(\mathfrak{A}, \mathfrak{C})$, there is some tuple $\bar{b}$ such that $\bar{a} \mapsto \bar{b} \in I^\kappa_x(\mathfrak{A}, \mathfrak{B})$ and $\bar{b} \mapsto \bar{c} \in I^\kappa_x(\mathfrak{B}, \mathfrak{C})$.

Given $\bar{a}$ of length $|\bar{a}| < \kappa$, we can find, by Lemma 4.9, some tuple $\bar{b}$ such that $\bar{a} \mapsto \bar{b} \in I^\kappa_x(\mathfrak{A}, \mathfrak{B})$. Since the maps $\bar{b} \mapsto \bar{a}$ and $\bar{a} \mapsto \bar{c}$ preserve all L_x-formulae it follows that so does $\bar{b} \mapsto \bar{c}$. Consequently, we also have $\bar{b} \mapsto \bar{c} \in I^\kappa_x(\mathfrak{B}, \mathfrak{C})$, as desired.

To prove the lemma, first note that the claim implies that $\langle\rangle \mapsto \langle\rangle \in I^\kappa_x(\mathfrak{A}, \mathfrak{C}) \neq \emptyset$. Therefore, it remains to check that $I^\kappa_x(\mathfrak{A}, \mathfrak{C})$ has the forth property with respect to itself. Let $\pi \in I^\kappa_x(\mathfrak{A}, \mathfrak{C})$ and $a \in A$. Then $\pi = q \circ p$, for some $p \in I^\kappa_x(\mathfrak{A}, \mathfrak{B})$ and $q \in I^\kappa_x(\mathfrak{B}, \mathfrak{C})$. Since these sets have the forth property, we can find elements $b \in B$ and $c \in C$ such that

$$p' := p \cup \{\langle a, b \rangle\} \in I^\kappa_x(\mathfrak{A}, \mathfrak{B})$$
$$\text{and} \quad q' := q \cup \{\langle b, c \rangle\} \in I^\kappa_x(\mathfrak{B}, \mathfrak{C})\,.$$

It follows that $\pi \cup \{\langle a, c \rangle\} = q' \circ p' \in I^\kappa_x(\mathfrak{A}, \mathfrak{C})$. □

Since the relations $\cong^\kappa_x$ are clearly symmetric we have the following corollaries.

Corollary 4.15. *Let κ be a cardinal and $x \in \{0, \text{FO}, \infty\}$.*

(a) *If* $\mathfrak{A} \cong^\kappa_x \mathfrak{B}$ *then* $\mathfrak{A} \cong^\kappa_x \mathfrak{A}$.

(b) *The relation* $\sqsubseteq^\kappa_x$ *is a preorder on the class*

$$\mathcal{C} := \{ \mathfrak{A} \mid \mathfrak{A} \cong^\kappa_x \mathfrak{A} \}\,.$$

5. *The theorems of Hanf and Gaifman*

In nontrivial applications the combinatorics involved in playing Ehrenfeucht-Fraïssé games quickly become unmanageable. Therefore, it is desirable to develop methods to simplify such games.

Definition 5.1. Let $\mathfrak{A}$ be a relational Σ-structure. The *Gaifman graph* of $\mathfrak{A}$ is the graph $\mathcal{G}(\mathfrak{A}) := \langle A, E \rangle$ with edge relation

$$E := \{ \langle a, b \rangle \in A^2 \mid a \neq b \text{ and } a, b \in \bar{c} \text{ for some } \bar{c} \in R^{\mathfrak{A}},\ R \in \Sigma \}\,.$$

Definition 5.2. Let $\mathfrak{A}$ be a relational structure. The following definitions will only be used in this section.

(a) We denote by $d(a,b)$ the distance between a and b in $\mathcal{G}(\mathfrak{A})$.

(b) For $X, Y \subseteq A$, we set $d(X,Y) := \min\{\, d(a,b) \mid a \in X,\ b \in Y \,\}$.

(c) The *r-neighbourhood* of $a \in A$ is the set

$$N(r,a) := \{\, b \in A \mid d(a,b) < r \,\}.$$

For $\bar{a} \in A^n$, we set $N(r,\bar{a}) := \bigcup_i N(r,a_i)$. In particular, we have $N(r,\langle\rangle) := \varnothing$. Finally,

$$\mathfrak{N}(r,\bar{a}) := \langle \mathfrak{A}|_{N(r,\bar{a})}, \bar{a} \rangle$$

is the substructure induced by $N(r,\bar{a})$.

(d) The *$N(r)$-type* of $\bar{a} \subseteq A$ is the isomorphism type of $\mathfrak{N}(r,\bar{a})$, i.e., the $\cong$-class of this structure.

(e) For a $N(r)$-type τ, let $\#_\tau(\mathfrak{A})$ be the number of tuples $\bar{a} \subseteq A$ that have $N(r)$-type τ.

(f) Finally, for $k, m, n < \omega$, recall that

$$m =_k n \quad :\text{iff} \quad m = n \text{ or } m, n \geq k\,.$$

Theorem 5.3 (Hanf). *Let $m < \omega$ and let $\mathfrak{A}$ and $\mathfrak{B}$ be relational structures such that every 3^m-neighbourhood in $\mathfrak{A}$ and $\mathfrak{B}$ has at most $k < \aleph_0$ elements. If*

$$\#_\tau(\mathfrak{A}) =_{mk} \#_\tau(\mathfrak{B})\,, \qquad \textit{for every } N(n)\textit{-type } \tau \textit{ with } n \leq 3^m,$$

then $\mathfrak{A} \equiv_m \mathfrak{B}$.

Proof. Let I_n be the set of all partial isomorphisms $\bar{a} \mapsto \bar{b}$ with $\bar{a} \in A^{m-n}$ and $\bar{b} \in B^{m-n}$ such that $\mathfrak{N}(3^n,\bar{a}) \cong \mathfrak{N}(3^n,\bar{b})$. We claim that $(I_n)_n : \mathfrak{A} \cong_m \mathfrak{B}$.

We have $\langle\rangle \mapsto \langle\rangle \in I_m$. By symmetry, we therefore only need to prove the forth property. Suppose that $\bar{a} \mapsto \bar{b} \in I_{n+1}$. By definition, there exists an isomorphism

$$\pi : \mathfrak{N}(3^{n+1},\bar{a}) \cong \mathfrak{N}(3^{n+1},\bar{b})\,.$$

Let $c \in A$. If $c \in N(2\cdot 3^n,\bar{a})$ then $N(3^n,\bar{a}c) \subseteq N(3^{n+1},\bar{a})$ and setting $d := \pi(c)$ we have

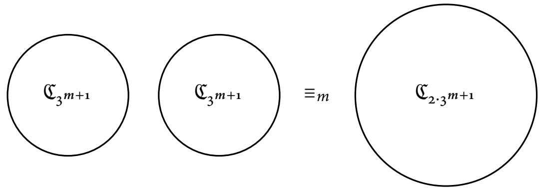
$$\pi : \mathfrak{N}(3^n,\bar{a}c) \cong \mathfrak{N}(3^n,\bar{b}d)\,, \quad \text{that is,} \quad \bar{a}c \mapsto \bar{b}d \in I_n\,.$$

If, on the other hand, $c \notin N(2\cdot 3^n,\bar{a})$ then $d(N(3^n,\bar{a}), N(3^n,c)) > 1$. Let τ be the $N(3^n)$-type of c. Since π is an isomorphism we have the same number of elements of 3^n-type τ in $N(2\cdot 3^n,\bar{a})$ and $N(2\cdot 3^n,\bar{b})$. This number is at most $|\bar{a}|\cdot k = (m-n-1)\cdot k < mk$. Since $\#_\tau(\mathfrak{A}) =_{mk} \#_\tau(\mathfrak{B})$ there exists some $d \in B \setminus N(2\cdot 3^n,\bar{b})$ of $N(3^n)$-type τ. Let $\sigma : \mathfrak{N}(3^n,c) \cong \mathfrak{N}(3^n,d)$ be the corresponding isomorphism of neighbourhoods. It follows that

$$\pi \cup \sigma : \mathfrak{N}(3^n,\bar{a}c) \cong \mathfrak{N}(3^n,\bar{b}d)\,,$$

which implies that $\bar{a}c \mapsto \bar{b}d \in I_n$. □

Example. (a) We have already seen in the example on page 518 that there is no first-order formula expressing that a graph is connected. The Theorem of Hanf allows an easy alternate proof. For a contradiction, suppose that there is such a formula φ and let m be its quantifier rank.

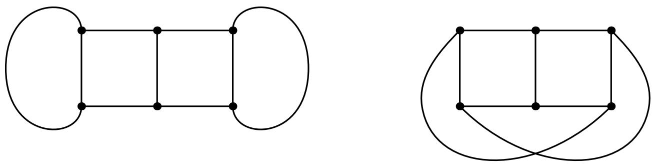


Let $\mathfrak{A} := \mathfrak{C}_{3^{m+1}} \cup \mathfrak{C}_{3^{m+1}}$ be the graph consisting of two disjoint copies of the cycle of length 3^{m+1} and let $\mathfrak{B} := \mathfrak{C}_{2\cdot 3^{m+1}}$ be the cycle of length $2\cdot 3^{m+1}$. Then we have

$$\#_\tau(\mathfrak{A}) = \#_\tau(\mathfrak{B})\,, \qquad \text{for every } N(r)\text{-type } \tau \text{ with } r \leq 3^m.$$

By the Theorem of Hanf, it follows that $\mathfrak{A} \equiv_m \mathfrak{B}$. In particular, $\mathfrak{A} \vDash \varphi$ iff $\mathfrak{B} \vDash \varphi$. Contradiction.

(b) In the same way we can prove that planarity of a graph is not expressible in first-order logic. If φ is a formula of quantifier rank m then, by the Theorem of Hanf, it cannot distinguish between the graphs



where each line represents a path of length 3^{m+1}. Since one of the graphs if planar while the other one is not it follows that φ does not define the class of planar graphs.

With the help of the Theorem of Hanf we can avoid playing Ehrenfeucht-Fraïssé games, but the theorem can only be applied to structures where the r-neighbourhoods are finite. If we want to drop this restriction we have to replace the isomorphism type of a neighbourhood by its α-equivalence type. This is the idea behind the Theorem of Gaifman below.

Remark. Let Σ be a finite signature. For all $n < \omega$, there exists a formula $\varphi_n(x, y) \in \mathrm{FO}[\Sigma]$ such that

$$\mathfrak{A} \vDash \varphi_n(a, b) \quad \text{iff} \quad d(a, b) < n\,, \quad \text{for every } \Sigma\text{-structure } \mathfrak{A}\,.$$

Definition 5.4. (a) A set $X \subseteq A$ is *r-scattered* if $d(a, b) \geq r$, for all distinct elements $a, b \in X$.

(b) For $\varphi(\bar{x}) \in \mathrm{FO}[\Sigma, X]$, we denote by $\varphi^{(r)}(\bar{x})$ the relativisation of φ to the (definable) set $N(r, \bar{x})$.

(c) A sentence of the form

$$\exists x_0 \cdots \exists x_{n-1} \bigwedge_{i<k} \big(d(x_i, x_k) \geq 2r \wedge \psi^{(r)}(x_i)\big)$$

is called *basic local.* A boolean combination of basic local sentences is called *local.*

Lemma 5.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures. We have $\mathfrak{A} \equiv_{\mathrm{FO}} \mathfrak{B}$ if and only if*

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad \mathfrak{B} \vDash \varphi \qquad \textit{for all basic local sentences } \varphi\,.$$

Proof. We have to show that $\mathfrak{A}|_{\Sigma_0} \equiv_m \mathfrak{B}|_{\Sigma_0}$, for all $m < \omega$ and all finite $\Sigma_0 \subseteq \Sigma$. Fix m and Σ_0 and let I_n be the set of all partial isomorphisms $\bar{a} \mapsto \bar{b} \in \mathrm{pIso}(\mathfrak{A}|_{\Sigma_0}, \mathfrak{B}|_{\Sigma_0})$ with $|\bar{a}| \in A^{m-n}$ and $|\bar{b}| \in B^{m-n}$ such that

$$\mathfrak{N}(7^n, \bar{a}) \equiv_{g(n)} \mathfrak{N}(7^n, \bar{b})\,,$$

where $g : \omega \to \omega$ is some function that will be specified below. We claim that $(I_n)_n : \mathfrak{A}|_{\Sigma_0} \equiv_m \mathfrak{B}|_{\Sigma_0}$.

Since $\langle\rangle \mapsto \langle\rangle \in I_m$ it remains to prove the forth property. Let $\bar{a} \mapsto \bar{b} \in I_{n+1}$ and $c \in A$. By Lemma 2.1, there exist formulae $\varphi^n_{\mathfrak{D},\bar{d}}$ such that

$$\mathfrak{C} \vDash \varphi^n_{\mathfrak{D},\bar{d}}(\bar{c}) \quad \text{iff} \quad \langle \mathfrak{C}, \bar{c} \rangle \equiv_n \langle \mathfrak{D}, \bar{d} \rangle\,.$$

If we define

$$\psi^n_{\bar{d}} := \big(\varphi^{g(n)}_{\mathfrak{N}(7^n,\bar{d})}\big)^{(7^n)}$$

then we have

$$\mathfrak{C} \vDash \psi^n_{\bar{d}}(\bar{c}) \quad \text{iff} \quad \mathfrak{N}(7^n, \bar{d}) \equiv_{g(n)} \mathfrak{N}(7^n, \bar{c})\,.$$

We distinguish two cases. If $c \in N(2 \cdot 7^n, \bar{a})$ then

$$\mathfrak{N}(7^{n+1}, \bar{a}) \vDash \exists z \big(d(\bar{a}, z) < 2 \cdot 7^n \wedge \psi^n_{\bar{a}c}(\bar{a}z)\big)\,.$$

Choose $g(n+1)$ such that it is larger than the quantifier rank of this formula. Then it follows that

$$\mathfrak{N}(7^{n+1}, \bar{b}) \vDash \exists z \big(d(\bar{b}, z) < 2 \cdot 7^n \wedge \psi^n_{\bar{a}c}(\bar{b}z)\big)\,.$$

Therefore, there is some $d \in N(7^{n+1}, \bar{b})$ such that

$$\mathfrak{N}(7^n, \bar{a}c) \equiv_{g(n)} \mathfrak{N}(7^n, \bar{b}d)\,, \quad \text{that is,} \quad \bar{a}c \mapsto \bar{b}d \in I_n\,.$$

It remains to consider the case that $c \notin N(2 \cdot 7^n, \bar{a})$. Then

$$d(N(7^n, \bar{a}), N(7^n, c)) > 1 .$$

The formula

$$\delta_s(\bar{x}) := \bigwedge_{l<k<s} d(x_l, x_k) \geq 4 \cdot 7^n \wedge \bigwedge_{l<s} \psi^n_c(x_l)$$

says that the set $\{x_0, \dots, x_{s-1}\}$ is $(4 \cdot 7^n)$-scattered and the 7^n-neighbourhood of every x_l is $g(n)$-equivalent to $\mathfrak{N}(7^n, c)$. Choose e maximal such that

$$\mathfrak{N}(7^{n+1}, \bar{a}) \vDash \chi_e := \exists x_0 \cdots \exists x_{e-1} \Big(\delta_e(\bar{x}) \wedge \bigwedge_{k<e} d(\bar{a}, x_k) < 2 \cdot 7^n \Big) .$$

Note that e is well-defined since $N(2 \cdot 7^n, \bar{a})$ does not contain a $(4 \cdot 7^n)$-scattered set of size greater than $|\bar{a}| = m - n - 1$. If we choose $g(n+1)$ large enough such that $\operatorname{qr}(\chi_e \wedge \neg\chi_{e+1}) \leq g(n+1)$ it follows that

$$\mathfrak{N}(7^{n+1}, \bar{b}) \vDash \chi_e \wedge \neg\chi_{e+1} .$$

Since the sentence $\vartheta_i := \exists x_0 \cdots \exists x_{i-1} \delta_i(\bar{x})$ is basic local we have

$$\mathfrak{B} \vDash \vartheta_i \quad \text{iff} \quad \mathfrak{A} \vDash \vartheta_i .$$

If $\mathfrak{B} \vDash \vartheta_{e+1}$ then there exists some $d \in B \smallsetminus N(2 \cdot 7^n, \bar{b})$ such that $\mathfrak{B} \vDash \psi^n_c(d)$. It follows that $\mathfrak{N}(7^n, c) \equiv_{g(n)} \mathfrak{N}(7^n, d)$ and $\bar{a}c \mapsto \bar{b}d \in I_n$.

It remains to consider the case that $\mathfrak{B} \nvDash \vartheta_{e+1}$. Then the distance between $\bar{a}$ and every element satisfying $\psi^n_c(x)$ is less than

$$4 \cdot 7^n + 2 \cdot 7^n = 6 \cdot 7^n < 7^{n+1}.$$

Since $c \notin N(2 \cdot 7^n, \bar{a})$ we have

$$\mathfrak{N}(7^{n+1}, \bar{a}) \vDash \exists z \big[2 \cdot 7^n \leq d(\bar{a}, z) < 6 \cdot 7^n \wedge \psi^n_c(z) \wedge \psi^n_{\bar{a}}(\bar{a}) \big]$$

which implies that

$$\mathfrak{N}(7^{n+1}, \bar{b}) \vDash \exists z \big[2 \cdot 7^n \leq d(\bar{b}, z) < 6 \cdot 7^n \wedge \psi^n_c(z) \wedge \psi^n_{\bar{a}}(\bar{b}) \big]$$

if we choose $g(n+1)$ larger than the quantifier rank of this formula. Therefore, there exists some element $d \in N(7^{n+1}, \bar{b})$ with

$$2 \cdot 7^n \leq d(\bar{b}, d) < 6 \cdot 7^n$$

such that

$$\mathfrak{N}(7^n, c) \equiv_{g(n)} \mathfrak{N}(7^n, d) .$$

It follows that $\mathfrak{N}(7^n, \bar{a}c) \equiv_{g(n)} \mathfrak{N}(7^n, \bar{b}d)$ and $\bar{a}c \mapsto \bar{b}d \in I_n$, as desired. □

The preceding lemma implies that every sentence is equivalent to a local one.

Theorem 5.6 (Gaifman). *Every sentence $\varphi \in \mathrm{FO}^0$ is equivalent to some local sentence.*

Proof. Let $\Phi := \{ \psi \mid \psi \text{ is local and } \varphi \vDash \psi \}$. We claim that $\Phi \vDash \varphi$. By the Compactness Theorem, it then follows that $\Phi_0 \vDash \varphi$, for some finite subset $\Phi_0 \subseteq \Phi$. This implies that $\varphi \equiv \bigwedge \Phi_0$.

Suppose that $\mathfrak{A} \vDash \Phi$. We have to show that $\mathfrak{A} \vDash \varphi$. Set

$$\Psi := \{ \psi \mid \psi \text{ is local and } \mathfrak{A} \vDash \psi \} .$$

If $\Psi \cup \{\varphi\}$ has some model $\mathfrak{B}$ then, since $\mathfrak{B} \vDash \Psi$ and local sentences are closed under negation, it follows by the preceding lemma that $\mathfrak{B} \equiv_{\mathrm{FO}} \mathfrak{A}$ and

$$\mathfrak{B} \vDash \varphi \quad \text{implies} \quad \mathfrak{A} \vDash \varphi .$$

Therefore, it is sufficient to show that $\Psi \cup \{\varphi\}$ is satisfiable. Suppose otherwise. Then, by the Compactness Theorem, there are finitely many formulae $\psi_0, \dots, \psi_n \in \Psi$ such that

$$\psi_0 \wedge \cdots \wedge \psi_n \vDash \neg\varphi .$$

Hence, we have $\neg\psi_0 \vee \dots \vee \neg\psi_n \in \Phi$ which implies that $\mathfrak{A} \vDash \neg\psi_0 \vee \dots \vee \neg\psi_n$. It follows that there is some $i \leq n$ with $\psi_i \notin \Psi$. Contradiction. □

c5. General model theory

1. Classifying logical systems

In this chapter we start with a more systematic investigation of the various extensions of first-order logic. Let us isolate some desirable properties a logic may have.

Definition 1.1. Let L and L' be logics. We write $L \leq L'$ if, for every $\varphi \in L$, there exists a formula $\varphi' \in L'$ such that $\mathrm{Mod}_{L'}(\varphi') = \mathrm{Mod}_L(\varphi)$.

Similarly, if $\mathcal{L} : \mathfrak{S} \to \mathfrak{Logic}$ and $\mathcal{L}' : \mathfrak{S}' \to \mathfrak{Logic}$ are logical systems then we write $\mathcal{L} \leq \mathcal{L}'$ if there exists a functor $F : \mathfrak{S} \to \mathfrak{S}'$ such that

$$\mathcal{L}[s] \leq \mathcal{L}'[F(s)]\,, \quad \text{for all } s \in \mathfrak{S}\,.$$

We write $L \equiv L'$ if $L \leq L'$ and $L \geq L'$. By $L < L'$ we denote the fact that $L \leq L'$ and $L \not\equiv L'$. The same notation is used for logical systems.

Definition 1.2. Let L be a logical system.

(a) L has the *finite occurrence property* if L is algebraic and, for every $\varphi \in L[\Sigma]$, there exists a finite set S of sorts and a finite S-sorted signature $\Sigma_0 \subseteq \Sigma$ such that φ is equivalent to some formula in $L[\Sigma_0]$.

(b) L is *compact* if every inconsistent set $\Phi \subseteq L[s]$ has a finite subset that is already inconsistent. Similarly, we call L *countably compact* if every countable inconsistent set $\Phi \subseteq L[s]$ has a finite inconsistent subset.

(c) L has the *Löwenheim-Skolem property* if it is algebraic and every formula $\varphi \subseteq L[\Sigma]$ that is satisfiable has a countable model.

(d) L has the *Karp property* if it is algebraic and

$$\mathfrak{A} \cong_\infty \mathfrak{B} \quad \text{implies} \quad \mathfrak{A} \equiv_L \mathfrak{B}\,.$$

(e) L is *closed under relativisations* if it is algebraic and, for all formulae $\varphi \in L[\Sigma]$ and $\chi_i \in L^{s_i}[\Sigma \cup \Gamma]$, for $i < n$, there exists a formula $\varphi^{(\bar{\chi})} \in L[\Sigma]$ such that we have

$$\mathfrak{A} \vDash \varphi^{(\bar{\chi})} \quad \text{iff} \quad (\mathfrak{A}|_\Sigma)|_{\bigcup_i \chi_i^{\mathfrak{A}}} \vDash \varphi\,,$$

whenever $\mathfrak{A}$ is a $(\Sigma \cup \Gamma)$-structure such that the set $\bigcup_i \chi_i^{\mathfrak{A}}$ induces a substructure of $\mathfrak{A}|_\Sigma$.

(f) L is *closed under substitutions* if it is algebraic and, for all formulae $\varphi \in L[\Sigma \cup \{R\}]$ and $\chi \in L^{\bar{s}}[\Sigma]$ where R is a relation symbol of type $\bar{s}$, there exists a formula $\varphi' \in L[\Sigma]$ such that

$$\mathfrak{A} \vDash \varphi' \quad \text{iff} \quad \langle \mathfrak{A}, \chi^{\mathfrak{A}} \rangle \vDash \varphi\,, \quad \text{for every } \Sigma\text{-structure } \mathfrak{A}\,.$$

(g) L has the *Tarski union property* if it is algebraic and, for every L-chain $(\mathfrak{A}_\alpha)_{\alpha<\delta}$, we have $\mathfrak{A}_\beta \preceq_L \bigcup_{\alpha<\delta} \mathfrak{A}_\alpha$, for all $\beta < \delta$.

(h) Let us define the following abbreviations:

- (A) L is algebraic.
- (B) L is boolean closed.
- (B_+) L is closed under finite conjunctions and disjunctions.
- (C) L is compact.
- (CC) L is countably compact.
- (FOP) L has the finite occurrence property.
- (KP) L has the Karp property.
- (LSP) L has the Löwenheim-Skolem property.
- (REL) L is closed under relativisations.
- (SUB) L is closed under substitutions.
- (TUP) L has the Tarski union property.

(i) L is called *weakly regular* if it satisfies (A), (B_+), and (FOP). If L satisfies (A), (B), (FOP), (REL), and (SUB) then it is called *regular.*

Example. FO° has all of the above properties but, if $\kappa > \aleph_0$ then $\text{FO}^\circ_{\kappa\aleph_0}$ satisfies only (A), (B), (B_+), (KP), (REL), and (SUB).

Exercise 1.1. Prove that SO does not have the Karp property.

Lemma 1.3. *Suppose that $L_0 \leq L_1$. If L_1 satisfies (C), (CC), (LSP), or (KP) then so does L_0.*

Exercise 1.2. (a) Suppose that L is closed under disjunction. Prove that L is compact if and only if the type space $S(L)$ is compact.

(b) Suppose that the logic L is compact and closed under negation. Let $\Phi \subseteq L$ and $\varphi \in L$. Prove that $\Phi \vDash \varphi$ if and only if $\Phi_0 \vDash \varphi$, for some finite subset $\Phi_0 \subseteq \Phi$.

The following lemmas summarise some consequences of compactness.

Lemma 1.4. *Let L be a logic with (B) and (C). If*

$$\varphi \equiv \bigvee_{i \in I} \bigwedge \Phi_i\,, \quad \text{for } \varphi \in L \text{ and } \Phi_i \subseteq L\,, i \in I\,,$$

then there exist finite sets $I_0 \subseteq I$ and $\Phi_i^0 \subseteq \Phi_i$ such that

$$\varphi \equiv \bigvee_{i \in I_0} \bigwedge \Phi_i^0\,.$$

Proof. For every $i \in I$, we have $\Phi_i \vDash \varphi$ which implies that $\Phi_i \cup \{\neg\varphi\}$ is inconsistent. Since L is compact it follows that there exists a finite subset $\Phi_i^0 \subseteq \Phi_i$ such that $\Phi_i^0 \cup \{\neg\varphi\}$ is inconsistent, i.e., $\Phi_i^0 \vDash \varphi$. Set $\psi_i := \bigwedge \Phi_i^0$ and let $\Psi := \{\, \psi_i \mid i \in I \,\}$. If the set

$$\Gamma := \{\varphi\} \cup \{\, \neg\psi \mid \psi \in \Psi \,\}$$

has a model $\mathfrak{I}$ then $\mathfrak{I} \vDash \varphi$ implies that $\mathfrak{I} \vDash \Phi_i$, for some i. In particular, we have $\mathfrak{I} \vDash \psi_i$ in contradiction to $\mathfrak{I} \vDash \neg\psi_i$.

Consequently, Γ is inconsistent and there exists a finite subset $\Psi_0 \subseteq \Psi$ such that

$$\{\varphi\} \cup \{\, \neg\psi \mid \psi \in \Psi_0 \,\}$$

is inconsistent. Set $\vartheta := \bigvee \Psi_0$. It follows that $\varphi \vDash \vartheta$.

Conversely, if $\mathfrak{J} \models \vartheta$ then $\mathfrak{J} \models \psi_i$, for some i, and $\psi_i \models \varphi$ implies that $\mathfrak{J} \models \varphi$. Hence, we also have $\vartheta \models \varphi$. Let $I_0 := \{ i \in I \mid \psi_i \in \Psi_0 \}$. Then we have

$$\varphi \equiv \vartheta \equiv \bigvee_{i \in I_0} \bigwedge \Phi_i^0 . \qquad \square$$

Lemma 1.5. *Let $L_0 \leq L_1$ be logics where L_0 satisfies* (B$_+$) *and L_1 satisfies* (B) *and* (C). *If*

$$\mathfrak{A} \equiv_{L_0} \mathfrak{B} \quad \textit{implies} \quad \mathfrak{A} \equiv_{L_1} \mathfrak{B}$$

then $L_0 \equiv L_1$.

Proof. Let φ be an L_1-formula. Then

$$\varphi \equiv \bigvee \{ \bigwedge \mathrm{Th}_{L_0}(\mathfrak{J}) \mid \mathfrak{J} \in \mathrm{Mod}_{L_1}(\varphi) \} .$$

By Lemma 1.4, we can find finitely many interpretations $\mathfrak{J}_0, \dots, \mathfrak{J}_n$ and finite subsets $\Phi_i \subseteq \mathrm{Th}_{L_0}(\mathfrak{J}_i)$ such that

$$\varphi \equiv \bigwedge \Phi_0 \vee \dots \vee \bigwedge \Phi_n .$$

Since L_0 satisfies (B$_+$) it follows that there is an L_0-formula $\psi \equiv \varphi$. $\square$

Lemma 1.6. *Let L be an algebraic logic with* (B) *and $\forall \leq L$. If L has the compactness property then it has the finite occurrence property.*

Proof. Suppose that $\varphi \in L[\Sigma]$. Let $\Sigma' := \{ \xi' \mid \xi \in \Sigma \}$ be a disjoint copy of Σ and let $\mu : \Sigma \to \Sigma' : \xi \mapsto \xi'$ be the corresponding bijection. Consider the set of first-order formulae

$$\begin{aligned} \Phi := &\{ \forall \bar{x}(R\bar{x} \leftrightarrow R'\bar{x}) \mid R \in \Sigma \text{ a relation symbol} \} \\ &\cup \{ \forall \bar{x}(f\bar{x} = f'\bar{x}) \mid f \in \Sigma \text{ a function symbol} \} . \end{aligned}$$

Since $\forall \leq L$ there exists an equivalent set $\tilde{\Phi} \subseteq L[\Sigma \cup \Sigma']$ of L-formulae. If $\varphi' := L[\mu](\varphi)$ then

$$\tilde{\Phi} \cup \{\varphi\} \models \varphi' \quad \text{and} \quad \tilde{\Phi} \cup \{\varphi'\} \models \varphi .$$

By (C), we can find finite subsets $\tilde{\Phi}_0, \tilde{\Phi}_1 \subseteq \tilde{\Phi}$ such that

$$\tilde{\Phi}_0 \cup \{\varphi\} \models \varphi' \quad \text{and} \quad \tilde{\Phi}_1 \cup \{\varphi'\} \models \varphi .$$

Let Φ_0 and Φ_1 be the subsets of Φ corresponding to $\tilde{\Phi}_0$ and $\tilde{\Phi}_1$. Fix a finite signature Γ such that $\Phi_0, \Phi_1 \subseteq \mathrm{FO}[\Gamma \cup \Gamma']$. For a Σ-structure $\mathfrak{A}$, we denote by $\mathfrak{A}_+$ the $(\Sigma \cup \Gamma')$-expansion of $\mathfrak{A}$ where $(\xi')^{\mathfrak{A}_+} = \xi^{\mathfrak{A}}$, for all $\xi \in \Gamma$. We claim that

$$\mathfrak{A}|_\Gamma \equiv_L \mathfrak{B}|_\Gamma \quad \text{implies} \quad \mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi ,$$

for all Σ-structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that $\mathfrak{A} \models \varphi$. Then $\mathfrak{A}_+ \models \tilde{\Phi}_0 \cup \varphi$, which implies that $\mathfrak{A}_+ \models \varphi'$. Note that $\mathfrak{A}|_\Gamma \equiv_L \mathfrak{B}|_\Gamma$ implies that $\mathfrak{A}_+|_{\Gamma'} \equiv_L \mathfrak{B}_+|_{\Gamma'}$. Consequently, it follows that $\mathfrak{B}_+ \models \varphi'$. Since $\mathfrak{B}_+ \models \tilde{\Phi}_1$ we obtain $\mathfrak{B} \models \varphi$, as desired.

For $\mathfrak{A} \in \mathrm{Str}[\Sigma]$, let $\Phi_{\mathfrak{A}} := \mathrm{Th}_{L[\Gamma]}(\mathfrak{A}|_\Gamma)$. By the above claim it follows that

$$\varphi \equiv \bigvee \{ \bigwedge \Phi_{\mathfrak{A}} \mid \mathfrak{A} \in \mathrm{Mod}_{L[\Sigma]}(\varphi) \} .$$

By Lemma 1.4, there are finitely many structures $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ and finite subsets $\Psi_i \subseteq \Phi_{\mathfrak{A}_i}$ such that

$$\varphi \equiv \bigwedge \Psi_0 \vee \dots \vee \bigwedge \Psi_n \in L[\Gamma] . \qquad \square$$

2. *Hanf and Löwenheim numbers*

The Compactness Theorem and the Upward and Downward Löwenheim-Skolem Theorems are central results in first-order model theory. While the Compactness Theorem fails for many natural logics, we can generalise the Löwenheim-Skolem theorems to most of them. The *Hanf* and the *Löwenheim number* of a logic measure the extend to which a logic satisfies these theorems. For their definition we need the following notions.

Definition 2.1. Let L be an algebraic logic and $\Phi \subseteq L[\Sigma]$ a set of L-formulae.

(a) We say that Φ *pins down* a cardinal κ if there is a unary predicate $P \in \Sigma$ such that Φ has a model $\mathfrak{A}$ with $|P^{\mathfrak{A}}| = \kappa$ but Φ does not have models $\mathfrak{A}$ where $P^{\mathfrak{A}}$ has arbitrarily high cardinality.

(b) Φ *pins down* an ordinal α if there exists a binary relation $< \in \Sigma$ such that

- in every model of Φ the relation $<$ is a well-order of its field and
- there exists a model of Φ such that $<$ is of order type α.

Definition 2.2. Let L be an algebraic logic and κ a cardinal.

(a) The *Hanf number* $\text{hn}_\kappa(L)$ of L is the supremum of all cardinals that can be pinned down by a set of L-formulae of size at most κ. If the supremum is undefined we set $\text{hn}_\kappa(L) := \infty$.

(b) The *Löwenheim number* $\text{ln}_\kappa(L)$ of L is the least cardinal λ such that every satisfiable set of L-formulae of size at most κ has a model of cardinality at most λ. If there is no such cardinal then we set $\text{ln}_\kappa(L) := \infty$.

(c) The *well-ordering number* $\text{wn}_\kappa(L)$ of L is the supremum of all ordinals α that can be pinned down by a set of L-formulae of size at most κ. If the supremum is undefined we set $\text{wn}_\kappa(L) := \infty$. If $\text{wn}_1(L) < \infty$ then L is called *bounded*.

(d) The *occurrence number* $\text{occ}(L)$ of L is the least cardinal κ such that, for every signature Σ and all formulae $\varphi \in L[\Sigma]$, there exists a signature $\Sigma_0 \subseteq \Sigma$ and a formula $\psi \in L[\Sigma_0]$ such that $|\Sigma_0| \leq \kappa$ and $\psi \equiv \varphi$. Again, if there is no such cardinal then we set $\text{occ}(L) := \infty$.

Remark. A logic L has (LSP) iff $\text{ln}_1(L) = \aleph_0$.

Hanf and Löwenheim numbers for first-order logic were already computed in Theorems C2.4.12 and C2.3.7.

Theorem 2.3. $\text{hn}_\kappa(\text{FO}) = \aleph_0$ *and* $\text{ln}_\kappa(\text{FO}) = \kappa \oplus \aleph_0$, *for all* κ.

Theorem 2.4. $\text{ln}_\kappa(\text{FO}_{\kappa^+\aleph_0}) = \kappa$.

Lemma 2.5. *For every regular cardinal* κ, *we have* $\text{wn}_1(\text{FO}_{\kappa\aleph_0}) \geq \kappa$ *and* $\text{occ}(\text{FO}_{\kappa\aleph_0}) = \kappa^- := \sup\{\lambda \mid \lambda < \kappa\}$.

Proof. We have already seen in Lemma C1.1.7 that every ordinal $\alpha < \kappa$ is finitely $\text{FO}_{\kappa\aleph_0}$-axiomatisable.

For the occurrence number note that $\text{occ}(\text{FO}_{\kappa\aleph_0}) < \kappa$ since each $\text{FO}_{\kappa\aleph_0}$-formula has less than κ subformulae. Conversely, for every $\lambda < \kappa$, we have the formula

$$\bigwedge_{i<\lambda} P_i x$$

with λ different relation symbols. □

Lemma 2.6. $\text{wn}_1(\text{MSO}) = \infty$.

Proof. The example on page 484 shows that the class of all well-orders is finitely MSO-axiomatisable. □

In general the Hanf numbers of $\text{FO}_{\kappa^+\aleph_0}$ depend on the model of set theory. In ZFC we can only prove the following bounds.

Theorem 2.7. $\beth_{\kappa^+} \leq \text{hn}_1(\text{FO}_{\kappa^+\aleph_0}) < \beth_{(2^\kappa)^+}$.

For the special case of $\text{FO}_{\aleph_1\aleph_0}$ the exact value can be computed. (The proof is based on the study of Borel subsets of the type space and employs Corollary C4.2.5.)

Theorem 2.8 (Hanf). $\text{hn}_1(\text{FO}_{\aleph_1\aleph_0}) = \beth_{\omega_1}$.

(Note that $\text{hn}_1(\text{FO}_{\kappa^+\aleph_0}) = \text{hn}_\kappa(\text{FO}_{\kappa^+\aleph_0})$ since we can take conjunctions over sets of size κ.) We will prove the lower bound in Corollary 2.12 below. The computation of the upper bound is deferred to Corollary E7.1.13 (where we only prove the weaker statement that $\text{hn}_1(\text{FO}_{\kappa^+\aleph_0}) \leq \beth_{(2^\kappa)^+}$).

Lemma 2.9. *Let* L *be a logical system with* $\forall\exists \leq L$.

(a) *If* $\text{hn}_\kappa(L) < \infty$ *then* $\text{hn}_\kappa(L)$ *is a limit cardinal and a cardinal* λ *can be pinned down by a set* $\Phi \subseteq L$ *of size* κ *if and only if* $\lambda < \text{hn}_\kappa(L)$.

(b) *If* $\text{wn}_\kappa(L) < \infty$ *then* $\text{wn}_\kappa(L)$ *is a limit ordinal and an ordinal* α *can be pinned down by a set* $\Phi \subseteq L$ *of size* κ *if and only if* $\alpha < \text{wn}_\kappa(L)$.

Proof. (a) Let Φ be a set of size at most κ pinning down the cardinal μ via the relation symbol P. We construct a set Ψ of the same size pinning down μ^+. Let S be the set of sorts appearing in Φ. Choose new binary relation symbols $<$ and R_s, for $s \in S$, a new unary relation symbol Q, and a new binary function symbol f. Ψ consists of formulae expressing the following properties.

- $<$ is a linear order of the set Q.
- For every $u \in Q$, the set $R(u) := \{ x \mid \langle u, x\rangle \in R_s \text{ for some } s \in S \}$ induces a substructure satisfying Φ.
- For every $u \in Q$, the function $x \mapsto f(u,x)$ is an injective map from $\Downarrow u$ into $R(u) \cap P$.

It follows that Ψ has a model where $<$ has the order type μ^+. To see that $|Q|$ cannot become arbitrarily large let λ be some cardinal such that Φ has no models with $|P| = \lambda$. Given any model of Ψ fix a strictly increasing cofinal map $f : \alpha \to Q$. By the third condition above we have $|\Downarrow f(i)| < \lambda$, for all $i < \alpha$. Consequently,

$$Q = \bigcup_{i<\alpha} \Downarrow f(i)$$

implies that $|Q| \leq \lambda$.

(b) The statement that $\mathfrak{A} = \langle A, \leq\rangle$ is a linear order with exactly $n < \omega$ elements can be expressed in $\forall$. Since $\forall\exists \leq L$ it follows that $\mathrm{wn}_\kappa(L) \geq \omega$.

To prove the claim we show that if α is pinned down by some $\Phi \subseteq L$ of size $|\Phi| \leq \kappa$ then so is $\alpha + 1$ and every ordinal $\beta \leq \alpha$.

Suppose that $\Phi \subseteq L$ pins down $\alpha \geq \omega$ via the relation symbol $<$. Let P be a new unary relation symbol and $\sqsubset$ a new binary one.

We can construct a set $\Phi \cup \{\psi\}$ pinning down every ordinal $\beta \leq \alpha$ via $\sqsubset$ by defining

$$\psi := \forall x \forall y (x \sqsubset y \leftrightarrow (Px \wedge Py \wedge x < y)),$$

which expresses that $\sqsubset = <|_P$.

Similarly, we can define a set $\Phi \cup \{\psi\}$ pinning down $\alpha + 1$ via $\sqsubset$ by defining

$$\psi := \forall x \forall y \big[x \sqsubset y \leftrightarrow \big[(x < y \wedge \exists z(z < x)) \vee (y < x \wedge \neg\exists z(z < y))\big]\big],$$

which states that $\sqsubset$ is the order obtained from $<$ by moving the least element to the end. □

Under very general conditions, we can show that a logical system L has a Hanf number and a Löwenheim number.

Proposition 2.10. *Let L be an algebraic logic such that $L[\Sigma]$ is a set, for all Σ. If $\mathrm{occ}(L) < \infty$ then we have $\mathrm{hn}_\kappa(L) < \infty$ and $\mathrm{ln}_\kappa(L) < \infty$, for all κ.*

Proof. Set $\mu := \kappa \otimes \mathrm{occ}(L)$ and fix an *universal* signature Σ of size μ, that is, Σ is S-sorted, for some set of sorts with $|S| = \mu$, and Σ contains, for all sorts $\bar{s}$ and t, μ relation symbols of type $\bar{s}$ and μ function symbols of type $\bar{s} \to t$. It is sufficient to consider sets $\Phi \subseteq L[\Sigma]$ since every signature of size μ can be embedded into Σ and, by definition of a logical system, L-formulae are invariant under such changes of the signature.

For every set $\Phi \subseteq L[\Sigma]$ of size $|\Phi| \leq \kappa$ and every unary predicate $P \in \Sigma$, we define two cardinals $\nu_{\Phi,P}$ and λ_Φ as follows. If Φ has models $\mathfrak{A}$ where $P^{\mathfrak{A}}$ can be arbitrarily large then we set $\nu_{\Phi,P} := 0$. Otherwise, let $\nu_{\Phi,P}$ be the least cardinal such that Φ has only models $\mathfrak{A}$ with $|P^{\mathfrak{A}}| \leq \nu_\Phi$. Similarly, if Φ is satisfiable then we set

$$\lambda_\Phi := \min \{ |A| \mid \mathfrak{A} \vDash \Phi \}.$$

Otherwise, we let λ_Φ undefined. It follows that

$$\mathrm{hn}_\kappa(L) = \sup \{ \nu_{\Phi,P} \mid P \in \Sigma,\ \Phi \subseteq L[\Sigma] \text{ of size } |\Phi| \leq \kappa \},$$

and $$\mathrm{ln}_\kappa(L) = \sup \{ \lambda_\Phi \mid \Phi \subseteq L[\Sigma] \text{ satisfiable and of size } |\Phi| \leq \kappa \}.$$

Note that the supremum on the right-hand side exists since, by the Axiom of Replacement, it is taken over a set of cardinals. □

Theorem 2.11. *Let L be a regular logical system with* FO $\leq L$ *such that, for every ordinal $\alpha < \mathrm{wn}_\kappa(L)$, there exists a set $\Phi_\alpha \subseteq L[\Sigma_\alpha]$ of size $|\Phi_\alpha| < \kappa$ pinning down α in a model of size at most $\mathrm{hn}_\kappa(L)$. Then we have*

$$\mathrm{hn}_\kappa(L) \geq \beth_{\mathrm{wn}_\kappa(L)}(\lambda), \quad \textit{for all } \lambda < \mathrm{hn}_\kappa(L).$$

Proof. Let X be a set of size λ. We define inductively a variant of the cumulative hierarchy by

$$\begin{aligned} P_0(X) &:= X, \\ P_{\alpha+1}(X) &:= \wp(P_\alpha(X)), \\ P_\delta(X) &:= \bigcup_{\alpha<\delta} P_\alpha(X), \quad \text{for limit ordinals } \delta. \end{aligned}$$

Then $|P_\alpha(X)| = \beth_\alpha(\lambda)$.

Since $\lambda < \mathrm{hn}_\kappa(L)$ we can find a set $\Psi \subseteq L[\Gamma]$ of size $|\Psi| \leq \kappa$ pinning down λ via a predicate Q. Suppose that Σ_α is S-sorted and Γ is T-sorted with $S \cap T = \varnothing$ and let $p \notin S \cup T$ be a new sort. Choose new unary predicates O, U, a binary relation symbol E, unary functions ρ, ζ, and a constant 0. We define a set Θ_α of formulae that is meant to describe a structure $\mathfrak{A}$ of the following form. We have $\mathfrak{A}|_S \vDash \Phi_\alpha$ and $\mathfrak{A}|_T \vDash \Psi$. Furthermore, $U \subseteq A_p \subseteq P_\alpha(U)$ for the ordinal α encoded in $\mathfrak{A}|_S$. The relation E is the membership relation of sets, $\rho : A_p \to O$ maps every set in $P_\beta(U)$ to the ordinal β, and $\zeta : Q \to U$ is a bijection. Formally, Θ_α consists of the union $\Psi \cup \Phi_\alpha$ together with the following formulae.

- The domains with sort T form a model of Φ_α and 0 is the least element of $<$.

  $$\forall x(Ox \leftrightarrow x \leq x)$$
  $$(\forall x.Ox)(0 \leq x)$$

- $\zeta : Q \to U$ is a bijection and ρ maps A_p to the field of $<$.

  $$\forall x(Qx \leftrightarrow U\zeta x)$$
  $$\forall x \forall y(\zeta x = \zeta y \to x = y)$$
  $$\forall x O \rho x$$

  (In the last formula x is of sort p.)

- $\rho^{-1}(\alpha) \subseteq P_\alpha(U)$ and E is the element relation.

  $$\forall x(Ux \leftrightarrow \rho x = 0)$$
  $$(\forall x.\neg Ux)[\forall y.\neg Uy)(\forall z(Ezx \leftrightarrow Ezy) \to x = y]$$
  $$\begin{aligned}\forall x(\forall u.Ou)\big[\rho x = u \leftrightarrow [&(\forall y.Eyx)(\rho y < \rho x) \\ &\wedge (\forall v.v < u)(\exists y.Eyx)(\rho y \geq v)]\big]\end{aligned}$$

If $\mathfrak{A}$ is a model of Θ_α then $<^{\mathfrak{A}}$ is a well-order of type $\beta < \mathrm{wn}_\kappa(L)$ and there exists an injective function $A \to P_\beta(U^{\mathfrak{A}})$. Consequently,

$$|A_p| \leq \beth_\beta(|U^{\mathfrak{A}}|).$$

Since Ψ pins down a cardinal we further have

$$|U^{\mathfrak{A}}| = |Q^{\mathfrak{A}}| \leq \mathrm{hn}_\kappa(L).$$

Therefore, Θ_α does not have models where A_p is arbitrarily large, but it does have a model $\mathfrak{A}$ with $|A_p| = \beth_\alpha(\lambda)$. □

We have shown in Lemma c1.1.7 that every ordinal $\alpha < \kappa^+$ can be defined in $\mathrm{FO}_{\kappa^+\aleph_0}$. Consequently, we obtain the following lower bound on the Hanf number.

Corollary 2.12. $\mathrm{hn}_\kappa(\mathrm{FO}_{\kappa^+\aleph_0}) \geq \beth_{\kappa^+}$

Lemma 2.13. *Suppose that L is a regular logical system with* FO $\leq L$*. Then L is countably compact if and only if $\mathrm{wn}_{\aleph_0}(L) = \omega$.*

Proof. A standard compactness argument shows that if L is countably compact and $\Phi \subseteq L$ has a model such that $<$ is of order type ω then there also is a model where $<$ contains an infinite descending chain. Consequently, (cc) implies $\mathrm{wn}_{\aleph_0}(L) \leq \omega$.

For the converse, assume that there exists a countable inconsistent set $\{ \varphi_n \mid n < \omega \} \subseteq L$ every finite subset of which is satisfiable. By

Lemma 2.9 (b) we can prove that $\mathrm{wn}_{\aleph_0}(L) > \omega$ by constructing a countable set $\Phi \subseteq L$ pinning down ω.

Let S be the set of sorts appearing in some φ_n and choose new binary relation symbols $<$ and R_s, for $s \in S$. The set Φ consists of the following statements all of which can be expressed in first-order logic:

- $<$ is a linear ordering of its field.
- For all elements a of the field of $<$ there is some element b with $\langle a, b\rangle \in R$.
- If there are at least n elements $<$-less than a then the set $\{ b \mid \langle a, b\rangle \in R_s$ for some $s \in S \}$ induces a substructure satisfying φ_n.

It follows that if $\mathfrak{A}$ is a model of Φ then every element in the field of $<^{\mathfrak{A}}$ has only finitely many elements below. Consequently, Φ pins down all ordinals $\alpha \leq \omega$. □

3. The Theorem of Lindström

We have seen that first-order logic has many pleasant properties like compactness and the Löwenheim-Skolem property. On the other hand, its expressive power is rather restricted as far as certain aspects like counting and recursion are concerned. The question naturally arises of whether there is a stronger logic that shares the good properties of first-order logic. Surprisingly, it turns out that one can prove that such a logic does not exist.

In many of the following proofs we consider a structure containing two other structures, say, specified by unary predicates P and Q. We use additional relations to encode a back-and-forth systems between these substructures.

Definition 3.1. Suppose that Σ and Γ are signatures and $\mu : \Sigma \to \Gamma$ is an isomorphism of $\mathfrak{Sig}$. Let $\mathfrak{A}$ be a $(\Sigma \cup \Gamma)$-structure and $P, Q \subseteq A$ subsets of A.

(a) A *partial isomorphism modulo μ from P to Q* is a function $p : \bar{a} \mapsto \bar{b}$ with $\bar{a} \subseteq P$ and $\bar{b} \subseteq Q$ such that, for all term-reduced atomic first-order formulae $\varphi(\bar{x})$, we have

$$\mathfrak{A} \vDash \varphi(\bar{a}) \quad \text{iff} \quad \mathfrak{A} \vDash \mathrm{FO}[\mu](\varphi)(\bar{b}) .$$

(b) A *pseudo back-and-forth system* (*modulo μ from P to Q*) is a sequence $(I_\alpha)_{\alpha \in U}$ where

- each I_α is a set of partial isomorphisms modulo μ from P to Q,
- U is a nonempty linear order such that every element $\alpha \in U$ has an immediate successor $\alpha + 1$, except possibly for the last element,
- we have $I_\delta := \bigcap_{\alpha<\delta} I_\alpha$, for elements $\delta \in U$ without immediate predecessor, and
- every $I_{\alpha+1}$ has the back-and-forth property restricted to P and Q with respect to I_α, that is,
  - if $\bar{a} \mapsto \bar{b} \in I_{\alpha+1}$ and $c \in P$ then there is some $d \in Q$ with $\bar{a}c \mapsto \bar{b}d \in I_\alpha$, and
  - if $\bar{a} \mapsto \bar{b} \in I_{\alpha+1}$ and $d \in Q$ then there is some $c \in P$ with $\bar{a}c \mapsto \bar{b}d \in I_\alpha$.

(c) We say that a tuple $\langle U, <, \bar{P}, \bar{Q}, I, F, \bar{G}\rangle$ *encodes* a pseudo back-and-forth system $(I_\alpha)_{\alpha \in X}$ modulo μ from P to Q if there exist a finite set of sorts S and sorts u and f such that

- $\bar{P} = (P_s)_{s \in S}$, $\bar{Q} = (Q_s)_{s \in S}$, and $\bar{G} = (G_s)_{s \in S}$,
- $P = \bigcup_s P_s$ and $Q = \bigcup_s Q_s$,
- $U \subseteq A_u$, $F \subseteq A_f$, $P_s \subseteq A_s$, $Q_s \subseteq A_{\mu(s)}$,
  $I \subseteq U \times F$, $< \subseteq U \times U$, $G_s \subseteq F \times P_s \times Q_s$,
- there exists an isomorphism $\iota : \langle U, <\rangle \cong \langle X, <\rangle$,
- there exists a bijection $\pi : F \to \bigcup_\alpha I_\alpha$,
- $I = \{ \langle u, p\rangle \in U \times F \mid \pi p \in I_{\iota u} \}$,
- $G_s = \{ \langle p, a, b\rangle \in F \times P_s \times Q_s \mid (\pi p)(a) = b \}$.

Lemma 3.2. *Suppose that Σ and Γ are finite signatures and $\mu : \Sigma \to \Gamma$ an isomorphism of $\mathfrak{Sig}$. There exists a first-order formula*

$$\beta_\mu(U, <, \bar{P}, \bar{Q}, I, F, \bar{G})$$

that holds if and only if $\langle U, <, \bar{P}, \bar{Q}, I, F, \bar{G}\rangle$ encodes a pseudo back-and-forth system modulo μ from P to Q.

Proof. We have to express the following properties:

(a) $\langle U, <\rangle$ is a nonempty linear order and every element has an immediate successor, except for the last one.

$$\begin{aligned}&\exists u Uu\\&\forall u\forall v(u < v \to Uu \wedge Uv)\\&\forall u(\neg u < u)\\&\forall u\forall v\forall w(u < v \wedge v < w \to u < w)\\&(\forall u.Uu)(\forall v.Uv)(u < v \vee u = v \vee u > v)\\&\forall u[\exists v(u < v) \to \exists v(u < v \wedge \neg\exists w(u < w \wedge w < v))]\end{aligned}$$

(b) $G_s \subseteq F \times P_s \times Q_s$ encodes a set of partial isomorphisms modulo μ.

$$\begin{aligned}&\forall p\forall a\forall b(G_s pab \to Fp \wedge P_s a \wedge Q_s b)\\&\forall p\forall a_0\forall a_1\forall b_0\forall b_1[G_s pa_0b_0 \wedge G_s pa_1b_1 \to (a_0 = a_1 \leftrightarrow b_0 = b_1)]\end{aligned}$$

For all n-ary relation symbols $R \in \Sigma$,

$$\forall p\forall \bar{a}\forall \bar{b}[G_{s_0} pa_0b_0 \wedge \dots \wedge G_{s_{n-1}} pa_{n-1}b_{n-1} \to (R\bar{a} \leftrightarrow \mu(R)\bar{b})]\,.$$

For all n-ary function symbols $f \in \Sigma$,

$$\forall p\forall \bar{a}\forall c\forall \bar{b}\forall d[G_{s_0} pa_0b_0 \wedge \dots \wedge G_{s_{n-1}} pa_{n-1}b_{n-1} \wedge G_t pcd \to (f\bar{a} = c \leftrightarrow \mu(f)\bar{b} = d)]\,.$$

(c) $I \subseteq U \times F$ encodes a sequence of nonempty sets with the back-and-forth property.

$$\begin{aligned}&\forall u\forall p(Iup \to Uu \wedge Fp)\\&\forall u\exists p Iup\\&\forall u\forall v\forall p[Iup \wedge v < u \to (\forall c.P_s c)\exists d\exists q\eta_s)]\\&\forall u\forall v\forall p[Iup \wedge v < u \to (\forall d.Q_s d)\exists c\exists q\eta_s)]\end{aligned}$$

where $\eta_s := Ivq \wedge G_s qcd \wedge \bigwedge_t \forall a\forall b(G_t pab \to G_t qab)$. □

Lemma 3.3. *Let Σ and Γ be finite signatures and $\mu : \Sigma \to \Gamma$ an isomorphism of $\mathfrak{Sig}$. Let $\mathfrak{A}$ be a $(\Sigma \cup \Gamma)$-structure and $P, Q \subseteq A$. Suppose that P and Q induce substructures of, respectively, $\mathfrak{A}|_\Sigma$ and $\mathfrak{A}|_\Gamma$.*

If there exists a pseudo back-and-forth system $(I_\alpha)_{\alpha\in U}$ modulo μ from P to Q where U is not well-ordered then

$$\mathfrak{A}|_\Sigma|_P \cong_\infty \mathfrak{A}|_\Gamma|_Q|_\mu\,.$$

Proof. Fix an infinite descending sequence $\alpha_0 > \alpha_1 > \dots$ in U. We claim that $J = \bigcup_n I_{\alpha_n}$ has the back-and-forth property with respect to itself. If $p \in J$ then $p \in I_{\alpha_n}$, for some n. Hence, for every $c \in P$ or $d \in Q$, we can find a suitable extension $q \in I_{\alpha_{n+1}} \subseteq J$ with, respectively, $c \in \operatorname{dom} q$ or $d \in \operatorname{rng} q$. Consequently,

$$J : \mathfrak{A}|_\Sigma|_P \cong_\infty \mathfrak{A}|_\Gamma|_Q|_\mu\,.$$ □

Definition 3.4. Let L and L' be logical systems and $\varphi, \psi \in L[s]$.

(a) φ and ψ are *contradictory* if

$$\operatorname{Mod}_L(\varphi) \cap \operatorname{Mod}_L(\psi) = \varnothing\,.$$

(b) A formula $\chi \in L'[t]$ *separates* φ from ψ if

$$\operatorname{Mod}_{L'}(\chi) \supseteq \operatorname{Mod}_L(\varphi) \quad\text{and}\quad \operatorname{Mod}_{L'}(\chi) \cap \operatorname{Mod}_L(\psi) = \varnothing\,.$$

We start by investigating logical systems containing first-order logic that have the Löwenheim-Skolem property. First, we show that if the logic is strictly more expressive than first-order logic then it can express finiteness.

Lemma 3.5. *Let L be a weakly regular logical system with* $\text{FO}^\circ \leq L$ *and* (LSP).

If there are contradictory formulae $\varphi, \psi \in L[\Sigma]$ that are not separated by any first-order formula $\chi \in \text{FO}[\Sigma]$ then there exists a signature Γ, a unary predicate $U \in \Gamma$, and a formula $\vartheta \in L[\Gamma]$ satisfying the following conditions:

(1) *If $\mathfrak{A} \vDash \vartheta$ then $U^{\mathfrak{A}}$ is finite and nonempty.*

(2) *For every $0 < n < \omega$, there exists a model $\mathfrak{A} \vDash \vartheta$ with $|U^{\mathfrak{A}}| = n$.*

Proof. For a contradiction, suppose that $\varphi, \psi \in L[\Sigma]$ are not separated by any first-order formula but there is no formula ϑ satisfying (1) and (2). By (FOP), we may assume that Σ is finite. We proceed in several steps.

(a) First, we prove that every formula $\chi \in L[\Gamma]$ that is not equivalent to a first-order formula has a model of cardinality $\aleph_0$. Let χ be such a formula. If χ has infinite models then choose a new unary function symbol $f \notin \Gamma$ and consider the formula

$$\chi' := \chi \wedge \text{“}f \text{ is injective but not surjective”}\,.$$

Since χ has infinite models it follows that χ' is satisfiable. By (LSP), there exists a countable model of χ'. Since there are no finite models of χ' it follows that this model is countably infinite.

It remains to consider the case that χ has only finite models. By (FOP), we may assume that Γ is finite. Thus, for every $n < \omega$, there are only finitely many non-isomorphic Γ-structures $\mathfrak{A}$ of cardinality n and each of them can be axiomatised by a first-order formula $\eta_{\mathfrak{A}}$. Consequently, χ must have models of arbitrarily large finite cardinality since, otherwise, χ would be equivalent to a finite disjunction of first-order formulae $\eta_{\mathfrak{A}}$. If $U \notin \Gamma$ is a new unary relation symbol then the formula

$$\vartheta := \chi \wedge \exists x Ux$$

satisfies (1) and (2). A contradiction.

(b) Second, we prove that, for every $n < \omega$, there are countably infinite structures $\mathfrak{A}_n$ and $\mathfrak{B}_n$ such that

$$\mathfrak{A}_n \vDash \varphi\,, \quad \mathfrak{B}_n \vDash \psi\,, \quad \text{and} \quad \mathfrak{A}_n \equiv_n \mathfrak{B}_n\,.$$

Let $\eta^n_{\mathfrak{A}}$ be the Hintikka-formula of $\mathfrak{A}$ of quantifier-rank n and set

$$\chi_n := \bigvee \{\, \eta^n_{\mathfrak{A}} \mid \mathfrak{A} \vDash \varphi \,\}\,.$$

Since Σ is finite we have $\eta^n_{\mathfrak{A}} \in \text{FO}[\Sigma]$ and there are only finitely many different Hintikka-formulae of quantifier rank n. Consequently, $\chi_n \in \text{FO}[\Sigma]$.

Since $\varphi \vDash \chi_n$ we have $\text{Mod}(\varphi) \subseteq \text{Mod}(\chi_n)$. As φ and ψ cannot be separated it follows that

$$\text{Mod}(\chi_n) \cap \text{Mod}(\psi) \neq \varnothing\,.$$

Hence, $\psi \wedge \chi_n$ is satisfiable and it is not equivalent to any first-order formula. By (a), there exists a countably infinite model $\mathfrak{B}_n \vDash \psi \wedge \chi_n$. In particular, we have $\mathfrak{B}_n \vDash \eta_{\mathfrak{A}}$, for some $\mathfrak{A} \vDash \varphi$. Moreover, $\varphi \wedge \eta_{\mathfrak{A}}$ is satisfiable and not equivalent to any first-order formula. Thus, by (a), we can find a countably infinite model $\mathfrak{A}_n \vDash \varphi \wedge \eta_{\mathfrak{A}}$. Note that $\mathfrak{A}_n \equiv_n \mathfrak{B}_n$ because both $\mathfrak{A}_n$ and $\mathfrak{B}_n$ satisfy $\eta_{\mathfrak{A}}$. Hence, $\mathfrak{A}_n$ and $\mathfrak{B}_n$ have the desired properties.

(c) Finally, we derive a contradiction as follows. Let Σ' be a disjoint copy of Σ and let $\mu : \Sigma \to \Sigma'$ be the corresponding bijection. If $\mathfrak{C}$ is a model of the L-formula

$$\begin{aligned}
\vartheta := {} & \varphi \wedge L[\mu](\psi) \\
& \wedge \beta_\mu(U, <, \bar{P}, \bar{Q}, I, F, \bar{G}) \\
& \wedge \textstyle\bigwedge_s \forall x(P_s x \wedge Q_s x) \\
& \wedge \exists x(\forall y.Uy)(y = x \vee x < y) \\
& \wedge \exists x(\forall y.Uy)(y = x \vee y < x) \\
& \wedge \forall x[\exists y(y < x) \to (\exists y.y < x)\neg\exists z(y < z \wedge z < x)]
\end{aligned}$$

then $\langle U, <, \bar{P}, \bar{Q}, I, F, \bar{G}\rangle$ encodes a pseudo back-and-forth system modulo μ from C to C where $\langle U, <\rangle$ is a discrete linear order with a least and greatest element. Furthermore, the Σ-reduct of $\mathfrak{C}$ satisfies φ and its Σ'-reduct satisfies $\mu(\psi)$.

For every $n < \omega$, we can find a model $\mathfrak{C}_n$ of ϑ with $|U^{\mathfrak{C}_n}| = n + 1$ as follows. By (b), there are countably infinite structures $\mathfrak{A}_n$ and $\mathfrak{B}_n$ with $\mathfrak{A}_n \equiv_n \mathfrak{B}_n$ such that $\mathfrak{A}_n \vDash \varphi$ and $\mathfrak{B}_n \vDash \psi$. Since $|A_n| = |B_n|$ we may w.l.o.g. assume that $A_n = B_n$. We form the structure $\mathfrak{C}_n$ with universe $A_n = B_n$ where, for every $\xi \in \Sigma$, we have two relations or functions

$$\xi^{\mathfrak{C}_n} := \xi^{\mathfrak{A}_n} \quad \text{and} \quad \mu(\xi)^{\mathfrak{C}_n} := \xi^{\mathfrak{B}_n}.$$

Hence, the Σ-reduct of $\mathfrak{C}_n$ equals $\mathfrak{A}_n$ and its Σ'-reduct equals $\mu(\mathfrak{B}_n)$. Furthermore, since $\mathfrak{A}_n \equiv_n \mathfrak{B}_n$ we can add relations $U, <, I, \bar{P}, \bar{Q}, F, \bar{G}$ encoding some back-and-forth system modulo μ where $|U| = n + 1$.

Consequently, the formula $\vartheta \wedge |U| = n$ is satisfiable, for all $0 < n < \omega$. This concludes the proof of (2). For (1), assume that ϑ has models where U is infinite. If f is a new unary function symbol then it follows that the formula

$$\vartheta' := \vartheta \wedge \text{“}f \text{ is injective but not surjective”}$$

is satisfiable. By (LSP), ϑ' has a countable model $\mathfrak{C}$. Let u_0 be the greatest element of $U^{\mathfrak{C}}$. Since every element of $U^{\mathfrak{C}}$ has an immediate predecessors we obtain an infinite descending sequence $u_0 > u_1 > \dots$. Hence, Lemma 3.3 implies that

$$\mathfrak{A} \cong \mathfrak{C}|_{\Sigma} \cong_{\infty} \mathrm{Str}[\mu](\mathfrak{C}|_{\Sigma'}) \cong \mathfrak{B}\,.$$

Since $\mathfrak{A}$ and $\mathfrak{B}$ are countable structures it follows by Corollary c4.4.11 that $\mathfrak{A} \cong \mathfrak{B}$. But $\mathfrak{A} \vDash \varphi$ and $\mathfrak{B} \vDash \psi$. This contradicts the fact that φ and ψ are contradictory. □

Lemma 3.6. *If L is a regular logic with* FO° $\leq L$ *then* (LSP) *implies* (KP).

Proof. For a contradiction, suppose that L is a regular logic with the Löwenheim-Skolem property but there are structures $\mathfrak{A} \cong_{\infty} \mathfrak{B}$ such that $\mathfrak{A} \vDash \varphi$ and $\mathfrak{B} \nvDash \varphi$, for some L-formula φ. By (FOP) we may assume that the signature of φ is finite.

Let $U, <, \bar{P}, \bar{Q}, I, F, \bar{G}$ be new relation symbols. By Lemma 3.2, there exists a formula $\beta_{\mathrm{id}}(U, <, \bar{P}, \bar{Q}, I, F, \bar{G})$ saying that $\langle U, <, \bar{P}, \bar{Q}, I, F, \bar{G}\rangle$ encodes a pseudo back-and-forth system from $P := \bigcup_s P_s$ to $Q := \bigcup_s Q_s$. The formula

$$\chi := \beta_{\mathrm{id}} \wedge \varphi^{(\bar{P})} \wedge (\neg\varphi)^{(\bar{Q})} \wedge (\forall x.Ux)\exists y(y < x)$$

has a model $\mathfrak{C}$ where $\langle U, <\rangle$ is an arbitrary discrete order without least element, $\mathfrak{C}|_P \cong \mathfrak{A}$, and $\mathfrak{C}|_Q \cong \mathfrak{B}$. (Note that, if there exists a pseudo back-and-forth system $(I_u)_{u \in U}$ from P to Q and the ordering U has arbitrarily large finite increasing chains then P and Q are closed under the functions of Σ. Hence, the formula implies that the sets P and Q induce substructures of $\mathfrak{C}|_{\Sigma}$.)

By (LSP), it follows that χ has a countable model $\mathfrak{C}$. Since $\langle U^{\mathfrak{C}}, <^{\mathfrak{C}}\rangle$ is not well-ordered we have $\mathfrak{C}|_P \cong_{\infty} \mathfrak{C}|_Q$, by Lemma 3.3. Because these substructures are countable it follows that $\mathfrak{C}|_P \cong \mathfrak{C}|_Q$. But $\mathfrak{C}|_P \vDash \varphi$ and $\mathfrak{C}|_Q \nvDash \varphi$. Contradiction. □

Lemma 3.7. *Let L be a weakly regular logical system with* FO° $\leq L$.

If L is countably compact and L has the Löwenheim-Skolem property then every pair of contradictory L-formulae can be separated by some FO°*-formula.*

Proof. Suppose that L satisfies (LSP) but there exists a pair of contradictory L-formulae that cannot be separated by any first-order formula. By Lemma 3.5, there exists a formula $\vartheta \in L[\Gamma]$ and a unary predicate $U \in \Gamma$ such that in models $\mathfrak{A}$ of ϑ the set $U^{\mathfrak{A}}$ can have any finite cardinality, but no infinite one. Let $\varphi_n \in L[\Gamma]$ be the L-formula equivalent to the first-order formula

$$\exists x_0 \cdots \exists x_{n-1}\Big(\bigwedge_i Ux_i \wedge \bigwedge_{i \neq k} x_i \neq x_k\Big)$$

which expresses that $|U| \geq n$. By construction, the set

$$\{\vartheta\} \cup \{\, \varphi_n \mid n < \omega \,\}$$

is inconsistent, but each of its finite subsets is satisfiable. Consequently, L is not countably compact. ☐

Combining the preceding technical lemmas we can prove that there does not exist a proper extension of first-order logic that has the Löwenheim-Skolem property and that is countably compact.

Theorem 3.8 (Lindström). *Let L be a weakly regular logical system with* (B) *and* $\mathrm{FO}^{\circ} \leq L$. *If L has the Löwenheim-Skolem property and L is countably compact then $L \equiv \mathrm{FO}^{\circ}$.*

Proof. Let $\varphi \in L[\Sigma]$. By Lemma 3.7, there exists a first-order formula χ separating φ from $\neg\varphi$. It follows that $\mathrm{Mod}(\chi) = \mathrm{Mod}(\varphi)$. ☐

We conclude this section with several variants of the Theorem of Lindström where (LSP) and (CC) are replaced by other properties.

Lemma 3.9. *Let L be a regular logical system with* $\mathrm{FO}^{\circ} < L$. *If L has the Karp property then there exists a satisfiable formula $\varphi(U, <) \in L$ such that, for all models $\mathfrak{A} \vDash \varphi$, we have*

$$\langle U^{\mathfrak{A}}, <^{\mathfrak{A}} \rangle \cong \langle \omega, < \rangle\,.$$

Proof. Fix a formula $\varphi \in L[\Sigma]$ that is not equivalent to any first-order formula. By (FOP), we may assume that Σ is finite. For every $n < \omega$, there are structures $\mathfrak{A}_n \equiv_n \mathfrak{B}_n$ such that

$$\mathfrak{A}_n \vDash \varphi \quad \text{and} \quad \mathfrak{B}_n \nvDash \varphi\,.$$

Let $U, <, \bar{P}, \bar{Q}, I, F, \bar{G} \notin \Sigma$ be new relation symbols where U is unary, $<, \bar{P}, \bar{Q}, F$ are binary, I is ternary, and $\bar{G}$ are of arity four. We modify

the formula $\beta_{\mathrm{id}}(U, <, \bar{P}, \bar{Q}, I, F, \bar{G})$ of Lemma 3.2 as follows. Let x be a variable not occurring in β_{id} and set

$$\begin{aligned} &\alpha(x, U, <, \bar{P}, \bar{Q}, I, F, \bar{G}) := \\ &\quad \beta_{\mathrm{id}}(U, <, (P_s x\_)_s, (Q_s x\_)_s, Ix\_\_, Fx\_, (G_s x\_\_\_)_s)\,, \end{aligned}$$

that is, we add x as new argument to every atom containing P_s, Q_s, I, F, G_s. The formula α states that these relations encode a sequence of pseudo back-and-forth systems indexed by x. Define

$$\chi := \exists x.Ux \wedge (\forall x.Ux)\big[\vartheta(x) \wedge \alpha(x) \wedge \varphi^{(\bar{P}x\_)} \wedge (\neg\varphi)^{(\bar{Q}x\_)}\big]\,,$$

where

$$\begin{aligned} \vartheta(x) := {} & \exists y(x < y) \\ & \wedge (\exists y(y < x) \rightarrow (\exists y.y < x)\neg\exists z(y < z \wedge z < x)) \end{aligned}$$

says that x has a successor and, if it is not the first element then it also has an immediate predecessor. The formula χ says that

- U is a nonempty discrete linear order without last element,
- for every $u \in U$, there is a pseudo back-and-forth system $(I_\alpha)_{\alpha<u}$ from $A_u := \{\, a \mid \langle u, a\rangle \in \bigcup_s P_s \,\}$ to $B_u := \{\, b \mid \langle u, b\rangle \in \bigcup_s Q_s \,\}$ of length ${\downarrow}u$,
- A_u induces a substructure that satisfies φ while B_u induces a substructure that does not satisfy φ.

Consequently, χ has a model where $\langle U, <\rangle \cong \langle \omega, <\rangle$ and the substructures induced by A_n and B_n, for $n < \omega$, are isomorphic to $\mathfrak{A}_n$ and $\mathfrak{B}_n$, respectively. Let $\mathfrak{C}$ be an arbitrary model of χ. We have to show that the order type of $\langle U^{\mathfrak{C}}, <^{\mathfrak{C}}\rangle$ is ω. Suppose otherwise. Then there exists some element $u \in U$ such that ${\downarrow}u$ is infinite. Since every element except for the first one has an immediate predecessor it follows that $<$ is not a well-order. By Lemma 3.3, we can conclude that $\mathfrak{C}|_{A_u} \cong_\infty \mathfrak{C}|_{B_u}$. Hence, $\mathfrak{C}|_{A_u} \vDash \varphi$ and $\mathfrak{C}|_{B_u} \nvDash \varphi$ contradicts (KP). ☐

Theorem 3.10. *Let L be a regular logical system with* $\mathrm{FO}^0 \leq L$. *If L has the Karp property and L is countably compact then* $L \equiv \mathrm{FO}^0$.

Proof. The claim follows immediately from the Lemma 3.9 since $\langle \omega, < \rangle$ cannot be axiomatised in a countably compact logic. □

For the next theorem we need the following variant of the Diagram Lemma.

Lemma 3.11. *Suppose that L is a regular logical system such that L is compact and* $\mathrm{FO} \leq L$. *Let $\mathfrak{A}$ be a structure and* $\Phi \subseteq L$.

There exists an elementary extension $\mathfrak{B} \succeq_{\mathrm{FO}} \mathfrak{A}$ *with* $\mathfrak{B} \vDash \Phi$ *if and only if* $\mathrm{Th}_{\mathrm{FO}}(\mathfrak{A}) \cup \Phi$ *is satisfiable.*

Proof. ($\Rightarrow$) Clearly, $\mathfrak{B} \succeq_{\mathrm{FO}} \mathfrak{A}$ and $\mathfrak{B} \vDash \Phi$ implies that $\mathfrak{B} \vDash \mathrm{Th}_{\mathrm{FO}}(\mathfrak{A}) \cup \Phi$.

($\Leftarrow$) Let $\Gamma := \mathrm{Th}_{\mathrm{FO}}(\mathfrak{A}_A)$. If $\mathfrak{B} \vDash \Gamma \cup \Phi$ then $\mathfrak{B}$ is the desired elementary extension of $\mathfrak{A}$. Hence, it is sufficient to show that $\Gamma \cup \Phi$ is satisfiable. For a contradiction, suppose otherwise. Since L is compact there exist finite subsets $\Gamma_0 \subseteq \Gamma$ and $\Phi_0 \subseteq \Phi$ such that $\Gamma_0 \cup \Phi_0$ is inconsistent. Let $\gamma(\bar{a}) := \bigwedge \Gamma_0$ where $\bar{a}$ are the constant symbols appearing in Γ_0. Then $\mathfrak{A} \vDash \exists \bar{x} \gamma(\bar{x})$. Hence, $\Phi_0 \cup \{\exists \bar{x} \gamma(\bar{x})\} \subseteq \mathrm{Th}_{\mathrm{FO}}(\mathfrak{A}) \cup \Phi$. This contradicts the assumption that the latter set is satisfiable. □

Theorem 3.12. *Let L be a regular logical system with* $\mathrm{FO}^0 \leq L$. *If L has the Tarski union property and L is compact then* $L \equiv \mathrm{FO}^0$.

Proof. Suppose that $\mathrm{FO}^0 < L$. By Lemma 1.5, there are structures $\mathfrak{A} \equiv \mathfrak{B}$ such that $\mathfrak{A} \vDash \varphi$ and $\mathfrak{B} \vDash \neg\varphi$, for some L-formula φ. We construct an elementary chain $(\mathfrak{A}^n)_{n<\omega}$ such that

- $\mathfrak{A}^n \preceq_L \mathfrak{A}^{n+2}$, for all n, and
- $\mathfrak{A}^n \vDash \varphi$ iff n is even.

Then, $\mathfrak{C} := \bigcup_n \mathfrak{A}^n = \bigcup_n \mathfrak{A}^{2n} = \bigcup_n \mathfrak{A}^{2n+1}$. By (TUP) it follows that $\mathfrak{A}^0 \preceq_L \mathfrak{C}$ and $\mathfrak{A}^1 \preceq_L \mathfrak{C}$. Consequently, we have

$$\mathfrak{A}^0 \vDash \varphi \quad \text{iff} \quad \mathfrak{C} \vDash \varphi \quad \text{iff} \quad \mathfrak{A}^1 \vDash \varphi\,.$$

A contradiction.

It remains to define the chain $(\mathfrak{A}^n)_n$. Let $\mathfrak{A}^0 := \mathfrak{A}$. Since

$$\mathrm{Th}_{\mathrm{FO}}(\mathfrak{A}) \cup \{\neg\varphi\} = \mathrm{Th}_{\mathrm{FO}}(\mathfrak{B}) \cup \{\neg\varphi\}$$

is satisfiable we can use Lemma 3.11 to find an elementary extension $\mathfrak{A}^1 \succeq \mathfrak{A}^0$ with $\mathfrak{A}^1 \vDash \neg\varphi$. Suppose that $\mathfrak{A}^n$ has already been defined. Since

$$\mathrm{Th}_{\mathrm{FO}}(\mathfrak{A}^n_{A_{n-1}}) = \mathrm{Th}_{\mathrm{FO}}(\mathfrak{A}^{n-1}_{A_{n-1}}) \subseteq \mathrm{Th}_L(\mathfrak{A}^{n-1}_{A_{n-1}})$$

it follows that

$$\mathrm{Th}_{\mathrm{FO}}(\mathfrak{A}^n_{A_{n-1}}) \cup \mathrm{Th}_L(\mathfrak{A}^{n-1}_{A_{n-1}})$$

is a satisfiable set of L-formulae. By Lemma 3.11, there exists an elementary extension $\mathfrak{A}^{n+1} \succeq \mathfrak{A}^n$ with $\mathfrak{A}^{n+1} \succeq_L \mathfrak{A}^{n-1}$, as desired. □

Theorem 3.13. *Let L be a regular logical system with* $\mathrm{FO}^0 \leq L$. *If L has the Karp property and L is bounded then* $L \leq \mathrm{FO}_{\infty\aleph_0}$.

Proof. For a contradiction, suppose that there exists an L-formula φ that is not equivalent to any $\mathrm{FO}_{\infty\aleph_0}$-formula.

First, we show that there are structures $\mathfrak{A}_\alpha \equiv_\alpha \mathfrak{B}_\alpha$, for $\alpha \in \mathrm{On}$, such that $\mathfrak{A}_\alpha \vDash \varphi$ and $\mathfrak{B}_\alpha \nvDash \varphi$. Set

$$\psi_\alpha := \bigvee \{ \eta^\alpha_{\mathfrak{A}} \mid \mathfrak{A} \vDash \varphi \}\,,$$

where $\eta^\alpha_{\mathfrak{A}}$ is the Hintikka-formula of $\mathfrak{A}$ of quantifier rank α. Then $\varphi \vDash \psi_\alpha$ and, by assumption, $\psi_\alpha \nvDash \varphi$. Hence, there exist structures $\mathfrak{B}_\alpha \vDash \psi_\alpha$ with $\mathfrak{B}_\alpha \nvDash \varphi$. By definition of ψ_α, it follows that $\mathfrak{B}_\alpha \equiv_\alpha \mathfrak{A}_\alpha$, for some $\mathfrak{A}_\alpha \vDash \varphi$.

As in Lemma 3.9 we can define a formula χ stating that,

- U is a discrete linear order without last element,
- for every $u \in U$, there exists a pseudo back-and-forth system from $A_u := \{ a \mid \langle u, a \rangle \in \bigcup_s P_s \}$ to $B_u := \{ b \mid \langle u, b \rangle \in \bigcup_s Q_s \}$ of length $\downarrow u$,

- A_u induces a substructure that satisfies φ while B_u induces a substructure that does not satisfy φ.

For every ordinal α, we can define a model $\mathfrak{C}_\alpha$ of χ where $\langle U, < \rangle$ is of order type α, $\mathfrak{C}_\alpha|_{A_\beta} \cong \mathfrak{A}_\beta$, and $\mathfrak{C}_\alpha|_{B_\beta} \cong \mathfrak{B}_\beta$. Since L is bounded it follows that χ has a model $\mathfrak{C}$ where $\langle U, < \rangle$ is not well-founded. By Lemma 3.3, it follows that $\mathfrak{C}|_{A_u} \cong_\infty \mathfrak{C}|_{B_u}$, for some $u \in U$. But $\mathfrak{C}|_{A_u} \vDash \varphi$ and $\mathfrak{C}|_{B_u} \nvDash \varphi$ contradicts (KP). □

4. Projective classes

The common idea behind Skolemisation and Chang's Reduction consists in constructing a theory T such that every structure in a given class has an expansion to a model of T. This section contains a more systematic investigation of such reductions.

Definition 4.1. (a) Let $\mathcal{K}$ be a class of Σ-structures and let $\Gamma \subseteq \Sigma$ be a subsignature. The *Γ-projection* of $\mathcal{K}$ is the class

$$\mathrm{pr}_\Gamma(\mathcal{K}) := \{ \mathfrak{A}|_\Gamma \mid \mathfrak{A} \in \mathcal{K} \}$$

of all Γ-reducts of structures in $\mathcal{K}$.

(b) Let L be an algebraic logic and κ either a cardinal or ∞. A class $\mathcal{K}$ of Σ-structures is a *κ-projective L-class* if there exists a signature $\Sigma_+ \supseteq \Sigma$ and a set $\Psi \subseteq L[\Sigma_+]$ of size $|\Psi| \leq \kappa$ such that

$$\mathcal{K} = \mathrm{pr}_\Sigma(\mathrm{Mod}_{L[\Sigma_+]}(\Psi)) .$$

The class of all such classes $\mathcal{K}$ is denoted by $\mathrm{PC}_\kappa(L, \Sigma)$. Furthermore, we set

$$\mathrm{PC}_{<\kappa}(L, \Sigma) := \bigcup_{\lambda < \kappa} \mathrm{PC}_\lambda(L, \Sigma) .$$

Projective FO-classes are also called *pseudo-elementary.*

(c) Let L_0 and L_1 be algebraic logics and κ a cardinal or ∞. We say that L_0 is *κ-projectively reducible* to L_1 and we write $L_0 \leq^\kappa_{\mathrm{pc}} L_1$ if

$$\mathrm{Mod}_{L_0[\Sigma]}(\varphi) \in \mathrm{PC}_\kappa(L_1, \Sigma) , \quad \text{for all } \Sigma \text{ and every } \varphi \in L_0[\Sigma] .$$

Example. The class of all ordered abelian groups is first-order axiomatisable. It follows that the class of all abelian groups that can be ordered is pseudo-elementary.

Exercise 4.1. Prove that $L \leq^1_{\mathrm{pc}} \mathrm{SO}_{\kappa\aleph_0}$ implies $L \leq \mathrm{SO}_{\kappa\aleph_0}$.

The results of Section C2.3 can be restated in the following form.

Lemma 4.2. $\mathrm{FO}_{\kappa\aleph_0} \leq^1_{\mathrm{pc}} \forall_{\kappa\aleph_0}$.

Proof. For every formula $\varphi \in \mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ we can use Lemma C2.3.3 to find a formula $\varphi^* \in \forall_{\kappa\aleph_0}[\Sigma^*, X]$ with $\varphi^* \vDash \varphi$ such that we can expand every model $\mathfrak{A}$ of φ to a model $\mathfrak{A}^*$ of φ^*. Consequently,

$$\mathrm{Mod}(\varphi) = \mathrm{pr}_\Sigma(\mathrm{Mod}(\varphi^*)) .$$ □

Lemma 4.3. *If $L_0 \leq^\kappa_{\mathrm{pc}} L_1$ then*

$$\mathrm{Mod}_{L_0[\Sigma]}(\Phi) \in \mathrm{PC}_{\kappa \oplus |\Phi|}(L_1, \Sigma) , \quad \textit{for all } \Phi \subseteq L_0[\Sigma] .$$

Proof. For every $\varphi \in \Phi$, there exists a signature $\Sigma(\varphi) \supseteq \Sigma$ and a set $\Psi(\varphi) \subseteq L_1[\Sigma(\varphi)]$ of size at most κ such that

$$\mathrm{Mod}(\varphi) = \mathrm{pr}_\Sigma(\mathrm{Mod}_{L[\Sigma(\varphi)]}(\Psi(\varphi))) .$$

We can choose these signatures such that $\Sigma(\varphi) \cap \Sigma(\psi) = \Sigma$, for $\varphi \neq \psi$. Setting $\Psi := \bigcup_{\varphi \in \Phi} \Psi(\varphi)$ it follows that

$$\mathrm{Mod}(\Phi) = \mathrm{pr}_\Sigma(\mathrm{Mod}(\Psi)) .$$ □

Lemma 4.4. *$L_0 \leq^\kappa_{\mathrm{pc}} L_1$ implies that*

(a) $\mathrm{hn}_\kappa(L_0) \leq \mathrm{hn}_\kappa(L_1)$,

(b) $\mathrm{wn}_\kappa(L_o) \leq \mathrm{wn}_\kappa(L_1)$,

(c) $\ln_\kappa(L_o) \leq \ln_\kappa(L_1)$.

Proof. For (a) and (b), note that if there is a set $\Phi \subseteq L_o[\Sigma]$ of size $|\Phi| \leq \kappa$ that pins down a cardinal λ or an ordinal α then we can find a signature $\Sigma_+ \supseteq \Sigma$ and a set $\Phi_+ \subseteq L_1[\Sigma_+]$ of size $|\Phi_+| \leq |\Phi| \oplus \kappa = \kappa$ that does the same.

(c) Let λ be a cardinal such that every set Φ of L_1-formulae of size $|\Phi| \leq \kappa$ has a model of size at most λ. We claim that $\ln_\kappa(L_o) \leq \lambda$. For each $\Psi \subseteq L_o[\Sigma]$ of size at most κ we can find a set $\Psi_+ \subseteq L_1[\Sigma_+]$ of size $|\Psi_+| \leq |\Psi| \oplus \kappa = \kappa$ such that $\mathrm{Mod}(\Psi) = \mathrm{pr}_\Sigma(\mathrm{Mod}(\Psi_+))$. Consequently, $\mathrm{Mod}(\Phi)$ contains a structure of size at most λ. ☐

Lemma 4.5. *Let L_o and L_1 be algebraic logics.*

(a) *If $L_o \leq^\infty_{\mathrm{pc}} L_1$ and L_1 is compact then so is L_o.*

(b) *If $L_o \leq^{\aleph_0}_{\mathrm{pc}} L_1$ and L_1 is countably compact then so is L_o.*

Proof. Both claims can be proved in the same way. Suppose that every finite subset of $\Phi \subseteq L_o[\Sigma]$ is satisfiable. For every finite $\Phi_o \subseteq \Phi$, fix a signature $\Sigma(\Phi_o) \supseteq \Sigma$ and a set $\Phi_o^+ \subseteq L_1[\Sigma(\Phi_o)]$ such that

$$\mathrm{Mod}(\Phi_o) = \mathrm{pr}_\Sigma(\mathrm{Mod}(\Phi_o^+)) .$$

For (b), we can choose Φ_o^+ to be countable. By replacing

$$\Sigma(\Phi_o) \quad \text{by} \quad \bigcup\{\, \Sigma(\Psi) \mid \Psi \subseteq \Phi_o \,\}$$

and $\quad \Phi_o^+ \quad$ by $\quad \bigcup\{\, \Psi^+ \mid \Psi \subseteq \Phi_o \,\}$

we may assume that $\Phi_o \subseteq \Phi_1$ implies $\Sigma(\Phi_o) \subseteq \Sigma(\Phi_1)$ and $\Phi_o^+ \subseteq \Phi_1^+$.

We claim that the set

$$\Phi^+ := \bigcup\{\, \Phi_o^+ \mid \Phi_o \subseteq \Phi \text{ finite} \,\}$$

is satisfiable. Note that, in case (b), Φ^+ is a countable union of countable sets. Since L_1 is, respectively, compact and countably compact it is sufficient to prove that every finite subset of Φ^+ is satisfiable.

For every finite subset $\Psi \subseteq \Phi^+$ we can find finitely many finite subsets $\Phi_o, \dots, \Phi_n \subseteq \Phi$ such that $\Psi \subseteq \Phi_o^+ \cup \dots \cup \Phi_n^+$. Setting $\Gamma := \Phi_o \cup \dots \cup \Phi_n$ it follows that $\Psi \subseteq \Gamma^+$. Hence, $\mathrm{Mod}(\Gamma) \neq \emptyset$ implies that $\mathrm{Mod}(\Psi) \neq \emptyset$, as desired.

Consequently, there exists a model $\mathfrak{A}^+ \vDash \Phi^+$. Let $\mathfrak{A} := \mathfrak{A}^+|_\Sigma$. Then we have $\mathfrak{A} \vDash \Phi_o$, for all finite subsets $\Phi_o \subseteq \Phi$. This implies that $\mathfrak{A} \vDash \Phi$. ☐

Lemma 4.6. *Let L_o, L_1 be algebraic logics and $\langle \alpha, \beta \rangle : L_o[\Sigma_o] \to L_1[\Sigma_1]$ a comorphism such that, for every signature $\Gamma_o \supseteq \Sigma_o$, there exist a signature $\Gamma_1 \supseteq \Sigma_1$ an epimorphism $\langle \alpha_+, \beta_+ \rangle : L_1[\Gamma_1] \to L_o[\Gamma_o]$, and a set $\Psi \subseteq L_1[\Gamma_1]$ such that*

$$\beta_+(\mathfrak{A})|_{\Sigma_1} = \beta(\mathfrak{A}|_{\Sigma_1}) , \quad \textit{for all } \Gamma_o\textit{-sructures } \mathfrak{A} ,$$

and $\mathrm{rng}\,\beta_+ = \mathrm{Mod}_{L_1[\Gamma_1]}(\Psi)$.

$$\begin{array}{ccc} \mathrm{Str}[\Gamma_o] & \xrightarrow{\ \beta_+\ } & \mathrm{Str}[\Gamma_1] \\ \big\downarrow{\scriptstyle \mathrm{pr}_{\Sigma_o}} & & \big\downarrow{\scriptstyle \mathrm{pr}_{\Sigma_1}} \\ \mathrm{Str}[\Sigma_o] & \xrightarrow[\ \beta\ ]{} & \mathrm{Str}[\Sigma_1] \end{array}$$

Then $\mathcal{K} \in \mathrm{PC}_\kappa(L_o, \Sigma_o)$ implies $\beta[\mathcal{K}] \in \mathrm{PC}_\kappa(L_1, \Sigma_1)$.

Proof. Suppose that $\mathcal{K} = \mathrm{pr}_{\Sigma_o}(\mathrm{Mod}(\Phi_o))$, for some $\Phi_o \subseteq L_o[\Gamma_o]$. Let $\langle \alpha_+, \beta_+ \rangle : L_1[\Gamma_1] \to L_o[\Gamma_o]$ be the corresponding epimorphism of the expansion and $\langle \gamma, \delta \rangle : L_o[\Gamma_o] \to L_1[\Gamma_1]$ its right inverse. We set

$$\Phi_1 := \gamma[\Phi_o] \cup \Psi .$$

Then we have

$$\begin{array}{lll} \mathfrak{B} \vDash \Phi_1 & \text{iff} & \mathfrak{B} \vDash \gamma[\Phi_o] \text{ and } \mathfrak{B} = \beta_+(\mathfrak{A}) \text{ for some } \mathfrak{A} \\ & \text{iff} & \mathfrak{B} = \beta_+(\mathfrak{A}) \text{ for some } \mathfrak{A} \text{ with } \beta_+(\mathfrak{A}) \vDash \gamma[\Phi_o] \\ & \text{iff} & \mathfrak{B} = \beta_+(\mathfrak{A}) \text{ for some } \mathfrak{A} \text{ with } \mathfrak{A} \vDash (\alpha \circ \gamma)[\Phi_o] \\ & \text{iff} & \mathfrak{B} = \beta_+(\mathfrak{A}) \text{ for some } \mathfrak{A} \text{ with } \mathfrak{A} \vDash \Phi_o . \end{array}$$

Hence, $\mathrm{Mod}_{L_1}(\Phi_1) = \beta_+[\mathrm{Mod}_{L_0}(\Phi_0)]$ and it follows that

$$\begin{aligned}\mathfrak{A} \in \beta[\mathcal{K}] \quad &\text{iff} \quad \mathfrak{A} = \beta(\mathfrak{A}'|_{\Sigma_0}) \quad \text{for some } \mathfrak{A}' \vDash \Phi_0 \\ &\text{iff} \quad \mathfrak{A} = \beta_+(\mathfrak{A}')|_{\Sigma_1} \quad \text{for some } \mathfrak{A}' \vDash \Phi_0 \\ &\text{iff} \quad \mathfrak{A} = \mathfrak{A}'|_{\Sigma_1} \quad \text{for some } \mathfrak{A}' \vDash \Phi_1 .\end{aligned}$$

Consequently, we have $\beta[\mathcal{K}] = \mathrm{pr}_{\Sigma_1}(\mathrm{Mod}(\Phi_1))$. □

Corollary 4.7. *Suppose that $\Sigma_0 \subseteq \Sigma_1$ are signatures and $(\varphi_\xi)_{\xi \in \Sigma_1 \setminus \Sigma_0}$ is a sequence of $\mathrm{FO}_{\kappa\aleph_0}[\Sigma_0]$-formulae. Let $\langle \alpha, \beta \rangle : \mathrm{FO}_{\kappa\aleph_0}[\Sigma_0] \to \mathrm{FO}_{\kappa\aleph_0}[\Sigma_1]$ be the comorphism where β maps a structure $\mathfrak{A}$ to its expansion defined by $(\varphi_\xi)_\xi$. If $\mathcal{K} \in \mathrm{PC}_\kappa(\mathrm{FO}_{\kappa\aleph_0}, \Sigma_0)$ then $\beta[\mathcal{K}] \in \mathrm{PC}_\kappa(\mathrm{FO}_{\kappa\aleph_0}, \Sigma_1)$.*

Proof. We have to show that $\langle \alpha, \beta \rangle$ satisfies the condition of the preceding lemma. Given Γ_0 set $\Gamma_1 := \Sigma_1 \cup \Gamma_0$. We define $\langle \alpha_+, \beta_+ \rangle$ as follows. The function β_+ maps a Γ_0-structure $\mathfrak{A}$ to the Γ_1-structure $\mathfrak{B}$ such that $\mathfrak{B}|_{\Gamma_0} = \mathfrak{A}$ and $\mathfrak{B}|_{\Sigma_1} = \beta(\mathfrak{A}|_{\Sigma_0})$. Then $\langle \alpha_+, \beta_+ \rangle$ is an epimorphism whose right inverse is given by the reduct operation. By definition, it satisfies $\beta_+(\mathfrak{A})|_{\Sigma_1} = \beta(\mathfrak{A}|_{\Sigma_0})$. Furthermore, we can define the range of β_+ by formulae of the form

$$\forall \bar{x}[R\bar{x} \leftrightarrow \varphi_R(\bar{x})] \quad \text{and} \quad \forall \bar{x} \forall y[f\bar{x} = y \leftrightarrow \varphi_f(\bar{x}, y)] . \qquad □$$

We can generalise the notion of a projective class by replacing the reduct operation by a combination of a reduct and a domain restriction.

Definition 4.8. Let Σ be an S-sorted signature.

(a) Let $\mathfrak{A}$ be a Σ-structure. A *relativised reduct* of $\mathfrak{A}$ is a structure of the form $\mathfrak{A}|_{\Sigma_0}|_P$ where $\Sigma_0 \subseteq \Sigma$ and $P \subseteq A$ induces a substructure of $\mathfrak{A}|_{\Sigma_0}$.

(b) Let L be an algebraic logic and κ either a cardinal or ∞. A class $\mathcal{K}$ of Σ-structures is a *relativised κ-projective L-class* if there exists a signature $\Sigma_+ \supseteq \Sigma$, a set $\Psi \subseteq L[\Sigma_+]$ of size $|\Psi| \leq \kappa$, and unary predicates $P_s \in \Sigma_+$, for $s \in S$, such that

$$\mathcal{K} = \{ \mathfrak{A}|_\Sigma|_{\bigcup_s P_s^{\mathfrak{A}}} \mid \mathfrak{A} \in \mathrm{Mod}_{L[\Sigma_+]}(\Psi) \} .$$

The class of all such classes is denoted by $\mathrm{RPC}_\kappa(L, \Sigma)$.

(c) Let L_0 and L_1 be algebraic logics and κ a cardinal or ∞. We say that L_0 is *relativised κ-projectively reducible* to L_1 and we write $L_0 \leq^\kappa_{\mathrm{rpc}} L_1$, if

$$\mathrm{Mod}_{L_0[\Sigma]}(\varphi) \in \mathrm{RPC}_\kappa(L_1, \Sigma) , \quad \text{for all } \Sigma \text{ and every } \varphi \in L_0[\Sigma] .$$

Lemma 4.9. *$L_0 \leq^\kappa_{\mathrm{rpc}} L_1$ implies that $\mathrm{ln}_\kappa(L_0) \leq \mathrm{ln}_\kappa(L_1)$.*

Proof. Let λ be a cardinal such that each satisfiable set Φ of L_1-formulae of size $|\Phi| \leq \kappa$ has a model of size at most λ. We claim that $\mathrm{ln}_\kappa(L_0) \leq \lambda$. For each $\Phi \subseteq L_0[\Sigma]$ of size at most κ we can find a set $\Phi_+ \subseteq L_1[\Sigma_+]$ of size $|\Phi_+| \leq |\Phi| \oplus \kappa = \kappa$ such that

$$\mathrm{Mod}(\Phi) = \{ \mathfrak{A}|_\Sigma|_{\bigcup_s P_s^{\mathfrak{A}}} \mid \mathfrak{A} \in \mathrm{Mod}_{L[\Sigma_+]}(\Phi_+) \} .$$

Consequently, if Φ is satisfiable then $\mathrm{Mod}(\Phi)$ contains a structure of size at most λ. □

Example. Let us show that $\mathrm{SO} \leq^1_{\mathrm{rpc}} \mathrm{MSO}$. Suppose that $\varphi \in \mathrm{SO}[\Sigma, X]$ where Σ is S-sorted for a finite set S. W.l.o.g. we may assume that φ contains no quantifiers over functions. Fix a number $n < \omega$ such that every second-order quantifier in φ ranges over a relation of arity at most n. For every sequence $\bar{s} \in S^{\leq n}$ of sorts of length at least 2, we add to Σ a new sort $p_{\bar{s}}$ and a function $g_{\bar{s}}$ of type $\bar{s} \to p_{\bar{s}}$. Let $\chi_{\bar{s}}$ be the formula stating that $g_{\bar{s}} : A_{s_0} \times \dots \times A_{s_{k-1}} \to A_{p_{\bar{s}}}$ is bijective. We construct a formula φ' by replacing in φ

- every second-order quantifier over a relation R of type $\bar{s}$ by a quantifier over a set X_R of sort $p_{\bar{s}}$,
- every atom $R\bar{t}$ where R is such a relation by the formula $X_R g_{\bar{s}} \bar{t}$.

Setting $\psi := \varphi' \wedge \bigwedge_{\bar{s} \in S^{\leq n}} \chi_{\bar{s}}$ it follows that

$$\mathrm{Mod}(\varphi) = \{ \mathfrak{A}|_\Sigma|_S \mid \mathfrak{A} \in \mathrm{Mod}(\psi) \} .$$

Exercise 4.2. State and prove a version of Lemma 4.6 for relativised projective classes and use it to show that the image of a relativised projective class $\mathcal{K}$ under an interpretation is again a relativised projective class.

Below we will show that for first-order logic there is no difference between projective and relativised projective classes. To do so we need some technical results about recovering a structure from a substructure.

Definition 4.10. Let $\mathfrak{A}$ be a Σ-structure and $C \subseteq A$.

(a) Let $\Gamma_0(\Sigma)$ be the signature consisting of n-ary relation symbols R_φ, for every atomic formula $\varphi \in \mathrm{FO}^{<\omega}[\Sigma]$ with $\mathrm{free}(\varphi) = \{x_0, \dots, x_{n-1}\}$. We assume that $\Gamma_0(\Sigma) \cap \Sigma = \varnothing$ and we set $\Gamma(\Sigma) := \Sigma \cup \Gamma_0(\Sigma)$.

(b) By $\langle\!\langle C \rangle\!\rangle^+_{\mathfrak{A}}$ we denote the $\Gamma(\Sigma)$-expansion of $\langle\!\langle C \rangle\!\rangle_{\mathfrak{A}}$ by the relations

$$R_\varphi := \{\, \bar a \in C^n \mid \mathfrak{A} \vDash \varphi(\bar a) \,\},$$

and we define

$$\langle\!\langle C \rangle\!\rangle^\circ_{\mathfrak{A}} := \langle\!\langle C \rangle\!\rangle^+_{\mathfrak{A}}|_{\Gamma_0(\Sigma)}|_C .$$

(c) Let $\Xi(\Sigma)$ be the first-order theory of the class

$$\mathcal{K}(\Sigma) := \big\{\, \langle\!\langle C \rangle\!\rangle^\circ_{\mathfrak{A}} \;\big|\; \mathfrak{A} \text{ a } \Sigma\text{-structure with } C \subseteq A \,\big\}.$$

Remark. Note that

$$\langle\!\langle C \rangle\!\rangle^\circ_{\mathfrak{A}} \cong \langle\!\langle D \rangle\!\rangle^\circ_{\mathfrak{B}} \quad \text{implies} \quad \langle\!\langle C \rangle\!\rangle^+_{\mathfrak{A}} \cong \langle\!\langle D \rangle\!\rangle^+_{\mathfrak{B}} .$$

Lemma 4.11. *If $\mathfrak{C} \vDash \Xi(\Sigma)^{\vDash}_{\forall}$ then there exists a Σ-structure $\mathfrak{A}$ with $A \supseteq C$ such that C generates $\mathfrak{A}$ and $\mathfrak{C} = \langle\!\langle C \rangle\!\rangle^\circ_{\mathfrak{A}}$.*

Proof. We define an equivalence relation $\sim$ on the set

$$Z := \big\{\, t(\bar c) \;\big|\; t \text{ a } \Sigma\text{-term and } \bar c \subseteq C \,\big\}$$

by $\quad s(\bar a) \sim t(\bar b) \quad :\text{iff} \quad \bar a \bar b \in R^{\mathfrak{C}}_\varphi$ where $\varphi := s(\bar x) = t(\bar y)$.

Note that $C \subseteq Z$ since we can choose $t = x$. Set $A := Z/{\sim}$. If $a, b \in C$ are elements with $a \neq b$ then $\langle a, b\rangle \notin R^{\mathfrak{C}}_{x=y}$ since

$$\forall x \forall y (R_{x=y} xy \to x = y) \in \Xi(\Sigma)_\forall .$$

This implies that $[a]_\sim \neq [b]_\sim$. Hence, the function $e : C \to A : a \mapsto [a]_\sim$ is an embedding. Let D be the range of this function. We construct a $\Gamma(\Sigma)$-structure $\mathfrak{A}$ with universe A such that $\langle\!\langle D \rangle\!\rangle^\circ_{\mathfrak{A}} \cong \mathfrak{C}$.

For $R_\varphi \in \Gamma(\Sigma)_0$, we define

$$R^{\mathfrak{A}}_\varphi := \{\, e(\bar a) \mid \bar a \in R^{\mathfrak{C}}_\varphi \,\}.$$

For atomic formulae $\psi \in \mathrm{FO}^{<\omega}[\Sigma]$, we define

$$\mathfrak{A} \vDash \psi\big([t_0(\bar a_0)]_\sim, \dots, [t_{n-1}(\bar a_{n-1})]_\sim\big) \quad :\text{iff} \quad \bar a_0 \dots \bar a_{n-1} \in R^{\mathfrak{C}}_\varphi ,$$

where $\varphi(\bar x_0, \dots, \bar x_{n-1}) := \psi(t_0(\bar x_0), \dots, t_{n-1}(\bar x_{n-1}))$.

It remains to show that D generates $\mathfrak{A}$ and that $\langle\!\langle D \rangle\!\rangle^\circ_{\mathfrak{A}} \cong \mathfrak{C}$. Let $t(\bar x)$ be a Σ-term and $\bar a \in C^n$. Then

$$t^{\mathfrak{A}}(e(\bar a)) = [t(\bar a)]_\sim$$

since setting $\psi(\bar x, y) := t(\bar x) = y$ and $\varphi := t(\bar x) = t(\bar y)$ we have

$$\begin{aligned} & \mathfrak{A} \vDash t(e(\bar a)) = [t(\bar a)]_\sim \\ \text{iff}\quad & \mathfrak{A} \vDash \psi\big([a_0]_\sim, \dots, [a_{n-1}]_\sim, [t(\bar a)]_\sim\big) \\ \text{iff}\quad & \bar a \bar a \in R^{\mathfrak{C}}_\varphi , \end{aligned}$$

and $\forall \bar x R_\varphi \bar x \bar x \in \Xi(\Sigma)_\forall$. In particular, D generates $\mathfrak{A}$.

If $\varphi(\bar x) \in \mathrm{FO}[\Sigma]$ is an atomic formula and $\bar a \in C^n$ then

$$\langle\!\langle D \rangle\!\rangle^\circ_{\mathfrak{A}} \vDash R_\varphi e(\bar a) \quad \text{iff} \quad \mathfrak{A} \vDash R_\varphi e(\bar a) \quad \text{iff} \quad \mathfrak{C} \vDash R_\varphi \bar a$$

implies that $e : \mathfrak{C} \cong \langle\!\langle D \rangle\!\rangle^\circ_{\mathfrak{A}}$. By taking isomorphic copies we may assume that $D = C \subseteq A$. □

Definition 4.12. For every model $\mathfrak{C} \vDash \Xi(\Sigma)^{\vDash}_{\forall}$, we denote by $\widehat{\mathfrak{C}}$ some structure as in the preceding lemma. Note that, up to isomorphism, $\widehat{\mathfrak{C}}$ is unique.

Lemma 4.13. *For every theory $T \subseteq \forall[\Sigma]$, there exists a theory $\widehat{T} \subseteq$ $\mathrm{FO}[\Gamma(\Sigma)_0]$ such that*

$$\widehat{\mathfrak{A}} \models T \quad \text{iff} \quad \mathfrak{A} \models \widehat{T}\,.$$

Proof. For every universal sentence φ we will construct a set $\Phi(\varphi)$ of $\mathrm{FO}^{<\omega}[\Gamma(\Sigma)_0]$-sentences such that

$$\widehat{\mathfrak{A}} \models \varphi \quad \text{iff} \quad \mathfrak{A} \models \Phi(\varphi)\,.$$

Then we can set $\widehat{T} := \bigcup\{\, \Phi(\varphi) \mid \varphi \in T \,\}$.

W.l.o.g. assume that $\varphi = \forall \bar{x} \bigwedge_i \bigvee_k \vartheta_{ik}$ where the quantifier-free part is in conjunctive normal form. For an atomic formula $\vartheta(x_0, \dots, x_{n-1})$, we have

$$\widehat{\mathfrak{A}} \models \vartheta\big([t_0(\bar{a}_0)]_\sim, \dots, [t_{n-1}(\bar{a}_{n-1})]_\sim\big)$$

iff $\quad \mathfrak{A} \models R_\psi(\bar{a}_0, \dots, \bar{a}_{n-1})$

where $\psi := \vartheta(t_0(\bar{x}_0), \dots, t_{n-1}(\bar{x}_{n-1}))$. Consequently, if, for each tuple $\bar{t}$ of Σ-terms, we define

$$\hat{\vartheta}_{ik}[\bar{t}] := \begin{cases} R_{\psi_{ik}} \bar{x}_0 \dots \bar{x}_{n-1} & \text{if } \vartheta_{ik} \text{ is an atom}\,, \\ \neg R_{\psi_{ik}} \bar{x}_0 \dots \bar{x}_{n-1} & \text{if } \vartheta_{ik} \text{ is a negated atom}\,, \end{cases}$$

where $\psi_{ik} := \vartheta_{ik}(t_0(\bar{x}_0), \dots, t_{n-1}(\bar{x}_{n-1}))$, then it follows that

$$\widehat{\mathfrak{A}} \models \bigwedge_i \bigvee_k \vartheta_{ik}\big([t_0(\bar{a}_0)]_\sim, \dots, [t_{n-1}(\bar{a}_{n-1})]_\sim\big)$$

iff $\quad \mathfrak{A} \models \bigwedge_i \bigvee_k \hat{\vartheta}_{ik}[\bar{t}](\bar{a}_0, \dots, \bar{a}_{n-1})\,.$

Hence, we can set

$$\Phi(\varphi) := \big\{\, \forall \bar{x} \textstyle\bigwedge_i \bigvee_k \hat{\vartheta}[\bar{t}] \;\big|\; \bar{t} \text{ a tuple of } \Sigma\text{-terms} \,\big\}\,.$$

(Note that every element of $\widehat{\mathfrak{A}}$ is denoted by a term with parameters from A.) ☐

Exercise 4.3. Let κ be an infinite cardinal with $\kappa > |\Sigma|$. Prove that, for every $\forall_{\kappa\aleph_0}[\Sigma]$-theory T, there exists an $\mathrm{FO}_{\kappa\aleph_0}[\Gamma(\Sigma)_0]$-theory $\widehat{T}$ as above.

Theorem 4.14. *Let $\Sigma_- \subseteq \Sigma_+$ be signatures and $P \in \Sigma_+$ a unary predicate. Suppose that*

$$\mathcal{K} = \big\{\, \mathfrak{A}|_{\Sigma_-}|_{\bigcup_s P_s^{\mathfrak{A}}} \;\big|\; \mathfrak{A} \in \mathrm{Mod}(\Phi) \,\big\}\,, \quad \textit{for some } \Phi \subseteq \mathrm{FO}^0[\Sigma_+]\,.$$

(a) *There exists a signature $\Gamma \supseteq \Sigma_-$ of size $|\Gamma| \leq |\Sigma_+| \oplus \aleph_0$ and a theory $\Psi \subseteq \mathrm{FO}^0[\Gamma]$ such that*

$$\mathcal{K} = \mathrm{pr}_{\Sigma_-}(\mathrm{Mod}(\Psi))\,.$$

(b) *If Φ is finite and every structure in $\mathcal{K}$ is infinite then we can choose a finite set Ψ as above.*

(c) *$\mathcal{K}$ is a pseudo-elementary class.*

Proof. W.l.o.g. we may assume that $\Sigma_- = \Sigma_+$. Hence, we drop the subscripts and just write Σ.

(b) Since Φ is finite we may assume that the signature Σ is finite. By the Theorem of Löwenheim and Skolem, it follows that, for every structure $\mathfrak{A} \in \mathcal{K}$, we can find a structure $\mathfrak{B} \in \mathrm{Mod}(\Phi)$ of cardinality $|B| = |A|$ such that $\mathfrak{A} = \mathfrak{B}|_{\bigcup_s P_s^{\mathfrak{B}}}$. Let $\Sigma' = \{\, \xi' \mid \xi \in \Sigma \,\}$ be a disjoint copy of Σ, and set $\Gamma := \Sigma \cup \Sigma' \cup \{f\}$, where f is a new unary function symbol. Since Φ is finite there exists a sentence $\psi \in \mathrm{FO}[\Gamma]$ expressing that

- the Σ'-reduct of the given structure is a model of Φ,
- f is a bijection between the whole universe and P.

It follows that $\mathcal{K} = \{\, \mathfrak{A}|_\Sigma \mid \mathfrak{A} \models \psi \,\}$.

(c) follows immediately from (a).

(a) By Skolemising we may assume that $\Phi \subseteq \forall$. Let $\Psi \subseteq \mathrm{FO}^0[\Gamma(\Sigma)]$ consist of $\widehat{\Phi} \cup \Xi(\Sigma)_\forall^\models$ together with with the sentences

$$\begin{aligned} &\forall \bar{x}[\varphi(\bar{x}) \leftrightarrow R_\varphi(\bar{x})] && \text{for atomic } \varphi \in \mathrm{FO}^{<\omega}[\Sigma]\,, \\ &\forall \bar{x}[R_{Pt\bar{x}}\bar{x} \rightarrow \exists y R_{t\bar{x}=y}\bar{x}y]\,, && \text{for every } \Sigma\text{-term } t\,. \end{aligned}$$

We claim that

$$\mathcal{K} = \{ \mathfrak{A}|_\Sigma \mid \mathfrak{A} \in \mathrm{Mod}(\Psi) \}.$$

($\subseteq$) If $\mathfrak{C} \in \mathcal{K}$ then $\mathfrak{C} = \mathfrak{A}|_{\bigcup_s P_s^{\mathfrak{A}}}$, for some $\mathfrak{A} \vDash \Phi$. Since Φ is a Skolem theory we can assume that $P^{\mathfrak{A}}$ generates $\mathfrak{A}$. Hence, $\langle\!\langle X \rangle\!\rangle^+_{\mathfrak{A}}$ is defined and $\mathfrak{C} \cong \langle X \rangle^\circ_{\mathfrak{A}}|_\Sigma$. Furthermore, $\langle\!\langle X \rangle\!\rangle^\circ_{\mathfrak{A}} \vDash \Psi$, as desired.

($\supseteq$) Let $\mathfrak{A} \vDash \Psi$. Since $\mathfrak{A} \vDash \Phi \cup \Xi(\Sigma)^{\vDash}_{\forall}$ it follows that $\widehat{\mathfrak{A}}$ exists and $\langle\!\langle X \rangle\!\rangle^\circ_{\widehat{\mathfrak{A}}} = \mathfrak{A}$. Since $\mathfrak{A} \vDash \widehat{\Phi}$ we have $\widehat{\mathfrak{A}} \vDash \Phi$. Consequently, $\widehat{\mathfrak{A}}|_{P^{\widehat{\mathfrak{A}}}} \in \mathcal{K}$. We claim that $\widehat{\mathfrak{A}}|_{P^{\widehat{\mathfrak{A}}}} = \mathfrak{A}$. On the one hand, $\Xi(\Sigma)^{\vDash}_{\forall} \vDash \forall x R_{Px} x$ implies that $\mathfrak{A} \vDash Pa$, for every $a \in A$. Hence, $A \subseteq P^{\widehat{\mathfrak{A}}}$. Conversely, suppose that $a \in P^{\widehat{\mathfrak{A}}}$. Then $a = t(\bar{b})$, for some term t and parameters $\bar{b} \subseteq A$. Then

$$\mathfrak{A} \vDash R_{Pt\bar{x}}\bar{b} \wedge \exists y R_{t\bar{x}=y}\bar{b}y$$

which implies that $a \in A$. □

Corollary 4.15. $L \leq^\infty_{\mathrm{rpc}} \mathrm{FO}$ *iff* $L \leq^\infty_{\mathrm{pc}} \mathrm{FO}$.

5. *Interpolation*

For most logics L there are projective L-classes that are not L-axiomatisable. In this section we study how this additional power affects the entailment relation. Surprisingly we can find many logics where it has no effect at all.

Definition 5.1. Let L be an algebraic logic.

(a) L has the *interpolation property* if, for all finite sets $\Phi_i \subseteq L[\Sigma_i]$, $i < 2$, with $\Phi_0 \vDash \Phi_1$, there exists a finite set $\Psi \subseteq L[\Sigma_0 \cap \Sigma_1]$ such that

$$\Phi_0 \vDash \Psi \quad \text{and} \quad \Psi \vDash \Phi_1 .$$

(b) L has the *Δ-interpolation property* if every class $\mathcal{K} \in \mathrm{PC}_{<\aleph_0}(L, \Sigma)$ with $\mathrm{Str}[\Sigma] \setminus \mathcal{K} \in \mathrm{PC}_{<\aleph_0}(L, \Sigma)$ is finitely L-axiomatisable.

Remark. If L is boolean closed then the interpolation property implies the Δ-interpolation property since, if

$$\mathcal{K} = \mathrm{pr}_\Sigma(\mathrm{Mod}(\Phi_+)) \quad \text{and} \quad \mathrm{Str}[\Sigma] \setminus \mathcal{K} = \mathrm{pr}_\Sigma(\mathrm{Mod}(\Phi_-))$$

then we have

$$\Phi_+ \vDash \neg \bigwedge \Phi_-$$

and any set $\Psi \subseteq L[\Sigma]$ with

$$\Phi_+ \vDash \Psi \quad \text{and} \quad \Psi \vDash \neg \bigwedge \Phi_-$$

is an axiom system for $\mathcal{K}$.

Theorem 5.2. FO *has the interpolation property.*

Proof. Since FO is closed under conjunctions it is sufficient to consider single formulae. Hence, suppose that $\varphi_0 \vDash \varphi_1$ where $\varphi_i \in \mathrm{FO}^\circ[\Gamma_i]$, for $i < 2$. Let

$$\Psi := (\varphi_0)^{\vDash} \cap \mathrm{FO}^\circ[\Sigma], \quad \text{where} \quad \Sigma := \Gamma_0 \cap \Gamma_1 .$$

We claim that $\Psi \cup \{\neg\varphi_1\}$ is inconsistent. By compactness, it then follows that there is a finite subset $\Psi_0 \subseteq \Psi$ such that $\Psi_0 \cup \{\neg\varphi_1\}$ is inconsistent. Setting $\psi := \bigwedge \Psi_0$ we have $\varphi_0 \vDash \psi$ and $\psi \vDash \varphi_1$, as desired.

It remains to prove the claim. For a contradiction, suppose that the set $\Psi \cup \{\neg\varphi_1\}$ has a model $\mathfrak{A}$. By Corollary c2.5.9, there exists a model $\mathfrak{B}$ of φ_0 such that $\mathfrak{A}|_\Sigma \preceq \mathfrak{B}|_\Sigma$. Since $\mathfrak{A}|_\Sigma \equiv \mathfrak{B}|_\Sigma$ we can apply Theorem c2.5.8 to obtain a $(\Gamma_0 \cup \Gamma_1)$-structure $\mathfrak{C}$ with $\mathfrak{B} \preceq \mathfrak{C}|_{\Gamma_0}$ and an elementary embedding $g : \mathfrak{A} \to \mathfrak{C}|_{\Gamma_1}$. In particular, we have $\mathfrak{C}|_{\Gamma_0} \vDash \varphi_0$ and $\mathfrak{C}|_{\Gamma_1} \vDash \neg\varphi_1$. Hence, $\mathfrak{C} \vDash \varphi_0 \wedge \neg\varphi_1$ and $\varphi_0 \nvDash \varphi_1$. Contradiction. □

Definition 5.3. Let L be an algebraic logic, Σ a signature, $R \notin \Sigma$ an n-ary relation symbol, and $\Phi(R) \subseteq L^\circ[\Sigma \cup \{R\}]$ a set of formulae.

(a) We say that R is *implicitly defined* by Φ if, for all models $\mathfrak{A}$ and $\mathfrak{B}$ of Φ with $\mathfrak{A}|_\Sigma = \mathfrak{B}|_\Sigma$, we have $R^{\mathfrak{A}} = R^{\mathfrak{B}}$.

(b) We say that R is *explicitly defined* by a set $\Psi \subseteq L^n[\Sigma]$ with respect to Φ if $R^{\mathfrak{A}} = \Psi^{\mathfrak{A}}$, for every model $\mathfrak{A}$ of Φ.

(c) L has the *Beth property* if, for all finite sets $\Phi \subseteq L^{\circ}[\Sigma \cup \{R\}]$ that define R implicitly, there exists a finite set $\Psi \subseteq L^n[\Sigma]$ explicitly defining R with respect to Φ.

Lemma 5.4. *Every boolean closed logic L with the interpolation property has the Beth property.*

Proof. Suppose that R is implicitly defined by $\Phi(R) \subseteq L^{\circ}[\Sigma \cup \{R\}]$. Let R' be a new relation symbol. It follows that

$$\bigwedge \Phi(R) \to R\bar{x} \vDash \bigwedge \Phi(R') \to R'\bar{x}\,.$$

By the interpolation property we can find a finite set $\Psi(\bar{x})$ such that

$$\bigwedge \Phi(R) \to R\bar{x} \vDash \bigwedge \Psi(\bar{x}) \quad \text{and} \quad \bigwedge \Psi(\bar{x}) \vDash \bigwedge \Phi(R') \to R'\bar{x}\,.$$

It follows that

$$\Phi(R) \vDash R\bar{x} \leftrightarrow \bigwedge \Psi(\bar{x})\,,$$

that is, Ψ explicitly defines R with respect to Φ. □

There is a general way to extend a given logic to one that has the Δ-interpolation property.

Definition 5.5. Let L be an algebraic logic. The *interpolation closure* $\Delta(L)$ of L is the logic where $\Delta(L)[\Sigma]$ consists of all pairs

$$\langle \varphi_0, \varphi_1 \rangle \in L[\Sigma_0] \times L[\Sigma_1]$$

with $\Sigma_i \supseteq \Sigma$ and

$$\mathrm{pr}_\Sigma(\mathrm{Mod}(\varphi_1)) = \mathrm{Str}[\Sigma] \setminus \mathrm{pr}_\Sigma(\mathrm{Mod}(\varphi_0))\,.$$

The semantics of such a formula is defined by

$$\mathfrak{A} \vDash \langle \varphi_0, \varphi_1 \rangle \quad :\text{iff} \quad \mathfrak{A} \in \mathrm{pr}_\Sigma(\mathrm{Mod}(\varphi_0))\,.$$

Lemma 5.6. *Let L be an algebraic logic.*

(a) *If L satisfies* (B$_+$) *then $\Delta(L)$ is boolean closed.*

(b) *If L_0 is closed under negation and $L_0 \leq^1_{\mathrm{pc}} L_1$, then $L_0 \leq \Delta(L_1)$.*

(c) *If L is closed under negation then $L \leq \Delta(L)$.*

(d) *$L_0 \leq^1_{\mathrm{pr}} L_1$ implies $\Delta(L_0) \leq \Delta(L_1)$.*

(e) *$\Delta(\Delta(L)) \leq \Delta(L)$.*

(f) *$\Delta(L)$ has the Δ-interpolation property.*

(g) *If L_1 has the Δ-interpolation property then*

$$L_0 \leq^1_{\mathrm{pr}} L_1 \quad \text{implies} \quad \Delta(L_0) \leq L_1\,.$$

(h)
$$\mathrm{occ}(\Delta(L)) = \mathrm{occ}(L)\,,$$
$$\mathrm{ln}_\kappa(\Delta(L)) = \mathrm{ln}_\kappa(L)\,,$$
$$\mathrm{wn}_\kappa(\Delta(L)) = \mathrm{wn}_\kappa(L)\,.$$

Proof. (a) We have

$$\langle \varphi, \psi \rangle \wedge \langle \varphi', \psi' \rangle \equiv \langle \varphi \wedge \varphi',\ \psi \vee \psi' \rangle\,,$$
$$\langle \varphi, \psi \rangle \vee \langle \varphi', \psi' \rangle \equiv \langle \varphi \vee \varphi',\ \psi \wedge \psi' \rangle\,,$$

and
$$\neg \langle \varphi, \psi \rangle \equiv \langle \psi, \varphi \rangle\,.$$

(b) For every $\varphi \in L_0[\Sigma]$, there exist a signature $\Sigma_0 \supseteq \Sigma$ and a formula $\psi_0 \in L_1[\Sigma_0]$ such that

$$\mathrm{Mod}(\varphi) = \mathrm{pr}_\Sigma(\mathrm{Mod}(\psi_0))\,.$$

Similarly, there exist a signature $\Sigma_1 \supseteq \Sigma$ and a formula $\psi_1 \in L_1[\Sigma_1]$ such that

$$\mathrm{Mod}(\neg\varphi) = \mathrm{pr}_\Sigma(\mathrm{Mod}(\psi_1))\,.$$

It follows that $\varphi \equiv \langle \psi_0, \psi_1 \rangle \in \Delta(L_1)$.

(c) follows immediately from (b).

(d) Let $\langle\varphi_0, \psi_0\rangle \in \Delta(L_0)$ where $\varphi_0 \in L_0[\Sigma_0]$ and $\psi_0 \in L_0[\Gamma_0]$. Since $L_0 \leq^1_{\mathrm{pr}} L_1$ we can find formulae $\varphi_1 \in L_1[\Sigma_1]$ and $\psi_1 \in L_1[\Gamma_1]$ such that

$$\mathrm{Mod}(\varphi_0) = \mathrm{pr}_{\Sigma_0}(\mathrm{Mod}(\varphi_1))$$
and $$\mathrm{Mod}(\psi_0) = \mathrm{pr}_{\Gamma_0}(\mathrm{Mod}(\psi_1))\,.$$

Hence, $\langle\varphi_0, \psi_0\rangle \equiv \langle\varphi_1, \psi_1\rangle \in \Delta(L_1)$.

(e) Let $\langle\langle\varphi_0, \psi_0\rangle, \langle\varphi_1, \psi_1\rangle\rangle \in \Delta(\Delta(L))[\Sigma]$. Then

$$\begin{aligned}\mathrm{pr}_\Sigma(\mathrm{Mod}(\varphi_0)) &= \mathrm{pr}_\Sigma(\mathrm{pr}_{\Sigma_0}(\mathrm{Mod}(\varphi_0))) \\ &= \mathrm{pr}_\Sigma(\mathrm{Mod}(\langle\varphi_0, \psi_0\rangle)) \\ &= \mathrm{Str}[\Sigma] \smallsetminus \mathrm{pr}_\Sigma(\mathrm{Mod}(\langle\varphi_1, \psi_1\rangle)) \\ &= \mathrm{Str}[\Sigma] \smallsetminus \mathrm{pr}_\Sigma(\mathrm{pr}_{\Sigma_1}(\mathrm{Mod}(\varphi_1))) \\ &= \mathrm{Str}[\Sigma] \smallsetminus \mathrm{pr}_\Sigma(\mathrm{Mod}(\varphi_1))\,.\end{aligned}$$

Consequently, $\langle\langle\varphi_0, \psi_0\rangle, \langle\varphi_1, \psi_1\rangle\rangle \equiv \langle\varphi_0, \varphi_1\rangle \in \Delta(L)[\Sigma]$.

(f) Let $\langle\varphi_0, \psi_0\rangle, \langle\varphi_1, \psi_1\rangle \in \Delta(L)$ be formulae such that

$$\mathrm{pr}_\Sigma(\mathrm{Mod}(\langle\varphi_0, \psi_0\rangle)) = \mathrm{Str}[\Sigma] \smallsetminus \mathrm{pr}_\Sigma(\mathrm{Mod}(\langle\varphi_1, \psi_1\rangle))\,.$$

Then $\langle\langle\varphi_0, \psi_0\rangle, \langle\varphi_1, \psi_1\rangle\rangle \in \Delta(\Delta(L))$ and, by (e), there is a formula $\langle\vartheta, \chi\rangle \in \Delta(L)$ such that

$$\langle\vartheta, \chi\rangle \equiv \langle\langle\varphi_0, \psi_0\rangle, \langle\varphi_1, \psi_1\rangle\rangle\,.$$

(g) Let $\langle\varphi_0, \psi_0\rangle \in \Delta(L_0)[\Sigma]$ where $\varphi_0 \in L_0[\Gamma_0]$ and $\psi_0 \in L_0[\Gamma'_0]$. Since $L_0 \leq^1_{\mathrm{pr}} L_1$ we can find formulae $\varphi_1 \in L_1[\Gamma_1]$ and $\psi_1 \in L_1[\Gamma'_1]$ such that

$$\mathrm{Mod}(\varphi_0) = \mathrm{pr}_{\Gamma_0}(\mathrm{Mod}(\varphi_1))$$
and $$\mathrm{Mod}(\psi_0) = \mathrm{pr}_{\Gamma'_0}(\mathrm{Mod}(\psi_1))\,.$$

Since

$$\begin{aligned}\mathrm{pr}_\Sigma(\mathrm{Mod}(\varphi_1)) &= \mathrm{pr}_\Sigma(\mathrm{Mod}(\varphi_0)) \\ &= \mathrm{Str}[\Sigma] \smallsetminus \mathrm{pr}_\Sigma(\mathrm{Mod}(\psi_0)) \\ &= \mathrm{Str}[\Sigma] \smallsetminus \mathrm{pr}_\Sigma(\mathrm{Mod}(\psi_1))\end{aligned}$$

and L_1 has the Δ-interpolation property we can find a formula $\vartheta \in L_1[\Sigma]$ such that

$$\mathrm{Mod}(\vartheta) = \mathrm{pr}_\Sigma(\mathrm{Mod}(\varphi_1))\,.$$

It follows that $\vartheta \equiv \langle\varphi_0, \psi_0\rangle$.

(h) We only prove the first equation. The other ones are left as an exercise. Let $\langle\varphi, \psi\rangle \in \Delta(L)[\Sigma]$ where $\varphi \in L[\Sigma_0]$ and $\psi \in L[\Sigma_1]$. Then there exist formulae $\varphi' \in L[\Gamma_0]$ and $\psi' \in L[\Gamma_1]$ where $\Gamma_i \subseteq \Sigma_i$ are subsignatures of size $|\Gamma_i| \leq \mathrm{occ}(L)$ such that $\varphi' \equiv \varphi$ and $\psi' \equiv \psi$. Let $\Gamma := \Sigma \cap (\Gamma_0 \cup \Gamma_1)$. It follows that $\langle\varphi, \psi\rangle \equiv \langle\varphi', \psi'\rangle \in L[\Gamma]$ where $|\Gamma| \leq \mathrm{occ}(L)$. ☐

Proposition 5.7. $\mathrm{FO}_{\kappa^+\aleph_0}(\exists^\kappa) \leq^\kappa_{\mathrm{pr}} \mathrm{FO}_{\kappa^+\aleph_0}$.

Proof. Let $\varphi \in \mathrm{FO}_{\kappa^+\aleph_0}(\exists^\kappa)$. Following Chang's Reduction we introduce a new relation symbol R_ψ, for every subformula $\psi(\bar{x})$ of φ, and we write down formulae ensuring that R_ψ is the set of all tuples satisfying ψ. For the operations of $\mathrm{FO}_{\kappa^+\aleph_0}$ this can be done in the same way as in Chang's reduction. For a subformula $\exists^\kappa y\psi(\bar{x}, y)$, we introduce a new relation symbol $<_\psi$ and κ new function symbols f^α_ψ, $\alpha < \kappa$, and we add the formulae

$$\forall\bar{x}\Big(R_{\exists^\kappa y\psi}\bar{x} \leftrightarrow \bigwedge_{\alpha\neq\beta} f^\alpha_\psi\bar{x} \neq f^\beta_\psi\bar{x}\Big) \wedge \forall\bar{x} \bigwedge_{\alpha<\kappa} R_\psi\bar{x}f^\alpha_\psi\bar{x}\,,$$
$$\forall\bar{x}\Big(\neg R_{\exists^\kappa y\psi}\bar{x} \leftrightarrow \bigvee_{\alpha<\kappa} \chi_\alpha(\bar{x})\Big)\,,$$

where $\chi_\alpha(\bar{c})$ is the formula of Lemma c1.1.7 stating that the relation $\{\langle a, b\rangle \mid \bar{c}a <_\psi \bar{c}b\}$ is a well-order of order type α on the set defined by $\psi(\bar{c}, y)$. Note that the first formula ensures that $R_{\exists^\kappa y\psi}$ contains only tuples $\bar{c}$ such that there are at least κ elements satisfying $\psi(\bar{c}, y)$, while the second formula ensures that all such tuples $\bar{c}$ are contained in $R_{\exists^\kappa y\psi}$. Finally, note that we have introduced at most κ formulae since φ has at most that many subformulae. □

Proposition 5.8. $\mathrm{FO}_{\aleph_2\aleph_0}(\exists^{\aleph_1})$ *does not have the Karp property.*

Proof. We consider the structures $\mathfrak{A} := \langle A \rangle$ and $\mathfrak{B} := \langle B \rangle$ over the empty signature with $|A| = \aleph_0$ and $|B| = \aleph_1$. Then we have

$$\mathrm{pIso}_{\aleph_0}(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \cong_\infty \mathfrak{B} .$$

But $\quad \mathfrak{A} \not\models \exists^{\aleph_1} x(x = x) \quad$ and $\quad \mathfrak{B} \models \exists^{\aleph_1} x(x = x)$

implies that $\mathfrak{A} \not\equiv_{\mathrm{FO}_{\aleph_2\aleph_0}(\exists^{\aleph_1})} \mathfrak{B}$. □

Corollary 5.9. $\Delta(\mathrm{FO}_{\aleph_2\aleph_0})$ *does not have the Karp property.*

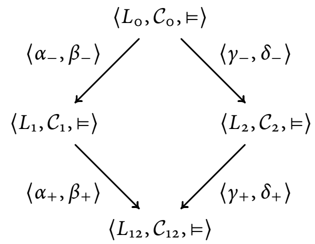
Proof. Note that

$$\mathrm{FO}_{\kappa^+\aleph_0}(\exists^\kappa) \leq^\kappa_{\mathrm{pr}} \mathrm{FO}_{\kappa^+\aleph_0} \quad \text{implies} \quad \mathrm{FO}_{\kappa^+\aleph_0}(\exists^\kappa) \leq^1_{\mathrm{pr}} \mathrm{FO}_{\kappa^+\aleph_0}$$

since $\mathrm{FO}_{\kappa^+\aleph_0}$ is closed under conjunctions of size κ. By Lemma 5.6 (b), it follows that $\mathrm{FO}_{\aleph_2\aleph_0}(\exists^{\aleph_1}) \leq \Delta(\mathrm{FO}_{\aleph_2\aleph_0})$. Since the former does not have the Karp property it follows that the latter does not have it either. □

For many logics that can be characterised via a preservation theorem we can derive the interpolation property from a general theorem which we will present below. Instead of considering the entailment relation $\Phi_0 \models \Phi_1$ for a single logic, we allow Φ_0 and Φ_1 to belong to different logics L_1 and L_2, and we look for an interpolant $\Phi_0 \models \Psi \models \Phi_1$ in a third logic L_0.

Definition 5.10. (a) A *weak amalgamation square* is a commuting diagram



in the category $\mathfrak{Logic}$ such that, for every pair $\mathfrak{J}_1 \in \mathcal{C}_1$ and $\mathfrak{J}_2 \in \mathcal{C}_2$ of interpretations with $\beta_-(\mathfrak{J}_1) = \delta_-(\mathfrak{J}_2)$, there exists an L_{12}-interpretation $\mathfrak{J}_{12}$ with

$$\beta_+(\mathfrak{J}_{12}) = \mathfrak{J}_1 \quad \text{and} \quad \delta_+(\mathfrak{J}_{12}) = \mathfrak{J}_2 .$$

(b) Given a weak amalgamation square as in (a) and sets $\Phi_1 \subseteq L_1$ and $\Phi_2 \subseteq L_2$ of formulae with $\alpha_+[\Phi_1] \models \gamma_+[\Phi_2]$, we call a set $\Phi_0 \subseteq L_0$ an *interpolant* of Φ_1 and Φ_2 if

$$\Phi_1 \models \alpha_-[\Phi_0] \quad \text{and} \quad \gamma_-[\Phi_0] \models \Phi_2 .$$

(c) Similarly, given a weak amalgamation square and classes $\mathcal{K}_1 \subseteq \mathcal{C}_1$ and $\mathcal{K}_2 \subseteq \mathcal{C}_2$ of interpretations with $\beta_+^{-1}[\mathcal{K}_1] \subseteq \delta_+^{-1}[\mathcal{K}_2]$ we call a class $\mathcal{K}_0 \subseteq \mathcal{C}_0$ an *interpolant* of $\mathcal{K}_1$ and $\mathcal{K}_2$ if

$$\mathcal{K}_1 \subseteq \beta_-^{-1}[\mathcal{K}_0] \quad \text{and} \quad \delta_-^{-1}[\mathcal{K}_0] \subseteq \mathcal{K}_2 .$$

Lemma 5.11. Φ_0 *is an interpolant of* Φ_1 *and* Φ_2 *if and only if* $\mathrm{Mod}(\Phi_0)$ *is an interpolant of* $\mathrm{Mod}(\Phi_1)$ *and* $\mathrm{Mod}(\Phi_2)$.

The next lemma shows that each pair of classes in a weak amalgamation square has an interpolant. For the interpolation property to hold we have to strengthen this result by proving that a pair of *axiomatisable* classes has an *axiomatisable* interpolant.

Lemma 5.12. *Consider a weak amalgamation square as above.*

(a) *$\mathcal{K}_0$ is an interpolant of $\mathcal{K}_1$ and $\mathcal{K}_2$ if and only if*

$$\beta_-[\mathcal{K}_1] \subseteq \mathcal{K}_0 \subseteq \mathcal{C}_0 \setminus \gamma_-[\mathcal{C}_2 \setminus \mathcal{K}_2] .$$

(b) *$\mathcal{K}_1$ and $\mathcal{K}_2$ have an interpolant.*

Proof. (a) We have

$$\mathcal{K}_1 \subseteq \beta_-^{-1}[\mathcal{K}_0] \quad \text{iff} \quad \beta_-[\mathcal{K}_1] \subseteq \mathcal{K}_0 ,$$

$$\text{and} \quad \mathcal{C}_2 \setminus \mathcal{K}_2 \subseteq \gamma_-^{-1}[\mathcal{C}_0 \setminus \mathcal{K}_0] \quad \text{iff} \quad \gamma_-[\mathcal{C}_2 \setminus \mathcal{K}_2] \subseteq \mathcal{C}_0 \setminus \mathcal{K}_0 .$$

(b) By (a) it is sufficient to show that

$$\beta_-[\mathcal{K}_1] \subseteq \mathcal{C}_0 \setminus \gamma_-[\mathcal{C}_2 \setminus \mathcal{K}_2] .$$

For a contradiction, suppose that there is some interpretation

$$\mathfrak{I}_0 \in \beta_-[\mathcal{K}_1] \setminus (\mathcal{C}_0 \setminus \gamma_-[\mathcal{C}_2 \setminus \mathcal{K}_2]) = \beta_-[\mathcal{K}_1] \cap \gamma_-[\mathcal{C}_2 \setminus \mathcal{K}_2] .$$

Choose interpretations $\mathfrak{I}_1 \in \mathcal{K}_1$ and $\mathfrak{I}_2 \in \mathcal{C}_2 \setminus \mathcal{K}_2$ with

$$\beta_-(\mathfrak{I}_1) = \mathfrak{I}_0 = \gamma_-(\mathfrak{I}_2) .$$

Since the diagram is a weak amalgamation square we can find an interpretation $\mathfrak{I}_{12} \in \mathcal{C}_{12}$ with $\beta_+(\mathfrak{I}_{12}) = \mathfrak{I}_1$ and $\gamma_+(\mathfrak{I}_{12}) = \mathfrak{I}_2$. It follows that

$$\mathfrak{I}_{12} \in \beta_+^{-1}(\mathfrak{I}_1) \subseteq \beta_+^{-1}[\mathcal{K}_1] \subseteq \gamma_+^{-1}[\mathcal{K}_2] .$$

Consequently, we have $\mathfrak{I}_2 = \gamma_+(\mathfrak{I}_{12}) \in \mathcal{K}_2$. Contradiction. □

If a logic can be characterised by a preservation theorem then a class of interpretation is axiomatisable if and only if it is a fixed point for the operations the logic is preserved under. Hence, to prove our interpolation theorem we consider fixed points of operations.

Definition 5.13. Let A and B be sets and $\alpha : \wp(A) \to \wp(A)$ and $\beta : \wp(B) \to \wp(B)$ functions on their power sets.

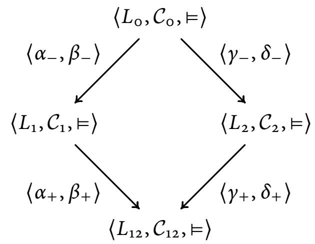
(a) A function $f : A \to B$ *preserves fixed points* of α and β if

$$C \in \operatorname{fix} \alpha \quad \text{implies} \quad f[C] \in \operatorname{fix} \beta .$$

(b) A function $f : A \to B$ *lifts α to β* if

$$(f^{-1} \circ \beta)[X] \subseteq (\alpha \circ f^{-1})[X] , \quad \text{for all } X \subseteq B .$$

Theorem 5.14 (Popescu, Roşu, Şerbănuţă). *Consider a weak amalgamation square*



Suppose that there are functions

$$\mu_i : \wp(\mathcal{C}_i) \to \wp(\mathcal{C}_i) , \quad \textit{for } i \in \{0, 1\} ,$$

$$\textit{and} \quad \nu_j : \wp(\mathcal{C}_j) \to \wp(\mathcal{C}_j) , \quad \textit{for } j \in \{0, 2\} ,$$

satisfying the following conditions:

(1) $\mu_0 \circ \nu_0 \circ \mu_0 = \nu_0 \circ \mu_0$.

(2) *ν_0 and ν_2 are closure operators.*

(3) *β_- preserves fixed points of μ_1 and μ_0.*

(4) *γ_- lifts ν_2 to ν_0.*

Every pair of fixed points $\mathcal{K}_1 \in \operatorname{fix} \mu_1$ and $\mathcal{K}_2 \in \operatorname{fix} \nu_2$ with

$$\beta_+^{-1}[\mathcal{K}_1] \subseteq \gamma_+^{-1}[\mathcal{K}_2]$$

has an interpolant $\mathcal{K}_0 \in \operatorname{fix} \mu_0 \cap \operatorname{fix} \nu_0$.

Proof. We claim that $\mathcal{K}_0 := (\nu_0 \circ \beta_-)[\mathcal{K}_1]$ is the desired interpolant. We have $\mathcal{K}_0 \in \operatorname{fix} \nu_0$ since $\nu_0 \circ \nu_0 = \nu_0$. Furthermore, $\beta_-[\mathcal{K}_1] \in \operatorname{fix} \mu_0$ as $\mathcal{K}_1 \in \operatorname{fix} \mu_1$ and β_- preserves fixed points. It follows that

$$\begin{aligned}\mu_0[\mathcal{K}_0] &= (\mu_0 \circ \nu_0 \circ \beta_-)[\mathcal{K}_1] \\ &= (\mu_0 \circ \nu_0 \circ \mu_0 \circ \beta_-)[\mathcal{K}_1] \\ &= (\nu_0 \circ \mu_0 \circ \beta_-)[\mathcal{K}_1] \\ &= (\nu_0 \circ \beta_-)[\mathcal{K}_1] = \mathcal{K}_0\,,\end{aligned}$$

and, therefore, $\mathcal{K}_0 \in \operatorname{fix} \mu_0$.

It remains to prove that $\mathcal{K}_0$ is an interpolant. Since ν_0 is a closure operator we have $\beta_-[\mathcal{K}_1] \subseteq (\nu_0 \circ \beta_-)[\mathcal{K}_1] = \mathcal{K}_0$ which implies that $\mathcal{K}_1 \subseteq \beta_-^{-1}[\mathcal{K}_0]$. For the other inclusion, note that we have

$$\gamma_-^{-1}[\mathcal{K}_0] = (\gamma_-^{-1} \circ \nu_0 \circ \beta_-)[\mathcal{K}_1] \subseteq (\nu_2 \circ \gamma_-^{-1} \circ \beta_-)[\mathcal{K}_1]$$

since γ_- lifts ν_2 to ν_0. Furthermore, $(\gamma_-^{-1} \circ \beta_-)[\mathcal{K}_1] \subseteq \mathcal{K}_2$ since we have shown in Lemma 5.12 that $\beta_-[\mathcal{K}_1]$ is an interpolant of $\mathcal{K}_1$ and $\mathcal{K}_2$. As ν_2 is a closure operator it follows that

$$(\nu_2 \circ \gamma_-^{-1} \circ \beta_-)[\mathcal{K}_1] \subseteq \nu_2[\mathcal{K}_2] = \mathcal{K}_2\,.$$

Consequently, we have

$$\gamma_-^{-1}[\mathcal{K}_0] \subseteq (\nu_2 \circ \gamma_-^{-1} \circ \beta_-)[\mathcal{K}_1] \subseteq \mathcal{K}_2\,,$$

as desired. □

We can use this theorem to obtain interpolation results for logics that can be characterised via preservation theorems.

Corollary 5.15. *Consider a weak amalgamation square as above and functions*

$$\mu_i : \wp(\mathcal{C}_i) \to \wp(\mathcal{C}_i)\,, \quad \text{for } i \in \{0,1\}\,,$$
$$\text{and} \quad \nu_j : \wp(\mathcal{C}_j) \to \wp(\mathcal{C}_j)\,, \quad \text{for } j \in \{0,2\}\,,$$

satisfying the conditions of the preceding theorem. Furthermore, suppose that

(1) *a class $\mathcal{K}_1 \subseteq \mathcal{C}_1$ is L_1-axiomatisable if and only if $\mathcal{K}_1 \in \operatorname{fix} \mu_1$;*

(2) *a class $\mathcal{K}_2 \subseteq \mathcal{C}_2$ is L_2-axiomatisable if and only if $\mathcal{K}_2 \in \operatorname{fix} \nu_2$;*

(3) *a class $\mathcal{K}_0 \subseteq \mathcal{C}_0$ is L_0-axiomatisable iff $\mathcal{K}_0 \in \operatorname{fix} \mu_0 \cap \operatorname{fix} \nu_0$.*

Then every pair of sets $\Phi_1 \subseteq L_1$ and $\Phi_2 \subseteq L_2$ with

$$\alpha_+[\Phi_1] \vDash \delta_+[\Phi_2]$$

has an interpolant $\Phi_0 \subseteq L_0$.

Unfortunately applications of this theorem will have to wait till Chapter D2 since at the moment we still lack the required preservation theorems.

6. Fixed-point logics

As an example we investigate extensions of first-order logic by fixed-point operators. Let $\mathfrak{A}$ be a structure and $f : \wp(A^n) \to \wp(A^n)$ a function. A fixed point of f is an n-ary relation on A. We are interested in operators that compute such fixed points for definable functions f.

Note that the partial order $\wp(A^n)$ is complete. Hence, if f is increasing then, by Theorem A2.4.3, it has a least fixed point $\operatorname{lfp} f$ and a greatest fixed point $\operatorname{gfp} f$. Similarly, if f is inflationary then we can use Theorem A3.3.14 to obtain the inductive fixed point $\operatorname{ifp} f$ of f over $\varnothing$.

If f is neither increasing nor inflationary then none of these fixed points need to exist. But we still would like to define a fixed point operator for such functions. Instead of asking for a real fixed point we will present two ways to compute an approximate one.

Firstly, we can artificially make f inflationary by replacing it with the function $x \mapsto x \cup f(x)$. Secondly, we can compute the 'fixed-point induction' $\varnothing, f(\varnothing), f^2(\varnothing), \dots$ (which generally is not increasing) and take some kind of limit.

Definition 6.1. Let X be a set and $f : \wp(X) \to \wp(X)$ an arbitrary function.

(a) The *inductive fixed point* $\operatorname{ifp} f$ of f is the inductive fixed point of the function $f' : x \mapsto x \cup f(x)$ over $\varnothing$. Correspondingly, by the *inductive fixed-point induction* of f we mean the fixed-point induction $F : \mathrm{On} \to \wp(X)$ of f' over $\varnothing$.

(b) The *lower fixed-point induction* of f is the map $F_- : \mathrm{On} \to \wp(X)$ defined by

$$\begin{aligned} F_-(0) &:= \varnothing\,, \\ F_-(\alpha+1) &:= f(F_-(\alpha))\,, \\ F_-(\delta) &:= \bigcup_{\alpha<\delta} \bigcap_{\alpha\leq\beta<\delta} F_-(\beta)\,, \quad \text{for limits } \delta\,. \end{aligned}$$

Analogously, we define the *upper fixed-point induction* F_+ by

$$\begin{aligned} F_+(0) &:= \varnothing\,, \\ F_+(\alpha+1) &:= f(F_+(\alpha))\,, \\ F_+(\delta) &:= \bigcap_{\alpha<\delta} \bigcup_{\alpha\leq\beta<\delta} F_+(\beta)\,, \quad \text{for limits } \delta\,. \end{aligned}$$

(c) The *least partial fixed point* $\liminf f$ of f is the set

$$F_-(\infty) := \bigcup_{\alpha} \bigcap_{\alpha\leq\beta} F_-(\beta)\,.$$

and its *greatest partial fixed point* $\limsup f$ is

$$F_+(\infty) := \bigcap_{\alpha} \bigcup_{\alpha\leq\beta} F_+(\beta)\,.$$

Remark. Note that, in general, $\operatorname{ifp} f$, $\liminf f$, and $\limsup f$ are *no* fixed points of f. But, if f is increasing then $\operatorname{ifp} f = \liminf f = \limsup f = \operatorname{lfp} f$.

Before defining logics with these fixed-point operators let us compute their closure ordinals.

Definition 6.2. Let $f : \wp(X) \to \wp(X)$ be a function.

(a) The *closure ordinal* for the inductive fixed-point induction F of f is the least ordinal α such that $F(\alpha) = F(\alpha+1)$.

(b) The *closure ordinal* for the lower fixed-point induction F_- of f is the least ordinal α such that

$$F_-(\alpha) = F_-(\infty) \quad \text{and} \quad F_-(\beta) \supseteq F_-(\alpha)\,, \quad \text{for all } \beta \geq \alpha\,.$$

Similarly, we define the closure ordinal for the upper fixed-point induction F_+ as the least ordinal α such that

$$F_+(\alpha) = F_+(\infty) \quad \text{and} \quad F_+(\beta) \subseteq F_+(\alpha)\,, \quad \text{for all } \beta \geq \alpha\,.$$

Since the inductive fixed-point induction of a function is increasing we obtain the same bound on the closure ordinal as for least fixed points.

Lemma 6.3. *Let $f : \wp(X) \to \wp(X)$. The closure ordinal of* $\operatorname{ifp} f$ *is less than $|X|^+$.*

For partial fixed points the situation is different. The following sequence of lemmas shows that in this case the bound is $(2^{|X|})^+$. We will only consider the case of upper fixed-point inductions. The closure ordinal of a least partial fixed point can be computed in exactly the same way.

Lemma 6.4. *Let F_+ be the upper fixed-point induction of the function $f : \wp(X) \to \wp(X)$.*

(a) *If $F_+(\alpha) = F_+(\beta)$ then $F_+(\alpha+\gamma) = F_+(\beta+\gamma)$, for all γ.*

(b) *If $F_+(\alpha) = F_+(\alpha+\beta)$ then*

$$\begin{aligned} F_+(\alpha+\beta n) &= F_+(\alpha)\,, \qquad \textit{for all } n < \omega\,, \\ \textit{and}\ \ F_+(\alpha+\beta\omega) &= \bigcup_{\gamma<\beta} F_+(\alpha+\gamma)\,. \end{aligned}$$

(c) *If $F_+(\alpha) = F_+(\alpha+\beta) = F_+(\alpha+\gamma)$ and $\beta \leq \gamma$ then*

$$F_+(\alpha+\beta\omega) \subseteq F_+(\alpha+\gamma\omega)\,.$$

(d) *If* $F_+(\alpha) = F_+(\alpha + \beta) = F_+(\alpha + \beta\omega)$ *then*

$$F_+(\alpha + \beta\gamma) = F_+(\alpha)\,, \qquad \textit{for all } \gamma\,,$$
$$\textit{and} \qquad F_+(\infty) = F_+(\alpha)\,.$$

Proof. (a) is proved by a straightforward induction on γ. For $\gamma = 0$, there is nothing to do. If $\gamma = \eta + 1$ then

$$F_+(\alpha + \eta + 1) = f(F_+(\alpha + \eta)) = f(F_+(\beta + \eta)) = F_+(\beta + \eta + 1)\,.$$

Finally, for limit ordinals γ we have

$$\begin{aligned} F_+(\alpha + \gamma) &= \bigcap_{i<\alpha+\gamma} \bigcup_{i\leq k<\alpha+\gamma} F_+(k) \\ &= \bigcap_{\alpha\leq i<\alpha+\gamma} \bigcup_{i\leq k<\alpha+\gamma} F_+(k) \\ &= \bigcap_{i<\gamma} \bigcup_{i\leq k<\gamma} F_+(\alpha + k) \\ &= \bigcap_{i<\gamma} \bigcup_{i\leq k<\gamma} F_+(\beta + k) \\ &= \bigcap_{i<\beta+\gamma} \bigcup_{i\leq k<\beta+\gamma} F_+(k) = F_+(\beta + \gamma)\,. \end{aligned}$$

(b) The first equation follows by induction on n. For $n = 0$ there is nothing to do. For $n > 0$, it follows from (a) and the inductive hypothesis that

$$F_+(\alpha + \beta n) = F_+(\alpha + \beta(n-1) + \beta) = F_+(\alpha + \beta) = F_+(\alpha)\,.$$

For the second equation, we have

$$\begin{aligned} F_+(\alpha + \beta\omega) &= \bigcap_{n<\omega} \bigcup_{n\leq k<\omega} \bigcup_{\gamma<\beta} F_+(\alpha + \beta k + \gamma) \\ &= \bigcap_{n<\omega} \bigcup_{n\leq k<\omega} \bigcup_{\gamma<\beta} F_+(\alpha + \gamma) = \bigcup_{\gamma<\beta} F_+(\alpha + \gamma)\,. \end{aligned}$$

(c) By (b), we have

$$F_+(\alpha + \beta\omega) = \bigcup_{i<\beta} F_+(\alpha + i) \subseteq \bigcup_{i<\gamma} F_+(\alpha + i) = F_+(\alpha + \gamma\omega)\,.$$

(d) Again, we use induction on γ. For $\gamma = 0$ there is nothing to do and the inductive step follows as in (b). If γ is a limit ordinal then we have

$$\begin{aligned} F_+(\alpha + \beta\gamma) &= \bigcap_{i<\gamma} \bigcup_{i\leq k<\gamma} \bigcup_{l<\beta} F_+(\alpha + \beta k + l) \\ &= \bigcap_{i<\gamma} \bigcup_{i\leq k<\gamma} \bigcup_{l<\beta} F_+(\alpha + l) \\ &= \bigcup_{l<\beta} F_+(\alpha + l) = F_+(\alpha + \beta\omega) = F_+(\alpha)\,, \end{aligned}$$

by inductive hypothesis and (b).

The second claim follows from (b) and the first claim. For one direction, note that we have

$$F_+(\alpha) = F_+(\alpha + \beta\eta\omega) = \bigcup_{\gamma<\beta\eta} F_+(\alpha + \gamma) \supseteq F_+(\eta)\,,$$

which implies that

$$F_+(\infty) = \bigcap_{\gamma\geq\alpha} \bigcup_{\eta\geq\gamma} F_+(\eta) \subseteq \bigcap_{\gamma\geq\alpha} \bigcup_{\eta\geq\gamma} F_+(\alpha) = F_+(\alpha)\,.$$

Conversely, $F_+(\alpha + \beta\gamma) \subseteq \bigcup_{\eta\geq\gamma} F_+(\eta)$ implies that

$$F_+(\alpha) = \bigcap_{\gamma} F_+(\alpha + \beta\gamma) \subseteq \bigcap_{\gamma} \bigcup_{\eta\geq\gamma} F_+(\eta) = F_+(\infty)\,.$$ ☐

In order to prove that there exist ordinals α and β with $F_+(\alpha) = F_+(\alpha + \beta) = F_+(\alpha + \beta\omega)$ we need some results about closed unbounded sets.

Lemma 6.5. *Let* F_+ *be the upper fixed-point induction of the function* $f : \wp(X) \to \wp(X)$. *Set* $\kappa := |X|$ *and* $\lambda := (2^\kappa)^+ \oplus \aleph_1$.

(a) *Suppose that $\alpha < \lambda$ and $S \subseteq \lambda$ is a cofinal set such that*

$$F_+(\alpha + \beta) = F_+(\alpha)\,, \quad \text{for all } \beta \in S\,.$$

If there is no $\beta \in S$ such that $F_+(\alpha + \beta\omega) = F_+(\alpha)$ then there exists an ordinal α' and a cofinal set $S' \subseteq \lambda$ such that

$$F_+(\alpha') \supset F_+(\alpha) \quad \text{and} \quad F_+(\alpha' + \beta') = F_+(\alpha') \text{ for all } \beta' \in S'\,.$$

(b) *There exist ordinals $\alpha, \beta < \lambda$ such that*

$$F_+(\alpha) = F_+(\alpha + \beta) = F_+(\alpha + \beta\omega)\,.$$

Proof. (a) Since λ is regular we have $|S| = \lambda$. Let $(\beta_i)_{i<\lambda}$ be an increasing enumeration of $S \setminus \{0\}$. By Lemma 6.4 (c) it follows that $F_+(\alpha + \beta_i\omega) \subseteq F_+(\alpha + \beta_k\omega)$, for all $i \leq k$. Consequently, there is some index $m < \lambda$ such that $F_+(\alpha + \beta_i\omega) = F_+(\alpha + \beta_m\omega)$, for all $i \geq m$. Set $\alpha' := \alpha + \beta_m\omega$ and

$$S' := \{\, \beta_i\omega \mid i \geq m \,\}\,.$$

By assumption, we have

$$F_+(\alpha) \neq F_+(\alpha + \beta_m\omega) = \bigcup_{\gamma<\beta_m} F_+(\alpha + \gamma) \supseteq F_+(\alpha)\,,$$

which implies that $F_+(\alpha') \supset F_+(\alpha)$.

(b) For $Z \subseteq X$, let

$$S(Z) := \{\, \alpha < \lambda \mid Z \subseteq F_+(\alpha) \,\}\,.$$

We construct a strictly increasing sequence of sets $(Z_i)_{i<\eta}$ such that each set $S(Z_i)$ is closed unbounded in λ. Let $Z_0 := \varnothing$. Then $S(Z_0) = \lambda$. For limit ordinals δ, set $Z_\delta := \bigcup_{i<\delta} Z_i$. By Proposition A4.6.4, it follows that $S(Z_\delta) = \bigcap_{i<\delta} S(Z_i)$ is closed unbounded.

For the successor step, suppose that we have already defined Z_i. Since $|\wp(X)| < \lambda$ we can find a set $Y \supseteq Z_i$ such that the set

$$P := \{\, \alpha < \lambda \mid F_+(\alpha) = Y \,\}$$

is cofinal. Let α be the minimal element of P and set

$$Q := \{\, \beta \mid \alpha + \beta \in P,\ \beta > 0 \,\}\,.$$

If there is some $\beta \in Q$ with $F_+(\alpha + \beta\omega) = F_+(\alpha)$ then we are done. Otherwise, we can use (a) to find an ordinal α' and a cofinal subset $Q' \subseteq \lambda$ such that $F_+(\alpha') \supset F_+(\alpha)$ and $F_+(\alpha' + \beta') = F_+(\alpha')$, for all $\beta' \in Q'$. We set $Z_{i+1} := F_+(\alpha')$. It remains to show that $S(Z_{i+1})$ is closed unbounded.

By construction the set $S(Z_{i+1}) \supseteq \{\, \alpha + \beta \mid \beta \in Q' \,\}$ is cofinal. Let $X \subseteq S(Z_{i+1})$ be a subset with $\sup X < \lambda$. If $\sup X \in X$ then we are done. Otherwise, $\delta := \sup X$ is a limit ordinal and $F_+(\delta) = \bigcap_{\alpha<\delta} \bigcup_{\alpha\leq\beta<\delta} F_+(\beta)$. Since, for every $\beta < \delta$, there is some ordinal $\beta \leq \gamma < \delta$ with $F_+(\gamma) \supseteq Z_{i+1}$ it follows that $F_+(\delta) \supseteq Z_{i+1}$. Hence, $\delta \in S(Z_{i+1})$.

We continue this construction until we either find indices α and β such that $F_+(\alpha) = F_+(\alpha + \beta) = F_+(\alpha + \beta\omega)$ or we have defined Z_i, for all $i < \lambda$. In the former case we are done. The latter case cannot happen since $Z_i \subset Z_k$, for $i < k$ and there are less than λ subsets of X. □

Corollary 6.6. *Let X be a set of size $\kappa := |X|$ and let F_+ be the upper fixed-point induction of $f : \wp(X) \to \wp(X)$. Set $\lambda := (2^\kappa)^+ \oplus \aleph_1$.*

(a) *There exists some ordinal $\alpha < \lambda$ such that $F_+(\infty) = F(\alpha)$.*

(b) *$F_+(\infty) = F(\lambda)$.*

Proof. By the preceding lemma, we can find ordinals $\alpha, \beta < (2^\kappa)^+$ such that $F_+(\alpha) = F_+(\alpha + \beta) = F_+(\alpha + \beta\omega)$. It follows by Lemma 6.4 (d) that $F_+(\infty) = F_+(\alpha)$. Furthermore, since $\lambda := (2^\kappa)^+$ is regular we have $F_+(\lambda) = F_+(\alpha + \beta\lambda) = F_+(\alpha) = F_+(\infty)$. □

In order to add these fixed-point operators to first-order logic we start by looking at definable functions $\wp(A^n) \to \wp(A^n)$.

Definition 6.7. Let $\varphi(R, \bar{x}, \bar{y}) \in L[\Sigma \cup \{R\}, X]$ be a formula where R is an n-ary relation symbol and $\bar{x}$ a tuple of n variables.

(a) Given a Σ-structure $\mathfrak{A}$ and parameters $\bar{c} \subseteq A$, φ defines the function

$$f_\varphi : \wp(A^n) \to \wp(A^n) : R \mapsto \{ \bar{a} \in A^n \mid \mathfrak{A} \vDash \varphi(R, \bar{a}, \bar{c}) \} .$$

(b) We say that the relation symbol R occurs *positively* in φ if every occurrence of R is in the scope of an even number of negation symbols. If R only appears in the scope of odd numbers of negation symbols, we say that it occurs *negatively* in φ.

Depending on which fixed-point operators we add we obtain several extensions of first-order logic.

Definition 6.8. (a) *Least fixed-point logic* $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$ is the extension of $\mathrm{FO}_{\kappa\aleph_0}$ by formulae of the form

$$[\mathrm{lfp}\, R\bar{x} : \varphi(R, \bar{x}, \bar{y})](\bar{z}) \quad \text{and} \quad [\mathrm{gfp}\, R\bar{x} : \varphi(R, \bar{x}, \bar{y})](\bar{z})$$

where we require that the relation R appears *positively* in φ. The semantics is defined by

$$\begin{aligned} \mathfrak{A} \vDash [\mathrm{lfp}\, R\bar{x} : \varphi(R, \bar{x}, \bar{c})](\bar{a}) \quad &:\text{iff} \quad \bar{a} \in \mathrm{lfp}\, f_\varphi ,\\ \mathfrak{A} \vDash [\mathrm{gfp}\, R\bar{x} : \varphi(R, \bar{x}, \bar{c})](\bar{a}) \quad &:\text{iff} \quad \bar{a} \in \mathrm{gfp}\, f_\varphi . \end{aligned}$$

(b) *Inflationary fixed-point logic* $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{IFP})$ is the extension of $\mathrm{FO}_{\kappa\aleph_0}$ by formulae of the form

$$[\mathrm{ifp}\, R\bar{x} : \varphi(R, \bar{x}, \bar{y})](\bar{z})$$

with the semantics

$$\mathfrak{A} \vDash [\mathrm{ifp}\, R\bar{x} : \varphi(R, \bar{x}, \bar{c})](\bar{a}) \quad :\text{iff} \quad \bar{a} \in \mathrm{ifp}\, f_\varphi .$$

(c) *Partial fixed-point logic* $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{PFP})$ is the extension of $\mathrm{FO}_{\kappa\aleph_0}$ by formulae of the form

$$[\liminf R\bar{x} : \varphi(R, \bar{x}, \bar{y})](\bar{z})$$

$$\text{and} \quad [\limsup R\bar{x} : \varphi(R, \bar{x}, \bar{y})](\bar{z})$$

with the semantics

$$\begin{aligned} \mathfrak{A} \vDash [\liminf R\bar{x} : \varphi(R, \bar{x}, \bar{c})](\bar{a}) \quad &:\text{iff} \quad \bar{a} \in \liminf f_\varphi ,\\ \mathfrak{A} \vDash [\limsup R\bar{x} : \varphi(R, \bar{x}, \bar{c})](\bar{a}) \quad &:\text{iff} \quad \bar{a} \in \limsup f_\varphi . \end{aligned}$$

The requirement on φ in the definition of $[\mathrm{lfp}\, \bar{R}x : \varphi]$ ensures that the least fixed point of f_φ does exist.

Lemma 6.9. *If $\varphi(R, \bar{x}, \bar{y}) \in \mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$ is a formula where the relation symbol R appears only positively then f_φ is increasing.*

Proof. One can show by a trivial induction on φ that, if $R \subseteq R'$ then

$$\mathfrak{A} \vDash \varphi(R, \bar{a}, \bar{c}) \quad \text{implies} \quad \mathfrak{A} \vDash \varphi(R', \bar{a}, \bar{c}) . \qquad \square$$

Example. We can express in $\mathrm{FO}(\mathrm{LFP})$ that a relation $<$ is well-founded by the formula

$$\varphi_{\mathrm{wf}} := \forall x[\mathrm{lfp}\, Px : (\forall y. y < x)Py](x) .$$

The α-th stage of the fixed-point induction of this formula contains all elements of foundation rank less than α.

Remark. Note that, by duality, we have

$$[\mathrm{gfp}\, R\bar{x} : \varphi(R, \bar{x})](\bar{z}) \equiv \neg[\mathrm{lfp}\, R\bar{x} : \neg\varphi(\neg R, \bar{x})](\bar{z}) ,$$

where $\varphi(\neg R)$ is the formula obtained form φ by negating every atom of the form $R\bar{t}$.

Lemma 6.10. $\mathrm{FO}_{\kappa\aleph_0} \leq \mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP}) \leq \mathrm{FO}_{\kappa\aleph_0}(\mathrm{IFP}) \leq \mathrm{FO}_{\kappa\aleph_0}(\mathrm{PFP})$

Proof. Clearly, $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$ is as least as expressive as $\mathrm{FO}_{\kappa\aleph_0}$. For the second inclusion note that, if f is an increasing function then $\mathrm{ifp}\, f = \mathrm{lfp}\, f$. Hence, we can simulate each least fixed point $[\mathrm{lfp}\, R\bar{x} : \varphi]$ by the formula $[\mathrm{ifp}\, R\bar{x} : \varphi]$. Similarly, we have

$$[\mathrm{ifp}\, R\bar{x} : \varphi](\bar{z}) \equiv [\liminf R\bar{x} : R\bar{x} \vee \varphi](\bar{z}) ,$$

since the fixed-point inductions of both fixed points coincide. $\square$

In order to compare fixed-point logics with infinitary first-order logic we construct formulae defining the various stages of a fixed point.

Definition 6.11. Let $\varphi(R, \bar{x})$ be a formula and α an ordinal.

(a) The α-th lfp-*approximation* of φ is defined by induction on α as

$$\begin{aligned} \varphi^0(\bar{x}) &:= \text{false}, \\ \varphi^{\alpha+1}(\bar{x}) &:= \varphi[R/\varphi^\alpha], \\ \varphi^\delta(\bar{x}) &:= \bigvee_{\alpha<\delta} \varphi[R/\varphi^\alpha], \quad \text{for limits } \delta, \end{aligned}$$

where $\varphi[R/\psi]$ denotes the formula obtained from φ by replacing every atom $R\bar{t}$ by the formula $\psi(\bar{t})$.

(b) The α-th ifp-*approximation* of φ is the same as the α-th lfp-approximation of the formula $R\bar{x} \lor \varphi$.

(c) The α-th lim inf-*approximation* of φ is the formula defined by

$$\begin{aligned} \varphi^0(\bar{x}) &:= \text{false}, \\ \varphi^{\alpha+1}(\bar{x}) &:= \varphi[R/\varphi^\alpha], \\ \varphi^\delta(\bar{x}) &:= \bigvee_{\alpha<\delta} \bigwedge_{i<\alpha} \varphi[R/\varphi^i], \quad \text{for limits } \delta. \end{aligned}$$

(d) The α-th lim sup-*approximation* of φ is the formula defined by

$$\begin{aligned} \varphi^0(\bar{x}) &:= \text{false}, \\ \varphi^{\alpha+1}(\bar{x}) &:= \varphi[R/\varphi^\alpha], \\ \varphi^\delta(\bar{x}) &:= \bigwedge_{\alpha<\delta} \bigvee_{i<\alpha} \varphi[R/\varphi^i], \quad \text{for limits } \delta. \end{aligned}$$

Lemma 6.12. *Let φ^α be the α-th* fp*-approximation of a formula φ where* fp *is one of* lfp, ifp, lim inf, *or* lim sup. *Let $\mathfrak{A}$ be a structure and F the fixed-point induction of* $[\text{fp} R\bar{x} : \varphi]$ *on $\mathfrak{A}$. Then we have*

$$(\varphi^\alpha)^{\mathfrak{A}} = F(\alpha).$$

Lemma 6.13. *Let $\varphi \in \text{FO}_{\kappa\aleph_0}(\text{PFP})$. For every regular cardinal μ, there exists a formula $\psi \in \text{FO}_{\lambda\aleph_0}$ where $\lambda := (2^\mu)^{++} \oplus \kappa \oplus \aleph_2$ such that*

$$\mathfrak{A} \vDash \varphi \leftrightarrow \psi, \quad \textit{for every structure } \mathfrak{A} \textit{ of size } |A| \leq \mu.$$

Proof. We prove the claim by induction on φ. Hence, we may assume that $\varphi = [\limsup R\bar{x} : \chi](\bar{x})$ with $\chi \in \text{FO}_{\lambda\aleph_0}$. Let χ^α be the α-th lim sup-approximation of χ. Let $\lambda_0 := (2^\mu)^+ \oplus \aleph_1$. By Corollary 6.6 and the preceding lemma it follows that the formula χ^{λ_0} defines the partial fixed point of χ on all structures of cardinality $|A| < \mu$. Finally, note that $\chi^{\lambda_0} \in \text{FO}_{\lambda\aleph_0}$. □

Corollary 6.14. $\text{FO}_{\infty\aleph_0}(\text{PFP})$ *has the Karp property.*

In some cases the closure ordinal of a least fixed point is independent of the size of the structure.

Lemma 6.15. *Let $\varphi(R, \bar{x}, \bar{y})$ be an existential first-order formula where R occurs only positively. On every structure $\mathfrak{A}$ the least fixed point* $[\text{lfp} R\bar{x} : \varphi(R, \bar{x}, \bar{y})]$ *is reached after at most ω steps.*

Proof. The corresponding function $f_\varphi : \mathscr{P}(A)^n \to \mathscr{P}(A)^n$ is continuous since $f_\varphi(R) = \bigcup\{ f_\varphi(R_0) \mid R_0 \subseteq R \text{ finite} \}$. Hence, the claim follows from Lemma A3.3.12 (c). □

In Chapter E1 we will study saturated structures. One of their many properties is the fact that, for such structures, the preceding lemma holds for all first-order formulae, not only for existential ones.

Definition 6.16. A structure $\mathfrak{A}$ is $\aleph_0$-*saturated* if $\mathfrak{A}$ realises every type $\mathfrak{p} \in S^1(U)$ where $U \subseteq A$ is finite.

Lemma 6.17. *Let $\varphi(R, \bar{x})$ be a first-order formula where R occurs only positively and let $\mathfrak{A}$ be an $\aleph_0$-saturated structure. The least fixed point of φ on $\mathfrak{A}$ is reached after at most ω steps.*

Proof. Let F be the fixed-point induction of φ on $\mathfrak{A}$. If we can show that

$$\varphi(F(\omega))^{\mathfrak{A}} = \bigcup_{n<\omega} \varphi(F(n))^{\mathfrak{A}}$$

then it follows that

$$F(\omega+1) = \varphi(F(\omega))^{\mathfrak{A}} = \bigcup_{n<\omega} F(n+1) = F(\omega)\,,$$

as desired. Note that each set $F(n)$ with $n < \omega$ is definable by the n-th approximation of φ.

For the induction below we prove a slightly more general statement. We consider formulae $\varphi(R, \bar{x})$ where the relation R occurs only positively, but where we do not require the arity of R to be equal to the number of variables $\bar{x}$. With every such formula $\varphi(R, \bar{x})$ we associate the function

$$f_\varphi(R) := \{\, \bar{a} \subseteq A \mid \mathfrak{A} \vDash \varphi(R, \bar{a}) \,\}\,,$$

and we prove by induction on φ that

$$f_\varphi(\bigcup_{n<\omega} R_n) = \bigcup_{n<\omega} f_\varphi(R_n)\,,$$

for every increasing sequence $(R_n)_{n<\omega}$ of FO-definable relations.

W.l.o.g. we may assume that φ is in negation normal form. As φ is monotone in R we have

$$f_\varphi(R_n) \subseteq f_\varphi(\bigcup_n R_n)\,, \quad \text{for all } n < \omega\,.$$

This implies that

$$\bigcup_n f_\varphi(R_n) \subseteq f_\varphi(\bigcup_n R_n)\,.$$

Hence, we only need to prove the converse inclusion.

First, suppose that φ is atomic. If R does not occur in φ then there is nothing to do. Hence, assume that $\varphi = Rt_0 \dots t_{m-1}$. Then we have

$$\begin{aligned} & \bar{a} \in f_\varphi(\bigcup_n R_n) \\ \Rightarrow\ & \langle t_0(\bar{a}), \dots, t_{m-1}(\bar{a})\rangle \in \bigcup_n R_n \\ \Rightarrow\ & \langle t_0(\bar{a}), \dots, t_{m-1}(\bar{a})\rangle \in R_n\,, \quad \text{for some } n < \omega \\ \Rightarrow\ & \bar{a} \in \bigcup_n f_\varphi(R_n)\,. \end{aligned}$$

If φ is the negation of an atom the proof is analogous.

For $\varphi = \psi \wedge \vartheta$ or $\varphi = \psi \vee \vartheta$ the claim follows immediately from inductive hypothesis.

Suppose that $\varphi = \exists y \psi(R, \bar{x}, y)$. Then we have

$$\begin{aligned} & \bar{a} \in f_\varphi(\bigcup_n R_n) \\ \Rightarrow\ & \bar{a}b \in f_\psi(\bigcup_n R_n)\,, \quad \text{for some } b \in A \\ \Rightarrow\ & \bar{a}b \in \bigcup_n f_\psi(R_n)\,, \quad \text{for some } b \in A \\ \Rightarrow\ & \bar{a} \in \bigcup_n f_\varphi(R_n)\,. \end{aligned}$$

Finally, we consider the case that $\varphi = \forall y \psi(R, \bar{x}, y)$. For a contradiction, suppose that there is some tuple

$$\bar{a} \in f_\varphi(\bigcup_n R_n) \setminus \bigcup_n f_\varphi(R_n)\,.$$

Since $\mathfrak{A} \nvDash \forall y \psi(R_n, \bar{a}, b)$ we can find elements $b_n \in A$ such that

$$\mathfrak{A} \nvDash \psi(R_n, \bar{a}, b_n)\,.$$

Let $\vartheta_n(\bar{z})$ be the formula defining R_n. We define

$$\Phi := \{\, \neg\psi(\vartheta_n, \bar{a}, y) \mid n < \omega \,\}\,,$$

where $\psi(\vartheta_n, \bar{x}, y)$ is the formula obtained from ψ by replacing every atom $R\bar{t}$ by $\vartheta_n(\bar{t})$. Since $R_k \subseteq R_n$, for $k \leq n$, we have $\vartheta_k \vDash \vartheta_n$. As ψ is monotone in R it follows that

$$\neg\psi(\vartheta_n, \bar{a}, y) \vDash \neg\psi(\vartheta_k, \bar{a}, y)\,, \quad \text{for all } k \leq n\,.$$

Therefore, every finite subset of Φ is satisfiable. Hence, Φ is a partial type over $\bar{a}$. Since $\mathfrak{A}$ is $\aleph_0$-saturated we can find some element $b_* \in A$ realising Φ. Consequently, we have

$$\bar{a}b_* \notin \bigcup_n f_\psi(R_n) = f_\psi(\bigcup_n R_n)\,.$$

Hence, $\mathfrak{A} \vDash \exists y \neg\psi(\bigcup_n R_n, \bar{a}, y)$ which implies that $\bar{a} \notin f_\varphi(\bigcup_n R_n)$. Contradiction. □

Theorem 6.18 (Barwise, Moschovakis). *Suppose that $\mathcal{K}$ is a pseudo-elementary class and $\varphi(R, \bar{x})$ a first-order formula. The following statements are equivalent:*

(1) *There exists a formula $\psi(\bar{x}) \in$ FO such that*

$$\mathfrak{A} \vDash \forall\bar{x}\big[\psi(\bar{x}) \leftrightarrow [\mathrm{lfp}\, R\bar{x} : \varphi](\bar{x})\big]\,, \quad \text{for all } \mathfrak{A} \in \mathcal{K}\,.$$

(2) *For every $\mathfrak{A} \in \mathcal{K}$, there exists a formula $\psi(\bar{x}) \in$ FO such that*

$$\mathfrak{A} \vDash \forall\bar{x}\big[\psi(\bar{x}) \leftrightarrow [\mathrm{lfp}\, R\bar{x} : \varphi](\bar{x})\big]\,.$$

(3) *On every $\mathfrak{A} \in \mathcal{K}$ the least fixed-point of φ is reached after finitely many steps.*

(4) *There is a contant $n < \omega$ such that, on each $\mathfrak{A} \in \mathcal{K}$ the least fixed-point of φ is reached after at most n steps.*

Proof. Let $\mathcal{K}^+$ be a class such that $\mathcal{K} = \mathrm{pr}_\Sigma(\mathcal{K}^+)$ and fix a first-order theory T axiomatising $\mathcal{K}^+$. Let φ^n be the n-th approximation of φ.

(4) ⇒ (1) If, on every structure of $\mathcal{K}$, the fixed point is reached after at most n steps then we have

$$\mathfrak{A} \vDash \varphi^n(\bar{a}) \leftrightarrow [\mathrm{lfp}\, R\bar{x} : \varphi](\bar{a})\,, \quad \text{for all } \mathfrak{A} \in \mathcal{K} \text{ and all } \bar{a} \subseteq A\,.$$

Hence, we can set $\psi := \varphi^n$.

(1) ⇒ (2) is trivial.

(2) ⇒ (3) For a contradiction, suppose that on some structure $\mathfrak{A} \in \mathcal{K}^+$ the fixed point of φ is not reached in finitely many steps. Fix some $\aleph_0$-saturated elementary extension $\mathfrak{B} \succeq \mathfrak{A}$. Since

$$\mathfrak{A}|_\Sigma \vDash \exists\bar{x}[\varphi^{n+1}(\bar{x}) \wedge \neg\varphi^n(\bar{x})]\,, \quad \text{for all } n < \omega\,,$$

it follows that, on the structure $\mathfrak{B}$, the fixed point is also not reached in finitely many steps. By assumption there is a first-order formula $\psi(\bar{x})$ defining the fixed point on $\mathfrak{B}$. Hence,

$$\mathfrak{B}|_\Sigma \vDash \exists\bar{x}[\psi(\bar{x}) \wedge \neg\varphi^n(\bar{x})]\,, \quad \text{for all } n < \omega\,.$$

As $\mathfrak{B}$ is $\aleph_0$-saturated we can find some tuple $\bar{b} \subseteq B$ such that

$$\mathfrak{B}|_\Sigma \vDash \psi(\bar{b}) \wedge \bigwedge_{n<\omega} \neg\varphi^n(\bar{b})\,.$$

Note that $\mathfrak{B} \in \mathcal{K}^+$. Hence, $\psi^{\mathfrak{B}}$ is the fixed point of φ. Since the tuple $\bar{b}$ enters the fixed point at an infinite stage it follows that the fixed point is not reached in ω steps. (Note that no tuple enters the fixed point at stage ω.) This contradicts Lemma 6.17.

(3) ⇒ (4) For a contradiction, suppose that, for each $n < \omega$, there is a structure $\mathfrak{A}_n \in \mathcal{K}^+$ such that on $\mathfrak{A}_n$ the fixed-point of φ is reached after more than n steps. Setting

$$\vartheta_n := \exists\bar{x}[\varphi^{n+1}(\bar{x}) \wedge \neg\varphi^n(\bar{x})]$$

we have

$$\mathfrak{A}_n|_\Sigma \vDash \vartheta_n\,.$$

It follows that $T \nvDash \neg\vartheta_n$, for all $n < \omega$. Let $\Theta := \{\, \vartheta_n \mid n < \omega \,\}$. The theory $T \cup \Theta$ is consistent since, for every finite subset $\Theta_0 \subseteq \Theta$, we can find some n such that $\mathfrak{A}_n|_\Sigma \vDash T \cup \Theta_0$. Let $\mathfrak{B}$ be a model of $T \cup \Theta$. It follows that on $\mathfrak{B}$ the fixed-point of φ is not reached after finitely many steps. Contradiction. □

As an example of the expressive power of fixed-point logics we consider linear orders.

Lemma 6.19. *There exists a formula* $\varphi(x,y,z) \in \mathrm{FO}^3(\mathrm{LFP})[<]$ *such that, for every infinite cardinal* κ*,* φ *defines in the structure* $\langle\kappa,<\rangle$ *a bijection* $\kappa\times\kappa\to\kappa$*.*

Proof. We have shown in the proof of Theorem A4.3.8 that the formula

$$\begin{aligned}\psi(x_0x_1,y_0y_1) := {} & [(x_0<y_0 \lor x_0<y_1)\land(x_1<y_0\lor x_1<y_1)] \\ & \lor [x_0<y_0 \land x_1=y_1 \land y_0\leq y_1] \\ & \lor [x_0<y_0 \land x_1=y_0 \land y_1\leq y_0] \\ & \lor [x_0=y_0 \land x_1<y_1 \land y_1\leq y_0]\end{aligned}$$

defines a linear order on $\kappa\times\kappa$ of order type κ. The fixed-point formula

$$\begin{aligned}&\varphi(x,y,z) := \\ &\quad [\mathrm{lfp}\, Ru_0u_1w : (\forall v_0v_1.\psi(\bar v,\bar u))(\exists w'.w'<w)R\bar v w' \\ &\qquad\qquad \land (\forall w'.w'<w)(\exists v_0v_1.\psi(\bar v,\bar u))R\bar v w'](x,y,z)\,.\end{aligned}$$

defines the corresponding bijection. □

Exercise 6.1. Let $\mathfrak{N}=\langle\mathbb{N},<\rangle$. Construct FO(LFP)-formulae $\varphi_+(x,y,z)$ and $\varphi_\cdot(x,y,z)$ defining addition and multiplication on $\mathfrak{N}$.

To facilitate the investigation of model theoretic properties of fixed point logics we reduce them to a simpler logic, the extension of first-order logic by well-ordering quantifiers.

Lemma 6.20. $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP}) =^1_{\mathrm{pc}} \mathrm{FO}_{\kappa\aleph_0}(\mathrm{wo})$.

Proof. We have seen in the example above that, for every $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$-formula $\varphi(\bar x,\bar y,\bar z)$, we can construct a formula $\psi(\bar z)\in\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$ expressing that φ defines a well-order. Hence, $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{wo})\leq\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$.

For the converse, let $\varphi\in\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$. For every subformula ψ of φ, we introduce a new relation symbol R_ψ and we construct a set of sentences Φ such that

$$R_\psi^{\mathfrak{A}} = \psi^{\mathfrak{A}}\,, \qquad \text{for every model } \mathfrak{A}\vDash\Phi\,.$$

The construction of Φ proceeds by induction on φ. For atomic subformulae ψ, we add the formula

$$\forall\bar x(R_\psi\bar x\leftrightarrow\psi(\bar x))$$

to Φ. For the inductive step we use the same formulae as in the proof of Chang's Reduction (Lemma C1.4.12), e.g., for conjuctions we use

$$\forall\bar x(R_{\bigwedge\Psi}\bar x\leftrightarrow\bigwedge_{\psi\in\Psi}R_\psi\bar x)\,.$$

The only nontrivial case is the case that $\psi=[\mathrm{lfp}\,P\bar x:\vartheta](\bar x)$ is a fixed-point formula.

Let $<$ be a new binary relation symbol, s a new unary function symbol, and 0 a new constant symbol. We add to Φ the sentences

$$\begin{aligned}&\forall u(u=0\lor 0<u)\\ &\forall u(u<su\land\neg\exists v(u<v\land v<su))\\ &\mathsf{W}uv(u<v)\end{aligned}$$

which express that $<$ is a well-order of the universe, s is the successor function, and 0 is the minimal element. Furthermore, we add the formulae

$$\begin{aligned}&\forall\bar x(\neg S_\varphi 0\bar x)\\ &\forall u\forall\bar x\big(S_\varphi su\bar x\leftrightarrow\chi[P/S_\varphi u]\big)\,,\\ &\forall u\forall\bar x\big[\forall v(sv\neq u)\to(S_\varphi u\bar x\leftrightarrow\exists v(v<u\land S_\varphi v\bar x))\big]\,,\\ &\forall\bar x(R_\varphi\bar x\leftrightarrow\exists uS_\varphi u\bar x)\,,\end{aligned}$$

which express that $S_\varphi = \{ \langle \alpha, \bar{a} \rangle \mid \bar{a} \in F(\alpha) \}$. Finally, we need the formula

$$\exists u \forall \bar{x} (S_\varphi s u \bar{x} \leftrightarrow S_\varphi u \bar{x}) ,$$

which expresses that the fixed point is actually reached. For the correctness of this construction note that the closure ordinal α of every $\mathrm{FO}_{\infty\aleph_0}(\mathrm{LFP})$-induction on a structure $\mathfrak{A}$ is less than $|A|^+$. Hence, we can really choose an ordering $<$ of A of order type α. □

For partial fixed points we have an analogous result where the projective reduction is replaced by a relativised reduct.

Lemma 6.21. $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{PFP}) =^1_{\mathrm{rpc}} \mathrm{FO}_{\kappa\aleph_0}(\mathrm{wo})$.

Proof. We can basically use the same construction as in the proof of Lemma 6.20. The only difference is that the closure ordinal for partial fixed points is not bounded by the size of the structure. Therefore, we cannot choose a sufficiently long well-ordering of the universe. Instead, we add a new sort w to the given structure $\mathfrak{A}$ and we choose the domain A_w large enough to contain a well-ordering $<$ of length $(2^{|A|})^+$. After performing the same construction as above in the larger structure we can take a relativised reduct to obtain the original structure $\mathfrak{A}$. □

Using this reduction we can use the Löwenheim-Skolem theorem for $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{wo})$ to derive a corresponding theorem for $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{PFP})$.

Theorem 6.22. *Let $\Delta \subseteq \mathrm{FO}^{<\omega}_{\kappa\aleph_0}(\mathrm{PFP})[\Sigma]$, for a regular cardinal κ, and set $\mu := |\Sigma| \oplus |\Delta| \oplus \kappa^-$ where $\kappa^- := \sup \{ \lambda \mid \lambda < \kappa \}$.*

For each Σ-structure $\mathfrak{A}$, every subset $X \subseteq A$, and all cardinals λ with $|X| \oplus \mu \leq \lambda \leq |A|$, there exists a Δ-substructure $\mathfrak{B} \preceq_\Delta \mathfrak{A}$ of size $|B| = \lambda$ with $X \subseteq B$.

Proof. This follows immediately by Theorem c2.3.10 and Lemma 6.21. □

We conclude this section with a proof that the logics $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$ and $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{IFP})$ have the same expressive power.

Definition 6.23. Let $\mathfrak{A}$ be a structure, $\varphi(R, \bar{x})$ an $\mathrm{FO}_{\infty\aleph_0}(\mathrm{IFP})$-formula, and F the fixed-point induction of $[\mathrm{ifp}\, R\bar{x} : \varphi]$.

(a) The *inductive fixed-point rank* $\mathrm{rk}_\varphi(\bar{a})$ of a tuple $\bar{a} \in [\mathrm{ifp}\, R\bar{x} : \varphi]^{\mathfrak{A}}$ is the ordinal α such that $\bar{a} \in F(\alpha + 1) \setminus F(\alpha)$. For $\bar{a} \notin [\mathrm{ifp}\, R\bar{x} : \varphi]^{\mathfrak{A}}$, we set $\mathrm{rk}_\varphi(\bar{a}) := \infty$.

(b) The *stage comparison relation* $\lhd_\varphi$ of φ is defined by

$$\bar{a} \lhd_\varphi \bar{b} \quad :\text{iff} \quad \mathrm{rk}_\varphi(\bar{a}) < \mathrm{rk}_\varphi(\bar{b}) .$$

Lemma 6.24. *Let $\varphi(P, \bar{x})$ be an $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{IFP})$-formula. The stage comparison relation $\lhd_\varphi$ for $[\mathrm{ifp}\, P\bar{x} : \varphi]$ is $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{IFP})$-definable.*

Proof. Let $\hat{\varphi}(\bar{x}, \bar{z})$ be the formula obtained from $P\bar{x} \vee \varphi(P, \bar{x})$ by replacing every atom of the form $P\bar{t}$ by the formula $R\bar{t}\bar{z}$. We claim that $\lhd_\varphi$ is defined by the formula where

$$[\mathrm{ifp}\, R\bar{x}\bar{y} : \hat{\varphi}(\bar{x}, \bar{x}) \wedge \neg\hat{\varphi}(\bar{y}, \bar{x})](\bar{x}\bar{y}) .$$

Let $(R^\alpha)_\alpha$ be the fixed-point induction of this formula. A straightforward induction on α shows that

$$\langle \bar{a}, \bar{b} \rangle \in R^\alpha \quad \text{iff} \quad \bar{a} \lhd_\varphi \bar{b} \text{ and } \mathrm{rk}_\varphi(\bar{a}) < \alpha .$$

Hence, the result follows. □

Proposition 6.25. *Let $\varphi(P, \bar{x})$ be an $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$-formula. The stage comparison relation $\lhd_\varphi$ for $[\mathrm{ifp}\, P\bar{x} : \varphi]$ is $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$-definable.*

Proof. By $\varphi[P\bar{z}/\psi(\bar{z})/\vartheta(\bar{z})]$ we denote the formula obtained from the formula $P\bar{x} \vee \varphi(P, \bar{x})$ by replacing every atom of the form $P\bar{t}$ by

- $\psi(\bar{t})$, if this atom occurs positively in φ,
- $\vartheta(\bar{t})$, if it occurs negatively in φ.

As in the proof of the preceding lemma we would like to compute $\lhd_\varphi$ by the formula

$$\big[\mathrm{lfp}\, R\bar{x}\bar{y} : \varphi[P\bar{z}/R\bar{z}\bar{x}/R\bar{z}\bar{x}](\bar{x}) \wedge \neg\varphi[P\bar{z}/R\bar{z}\bar{x}/R\bar{z}\bar{x}](\bar{y})\big](\bar{x}\bar{y})\,.$$

Unfortunately, this does not work since we can use R only positively in φ and only negatively in $\neg\varphi$. Instead, we construct another formula ψ computing $\lhd_\varphi$ that we can substitute for R at those places where we cannot use it. Again the obvious definition

$$\psi(\bar{x},\bar{y}) := \big[\mathrm{lfp}\, S\bar{x}\bar{y} : \varphi[P\bar{z}/S\bar{z}\bar{x}/S\bar{z}\bar{x}](\bar{x}) \wedge \neg\varphi[P\bar{z}/S\bar{z}\bar{x}/S\bar{z}\bar{x}](\bar{y})\big](\bar{x}\bar{y})$$

does not work. But, since ψ is used in the above formula at those places where R occurs negatively we can use R inside of ψ provided its occurrence is also negative. These considerations lead to following attempt to define $\lhd_\varphi$:

$$\big[\mathrm{lfp}\, R\bar{x}\bar{y} : \varphi[P\bar{z}/R\bar{z}\bar{x}/\psi(\bar{z},\bar{x})](\bar{x}) \wedge \wedge \neg\varphi[P\bar{z}/\psi(\bar{z},\bar{x})/R\bar{z}\bar{x}](\bar{y})\big](\bar{x}\bar{y})\,,$$

where $\psi(\bar{x},\bar{y})$ is the formula

$$\big[\mathrm{lfp}\, S\bar{x}\bar{y} : \varphi[P\bar{z}/S\bar{z}\bar{x}/R\bar{z}\bar{x}](\bar{x}) \wedge \neg\varphi[P\bar{z}/R\bar{z}\bar{x}/S\bar{z}\bar{x}](\bar{y})\big](\bar{x}\bar{y})\,.$$

This definition is still not correct but we can repair it as follows. We claim that $\lhd_\varphi$ is defined by the formula $[\mathrm{lfp}\, R\bar{x}\bar{y} : \chi](\bar{x}\bar{y})$ where

$$\begin{aligned}\chi(\bar{x},\bar{y}) &:= \varphi[P\bar{z}/R\bar{z}\bar{x}/\psi(\bar{z},\bar{x})](\bar{x})\\ &\quad\wedge \neg\varphi[P\bar{z}/\psi(\bar{z},\bar{x})/R\bar{z}\bar{x}](\bar{y})\\ &\quad\wedge \forall\bar{z}(\psi(\bar{z},\bar{x}) \rightarrow R\bar{z}\bar{x})\,,\\ \psi(\bar{x},\bar{y}) &:= [\mathrm{lfp}\, S\bar{x}\bar{y} : \vartheta](\bar{x}\bar{y})\,,\\ \vartheta(\bar{x},\bar{y}) &:= \varphi[P\bar{z}/S\bar{z}\bar{x}/R\bar{z}\bar{x}](\bar{x})\\ &\quad\wedge \neg\varphi[P\bar{z}/R\bar{z}\bar{x}/S\bar{z}\bar{x} \wedge S\bar{z}\bar{y}](\bar{y})\\ &\quad\wedge (\forall\bar{z}.R\bar{z}\bar{x})(S\bar{z}\bar{x} \wedge S\bar{z}\bar{y})\,.\end{aligned}$$

First, note that $[\mathrm{lfp}\, R\bar{x}\bar{y} : \chi] \in \mathrm{FO(LFP)}$ since S occurs only positively in ϑ and R occurs only negatively in ψ. Let $(P^\alpha)_\alpha$ be the fixed-point induction of $[\mathrm{ifp}\, P\bar{x} : \varphi]$. For $\alpha \in \mathrm{On}$, define

$$R^\alpha := \{\, \langle\bar{a},\bar{b}\rangle \mid \bar{a} \lhd_\varphi \bar{b} \text{ and } \mathrm{rk}_\varphi(\bar{a}) < \alpha \,\}\,.$$

We will show that the sequence $(R^\alpha)_\alpha$ is the fixed-point induction of χ.

Claim. *Let $(S^\beta)_\beta$ be the fixed-point induction of ψ where R is interpreted by R^α, and set $S^\infty := \bigcup_\beta S^\beta$.*

(a) *For $\beta \leq \alpha$, we have*

$$\langle\bar{a},\bar{b}\rangle \in S^\beta \quad\text{iff}\quad \bar{a} \lhd_\varphi \bar{b} \text{ and } \mathrm{rk}_\varphi(\bar{a}) < \beta\,.$$

(b) *For all tuples $\bar{b}$ with rank $\mathrm{rk}_\varphi(\bar{b}) > \alpha$, there exist a tuple $\bar{a}$ with $\mathrm{rk}_\varphi(\bar{a}) = \alpha$ such that $\langle\bar{a},\bar{b}\rangle \in S^\infty$.*

(c) *If $R^{\alpha+1} = R^\alpha$ then $S^\infty = R^\alpha$.*

(d) *If $\mathrm{rk}_\varphi(\bar{a}) < \alpha$ or $\mathrm{rk}_\varphi(\bar{b}) < \alpha$ then we have*

$$\langle\bar{a},\bar{b}\rangle \in S^\infty \quad\text{iff}\quad \bar{a} \lhd_\varphi \bar{b}\,.$$

(a) We prove the claim by induction on β. The case that $\beta = 0$ is trivial and the limit step follows immediately from the inductive hypothesis. For the successor step, note that

$$\langle\bar{a},\bar{b}\rangle \in S^{\beta+1} \quad\text{iff}\quad \langle\mathfrak{A}, R^\alpha, S^\beta\rangle \vDash \vartheta(\bar{a},\bar{b})\,.$$

First, suppose that $\gamma := \mathrm{rk}_\varphi(\bar{a}) < \beta$. By inductive hypothesis, it follows that

$$\langle\bar{c},\bar{a}\rangle \in S^\beta \quad\text{iff}\quad \bar{c} \lhd_\varphi \bar{a} \quad\text{iff}\quad \bar{c} \in P^\gamma.$$

Since $\beta \leq \alpha$ we further have that

$$\langle\bar{c},\bar{a}\rangle \in R^\alpha \quad\text{iff}\quad \bar{c} \lhd_\varphi \bar{a} \quad\text{iff}\quad \bar{c} \in P^\gamma.$$

Consequently, $\mathfrak{A} \vDash \varphi(P^\gamma, \bar{a})$ implies that

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \vDash \varphi[P\bar{z}/S\bar{z}\bar{a}/R\bar{z}\bar{a}](\bar{a}) .$$

If $\mathrm{rk}_\varphi(\bar{a}) < \mathrm{rk}_\varphi(\bar{b})$ then, by inductive hypothesis, $\langle \bar{c}, \bar{a} \rangle \in S$ implies $\langle \bar{c}, \bar{b} \rangle \in S$. Since $\beta \leq \alpha$ it follows that we have $\langle \bar{c}, \bar{a} \rangle \in S^\beta$ iff $\langle \bar{c}, \bar{a} \rangle \in R^\alpha$. Consequently, there is no tuple $\bar{c}$ such that

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \vDash R\bar{c}\bar{a} \wedge \neg(S\bar{c}\bar{a} \wedge S\bar{c}\bar{b}) .$$

Finally, we have

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \nvDash \varphi[P\bar{z}/R\bar{z}\bar{a}/S\bar{z}\bar{a} \wedge S\bar{z}\bar{b}](\bar{b})$$

since, otherwise, $\mathrm{rk}_\varphi(\bar{b}) \leq \mathrm{rk}_\varphi(\bar{a})$. It follows that $\langle \bar{a}, \bar{b} \rangle \in S^{\beta+1}$.

Next, consider the case that $\mathrm{rk}_\varphi(\bar{a}) > \mathrm{rk}_\varphi(\bar{b})$. By inductive hypothesis, $S\bar{c}\bar{a} \wedge S\bar{c}\bar{b}$ is equivalent to $S\bar{c}\bar{b}$. Consequently, choosing $\bar{c} := \bar{b}$ we can find a tuple $\bar{c}$ with

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \vDash R\bar{c}\bar{a} \wedge \neg(S\bar{c}\bar{a} \wedge S\bar{c}\bar{b}) .$$

Hence, $\langle \bar{a}, \bar{b} \rangle \notin S^{\beta+1}$.

Finally, suppose that $\mathrm{rk}_\varphi(\bar{a}) = \mathrm{rk}_\varphi(\bar{b})$. Then

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \vDash \varphi[P\bar{z}/R\bar{z}\bar{a}/S\bar{z}\bar{a} \wedge S\bar{z}\bar{b}](\bar{b})$$

and $\langle \bar{a}, \bar{b} \rangle \notin S^{\beta+1}$.

It remains to consider the case that $\mathrm{rk}_\varphi(\bar{a}) \geq \beta$. By inductive hypothesis, we have $\langle \bar{c}, \bar{a} \rangle \in S^\beta$ iff $\mathrm{rk}(\bar{c}) < \beta$. Since $\mathfrak{A} \nvDash \varphi(P^\beta, \bar{a})$ it follows that

$$\langle \mathfrak{A}, S^\beta \rangle \nvDash \varphi[P\bar{z}, S\bar{z}\bar{a}, S\bar{z}\bar{a}](\bar{a}) .$$

Note that

$$\langle \bar{c}, \bar{a} \rangle \in S^\beta \quad \text{implies} \quad \langle \bar{c}, \bar{a} \rangle \in R^\alpha .$$

Since P'' occurs only negatively in $\varphi[P\bar{z}/P'\bar{z}/P''\bar{z}]$ it therefore follows that

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \nvDash \varphi[P\bar{z}/S\bar{z}\bar{a}/R\bar{z}\bar{a}](\bar{a}) .$$

(b) By (a), we have $S^\alpha = R^\alpha$. Let $\mathrm{rk}_\varphi(\bar{a}) \leq \alpha$. Then $\bar{a} \in P^{\alpha+1}$ implies that

$$\langle \mathfrak{A}, R^\alpha, S^\alpha \rangle \vDash \varphi[P\bar{z}/S\bar{z}\bar{a}/R\bar{z}\bar{a}](\bar{a}) .$$

If $\mathrm{rk}_\varphi(\bar{b}) \geq \mathrm{rk}_\varphi(\bar{a})$ then $S\bar{c}\bar{a} \wedge S\bar{c}\bar{b}$ is equivalent to $S\bar{c}\bar{a}$ and, hence, to $R\bar{c}\bar{a}$. Consequently, it follows in this case that

$$\langle \mathfrak{A}, R^\alpha, S^\alpha \rangle \vDash \neg\varphi[P\bar{z}, R\bar{z}\bar{b}, S\bar{z}\bar{a} \wedge S\bar{z}\bar{b}](\bar{b})$$

iff $\quad \mathrm{rk}_\varphi(\bar{b}) > \mathrm{rk}_\varphi(\bar{a})$.

If, on the other hand, $\mathrm{rk}_\varphi(\bar{b}) < \mathrm{rk}_\varphi(\bar{a})$ then $\mathrm{rk}_\varphi(\bar{b}) < \alpha$ and setting $\bar{c} := \bar{b}$ we obtain a tuple such that

$$\langle \mathfrak{A}, R^\alpha, S^\alpha \rangle \vDash R\bar{c}\bar{a} \wedge \neg(S\bar{c}\bar{a} \wedge S\bar{c}\bar{b}) .$$

Consequently, $\langle \mathfrak{A}, R^\alpha, S^\alpha \rangle \nvDash \vartheta(\bar{a}, \bar{b})$.

Finally, suppose that $\mathrm{rk}_\varphi(\bar{a}) > \alpha$. Then

$$\langle \mathfrak{A}, R^\alpha, S^\alpha \rangle \nvDash \varphi[P\bar{z}/S\bar{z}\bar{a}/R\bar{z}\bar{a}](\bar{a})$$

since $\langle \bar{c}, \bar{a} \rangle \in S^\alpha$ iff $\mathrm{rk}_\varphi(\bar{c}) < \alpha$ and $\langle \bar{c}, \bar{a} \rangle \in R^\alpha$ iff $\mathrm{rk}_\varphi(\bar{c}) < \alpha$.

It follows that $S^{\alpha+1}$ contains all pairs $\langle \bar{a}, \bar{b} \rangle$ with $\mathrm{rk}_\varphi(\bar{a}) \leq \alpha$ and $\mathrm{rk}_\varphi(\bar{a}) < \mathrm{rk}_\varphi(\bar{b})$. If $\mathrm{rk}_\varphi(\bar{b}) > \alpha$ then there exists some tuple $\bar{a}$ with $\mathrm{rk}_\varphi(\bar{a}) = \alpha$ and it follows that $\langle \bar{a}, \bar{b} \rangle \in S^{\alpha+1}$.

(c) If $R^{\alpha+1} = R^\alpha$ then there are no tuples $\bar{a}$ with $\mathrm{rk}_\varphi(\bar{a}) = \alpha$. By (b) it follows that $S^{\alpha+1} = S^\alpha$. Consequently, $S^\alpha = S^\infty$.

(d) If $R^{\alpha+1} = R^\alpha$ then the claim follows from (c). Hence, we may assume that $R^\alpha \subset R^{\alpha+1}$. We show that, for every $\gamma \geq \alpha$, if $\mathrm{rk}_\varphi(\bar{a}), \mathrm{rk}_\varphi(\bar{b}) < \alpha$ then $\langle \bar{a}, \bar{b} \rangle \in S^{\gamma+1}$ implies $\langle \bar{a}, \bar{b} \rangle \in S^{\alpha+1}$. Suppose otherwise and let

γ be the minimal ordinal such that there exists a counterexample. Then we obtain a contradiction as in the proof of (a).

It remains to show that there are no tuples with $\mathrm{rk}_\varphi(\bar{a}) \geq \alpha$ and $\mathrm{rk}_\varphi(\bar{b}) < \alpha$ such that $\langle \bar{a}, \bar{b} \rangle \in S^{\beta+1}$, for some $\beta \geq \alpha$. Suppose otherwise and let β be the minimal ordinal such that there exists a counterexample $\langle \bar{a}, \bar{b} \rangle \in S^{\beta+1}$. Then

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \vDash \vartheta(\bar{a}, \bar{b}) .$$

Since $\mathrm{rk}_\varphi(\bar{a}) \geq \alpha$, we have $\langle \bar{c}, \bar{a} \rangle \in R^\alpha$ iff $\mathrm{rk}_\varphi(\bar{c}) < \alpha$. By minimality of β, it follows that we have

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \vDash S\bar{c}\bar{a} \wedge S\bar{c}\bar{b} \quad \text{iff} \quad \mathrm{rk}_\varphi(\bar{c}) < \mathrm{rk}_\varphi(\bar{b}) .$$

If $\mathrm{rk}_\varphi(\bar{b}) < \alpha$ then setting $\bar{c} := \bar{b}$ we obtain a tuple $\bar{c}$ such that

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \vDash R\bar{c}\bar{a} \wedge \neg(S\bar{c}\bar{a} \wedge S\bar{c}\bar{b}) .$$

Consequently, $\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \nvDash \vartheta(\bar{a}, \bar{b})$. Contradiction.

Similarly, if $\mathrm{rk}_\varphi(\bar{b}) = \alpha$ then

$$\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \vDash \varphi[P\bar{z}/R\bar{z}\bar{a}/S\bar{z}\bar{a} \wedge S\bar{z}\bar{b}](\bar{b}) ,$$

and again $\langle \mathfrak{A}, R^\alpha, S^\beta \rangle \nvDash \vartheta(\bar{a}, \bar{b})$. This contradiction concludes the proof of the claim.

To finish the proof of the lemma we still have to show that $(R^\alpha)_\alpha$ is the fixed-point induction of χ. We prove this statement by induction on α. For $\alpha = 0$ and for limit ordinals the proof is trivial. For the successor step we show that

$$\langle \mathfrak{A}, R^\alpha \rangle \vDash \chi(\bar{a}, \bar{b}) \quad \text{iff} \quad \bar{a} \lhd_\varphi \bar{b} \text{ and } \mathrm{rk}_\varphi(\bar{a}) \leq \alpha .$$

First, we consider the case that $\mathrm{rk}_\varphi(\bar{a}) \leq \alpha$. Then we have $\langle \bar{c}, \bar{a} \rangle \in R^\alpha$ iff $\bar{c} \lhd_\varphi \bar{a}$. By statement (d) above, it follows that $\langle \bar{c}, \bar{a} \rangle \in R^\alpha$ iff $\psi(\bar{c}, \bar{a})$. Consequently, we have

$$\langle \mathfrak{A}, R^\alpha \rangle \vDash \varphi[P\bar{z}/R\bar{z}\bar{a}/\psi(\bar{z}, \bar{a})](\bar{a})$$

and every tuple $\bar{c}$ satisfies $\psi(\bar{c}, \bar{a}) \rightarrow R^\alpha \bar{c}\bar{a}$. Since

$$\langle \mathfrak{A}, R^\alpha \rangle \vDash \neg\varphi[P\bar{z}/\psi(\bar{z}, \bar{a})/R\bar{z}\bar{a}](\bar{b})$$

it follows that

$$\langle \mathfrak{A}, R^\alpha \rangle \vDash \chi(\bar{a}, \bar{b}) \quad \text{iff} \quad \bar{a} \lhd_\varphi \bar{b} .$$

It remains to consider the case that $\mathrm{rk}_\varphi(\bar{a}) > \alpha$. Then we have $\langle \bar{c}, \bar{a} \rangle \in R^\alpha$ iff $\mathrm{rk}_\varphi(\bar{c}) < \alpha$. If $P^\alpha = P^{\alpha+1}$ then, by statement (c) above, it follows that $\psi(\bar{x}, \bar{y})$ defines R^α and

$$\langle \mathfrak{A}, R^\alpha \rangle \nvDash \varphi[P\bar{z}/R\bar{z}\bar{a}/\psi(\bar{z}, \bar{a})](\bar{a}) .$$

Hence, $\langle \mathfrak{A}, R^\alpha \rangle \nvDash \chi(\bar{a}, \bar{b})$.

If, on the other hand, $P^\alpha \subset P^{\alpha+1}$ then, by (b), there is a tuple $\bar{c} \lhd_\varphi \bar{a}$ with $\mathrm{rk}_\varphi(\bar{c}) = \alpha$. Consequently,

$$\langle \mathfrak{A}, R^\alpha \rangle \nvDash \forall \bar{z}(\psi(\bar{z}, \bar{a}) \rightarrow R\bar{z}\bar{a}) . \qquad \square$$

Theorem 6.26 (Gurevich, Kreutzer, Shelah). $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP}) = \mathrm{FO}_{\kappa\aleph_0}(\mathrm{IFP})$.

Proof. Let $[\mathrm{ifp}\, R\bar{x} : \varphi]$ be an $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{IFP})$-formula. By induction we may assume that $\varphi \in \mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$. By Proposition 6.25, there is an $\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$-formula defining the stage comparison relation $\lhd_\varphi$. Note that we have

$$\mathrm{ifp}\, f = f(\mathrm{dom} \lhd_f) \cup \mathrm{dom} \lhd_f , \quad \text{for every function } f .$$

Hence, it follows that

$$[\mathrm{ifp}\, R\bar{x} : \varphi](\bar{x}) \equiv \varphi[R\bar{z}/\exists\bar{y}(\bar{z} \lhd_\varphi \bar{y})](\bar{x}) ,$$

where $\varphi[R\bar{z}/\vartheta(\bar{z})]$ denotes the formula obtained from $R\bar{x} \vee \varphi(R, \bar{x})$ by replacing every atom of the form $R\bar{t}$ by the formula $\vartheta(\bar{t})$. $\square$

The following two theorems summarise the results of this section.

Theorem 6.12 (Cohen, Shelah). *Let T be a complete first-order theory. The following conditions are equivalent:*

(1) *T is stable.*

(2) *T has $\mathrm{Un}(\kappa, \lambda)$-representations, for some cardinals κ and λ.*

(3) *T has $\mathrm{Wf}(0, |T|)$-representations.*

(4) *T has $\mathrm{Wf}(|T|, |T|)$-representations.*

Proof. (2) ⇒ (1) has been shown in Proposition 6.8 (a), the implications (4) ⇒ (3) ⇒ (2) follow from Lemmas 6.5 and 6.2, and (1) ⇒ (4) follows by Proposition 6.11. □

Theorem 6.13 (Cohen, Shelah). *Let T be a complete first-order theory. The following conditions are equivalent:*

(1) *T is $\aleph_0$-stable.*

(2) *T has $\mathrm{Lf}(\aleph_0, \aleph_0)$-representations.*

Proof. (2) ⇒ (1) follows by Proposition 6.8 (b) and (1) ⇒ (2) follows by Proposition 6.11. □

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A	*a*	𝔄	𝔞	*N*	*n*	𝔑	𝔫
B	*b*	𝔅	𝔟	*O*	*o*	𝔒	𝔬
C	*c*	ℭ	𝔠	*P*	*p*	𝔓	𝔭
D	*d*	𝔇	𝔡	*Q*	*q*	𝔔	𝔮
E	*e*	𝔈	𝔢	*R*	*r*	ℜ	𝔯
F	*f*	𝔉	𝔣	*S*	*s*	𝔖	ſ 𝔰
G	*g*	𝔊	𝔤	*T*	*t*	𝔗	𝔱
H	*h*	ℌ	𝔥	*U*	*u*	𝔘	𝔲
I	*i*	ℑ	𝔦	*V*	*v*	𝔙	𝔳
J	*j*	𝔍	𝔧	*W*	*w*	𝔚	𝔴
K	*k*	𝔎	𝔨	*X*	*x*	𝔛	𝔵
L	*l*	𝔏	𝔩	*Y*	*y*	𝔜	𝔶
M	*m*	𝔐	𝔪	*Z*	*z*	ℨ	𝔷

The Greek alphabet					
Α	*α*	alpha	*Ν*	*ν*	nu
Β	*β*	beta	*Ξ*	*ξ*	xi
Γ	*γ*	gamma	*Ο*	*ο*	omicron
Δ	*δ*	delta	*Π*	*π*	pi
Ε	*ε*	epsilon	*Ρ*	*ρ*	rho
Ζ	*ζ*	zeta	*Σ*	*σ*	sigma
Η	*η*	eta	*Τ*	*τ*	tau
Θ	*ϑ*	theta	*Υ*	*υ*	upsilon
Ι	*ι*	iota	*Φ*	*ϕ*	phi
Κ	*κ*	kappa	*Χ*	*χ*	chi
Λ	*λ*	lambda	*Ψ*	*ψ*	psi
Μ	*μ*	mu	*Ω*	*ω*	omega