Achim Blumensath
blumens@fi.muni.cz

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www.fi.muni.cz/~blumens

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Part D.

Axiomatisation and Definability

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1. *Preservation theorems*

In Section C2.1 we have seen that several fragments of first-order logic are preserved under various operations. In this section we will show the converse. A preservation theorem is a result that characterises a semantic property of a formula by a syntactic condition. The general form of such a theorem is the statement:

> Let $\varphi \in L_+$. The class $\mathrm{Mod}_{L_+}(\varphi)$ has the property P if and only if there exists a formula $\psi \in L_-$ such that $\varphi \equiv \psi$.

Here P is an arbitrary property and L_+ and L_- are logics where usually we have $L_- \subset L_+$.

We will mostly be interested in closure properties. We consider a relation $\sqsubseteq$ with the property that $\mathrm{Mod}_{L_-}(\psi)$ is closed under $\sqsubseteq$, for every L_--formula ψ, i.e.,

$$\mathfrak{A} \vDash \psi \text{ and } \mathfrak{A} \sqsubseteq \mathfrak{B} \quad \text{implies} \quad \mathfrak{B} \vDash \psi\,.$$

Further, we assume that φ is an L_+-formula such that $\mathrm{Mod}_{L_+}(\varphi)$ is closed under $\sqsubseteq$. We want to find a formula $\psi \in L_-$ with $\psi \equiv \varphi$.

One way to prove that such a formula exists is the following. For a contradiction, we suppose that we can find structures $\mathfrak{A} \equiv_{L_-} \mathfrak{B}$ such that $\mathfrak{A} \vDash \varphi$ and $\mathfrak{B} \nvDash \varphi$. Given $\mathfrak{A}$ and $\mathfrak{B}$ we construct a structure $\mathfrak{C}$ such that

$$\mathfrak{A} \sqsubseteq \mathfrak{C} \quad \text{and} \quad \mathfrak{B} \equiv_{L_+} \mathfrak{C}\,.$$

This leads to a contradiction since, on the one hand, $\mathfrak{A} \vDash \varphi$ and $\mathfrak{A} \sqsubseteq \mathfrak{C}$ implies that $\mathfrak{C} \vDash \varphi$. But, on the other hand, $\mathfrak{B} \nvDash \varphi$ and $\mathfrak{B} \equiv_{L_+} \mathfrak{C}$ implies that $\mathfrak{C} \nvDash \varphi$.

Lemma 1.1. *Let T be a first-order theory and $\mathfrak{A}$ a structure.*

$$\mathfrak{A} \vDash T_\forall^\vDash \quad \text{iff} \quad \text{there exists an embedding } \mathfrak{A} \to \mathfrak{B} \text{ into some model } \mathfrak{B} \vDash T\,.$$

Proof. ($\Leftarrow$) Let $\mathfrak{A} \to \mathfrak{B}$ be an embedding and $\mathfrak{B} \vDash T$. Replacing $\mathfrak{A}$ by an isomorphic copy we may assume that $\mathfrak{A} \subseteq \mathfrak{B}$. Let $\varphi \in T_\forall^\vDash$. Since $T \vDash T_\forall^\vDash$ we have $\mathfrak{B} \vDash \varphi$. By Lemma C2.1.6, it follows that $\mathfrak{A} \vDash \varphi$.

($\Rightarrow$) Note that every function preserving $\exists$-formulae is an embedding. Therefore, this direction follows from Corollary C2.5.6 if we set $\Delta := \forall$. □

Theorem 1.2 (Łoś, Tarski). *For a first-order theory T and a set Φ of sentences, the following statements are equivalent:*

(1) $\mathfrak{B} \vDash \Phi$ *implies* $\mathfrak{A} \vDash \Phi$, *for all models* $\mathfrak{A} \subseteq \mathfrak{B}$ *of* T.

(2) Φ *is equivalent modulo T to a set of first-order $\forall$-formulae.*

Proof. The implication (2) ⇒ (1) was proved in Lemma C2.1.6. For the other direction, we claim that $\Psi := (T \cup \Phi)_\forall^\vDash$ is equivalent to Φ. Clearly, if $\mathfrak{A} \vDash \Phi$ and $\mathfrak{A} \vDash T$ then $\mathfrak{A} \vDash \Psi$. On the other hand, by Lemma 1.1, we have

$$\mathfrak{A} \vDash \Psi \quad \text{iff} \quad \mathfrak{A} \subseteq \mathfrak{B} \text{ for some } \mathfrak{B} \vDash T \cup \Phi\,.$$

By (1), it follows that $\mathfrak{A} \vDash \Psi$ implies $\mathfrak{A} \vDash \Phi$. Therefore, $\Phi \equiv \Psi$ modulo T. □

Dualising the statement of the Theorem of Łoś and Tarski we obtain a characterisation of formulae preserved by embeddings.

Corollary 1.3. *Let T be a first-order theory. A formula $\varphi \in$ FO is preserved by embeddings between models of T if and only if φ is equivalent modulo T to an $\exists$-formula.*

Proof. Since $\neg\varphi$ is preserved in substructures it follows by Theorem 1.2 that we can find a set Φ of $\forall$-formulae with $\Phi \equiv \neg\varphi$. By the Compactness Theorem, there exists a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \vDash \neg\varphi$. Hence, $\neg\varphi \equiv \bigwedge \Phi_0$ and $\varphi \equiv \neg \bigwedge \Phi_0$. The latter is equivalent to an $\exists$-formula. □

We can extend the Theorem of Łoś and Tarski to pseudo-elementary classes.

Theorem 1.4. *If a class $\mathcal{K} \in \mathrm{RPC}_\infty(\mathrm{FO}, \Sigma)$ is closed under substructures then $\mathcal{K}$ is $\forall[\Sigma]$-axiomatisable.*

Proof. By Theorem C5.4.14, there exists a set $\Phi \subseteq \mathrm{FO}[\Gamma]$ such that

$$\mathcal{K} = \mathrm{pr}_\Sigma(\mathrm{Mod}(\Phi)) .$$

Let $T := \Phi^{\vDash}_{\forall} \cap \mathrm{FO}[\Sigma]$. Clearly, $\mathcal{K} \subseteq \mathrm{Mod}(T)$. It remains to prove the converse. Suppose that $\mathfrak{A} \vDash T$. Let $\Delta := \mathrm{QF}^{<\omega}[\Sigma]$ and set

$$\Psi := \mathrm{Th}_\Delta(\mathfrak{A}_A) \cup \Phi .$$

We show that Ψ is satisfiable. Suppose otherwise. Then there is some quantifier-free formula $\psi(\bar{a})$ with parameters $\bar{a} \subseteq A$ such that

$$\mathfrak{A} \vDash \psi(\bar{a}) \quad \text{and} \quad \Phi \vDash \neg\psi(\bar{a}) .$$

Consequently, $\Phi \vDash \forall\bar{x}\neg\psi(\bar{x})$. Since this sentence is in T it follows that $\mathfrak{A} \vDash \forall\bar{x}\neg\psi(\bar{x})$. Contradiction.

Let $\mathfrak{B}$ be a model of Ψ. Since $\mathfrak{B} \vDash \mathrm{Th}_\Delta(\mathfrak{A}_A)$ there exists an embedding $\mathfrak{A} \to \mathfrak{B}$. Furthermore, we have $\mathfrak{B} \in \mathcal{K}$. As $\mathcal{K}$ is closed under substructures and isomorphisms, it follows that $\mathfrak{A} \in \mathcal{K}$. □

Example. As an application we consider representable groups. Let $0 < n < \omega$. We say that a group $\mathfrak{G}$ has a *faithful n-linear representation* if it can be embedded into $\mathrm{GL}_n(\mathfrak{K})$, the group of all invertible $n \times n$ matrices over some field $\mathfrak{K}$.

Claim. *A group $\mathfrak{G}$ has a faithful n-linear representation if and only if every finitely generated subgroup of $\mathfrak{G}$ has such a representation.*

($\Rightarrow$) Clearly, if $\mathfrak{G}$ can be embedded into $\mathrm{GL}_n(\mathfrak{K})$ then the same is true for all subgroups of $\mathfrak{G}$.

($\Leftarrow$) Let $\mathcal{K}_n$ be the class of all groups with a faithful n-linear representation. Then $\mathcal{K}_n$ is closed under substructures. Furthermore, we have $\mathcal{K}_n \in \mathrm{PC}_1(\mathrm{FO}, \{\cdot, {}^{-1}, e\})$. By the preceding lemma, it follows that $\mathcal{K}_n = \mathrm{Mod}(T)$, for some $T \subseteq \forall$.

Suppose that $\mathfrak{G} \notin \mathcal{K}_n$. Then there is some formula $\forall \bar{x} \varphi(\bar{x}) \in T$ such that $\mathfrak{G} \vDash \neg\forall \bar{x}\varphi$. Fix some $\bar{a} \subseteq G$ with $\mathfrak{G} \vDash \neg\varphi(\bar{a})$. Setting $\mathfrak{G}_0 := \langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{G}}$ it follows that $\mathfrak{G}_0 \vDash \neg\varphi(\bar{a})$. Hence, we have found a finitely generated subgroup with $\mathfrak{G}_0 \notin \mathcal{K}_n$.

We conclude this section with a characterisation of classes closed under unions of chains.

Theorem 1.5 (Chang, Łoś, Suszko). *For a first-order theory T and a set Φ of sentences, the following statements are equivalent:*

(1) *If $(\mathfrak{A}_i)_{i<\alpha}$ is a chain such that $\bigcup_i \mathfrak{A}_i \vDash T$ and $\mathfrak{A}_i \vDash T \cup \Phi$, for all $i < \alpha$, then $\bigcup_i \mathfrak{A}_i \vDash \Phi$.*

(2) *Φ is equivalent modulo T to a set of first-order $\forall\exists$-formulae.*

Proof. (2) $\Rightarrow$ (1) was already proved in Lemma C2.1.8. For the other direction, set $\Psi := (T \cup \Phi)^{\vDash}_{\forall\exists}$. It is sufficient to show that $T \cup \Psi \vDash \Phi$.

We prove that every model $\mathfrak{D} \vDash T \cup \Psi$ is elementary equivalent to the union $\mathfrak{C} := \bigcup_{i<\omega} \mathfrak{A}_i$ of a chain $(\mathfrak{A}_i)_{i<\omega}$ where $\mathfrak{C} \vDash T$ and $\mathfrak{A}_i \vDash T \cup \Phi$, for all $i < \omega$. Since Φ is closed under unions of chains it follows that $\mathfrak{C} \vDash \Phi$, which implies that $\mathfrak{D} \vDash \Phi$.

Fix an arbitrary model $\mathfrak{D} \vDash T \cup \Psi$. By induction on i, we construct an elementary chain $(\mathfrak{A}_i)_{i<\omega}$, extensions $\mathfrak{B}_i \supseteq \mathfrak{A}_i$, and embeddings $g_i : \mathfrak{B}_i \to \mathfrak{A}_{i+1}$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathfrak{B}_0 \arrow[dr, "g_0"] & & \mathfrak{B}_1 \arrow[dr, "g_1"] & & \mathfrak{B}_2 \arrow[dr, "g_2"] & \\
\mathfrak{A}_0 \arrow[ur, "\subseteq"] \arrow[rr, "\preceq"'] & & \mathfrak{A}_1 \arrow[ur, "\subseteq"] \arrow[rr, "\preceq"'] & & \mathfrak{A}_2 \arrow[ur, "\subseteq"] \arrow[rr, "\preceq"'] & & \cdots
\end{tikzcd}$$

Furthermore, we ensure that

$$\mathfrak{B}_i \vDash T \cup \Phi \quad \text{and} \quad \langle \mathfrak{B}_i, \bar{a}^i \rangle \leq_{\forall\exists} \langle \mathfrak{A}_i, \bar{a}^i \rangle ,$$

where $\bar{a}^i$ is some enumeration of A_i.

We start with $\mathfrak{A}_0 := \mathfrak{D}$. Suppose that $\mathfrak{A}_i$ has already been defined. $\mathfrak{A}_0 \preceq \mathfrak{A}_i$ implies that $\mathfrak{A}_i \vDash \Psi$. If we set $\Delta := \forall\exists$ in Corollary c2.5.6 then we obtain an extension $\mathfrak{B}_i \supseteq \mathfrak{A}_i$ such that

$$\mathfrak{B}_i \vDash T \cup \Phi \quad \text{and} \quad \langle \mathfrak{A}_i, \bar{a}^i \rangle \leq_{\exists\forall} \langle \mathfrak{B}_i, \bar{a}^i \rangle ,$$

that is, $\langle \mathfrak{B}_i, \bar{a}^i \rangle \leq_{\forall\exists} \langle \mathfrak{A}_i, \bar{a}^i \rangle$. Since $\exists \subseteq \forall\exists$, we can use Corollary c2.5.4 to find an elementary extension $\mathfrak{A}_{i+1} \succeq \mathfrak{A}_i$ and an embedding $g_i : \mathfrak{B}_i \to \mathfrak{A}_{i+1}$ with $g_i \upharpoonright A_i = \mathrm{id}_{A_i}$.

Let $\mathfrak{C} := \bigcup_{i<\omega} \mathfrak{A}_i = \bigcup_{i<\omega} g_i(\mathfrak{B}_i)$. Since $(\mathfrak{A}_i)_i$ is an elementary chain it follows that $\mathfrak{A}_0 \preceq \mathfrak{C}$. Hence, we have found a model $\mathfrak{C} \vDash T$ that is the union of a chain of models of $T \cup \Phi$. $\square$

2. *Quantifier elimination*

Some theories, like the theory of dense linear orders or the theory of algebraically closed fields, have the pleasant property that every formula is equivalent to a quantifier-free one. We can use this fact to deduce some useful information about the theory.

First of all, we gain a better understanding of which relations are definable since we only need to consider relations definable by quantifier-free formulae. For instance, every definable relation of an algebraically closed field is given by finitely many equations and inequations between polynomials.

Secondly, we can sometimes use this fact to prove that a theory is complete. Since every sentence is equivalent to a quantifier-free one we only have to check that, for every quantifier-free sentence φ, the theory determines whether φ does hold or not. In particular, if the signature contains neither constant symbols nor 0-ary relation symbols then the only quantifier-free sentences are true and false and this question becomes trivial.

Definition 2.1. (a) Let L be a logic, $\Delta, \Gamma \subseteq L$, and $\mathcal{K}$ a class of L-interpretations. We say that Γ is a *Δ-elimination set over $\mathcal{K}$* if, for all sets $\Phi \subseteq \Delta$ there exists a set $\Psi \subseteq \Gamma$ such that

$$\mathfrak{J} \vDash \Phi \quad \text{iff} \quad \mathfrak{J} \vDash \Psi\,, \qquad \text{for all } \mathfrak{J} \in \mathcal{K}\,.$$

(b) We say that a class of Σ-structures $\mathcal{K}$ admits *quantifier elimination for* $\mathrm{FO}_{\kappa\aleph_0}$ if $\mathrm{QF}^{<\omega}_{\kappa\aleph_0}[\Sigma]$ is an $\mathrm{FO}^{<\omega}_{\kappa\aleph_0}[\Sigma]$-elimination set over $\mathcal{K}$. In particular, we say that a first-order theory T admits *quantifier elimination* if $\mathrm{Mod}(T)$ admits quantifier elimination for FO.

In terms of type spaces we obtain the following characterisation.

Lemma 2.2. *Let L be a logic, $T \subseteq L$ a theory, and $\Gamma \subseteq \Delta \subseteq L$ fragments of L/T that are both closed under disjunctions. The following statements are equivalent.*

(1) *Γ is an Δ-elimination set over T.*
(2) *The function $\mathfrak{S}(i) : \mathfrak{S}((L/T)|_\Delta) \to \mathfrak{S}((L/T)|_\Gamma)$ corresponding to the inclusion map $i : L|_\Gamma \to L|_\Delta$ is a homeomorphism.*

Proof. Replacing L by L/T we may w.l.o.g. assume that $T = \varnothing$. Further, note that $S(i)(\mathfrak{p}) = \mathfrak{p} \cap \Gamma$ and that, according to Lemma C3.2.2, the closed sets of $\mathfrak{S}(L|_\Delta)$ and $\mathfrak{S}(L|_\Gamma)$ are of the form $\langle \Phi \rangle_{L|_\Delta}$ and $\langle \Psi \rangle_{L|_\Gamma}$, for $\Phi \subseteq \Delta$ and $\Psi \subseteq \Gamma$.

(1) $\Rightarrow$ (2) Suppose that Γ is a Δ-elimination set. We have to prove that $S(i)$ is continuous and that it has a continuous inverse. It follows from Proposition C3.2.11 that $S(i)$ is a continuous surjection. To prove that it

is also injective suppose that $\mathfrak{p}, \mathfrak{q} \in S(\Delta)$ are two types with $S(i)(\mathfrak{p}) = S(i)(\mathfrak{q})$. By assumption there exist sets $\Phi, \Psi \subseteq \Gamma$ such that $\mathfrak{p} \equiv \Phi$ and $\mathfrak{q} \equiv \Psi$. Consequently, we have

$$\Phi \subseteq \mathfrak{p}^{\vDash}_{\Gamma} = \mathfrak{p} \cap \Gamma = S(i)(\mathfrak{p}) = S(i)(\mathfrak{q}) = \mathfrak{q} \cap \Gamma \subseteq \mathfrak{q}\,.$$

Hence, $\mathfrak{p} = \Phi^{\vDash}_{\Delta} \subseteq \mathfrak{q}^{\vDash}_{\Delta} = \mathfrak{q}$. By symmetry, we also have $\mathfrak{q} \subseteq \mathfrak{p}$. It follows that $\mathfrak{p} = \mathfrak{q}$, as desired.

We have shown that $S(i)$ has an inverse. It remains to prove that $S(i)^{-1}$ is continuous. Let $\langle \Phi \rangle$ be a closed subset of $S(\Delta)$. We have to show that $(S(i)^{-1})^{-1}[\langle \Phi \rangle] = S(i)[\langle \Phi \rangle]$ is closed in $S(\Gamma)$. By assumption there is a set $\Psi \subseteq \Gamma$ with $\Phi \equiv \Psi$. We claim that $S(i)[\langle \Phi \rangle] = \langle \Psi \rangle$.

First, suppose that $\mathfrak{p} \in \langle \Phi \rangle$. Then $\Psi \subseteq \mathfrak{p}$ and

$$S(i)(\mathfrak{p}) = p \cap \Gamma \supseteq \Psi\,.$$

Hence, $S(i)(\mathfrak{p}) \in \langle \Psi \rangle$. Conversely, suppose that $\mathfrak{p} \in \langle \Psi \rangle$. Then $\Psi \subseteq \mathfrak{p} \subseteq S(i)^{-1}(\mathfrak{p})$ implies that $\Phi \subseteq S(i)^{-1}(\mathfrak{p})$. Hence, $S(i)^{-1}(\mathfrak{p}) \in \langle \Phi \rangle$, i.e., $\mathfrak{p} \in S(i)[\langle \Phi \rangle]$

(2) $\Rightarrow$ (1) Suppose that $S(i)$ is a homeomorphism. To show that Γ is a Δ-elimination set let $\Phi \subseteq \Delta$. Since $\langle \Phi \rangle$ is a closed subset of $S(\Delta)$ it follows that $C := S(i)[\langle \Phi \rangle]$ is a closed subset of $S(\Gamma)$. Hence, there exists a set $\Psi \subseteq \Gamma$ such that $C = \langle \Psi \rangle$. We claim that $\Phi \equiv \Psi$.

First, suppose that $\mathfrak{J} \vDash \Phi$ and let $\mathfrak{p} := \mathrm{Th}_{\Delta}(\mathfrak{J})$. Then $\mathfrak{p} \in \langle \Phi \rangle$ implies that

$$\mathrm{Th}_{\Gamma}(\mathfrak{J}) = \mathfrak{p} \cap \Gamma = S(i)(\mathfrak{p}) \in \langle \Psi \rangle\,.$$

Hence, $\mathfrak{J} \vDash \Psi$. Conversely, suppose that $\mathfrak{J} \vDash \Psi$ and let $\mathfrak{p} := \mathrm{Th}_{\Delta}(\mathfrak{J})$. Then $S(i)(\mathfrak{p}) = \mathfrak{p} \cap \Gamma \in \langle \Psi \rangle$. Hence, we have $\mathfrak{p} = S(i)^{-1}(\mathfrak{p} \cap \Gamma) \in \langle \Phi \rangle$ and, therefore, $\mathfrak{J} \vDash \Phi$. □

For first-order logic we can get a slightly stronger result.

Lemma 2.3. *Let $T \subseteq \mathrm{FO}^{\bar{s}}[\Sigma]$ be a first-order theory and $\Delta \subseteq \Phi \subseteq \mathrm{FO}^{\bar{s}}[\Sigma]$ sets of formulae. If*

$$\mathfrak{p}|_\Delta = \mathfrak{q}|_\Delta \quad \textit{implies} \quad \mathfrak{p}|_\Phi = \mathfrak{q}|_\Phi\,, \quad \textit{for all } \mathfrak{p}, \mathfrak{q} \in S^{\bar{s}}(T)\,,$$

then every formula of Φ is equivalent modulo T to a finite boolean combination of formulae of Δ.

Proof. Let Δ_+ and Φ_+ be the boolean closures of, respectively, Δ and Φ. The inclusion $i : \Delta_+ \to \Phi_+$ induces an injective homomorphism

$$f : \mathfrak{Lb}(\Delta_+/T) \to \mathfrak{Lb}(\Phi_+/T)\,.$$

By Corollary B5.6.11, we obtain a surjective continuous map

$$\mathrm{spec}(f) : \mathfrak{S}_{\Phi_+}(T) \to \mathfrak{S}_{\Delta_+}(T) : \mathfrak{p} \mapsto \mathfrak{p}|_{\Delta_+}\,.$$

By assumption, this map is injective. Hence, $\mathrm{spec}(f)$ is in fact an isomorphism. By Corollary B5.6.11 it follows that so is f. Consequently, for every formula $\varphi \in \Phi_+$, there is some formula $\delta \in \Delta_+$ with

$$[\varphi]_{\equiv_T} = f([\delta]_{\equiv_T}) = [i(\delta)]_{\equiv_T} = [\delta]_{\equiv_T}\,.$$

It follows that $\varphi \equiv \delta$ modulo T, as desired. □

If Γ is a Δ-elimination set and the logic in question is compact then it follows that every Δ-formula is equivalent to a single Γ-formula. In particular, if a theory T admits quantifier elimination then every first-order formula is equivalent modulo T to a quantifier-free one.

Lemma 2.4. *Let $\Delta, \Gamma, T \subseteq \mathrm{FO}$ sets of first-order formulae where Γ is closed under conjunctions. Γ is a Δ-elimination set over T if and only if, for every formula $\varphi \in \Delta$, there exists a formula $\psi \in \Gamma$ such that $\varphi \equiv \psi$ modulo T.*

Proof. ($\Leftarrow$) is trivial. For ($\Rightarrow$), let $\varphi \in \Delta$. By assumption, there exists a set $\Psi \subseteq \Gamma$ such that $\varphi \equiv \Psi$ modulo T. By compactness, we can find a finite subset $\Psi_0 \subseteq \Psi$ such that $T \cup \Psi_0 \vDash \varphi$. If we set $\psi := \bigwedge \Psi_0 \in \Gamma$ then we have $T \vDash \varphi \leftrightarrow \psi$. □

Lemma 2.5. *Let T be a first-order theory and $\varphi(\bar{x})$ a formula. The following statements are equivalent:*

(1) *There exists a quantifier-free formula $\psi(\bar{x})$ that is equivalent to φ modulo T.*

(2) *For all models $\mathfrak{A}$ and $\mathfrak{B}$ of T and all $\bar{a} \in A^{<\omega}$ and $\bar{b} \in B^{<\omega}$ with $\langle \mathfrak{A}, \bar{a} \rangle \equiv_0 \langle \mathfrak{B}, \bar{b} \rangle$, we have*

$$\mathfrak{A} \vDash \varphi(\bar{a}) \quad \text{iff} \quad \mathfrak{B} \vDash \varphi(\bar{b}) .$$

Proof. (1) ⇒ (2) Suppose that $\langle \mathfrak{A}, \bar{a} \rangle \equiv_0 \langle \mathfrak{B}, \bar{b} \rangle$. By (1), there exists a quantifier-free formula $\psi(\bar{x}) \equiv \varphi$ modulo T. It follows that

$$\begin{aligned} \mathfrak{A} \vDash \varphi(\bar{a}) \quad &\text{iff} \quad \mathfrak{A} \vDash \psi(\bar{a}) \\ &\text{iff} \quad \mathfrak{B} \vDash \psi(\bar{b}) \quad \text{iff} \quad \mathfrak{B} \vDash \varphi(\bar{b}) . \end{aligned}$$

(2) ⇒ (1) Let Φ the closure of QF $\cup \{\varphi\}$ under boolean operations. Condition (2) can be written as

$$\mathfrak{p}|_{\mathrm{QF}} = \mathfrak{q}|_{\mathrm{QF}} \quad \text{implies} \quad \mathfrak{p}|_{\Phi} = \mathfrak{q}|_{\Phi} , \qquad \text{for all } \mathfrak{p}, \mathfrak{q} \in S^n(T) .$$

Consequently the claim follows by Lemma 2.3. □

Theorem 2.6. *Let T be a first-order theory. The following statements are equivalent:*

(1) *T admits quantifier elimination.*

(2) *For all models $\mathfrak{A}$ and $\mathfrak{B}$ of T and all $\bar{a} \in A^{<\omega}$ and $\bar{b} \in B^{<\omega}$, we have*

$$\langle \mathfrak{A}, \bar{a} \rangle \equiv_0 \langle \mathfrak{B}, \bar{b} \rangle \quad \text{iff} \quad \langle \mathfrak{A}, \bar{a} \rangle \equiv_{\mathrm{FO}} \langle \mathfrak{B}, \bar{b} \rangle .$$

(3) *For all models $\mathfrak{A}$ and $\mathfrak{B}$ of T, each quantifier-free formula $\varphi(\bar{x}, y)$, and all elements $\bar{a} \in A^{<\omega}$ and $\bar{b} \in B^{<\omega}$ with $\langle \mathfrak{A}, \bar{a} \rangle \equiv_0 \langle \mathfrak{B}, \bar{b} \rangle$ we have*

$$\mathfrak{A} \vDash \exists y \varphi(\bar{a}, y) \quad \textit{implies} \quad \mathfrak{B} \vDash \exists y \varphi(\bar{b}, y) .$$

Proof. (1) ⇔ (2) follows from Lemma 2.5 and (2) ⇒ (3) is trivial.

(3) ⇒ (1) W.l.o.g. we may assume that φ is written without universal quantifiers. By induction on φ, we construct a quantifier-free formula φ° with $\varphi^\circ \equiv \varphi$ modulo T. If φ is quantifier-free we are done. For boolean combinations we can set

$$(\neg\varphi)^\circ := \neg\varphi^\circ, \quad (\varphi \vee \psi)^\circ := \varphi^\circ \vee \psi^\circ, \quad (\varphi \wedge \psi)^\circ := \varphi^\circ \wedge \psi^\circ.$$

Finally, suppose that $\varphi = \exists y \psi(\bar{x}, y)$. By (3) and Lemma 2.5, we can find a quantifier-free formula φ° such that $\varphi^\circ \equiv \exists y \psi^\circ(\bar{x}, y)$ modulo T. □

A useful simple criterion for quantifier elimination is the following one.

Definition 2.7. (a) Let T be a theory and $\mathfrak{A}$ a model of $T_\forall^\vDash$. An *algebraic prime model* of T over $\mathfrak{A}$ is an embedding $f : \mathfrak{A} \to \mathfrak{B}$ into a model of T such that any other embedding $g : \mathfrak{A} \to \mathfrak{C}$ into a model of T factorises as $g = h \circ f$, for some embedding $h : \mathfrak{B} \to \mathfrak{C}$. We say that T has *algebraic prime models* if, for every $\mathfrak{A} \vDash T_\forall^\vDash$, there is an algebraic prime model of T over $\mathfrak{A}$.

(b) Let $\mathfrak{A} \subseteq \mathfrak{B}$. We say that $\mathfrak{A}$ is *simply closed* in $\mathfrak{B}$ if, for every quantifier-free formula $\varphi(\bar{x}, y)$ and all elements $\bar{a} \subseteq A$

$$\mathfrak{B} \vDash \exists y \varphi(\bar{a}, y) \quad \text{implies} \quad \mathfrak{A} \vDash \exists y \varphi(\bar{a}, y)\,.$$

Proposition 2.8. *Let T be a first-order theory with algebraic prime models such that, whenever $\mathfrak{A} \subseteq \mathfrak{B}$ are both models of T then $\mathfrak{A}$ is simply closed in $\mathfrak{B}$. Then T admits quantifier elimination.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of T and suppose that

$$\langle \mathfrak{A}, \bar{a} \rangle \equiv_0 \langle \mathfrak{B}, \bar{b} \rangle\,.$$

By Theorem 2.6, it is sufficient to show that

$$\mathfrak{A} \vDash \exists y \varphi(\bar{a}, y) \quad \text{implies} \quad \mathfrak{B} \vDash \exists y \varphi(\bar{b}, y)\,,$$

for every quantifier-free formula φ. Let $f : \langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}} \to \mathfrak{C}$ be the algebraic prime model of T over $\langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}}$. Since $\langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}} \cong \langle\!\langle \bar{b} \rangle\!\rangle_{\mathfrak{B}}$ we obtain an embedding $g : \langle\!\langle \bar{b} \rangle\!\rangle \to \mathfrak{C}$ with $g(\bar{b}) = f(\bar{a})$. By definition of an algebraic prime model there exist embeddings $h : \mathfrak{C} \to \mathfrak{A}$ and $k : \mathfrak{C} \to \mathfrak{B}$ such that $h(f(\bar{a})) = \bar{a}$ and $k(g(\bar{b})) = \bar{b}$.

Suppose that $\mathfrak{A} \vDash \varphi(\bar{a}, b)$. By assumption $\mathfrak{C}$ is simply closed in $\mathfrak{A}$. Hence,

$$\mathfrak{C} \vDash \varphi(f(\bar{a}), c), \quad \text{for some } c \in C.$$

It follows that $\mathfrak{B} \vDash \varphi(k(f(\bar{a})), k(c))$. Since $k(f(\bar{a})) = k(g(\bar{b})) = \bar{b}$ this implies that

$$\mathfrak{B} \vDash \varphi(\bar{b}, k(c)),$$

as desired. □

Similar to the characterisation of Theorem 2.6 above we can describe infinitary first-order theories admitting quantifier elimination.

Theorem 2.9. *Let $\mathcal{K}$ be a class of structures. The following statements are equivalent:*

(1) *$\mathcal{K}$ admits quantifier elimination for* $\mathrm{FO}_{\infty\aleph_0}$.

(2) $\mathfrak{A} \sqsubseteq_0^{\aleph_0} \mathfrak{B}$ *for all structures* $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$.

(3) $\mathfrak{A} \cong_0^{\aleph_0} \mathfrak{B}$ *for all structures* $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$.

(4) *For all structures $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and all $\bar{a} \in A^{<\omega}$ and $\bar{b} \in B^{<\omega}$, we have*

$$\langle \mathfrak{A}, \bar{a} \rangle \equiv_0 \langle \mathfrak{B}, \bar{b} \rangle \quad \text{iff} \quad \langle \mathfrak{A}, \bar{a} \rangle \equiv_\infty \langle \mathfrak{B}, \bar{b} \rangle .$$

Proof. (1) ⇒ (4) Suppose that $\langle \mathfrak{A}, \bar{a} \rangle \equiv_0 \langle \mathfrak{B}, \bar{b} \rangle$. By (1), there exists a set $\Phi(\bar{x}) \subseteq \mathrm{QF}^{<\omega}_{\infty\aleph_0}$ such that $\Phi(\bar{a})$ is equivalent to $\mathrm{tp}_{\mathrm{FO}_{\infty\aleph_0}}(\bar{a}/\mathfrak{A})$ on structures of $\mathcal{K}$. Hence, $\mathfrak{A} \vDash \Phi(\bar{a})$ implies that $\mathfrak{B} \vDash \Phi(\bar{b})$, and it follows that $\mathrm{tp}_{\mathrm{FO}_{\infty\aleph_0}}(\bar{b}/\mathfrak{B}) = \mathrm{tp}_{\mathrm{FO}_{\infty\aleph_0}}(\bar{a}/\mathfrak{A})$.

(4) ⇒ (1) Let $\varphi(\bar{x}) \in \mathrm{FO}_{\infty\aleph_0}$. For each pair of types $\mathfrak{p} \in \langle\varphi\rangle$ and $\mathfrak{q} \in \langle\neg\varphi\rangle$ there exists a quantifier-free formula $\psi_{\mathfrak{pq}}$ such that $\psi_{\mathfrak{pq}} \in \mathfrak{p}$ and $\neg\psi_{\mathfrak{pq}} \in \mathfrak{q}$. It follows that the formula

$$\bigvee_{\mathfrak{p}\in\langle\varphi\rangle} \bigwedge_{\mathfrak{q}\in\langle\neg\varphi\rangle} \psi_{\mathfrak{pq}}$$

is equivalent to φ on structures of $\mathcal{K}$. (Note that the above disjunction and the conjunctions are over sets of formulae since, up to logical equivalence, the number of quantifier-free formulae with a given number of free variables can be bounded in terms of the size of the signature.)

(2) ⇒ (3) $\mathfrak{A} \sqsubseteq_0^{\aleph_0} \mathfrak{B}$ and $\mathfrak{B} \sqsubseteq_0^{\aleph_0} \mathfrak{A}$ implies that $\mathfrak{A} \cong_0^{\aleph_0} \mathfrak{B}$.

(3) ⇒ (4) Suppose that $\mathfrak{A} \cong_0^{\aleph_0} \mathfrak{B}$. Then $\mathrm{pIso}_{\aleph_0}(\mathfrak{A},\mathfrak{B}) = I_\infty^{\aleph_0}(\mathfrak{A},\mathfrak{B})$. Hence,

$$\langle\mathfrak{A},\bar{a}\rangle \equiv_0 \langle\mathfrak{B},\bar{b}\rangle \quad \text{implies} \quad \langle\mathfrak{A},\bar{a}\rangle \equiv_\infty \langle\mathfrak{B},\bar{b}\rangle\,.$$

(4) ⇒ (2) We have to show that $\mathrm{pIso}_{\aleph_0}(\mathfrak{A},\mathfrak{B})$ has the forth property with respect to itself. Since $\mathrm{pIso}_{\aleph_0}(\mathfrak{A},\mathfrak{B}) = I_\infty^{\aleph_0}(\mathfrak{A},\mathfrak{B})$ and the latter set has the back-and-forth property with respect to itself the claim follows. □

Corollary 2.10. *Let T be a first-order theory. If T admits quantifier elimination for* $\mathrm{FO}_{\infty\aleph_0}$ *then it also admits quantifier elimination for* FO.

Proof. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are models of T with

$$\langle\mathfrak{A},\bar{a}\rangle \equiv_0 \langle\mathfrak{B},\bar{b}\rangle\,.$$

If T admits quantifier elimination for $\mathrm{FO}_{\infty\aleph_0}$ then it follows that

$$\langle\mathfrak{A},\bar{a}\rangle \equiv_\infty \langle\mathfrak{B},\bar{b}\rangle\,.$$

In particular, we have

$$\langle\mathfrak{A},\bar{a}\rangle \equiv_{\mathrm{FO}} \langle\mathfrak{B},\bar{b}\rangle\,.$$

By Theorem 2.6 it follows that T admits quantifier elimination. □

Example. (a) In Corollary c4.4.7 we have shown that we have $\mathfrak{A} \cong_0^{\aleph_0} \mathfrak{B}$ for all open dense linear orders $\mathfrak{A}$ and $\mathfrak{B}$. By the preceding theorem, it follows that the class of open dense linear orders admits quantifier elimination for $\mathrm{FO}_{\infty\aleph_0}$.

(b) Further examples like the theory of algebraically closed fields will be treated in the sections below.

Exercise 2.1. Let $\mathfrak{Z} := \langle \mathbb{Z}, s \rangle$ where $s : x \mapsto x+1$ is the successor function. Prove that $\mathrm{Th}(\mathfrak{Z})$ admits quantifier-elimination.

To check whether a theory T admits quantifier elimination for $\mathrm{FO}_{\infty\aleph_0}$ the most useful characterisation is statement (2) of Theorem 2.9. In fact, we do not need to consider all models of T, only sufficiently large ones.

Lemma 2.11. *Let L be a logic and $\Gamma, \Delta \subseteq L$ sets such that Γ is a Δ-elimination set over $\mathcal{K}_0$. If $\mathcal{K}$ is a class of L-interpretations such that, for every $\mathfrak{I} \in \mathcal{K}$, there exists some $\mathfrak{I}_0 \in \mathcal{K}_0$ with $\mathfrak{I}_0 \equiv_L \mathfrak{I}$ then Γ is a Δ-elimination set over $\mathcal{K}$.*

Proof. Given $\Phi \subseteq \Delta$ there exists a set $\Psi \subseteq \Gamma$ such that

$$\mathfrak{I} \vDash \Phi \quad \text{iff} \quad \mathfrak{I} \vDash \Psi, \qquad \text{for all } \mathfrak{I} \in \mathcal{K}_0 .$$

We claim that these sets are also equivalent for all interpretations in $\mathcal{K}$. Let $\mathfrak{I} \in \mathcal{K}$. By assumption, there exists an interpretation $\mathfrak{I}_0 \in \mathcal{K}_0$ with $\mathfrak{I}_0 \equiv_L \mathfrak{I}$. Consequently, we have

$$\mathfrak{I} \vDash \Phi \quad \text{iff} \quad \mathfrak{I}_0 \vDash \Phi \quad \text{iff} \quad \mathfrak{I}_0 \vDash \Psi \quad \text{iff} \quad \mathfrak{I} \vDash \Psi . \qquad \square$$

Corollary 2.12. *Let T be a first-order theory and $\mathcal{K} \subseteq \mathrm{Mod}(T)$ a class such that, for every model $\mathfrak{A} \vDash T$, there is some structure $\mathfrak{B} \in \mathcal{K}$ with $\mathfrak{A} \preceq \mathfrak{B}$. If $\mathfrak{A} \sqsubseteq_0^{\aleph_0} \mathfrak{B}$, for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$, then T admits quantifier elimination.*

If we replace in the proof of Theorem 2.9 all quantifier-free formulae by arbitrary first-order formulae we obtain the following result.

Theorem 2.13. *Let $\mathcal{K}$ be a class of structures. The following statements are equivalent:*

(1) *Over the class $\mathcal{K}$ every $\mathrm{FO}^{<\omega}_{\infty\aleph_0}$-formula is equivalent to an infinite boolean combination of first-order formulae.*
(2) $\mathfrak{A} \sqsubseteq^{\aleph_0}_{\mathrm{FO}} \mathfrak{B}$ *for all structures* $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$.
(3) $\mathfrak{A} \cong^{\aleph_0}_{\mathrm{FO}} \mathfrak{B}$ *for all structures* $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$.
(4) *For all structures* $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ *and all* $\bar{a} \in A^{<\omega}$ *and* $\bar{b} \in B^{<\omega}$, *we have*

$$\langle \mathfrak{A}, \bar{a} \rangle \equiv_{\mathrm{FO}} \langle \mathfrak{B}, \bar{b} \rangle \quad \text{iff} \quad \langle \mathfrak{A}, \bar{a} \rangle \equiv_{\infty} \langle \mathfrak{B}, \bar{b} \rangle .$$

We conclude this section with a closer look at quantifier elimination for the quantifier $\exists^{\aleph_0}$.

Definition 2.14. A first-order theory $T \subseteq \mathrm{FO}^{\mathrm{o}}[\Sigma]$ is *graduated* if, for every formula $\varphi(\bar{x}, y) \in \mathrm{FO}^{<\omega}[\Sigma]$, there exists a number $k < \omega$ such that, for every model $\mathfrak{A}$ of T and all parameters $\bar{a} \subseteq A$,

$$\left|\varphi(\bar{a}, y)^{\mathfrak{A}}\right| < \aleph_0 \quad \text{implies} \quad \left|\varphi(\bar{a}, y)^{\mathfrak{A}}\right| \leq k .$$

Theorem 2.15. *A theory $T \subseteq \mathrm{FO}^{\mathrm{o}}[\Sigma]$ is graduated if and only if* FO *is an* $\mathrm{FO}(\exists^{\aleph_0})$*-elimination set over T.*

Proof. ($\Rightarrow$) For every formula $\varphi \in \mathrm{FO}(\exists^{\aleph_0})$, we construct an equivalent first-order formula by induction on φ. Suppose that $\varphi = \exists^{\aleph_0} y \psi(\bar{x}, y)$. By inductive hypothesis, we may assume that ψ is a first-order formula. Since T is graduated there exists a number $k < \omega$ such that

$$\varphi(\bar{x}) \equiv \exists y_0 \cdots \exists y_k \Big[\bigwedge_{0 \leq i < l \leq k} y_i \neq y_l \wedge \bigwedge_{i \leq k} \psi(\bar{x}, y_i) \Big] .$$

($\Leftarrow$) For a contradiction, suppose that T is not graduated but FO is an $\mathrm{FO}(\exists^{\aleph_0})$-elimination set over T. Then there exists a formula $\varphi(\bar{x}, y)$ such that, for every $n < \omega$, there is a model $\mathfrak{A}_n$ of T and parameters $\bar{a}_n \subseteq A_n$ such that

$$n < \left|\varphi(\bar{a}_n, y)^{\mathfrak{A}_n}\right| < \aleph_0 .$$

By assumption there exists a set $\Phi \subseteq \mathrm{FO}$ such that $\neg\exists^{\aleph_0} y\varphi \equiv \Phi$. Then the set

$$\Psi := \Phi \cup \{\, \exists y_0 \cdots \exists y_n [\bigwedge_{i<l} y_i \neq y_l \wedge \bigwedge_i \varphi(\bar{x}, y_i)] \mid n < \omega \,\}$$

is inconsistent. On the other hand, for every finite subset $\Psi_0 \subseteq \Psi$, there is some number $m < \omega$ such that

$$\Psi_0 \subseteq \Phi \cup \{\, \exists y_0 \cdots \exists y_n [\bigwedge_{i<l} y_i \neq y_l \wedge \bigwedge_i \varphi(\bar{x}, y_i)] \mid n < m \,\}.$$

Consequently, $\mathfrak{A}_m \vDash \Psi_0(\bar{a}_m)$. By the Compactness Theorem, it follows that Ψ is satisfiable. Contradiction. □

3. Existentially closed structures

In this section we study classes where each structure passes the Tarski-Vaught Test.

Definition 3.1. (a) A first-order formula is *primitive* if it is of the form

$$\varphi(\bar{x}) = \exists \bar{y} \bigwedge_{i<n} \psi_i(\bar{x}, \bar{y}),$$

where each ψ_i is a literal.

(b) Let $\mathcal{K}$ be a class of structures. A structure $\mathfrak{A} \in \mathcal{K}$ is *existentially closed* (in $\mathcal{K}$) if, for every extension $\mathfrak{B} \supseteq \mathfrak{A}$ with $\mathfrak{B} \in \mathcal{K}$, we have

$$\mathfrak{B} \vDash \varphi(\bar{a}) \quad \text{implies} \quad \mathfrak{A} \vDash \varphi(\bar{a}),$$

for each primitive formula $\varphi(\bar{x})$ and all parameters $\bar{a} \subseteq A$.

(c) We call a theory T *existentially closed,* or *model-complete,* if every model of T is existentially closed in $\mathrm{Mod}(T)$. A theory T_{ec} is the *existential closure,* or *model companion,* of the theory T if

$$\mathrm{Mod}(T_{\mathrm{ec}}) = \{\, \mathfrak{A} \in \mathrm{Mod}(T) \mid \mathfrak{A} \text{ is existentially closed in } \mathrm{Mod}(T) \,\}.$$

Remark. The existential closure of a theory does not necessarily exist since the class

$$\mathcal{K} := \{ \mathfrak{A} \in \mathrm{Mod}(T) \mid \mathfrak{A} \text{ is existentially closed} \}$$

does not need to be axiomatisable. But if it exists then it is unique since $\mathrm{Mod}(T_0) = \mathcal{K} = \mathrm{Mod}(T_1)$ implies that $T_0 \equiv T_1$.

Theorem 3.2. *Let T be a first-order theory. The following statements are equivalent:*

(1) *T is existentially closed.*

(2) *$\mathfrak{B} \vDash \varphi(\bar{a})$ implies $\mathfrak{A} \vDash \varphi(\bar{a})$, for all models $\mathfrak{A} \subseteq \mathfrak{B}$ of T, all parameters $\bar{a} \subseteq A$, and every first-order formula φ.*

(3) *Every embedding between models of T is elementary.*

(4) *For every formula φ, there exists a universal formula ψ such that $T \vDash \varphi \leftrightarrow \psi$.*

(5) *For every primitive formula φ, there exists a universal formula ψ such that $T \vDash \varphi \leftrightarrow \psi$.*

Proof. (4) $\Rightarrow$ (3) follows from the fact that universal formulae are preserved in substructures.

(3) $\Rightarrow$ (2) If $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B} \vDash \varphi(\bar{a})$, for $\bar{a} \subseteq A$, then $\mathfrak{A} \preceq \mathfrak{B}$ implies that $\mathfrak{A} \vDash \varphi(\bar{a})$.

(2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (5) Let φ be a primitive formula. By (1), the negation $\neg\varphi$ is preserved by embeddings between models of T. Hence, we can use Corollary 1.3 to find an existential formula ψ equivalent to $\neg\varphi$ modulo T. The negation $\neg\psi$ is the desired universal formula.

(5) $\Rightarrow$ (4) W.l.o.g. we may assume that φ is in prenex normal form, say, $\varphi = Q_0 x_0 \cdots Q_{n-1} x_{n-1} \psi$ with ψ quantifier-free. We prove the claim by induction on n. By inductive hypothesis, there exists a universal formula $\forall \bar{y} \vartheta$ equivalent to $Q_1 x_1 \cdots Q_{n-1} x_{n-1} \psi$. If $Q_0 = \forall$ then $\forall x_0 \forall \bar{y} \vartheta$ is the desired formula. Suppose that $Q_0 = \exists$. Let $\bigvee_i \chi_i$ be the disjunctive

normal form of $\neg\vartheta$. By (5), there exists a universal formula $\forall \bar z^i \eta_i$ that is equivalent to $\exists \bar y \chi_i$. Consequently, we have

$$\exists \bar y \neg\vartheta \equiv \bigvee_i \exists \bar y \chi_i \equiv \bigvee_i \forall \bar z^i \eta_i \equiv \forall \bar z^0 \cdots \forall \bar z^m \bigvee_i \eta_i .$$

Let $\bar z = \bar z^0 \dots \bar z^m$ and let $\bigvee_i \beta_i$ be the disjunctive normal form of $\bigwedge_i \neg\eta_i$. It follows that

$$\varphi = \exists x_0 \forall \bar y \vartheta \equiv \exists x_0 \exists \bar z \bigwedge_i \neg\eta_i \equiv \bigvee_i \exists x_0 \exists \bar z \beta_i .$$

Applying (5) again, we obtain universal formula $\forall \bar y^i \gamma_i$ equivalent to $\exists x_0 \exists \bar z \beta_i$. Hence,

$$\varphi \equiv \bigvee_i \forall \bar y^i \gamma_i \equiv \forall \bar y^0 \cdots \forall \bar y^k \bigvee_i \gamma_i ,$$

as desired. □

Corollary 3.3. *Let T be a first-order theory.*

(a) *If T admits quantifier elimination then it is existentially closed.*

(b) *If T has algebraic prime models then it is existentially closed if and only if it admits quantifier elimination.*

(c) *If T is a Skolem theory then it is existentially closed.*

Proof. (a) and (c) follow from Theorem 3.2 (4). (b) follows from (a) and Proposition 2.8. □

Example. The theory of open dense linear orders is existentially closed. Other examples such as the theory of divisible abelian groups and the theory of algebraically closed fields will be treated below.

Let us give some basic properties of existentially closed theories. We start with a partial converse of Corollary 3.3 (a).

Lemma 3.4. *Let T be a theory such that* $\mathrm{Mod}(T)$ *is closed under substructures. Then T is existentially closed if and only if T admits quantifier elimination.*

Proof. We have already seen that every theory admitting quantifier elimination is existentially closed. For the converse, suppose that T is existentially closed. We apply Theorem 2.6 (3). Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are models of T with elements $\bar{a} \subseteq A$ and $\bar{b} \subseteq B$ such that

$$\langle \mathfrak{A}, \bar{a} \rangle \equiv_0 \langle \mathfrak{B}, \bar{b} \rangle .$$

Let $\varphi(\bar{x}, y)$ be a quantifier-free formula such that

$$\mathfrak{A} \vDash \exists y \varphi(\bar{a}, y) .$$

Since $\mathrm{Mod}(T)$ is closed under substructures, we have $\langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}} \vDash T$. By Theorem 3.2, it follows that $\langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}} \preceq \mathfrak{A}$. Hence,

$$\langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}} \vDash \exists y \varphi(\bar{a}, y) .$$

Fix some element $c \in \langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}}$ such that $\langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}} \vDash \varphi(\bar{a}, c)$. There exists some term t such that $c = t^{\langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}}}(\bar{a})$. Therefore, we have

$$\langle\!\langle \bar{a} \rangle\!\rangle_{\mathfrak{A}} \vDash \varphi(\bar{a}, t(\bar{a})) .$$

It follows that

$$\langle\!\langle \bar{b} \rangle\!\rangle_{\mathfrak{B}} \vDash \varphi(\bar{b}, t(\bar{b})) .$$

Consequently, $\mathfrak{B} \vDash \exists y \varphi(\bar{b}, y)$. □

Lemma 3.5. *Let T be an existentially closed theory. Then T is the existential closure of $T_\forall^\vDash$.*

Proof. Consider structures $\mathfrak{A} \subseteq \mathfrak{B}$ where $\mathfrak{A}$ is a model of T and $\mathfrak{B}$ a model of $T_\forall^\vDash$. Suppose that $\mathfrak{B} \vDash \varphi(\bar{a})$ where $\varphi(\bar{x})$ is a primitive formula and $\bar{a} \subseteq A$. We have to show that $\mathfrak{A} \vDash \varphi(\bar{a})$. By Lemma 1.1, we can find a model $\mathfrak{C}$ of T with $\mathfrak{B} \subseteq \mathfrak{C}$. Since existential formulae are preserved in extensions it follows that $\mathfrak{C} \vDash \varphi(\bar{a})$. As T is existentially closed and we have $\mathfrak{A} \subseteq \mathfrak{C}$, it follows that $\mathfrak{A} \preceq \mathfrak{C}$. Hence, $\mathfrak{A} \vDash \varphi(\bar{a})$, as desired. □

Lemma 3.6. *If T is existentially closed then $T \equiv T^{\vDash}_{\forall\exists}$.*

Proof. If T is existentially closed then every chain is elementary. Hence, $\mathrm{Mod}(T)$ is closed under unions of chains and the claim follows by Theorem 1.5. □

For $\forall\exists$-theories, one can embed every model into an existentially closed one.

Proposition 3.7. *Let $T \subseteq \forall\exists$ be a first-order theory and $\mathfrak{A}$ an infinite Σ-structure with $\mathfrak{A} \vDash T^{\vDash}_{\forall}$. Then there exists an existentially closed model $\mathfrak{B}$ of T of size $|B| = |A| \oplus |\Sigma|$ such that $\mathfrak{A} \subseteq \mathfrak{B}$.*

Proof. By Lemma 1.1, there exists a model $\mathfrak{C}$ of T with $\mathfrak{A} \subseteq \mathfrak{C}$. By the Theorem of Löwenheim and Skolem we may choose $\mathfrak{C}$ of size $|C| = |A| \oplus |\Sigma|$. To conclude the proof we construct an existentially closed elementary extension $\mathfrak{B} \succeq \mathfrak{C}$ of size $|B| = |C|$. The construction is similar to the one used in Theorem C2.3.6 to find a Skolem theory.

Claim. *For every infinite model $\mathfrak{A} \vDash T$, there exists an extension $\mathfrak{A}^+ \supseteq \mathfrak{A}$ of size $|A^+| = |A| \oplus |\Sigma|$ such that $\mathfrak{A}^+ \vDash T$ and, for every $\exists$-formula $\varphi(\bar{x})$ and all $\bar{a} \subseteq A$,*

$$\mathfrak{A}^+ \nvDash \varphi(\bar{a}) \quad \textit{implies} \quad \mathfrak{B} \nvDash \varphi(\bar{a}), \quad \textit{for all } \mathfrak{B} \supseteq \mathfrak{A}^+.$$

When we have proved the claim then we can find the desired existentially closed structure $\mathfrak{B} \succeq \mathfrak{C}$ as follows. We define an increasing chain $(\mathfrak{B}_n)_{n<\omega}$ by

$$\mathfrak{B}_0 := \mathfrak{C} \quad \text{and} \quad \mathfrak{B}_{n+1} := (\mathfrak{B}_n)^+.$$

Since $T \subseteq \forall\exists$ it follows that $\mathfrak{B} := \bigcup_n \mathfrak{B}_n$ is a model of T. By definition, we have $\mathfrak{C} \subseteq \mathfrak{B}$ and

$$|B| = \sup_n |B_n| \leq \aleph_0 \oplus |C| = |C|.$$

It remains to show that $\mathfrak{B}$ is existentially closed. If $\varphi(\bar{x})$ is an $\exists$-formula and $\bar{a} \subseteq B$ then there is some index $n < \omega$ such that $\bar{a} \subseteq B_n$. Consequently, if there exists a model $\mathfrak{D} \supseteq \mathfrak{B}$ of T with $\mathfrak{D} \vDash \varphi(\bar{a})$ then, by construction of $\mathfrak{B}_{n+1} = \mathfrak{B}_n^+$, we have $\mathfrak{B}_{n+1} \vDash \varphi(\bar{a})$. Since φ is existential and $\mathfrak{B}_{n+1} \subseteq \mathfrak{B}$ it follows that $\mathfrak{B} \vDash \varphi(\bar{a})$, as desired.

It remains to prove the above claim. Let $\kappa := |A| \oplus |\Sigma|$ and fix an enumeration $\langle \varphi_\alpha, \bar{a}_\alpha \rangle_{\alpha<\kappa}$ of all pairs $\langle \varphi, \bar{a} \rangle$ where $\varphi \in \exists$ and $\bar{a} \in A^{<\omega}$. We define an increasing sequence $(\mathfrak{A}_\alpha)_{\alpha<\kappa}$ of models of T as follows. We start with $\mathfrak{A}_0 := \mathfrak{A}$ and, for limit ordinals δ, we set $\mathfrak{A}_\delta := \bigcup_{\alpha<\delta} \mathfrak{A}_\alpha$. For the successor step, we distinguish two cases. If there is some extension $\mathfrak{B} \supseteq \mathfrak{A}_\alpha$ with $\mathfrak{B} \vDash \varphi_\alpha(\bar{a}_\alpha)$ then, by the Theorem of Löwenheim and Skolem, we can choose such an extension of size $|B| \leq |A_\alpha| \oplus |\Sigma|$ and we set $\mathfrak{A}_{\alpha+1} := \mathfrak{B}$. Otherwise, we set $\mathfrak{A}_{\alpha+1} := \mathfrak{A}_\alpha$.

We claim that $\mathfrak{A}^+ := \bigcup_\alpha \mathfrak{A}_\alpha$ is the desired structure. By induction on α, it follows that $|A_\alpha| \leq \kappa$. Hence, $|A^+| \leq \kappa$. Furthermore, if there exists an extension $\mathfrak{B} \supseteq \mathfrak{A}$ such that $\mathfrak{B} \vDash \varphi(\bar{a})$, for some $\varphi \in \exists$ and $\bar{a} \subseteq A$, then there exits an index α with $\varphi = \varphi_\alpha$ and $\bar{a} = \bar{a}_\alpha$. Hence, $\mathfrak{A}_{\alpha+1}$ is some extension of $\mathfrak{A}_\alpha$ with $\mathfrak{A}_{\alpha+1} \vDash \varphi(\bar{a})$. Since φ is existential and $\mathfrak{A}_{\alpha+1} \subseteq \mathfrak{A}^+$ it follows that $\mathfrak{A}^+ \vDash \varphi(\bar{a})$, as desired. □

Example. A field is existentially closed if and only if it is algebraically closed. Since the theory of fields is $\forall\exists$-axiomatisable it follows that every field has an algebraically closed extension.

Lemma 3.8. *Let $T \subseteq \forall\exists$ be a theory with existential closure T_{ec}.*

(a) *Every model of T_{ec} is a model of T.*

(b) *Every model of T has an extension that is a model of T_{ec}.*

Proof. (a) holds by definition of an existential closure and (b) follows from Proposition 3.7. □

Corollary 3.9. *If T_{ec} is the existential closure of a theory $T \subseteq \forall\exists$ then*

$$T_\forall^\vDash = (T_{\mathrm{ec}})_\forall^\vDash \quad \text{and} \quad (T_{\mathrm{ec}})_\forall^\vDash \subseteq T \subseteq T_{\mathrm{ec}}\,.$$

Proof. The equation $T_\forall^\vDash = (T_{ec})_\forall^\vDash$ follows by the preceding lemma and Lemma 1.1. Hence, we have $(T_{ec})_\forall^\vDash = T_\forall^\vDash \subseteq T$. Finally, $\mathrm{Mod}(T_{ec}) \subseteq \mathrm{Mod}(T)$ implies $T \subseteq T_{ec}$. ☐

4. Abelian groups

As a simple example of existentially closed theories we consider theories of abelian groups.

Definition 4.1. Let $\mathfrak{G} = \langle G, \cdot, ^{-1}, e \rangle$ be a group. A *torsion element* of $\mathfrak{G}$ is an element $a \neq e$ such that $a^n = e$, for some $0 < n < \omega$. The set of all torsion elements of $\mathfrak{G}$ (including e) is denoted by

$$\operatorname{tor}(\mathfrak{G}) := \{ a \in G \mid a^n = e \text{ for some } n > 0 \} .$$

We say that $\mathfrak{G}$ is *torsion-free* if $\operatorname{tor}(\mathfrak{G}) = \{e\}$.

Example. $\operatorname{tor}\langle \mathbb{R}/\mathbb{Z}, +, -, 0 \rangle = \langle \mathbb{Q}/\mathbb{Z}, +, -, 0 \rangle$.

Lemma 4.2. *If $\mathfrak{G}$ is an abelian group then* $\operatorname{tor}(\mathfrak{G})$ *is a normal subgroup of $\mathfrak{G}$.*

Proof. In an abelian group every subgroup is normal. Hence, we only need to show that $\operatorname{tor}(\mathfrak{G})$ is closed under the group operations. Let $a, b \in \operatorname{tor}(\mathfrak{G})$. Then there are numbers $m, n > 0$ such that $a^m = e$ and $b^n = e$. Consequently, we have

$$(ab^{-1})^{mn} = a^{mn}(b^{mn})^{-1} = e^n(e^m)^{-1} = e ,$$

which implies that $ab^{-1} \in \operatorname{tor}(\mathfrak{G})$. ☐

Corollary 4.3. *Every abelian group $\mathfrak{G}$ can we written as direct sum*

$$\mathfrak{G} \cong \mathfrak{H} \oplus \operatorname{tor}(\mathfrak{G}) \quad \textit{where } \mathfrak{H} \textit{ is torsion-free.}$$

Definition 4.4. An *ordered group* is a structure $\mathfrak{G} = \langle G, \circ, {}^{-1}, e, < \rangle$ such that $\langle G, \circ, {}^{-1}, e \rangle$ forms a group, $<$ is a linear order on G, and we have

$$a < b \quad \text{implies} \quad ac < bc \quad \text{and} \quad ca < cb\,, \quad \text{for all } a, b, c \in G\,.$$

Exercise 4.1. Prove that there are exactly two orderings $\sqsubseteq$ on $\mathbb{Q}$ such that $\langle \mathbb{Q}, +, \sqsubseteq \rangle$ is an ordered group.

Lemma 4.5. *Every ordered group is torsion-free.*

Proof. For a contradiction, suppose that there is some element $a \neq e$ such that $a^n = e$, for some $n > 0$. If $a > e$ then we have $a^{k+1} > a^k$, for all k. It follows that $e < a < \cdots < a^n = e$. Contradiction. Similarly, $a < e$ implies that $e > a > \cdots > a^n = e$. □

Definition 4.6. (a) An abelian group $\mathfrak{G} = \langle G, +, -, 0 \rangle$ is *divisible* if, for every element $a \in G$ and all numbers $0 < n < \omega$, there exists an element $b \in G$ with $nb = a$. We denote this element by a/n.

(b) Let DAG be the theory of all divisible torsion-free abelian groups with more than one element. Let ODAG be the theory of all ordered divisible abelian groups with more than one element.

If $\mathfrak{G}$ is divisible and torsion-free we can define an action $\mathbb{Q} \times G \to G$ by setting $\frac{m}{n} \cdot a := m(a/n)$.

Lemma 4.7. *Every divisible torsion-free abelian group $\mathfrak{G}$ is a $\mathbb{Q}$-module.*

Exercise 4.2. Let $\mathfrak{G}$ be a divisible abelian group that is not torsion-free. Show that $\mathfrak{G}$ is no $\mathbb{Q}$-module under the above action.

Theorem 4.8. *For every divisible torsion-free abelian group $\mathfrak{G}$ there is a cardinal κ such that $\mathfrak{G} \cong \langle \mathbb{Q}, + \rangle^{(\kappa)}$.*

Proof. $\mathfrak{G}$ is a $\mathbb{Q}$-module, that is, a $\mathbb{Q}$-vector space. By Theorem B6.4.12, we have $\mathfrak{G} \cong \mathbb{Q}^{(\kappa)}$ where κ is the dimension of $\mathfrak{G}$. □

Corollary 4.9. *For every divisible torsion-free abelian group $\mathfrak{G}$ there exists a linear order $<$ such that $\langle \mathfrak{G}, < \rangle$ is an ordered group.*

Proof. We can take the lexicographic order on $\mathbb{Q}^{(\kappa)}$. □

Every abelian group can be embedded into a divisible one.

Definition 4.10. Let $\mathfrak{G}$ be an abelian group. The *divisible closure* of $\mathfrak{G}$ is the group $\operatorname{div}(\mathfrak{G})$ with universe

$$\operatorname{div}(G) := \{\, \langle a, n \rangle \mid a \in G,\ 0 < n < \omega \,\}/{\sim}$$

where

$$\langle a, m \rangle \sim \langle b, n \rangle \quad :\text{iff} \quad na = mb\,.$$

We denote the $\sim$-class of $\langle a, n \rangle$ by a/n. The group operations of $\operatorname{div}(\mathfrak{G})$ are given by

$$a/m + b/n := (na + mb)/mn \quad \text{and} \quad -(a/m) := (-a)/m\,.$$

Theorem 4.11. *Let $\mathfrak{G}$ be an abelian group.*

(a) *The divisible closure* $\operatorname{div}(\mathfrak{G})$ *of $\mathfrak{G}$ is a divisible abelian group.*

(b) *If $\mathfrak{G}$ is torsion-free then so is* $\operatorname{div}(\mathfrak{G})$.

(c) *If $\mathfrak{G}$ is ordered then so is* $\operatorname{div}(\mathfrak{G})$.

(d) *The embedding $\mathfrak{G} \to \operatorname{div}(\mathfrak{G}) : a \mapsto a/1$ is an algebraic prime model for the theory* DAG *and* ODAG*, respectively.*

Proof. (a) If $a/m = a'/m'$ then we have $a/m + b/n = a'/m' + b/n$ since $m'a = ma'$ implies that

$$\begin{aligned} m'n(na + mb) &= m'n^2 a + mm'nb \\ &= mn^2 a' + mm'nb = mn(na' + m'b)\,. \end{aligned}$$

Hence, + is well-defined. In a similar way one shows that − is also well-defined and that $\operatorname{div}(\mathfrak{G})$ forms an abelian group with unit $0/1$.

Note that $\operatorname{div}(\mathfrak{G})$ is divisible since $n(a/mn) = (na/mn) = a/m$.

(b) Suppose that $n(a/m) = 0/1$. Then we have $na = m0 = 0$, which implies that $a = 0$ since $\mathfrak{G}$ is torsion-free.

(c) We define the order on $\operatorname{div}(\mathfrak{G})$ by setting

$$a/m < b/n \quad :\text{iff} \quad na < mb\,.$$

To see that this definition turns $\operatorname{div}(\mathfrak{G})$ into an ordered group note that $na < mb$ implies

$$nk(ka + mc) < mk(kb + nc)\,.$$

Consequently,

$$a/m < b/n \quad \text{implies} \quad a/m + c/k < b/n + c/k\,.$$

(d) Let $g : \mathfrak{G} \to \mathfrak{H}$ be some embedding of $\mathfrak{G}$ into a model of DAG or ODAG. Then we obtain an embedding $\operatorname{div}(\mathfrak{G}) \to \mathfrak{H}$ by mapping $a/n \in \operatorname{div}(G)$ to the unique element $b \in H$ with $nb = g(a)$. □

Corollary 4.12. *Every abelian group can be embedded into a divisible abelian group.*

Corollary 4.13. *For every torsion-free abelian group $\mathfrak{G}$, there exists a cardinal κ such that $\mathfrak{G}$ can be embedded into $\mathbb{Q}^{(\kappa)}$.*

Corollary 4.14. DAG *and* ODAG *have algebraic prime models.*

In order to prove that DAG and ODAG admit quantifier elimination it remains to check that subgroups are simply closed.

Lemma 4.15. *If $\mathfrak{G} \subseteq \mathfrak{H}$ are torsion-free divisible abelian groups then $\mathfrak{G}$ is simply closed in $\mathfrak{H}$. The same holds if $\mathfrak{G}$ and $\mathfrak{H}$ are ordered.*

Proof. We have to show that

$$\mathfrak{H} \vDash \exists y \varphi(\bar{a}, y) \quad \text{implies} \quad \mathfrak{G} \vDash \exists y \varphi(\bar{a}, y)\,,$$

for every quantifier-free formula φ and all $\bar{a} \subseteq G$. Suppose that $\varphi = \bigvee_i \bigwedge_k \psi_{ik}$ is in disjunctive normal form. If $\mathfrak{H} \vDash \varphi(\bar{a}, b)$ then there is some i such that $\mathfrak{H} \vDash \bigwedge_k \psi_{ik}(\bar{a}, b)$. Since each atomic formula can be written as

$$\sum_i m_i x_i + ny = 0 \quad \text{or} \quad \sum_i m_i x_i + ny < 0\,, \quad \text{for } m_i, n \in \mathbb{Z}\,,$$

we may therefore assume that

$$\begin{aligned}\varphi = \bigwedge_k \sum_i m_{ki} x_i + n_k y = 0 \wedge \bigwedge_k \sum_i m'_{ki} x_i + n'_k y < 0 \\ \wedge \bigwedge_k \sum_i m''_{ki} x_i + n''_k y \neq 0\,.\end{aligned}$$

Set $c_k := \sum_i m_{ki} a_i$, $c'_k := \sum_i m'_{ki} a_i$, and $c''_k := \sum_i m''_{ki} a_i$. These elements are in G and we have

$$\varphi \equiv \bigwedge_k c_k + n_k y = 0 \wedge \bigwedge_k c'_k + n'_k y < 0 \wedge \bigwedge_k c''_k + n''_k y \neq 0\,.$$

If there is some k with $n_k \neq 0$ then

$$\mathfrak{H} \vDash \varphi(\bar{a}, -c_k/n_k)\,.$$

Since $-c_k/n_k \in G$ we are done. Therefore, we may assume that $n_k = 0$, for all k. Then

$$\varphi \equiv \bigwedge_k c'_k + n'_k y < 0 \wedge \bigwedge_k c''_k + n''_k y \neq 0\,.$$

Suppose that $n'_0, \dots, n'_{s-1} < 0$ and $n'_s, \dots, n'_{t-1} > 0$. Then this formula simplifies to

$$\varphi \equiv \bigwedge_{k=0}^{s-1} y > -c'_k/n'_k \wedge \bigwedge_{k=s}^{t-1} y < -c'_k/n'_k \wedge \bigwedge_k y \neq -c''_k/n''_k\,.$$

Setting $d_0 := \max\{-c'_k/n'_k \mid k < s\}$ and $d_1 := \min\{-c'_k/n'_k \mid s \leq k < t\}$ we obtain

$$\varphi \equiv y > d_0 \wedge y < d_1 \wedge \bigwedge_k y \neq -c''_k/n''_k .$$

Since $\mathfrak{H} \vDash \exists y\varphi(\bar{a}, y)$ it follows that $d_0 < d_1$. Hence, $d_0, d_1 \in G$ implies that G contains infinitely many elements b with $d_0 < b < d_1$. Consequently, we can find an element $b \in G$ with $d_0 < b < d_1$ such that $b \neq -c''_k/n''_k$, for all k. It follows that $\mathfrak{G} \vDash \varphi(\bar{a}, b)$. □

Theorem 4.16. DAG *and* ODAG *admit quantifier elimination.*

Proof. This follows from the preceding lemmas by Proposition 2.8. □

Corollary 4.17. DAG *is the existential closure of the theory of torsion-free abelian groups.* ODAG *is the existential closure of the theory of ordered abelian groups.*

5. *Fields*

Further classes with a well-behaved model theory are the class of algebraically closed fields and the class of real closed fields.

Definition 5.1. (a) The axiom system for the theory of *fields* is the set F consisting of all ring axioms together with the formulae

$$0 \neq 1 \quad \text{and} \quad \forall x \exists y[x \neq 0 \rightarrow x \cdot y = 1] .$$

(b) The theory ACF of *algebraically closed fields* is obtained from F by adding, for every $1 < n < \omega$, the sentence

$$\forall y_0 \cdots \forall y_{n-1} \exists x[x^n + y_{n-1} \cdot x^{n-1} + \cdots + y_1 \cdot x + y_0 = 0] .$$

(c) For a prime number p, we obtain the theory ACF_p of *algebraically closed fields of characteristic p* by adding to ACF the sentence

$$\underbrace{1 + \dots + 1}_{p \text{ times}} = 0 .$$

Similarly, the theory ACF_0 of algebraically closed fields of characteristic 0 is obtained by adding all the sentences

$$\underbrace{1 + \dots + 1}_{n \text{ times}} \neq 0 , \qquad \text{for all } 0 < n < \omega .$$

(d) We denote by RCF the axiom system for the theory of *real closed fields*. It consists of the axioms for an ordered field and the formulae

$$\begin{aligned}
&\forall x \exists y [y \cdot y = x \vee y \cdot y = -x] , \\
&\forall x_0 \cdots \forall x_{n-1} [x_0 \cdot x_0 + \dots + x_{n-1} \cdot x_{n-1} + 1 \neq 0] , \\
&\forall y_0 \cdots \forall y_{2n} \exists x [x^{2n+1} + y_{2n} \cdot x^{2n} + \dots + y_1 \cdot x^1 + y_0 = 0] ,
\end{aligned}$$

for all $n < \omega$.

Remark. (a) If $\mathfrak{K} \vDash F$ is a field then every atomic formula has the form $p(\bar{x}) = q(\bar{x})$ or, equivalently, $p(\bar{x}) - q(\bar{x}) = 0$, for polynomials $p, q \in \mathbb{Z}[\bar{x}]$.

(b) In Theorem B6.5.5 we have seen that $F_\forall^\vDash$ is the theory of integral domains.

Since the axiom systems F, ACF, ACF_p, and RCF consist solely of $\forall\exists$-sentences it follows by Lemma C2.1.8 that their model classes are closed under unions of chains.

Lemma 5.2. *If $(\mathfrak{K}_\alpha)_{\alpha<\kappa}$ is a chain of fields then their union $\bigcup_{\alpha<\kappa} \mathfrak{K}_\alpha$ is also a field. If every $\mathfrak{K}_\alpha$ is algebraically closed then so is the union. The same holds for real closed fields.*

Proposition 5.3. *Let κ be an infinite cardinal and let $\mathfrak{K}$ and $\mathfrak{L}$ be algebraically closed fields of transcendence degree at least κ. If $\mathfrak{K}$ and $\mathfrak{L}$ have the same characteristic then $\mathfrak{K} \cong_0^\kappa \mathfrak{L}$.*

Proof. First, note that $\mathrm{pIso}_\kappa(\mathfrak{K}, \mathfrak{L}) \neq \emptyset$ since it contains $1 \mapsto 1$. By symmetry, we therefore only need to prove the forth property.

Let $\bar{a} \mapsto \bar{b} \in \mathrm{pIso}_\kappa(\mathfrak{K}, \mathfrak{L})$ and $c \in K$. We denote by $\mathfrak{A}$ the subfield of $\mathfrak{K}$ generated by $\bar{a}$ and $\mathfrak{B}$ is the subfield of $\mathfrak{L}$ generated by $\bar{b}$. The partial isomorphism $\bar{a} \mapsto \bar{b}$ extends to an isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$.

If $c \in A$ then $d := \pi(c) \in B$ and $\bar{a}c \mapsto \bar{b}d \in \mathrm{pIso}_\kappa(\mathfrak{K}, \mathfrak{L})$.

Next we consider the case that c is algebraic over A. Let $p \in A[x]$ be the minimal polynomial of c. Consider the canonical extension $\pi' : \mathfrak{A}[x] \to \mathfrak{B}[x]$ of π and set $q := \pi'(p)$. Since $\mathfrak{L}$ is algebraically closed, q has some root $d \in L$. It follows that

$$\mathfrak{A}(c) \cong \mathfrak{A}[x]/(p) \cong \mathfrak{B}[x]/(q) \cong \mathfrak{B}(d)$$

and, hence, $\bar{a}c \mapsto \bar{b}d \in \mathrm{pIso}_\kappa(\mathfrak{K}, \mathfrak{L})$.

Finally, suppose that c is not algebraic over A. Since $\mathfrak{L}$ has transcendence degree at least κ, there is some element $d \in L$ that is transcendental over B. It follows that $\mathfrak{A}(c) \cong \mathrm{FF}(\mathfrak{A}[x]) \cong \mathrm{FF}(\mathfrak{B}[x]) \cong \mathfrak{B}(d)$. □

Theorem 5.4. ACF *admits quantifier elimination.*

Proof. By Corollary 2.12 and the preceding proposition it is sufficient to show that every algebraically closed field $\mathfrak{K}$ has an elementary extension $\mathfrak{L}$ with infinite transcendence degree. Let Δ be the elementary diagram of $\mathfrak{K}$ and let C be a countable set of new constant symbols. We set

$$\Phi := \{\, p[\bar{c}] \neq 0 \mid p \in K[\bar{x}],\ \bar{c} \subseteq C \,\}\,.$$

If $\mathfrak{L} \models \Delta \cup \Phi$ then $\mathfrak{L} \geq \mathfrak{K}$ implies that $\mathfrak{L}$ is an algebraically closed extension of $\mathfrak{K}$. Furthermore, C is an infinite algebraically independent subset of L.

Hence, it remains to prove that $\Delta \cup \Phi$ is satisfiable. By the Compactness Theorem we only have to check that all finite subsets of $\Delta \cup \Phi$ are satisfiable. Let $\Phi_0 \subseteq \Phi$ be finite and let $p_0, \dots, p_{n-1}$ be the polynomials appearing in Φ_0. Suppose that $p_0, \dots, p_{n-1} \in K[x_0, \dots, x_{k-1}]$. By induction on i, we find elements $a_i \in K$ such that $p_l[\bar{a}] \neq 0$, for all l.

Suppose that we have already chosen $a_0, \dots, a_{i-1}$. We partition the polynomials $p_0, \dots, p_{n-1}$ into three classes.

(i) those containing only variables from $x_0, \dots, x_{i-1}$;

(ii) those not in class (i) that contain only variables from $x_0, \dots, x_i$;

(iii) those containing some variable from $x_{i+1}, \dots, x_{k-1}$.

We choose an arbitrary element $a_i \in K$ such that, for every polynomial p_l in class (ii), we have $p_l[a_0, \dots, a_{i-1}, a_i] \neq 0$. This is possible since K is infinite and, for every polynomial $p_l[a_0, \dots, a_{i-1}, x_i]$, there are only finitely many values for x_i that we cannot choose.

Interpreting the constants $\bar{c}$ in Φ by the elements $\bar{a}$ we obtain a model $\langle \mathfrak{K}, \bar{a} \rangle$ of $\Delta \cup \Phi_0$. □

Theorem 5.5. *If p is a prime number or $p = 0$ then the theory ACF_p is complete.*

Proof. Let $\varphi \in \mathrm{FO}$ be a sentence. We have to show that either $\mathrm{ACF}_p \vDash \varphi$ or $\mathrm{ACF}_p \vDash \neg\varphi$. Since ACF admits quantifier elimination there exists a quantifier-free sentence ψ such that

$$\mathrm{ACF}_p \vDash \varphi \leftrightarrow \psi .$$

ψ is a boolean combination of sentences of the form $\vartheta := 1 + \dots + 1 = 0$. But for each such sentence we either have $\mathrm{ACF}_p \vDash \vartheta$ or $\mathrm{ACF}_p \vDash \neg\vartheta$. □

After having seen that the theory of algebraically closed fields admits quantifier elimination we turn to real closed fields.

Proposition 5.6. $\mathrm{RCF}_\forall$ *is the theory of ordered integral domains.*

Proof. If $\mathfrak{R}$ is a substructure of a real closed field then it is a commutative ring without zero-divisors. Conversely, let $\mathfrak{R}$ be an ordered integral domain. We can order $\mathrm{FF}(\mathfrak{R})$ by

$$a/b > 0 \quad :\text{iff} \quad a, b > 0 \text{ or } a, b < 0 .$$

By Theorem B6.6.13, we can embed $\mathrm{FF}(\mathfrak{R})$ into a real closed field. □

Proposition 5.7. RCF *has algebraic prime models.*

Proof. Let $\mathfrak{R}$ be an ordered integral domain and let $\mathfrak{K}$ be the real closure of $\mathrm{FF}(\mathfrak{R})$. We claim that $\mathfrak{K}$ is the algebraic prime model of $\mathfrak{R}$.

Fix an arbitrary ordered real closed extension $\mathfrak{L}$ of $\mathfrak{R}$. Then $\mathrm{FF}(\mathfrak{R}) \subseteq \mathfrak{L}$. Let

$$L_0 := \{ a \in L \mid a \text{ is algebraic over } \mathrm{FF}(R) \} .$$

By Theorem B6.6.14, it follows that $\mathfrak{L}_0 \subseteq \mathfrak{L}$ is real closed. Since the order of $\mathfrak{L}_0$ extends the order of $\mathrm{FF}(\mathfrak{R})$, we can use Theorem B6.6.22 to find an isomorphism $\mathfrak{L}_0 \to \mathfrak{K}$. □

Lemma 5.8. *If $\mathfrak{K} \subseteq \mathfrak{L}$ are real closed fields then $\mathfrak{K}$ is simply closed in $\mathfrak{L}$.*

Proof. Let $\varphi(x, \bar{y})$ be quantifier-free and suppose that

$$\mathfrak{L} \vDash \varphi(a, \bar{b}) , \quad \text{for some } a \in L,\ \bar{b} \subseteq K .$$

Note that, for a polynomial $p \in \mathbb{Z}[\bar{x}]$,

$$\begin{aligned} p[\bar{c}] \neq 0 \quad &\text{iff} \quad p[\bar{c}] > 0 \lor -p[\bar{c}] > 0 , \\ p[\bar{c}] \leq 0 \quad &\text{iff} \quad p[\bar{c}] = 0 \lor -p[\bar{c}] > 0 . \end{aligned}$$

Therefore, we may assume that $\varphi(x, \bar{y}) = \bigvee_{k \leq n} \psi_k(x, \bar{y})$ where each ψ_k is a conjunction of formulae of the form $p[x, \bar{y}] = 0$ or $p[x, \bar{y}] > 0$, for some $p \in \mathbb{Z}[x, \bar{y}]$. Fix some k such that $\mathfrak{L} \vDash \psi_k(a, \bar{b})$ and suppose that

$$\psi_k(x, \bar{b}) = \bigwedge_{i<m} p_i[x] = 0 \land \bigwedge_{i<n} q_i[x] > 0 ,$$

for $p_i, q_i \in K[x]$. If any of the p_i is nonzero then $p_i[a] = 0$ implies that a is algebraic over $\mathfrak{K}$. Since $\mathfrak{K}$ is real closed, it has no proper algebraic extension that is real. Therefore, $a \in K$ and we are done.

Hence, we may assume that

$$\psi_k(x, \bar{b}) = \bigwedge_{i<n} q_i[x] > 0 .$$

The sign of $q_i[x]$ can only change at a root of q_i. As we have just seen each such root is an element of K. Therefore, there are elements $c_i, d_i \in K$ with $c_i < a < d_i$ and $q_i[x] > 0$, for all $x \in (c_i, d_i)$. Set

$$c := \max\{c_0, \dots, c_{n-1}\} \quad \text{and} \quad d := \min\{d_0, \dots, d_{n-1}\}.$$

Then $c < a < d$. Setting $a' := (c + d)/2 \in K$ it follows that $q_i[a'] > 0$, for all $i < n$. Hence, $\mathfrak{L} \vDash \psi_k(a', \bar{b})$. □

Theorem 5.9. RCF *admits quantifier elimination.*

Proof. We have shown that RCF has algebraic prime models and that real closed subfields are simply closed. Therefore, the claim follows by Proposition 2.8. □

Corollary 5.10. $\mathrm{RCF}^{\vDash} = \mathrm{Th}\langle \mathbb{R}, +, -, \cdot, 0, 1, < \rangle$ *is complete and existentially closed.*

Proof. Every theory that admits quantifier elimination is existentially closed. To show that RCF is complete note that every real closed field $\mathfrak{R}$ has characteristic 0. Hence, $\mathbb{Q} \subseteq \mathfrak{R}$. Let $\mathbb{R}_{\mathrm{alg}}$ be the real closure of $\mathbb{Q}$, that is, the field of algebraic real numbers. It follows that $\mathbb{R}_{\mathrm{alg}}$ can be embedded into every real closed field $\mathfrak{R}$. Since RCF is existentially closed this embedding is elementary. Therefore, $\mathfrak{R} \equiv \mathbb{R}_{\mathrm{alg}}$. □

D2. *Products and varieties*

1. *Ultraproducts*

In Section D1.1 we have studied operations that preserve various fragments of first-order logic. But we have found no operation so far that preserves all first-order formulae. In this section we will show that ultraproducts have this property.

We generalise the notation of Section B3.2 as follows. Let $(\mathfrak{A}^i)_{i \in I}$ be a sequence of Σ-structures. For every sort s, we set

$$I_s := \{ i \in I \mid A^i_s \neq \varnothing \} .$$

If $\varphi(\bar{x})$ is a formula and $a_k \in \prod_{i \in I_{s_k}} A^i_{s_k}$, for $k < n$, are parameters then we define

$$[\![\varphi(\bar{a})]\!] := \{ i \in I_{s_0} \cap \dots \cap I_{s_{n-1}} \mid \mathfrak{A}^i \vDash \varphi(\bar{a}^i) \} .$$

Recall that, for a filter $\mathfrak{u}$ on I, we write

$$\bar{a} \sim_{\mathfrak{u}} \bar{b} \quad :\text{iff} \quad [\![\bar{a} = \bar{b}]\!] \in \mathfrak{u} ,$$

and $[\bar{a}]$ denotes the $\sim_{\mathfrak{u}}$-class of $\bar{a}$.

Theorem 1.1 (Łoś). *Let $(\mathfrak{A}^i)_{i \in I}$ be a sequence of Σ-structures and $\mathfrak{u}$ an ultrafilter on I. For every first-order formula $\varphi(\bar{x})$ and all parameters $a_k \in \prod_{i \in I_{s_k}} A^i_{s_k}$ we have*

$$\prod_i \mathfrak{A}^i / \mathfrak{u} \vDash \varphi([\bar{a}]) \quad \text{iff} \quad [\![\varphi(\bar{a})]\!] \in \mathfrak{u} .$$

Proof. Let $\mathfrak{A} := \prod_i \mathfrak{A}^i$ and $\mathfrak{B} := \prod_i \mathfrak{A}^i/\mathfrak{u}$. We prove the claim by induction on φ. If $\varphi = s = t$ then we have

$$\begin{aligned} \mathfrak{B} \vDash (s = t)([\bar{a}]) \quad & \text{iff} \quad s^{\mathfrak{B}}([\bar{a}]) = t^{\mathfrak{B}}([\bar{b}]) \\ & \text{iff} \quad s^{\mathfrak{A}}(\bar{a}) \sim_{\mathfrak{u}} t^{\mathfrak{A}}(\bar{a}) \\ & \text{iff} \quad [\![s(\bar{a}) = t(\bar{a})]\!] \in \mathfrak{u}\,. \end{aligned}$$

Similarly, if $\varphi = Rt_0 \dots t_{m-1}$ then

$$\begin{aligned} \mathfrak{B} \vDash (R\bar{t})([\bar{a}]) \quad & \text{iff} \quad \langle t_0^{\mathfrak{B}}([\bar{a}]), \dots, t_{m-1}^{\mathfrak{B}}([\bar{a}]) \rangle \in R^{\mathfrak{B}} \\ & \text{iff} \quad [\![(R\bar{t})(\bar{a})]\!] \in \mathfrak{u}\,. \end{aligned}$$

For the boolean operators, we have, by inductive hypothesis,

$$\begin{aligned} \mathfrak{B} \vDash \neg\varphi([\bar{a}]) \quad & \text{iff} \quad \mathfrak{B} \nvDash \varphi([\bar{a}]) \\ & \text{iff} \quad [\![\varphi(\bar{a})]\!] \notin \mathfrak{u} \\ & \text{iff} \quad [\![\neg\varphi(\bar{a})]\!] \in \mathfrak{u} \end{aligned}$$

and
$$\begin{aligned} \mathfrak{B} \vDash (\varphi \wedge \psi)([\bar{a}]) \quad & \text{iff} \quad \mathfrak{B} \vDash \varphi([\bar{a}]) \text{ and } \mathfrak{B} \vDash \psi([\bar{a}]) \\ & \text{iff} \quad [\![\varphi(\bar{a})]\!] \in \mathfrak{u} \text{ and } [\![\psi(\bar{a})]\!] \in \mathfrak{u} \\ & \text{iff} \quad [\![\varphi(\bar{a})]\!] \cap [\![\psi(\bar{a})]\!] \in \mathfrak{u} \\ & \text{iff} \quad [\![\varphi(\bar{a}) \wedge \psi(\bar{a})]\!] \in \mathfrak{u}\,. \end{aligned}$$

It remains to consider the case that $\varphi = \exists y\psi$. Let s be the sort of y. We have

$$\begin{aligned} & \mathfrak{B} \vDash \exists y\psi([\bar{a}], y) \\ \text{iff} \quad & I_s \in \mathfrak{u} \text{ and there is some } b \in \textstyle\prod_{i \in I_s} A_s^i \text{ such that } \mathfrak{B} \vDash \psi([\bar{a}], [b]) \\ \text{iff} \quad & \text{there is some } b \in \textstyle\prod_{i \in I_s} A_s^i \text{ such that } [\![\psi(\bar{a}, b)]\!] \in \mathfrak{u} \\ \text{iff} \quad & [\![\exists y\psi(\bar{a}, y)]\!] \in \mathfrak{u}\,. \end{aligned}$$

For the last step note that, on the one hand, we have

$$[\![\psi(\bar{a}, b)]\!] \subseteq [\![\exists y\psi(\bar{a}, y)]\!]\,.$$

Conversely, we can fix, for every $i \in [\![\exists y \psi(\bar{a}, y)]\!]$, some $b^i \in A^i_s$ such that $\mathfrak{A}^i \vDash \psi(\bar{a}^i, b^i)$. For $i \in I_s \setminus [\![\exists y \psi(\bar{a}, y)]\!]$, we choose an arbitrary element $b^i \in A^i_s$. With these choices we have

$$[\![\exists y \psi(\bar{a}, y)]\!] \subseteq [\![\psi(\bar{a}, b)]\!] . \qquad \square$$

Corollary 1.2. $\mathfrak{A} \leq \mathfrak{A}^{\mathfrak{u}}$, *for all structures* $\mathfrak{A}$ *and every ultrafilter* $\mathfrak{u}$.

For the constructions below we frequently need a special kind of ultrafilter.

Definition 1.3. A filter $\mathfrak{u}$ on a set I is *regular* if there exists a sequence $(s_i)_{i \in I}$ of sets $s_i \in \mathfrak{u}$ such that, for every $k \in I$, the set $\{ i \mid k \in s_i \}$ is finite.

Lemma 1.4. *For every infinite set I, there exists a regular ultrafilter $\mathfrak{u}$ on I.*

Proof. Let $J := \{ s \subseteq I \mid |s| < \aleph_0 \}$. As I is infinite we have $|J| = |I|$ and there exists a bijection $f : J \to I$. Therefore, it is sufficient to construct a regular ultrafilter $\mathfrak{u}$ on J. Its image under f will be the desired regular ultrafilter on I.

For $i \in J$, set $s_i := \{ k \in J \mid i \subseteq k \}$. Since

$$s_i \cap s_j = \{ k \in J \mid i \cup j \subseteq k \} = s_{i \cup j}$$

it follows that $\mathfrak{v} := \{ s_i \mid i \in J \}$ has the finite intersection property. By Corollary B2.4.10, we can therefore find an ultrafilter $\mathfrak{u} \supseteq \mathfrak{v}$. Furthermore, $\mathfrak{u}$ is regular since, for every $k \in J$, the set

$$\{ i \in J \mid k \in s_i \} = \{ i \in J \mid i \subseteq k \}$$

is finite. $\square$

For ultrafilters over countable sets, regularity and non-principality coincide.

Lemma 1.5. *An ultrafilter $\mathfrak{u}$ over ω is regular if and only if it is non-principal.*

Proof. ($\Rightarrow$) Suppose that $\mathfrak{u}$ is principal, that is, $\mathfrak{u} = \Uparrow\{k\}$, for some k. If $(s_n)_{n<\omega}$ is a sequence of sets $s_n \in \mathfrak{u}$ then we have $k \in s_n$, for all n. Hence, $\mathfrak{u}$ cannot be regular.

($\Leftarrow$) Suppose that $\mathfrak{u}$ is non-principal. For $n < \omega$, set $s_n := \Uparrow n$. Then we have $s_n \in \mathfrak{u}$ since $\omega \smallsetminus s_n = [n] \notin \mathfrak{u}$. Furthermore, the set

$$\{ n < \omega \mid k \in s_n \} = \{ n < \omega \mid n \leq k \} = [k+1]$$

is finite, for every $k < \omega$. $\square$

We use regular ultrafilters for the following alternative proof of the compactness theorem.

Proposition 1.6. *A set $\Phi \subseteq \mathrm{FO}[\Sigma, X]$ is satisfiable if and only if every finite subset $\Phi_0 \subseteq \Phi$ is satisfiable.*

Proof. Suppose that every finite subset of Φ is satisfiable. By replacing each free variable in Φ by a constant symbol we may assume that every formula in Φ is a sentence. We have to construct a model of Φ.

Let $\mathfrak{u}$ be a regular ultrafilter on Φ and fix a sequence $(s_\varphi)_{\varphi\in\Phi}$ with $s_\varphi \in \mathfrak{U}$ such that the sets

$$\Psi_\psi := \{ \varphi \in \Phi \mid \psi \in s_\varphi \}, \quad \text{for } \psi \in \Phi,$$

are finite. By assumption we can find models $\mathfrak{A}^\psi \vDash \Psi_\psi$, for every $\psi \in \Phi$. We claim that

$$\prod_{\psi\in\Phi} \mathfrak{A}^\psi/\mathfrak{u} \vDash \Phi$$

is the desired model of Φ. Let $\varphi \in \Phi$. Then

$$[\![\varphi]\!] \supseteq \{ \psi \in \Phi \mid \varphi \in \Psi_\psi \} = \{ \psi \in \Phi \mid \psi \in s_\varphi \} = s_\varphi \in \mathfrak{u}.$$

By Łoś' theorem it follows that $\prod_s \mathfrak{A}^s/\mathfrak{u} \vDash \varphi$. $\square$

Lemma 1.7. *Let $\mathfrak{A}$ be a structure, κ an infinite cardinal, and $\mathfrak{u}$ a regular ultrafilter over a set I of size κ. If $\varphi(x)$ is a first-order formula such that $\varphi^{\mathfrak{A}}$ is infinite then*

$$\left|\varphi^{\mathfrak{A}^{\mathfrak{u}}}\right| = \left|\varphi^{\mathfrak{A}}\right|^{\kappa}.$$

Proof. By the Theorem of Łoś we have

$$\varphi^{\mathfrak{A}^{\mathfrak{u}}} = \{ [a] \in A^I/\mathfrak{u} \mid [\![\varphi(a)]\!] \in \mathfrak{u} \}.$$

Since $\varphi^{\mathfrak{A}} \neq \varnothing$, we can fix some element $c \in \varphi^{\mathfrak{A}}$. For every element $[a] \in \varphi^{\mathfrak{A}^{\mathfrak{u}}}$ with $s_a := [\![\varphi(a)]\!] \in \mathfrak{u}$, we define

$$a'_i := \begin{cases} a_i & \text{if } i \in s_a, \\ c & \text{otherwise}. \end{cases}$$

Note that we have $[a'] = [a]$ since $s_a \subseteq [\![a = a']\!] \in \mathfrak{u}$. Furthermore, $[\![\varphi(a')]\!] = I$. Consequently, we can define a function $f : \varphi^{\mathfrak{A}^{\mathfrak{u}}} \to (\varphi^{\mathfrak{A}})^I$ by mapping an element $[a] \in \varphi^{\mathfrak{A}^{\mathfrak{u}}}$ to some representative $a' \in [a]$ with $[\![\varphi(a')]\!] = I$. Note that f is injective since, for $[a] \neq [b]$, $f(a) \in [a]$ and $f(b) \in [b]$ implies that $f(a) \neq f(b)$. Therefore, we have $|\varphi^{\mathfrak{A}^{\mathfrak{u}}}| \leq |\varphi^{\mathfrak{A}}|^{\kappa}$.

It remains to prove that $|\varphi^{\mathfrak{A}^{\mathfrak{u}}}| \geq |\varphi^{\mathfrak{A}}|^{\kappa}$. Since $\mathfrak{u}$ is regular we can find sets $(s_i)_{i \in I}$ in $\mathfrak{u}$ such that the sets

$$w_k := \{ i \in I \mid k \in s_i \}$$

are finite. Since $\varphi^{\mathfrak{A}}$ is infinite we can fix bijections $\mu_k : (\varphi^{\mathfrak{A}})^{w_k} \to \varphi^{\mathfrak{A}}$, for $k \in I$. For $a \in (\varphi^{\mathfrak{A}})^I$, we define a sequence $a^{\mu} \in (\varphi^{\mathfrak{A}})^I$ by

$$a^{\mu}_i := \mu_i(a \restriction w_i), \quad \text{for } i \in I.$$

Then $[\![\varphi(a^{\mu})]\!] = I$ which implies, by the Theorem of Łoś, that $[a^{\mu}] \in \varphi^{\mathfrak{A}^{\mathfrak{u}}}$. To conclude the proof it is therefore sufficient to show that the mapping $a \mapsto [a^{\mu}]$ is injective. If $a \neq b$ then there is some index $i \in I$ with $a_i \neq b_i$. Hence, $a \restriction w_k \neq b \restriction w_k$, for every k with $i \in w_k$, that is, for every $k \in s_i$. Consequently, $s_i \subseteq [\![a^{\mu} \neq b^{\mu}]\!] \in \mathfrak{u}$. □

Corollary 1.8. *Let κ be an infinite cardinal. Every structure $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ such that, for every first-order formula $\varphi(\bar{x})$, either*

$$\left|\varphi^{\mathfrak{B}}\right| < \aleph_0 \quad \text{or} \quad \left|\varphi^{\mathfrak{B}}\right| = \left|\varphi^{\mathfrak{A}}\right|^{\kappa}.$$

Forming an ultraproduct of a sequence of structures corresponds to taking the limit of their theories in the type space.

Lemma 1.9. *Let $T \subseteq \mathrm{FO}$ and $X \subseteq S^{\bar{s}}(T)$ a set of $\bar{s}$-types. For every accumulation point $\mathfrak{p}$ of X, there exist an ultrafilter $\mathfrak{u}$ on I, a sequence of structures $(\mathfrak{A}_i)_{i\in I}$, and parameters $\bar{a}^i \subseteq A_i$, $i \in I$, with* $\mathrm{tp}(\bar{a}^i/\mathfrak{A}_i) \in X$ *such that*

$$\mathfrak{p} = \mathrm{tp}\big([(\bar{a}^i)_i] \,\big/\, \textstyle\prod_i \mathfrak{A}_i/\mathfrak{u}\big)\,.$$

Proof. Let $I := \mathfrak{p}$ and fix a regular ultrafilter $\mathfrak{u}$ over $\mathfrak{p}$. Then there exists a sequence $(s_\varphi)_{\varphi\in\mathfrak{p}}$ of sets $s_\varphi \in \mathfrak{u}$ such that, for every $i \in \mathfrak{p}$, the set $\Phi_i := \{\, \varphi \in \mathfrak{p} \mid i \in s_\varphi \,\}$ is finite. Since $\mathfrak{p}$ is an accumulation point of X we can find elements $\mathfrak{q}_i \in \langle \Phi_i \rangle \cap X \neq \varnothing$. Fix $\mathfrak{A}_i$ and $\bar{a}^i$ such that $\mathrm{tp}(\bar{a}^i/\mathfrak{A}_i) = \mathfrak{q}_i$, and set $\mathfrak{B} := \prod_{i\in I} \mathfrak{A}_i/\mathfrak{u}$ and $\bar{b} := [(\bar{a}^i)_i]$.

If $i \in s_\varphi$ then $\varphi \in \Phi_i$ which implies $\mathfrak{A}_i \vDash \varphi(\bar{a}^i)$. Therefore, we have $s_\varphi \subseteq [\![\varphi(\bar{a}^i)]\!] \in \mathfrak{u}$, for every $\varphi \in \mathfrak{p}$. By the Theorem of Łoś, it follows that $\mathfrak{B} \vDash \mathfrak{p}(\bar{b})$, that is, $\mathfrak{p} = \mathrm{tp}(\bar{b}/\mathfrak{B})$. ☐

2. *The theorem of Keisler and Shelah*

According to the Amalgamation Theorem any two elementary equivalent structures have a common elementary extension. In this section we prove the Theorem of Keisler and Shelah, which states that this extension can be taken as an ultrapower with respect to the same ultrafilter.

To construct such an ultrafilter $\mathfrak{u}$, we choose a sufficiently large cardinal λ. Starting with the trivial filter $\{\lambda\}$ on λ, we construct larger and larger filters until we have found the desired ultrafilter. In each step, we have to ensure that the filter we construct is general enough in the sense of being consistent with sufficiently many additional conditions. The precise definition are as follows.

Definition 2.1. Let λ be an infinite cardinal, $P \subseteq \wp(\lambda)$, and $C \subseteq \lambda$. Recall that $\mathrm{cl}_\uparrow(P)$ denotes the filter generated by P.

(a) P *forces* C if $C \in \mathrm{cl}_\uparrow(P)$.

(b) P is *consistent* with C if it does not force the complement $\lambda \setminus C$.

(c) P *decides* C if it forces C or $\lambda \setminus C$.

Remark. (a) Note that $\mathrm{cl}_\uparrow(P)$ is an ultrafilter if, and only if, for every set $C \subseteq \lambda$, P forces exactly one of C and $\lambda \setminus C$.

(b) P is not consistent with C if, and only if, there is a finite subset $P_0 \subseteq P$ such that $\bigcap P_0 \cap C = \varnothing$. Hence, P is consistent with C if, and only if, $P \cup \{C\}$ does have the finite intersection property.

Definition 2.2. Let λ be an infinite cardinal and let μ be the least cardinal such that $2^\mu > \lambda$.

(a) We denote by $(<\mu)^\lambda$ the set of all functions $\lambda \to \kappa$ for a cardinal $\kappa < \mu$.

(b) Let $m < \omega$ and $\gamma < \mu$ be ordinals, let $\bar{f} = (f_i)_{i<\gamma}$, $\bar{f}' = (f'_i)_{i<m}$ and $\bar{g} = (g_i)_{i<m}$ be sequences of functions $f_i, f'_i, g_i : \lambda \to \mu$, and let $\bar{\beta} = (\beta_i)_{i<\gamma}$ be a sequence of ordinals $\beta_i < \mu$. A *condition* is a set of the form

$$[\![\bar{f} = \bar{\beta},\ \bar{f}' = \bar{g}]\!] := \{\, \alpha < \lambda \mid f_i(\alpha) = \beta_i\,,\ \text{for all } i < \gamma\,,\ \text{and} \\ f'_i(\alpha) = g_i(\alpha)\,,\ \text{for all } i < m \,\}\,.$$

For $m = 0$, we simply write $[\![\bar{f} = \bar{\beta}]\!]$ instead of $[\![\bar{f} = \bar{\beta},\ \langle\rangle = \langle\rangle]\!]$.

(c) Let $F \subseteq \mu^\lambda$ and $G \subseteq (<\mu)^\lambda$. An (F,G)-*condition* is a condition $[\![\bar{f} = \bar{\beta},\ \bar{f}' = \bar{g}]\!]$ with $\bar{f}, \bar{f}' \subseteq F$ and $\bar{g} \subseteq G$. A set $P \subseteq \wp(\lambda)$ is (F,G)-*consistent* if it is consistent with every (F,G)-condition. For $G = \varnothing$, we simply speak of F-*conditions* and F-*consistency.*

Exercise 2.1. Let $P \subseteq \wp(\lambda)$ be F-consistent. Prove that every function $f \in F$ is surjective.

Exercise 2.2. Let $P \subseteq \wp(\lambda)$ be F-consistent. Show that there is no set $C \subseteq \lambda$ such that P forces both C and $\lambda \setminus C$.

Example. Let $\lambda = \aleph_0$ and let P be the set of all cofinite subsets of λ. Then $\mu = \aleph_0$ and a condition $C = [\![\bar{f} = \bar{\beta},\ \bar{f}' = \bar{g}]\!]$ is consistent with P if, and only if, C is infinite. It follows that P is F-consistent, where F is the set of all functions $f : \aleph_0 \to \aleph_0$ such that $f^{-1}(n)$ is infinite, for every $n < \aleph_0$.

Lemma 2.3. *Let $F \subseteq \mu^\lambda$ and $G \subseteq (<\mu)^\lambda$.*

(a) *If A and B are (F, G)-conditions, then $A \cap B$ is also an (F, G)-condition.*

(b) *If $(A_i)_{i<\gamma}$ is a sequence of F-conditions of length $\gamma < \mu$, then the intersection $\bigcap_{i<\gamma} A_i$ is also an F-condition.*

Proof. (a) Suppose that

$$A = [\![\bar{f}_0 = \bar{\beta}_0,\ \bar{f}'_0 = \bar{g}_0]\!] \quad \text{and} \quad B = [\![\bar{f}_1 = \bar{\beta}_1,\ \bar{f}'_1 = \bar{g}_1]\!] .$$

Then $A \cap B = [\![\bar{f}_0\bar{f}_1 = \bar{\beta}_0\bar{\beta}_1,\ \bar{f}'_0\bar{f}'_1 = \bar{g}_0\bar{g}_1]\!]$.

(b) follows as in (a) since F-conditions are closed under concatenations of length $\gamma < \mu$. ☐

Lemma 2.4. *Let I be a directed set and, for $i \in I$, let $P_i \subseteq \wp(\lambda)$, $F_i \subseteq \mu^\lambda$, and $G_i \subseteq (<\mu)^\lambda$ be sets such that $i \leq k$ implies $P_i \subseteq P_k$, $F_i \supseteq F_k$, and $G_i \subseteq G_k$. If P_i is (F_i, G_i)-consistent, for every $i \in I$, then $\bigcup_{i \in I} P_i$ is $(\bigcap_{i\in I} F_i, \bigcup_{i \in I} G_i)$-consistent.*

Proof. Let $C = [\![\bar{f} = \bar{\beta},\ \bar{f}' = \bar{g}]\!]$ be a $(\bigcap_i F_i, \bigcup_i G_i)$-condition. For a contradiction, suppose that $\bigcup_i P_i$ forces $\lambda \setminus C$. Then there exists a finite subset $Q \subseteq \bigcup_i P_i$ such that $\bigcap Q \cap C = \emptyset$. As I is directed, we can fix an index $k \in I$ such that $Q \subseteq P_k$.

Since $\bar{g}$ is a finite tuple, there exists an index $l \in I$ such that $\bar{g} \subseteq G_l$. Consequently, C is an (F_i, G_i)-condition, for all $i \geq l$. Fix $i \in I$ with $i \geq k, l$. Since $Q \subseteq P_i$, it follows that P_i forces $\lambda \setminus C$. Hence, P_i is not (F_i, G_i)-consistent. A contradiction. ☐

In the following sequence of lemmas, we will construct larger and larger sets $P \subseteq \wp(\lambda)$ that are (F,G)-consistent, for sufficiently large sets F and G, until we obtain a set P that decides every subset of λ.

Lemma 2.5. *There exists a set $F \subseteq \mu^\lambda$ of size $|F| = 2^\lambda$ such that $\{\lambda\}$ is F-consistent.*

Proof. Let H be the set of all pairs $\langle A, h\rangle$ such that $A \subseteq \lambda$ is a set of size $|A| < \mu$ and $h : S \to \mu$ is a function with domain $S \subseteq \wp(A)$ of size $|S| < \mu$.

Let us first show that $|H| = \lambda$. There are $\lambda^{<\mu} = \lambda$ sets $A \subseteq \lambda$ of size $|A| < \mu$. For each such A, the number of sets $S \subseteq \wp(A)$ of size $|S| < \mu$ is at most

$$(2^{|A|})^{<\mu} \leq (\lambda^{|A|})^{<\mu} = \lambda^{<\mu} = \lambda\,.$$

For each set S, there are $\mu^{|S|} \leq \lambda^{|S|} \leq \lambda^{<\mu} = \lambda$ functions $S \to \mu$. Therefore, $|H| \leq \lambda \otimes \lambda \otimes \lambda = \lambda$. As it is easy to find λ different elements of H, it follows that $|H| = \lambda$.

Fix an enumeration $\langle A_\alpha, h_\alpha\rangle_{\alpha<\lambda}$ of H. For $C \subseteq \lambda$, we define a function $f_C : \lambda \to \mu$ by

$$f_C(\alpha) := \begin{cases} h_\alpha(C \cap A_\alpha) & \text{if } C \cap A_\alpha \in \operatorname{dom} h_\alpha\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We claim that $F := \{\, f_C \mid C \subseteq \lambda \,\}$ has the desired properties.

To show that $\{\lambda\}$ is F-consistent, consider an F-condition $[\![\bar f = \bar\beta]\!]$ where the sequences $\bar f$ and $\bar\beta$ have length $\gamma < \mu$. Since λ is the only set forced by $\{\lambda\}$, it is sufficient to show that $[\![\bar f = \bar\beta]\!] \neq \emptyset$.

Let $C_i \subseteq \lambda$ be the set such that $f_i = f_{C_i}$, for $i < \gamma$. W.l.o.g. we may assume that $f_i \neq f_k$, for $i \neq k$. Then $C_i \neq C_k$, for $i \neq k$. Hence, there is a set $A \subseteq \lambda$ of size $|A| = |\gamma|$ such that $i \neq k$ implies $C_i \cap A \neq C_k \cap A$.

Set $S := \{\, C_i \cap A \mid i < \gamma \,\}$ and define $h : S \to \mu$ by

$$h(C_i \cap A) := \beta_i\,.$$

Then $\langle A, h\rangle \in H$. Hence, there is some $\alpha < \lambda$ such that $\langle A, h\rangle = \langle A_\alpha, h_\alpha\rangle$. For each $i < \gamma$, it follows that

$$f_i(\alpha) = f_{C_i}(\alpha) = h_\alpha(C_i \cap A_\alpha) = h(C_i \cap A) = \beta_i\,.$$

Therefore, $\alpha \in [\![\bar{f} = \bar{\beta}]\!] \neq \varnothing$. □

Lemma 2.6. *Suppose that $P \subseteq \wp(\lambda)$ is F-consistent. For every set $G \subseteq ({<}\mu)^\lambda$, there exists a set $F_o \subseteq F$ of size $|F_o| \leq |G| \otimes |P| \otimes \mu$ such that P is $(F \setminus F_o, G)$-consistent.*

Proof. We shall prove that, for every finite set $G_o \subseteq G$, there is some set $F(G_o) \subseteq F$ of size $|F(G_o)| \leq |P| \oplus \mu$ such that P is $(F \setminus F(G_o), G_o)$-consistent. By Lemma 2.4, it then follows that P is $(F \setminus F_o, G)$-consistent, where

$$F_o := \bigcup \{\, F(G_o) \mid G_o \subseteq G \text{ finite} \,\}$$

has size $|F_o| \leq |G| \otimes \aleph_0 \otimes |P| \otimes \mu$.

Fix a finite tuple $\bar{g} \in G^m$, $m < \omega$. By induction on α, we define a sequence of tuples $\bar{f}'_\alpha \in F^m$ as follows. Suppose we have already defined $\bar{f}'_i$, for $i < \alpha$. Set $F_\alpha := \bigcup_{i<\alpha} \bar{f}'_i$. If P is $(F \setminus F_\alpha, \bar{g})$-consistent, we stop. Otherwise, there is some $(F \setminus F_\alpha, \bar{g})$-condition $[\![\bar{f} = \bar{\beta},\ \bar{f}' = \bar{g}]\!]$ that is not consistent with P. We set $\bar{f}'_\alpha := \bar{f}'$.

Let $(\bar{f}'_\alpha)_{\alpha<\gamma}$ be the sequence constructed in this way. Obviously, we have $\gamma < |F|^+$. If $\gamma < \kappa := (|P| \oplus \mu)^+$, we can obtain the desired set as $F(\bar{g}) := \bigcup_{\alpha<\gamma} \bar{f}'_\alpha$.

Hence, assume that $\gamma \geq \kappa$. We will derive a contradiction as follows. For each $\alpha < \kappa$, fix a $(F \setminus F_\alpha, \bar{g})$-condition

$$A_\alpha := [\![\bar{f}_\alpha = \bar{\beta}_\alpha,\ \bar{f}'_\alpha = \bar{g}]\!]$$

such that P forces $\lambda \setminus A_\alpha$. Let P^+ be the closure of P under finite intersections. There are sets $S_\alpha \in P^+$ such that $S_\alpha \cap A_\alpha = \varnothing$. Since $|P^+| \leq |P| \oplus \aleph_0 < \kappa$, we can find a set $I \subseteq \kappa$ of size $|I| = \kappa$ such that

$$S_\alpha = S_{\alpha'}\,, \quad \text{for all } \alpha, \alpha' \in I\,.$$

Let S be the set such that $S = S_\alpha$, for $\alpha \in I$. Since each sequence $\bar{f}_\alpha$ has length less than $\mu < \kappa$, there is a subset $J \subseteq I$ of size $|J| = \kappa$ such that $|\bar{f}_\alpha| = |\bar{f}_{\alpha'}|$, for all $\alpha, \alpha' \in J$.

Set

$$\chi := \sup \{ |g_i(\alpha)|^+ \mid i < m,\ \alpha < \lambda \}$$

and let $(\bar{\gamma}_\alpha)_{\alpha<\chi}$ be an enumeration of χ^m. Note that $\chi < \mu$ since $\operatorname{rng} g_i \subseteq \nu_i$, for some $\nu_i < \mu$. Hence,

$$g_i(\alpha) < \nu_i < \mu \quad \text{implies} \quad |g_i(\alpha)|^+ \leq \nu_i < \mu\,.$$

Fix an injective function $h : \chi \to J$ and set

$$A := \bigcap_{i<\chi} [\![\bar{f}_{h(i)} = \bar{\beta}_{h(i)},\ \bar{f}'_{h(i)} = \bar{\gamma}_i]\!]\,.$$

Since $\chi < \mu$ it follows by Lemma 2.3 (b) that A is an F-condition. Hence, the F-consistency of P implies that P does not force $\lambda \setminus A$.

Consequently, $S \cap A \neq \varnothing$ and we can find some $\alpha \in S \cap A$. It follows that

$$\bar{f}_{h(i)}(\alpha) = \bar{\beta}_{h(i)}(\alpha) \quad \text{and} \quad \bar{f}'_{h(i)}(\alpha) = \bar{\gamma}_i(\alpha)\,, \quad \text{for all } i < \chi\,.$$

Fix $i < \chi$ such that $\bar{\gamma}_i = \bar{g}(\alpha)$. Then $\alpha \in A_{h(i)}$. Hence, $S_{h(i)} \cap A_{h(i)} = \varnothing$ implies that $\alpha \notin S_{h(i)} = S$. A contradiction. □

To extend the set P to an ultrafilter, we can use the following lemma and its corollary to ensure that P decides every set.

Lemma 2.7. *Let $P \subseteq \wp(\lambda)$ be (F,G)-consistent. For every set $A \subseteq \lambda$ there is some $F_0 \subseteq F$ of size $|F_0| < \mu$ such that at least one of $P \cup \{A\}$ and $P \cup \{\lambda \setminus A\}$ is $(F \setminus F_0, G)$-consistent.*

Proof. Suppose that $P \cup \{A\}$ is not (F,G)-consistent. Then there is an (F,G)-condition $C_0 := [\![\bar{f}_0 = \bar{\beta}_0,\ \bar{f}'_0 = \bar{g}_0]\!]$ such that $P \cup \{A\}$ forces $\lambda \setminus C_0$. Hence, there is some $S_0 \in \mathrm{cl}_\uparrow(P)$ such that

$$S_0 \cap A \cap C_0 = \varnothing\,.$$

Set $F_0 := \bar{f}_0 \cup \bar{f}'_0$. If $P \cup \{\lambda \smallsetminus A\}$ is $(F \smallsetminus F_0, G)$-consistent, we are done.

Hence, we may assume that this set is not $(F \smallsetminus F_0, G)$-consistent. Then there is an $(F \smallsetminus F_0, G)$-condition $C_1 := [\![\bar{f}_1 = \bar{\beta}_1,\ \bar{f}'_1 = \bar{g}_1]\!]$ such that $P \cup \{\lambda \smallsetminus A\}$ forces $\lambda \smallsetminus C_1$. Hence, there is some set $S_1 \in \mathrm{cl}_\uparrow(P)$ such that

$$S_1 \cap (\lambda \smallsetminus A) \cap C_1 = \varnothing\,.$$

It follows that $S_1 \cap C_1 \subseteq A$, which implies that

$$S_0 \cap S_1 \cap C_0 \cap C_1 \subseteq S_0 \cap C_0 \cap A = \varnothing\,.$$

As $S_0 \cap S_1 \in \mathrm{cl}_\uparrow(P)$, it follows that P forces $\lambda \smallsetminus (C_0 \cap C_1)$. Since $C_0 \cap C_1$ is an (F, G)-condition, P is not (F, G)-consistent. A contradiction. $\square$

Repeating this lemma for each set $A \in H$, we obtain the following statement.

Corollary 2.8. *Let $P \subseteq \wp(\lambda)$ be (F, G)-consistent. For every set $H \subseteq \wp(\lambda)$ there is some $F_0 \subseteq F$ of size $|F_0| \leq |H| \otimes \mu$ and some $Q \subseteq \wp(\lambda)$ of size $|Q| = |H|$ such that $P \cup Q$ is $(F \smallsetminus F_0, G)$-consistent and it decides every set $A \in H$.*

To prove the Theorem of Keisler and Shelah below, we will have to show that $\mathfrak{A}^{\mathfrak{u}} \cong \mathfrak{B}^{\mathfrak{u}}$, for certain structures $\mathfrak{A}$ and $\mathfrak{B}$. This is done via a back-and-forth argument where we construct an increasing chain of partial isomorphisms between the structures $\mathfrak{A}^{\mathfrak{u}}$ and $\mathfrak{B}^{\mathfrak{u}}$. Matters become slightly more complicated since we construct the ultrafilter $\mathfrak{u}$ at the same time. Hence, we do not yet know between which structures we should eventually construct partial isomorphisms. Therefore, we introduce a notion of a partial isomorphism between partially defined ultrapowers.

Definition 2.9. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures and let $P \subseteq \wp(\lambda)$ be a set with the finite intersection property. A partial function π from A^λ to B^λ is a *partial isomorphism modulo P* if, for every formula $\varphi(\bar{x}) \in \mathrm{FO}^{<\omega}[\Sigma]$ and every finite mapping $\bar{a} \mapsto \bar{b} \subseteq \pi$,

$$P \quad \text{forces} \quad \bigl\{\, k < \lambda \bigm| \mathfrak{A} \vDash \varphi(\bar{a}(k)) \Leftrightarrow \mathfrak{B} \vDash \varphi(\bar{b}(k)) \,\bigr\}\,,$$

and $\quad P \quad$ decides $\quad [\![\mathfrak{A} \vDash \varphi(\bar{a}(k))]\!]_{k<\lambda}$.

Exercise 2.3. Show that every partial isomorphism π from A^λ to B^λ modulo an ultrafilter $\mathfrak{u}$ induces an ordinary partial isomorphism between $\mathfrak{A}^\mathfrak{u}$ and $\mathfrak{B}^\mathfrak{u}$.

The back-and-forth step of the construction below is contained in the following two lemmas. The first one is a technical result which, intuitively, states that we can realise every partial type.

Lemma 2.10. *Let P be F-consistent, let $\mathfrak{M}$ be a Σ-structure of size $\kappa := |M| < \mu$, and let $\Phi \subseteq \mathrm{FO}^1[\Sigma_{M^\lambda}]$ be a set of first-order formulae over M^λ that is closed under conjunctions.*

If, for every $\varphi(x;\bar{a}) \in \Phi$,

$$P \quad \textit{forces} \quad [\![\mathfrak{M} \vDash \exists x \varphi(x;\bar{a}(\alpha))]\!]_{\alpha<\lambda},$$

there exist a sequence $b \in M^\lambda$ and sets $F_0 \subseteq F$ and $Q \subseteq \wp(\lambda)$ of size

$$|F_0| \leq |P| \oplus |\Phi| \oplus \mu \quad \textit{and} \quad |Q| \leq |\Phi|$$

such that $P \cup Q$ is $(F \setminus F_0)$-consistent and, for every $\varphi(x;\bar{a}) \in \Phi$,

$$P \cup Q \quad \textit{forces} \quad [\![\mathfrak{M} \vDash \varphi(b(\alpha);\bar{a}(\alpha))]\!]_{\alpha<\lambda}.$$

Proof. Fix enumerations $(c_i)_{i<\kappa}$ of M and $(\varphi_l(x;\bar{a}_l))_{l<\chi}$ of Φ. For each $l < \chi$, we fix a function $g_l : \lambda \to \kappa$ such that

$$\mathfrak{M} \vDash \exists x \varphi_l(x, \bar{a}_l(\alpha)) \quad \text{implies} \quad \mathfrak{M} \vDash \varphi_l(c_{g_l(\alpha)}, \bar{a}_l(\alpha)).$$

Set $G := \{ g_l \mid l < \chi \}$. By Lemma 2.6, there is a set $F_1 \subseteq F$ of size $|F_1| \leq |P| \oplus \chi \oplus \mu$ such that P is $(F \setminus F_1, G)$-consistent. Fix some $f \in F \setminus F_1$ and set

$$\begin{aligned} F_0 &:= F_1 \cup \{f\}, \\ b(\alpha) &:= \begin{cases} c_{f(\alpha)} & \text{if } f(\alpha) < \kappa, \\ c_0 & \text{otherwise}, \end{cases} \\ Q &:= \big\{ [\![\mathfrak{M} \vDash \varphi_l(b(\alpha), \bar{a}_l(\alpha))]\!]_{\alpha<\lambda} \bigm| l < \chi \big\}. \end{aligned}$$

We claim that F_o, Q, and b have the desired properties.

Since

$$[\![\mathfrak{M} \vDash \varphi_l(b(\alpha), \bar{a}_l(\alpha))]\!]_{\alpha<\lambda} \in Q \subseteq \mathrm{cl}_\uparrow(P \cup Q)\,, \quad \text{for all } l < \chi\,,$$

it remains to show that $P \cup Q$ is $(F \setminus F_o)$-consistent. For a contradiction, suppose otherwise. Then we can find an $(F \setminus F_o)$-condition $C := [\![\bar{f} = \bar{\beta}]\!]$ such that $P \cup Q$ forces $\lambda \setminus C$. Since Φ is closed under conjunctions, the set Q is closed under finite intersections. Therefore, there are sets $S \in \mathrm{cl}_\uparrow(P)$ and $T \in Q$ such that

$$S \cap T \cap C = \varnothing\,.$$

Let $l < \chi$ be the index such that $T = [\![\mathfrak{M} \vDash \varphi_l(b(\alpha), \bar{a}_l(\alpha))]\!]_{\alpha<\lambda}$. Then

$$S \cap [\![\mathfrak{M} \vDash \varphi_l(b(\alpha), \bar{a}_l(\alpha))]\!]_{\alpha<\lambda} \cap [\![\bar{f} = \bar{\beta}]\!] = \varnothing\,.$$

By choice of g_l, we have

$$\begin{aligned} &[\![\mathfrak{M} \vDash \exists x \varphi_l(x, \bar{a}_l(\alpha))]\!]_{\alpha<\lambda} \cap [\![\langle\rangle = \langle\rangle,\ f = g_l]\!] \\ \subseteq\ &[\![\mathfrak{M} \vDash \varphi_l(b(\alpha), \bar{a}_l(\alpha))]\!]_{\alpha<\lambda}\,. \end{aligned}$$

Therefore, it follows that

$$S \cap [\![\mathfrak{M} \vDash \exists x \varphi_l(x, \bar{a}_l(\alpha))]\!]_{\alpha<\lambda} \cap [\![\bar{f} = \bar{\beta},\ f = g_l]\!] = \varnothing\,.$$

Hence, P forces $[\![\bar{f} = \bar{\beta},\ f = g_l]\!]$ in contradiction to the $(F \setminus F_1, G)$-consistency of P. $\square$

Lemma 2.11. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures with $|\Sigma| \leq \lambda$, $P \subseteq \wp(\lambda)$ a set that is F-consistent, and π a partial isomorphism from A^λ to B^λ modulo P. For every element $c \in A^\lambda$, there exist an element $d \in B^\lambda$ and sets $Q \subseteq \wp(\lambda)$ and $F_o \subseteq F$ of size*

$$|Q| \leq |\pi| \oplus \lambda \quad \textit{and} \quad |F_o| \leq |P| \oplus |\pi| \oplus \lambda$$

such that $P \cup Q$ is $(F \setminus F_o)$-consistent and $\pi \cup \{\langle c, d\rangle\}$ is a partial isomorphism modulo $P \cup Q$.

Proof. Note that there are $|\pi|^{<\omega} = |\pi| \oplus \aleph_0$ finite tuples $\bar{a} \subseteq \mathrm{dom}(\pi)$ and there are at most λ formulae $\varphi \in \mathrm{FO}^{<\omega}[\Sigma]$. Hence, we can use Corollary 2.8 to find sets Q_1 and $F_1 \subseteq F$ of size

$$|Q_1| \leq \lambda \oplus |\pi| \quad \text{and} \quad |F_1| \leq \lambda \oplus |\pi| \oplus \mu = \lambda \oplus |\pi|$$

such that $P \cup Q_1$ is $(F \setminus F_1)$-consistent and

$$P \cup Q_1 \quad \text{decides} \quad [\![\mathfrak{A} \vDash \varphi(\bar{a}(\alpha), c(\alpha))]\!]_{\alpha<\lambda}\,,$$

for all $\varphi(\bar{x}, y) \in \mathrm{FO}^{<\omega}[\Sigma]$ and all finite $\bar{a} \subseteq \mathrm{dom}(\pi)$.

Suppose that $\pi = \bar{a} \mapsto \bar{b}$ and set

$$\Phi := \big\{\, \varphi(\bar{x}, y) \;\big|\; P \cup Q_1 \text{ forces } [\![\mathfrak{A} \vDash \varphi(\bar{a}(\alpha), c(\alpha))]\!]_{\alpha<\lambda} \,\big\}\,.$$

Note that $\varphi \notin \Phi$ implies $\neg\varphi \in \Phi$, by construction of Q_1. Since π is a partial isomorphism modulo P, it follows for $\varphi \in \Phi$ that

$$\begin{aligned}
&[\![\mathfrak{B} \vDash \exists y \varphi(\bar{b}(\alpha), y)]\!]_{\alpha<\lambda} \\
\supseteq\; &[\![\mathfrak{A} \vDash \exists y \varphi(\bar{a}(\alpha), y)]\!]_{\alpha<\lambda} \\
&\cap \big\{\, \alpha < \lambda \;\big|\; \mathfrak{A} \vDash \exists y \varphi(\bar{a}(\alpha), y) \Leftrightarrow \mathfrak{B} \vDash \exists y \varphi(\bar{b}(\alpha), y) \,\big\} \\
\supseteq\; &[\![\mathfrak{A} \vDash \varphi(\bar{a}(\alpha), c(\alpha))]\!]_{\alpha<\lambda} \\
&\cap \big\{\, \alpha < \lambda \;\big|\; \mathfrak{A} \vDash \exists y \varphi(\bar{a}(\alpha), y) \Leftrightarrow \mathfrak{B} \vDash \exists y \varphi(\bar{b}(\alpha), y) \,\big\} \\
\in\; &\mathrm{cl}_{\uparrow}(P \cup Q_1)\,.
\end{aligned}$$

Hence, $P \cup Q_1$ forces $[\![\mathfrak{B} \vDash \exists y \varphi(\bar{b}(\alpha), y)]\!]_{\alpha<\lambda}$, for all $\varphi \in \Phi$, and we can use Lemma 2.10 to find an element $d \in B^\lambda$ and sets Q_2 and $F_2 \subseteq F \setminus F_1$ of size

$$|Q_2| \leq |\Phi| = |\Sigma| \oplus |\pi| \oplus \aleph_0 = \lambda \oplus |\pi|$$

$$\text{and} \quad |F_2| \leq |P \cup Q_1| \oplus |\Phi| \oplus \mu = |P| \oplus |\pi| \oplus \lambda$$

such that $P \cup Q_1 \cup Q_2$ is $(F \setminus (F_1 \cup F_2))$-consistent and

$$P \cup Q_1 \cup Q_2 \quad \text{forces} \quad [\![\mathfrak{B} \vDash \varphi(\bar{b}(\alpha), d(\alpha))]\!]_{\alpha<\lambda}\,,$$

for all $\varphi \in \Phi$.

We claim that the extension $\pi \cup \{\langle c, d\rangle\}$ is a partial isomorphism modulo $P \cup Q_1 \cup Q_2$. We have already seen above that $P \cup Q_1$, and hence also $P \cup Q_1 \cup Q_2$, decides every set of the form $[\![\mathfrak{A} \vDash \varphi(\bar{a}(\alpha), c(\alpha))]\!]_{\alpha<\lambda}$ with $\bar{a} \subseteq \operatorname{dom}(\pi)$. To check the remaining condition, we distinguish two cases.

If $\varphi \in \Phi$, the fact that

$$[\![\mathfrak{A} \vDash \varphi(\bar{a}(\alpha), c(\alpha))]\!]_{\alpha<\lambda} \quad \text{and} \quad [\![\mathfrak{B} \vDash \varphi(\bar{b}(\alpha), d(\alpha))]\!]_{\alpha<\lambda}$$

are in $\mathrm{cl}_{\uparrow}(P \cup Q_1 \cup Q_2)$ implies that

$$\begin{aligned} &\big\{ \alpha < \lambda \bigm| \mathfrak{A} \vDash \varphi(\bar{a}(\alpha), c(\alpha)) \Leftrightarrow \mathfrak{B} \vDash \varphi(\bar{b}(\alpha), d(\alpha)) \big\} \\ \supseteq &\big\{ \alpha < \lambda \bigm| \mathfrak{A} \vDash \varphi(\bar{a}(\alpha), c(\alpha)) \text{ and } \mathfrak{B} \vDash \varphi(\bar{b}(\alpha), d(\alpha)) \big\} \\ = &[\![\mathfrak{A} \vDash \varphi(\bar{a}(\alpha), c(\alpha))]\!]_{\alpha<\lambda} \cap [\![\mathfrak{B} \vDash \varphi(\bar{b}(\alpha), d(\alpha))]\!]_{\alpha<\lambda} \\ \in\ &\mathrm{cl}_{\uparrow}(P \cup Q_1 \cup Q_2)\,. \end{aligned}$$

If $\varphi \notin \Phi$, we have noted above that $\neg\varphi \in \Phi$. Therefore,

$$\begin{aligned} &\big\{ \alpha < \lambda \bigm| \mathfrak{A} \vDash \varphi(\bar{a}(\alpha), c(\alpha)) \Leftrightarrow \mathfrak{B} \vDash \varphi(\bar{b}(\alpha), d(\alpha)) \big\} \\ = &\big\{ \alpha < \lambda \bigm| \mathfrak{A} \vDash \neg\varphi(\bar{a}(\alpha), c(\alpha)) \Leftrightarrow \mathfrak{B} \vDash \neg\varphi(\bar{b}(\alpha), d(\alpha)) \big\} \\ \in\ &\mathrm{cl}_{\uparrow}(P \cup Q_1 \cup Q_2)\,. \end{aligned}$$

Consequently, $P \cup Q_1 \cup Q_2$ forces

$$\big\{ \alpha < \lambda \bigm| \mathfrak{A} \vDash \neg\varphi(\bar{a}(\alpha), c(\alpha)) \Leftrightarrow \mathfrak{B} \vDash \neg\varphi(\bar{b}(\alpha), d(\alpha)) \big\}$$

for all formulae φ. $\square$

Theorem 2.12 (Keisler, Shelah). *Let λ be an infinite cardinal and let μ be the least cardinal such that $2^\mu > \lambda$. There exists an ultrafilter $\mathfrak{u}$ on λ such that*

$$\mathfrak{A} \equiv \mathfrak{B} \quad \textit{implies} \quad \mathfrak{A}^{\mathfrak{u}} \cong \mathfrak{B}^{\mathfrak{u}}\,,$$

for all structures $\mathfrak{A}$ and $\mathfrak{B}$ of size $|A|, |B| < \mu$.

Proof. Note that every Σ-structure $\mathfrak{M}$ of size $\kappa := |M| < \mu$ is interdefinable with a reduct $\mathfrak{M}|_{\Sigma_0}$ for some $\Sigma_0 \subseteq \Sigma$ of size $|\Sigma_0| \leq 2^\kappa \leq \lambda$ since there are only 2^κ distinct relations and functions on M. We may therefore w.l.o.g. assume that the signature Σ of every structure is contained in a fixed signature Σ_+ of size λ consisting, for all finite sequences $\bar{s}t$ of sorts, of λ relation symbols of type $\bar{s}$ and λ function symbols of type $\bar{s} \to t$. Furthermore, we may assume that all structures have universe κ, for some cardinal $\kappa < \mu$. Note that, by Lemma B1.1.5, there are, up to isomorphism, at most $2^{|\Sigma| \oplus \kappa} \leq 2^\lambda$ such Σ-structures.

Therefore, we can fix an enumeration $\langle \mathfrak{A}_i, \mathfrak{B}_i \rangle_{i<2^\lambda}$ of all pairs of Σ_i-structures with $\Sigma_i \subseteq \Sigma_+$ where the universe is some cardinal less than μ and such that $\mathfrak{A}_i \equiv \mathfrak{B}_i$. We also fix a surjective function

$$R : 2^\lambda \to [3] \times 2^\lambda \times 2^\lambda$$

and enumerations $(u_\alpha)_{\alpha<2^\lambda}$ of μ^λ and $(S_\alpha)_{\alpha<2^\lambda}$ of $\mathscr{P}(\lambda)$.

We will construct an ultrafilter $\mathfrak{u}$ such that $\mathfrak{A}_i^{\mathfrak{u}} \cong \mathfrak{B}_i^{\mathfrak{u}}$, for all i. By induction on $\gamma < 2^\lambda$, we construct

- an increasing sequence $(P_\gamma)_{\gamma<2^\lambda}$ of sets $P_\gamma \subseteq \mathscr{P}(\lambda)$,
- a decreasing sequence $(F_\gamma)_{\gamma<2^\alpha}$ of sets $F_\gamma \subseteq \mu^\lambda$, and
- for each $i < 2^\lambda$, an increasing sequence $(\pi^i_\gamma)_{\gamma<2^\lambda}$ of partial functions π^i_γ from $A_i^\lambda \subseteq (<\mu)^\lambda$ to $B_i^\lambda \subseteq (<\mu)^\lambda$

satisfying the following conditions:

(1) P_γ is F_γ-consistent;

(2) each π^i_γ is a partial isomorphism from A_i^λ to B_i^λ modulo P_γ;

(3) $$\begin{aligned} \Big|\bigcup_{i<2^\lambda} \operatorname{dom}(\pi^i_\gamma)\Big| &\leq |\gamma|, \\ |P_\gamma| &\leq \lambda \oplus |\gamma|, \\ |F_0| &= 2^\lambda, \\ |F_0 \smallsetminus F_\gamma| &\leq \lambda \oplus |\gamma|; \end{aligned}$$

(4) if $R(\gamma) = \langle 0, i, \alpha \rangle$ and $u_\alpha \in A_i^\lambda$, then $u_\alpha \in \operatorname{dom}(\pi^i_{\gamma+1})$;

(5) if $R(\gamma) = \langle 1, i, \beta \rangle$ and $u_\beta \in B_i^\lambda$, then $u_\beta \in \mathrm{rng}(\pi_{\gamma+1}^i)$;

(6) if $R(\gamma) = \langle 2, \alpha, \beta \rangle$, then $P_{\gamma+1}$ decides S_α.

First, let us show that, after having performed this construction, the limit $\mathfrak{u} := \bigcup_{\gamma<2^\lambda} P_\gamma$ is the desired ultrafilter. By (6) and the surjectivity of R, $\mathfrak{u}$ is an ultrafilter. Furthermore, by (2) $\pi^i := \bigcup_{\gamma<2^\lambda} \pi_\gamma^i$ is a partial isomorphism between $\mathfrak{A}_i^\mathfrak{u}$ and $\mathfrak{B}_i^\mathfrak{u}$. Finally, by (4), (5), and the surjectivity of R, π^i is bijective.

Hence, it remains to do the induction. We start with $P_0 := \{\lambda\}$ and $\pi_0^i := \langle\rangle \mapsto \langle\rangle$, for all $i < 2^\lambda$. According to Lemma 2.5, there exists a set F_0 of size $|F_0| = 2^\lambda$ such that P_0 is F_0-consistent. Note that Condition (2) is satisfied, since $\mathfrak{A}_i \equiv \mathfrak{B}_i$, while all other conditions are satisfied trivially.

For limit ordinals δ, we set

$$P_\delta := \bigcup_{\gamma<\delta} P_\gamma\,, \quad F_\delta := \bigcap_{\gamma<\delta} F_\gamma\,, \quad \text{and} \quad \pi_\delta^i := \bigcup_{\gamma<\delta} \pi_\gamma^i\,.$$

Then Condition (1) follows by Lemma 2.4, while Conditions (2)–(6) follow immediately from the inductive hypothesis.

For the successor step, suppose that we have already defined P_γ, F_γ, and π_γ^i. Depending on the value of $R(\gamma)$, we distinguish three cases. First, suppose that $R(\gamma) = \langle 0, i, \alpha \rangle$, for some $i, \alpha < 2^\lambda$. If $u_\alpha \notin A_i^\alpha$, we simply set $P_{\gamma+1} := P_\gamma$, $F_{\gamma+1} := F_\gamma$, and $\pi_{\gamma+1}^k := \pi_\gamma^k$, for all k. Hence, suppose that $u_\alpha \in A_i^\lambda$. By Lemma 2.11, there exist an element $v \in B_i^\lambda$ and sets Q' and $F' \subseteq F_\gamma$ of size

$$|Q'|, |F'| \leq \lambda \oplus |\gamma|$$

such that $P \cup Q'$ is $(F_\gamma \setminus F')$-consistent and $\pi_\gamma^i \cup \{\langle u_\alpha, v \rangle\}$ is a partial isomorphism modulo $P \cup Q'$.

We set $P_{\gamma+1} := P_\gamma \cup Q'$, $F_{\gamma+1} := F_\gamma \setminus F'$, and $\pi_{\gamma+1}^i := \pi_\gamma^i \cup \{\langle u_\alpha, v \rangle\}$. By construction, $\pi_{\gamma+1}^i$ satisfies Conditions (1), (2), and (4). Conditions (3), (5), and (6) are also satisfied.

If $R(\gamma) = \langle 1, i, \beta \rangle$, for some $i, \beta < 2^\lambda$, we proceed analogously to the first case applying Lemma 2.11 to $(\pi_\gamma^i)^{-1}$.

Finally, if $R(\gamma) = \langle 2, \alpha, \beta \rangle$, we use Corollary 2.8 to find sets Q' and $F' \subseteq F_\gamma$ of size $|Q'| = 1$ and $|F'| \leq \mu \leq \lambda$ such that $P_{\gamma+1} := P_\gamma \cup Q'$ is $(F \smallsetminus F')$-consistent and decides S_α. We set $F_{\gamma+1} := F_\gamma \smallsetminus F'$ and $\pi^i_{\gamma+1} := \pi^i_\gamma$, for all i. $\square$

Corollary 2.13. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures. We have*

$$\mathfrak{A} \equiv \mathfrak{B} \quad \text{iff} \quad \mathfrak{A}^{\mathfrak{u}} \cong \mathfrak{B}^{\mathfrak{u}} \quad \textit{for some ultrafilter } \mathfrak{u}\,.$$

The Theorem of Keisler and Shelah can be used to characterise first-order axiomatisable classes via their closure properties.

Definition 2.14. We say that a class $\mathcal{K}$ is *closed under reverse ultrapowers* if $\mathfrak{A}^{\mathfrak{u}} \in \mathcal{K}$ implies $\mathfrak{A} \in \mathcal{K}$, for every structure $\mathfrak{A}$ and all ultrafilters $\mathfrak{u}$.

Theorem 2.15. *A class $\mathcal{K}$ of Σ-structures is first-order axiomatisable if and only if $\mathcal{K}$ is closed under isomorphisms, ultraproducts, and reverse ultrapowers.*

Proof. One direction follows immediately from Corollary 1.2. For the other one, let $\Phi := \mathrm{Th}(\mathcal{K})$. We claim that $\mathrm{Mod}(\Phi) = \mathcal{K}$. Suppose otherwise. Then there exists a model $\mathfrak{B} \vDash \Phi$ such that $\mathfrak{B} \notin \mathcal{K}$. If we can show that $T := \mathrm{Th}(\mathfrak{B})$ is an accumulation point of the set $X := \{\, \mathrm{Th}(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{K} \,\}$ then we can apply Lemma 1.9 to find an ultraproduct $\mathfrak{C} := \prod_{i \in I} \mathfrak{A}_i / \mathfrak{u}$ of structures $\mathfrak{A}_i \in \mathcal{K}$ such that $\mathrm{Th}(\mathfrak{C}) = T = \mathrm{Th}(\mathfrak{B})$. Hence, by Corollary 2.13, there exists an ultrafilter $\mathfrak{v}$ such that $\mathfrak{B}^{\mathfrak{v}} \cong \mathfrak{C}^{\mathfrak{v}}$. But $\mathfrak{C} \in \mathcal{K}$ implies $\mathfrak{C}^{\mathfrak{v}} \in \mathcal{K}$ while $\mathfrak{B} \notin \mathcal{K}$ implies $\mathfrak{B}^{\mathfrak{v}} \notin \mathcal{K}$. Contradiction.

It remains to show that T is an accumulation point of X. Let $T \in \langle \varphi \rangle$. Then $\neg\varphi \notin \Phi \subseteq T$ and, by definition of Φ, there exists some structure $\mathfrak{A} \in \mathcal{K}$ such that $\mathfrak{A} \nvDash \neg\varphi$. Hence, $\mathrm{Th}(\mathfrak{A}) \in \langle \varphi \rangle \cap X \neq \varnothing$. $\square$

3. Reduced products and Horn formulae

In this section we study classes that are closed under arbitrary products and formulae that are preserved in products.

Definition 3.1. A formula φ is *preserved in reduced products* if, for every family $(\mathfrak{A}^i)_{i\in I}$ of structures and every filter $\mathfrak{u}$ over I, we have

$$\mathfrak{A}^i \vDash \varphi \text{ for all } i \quad \text{implies} \quad \prod_{i\in I} \mathfrak{A}^i/\mathfrak{u} \vDash \varphi\,.$$

If this holds only for $\mathfrak{u} = \{I\}$ then φ is *preserved in products.* Finally, we say that φ is *preserved in nonempty products* if the above is true only for $\mathfrak{u} = \{I\}$ and $I \neq \varnothing$.

Definition 3.2. (a) A *basic Horn formula* is a formula of the form

$$\varphi := \bigwedge \Phi \to \psi\,,$$

where ψ is an atomic formula or the formula false and Φ is a set (possibly empty) of atomic formulae. If $\psi \neq$ false then we say that φ is *strict.*

(b) A *Horn formula* is a formula of the form

$$\varphi = Q_0\bar{x}_0 \cdots Q_{n-1}\bar{x}_{n-1} \bigwedge \Phi$$

where Φ is a set of basic Horn formulae and the $Q_i \in \{\exists, \forall\}$ are quantifiers. We allow both Φ and the sequences $\bar{x}_i$ to be infinite. We call φ *strict* if Φ only contains strict basic Horn formulae. A Horn formula is *universal* if it is of the form $\forall\bar{x}\psi$ where ψ is a single basic Horn formula.

We denote the set of all Horn formulae by $\mathrm{HO}_\infty[\Sigma, X]$. $\mathrm{SH}_\infty[\Sigma, X]$ is the subset of all strict Horn formulae. The set of all universal (strict) Horn formulae is denoted by $\mathrm{H}\forall_\infty[\Sigma, X]$ and $\mathrm{SH}\forall_\infty[\Sigma, X]$, respectively. We write $\mathrm{HO}[\Sigma, X]$, $\mathrm{SH}[\Sigma, X]$, $\mathrm{H}\forall[\Sigma, X]$, and $\mathrm{SH}\forall[\Sigma, X]$, for the corresponding fragments of first-order logic.

(c) A formula is *positive primitive* if it is obtained from atomic formulae by (possibly infinite) conjunctions and existential quantifications. Again we allow quantifiers of the form $\exists\bar{x}$ where $\bar{x}$ is a possibly infinite sequence of variables.

Lemma 3.3. *Suppose that $\varphi(\bar{x})$ is a positive primitive formula, $(\mathfrak{A}^i)_{i\in I}$ a nonempty sequence of structures, and $\bar{a} \subseteq \prod_i A^i$. Then we have*

$$\textstyle\prod_i \mathfrak{A}^i \vDash \varphi(\bar{a}) \quad \text{iff} \quad \mathfrak{A}^i \vDash \varphi(\bar{a}^i)\,, \quad \textit{for all } i \in I\,.$$

Proof. W.l.o.g. we may assume that φ is term-reduced. We prove the claim by induction on φ. For atomic formulae φ, the claim holds by definition of a direct product. If φ is a conjunction then the claim follows immediately from the inductive hypothesis. Hence, we may assume that $\varphi(\bar{x}) = \exists\bar{y}\psi(\bar{x},\bar{y})$.

If $\prod_i \mathfrak{A}^i \vDash \varphi(\bar{a})$ then there exists a sequence $\bar{b} \subseteq \prod_i A^i$ such that $\prod_i \mathfrak{A}^i \vDash \psi(\bar{a},\bar{b})$. By inductive hypothesis, we therefore have

$$\mathfrak{A}^i \vDash \psi(\bar{a}^i,\bar{b}^i)\,, \quad \text{for all } i\,,$$

and it follows that $\mathfrak{A}^i \vDash \exists\bar{y}\psi(\bar{a}^i,\bar{y})$.

Conversely, suppose that $\mathfrak{A}^i \vDash \varphi(\bar{a}^i)$, for all i. Choose sequences $\bar{b}^i \subseteq A^i$ such that $\mathfrak{A}^i \vDash \psi(\bar{a}^i,\bar{b}^i)$. By inductive hypothesis, it follows that $\prod_i \mathfrak{A}^i \vDash \psi(\bar{a},\bar{b})$. This implies that $\prod_i \mathfrak{A}^i \vDash \varphi(\bar{a})$. $\square$

Theorem 3.4. *Let $\varphi(\bar{x})$ be a Horn formula, $(\mathfrak{A}^i)_{i\in I}$ a nonempty sequence of structures, and $\bar{a} \subseteq \prod_i A^i$. Then*

$$\mathfrak{A}^i \vDash \varphi(\bar{a}^i)\,, \textit{ for all } i\,, \quad \textit{implies} \quad \prod_i \mathfrak{A}^i \vDash \varphi(\bar{a})\,.$$

Proof. We prove the claim by induction on φ. Suppose that $\mathfrak{A}^i \vDash \varphi(\bar{a}^i)$, for all i. First, we consider the case that $\varphi = \bigwedge\Phi \rightarrow \psi$ is a basic Horn formula. If $\prod_i \mathfrak{A}^i \nvDash \Phi(\bar{a})$ then we are done. Hence we may assume that $\prod_i \mathfrak{A}^i \vDash \Phi(\bar{a})$. By Lemma 3.3, it follows that $\mathfrak{A}^i \vDash \Phi(\bar{a}^i)$, for all i. Since $\mathfrak{A}^i \vDash \varphi(\bar{a}^i)$ this implies that $\mathfrak{A}^i \vDash \psi(\bar{a}^i)$. In this case ψ cannot be false and we can use Lemma 3.3 to conclude that $\prod_i \mathfrak{A}^i \vDash \psi(\bar{a})$, as desired.

If φ is a conjunction then the claim follows immediately by inductive hypothesis. For $\varphi = \exists\bar{y}\psi(\bar{x},\bar{y})$ we can argue in the same way as in the proof of Lemma 3.3. Finally, assume that $\varphi = \forall\bar{y}\psi(\bar{x},\bar{y})$. Let $\bar{b} \subseteq \prod_i A^i$. Since $\mathfrak{A}^i \vDash \varphi(\bar{a}^i)$ we have $\mathfrak{A}^i \vDash \psi(\bar{a}^i,\bar{b}^i)$. By inductive hypothesis, this implies that $\prod_i \mathfrak{A}^i \vDash \psi(\bar{a},\bar{b})$. Since $\bar{b}$ was arbitrary it follows that $\prod_i \mathfrak{A}^i \vDash \forall\bar{y}\psi(\bar{a},\bar{y})$. $\square$

For first-order formulae we can generalise these results to reduced products.

Lemma 3.5. *Suppose that $\varphi(\bar{x})$ is a positive primitive first-order formula, $(\mathfrak{A}^i)_{i\in I}$ a nonempty sequence of structures, $\mathfrak{u}$ a filter over I, and $[\bar{a}]$ a tuple in $\prod_i \mathfrak{A}^i/\mathfrak{u}$. Then we have*

$$\textstyle\prod_i \mathfrak{A}^i/\mathfrak{u} \vDash \varphi([\bar{a}]) \quad \text{iff} \quad [\![\varphi(\bar{a})]\!] \in \mathfrak{u}\,.$$

Proof. The proof is analogous to those of Lemma 3.3 and Theorem 1.1. We assume that φ is term-reduced and we prove the claim by induction on φ.

For atomic formulae φ, the claim holds by the definition of a reduced product. If φ is a conjunction then the claim follows immediately from the inductive hypothesis and the fact that filters are closed under finite intersections. Hence, we may assume that $\varphi(\bar{x}) = \exists\bar{y}\psi(\bar{x},\bar{y})$.

If $\prod_i \mathfrak{A}^i/\mathfrak{u} \vDash \varphi([\bar{a}])$ then there exists a sequence $\bar{b} \subseteq \prod_i A^i$ such that $\prod_i \mathfrak{A}^i/\mathfrak{u} \vDash \psi([\bar{a}],[\bar{b}])$. By inductive hypothesis, we therefore have

$$[\![\psi(\bar{a},\bar{b})]\!] \in \mathfrak{u}\,.$$

Since $[\![\psi(\bar{a},\bar{b})]\!] \subseteq [\![\exists\bar{y}\psi(\bar{a},\bar{y})]\!]$ it follows that

$$[\![\exists\bar{y}\psi(\bar{a},\bar{b})]\!] \in \mathfrak{u}\,.$$

Conversely, suppose that $s := [\![\exists\bar{y}\psi(\bar{a},\bar{b})]\!] \in \mathfrak{u}$. For every $i \in s$, we choose sequences $\bar{b}^i \subseteq A^i$ such that $\mathfrak{A}^i \vDash \psi(\bar{a}^i,\bar{b}^i)$. For $i \in [\![\exists\bar{y}\,\text{true}]\!] \smallsetminus s$, we take an arbitrary tuple $\bar{b}^i \subseteq A^i$. Then $[\![\psi(\bar{a},\bar{b})]\!] = s \in \mathfrak{u}$ which implies that $\prod_i \mathfrak{A}^i/\mathfrak{u} \vDash \psi([\bar{a}],[\bar{b}])$. Consequently, we have $\prod_i \mathfrak{A}^i/\mathfrak{u} \vDash \varphi([\bar{a}])$. ☐

For first-order Horn formulae we can extend the Theorem of Łoś to arbitrary filters.

Theorem 3.6. *Suppose that $\varphi(\bar{x})$ is a first-order Horn formula, $(\mathfrak{A}^i)_{i\in I}$ a nonempty sequence of structures, $\mathfrak{u}$ a filter on I, and $[\bar{a}]$ a tuple in $\prod_i \mathfrak{A}^i/\mathfrak{u}$. Then*

$$[\![\varphi(\bar{a})]\!] \in \mathfrak{u} \quad \textit{implies} \quad \textstyle\prod_i \mathfrak{A}^i/\mathfrak{u} \vDash \varphi([\bar{a}])\,.$$

Proof. We prove the claim by induction on φ. Let $s := [\![\varphi(\bar{a})]\!] \in \mathfrak{u}$. First, we consider the case that $\varphi = \bigwedge \Phi \rightarrow \psi$ is a basic Horn formula. If $\prod_i \mathfrak{A}^i/\mathfrak{u} \not\models \Phi([\bar{a}])$ then we are done. Hence we may assume that $\prod_i \mathfrak{A}^i/\mathfrak{u} \models \Phi([\bar{a}])$. By Lemma 3.5, it follows that $w := [\![\bigwedge \Phi(\bar{a})]\!] \in \mathfrak{u}$. Consequently, we have $s \cap w \subseteq [\![\psi(\bar{a})]\!] \in \mathfrak{u}$. In this case ψ cannot be false and we can use Lemma 3.3 to conclude that $\prod_i \mathfrak{A}^i/\mathfrak{u} \models \psi([\bar{a}])$, as desired.

If φ is a conjunction then the claim follows immediately by inductive hypothesis. For $\varphi = \exists \bar{y}\psi(\bar{x}, \bar{y})$ we can argue in the same way as in the proof of Lemma 3.5. Finally, assume that $\varphi = \forall \bar{y}\psi(\bar{x}, \bar{y})$. Let $b_k \in \prod_{i \in I_{s_k}} A^i_{s_k}$. Then $s \subseteq [\![\psi(\bar{a}, \bar{b})]\!] \in \mathfrak{u}$. By inductive hypothesis, this implies that $\prod_i \mathfrak{A}^i/\mathfrak{u} \models \psi([\bar{a}], [\bar{b}])$. Since $\bar{b}$ was arbitrary it follows that $\prod_i \mathfrak{A}^i/\mathfrak{u} \models \forall \bar{y}\psi([\bar{a}], [\bar{y}])$. □

Corollary 3.7. *Let Σ be a signature and X a set of variables.*

(a) $\mathrm{HO}_\infty[\Sigma, X]$*-formulae are preserved in nonempty products.*

(b) $\mathrm{SH}_\infty[\Sigma, X]$*-formulae are preserved in products.*

(c) $\mathrm{HO}[\Sigma, X]$*-formulae are preserved in nonempty reduced products.*

(d) $\mathrm{SH}[\Sigma, X]$*-formulae are preserved in reduced products.*

Proof. (a) and (c) follow immediately from Theorem 3.4 and 3.6, respectively. For (b) and (d) it is sufficient to note that in the empty product 1 every n-ary relation contains the tuple $\langle\langle\rangle, \dots, \langle\rangle\rangle$. Hence, we have

$$1 \models \varphi(\langle\rangle, \dots, \langle\rangle)\,,$$

for every atomic formula φ. □

Example. Groups, rings, and modules are SH-axiomatisable. Hence, theses classes are closed under reduced products.

Lemma 3.8 (McKinsey). *Let $\mathcal{K}$ be a class of structures that is closed under nonempty products. If Φ is a set of Horn formulae and Ψ a nonempty set* *of atomic formulae (possibly including the formula* false) *such that*

$$\mathfrak{A} \vDash \forall \bar{x}\Big(\bigwedge \Phi \to \bigvee \Psi\Big), \quad \textit{for all } \mathfrak{A} \in \mathcal{K},$$

then there is some formula $\psi \in \Psi$ *such that*

$$\mathfrak{A} \vDash \forall \bar{x}\Big(\bigwedge \Phi \to \psi\Big), \quad \textit{for all } \mathfrak{A} \in \mathcal{K}.$$

Proof. For a contradiction, suppose that, for every formula $\psi \in \Psi$ there are a structure $\mathfrak{A}^\psi \in \mathcal{K}$ and parameters $\bar{a}^\psi \subseteq A^\psi$ such that

$$\mathfrak{A}^\psi \vDash \bigwedge \Phi(\bar{a}^\psi) \wedge \neg\psi(\bar{a}^\psi).$$

Set $\mathfrak{B} := \prod_{\psi\in\Psi} \mathfrak{A}^\psi$ and $\bar{b} := (\bar{a}^\psi)_\psi \subseteq B$. Since $\Psi \neq \varnothing$ we have $\mathfrak{B} \in \mathcal{K}$. Furthermore, it follows by Theorem 3.4 that

$$\mathfrak{B} \vDash \bigwedge \Phi(\bar{b}).$$

Hence, there is some $\psi \in \Psi$ such that $\mathfrak{B} \vDash \psi(\bar{b})$. By Lemma 3.3, this implies that $\mathfrak{A}^\psi \vDash \psi(\bar{a}^\psi)$. Contradiction. $\square$

The converse of Corollary 3.7 is given by the following preservation theorem.

Theorem 3.9. *A first-order sentence φ is preserved in nonempty reduced products if and only if it is equivalent to a first-order Horn sentence.*

4. *Quasivarieties*

Classes that are axiomatised by universal Horn formulae admit a nice algebraic characterisation.

Definition 4.1. Let $\mathcal{K}$ be a class of Σ-structures.

(a) A $\mathcal{K}$-*presentation* is a pair $\langle C; \Phi \rangle$ consisting of a set C of constant symbols disjoint from Σ and a set Φ of atomic sentences over the signature $\Sigma_C = \Sigma \cup C$. The constants in C are called the *generators* of the presentation.

(b) A *model* of a $\mathcal{K}$-presentation $\langle C; \Phi \rangle$ is a Σ_C-structure $\mathfrak{A}$ such that

$$\mathfrak{A} \vDash \Phi \quad \text{and} \quad \mathfrak{A}|_\Sigma \in \mathcal{K} .$$

(c) A model $\mathfrak{A}$ of a $\mathcal{K}$-presentation $\langle C; \Phi \rangle$ is *free* if

- $\mathfrak{A}$ is generated by the constants in C and
- for every model $\mathfrak{B}$ of $\langle C; \Phi \rangle$ there is a homomorphism $\mathfrak{A} \to \mathfrak{B}$.

(d) We say that $\mathcal{K}$ *has free models* if every $\mathcal{K}$-presentation has a free model.

Remark. Note that the homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ in (c) maps $c^{\mathfrak{A}}$ to $c^{\mathfrak{B}}$, for every $c \in C$. Since $\mathfrak{A}$ is generated by C it follows that h is unique.

Example. Let $\mathcal{K}$ be the class of all groups, $C := \{a, b\}$, and let Φ be the set consisting of the single formula $a \cdot b = b \cdot a$. Then $\langle C; \Phi \rangle$ is a $\mathcal{K}$-presentation. Its free model consists of the direct product

$$\langle \mathbb{Z}, +, -, 0 \rangle \times \langle \mathbb{Z}, +, -, 0 \rangle$$

with additional constants $a = \langle 0, 1 \rangle$ and $b = \langle 1, 0 \rangle$.

Example. Suppose that Σ is a signature without relation symbols. The class $\mathrm{Str}[\Sigma]$ of all Σ-structures has free models. Let $\langle C; \Phi \rangle$ be a $\mathrm{Str}[\Sigma]$-presentation. W.l.o.g. we may assume that Φ is closed under entailment. In particular, it is =-closed and, as in Lemma C2.4.4, we obtain a Herbrand model $\mathfrak{H}$ of Φ that is of the form $\mathfrak{H} = \mathfrak{T}[\Sigma_C; \varnothing]/{\sim}$ where

$$s \sim t \quad \text{iff} \quad s = t \in \Phi .$$

We claim that $\mathfrak{H}$ is a free model of $\langle C; \Phi \rangle$.

Suppose that $\mathfrak{B}$ is a model of $\langle C; \Phi \rangle$. We have to find a homomorphism $f : \mathfrak{H} \to \mathfrak{B}$. Let π be the canonical projection $\mathfrak{T}[\Sigma_C; \varnothing] \to \mathfrak{H}$. By Theorem B3.1.9, there exists a unique homomorphism $h : \mathfrak{T}[\Sigma_C; \varnothing] \to \mathfrak{B}$.

$$\begin{tikzcd} \mathfrak{T}[\Sigma_C;\varnothing] \arrow[r, "h"] \arrow[d, "\pi"'] & \mathfrak{B} \\ \mathfrak{T}[\Sigma_C;\varnothing]/{\sim} \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Since $\mathfrak{B}$ is a model of Φ it follows that $\ker\pi = {\sim} \subseteq \ker h$. Hence, we can use the Factorisation Lemma to find the desired homomorphism $f : \mathfrak{H} \to \mathfrak{B}$.

We start by giving conditions ensuring that $\mathcal{K}$ has free models.

Lemma 4.2. *Let $\langle C; \Phi\rangle$ be a $\mathcal{K}$-presentation and $\mathfrak{A}$ a Σ_C-structure with $\mathfrak{A}|_\Sigma \in \mathcal{K}$. Then $\mathfrak{A}$ is a free model of $\langle C; \Phi\rangle$ if and only if*

- *C generates $\mathfrak{A}$ and*
- *for every atomic formula φ over Σ_C, we have*

$$(*) \quad \mathfrak{A} \vDash \varphi \quad \text{iff} \quad \mathrm{Th}(\mathcal{K}) \vDash \bigwedge \Phi \to \varphi\,.$$

Proof. Let $\varphi_0(\bar{x}) \in \mathrm{FO}[\Sigma, X]$ and $\Phi_0(\bar{x}) \subseteq \mathrm{FO}[\Sigma, X]$ be the formulae obtained from φ and Φ by replacing the constant symbols in C by variables.

($\Rightarrow$) If every structure in $\mathcal{K}$ satisfies the sentence

$$\forall\bar{x}\big[\bigwedge \Phi_0(\bar{x}) \to \varphi_0(\bar{x})\big]$$

then, in particular, so does $\mathfrak{A}|_\Sigma$. Hence, $\mathfrak{A} \vDash \varphi$.

Conversely, suppose that $\mathfrak{A} \vDash \varphi_0(\bar{c})$ and let $\mathfrak{B} \in \mathcal{K}$ be a structure with $\mathfrak{B} \vDash \Phi_0(\bar{b})$, for some $\bar{b} \subseteq B$. Since $\mathfrak{A}$ is free there exists a homomorphism $h : \mathfrak{A} \to \langle\mathfrak{B}, \bar{b}\rangle$. Since $h(\bar{c}) = \bar{b}$ and atomic formulae are preserved under homomorphisms it follows that

$$\mathfrak{A} \vDash \varphi_0(\bar{c}) \quad \text{implies} \quad \mathfrak{B} \vDash \varphi_0(\bar{b})\,,$$

as desired.

($\Leftarrow$) For every $\varphi \in \Phi$ we have $\Phi \vDash \varphi$. By $(*)$, this implies that $\mathfrak{A} \vDash \varphi$. Consequently, $\mathfrak{A}$ is a model of $\langle C; \Phi \rangle$. If $\mathfrak{B}$ is another model of $\langle C; \Phi \rangle$ then we have $\mathfrak{B}|_\Sigma \in \mathcal{K}$ and $\mathfrak{B}|_\Sigma \vDash \Phi_0(\bar{c}^{\mathfrak{B}})$. By $(*)$ it follows that $\mathfrak{B} \vDash \psi$, for every atomic formula ψ with $\mathfrak{A} \vDash \psi$. Consequently, $\mathfrak{B}$ satisfies the atomic diagram of $\mathfrak{A}$ and we can use Corollary C2.2.4 to find a homomorphism $\mathfrak{A} \to \mathfrak{B}$. $\square$

Theorem 4.3. *Let $\mathcal{K}$ be a class of Σ-structures that is closed under isomorphic copies. The following statements are equivalent:*

(1) *Every $\mathcal{K}$-presentation with a model has a free model.*

(2) *$\mathcal{K}$ is closed under nonempty products and substructures.*

(3) *$\mathcal{K}$ is $\mathrm{H}\forall_\infty$-axiomatisable.*

Proof. (3) $\Rightarrow$ (2) follows from Corollary 3.7 and Lemma C2.1.6.

(2) $\Rightarrow$ (1) Let $\langle C; \Phi \rangle$ be a $\mathcal{K}$-presentation with a model and let Ψ be the set of all atomic formulae $\psi(\bar{x})$ (including false) such that

$$\mathrm{Th}(\mathcal{K}) \nvDash \bigwedge \Phi \to \psi(\bar{c}) .$$

If every model of $\langle C; \Phi \rangle$ would satisfy $\bigvee \Psi$ then it would follow by Lemma 3.8 that

$$\mathrm{Th}(\mathcal{K}) \vDash \bigwedge \Phi \to \psi(\bar{c}) ,$$

for some $\psi \in \Psi$. By choice of Ψ we can therefore find some structure $\mathfrak{A} \in \mathcal{K}$ and elements $\bar{c} \subseteq A$ such that

$$\langle \mathfrak{A}, \bar{c} \rangle \vDash \bigwedge \Phi \wedge \neg \bigvee \Psi .$$

It follows that

$$\langle \mathfrak{A}, \bar{c} \rangle \vDash \varphi \quad \text{iff} \quad \mathrm{Th}(\mathcal{K}) \vDash \bigwedge \Phi \to \varphi ,$$

for every atomic formula φ over Σ_C. Setting $\mathfrak{A}_0 := \langle\!\langle \bar{c} \rangle\!\rangle_{\mathfrak{A}}$ we still have

$$\langle \mathfrak{A}_0, \bar{c} \rangle \vDash \varphi \quad \text{iff} \quad \mathrm{Th}(\mathcal{K}) \vDash \bigwedge \Phi \to \varphi ,$$

for all such φ. Since $\mathcal{K}$ is closed under substructures we have $\mathfrak{A}_0 \in \mathcal{K}$. Hence, Lemma 4.2 implies that $\mathfrak{A}_0$ is a free model of $\langle C; \Phi \rangle$.

(1) $\Rightarrow$ (3) Set $T := \mathrm{Th}_{\mathrm{H}\forall_\infty}(\mathcal{K})$ and suppose that $\mathfrak{B}$ is a model of T. Let Φ be the atomic diagram of $\mathfrak{B}$. Because $\mathfrak{B}$ is a model of the $\mathcal{K}$-presentation $\langle B; \Phi \rangle$ there exists, by (1), a free model $\mathfrak{A}$ of $\langle B; \Phi \rangle$. By Corollary C2.2.4 there exists a homomorphism $h : \mathfrak{B} \to \mathfrak{A}$. Since B generates $\mathfrak{A}$ this homomorphism is surjective. If we can show that it is an embedding then it follows that $\mathfrak{B} \cong \mathfrak{A}|_\Sigma$ and, since $\mathcal{K}$ is closed under isomorphic copies, we have $\mathfrak{B} \in \mathcal{K}$, as desired.

Let $\Phi_0(\bar{x})$ be the set of formulae obtained from Φ by replacing the constants in B by variables. Let $\psi(\bar{x})$ be an atomic formula over Σ with $\mathfrak{A} \vDash \psi(\bar{b})$, for some $\bar{b} \subseteq B$. By Lemma 4.2, T contains the formula

$$\forall \bar{x}\Big(\bigwedge \Phi_0(\bar{x}) \to \psi(\bar{x})\Big).$$

Since $\mathfrak{B} \vDash T$ we have $\mathfrak{B} \vDash \bigwedge \Phi_0(\bar{b}) \to \psi(\bar{b})$. By definition of Φ this implies that $\mathfrak{B} \vDash \psi(\bar{b})$. Consequently, h is an embedding. □

Theorem 4.4. *Let $\mathcal{K}$ be a class of Σ-structures that is closed under isomorphic copies. The following statements are equivalent:*

(1) *$\mathcal{K}$ has free models.*

(2) *$\mathcal{K}$ is closed under products and substructures.*

(3) *$\mathcal{K}$ is* $\mathrm{SH}\forall_\infty$*-axiomatisable.*

Proof. (3) $\Rightarrow$ (2) follows from Corollary 3.7 and Lemma C2.1.6.

(2) $\Rightarrow$ (1) Note that the empty product is a model of every $\mathcal{K}$-presentation. Hence, the claim follows from Theorem 4.3.

(1) $\Rightarrow$ (3) By Theorem 4.3, we know that $\mathcal{K}$ has an $\mathrm{H}\forall_\infty$-axiomatisation T. We claim that $T \subseteq \mathrm{SH}\forall_\infty$. Suppose otherwise. Then T contains a formula of the form $\forall \bar{x}(\bigwedge \Phi \to \text{false})$. Let $X := \bar{x}$ be the set of variables appearing in Φ. The $\mathcal{K}$-presentation $\langle X; \Phi \rangle$ has a free model $\langle \mathfrak{A}, \bar{c} \rangle$, by (1). But then $\mathfrak{A} \vDash \Phi(\bar{c})$ would imply that $\mathfrak{A} \vDash$ false. A contradiction. □

Definition 4.5. Let $\mathcal{K}$ be a class of Σ-structures.

(a) $\mathcal{K}$ is a *quasivariety* if it is $\mathrm{SH}\forall$-axiomatisable.

(b) $\mathcal{K}$ is a *variety* if it can be axiomatised by a set of formulae of the form $\forall \bar{x} \varphi$ where φ is an atomic formula.

Example. The classes of all groups, all rings, and all modules are varieties. The class of lattices (with signature $\sqcap, \sqcup, \sqsubseteq$) is a quasivariety, but not a variety. If we omit $\sqsubseteq$ then the class becomes a variety. The class of all fields is not a quasivariety.

Definition 4.6. Let $\mathcal{K}$ be a class of Σ-structures. We define the following operations.

(a) $\mathrm{Prod}(\mathcal{K})$ is the class of all nonempty products of structures in $\mathcal{K}$.

(b) $\mathrm{Sub}(\mathcal{K})$ is the class of all substructures of structures in $\mathcal{K}$.

(c) $\mathrm{Iso}(\mathcal{K})$ is the class of all structures isomorphic to one in $\mathcal{K}$.

(d) $\mathrm{Hom}(\mathcal{K})$ is the class of all weak homomorphic images of structures in $\mathcal{K}$.

(e) $\mathrm{ERP}(\mathcal{K})$ is the class of all structures that can be embedded into a reduced product of structures in $\mathcal{K}$.

(f) Finally, we define the abbreviations

$$\mathrm{QV} := \mathrm{Iso} \circ \mathrm{Sub} \circ \mathrm{Prod}\,,$$
$$\mathrm{Var} := \mathrm{Hom} \circ \mathrm{Sub} \circ \mathrm{Prod}\,.$$

Theorem 4.7. *Let $\mathcal{K}$ be a class of Σ-structures.*

(a) $\mathrm{QV}(\mathcal{K})$ *is the smallest class of Σ-structures containing $\mathcal{K}$ that is closed under products, substructures, and isomorphic copies.*

(b) $\mathrm{QV}(\mathcal{K}) = \mathrm{Mod}(\mathrm{Th}_{\mathrm{SH}\forall_\infty}(\mathcal{K}))$.

(c) *If $\mathcal{K}$ or $\mathrm{QV}(\mathcal{K})$ is first-order axiomatisable then $\mathrm{QV}(\mathcal{K})$ is a quasivariety.*

Proof. Let $T := \mathrm{SH}\forall_\infty(\mathcal{K})$.

(a) and (b) Let $\mathcal{H}$ be the smallest class of Σ-structures containing $\mathcal{K}$ that is closed under products, substructures, and isomorphic copies. Then we have $\mathrm{QV}(\mathcal{K}) = (\mathrm{Iso} \circ \mathrm{Sub} \circ \mathrm{Prod})(\mathcal{K}) \subseteq \mathcal{H}$. Furthermore, by Lemma C2.1.6 and Corollary 3.7 it follows that every structure in $\mathcal{H}$ is a model of T. Consequently, we have

$$\mathrm{QV}(\mathcal{K}) \subseteq \mathcal{H} \subseteq \mathrm{Mod}(T),$$

and it remains to prove that $\mathrm{Mod}(T) \subseteq \mathrm{QV}(\mathcal{K})$.

Suppose that $\mathfrak{A} \vDash T$ and fix an enumeration $\bar{a}$ of A without repetitions. Let $\Phi(\bar{x})$ be the set of all atomic formulae $\varphi(\bar{x})$ with $\mathfrak{A} \vDash \varphi(\bar{a})$ and let $\Psi(\bar{x})$ be the set of all atomic formulae $\varphi(\bar{x})$ (including false) with $\mathfrak{A} \nvDash \varphi(\bar{a})$. Consider a formula $\psi \in \Psi$. Since $\mathfrak{A}$ is a model of T we have $\forall\bar{x}(\bigwedge\Phi \to \psi) \notin T$. Therefore, we can find a structure $\mathfrak{B}^\psi \in \mathcal{K}$ and parameters $\bar{b}^\psi \subseteq B$ such that

$$\mathfrak{B}^\psi \vDash \bigwedge \Phi(\bar{b}^\psi) \wedge \neg\psi(\bar{b}^\psi).$$

Let $\langle \mathfrak{C}, \bar{c}\rangle := \prod_{\psi\in\Psi}\langle\mathfrak{B}^\psi, \bar{b}^\psi\rangle$. Since the algebraic diagrams of $\langle\!\langle \bar{c} \rangle\!\rangle_{\langle\mathfrak{C},\bar{c}\rangle}$ and $\langle\mathfrak{A}, \bar{a}\rangle$ coincide we can use Corollary C2.2.4 to find an embedding $h : \mathfrak{A} \to \mathfrak{C}$ with $h(\bar{a}) = \bar{c}$. Hence, $\mathfrak{A}$ is isomorphic to a substructure of a product of structures in $\mathcal{K}$, i.e.,

$$\mathfrak{A} \in (\mathrm{Iso} \circ \mathrm{Sub} \circ \mathrm{Prod})(\mathcal{K}) = \mathrm{QV}(\mathcal{K}).$$

(c) Let T_0 be an axiomatisation of either $\mathcal{K}$ or $\mathrm{QV}(\mathcal{K})$. Note that in both cases we have $T = (T_0)^{\vDash}_{\mathrm{SH}\forall_\infty}$. For every formula $\varphi \in T$, we will construct a first-order formula $\varphi' \in T$ with $\varphi' \vDash \varphi$. This implies that $T \cap \mathrm{FO} \vDash T$. It follows that $\mathrm{Mod}(T \cap \mathrm{FO}) = \mathrm{QV}(\mathcal{K})$, as desired.

It remains to find φ'. Let $\forall\bar{x}(\bigwedge\Phi \to \psi) \in T$. Then $T_0 \cup \Phi \vDash \psi$. By the Compactness Theorem, we can find a finite subset $\Phi_0 \subseteq \Phi$ such that $T_0 \cup \Phi_0 \vDash \psi$. Setting $\varphi' := \forall\bar{x}(\bigwedge\Phi_0 \to \psi)$ it follows that $T_0 \vDash \varphi'$. Furthermore, since φ' is a universal strict Horn formula we have $\varphi' \in T$, as desired. $\square$

Corollary 4.8. *A class $\mathcal{K}$ is a quasivariety if and only if it is first-order axiomatisable and closed under products, substructures, and isomorphic copies.*

Lemma 4.9. *Let $\mathfrak{A}$ be a Σ-structure and $\mathcal{K}$ a nonempty class of Σ-structures.*

$$\mathfrak{A} \in \mathrm{ERP}(\mathcal{K}) \quad \text{iff} \quad \mathrm{Th}_{\mathrm{H}\forall}(\mathcal{K}) \subseteq \mathrm{Th}_{\mathrm{H}\forall}(\mathfrak{A}) .$$

Proof. ($\Rightarrow$) follows from the preservation properties of universal Horn formulae. For ($\Leftarrow$), suppose that $\mathrm{Th}_{\mathrm{H}\forall}(\mathcal{K}) \subseteq \mathrm{Th}_{\mathrm{H}\forall}(\mathfrak{A})$. Let Δ_+ be the set of all atomic first-order formulae and Δ_- the set of all negations of atomic first-order formulae. We set $\Phi_+ := \mathrm{Th}_{\Delta_+}(\mathfrak{A}_A)$ and $\Phi_- := \mathrm{Th}_{\Delta_-}(\mathfrak{A}_A)$.

First, we show that, for every finite subset $\Psi \subseteq \Phi_+$, there exists a structure $\mathfrak{B}^\Psi \in \mathrm{Prod}(\mathcal{K})$ and parameters $\bar{b}^\Psi \subseteq B$ such that

$$\langle \mathfrak{B}^\Psi, \bar{b}^\Psi \rangle \vDash \Psi \cup \Phi_- .$$

Suppose that $\Psi = \{\psi_0(\bar{a}), \dots, \psi_n(\bar{a})\}$. For every $\neg\varphi(\bar{a}) \in \Phi_-$, we have

$$\mathfrak{A} \vDash \psi_0(\bar{a}) \wedge \dots \wedge \psi_n(\bar{a}) \wedge \neg\varphi(\bar{a}) .$$

It follows that $\mathfrak{A} \nvDash \psi_0(\bar{a}) \wedge \dots \wedge \psi_n(\bar{a}) \to \varphi(\bar{a})$. By assumption this implies that

$$\forall \bar{x}[\psi_0(\bar{x}) \wedge \dots \wedge \psi_n(\bar{x}) \to \varphi(\bar{x})] \notin \mathrm{Th}_{\mathrm{H}\forall}(\mathcal{K}) .$$

Consequently, there is a structure $\mathfrak{C}^\varphi \in \mathcal{K}$ and elements $\bar{c}^\varphi \subseteq C$ such that

$$\mathfrak{C}^\varphi \vDash \psi_0(\bar{c}^\varphi) \wedge \dots \wedge \psi_n(\bar{c}^\varphi) \wedge \neg\varphi(\bar{c}^\varphi) .$$

Similarly, we have

$$\forall \bar{x}[\psi_0(\bar{x}) \wedge \dots \wedge \psi_n(\bar{x}) \to \mathrm{false}] \notin \mathrm{Th}_{\mathrm{H}\forall}(\mathcal{K}) ,$$

and there is a structure $\mathfrak{C}^\perp \in \mathcal{K}$ and elements $\bar{c}^\perp \subseteq C$ such that

$$\mathfrak{C}^\perp \vDash \psi_0(\bar{c}^\perp) \wedge \cdots \wedge \psi_n(\bar{c}^\perp).$$

We form the product

$$\langle \mathfrak{B}, \bar{b} \rangle := \prod_{\varphi \in \Phi_- \cup \{\perp\}} \langle \mathfrak{C}^\varphi, \bar{c}^\varphi \rangle.$$

By Lemma 3.3 it follows that

$$\begin{aligned} & \mathfrak{B} \vDash \psi_i(\bar{b}), && \text{for all } i, \\ \text{and} \quad & \mathfrak{B} \vDash \neg\varphi(\bar{b}), && \text{for all } \neg\varphi \in \Phi_-. \end{aligned}$$

Furthermore,

$$\mathfrak{B} = \prod_{\varphi \in \Phi_- \cup \{\perp\}} \mathfrak{C}^\varphi \in \mathrm{Prod}(\mathcal{K}),$$

as desired.

It remains to construct a model $\langle \mathfrak{D}, \bar{d} \rangle$ of $\Phi_+ \cup \Phi_-$ that is a reduced product of structures in $\mathcal{K}$. By the Diagram Lemma, this implies that $\mathfrak{A}$ can be embedded into the product $\mathfrak{D}$.

If Φ_+ is finite we can use the structure $\langle \mathfrak{B}^{\Phi_+}, \bar{b}^{\Phi_+} \rangle$. Hence, we may assume that Φ_+ is infinite. Let $\mathfrak{u}$ be a regular ultrafilter over Φ_+ and let $(s_\varphi)_{\varphi \in \Phi_+}$ be the corresponding sequence of sets $s_\varphi \in \mathfrak{u}$ such that, for every $i \in \Phi_+$, the set

$$w_i := \{ \varphi \in \Phi_+ \mid i \in s_\varphi \}$$

is finite. We claim that the reduced product

$$\langle \mathfrak{D}, \bar{d} \rangle := \prod_{i \in \Phi_+} \langle \mathfrak{B}^{w_i}, \bar{b}^{w_i} \rangle / \mathfrak{u}$$

is the desired model of $\Phi_+ \cup \Phi_-$.

First consider $\varphi(\bar{a}) \in \Phi_+$. For every $i \in s_\varphi$, we have $\mathfrak{B}^{w_i} \vDash \varphi(\bar{b}^{w_i})$. Therefore, $s_\varphi \subseteq [\![\varphi(\bar{d})]\!] \in \mathfrak{u}$ and it follows that $\mathfrak{D} \vDash \varphi(\bar{d})$. Furthermore, we have $\langle \mathfrak{D}, \bar{d} \rangle \vDash \Phi_-$, since $\langle \mathfrak{B}^{w_i}, \bar{b}^{w_i} \rangle \vDash \Phi_-$, for all i. Finally, note that $\mathfrak{D}$ is a reduced product of structures in $\mathrm{Prod}(\mathcal{K})$. Therefore, it can be written as a reduced product of structures in $\mathcal{K}$. □

Theorem 4.10. *Let $\mathcal{K}$ be a class of Σ-structures. The following statements are equivalent:*

(1) *$\mathcal{K}$ is closed under substructures, reduced products, and isomorphic copies.*

(2) *$\mathcal{K}$ is $\mathrm{H}\forall$-axiomatisable.*

Proof. (2) ⇒ (1) follows from the preservation properties of universal Horn formulae. For (1) ⇒ (2), let $T := \mathrm{Th}_{\mathrm{H}\forall}(\mathcal{K})$. By Lemma 4.9, we have

$$\mathrm{Mod}(T) \subseteq \mathrm{ERP}(\mathcal{K}) = \mathcal{K} \subseteq \mathrm{Mod}(T)\,,$$

as desired. □

Corollary 4.11. *Let T be a $\mathrm{H}\forall[\Sigma]$-theory and $\varphi \in \mathrm{FO}[\Sigma]$ a first-order formula. The following statements are equivalent:*

(1) *We have $\mathfrak{A} \vDash \varphi$, for every structure $\mathfrak{A} \in \mathrm{ERP}(\mathrm{Mod}(T \cup \{\varphi\}))$.*

(2) *φ is equivalent modulo T to a finite conjunction of $\mathrm{H}\forall[\Sigma]$-formulae.*

Proof. (2) ⇒ (1) follows from the preservation properties of universal Horn formulae. For (1) ⇒ (2), let $\Phi := (T \cup \{\varphi\})^{\vDash}_{\mathrm{H}\forall}$. Clearly, $T \cup \{\varphi\} \vDash \Phi$. If we can show that $\Phi \vDash T \cup \{\varphi\}$ then the claim follows by compactness.

Suppose that $\mathfrak{A} \vDash \Phi$. By Lemma 4.9, we have

$$\mathfrak{A} \in \mathrm{ERP}(\mathrm{Mod}(T \cup \{\varphi\}))\,,$$

which, by (1), implies that $\mathfrak{A} \vDash \varphi$. Furthermore, we have $\mathfrak{A} \vDash T$ since $T \subseteq \Phi$. □

Theorem 4.12. *Let $\mathcal{K}$ be a class of Σ-structures containing the empty product and set $T := \mathrm{Th}_{\mathrm{H}\forall}(\mathcal{K})$. Then*

$$\mathrm{QV}(\mathcal{K}) = \mathrm{ERP}(\mathcal{K}) = \mathrm{Mod}(T)\,.$$

Proof. Let $\mathcal{Q}$ be the class of all structures that can be embedded into a reduced product of structures in $\mathcal{K}$. Any quasivariety containing $\mathcal{K}$ must contain $\mathcal{Q}$. Hence, it is sufficient to show that $\mathcal{Q}$ is a quasivariety.

By Lemma 4.9, we have $\mathcal{Q} = \mathrm{Mod}(T)$. Every Horn formula in T is strict since $\mathcal{K}$ contains the empty product. Consequently, $T \subseteq \mathrm{SH}\forall[\Sigma]$ and $\mathcal{Q} = \mathrm{Mod}(T)$ is a quasivariety. □

We conclude this section with a analogous characterisations of varieties.

Definition 4.13. Let $\mathcal{K}$ be a class of structures. A element $\mathfrak{A} \in \mathcal{K}$ is *free* (in $\mathcal{K}$) if there exists a subset $C \subseteq A$ such that $\mathfrak{A}_C$ is a free model of $\langle C; \varnothing \rangle$. In this case we also say that $\mathfrak{A}$ is *freely generated by C*.

We can use Lemma 4.2 to obtain a characterisation of free structures.

Lemma 4.14. *Let $\mathcal{K}$ be a class of structures, $\mathfrak{A} \in \mathcal{K}$, and $C \subseteq A$. Then $\mathfrak{A}$ is freely generated by C if and only if $\mathfrak{A}$ is generated by C and, for every tuple $\bar{a} \subseteq C$ of distinct elements and each atomic formula $\varphi(\bar{x})$ with $\mathfrak{A} \vDash \varphi(\bar{a})$, we have*

$$\mathfrak{B} \vDash \forall \bar{x} \varphi\,, \quad \textit{for all } \mathfrak{B} \in \mathcal{K}\,.$$

Lemma 4.15. *Let $\mathcal{K}$ be a class of structures and $\mathfrak{A}$ and $\mathfrak{B}$ structures in $\mathcal{K}$ freely generated by, respectively, C and D. If $|C| = |D|$ then every bijection $C \to D$ extends to an isomorphism $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. Let $f : C \to D$ be a bijection. By definition of a free model, we can extend f to a homomorphism $g : \mathfrak{A} \to \mathfrak{B}$ and f^{-1} to a homomorphism $h : \mathfrak{B} \to \mathfrak{A}$. Since $h \circ g$ is a homomorphism $\mathfrak{A} \to \mathfrak{A}$ with $(h \circ g) \restriction C = \mathrm{id}_C$ it follows by uniqueness that $h \circ g = \mathrm{id}_A$. Similarly, we have $g \circ h = \mathrm{id}_B$. Hence, $g : \mathfrak{A} \to \mathfrak{B}$ is the desired isomorphism. □

Lemma 4.16. *Let $\mathcal{K}$ be a class of structures that is closed under nonempty products, substructures, and isomorphic copies.*

(a) *If a structure $\mathfrak{A} \in \mathcal{K}$ is generated by a set X of size $|X| = \kappa$ then $\mathcal{K}$ contains a structure $\mathfrak{F}_\kappa \in \mathcal{K}$ that is freely generated by a set of size κ. Furthermore, there exists a surjective homomorphism $\mathfrak{F}_\kappa \to \mathfrak{A}$.*

(b) *If $\mathcal{K}$ contains a structure with at least 2 elements then $\mathcal{K}$ contains, for every cardinal κ, a structure that is freely generated by a set of size κ.*

Proof. (a) Let C be a set of κ constant symbols. By Theorem 4.3, the $\mathcal{K}$-presentation $\langle C; \varnothing \rangle$ has a free model $\mathfrak{F}$. If we can show that $c^{\mathfrak{F}} \neq d^{\mathfrak{F}}$, for all distinct constants $c, d \in C$, then it follows that $\mathfrak{F}$ is freely generated by C.

For a contradiction, suppose that there are $c \neq d$ with $c^{\mathfrak{F}} = d^{\mathfrak{F}}$. By Lemma 4.14 it follows that every structure in $\mathcal{K}$ satisfies $\forall x \forall y (x = y)$. Hence, every structure in $\mathcal{K}$ has at most 1 element. This contradicts the fact that $\mathfrak{A}$ contains a subset $X \subseteq A$ of size κ.

Finally, note that we can extend any bijection $C \to X$ to a homomorphism $\mathfrak{F} \to \mathfrak{A}$. Since $\mathfrak{A}$ is generated by X this homomorphism is surjective and $\mathfrak{A}$ is a weak homomorphic image of $\mathfrak{F}$.

(b) follows from (a). If $\mathcal{K}$ contains a structure with at least 2 elements then $\mathcal{K}$ contains arbitrarily large structures since it is closed under products.

□

Theorem 4.17 (Birkhoff). *Let $\mathcal{K}$ be a class of Σ-structures. The following statements are equivalent:*

(1) *$\mathcal{K}$ is closed under nonempty products, substructures, and weak homomorphic images.*

(2) $\mathcal{K} = \mathrm{Var}(\mathcal{K})$

(3) *$\mathcal{K}$ is axiomatised by a set of formulae of the form $\forall \bar{x} \varphi$ where φ is an atomic formula.*

Proof. It is easy to see that (1) and (2) are equivalent. The implication (3) ⇒ (1) follows from Lemmas C2.1.6 and C2.1.3 (a), and Corollary 3.7. Hence, it remains to prove that (1) implies (3).

Set $\mathcal{H} := \mathrm{Mod}(T)$ where T is the set of all sentences $\forall\bar{x}\varphi \in \mathrm{Th}(\mathcal{K})$ where φ is an atomic formula. We have to show that $\mathcal{H} \subseteq \mathcal{K}$.

First, we consider the case that $\mathcal{K}$ contains a structure with at least 2 elements. Then $\mathcal{K}$ has arbitrarily large free structures $\mathfrak{F}$, by Lemma 4.16. Hence, $\mathfrak{F} \in \mathcal{K} \subseteq \mathcal{H}$. But the class $\mathcal{H}$ is closed under nonempty products, substructures, and weak homomorphic images, by (3) $\Rightarrow$ (1). By Lemma 4.14 it follows that $\mathfrak{F}$ is also a free structure of $\mathcal{H}$. Since free structures are uniquely determined by the cardinality of their set of generators we can conclude that $\mathcal{K}$ contains all free structures of $\mathcal{H}$. Since, by Lemma 4.16 (a) every structure of $\mathcal{H}$ is a weak homomorphic image of a free structure and $\mathcal{K}$ is closed under weak homomorphic images it follows that $\mathcal{H} \subseteq \mathcal{K}$.

It remains to consider the case that $\mathcal{K}$ only contains structures with at most 1 element. Then $\forall x \forall y(x = y) \in T$ and all structures of $\mathcal{H}$ contain at most 1 element. Since each such structure can be described up to isomorphism by formulae of the form $\forall\bar{x}\varphi$ it follows that $\mathcal{H} = \mathcal{K}$. $\square$

5. *The Theorem of Feferman and Vaught*

In general, first-order formulae are not preserved in products. Nevertheless the first-order theories of products are well behaved. We will prove below that the first-order theory of a product can be computed from the first-order theories of its factors. In fact, this result holds not only for ordinary direct products, but it can be extended to a quite general notion of a product.

Definition 5.1. Let S and T be disjoint sets of sorts, Σ an S-sorted signature, Γ a T-sorted one, and $\iota \in T$ a sort of T. Suppose that $(\mathfrak{A}^i)_{i \in I}$ is a sequence of Σ-structures and $\mathfrak{J}$ a Γ-structure whose domain of sort ι is $J_\iota = \mathscr{P}(I)$. For $s \in S$, let $I_s := \{\, i \in I \mid A^i_s \neq \varnothing \,\}$.

The *generalised product* of $(\mathfrak{A}^i)_i$ over $\mathfrak{J}$ is the structure

$$\prod_{i \in \mathfrak{J}} \mathfrak{A}^i := \big\langle U, \subseteq, E_=, (\zeta^{\mathfrak{J}})_{\zeta \in \Gamma}, (\xi')_{\xi \in \Sigma} \big\rangle,$$

with domains

$$U_s := \begin{cases} \prod_{i \in I_s} A^i_s & \text{for } s \in S\,, \\ J_s & \text{for } s \in T\,. \end{cases}$$

The relations and functions $\zeta^{\mathfrak{I}}$, for $\zeta \in \Gamma$, are taken from $\mathfrak{I}$, while the relations R', for $R \in \Sigma$, are defined by

$$R' := \left\{ \langle w, a_0, \dots, a_{n-1} \rangle \in \wp(I) \times U^n \;\middle|\; w = [\![R\bar{a}^i]\!]_{i \in I} \right\}.$$

As usual the functions f', for $f \in \Sigma$, are defined component wise

$$f'(\bar{a}) := (f^{\mathfrak{A}_i}(\bar{a}^i))_i\,.$$

Finally, $\subseteq$ is the subset relation on $J_\iota = \wp(I)$ and

$$E_= := \left\{ \langle w, a, b \rangle \in \wp(I) \times U^2 \;\middle|\; w = [\![a^i = b^i]\!]_{i \in I} \right\}.$$

Example. (a) Let $(\mathfrak{A}^i)_{i \in I}$ be a sequence of structures and $\mathfrak{u}$ a filter on I. The reduced product $\prod_i \mathfrak{A}^i/\mathfrak{u}$ can be interpreted in the generalised product $\prod_{i \in \mathfrak{I}} \mathfrak{A}^i$ with index structure $\mathfrak{I} := \langle \wp(I), \mathfrak{u} \rangle$. A relation R of $\prod_i \mathfrak{A}^i/\mathfrak{u}$ can be defined by the formula

$$\varphi_R(\bar{x}) := \exists z (Rz\bar{x} \wedge \mathfrak{u}z)\,.$$

(b) Suppose that $\mathfrak{G}_i = \langle V_i, E_i \rangle$, $i < 2$, are two directed graphs. Their *asynchronous product* is the graph $\mathfrak{H} = \langle V, E \rangle$ with universe $V := V_0 \times V_1$ and edge relation

$$E := (\mathrm{id}_{V_0} \times E_1) \cup (E_0 \times \mathrm{id}_{V_1})\,.$$

We can interpret $\mathfrak{H}$ in the generalised product over the index structure $\mathfrak{I} := \langle \wp[2] \rangle$ by the formula

$$\varphi_E(x, y) := \exists u \exists v [u \nsubseteq v \wedge v \nsubseteq u \wedge E_= uxy \wedge Evxy]\,,$$

which states that, for $x = \langle x_0, x_1 \rangle$ and $y = \langle y_0, y_1 \rangle$, there are sets $u = \{i\}$ and $v = \{k\}$ with $i \neq k$ such that $x_i = y_i$ and $\langle x_k, y_k \rangle \in E_k$.

Theorem 5.2 (Feferman-Vaught). *For every first-order formula $\varphi(\bar{x}, \bar{y})$, there exist a finite number of first-order formulae $\chi_0(\bar{x}), \dots, \chi_{m-1}(\bar{x})$ and $\psi(\bar{y}, \bar{z})$ such that,*

$$\prod_{i \in \mathfrak{J}} \mathfrak{A}^i \vDash \varphi(\bar{w}, \bar{a})$$

$$\text{iff} \quad \langle \mathfrak{J}, \subseteq \rangle \vDash \psi\big(\bar{w}, [\![\chi_0(\bar{a}^i)]\!]_i, \dots, [\![\chi_{m-1}(\bar{a}^i)]\!]_i\big),$$

for all sequences $(\mathfrak{A}^i)_{i \in I}$, index structures $\mathfrak{J}$, and tuples $\bar{a} \subseteq \prod_i A^i$ and $\bar{w} \subseteq J$.

Proof. We construct the formulae χ_i and ψ by induction on φ. If φ is an atomic formula whose free variables all range over J then we have

$$\prod_{i \in \mathfrak{J}} \mathfrak{A}^i \vDash \varphi(\bar{w}) \quad \text{iff} \quad \langle \mathfrak{J}, \subseteq \rangle \vDash \varphi(\bar{w}).$$

If $\varphi = Rst_0 \dots t_{n-1}$ where $R \in \Sigma$ then we have

$$\prod_{i \in \mathfrak{J}} \mathfrak{A}^i \vDash \varphi(\bar{w}, \bar{a}) \quad \text{iff} \quad [\![Rt_0 \dots t_{n-1}[\bar{a}^i]]\!]_i = s^{\mathfrak{J}}[\bar{w}].$$

Hence, we can set $\chi_0 := Rt_0 \dots t_{n-1}$ and $\psi := z_0 = s$.

Similarly, if $\varphi = E_= st_0 t_1$ then we define $\chi_0 := t_0 = t_1$ and $\psi := z_0 = s$. If φ is a boolean combination then we can take the corresponding boolean combination of the formulae obtained by inductive hypothesis.

Hence, it remains to consider the case that $\varphi = \exists z \varphi'(\bar{x}, \bar{y}, z)$. Let $\chi'_0, \dots, \chi'_{m-1}$ and ψ' be the formulae for φ' obtained from the inductive hypothesis. If z ranges over J then we have

$$\prod_{i \in \mathfrak{J}} \mathfrak{A}^i \vDash \varphi(\bar{w}, \bar{a})$$

iff there is some $w' \in J$ with

$$\langle \mathfrak{J}, \subseteq \rangle \vDash \psi'\big(\bar{w}, w', [\![\chi'_0(\bar{a}^i)]\!]_i, \dots, [\![\chi'_{m-1}(\bar{a}^i)]\!]_i\big)$$

iff $\langle \mathfrak{J}, \subseteq \rangle \vDash \exists z' \psi'\big(\bar{w}, z', [\![\chi'_0(\bar{a}^i)]\!]_i, \dots, [\![\chi'_{m-1}(\bar{a}^i)]\!]_i\big)$.

If, on the other hand, z ranges over sequences in $\prod_i A^i$ then we proceed as follows. As φ only mentions finitely many symbols of the signature we may assume that the signature is finite. Therefore, every first-order formula can be written as a finite disjunction of Hintikka-formulae. Let r be the maximal quantifier rank of the formulae χ'_l, $l < m$, and let $\chi''_0, \dots, \chi''_{p-1}$ be an enumeration of all Hintikka-formulae of this quantifier rank. We can find a formula ψ'' such that

$$\begin{aligned} &\langle \mathfrak{I}, \subseteq \rangle \vDash \psi'\big(\bar{w}, [\![\chi'_0(\bar{a}^i, b^i)]\!]_i, \dots, [\![\chi'_{m-1}(\bar{a}^i, b^i)]\!]_i\big) \\ \text{iff} \quad &\langle \mathfrak{I}, \subseteq \rangle \vDash \psi''\big(\bar{w}, [\![\chi''_0(\bar{a}^i, b^i)]\!]_i, \dots, [\![\chi''_{p-1}(\bar{a}^i, b^i)]\!]_i\big) . \end{aligned}$$

Therefore, we may w.l.o.g. assume that, for all elements $\bar{a}$ and b, the sets

$$[\![\chi'_0(\bar{a}^i, b^i)]\!]_i, \dots, [\![\chi'_{m-1}(\bar{a}^i, b^i)]\!]_i$$

form a partition of I. For $s \subseteq [m]$, let

$$\chi_s(\bar{x}) := \bigwedge_{l \in s} \exists z \chi'_l(\bar{x}, z) \wedge \forall z \bigvee_{l \in s} \chi'_l(\bar{x}, z) ,$$

and define

$$\begin{aligned} \psi(\bar{y}, \bar{z}) := \exists u_0 \cdots \exists u_{m-1}\Big(&\text{“}u_0, \dots, u_{m-1} \text{ form a partition of } I\text{”} \\ &\wedge \psi'(\bar{y}, \bar{u}) \wedge \bigwedge_{l<m} u_l \subseteq \bigcup_{s \ni l} z_s\Big) . \end{aligned}$$

We claim that the formulae ψ and χ_s, for $s \subseteq [m]$, have the desired properties. Note that

$$k \in [\![\chi_s(\bar{a}^i)]\!]_i \quad \text{iff} \quad s = \big\{\, l < m \bigm| k \in [\![\exists z \chi'_l(\bar{a}^i, z)]\!]_i \,\big\} ,$$

which implies that

$$\begin{aligned} k \in \bigcup_{s \ni l} [\![\chi_s(\bar{a}^i)]\!]_i \quad &\text{iff} \quad k \in [\![\chi_s(\bar{a}^i)]\!]_i \text{ for some } s \ni l \\ &\text{iff} \quad l \in \big\{\, l < m \bigm| k \in [\![\exists z \chi'_l(\bar{a}^i, z)]\!]_i \,\big\} \\ &\text{iff} \quad k \in [\![\exists z \chi'_l(\bar{a}^i, z)]\!]_i . \end{aligned}$$

First, suppose that there is some $b \in \prod_i A^i$ with

$$\prod_{i \in \mathfrak{I}} \mathfrak{A}^i \vDash \varphi'(\bar{a}, \bar{w}, b) .$$

Setting $u_l := [\![\chi'_l(\bar{a}^i, b^i)]\!]_i$ it follows by inductive hypothesis that

$$\langle \mathfrak{I}, \subseteq \rangle \vDash \psi'(\bar{w}, u_0, \dots, u_{m-1}) .$$

Furthermore, $u_l \subseteq [\![\exists z \chi'_l(\bar{a}^i, z)]\!]_i$ which, by the above remark, implies that $u_l \subseteq \bigcup_{s \ni l} [\![\chi_s(\bar{a}^i)]\!]_i$. Since, by assumption, $u_0, \dots, u_{m-1}$ form a partition of I, it follows that

$$\langle \mathfrak{I}, \subseteq \rangle \vDash \psi\big(\bar{w}, ([\![\chi_s(\bar{a}^i)]\!]_i)_{s \subseteq [m]}\big) .$$

Conversely, suppose that

$$\langle \mathfrak{I}, \subseteq \rangle \vDash \psi\big(\bar{w}, ([\![\chi_s(\bar{a}^i)]\!]_i)_{s \subseteq [m]}\big) .$$

Then there are sets $u_l \subseteq [\![\exists z \chi'_l(\bar{a}^i, z)]\!]_i$, $l < m$, forming a partition of I such that

$$\langle \mathfrak{I}, \subseteq \rangle \vDash \psi'(\bar{w}, u_0, \dots, u_{m-1}) .$$

For each $i \in u_l$, fix some element $b^i \in A^i$ with $\mathfrak{A}^i \vDash \chi'_l(\bar{a}^i, b^i)$. Since the u_l form a partition of I this defines an element $b \in \prod_i A^i$. By inductive hypothesis, we have

$$\prod_{i \in \mathfrak{I}} \mathfrak{A}^i \vDash \varphi'(\bar{a}, \bar{w}, b) .$$

□

Corollary 5.3. *Let $(\mathfrak{A}^i)_{i \in I}$ and $(\mathfrak{B}^i)_{i \in I}$ be two sequences of structures and suppose that $\mathfrak{I}$ is a suitable index structure.*

$$\mathfrak{A}^i \equiv \mathfrak{B}^i , \quad \textit{for all } i \in I , \quad \textit{implies} \quad \prod_{i \in \mathfrak{I}} \mathfrak{A}^i \equiv \prod_{i \in \mathfrak{I}} \mathfrak{B}^i .$$

D3. O-minimal structures

1. Ordered topological structures

In this chapter we study ordered algebraic structures where the definable relations have similar properties as those in real closed fields. We start with some general remarks concerning densely ordered structures and the order topology.

Definition 1.1. Let $\langle A, < \rangle$ be an open dense linear order.

(a) For convenience, we add to A a least element $-\infty$ and a greatest one $+\infty$. Let A_∞ denote the resulting order.

(b) An *interval* is a nonempty set of the form

$$\begin{aligned}
&(a, b) := {\uparrow}a \cap {\downarrow}b\,, \qquad &&[a, b) := {\Uparrow}a \cap {\downarrow}b\,,\\
\text{or}\quad &(a, b] := {\uparrow}a \cap {\Downarrow}b\,, \qquad &&[a, b] := {\Uparrow}a \cap {\Downarrow}b\,,
\end{aligned}$$

with $a, b \in A_\infty$. Intervals of the form (a, b) are called *open,* those of the form $[a, b]$ *closed.*

(c) For functions $f, g : D \to A_\infty$ with $D \subseteq A$, we define

$$\begin{aligned}
f < g \quad &:\text{iff} \quad f(c) < g(c) \quad \text{for all } c \in D\,,\\
f \leq g \quad &:\text{iff} \quad f(c) \leq g(c) \quad \text{for all } c \in D\,,
\end{aligned}$$

and we set

$$\begin{aligned}
(f, g) &:= \{\, \langle c, a \rangle \in D \times A \mid f(c) < a < g(c) \,\}\,,\\
[f, g] &:= \{\, \langle c, a \rangle \in D \times A_\infty \mid f(c) \leq a \leq g(c) \,\}\,.
\end{aligned}$$

(d) We equip A with the order topology and each product A^n with the corresponding product topology. For $\bar{a}, \bar{b} \in A^n$, we define

$$B(\bar{a}, \bar{b}) := (a_0, b_0) \times \cdots \times (a_{n-1}, b_{n-1}) \subseteq A^n .$$

Sets of this form are called *boxes.* Recall that the topological closure of a set $U \subseteq A$ is denoted by $\mathrm{cl}(U)$, its interior by $\mathrm{int}(U)$, and the boundary by ∂U.

Remark. For every $n < \omega$, the set of boxes forms an open base for the topology on A^n. This topology is Hausdorff.

Definition 1.2. A function $f : A \to B$ between linear orders is *monotone* if it is either increasing or decreasing. It is *strictly monotone* if it is strictly increasing or strictly decreasing.

The following lemma gives a criterion for a function defined on a direct product to be continuous. It will be used in Section 3.

Lemma 1.3. *Let $\mathfrak{X}$ be a topological space, $\langle A, < \rangle$ and $\langle B, < \rangle$ open dense linear orders, and $f : X \times A \to B$ a function such that*

(1) *for each $x \in X$, the function $f(x, \cdot) : A \to B$ is continuous and monotone, and*

(2) *for each $a \in A$, the function $f(\cdot, a) : X \to B$ is continuous.*

Then f is continuous.

Proof. Let $J \subseteq B$ be an open interval. To prove that $f^{-1}[J]$ is open we show that, for every pair $\langle x, a \rangle \in f^{-1}[J]$, there are open sets $O \subseteq X$ and $I \subseteq A$ with $\langle x, a \rangle \in O \times I$ and $f[O \times I] \subseteq J$.

By (1) there is an open interval $(b_0, b_1) \subseteq A$ with $a \in (b_0, b_1)$ such that $f[\{x\} \times (b_0, b_1)] \subseteq J$. We use (2) to obtain open sets $O_0, O_1 \subseteq X$ such that $f[O_i \times \{b_i\}] \subseteq J$, for $i < 2$. Let $O := O_0 \cap O_1$. We claim that $f[O \times (b_0, b_1)] \subseteq J$.

Let $y \in O$ and $b_0 < c < b_1$. By symmetry, we assume that the function $f(y, \cdot) : A \to B$ is increasing. Then $f(y, b_0) \leq f(y, c) \leq f(y, b_1)$. Since $f(y, b_0), f(y, b_1) \in J$, this implies that $f(y, c) \in J$. $\square$

We investigate the structure of definable relations in ordered structures. Throughout this chapter we will work with definitions with parameters.

Definition 1.4. Let $\mathfrak{A}$ be a structure.

(a) A relation $R \subseteq A^n$ is *parameter-definable* if there exists a first-order formula $\varphi(\bar{x}; \bar{y})$ and parameters $\bar{c} \subseteq A$ such that $R = \varphi(\bar{x}; \bar{c})^{\mathfrak{A}}$.

(b) A topology $\mathcal{C}$ on $\mathfrak{A}$ is *definable* if there exists a first-order formula $\varphi(x, \bar{y}; \bar{z})$ and parameters $\bar{c} \subseteq A$ such that the family $(\varphi(x, \bar{a}; \bar{c})^{\mathfrak{A}})_{\bar{a} \subseteq A}$ is a base of $\mathcal{C}$.

Lemma 1.5. *Let $\mathfrak{A} = \langle A, < \rangle$ be an open dense linear order and $n < \omega$.*

(a) *There exists a formula $\beta(\bar{x}; \bar{y}, \bar{z})$ such that*

$$\mathfrak{A} \vDash \beta(\bar{c}; \bar{a}, \bar{b}) \quad \text{iff} \quad \bar{c} \in B(\bar{a}, \bar{b}) .$$

(b) *If $X \subseteq A^n$ is parameter-definable then so are* $\mathrm{cl}(X)$ *and* $\mathrm{int}(X)$.

(c) *If $X \subseteq Y \subseteq A^n$ are parameter-definable sets and X is open in Y then there exists a parameter-definable open set O such that $X = Y \cap O$.*

Proof. (a) Set

$$\beta(\bar{x}; \bar{y}, \bar{z}) := \bigwedge_{i<n} (y_i < x_i \wedge x_i < z_i) .$$

(b) Let $\varphi(\bar{x})$ be the formula defining X. By (a), there exists a formula expressing that $\bar{c} \in B(\bar{a}, \bar{b})$. We can define $\mathrm{cl}(X)$ by the formula

$$\psi(\bar{x}) := \forall \bar{y}\bar{z}\big[\bar{x} \in B(\bar{y}, \bar{z}) \rightarrow (\exists \bar{u} \in B(\bar{y}, \bar{z}))\varphi(\bar{u})\big] ,$$

which expresses that every neighbourhood of $\bar{x}$ contains a point of X. Similarly, we can define $\mathrm{int}(X)$ by

$$\vartheta(\bar{x}) := \exists \bar{y}\bar{z}\big[\bar{x} \in B(\bar{y}, \bar{z}) \wedge (\forall \bar{u} \in B(\bar{y}, \bar{z}))\varphi(\bar{u})\big] .$$

(c) Let $\varphi(\bar{x})$ and $\psi(\bar{x})$ be the formulae defining X and Y, respectively and set

$$O := \bigcup \{ B(\bar{a}, \bar{b}) \mid B(\bar{a}, \bar{b}) \cap Y \subseteq X \} .$$

Then O is an open set with $Y \cap O = X$. It can be defined by the formula

$$\vartheta(\bar{x}) := \exists \bar{y}\bar{z}[\bar{x} \in B(\bar{y}, \bar{z}) \wedge (\forall \bar{u} \in B(\bar{y}, \bar{z}))(\psi(\bar{u}) \rightarrow \varphi(\bar{u}))] . \quad \square$$

We have seen that every parameter-definable relation in a real closed field is given by a boolean combination of polynomial equations and inequalities. As a consequence these relations are structurally quite tame. The next definition isolates the combinatorial core responsible for this simplicity.

Definition 1.6. A structure $\mathfrak{A}$ is *o-minimal* if there exists a parameter-definable open dense linear order $<$ on A such that every parameter-definable subset $X \subseteq A$ is a finite union of singletons $\{a\}$ and open intervals (a, b) with $a, b \in A_\infty$.

In this chapter $<$ will always denote the order with respect to which the given structure is o-minimal.

Example. (a) Every open dense linear order $\langle A, < \rangle$ is o-minimal since these structures admit quantifier elimination.

(b) As already mentioned above, real closed fields are another prominent example of o-minimal structures. Because of quantifier elimination each parameter-definable set in such a field is a boolean combination of sets defined by polynomial inequalities. To see that a real closed field is o-minimal it is therefore sufficient to note that every inequality $p[x] > 0$ defines a finite union of open intervals.

Lemma 1.7. *Let $\mathfrak{A}$ be an o-minimal structure and $X \subseteq A$ parameter-definable.*

(a) $\inf X$ *and* $\sup X$ *exist in* A_∞.

(b) *∂X is finite. Let $a_1 < \cdots < a_{n-1}$ be an increasing enumeration of ∂X and set $a_0 := -\infty$ and $a_n := \infty$. Each interval (a_i, a_{i+1}), $0 \leq i < n$, is either contained in X or disjoint from X.*

Proof. By definition of o-minimality, X is of the form

$$X = (a_0, b_0) \cup \cdots \cup (a_{n-1}, b_{n-1}) \cup \{c_0, \dots, c_{m-1}\}.$$

Consequently,

$$\begin{aligned} &\sup X = \max\{b_0, \dots, b_{n-1}, c_0, \dots, c_{m-1}\} \\ \text{and} \quad &\inf X = \min\{a_0, \dots, a_{n-1}, c_0, \dots, c_{m-1}\} \end{aligned}$$

exist. For the second claim, note that

$$\partial X \subseteq \{a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}, c_0, \dots, c_{m-1}\}$$

is finite. W.l.o.g. we may assume that the decomposition of X has been chosen such that

$$\begin{aligned} &(a_i, b_i) \cap (a_k, b_k) = \varnothing, && \text{for } i \neq k, \\ \text{and} \quad &c_i \notin (a_k, b_k), && \text{for all } i, k. \end{aligned}$$

If $d < e$ are consecutive elements of an increasing enumeration of X then we either have

$$\begin{aligned} &d = a_i, \quad e = b_i, && \text{and} \quad (d,e) = (a_i, b_i) \subseteq X, \\ &d = b_i, \quad e = a_{i+1}, && \text{and} \quad (d,e) \cap X = \varnothing, \\ &d = b_i, \quad e = c_k, && \text{and} \quad (d,e) \cap X = \varnothing, \\ &d = c_i, \quad e = a_k, && \text{and} \quad (d,e) \cap X = \varnothing, \\ \text{or} \quad &d = c_i, \quad e = c_k, && \text{and} \quad (d,e) \cap X = \varnothing. \end{aligned}$$

□

Definition 1.8. Let $\langle A, < \rangle$ be an open dense linear order and $n < \omega$. A set $X \subseteq A^n$ is *definably connected* if it is parameter-definable and there is no partition $X = Y_0 \cup Y_1$ of X into two disjoint nonempty parameter-definable subsets $Y_0, Y_1 \subseteq X$ that are open in X.

Lemma 1.9. *Let $\mathfrak{A}$ be an o-minimal structure.*

(a) *A subset $X \subseteq A$ is definably connected if and only if it is either empty or a single interval.*

(b) *The image of a definably connected set $X \subseteq A^m$ under a continuous parameter-definable function $f : X \to A^n$ is definably connected.*

(c) *Let $X, Y \subseteq A^n$ be parameter-definable. If $X \subseteq Y \subseteq \mathrm{cl}(X)$ and X is definably connected then so is Y.*

(d) *If $X, Y \subseteq A^n$ are definably connected and $X \cap Y \neq \varnothing$ then $X \cup Y$ is definably connected.*

Proof. (a) By definition of o-minimality, X is of the form

$$X = (a_0, b_0) \cup \cdots \cup (a_{n-1}, b_{n-1}) \cup \{c_0, \ldots, c_{m-1}\},$$

where we assume that n and m are chosen minimal. If $n > 1$, or $n = 1$ and $m > 0$, then we can decompose X into the sets

$$\begin{aligned} Y_0 &:= (a_0, b_0) \\ Y_1 &:= (a_1, b_1) \cup \cdots \cup (a_{n-1}, b_{n-1}) \cup \{c_0, \ldots, c_{m-1}\}. \end{aligned}$$

Similarly, if $n = 0$ and $m > 1$ then we can set $Y_0 := \{c_0\}$ and $Y_1 := \{c_1, \ldots, c_{m-1}\}$. Consequently, the pair $\langle n, m\rangle$ can only take the values $\langle 0, 0\rangle$, $\langle 0, 1\rangle$, or $\langle 1, 0\rangle$. In the first case $X = \varnothing$ and, otherwise, X is an interval.

(b) Suppose that $f[X]$ is not definably connected. Let $Y_0 \cup Y_1 = f[X]$ be the corresponding decomposition. Then we obtain a decomposition $f^{-1}[Y_0] \cup f^{-1}[Y_1] = X$ of X into two disjoint nonempty parameter-definable open subsets. Hence, X is not definably connected.

(c) Suppose that Y is not definably connected and let $Z_0 \cup Z_1 = Y$ be the corresponding decomposition. The sets $Z_0 \cap X$ and $Z_1 \cap X$ are disjoint, parameter-definable, and open in X. If we can show that they are nonempty then the result follows. Fix $a \in Z_i \subseteq \mathrm{cl}(X)$ and an open set O such that $O \cap Y \subseteq Z_i$. Since $a \in \mathrm{cl}(X)$ it follows that $O \cap X \neq \varnothing$. Hence, there is some element $b \in O \cap X \subseteq (O \cap Y) \cap X \subseteq Z_i \cap X$.

(d) Suppose that $X \cup Y$ is not definably connected and let $Z_0 \cup Z_1 = X \cup Y$ be a corresponding decomposition. If $Z_0 \cap X \neq \varnothing$ and $Z_1 \cap X \neq \varnothing$ then $Z_0 \cap X$ and $Z_1 \cap X$ witness the fact that X is not definably connected. Suppose that $Z_0 \cap X = \varnothing$, i.e., $X \subseteq Z_1$. Then we have $Y \cap Z_1 \supseteq (X \cap Y) \cap Z_1 = X \cap Y \neq \varnothing$ and $Y \cap Z_0 = Z_0 \neq \varnothing$. Consequently, Y is not definably connected. □

Corollary 1.10. *Let $\mathfrak{A}$ be an o-minimal structure and $f : [a, b] \to A$ parameter-definable and continuous. Then* $\operatorname{rng} f$ *contains every element between $f(a)$ and $f(b)$.*

2. O-minimal groups and rings

Before continuing to develop the theory of o-minimal structures let us give examples of o-minimal structures from algebra. We consider groups and rings.

Proposition 2.1. *Let $\mathfrak{M}$ be an o-minimal structure and suppose that $\cdot$ is a parameter-definable operation such that $\mathfrak{G} := \langle M, \cdot, < \rangle$ forms an ordered group.*

(a) *The only parameter-definable subgroups of $\mathfrak{G}$ are $\{e\}$ and M.*

(b) *$\mathfrak{G}$ is abelian, divisible, and torsion-free.*

Proof. (a) Let $H \subset M$ be a parameter-definable proper subgroup of $\mathfrak{G}$. First, we show that H is convex. For a contradiction, suppose otherwise. Then there are elements $h \in H$ and $a \in M \setminus H$ with $e < a < h$. This implies that $h^n < ah^n < h^{n+1}$, for all n. Consequently, we obtain a strictly increasing sequence

$$e < a < h < ah < h^2 < ah^2 < h^3 < \dots$$

where every second element belongs to H while the other elements belong to $M \setminus H$. Hence, H cannot be written as a finite union of intervals. A contradiction.

By Lemma 1.7, the supremum $c := \sup H$ exists. Because H is convex it follows that $(e, c) \subseteq H$. Suppose that $c > e$ and let $h \in (e, c)$. Then $h < c$ implies $e < h^{-1}c$ and $e < h$ implies $h^{-1} < e$ and $h^{-1}c < c$. Hence, $h^{-1}c \in (e, c) \subseteq H$ and it follows that $c = hh^{-1}c \in H$. Thus, we have $c < ch \in H$, in contradiction to the choice of c. Consequently, we have $c = e$ and $H = \{e\}$.

(b) We have already shown in Lemma D1.4.5 that all ordered groups are torsion-free.

For every $a \in M$, the centraliser $C(a) := \{ x \in M \mid ax = xa \}$ is a parameter-definable subgroup of $\mathfrak{G}$. Since $a \in C(a)$ it follows by (a) that $C(a) = M$. Consequently, every element a commutes with all other elements and $\mathfrak{G}$ is abelian.

Analogously, for $1 < n < \omega$, we can consider the non-trivial parameter-definable subgroup $D_n := \{ a^n \mid a \in M \}$. By (a), it follows that $D_n = M$. Hence, for every $a \in M$ there is some $b \in M$ with $a = b^n$. Consequently, $\mathfrak{G}$ is divisible. □

Theorem 2.2. *An ordered group $\mathfrak{G}$ is o-minimal if and only if it is abelian, divisible, and torsion-free.*

Proof. ($\Rightarrow$) was already shown in Proposition 2.1. For ($\Leftarrow$), suppose that $\mathfrak{G} = \langle G, +, -, 0, < \rangle$ is a model of ODAG. We have seen in Theorem D1.4.16 that this theory admits quantifier elimination. Hence, every parameter-definable subset $X \subseteq G$ is given as a boolean combination of inequalities $x < a$, for $a \in G$. It follows that X can be written as a finite union of intervals. □

Theorem 2.3. *Let $\mathfrak{A}$ be an o-minimal structure and suppose that $+$ and $\cdot$ are parameter-definable operations such that $\langle A, +, \cdot, < \rangle$ forms an ordered ring. Then $\langle A, +, \cdot, < \rangle$ is a real closed field.*

Proof. For every $a \in A$, there exists the parameter-definable additive subgroup $aA := \{ ax \mid x \in A \}$. If $a \neq 0$ then $a \in aA$ implies, by Proposition 2.1 (a), that $aA = A$. In particular, there is some element $b \in A$ with $ab = 1$.

Let $P := \{ a \in A \mid a > 0 \}$. Then P is closed under multiplication and, hence, forms an ordered group $\langle P, \cdot, < \rangle$. By Proposition 2.1 (b), it follows that this group is abelian. Since, for every element $a \in A$, we have $a \in P$, or $a = 0$, or $-a \in P$, it follows that $\cdot$ is commutative, for all elements of A. Consequently, $\langle A, +, \cdot, < \rangle$ is an ordered field.

It remains to prove that it is real closed. We use the characterisation of Proposition B6.6.17. Let $p \in A[x]$ be a polynomial over A. The corresponding polynomial function $A \to A : a \mapsto p[a]$ is parameter-definable. Suppose that $a < b$ are elements with $p[a] < 0 < p[b]$. By Corollary 1.10, there exists an element $c \in (a, b)$ with $p[c] = 0$. $\square$

Corollary 2.4. *An ordered ring is o-minimal if and only if it is a real closed field.*

Besides real closed fields and models of ODAG, let us also mention the following example of an o-minimal structure.

Theorem 2.5 (Wilkie). *The structure* $\langle \mathbb{R}, +, \cdot, 0, 1, \exp \rangle$ *is o-minimal where* $\exp(x) := e^x$ *is the exponential function.*

3. *Cell decompositions*

In this section we prove an important structure result on parameter-definable relations in o-minimal structures. We will show that each such relation can be decomposed into finitely many 'simple' parts.

We start by considering binary relations $R \subseteq M^2$. The general theorem below will then follow by induction on the arity.

Lemma 3.1. *Let $\mathfrak{M}$ be o-minimal and $f : (a, b) \to M$ parameter-definable.*

(a) *There exist elements $a \leq c < d \leq b$ such that $f \restriction (c, d)$ is either constant or injective.*

(b) *If f is injective then there are elements $a \leq c < d \leq b$ such that $f \restriction (c, d)$ is strictly monotone.*

(c) *If f is strictly monotone then there are elements $a \leq c < d \leq b$ such that $f \upharpoonright (c, d)$ is continuous.*

Proof. (a) If there is some $x \in M$ such that $f^{-1}(x)$ is infinite then, being parameter-definable, $f^{-1}(x)$ contains an open interval (c, d). Hence, $f \upharpoonright (c, d)$ is constant.

It remains to consider the case that all sets $f^{-1}(x)$, $x \in M$, are finite. Then $f[(a, b)]$ is an infinite parameter-definable subset of M. Hence, it contains some interval I. We define a function $g : I \to (a, b)$ by

$$g(z) := \min \{ c \mid f(c) = z \} .$$

The function g is injective since it has a left-inverse f. As above, we can conclude that $g[I]$ is infinite and it contains an interval (c, d). Setting $J := f[(c, d)]$ it follows that the restriction $g \upharpoonright J : J \to (c, d)$ is surjective. Consequently, $g \upharpoonright J$ is a bijection between J and (c, d) and f is its inverse. In particular, $f \upharpoonright (c, d)$ is injective.

(b) Let $x \in (a, b)$. Since f is injective, we have a partition

$$\begin{aligned} (a, x) = \{ y \in (a, x) \mid f(y) < f(x) \} \\ \cup \{ y \in (a, x) \mid f(y) > f(x) \} . \end{aligned}$$

One of these two sets must contain an interval (c, x), for some $a < c < x$. The same holds for the interval (x, b). For $\sigma, \rho \in \{+, -\}$, define

$$\begin{aligned} \varphi_{\sigma\rho}(x) := \exists y \exists z \big[a < y < x < z < b \\ \wedge \forall u [y < u < x \to f(x) <^{\sigma} f(u)] \\ \wedge \forall u [x < u < z \to f(x) <^{\rho} f(u)] \big] , \end{aligned}$$

where $<^{+} := <$ and $<^{-} := >$. It follows that every $x \in (a, b)$ satisfies exactly one of the formulae $\varphi_{++}, \varphi_{+-}, \varphi_{-+}, \varphi_{--}$.

Consequently, (a, b) contains an open interval all elements of which satisfy the same formula. Replacing (a, b) by this interval we may assume that all elements of (a, b) satisfy the same formula. By symmetry, we may further assume that this formula is either φ_{-+} or φ_{++}.

First, suppose that all elements in (a, b) satisfy φ_{-+}. For $x \in (a, b)$, let

$$s(x) := \sup \{ s \in (x, b) \mid f(x) < f(z) \text{ for all } z \in (x, s] \} .$$

Then we have $s(x) = b$ since $s(x) < b$ would contradict $\varphi_{-+}(s(x))$. Consequently, f is strictly increasing.

It remains to consider the case that all elements in (a, b) satisfy φ_{++}. Set

$$B := \{ x \in (a, b) \mid f(x) < f(z) \text{ for all } z \in (x, b) \} .$$

If B is infinite then it contains an open interval I. Hence, f is strictly increasing on I and we are done. Consequently, let us assume that B is finite. Replacing a by $\sup B$ we may assume that,

$(*)$ for every $x \in (a, b)$, there is some $x < y < b$ with $f(y) < f(x)$.

Fix $c \in (a, b)$. We claim that, for all sufficiently large elements $y \in (c, b)$, we have $f(y) < f(c)$. Otherwise, we would have $f(y) > f(c)$, for all sufficiently large $y \in (c, b)$. Let $d \in [c, b)$ be the minimal element such that $f(y) > f(c)$ for all $y \in (d, b)$. If $f(d) > f(c)$ then d would not be minimal since $\varphi_{++}(d)$ holds. Hence, $f(d) < f(c)$ and, by $(*)$, there is some $d < e < b$ such that $f(e) < f(d) < f(c)$. Contradiction.

Consequently, we have $f(y) < f(c)$, for all sufficiently large y. Set

$$y(c) := \inf \{ y \in [c, b) \mid f(z) < f(c) \text{ for all } z \in (y, b) \} .$$

Then $\varphi_{++}(c)$ implies that $c < y(c)$ and $f(y(c)) < f(c)$. Minimality of $y(c)$ implies that $y(c)$ satisfies the following formula:

$$\begin{aligned} \psi_{\searrow}(y) := \exists uv\big[a < u < y < v < b \\ \wedge \forall st[u < s < y < t < v \rightarrow f(s) > f(t)]\big] . \end{aligned}$$

Since c was arbitrary it follows that, for every element $c \in (a, b)$, there is some $y \in (c, b)$ satisfying $\psi_{\searrow}$.

Therefore, there is an interval $(d,b) \subseteq (a,b)$ such that $\psi_{\searrow}$ holds for all $y \in (d,b)$. Replacing a by d we may assume that all elements of (a,b) satisfy this formula.

Let $\psi_{\nearrow}$ be the formula obtained from $\psi_{\searrow}$ by replacing the inequality $f(s) > f(t)$ by $f(s) < f(t)$. An analogous argument shows that we may assume that every element of (a,b) satisfies $\psi_{\nearrow}$. But no element can simultaneously satisfy $\psi_{\searrow}$ and $\psi_{\nearrow}$. Contradiction.

(c) By symmetry, we may assume that f is strictly increasing. Since $\operatorname{rng} f$ is infinite it contains an open interval $I \subseteq \operatorname{rng} f$. Choose elements $x < y$ in I and set $c := f^{-1}(x)$ and $d := f^{-1}(y)$. Then f induces an order-preserving bijection $(c,d) \to (x,y)$. Every order-isomorphism is continuous since the topology is defined in terms of the order. Consequently, $f \upharpoonright (c,d)$ is continuous. □

Theorem 3.2 (Monotonicity Theorem). *Let $\mathfrak{M}$ be o-minimal and $f : (a,b) \to M$ parameter-definable. There exist elements*

$$a = a_0 < a_1 < \dots < a_n = b$$

such that, for every $i < n$, the restriction $f \upharpoonright (a_i, a_{i+1})$ is either constant, or strictly monotone and continuous.

Proof. Let X be the set of all elements $x \in (a,b)$ such that, for some $a \leq c < x < d \leq b$, the restriction $f \upharpoonright (c,d)$ is either constant, or strictly monotone and continuous. Note that $(a,b) \setminus X$ is finite since, otherwise, it would contain some interval I and we could use Lemma 3.1 to find an interval $I_0 \subseteq I$ such that $f \upharpoonright I_0$ is either constant, or strictly monotone and continuous. This would imply $I_0 \subseteq X$. A contradiction.

Let $b_1 < \dots < b_{m-1}$ be an enumeration of $(a,b) \setminus X$ and set $b_0 := a$ and $b_m := b$. It is sufficient to prove the theorem for $f \upharpoonright (b_i, b_{i+1})$. Hence, we may w.l.o.g. assume that $X = (a,b)$. There exist finitely many elements $a = a_0 < a_1 < \dots < a_{n-1} < a_n = b$ such that, for each interval (a_i, a_{i+1}), one of the following cases occurs:

(1) For all $x \in (a_i, a_{i+1})$, f is constant on some neighbourhood of x.

(2) For all $x \in (a_i, a_{i+1})$, f is strictly increasing on some neighbourhood of x.

(3) For all $x \in (a_i, a_{i+1})$, f is strictly decreasing on some neighbourhood of x.

We consider each case in turn.

(1) Fix some element $x \in (a_i, a_{i+1})$ and set

$$s := \sup \{\, y \in (x, a_{i+1}) \mid f \text{ is constant on } [x, y) \,\} .$$

Then we have $s = a_{i+1}$ since, if $s < a_{i+1}$, then $s \in (a_i, a_{i+1})$ and f would be constant on some neighbourhood of s. A contradiction. Therefore, f is constant on $[x, a_{i+1})$. In the same way we can show that f is constant on $(a_i, x]$. Hence, it is constant on the whole interval (a_i, a_{i+1}).

(2) Fix some $x \in (a_i, a_{i+1})$ and set

$$s := \sup \{\, y \in (x, a_{i+1}) \mid f \text{ is strictly increasing on } [x, y) \,\} .$$

As above, we have $s = a_{i+1}$ and f is strictly increasing on $[x, a_{i+1})$. Similarly, it is strictly increasing on $(a_i, x]$.

(3) This case follows in the same way as (2). □

Corollary 3.3. *Let $\mathfrak{M}$ be o-minimal and $f : (a, b) \to M$ parameter-definable.*

(a) *For every $c \in [a, b)$, the right sided limit $\lim_{x \downarrow c} f(x)$ exist in M_∞.*

(b) *For every $c \in (a, b]$, the left sided limit $\lim_{x \uparrow c} f(x)$ exist in M_∞.*

Corollary 3.4. *Let $\mathfrak{M}$ be o-minimal and $f : [a, b] \to M$ parameter-definable. Then f takes a maximum and a minimum value on $[a, b]$.*

The Cell Decomposition Theorem below is proved by an induction on the dimension. For the base case of this induction, we will need the following technical result.

Theorem 3.5. *Let $\mathfrak{M}$ be o-minimal and suppose that $R \subseteq M^2$ is a parameter-definable relation such that, for every $a \in M$, the fibre*

$$R_a := \{\, b \in M \mid \langle a, b\rangle \in R \,\}$$

is finite. Then there is a constant $n < \omega$ such that $|R_a| \leq n$, for all $a \in M$.

Proof. We call a pair $\langle a, b\rangle \in M^2_\infty$ *generic* if there exist open intervals $I, J \subseteq M_\infty$ with $\langle a, b\rangle \in I \times J$ such that either

- $R \cap I \times J = \emptyset$, or
- $\langle a, b\rangle \in R$ and $R \cap I \times J$ is the graph of a continuous function $I \to M$.

(In this definition we consider intervals of the form $(c, \infty]$ and $[-\infty, c)$ as open.) Note that the sets

$$\begin{aligned}
G_0 &:= \{\, \langle a, b\rangle \in M^2 \mid \langle a, b\rangle \text{ is generic} \,\}\,,\\
G_+ &:= \{\, a \in M \mid \langle a, \infty\rangle \text{ is generic} \,\}\,,\\
G_- &:= \{\, a \in M \mid \langle -\infty, b\rangle \text{ is generic} \,\}
\end{aligned}$$

are parameter-definable. For $n < \omega$, let s_n be the (parameter-definable) function with

$$\operatorname{dom} s_n = \{\, a \in M \mid |R_a| \geq n \,\}$$

such that $s_n(a) := b_n$ where $b_0 < b_1 < \dots < b_n < \dots$ is an enumeration of R_a.

For an element $a \in M$, let n be the maximal number such that the functions $s_0, \dots, s_{n-1}$ are defined and continuous on some neighbourhood of a. We call a *normal* if $a \notin \operatorname{cl}(\operatorname{dom} s_n)$. Otherwise, a is *special.* Let N be the set of normal points and S the set of special ones. Note that, if a is normal and n is the number from above then there is some open neighbourhood U of a such that $\operatorname{dom} s_n$ is disjoint from U. This implies that

$$|R_x| = n\,, \text{ for all } x \in U\,, \text{ and } \langle a, b\rangle \text{ is generic, for all } b \in M_\infty\,.$$

We claim that N and S are definable. It is sufficient to show that, for every special element a, there is some $b \in M_\infty$ such that $\langle a, b \rangle$ is not generic. Let $a \in S$ and let n be the number from above. We define

$$\lambda_-(a) := \begin{cases} \lim_{x \uparrow a} s_n(x) & \text{if } (t, a) \subseteq \operatorname{dom} s_n\,, \text{ for some } t\,, \\ \infty & \text{otherwise}\,, \end{cases}$$

$$\lambda_0(a) := \begin{cases} s_n(x) & \text{if } a \in \operatorname{dom} s_n\,, \\ \infty & \text{otherwise}\,, \end{cases}$$

$$\lambda_+(a) := \begin{cases} \lim_{x \downarrow a} s_n(x) & \text{if } (a, t) \subseteq \operatorname{dom} s_n\,, \text{ for some } t\,, \\ \infty & \text{otherwise}\,, \end{cases}$$

and $\quad \beta(a) := \min\{\lambda_-(a),\ \lambda_0(a),\ \lambda_+(a)\}\,.$

It follows that $\beta(a)$ is the least element $b \in M_\infty$ such that $\langle a, b \rangle$ is not generic.

To conclude the proof of the theorem we distinguish two cases. First, suppose that S is finite. Let $a_1 < \dots < a_{k-1}$ be an enumeration of S and set $a_0 := -\infty$ and $a_k := \infty$. We claim that $|R_x|$ is constant on each interval (a_i, a_{i+1}). Let

$$F_n := \{\, x \in (a_i, a_{i+1}) \mid |R_x| = n \,\}\,.$$

Since $|R_x|$ is constant on an open neighbourhood of each element $a \in N$ it follows that the sets F_n are open. As (a_i, a_{i+1}) is connected this implies that there is some n such that $F_n = (a_i, a_{i+1})$.

It remains to consider the case that S is infinite. Let

$$\begin{aligned} S_- &:= \{\, a \in S \mid \langle a, b \rangle \in R \text{ for some } b < \beta(a) \,\}\,, \\ S_+ &:= \{\, a \in S \mid \langle a, b \rangle \in R \text{ for some } b > \beta(a) \,\}\,, \\ \beta_-(a) &:= \max\{\, b \in R_a \mid b < \beta(a) \,\}\,, \\ \beta_+(a) &:= \max\{\, b \in R_a \mid b > \beta(a) \,\}\,. \end{aligned}$$

At least one of the sets $S_- \cap S_+$, $S_- \setminus S_+$, $S_+ \setminus S_-$, $S \setminus (S_- \cup S_+)$ is infinite.

Let us consider the case that $S_- \cap S_+$ is infinite. As β_-, β, β_+ are parameter-definable we can use the Monotonicity Theorem to find an open interval $I \subseteq S_- \cap S_+$ on which each of these functions is continuous. Note that $\beta_- < \beta < \beta_+$. We can partition I as

$$I = \{ a \in I \mid \langle a, \beta(a) \rangle \in R \} \cup \{ a \in I \mid \langle a, \beta(a) \rangle \notin R \} .$$

One of these two sets contains an open interval I_0. Hence, we have either $\beta \restriction I_0 \subseteq R$ or $\beta \restriction I_0 \cap R = \emptyset$. In both cases it follows that $\beta \restriction I_0 \subseteq G_0$ since $\beta_- \restriction I_0$, $\beta \restriction I_0$, and $\beta_+ \restriction I_0$ are continuous. But $\langle a, \beta(a) \rangle$ is never generic. Contradiction.

In a similar way one can show that the remaining three cases also lead to contradictions. □

In the preceding proof we have used the observation that the elements of a fibre R_a depend continuously on a. This is a consequence of the Monotonicity Theorem. Since this situation will occur several times in the following, we introduce some terminology.

Definition 3.6. Let $\mathfrak{M}$ be an ordered structure.

(a) For $D \subseteq M^n$, we denote by $\mathrm{Cn}(D)$ the set of all parameter-definable continuous functions $D \to M$. Furthermore, we set

$$\mathrm{Cn}_\infty(D) := \mathrm{Cn}(D) \cup \{-\infty, \infty\} ,$$

where we regard $-\infty$ and ∞ as the constant functions with the respective value.

(b) Let $R \subseteq M^{n+1}$ be a relation and suppose that $D \subseteq M^n$ is a set such that every fibre $R_{\bar{a}}$ with $\bar{a} \in D$ contains exactly k elements. We say that a family of parameter-definable functions $s_0, \dots, s_{k-1} : D \to M$ is a *local enumeration* of R over D if

$$s_0 < \dots < s_{k-1} \quad \text{and} \quad R_{\bar{a}} = \{s_0(\bar{a}), \dots, s_{k-1}(\bar{a})\} , \quad \text{for } \bar{a} \in D .$$

Note that we can write the last condition also as

$$R \cap (D \times M) = s_0 \cup \dots \cup s_{k-1} .$$

A local enumeration $s_0, \ldots, s_{k-1}$ is *continuous* if every s_i is continuous.

Corollary 3.7. *Let $R \subseteq M^2$ be a parameter-definable relation such that each fibre R_a, $a \in M$, is finite. There are finitely many elements*

$$-\infty = a_0 < a_1 < \cdots < a_{m-1} < a_m = \infty$$

such that over every interval (a_i, a_{i+1}) there exists a continuous local enumeration of R.

Proof. This follows immediately from the Monotonicity Theorem and Theorem 3.5. □

After having dealt with the case of binary relations, we turn to relations of larger arity. First, we define the 'simple parts' we want to decompose our relation into. These are generalisations of the notion of an interval to higher dimensions.

Definition 3.8. Let $\mathfrak{M}$ be an ordered structure.

(a) Let $\bar{\delta} \in [2]^n$. A *$\bar{\delta}$-cell* is a subset $C \subseteq M^n$ defined inductively as follows.

- The set M^0 is the unique $\langle\rangle$-cell.
- A $\bar{\delta}0$-cell is the graph of a function $f \in \mathrm{Cn}(D)$ where D is a $\bar{\delta}$-cell.
- A $\bar{\delta}1$-cell is a set of the form (f, g) where D is a $\bar{\delta}$-cell and $f, g \in \mathrm{Cn}_\infty(D)$ are functions with $f < g$.

A *cell* is a set that is a $\bar{\delta}$-cell for some $\bar{\delta}$. A cell is *open* if it is a $\langle 1, \ldots, 1\rangle$-cell. (We also consider the $\langle\rangle$-cell as open.)

(b) The *dimension* of a $\bar{\delta}$-cell C is is the number

$$\dim C := \delta_0 + \cdots + \delta_{n-1}\,.$$

Lemma 3.9. *Let $C \subseteq M^n$ be a cell.*

(a) *If C is not open then it has empty interior.*

(b) *C is locally closed, i.e., there is an open set O with $C = \mathrm{cl}(C) \cap O$.*

(c) *C is homeomorphic to an open cell $D \subseteq M^{\dim C}$ via a parameter-definable homeomorphism $p : C \to D$.*

(d) *C is definably connected.*

Proof. (a) If $\mathrm{int}(C) \neq \varnothing$ then there is some box B with $B \subseteq C$. This implies that C is a $\langle 1, \dots, 1 \rangle$-cell.

(b) We prove the claim by induction on n. For $n = 0$, $C = M^0$ is clopen. Suppose that $n > 0$ and let $D := \pi(C) \subseteq M^{n-1}$ be the projection of C to M^{n-1}. By inductive hypothesis, D is locally closed. Hence, $\mathrm{cl}(D) \setminus D$ is a closed set. If C is the graph of a function $f \in \mathrm{Cn}(D)$ then

$$\mathrm{cl}(C) \setminus C \subseteq (\mathrm{cl}(D) \setminus D) \times M\,.$$

Hence, C is open in the closed set $C \cup (\mathrm{cl}(D) \setminus D) \times M$.

If $C = (f, g)$, for $f, g \in \mathrm{Cn}(D)$, then

$$\mathrm{cl}(C) \setminus C \subseteq f \cup g \cup (\mathrm{cl}(D) \setminus D) \times M\,.$$

As above it follows that C is locally closed.

The cases that $f = -\infty$ or $g = \infty$ follow analogously.

(c) Suppose that C is a $\bar{\delta}$-cell and let $i_0 < \dots < i_{k-1}$ be an enumeration of all indices i with $\delta_i = 1$. We define a map $p : M^n \to M^{\dim C}$ by

$$p(\bar{a}) := \langle a_{i_0}, \dots, a_{i_{k-1}} \rangle\,.$$

By induction on n, we prove that that p is a homeomorphism from C to an open cell $p[C] \subseteq M^{\dim C}$.

If C is open then $p = \mathrm{id}_C$ and there is nothing to do. Hence, suppose that C is not open. Then $n > 0$ and we can distinguish two cases.

If C is the graph of some function $f \in \mathrm{Cn}(D)$ then we can use the inductive hypothesis to obtain a homeomorphism $q : D \to q[D]$ from D to an open cell $q[D]$. Let $\pi : M^n \to M^{n-1}$ be the projection to the first $n-1$ coordinates. Then $\pi \restriction C : C \to D$ is a homeomorphism. Hence, so is $p = q \circ \pi \restriction C : C \to q[D]$.

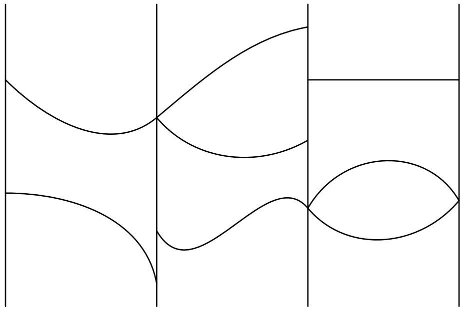


Figure 1.. A cell decomposition of $\mathbb{R}^2$.

It remains to consider the case that $C = (f, g)$, for $f, g \in \mathrm{Cn}_\infty(D)$. Then $p(\bar{a}b) = \langle q(\bar{a}), b \rangle$ where $q : D \to q[D]$ is the homeomorphism from the inductive hypothesis. Set $f' := f \circ q^{-1}$ and $g' := g \circ q^{-1}$. Then $f', g' \in \mathrm{Cn}_\infty(q[D])$ and $p : C \to (f', g')$ is a homeomorphism.

(d) We proceed by induction on n. Clearly, M^0 is definably connected. Suppose that $n > 0$. By inductive hypothesis, the projection D of C to M^{n-1} is definably connected. For a contradiction, suppose that $C = O_0 \cup O_1$ where O_0 and O_1 are disjoint parameter-definable open sets. Since each fibre $\pi^{-1}(a) \cap C$ is definably connected we have $\pi^{-1}(a) \subseteq O_i$, for some i. Hence, there are sets $U_0, U_1 \subseteq M^{n-1}$ such that $O_i = \pi^{-1}[U_i] \cap C$. Clearly, U_0 and U_1 are open and parameter-definable. Since D is definably connected it follows that one of them is empty. $\square$

We will show below that we can partition every definable relation into disjoint cells. In the same way we defined the notion of a cell by induction on the dimension, we also construct these partitions inductively.

Definition 3.10. (a) A *cell decomposition* of M^n is a partition $\mathcal{D}$ of M^n into finitely many pairwise disjoint cells where, for $n > 1$, we further require that the projection $\pi[\mathcal{D}]$ of $\mathcal{D}$ onto the first $n-1$ components is a cell decomposition of M^{n-1}.

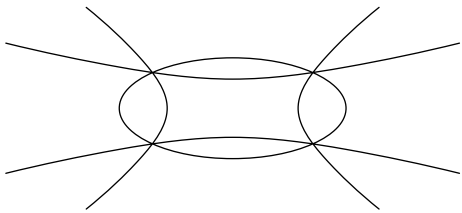
(b) A cell decomposition $\mathcal{D}$ *partitions* a relation $R \subseteq M^n$ if we have $R = C_0 \cup \dots \cup C_{k-1}$, for some cells $C_0, \dots, C_{k-1} \in \mathcal{D}$.

(c) A relation $R \subseteq M^{n+1}$ is *finite* over M^n if every fibre

$$R_{\bar{a}} := \{ b \in M \mid \bar{a}b \in R \}$$

is finite. We call R *uniformly finite* over M^n if there is a number $k < \omega$ such that $|R_{\bar{a}}| < k$, for all $\bar{a} \in M^n$.

Exercise 3.1. Find a cell decomposition of $\mathbb{R}^2$ partitioning the relation



which consists of all pairs $\langle x, y \rangle \in \mathbb{R}^2$ such that

$$\tfrac{4}{9}x^2 + \tfrac{9}{4}y^2 = 1 \quad \text{or} \quad \tfrac{4}{3}x^2 - \tfrac{9}{4}y^2 = 1 \quad \text{or} \quad \tfrac{27}{4}y^2 - \tfrac{4}{9}x^2 = 1\,.$$

Theorem 3.11 (Cell Decomposition Theorem). *Let $\mathfrak{M}$ be an o-minimal structure.*

(a) *For every finite family $R_0, \dots, R_{t-1} \subseteq M^n$ of parameter-definable relations there is a cell decomposition of M^n simultaneously partitioning each R_i.*

(b) *For every parameter-definable function $f : S \to M$ with $S \subseteq M^n$, there is a cell decomposition $\mathcal{D}$ of M^n partitioning S such that, for each cell $C \in \mathcal{D}$, the restriction $f \upharpoonright C : C \to M$ is continuous.*

(c) *Every parameter-definable relation $R \subseteq M^n$ that is finite over M^{n-1} is uniformly finite.*

Proof. We prove all statements simultaneously by induction on n. Note that, for $n = 1$, (a) holds since $\mathfrak{M}$ is o-minimal, (b) follows from the Monotonicity Theorem, and (c) holds trivially.

For the inductive step, suppose that $n > 1$ and we have proved (a), (b), and (c) already for subsets of M^{n-1}.

We start by proving (c). We call a box $B \subseteq M^{n-1}$ *R-normal* if, for every point $\bar{a}b \in R$ with $\bar{a} \in B$, there exists an open interval I with $b \in I$ such that $R \cap (B \times I)$ is the graph of some continuous function $f : B \to M$. (Note that this function f is then necessarily parameter-definable.) A point $\bar{a} \in M^{n-1}$ is called R-normal if it is contained in some R-normal box. Below we will establish the following claims.

(1) If B is R-normal then there exists a continuous local enumeration of R over B.

(2) If $S \subseteq M^{n-1}$ is definably connected and all elements of S are R-normal then there exists a continuous local enumeration of R over S.

(3) Every open cell $C \subseteq M^n$ contains an R-normal point.

First, let us show how (c) follows from (1)–(3). By inductive hypothesis, there exists a cell decomposition $\mathcal{D}$ of M^{n-1} partitioning the set of R-normal points. If a cell $C \in \mathcal{D}$ is open then, by (3), it contains an R-normal point. Hence, all points of C are R-normal and, by (2), there is a number $k(C)$ such that $|R_{\bar{a}}| < k(C)$, for all $\bar{a} \in C$. For cells $C \in \mathcal{D}$ that are not open, we can use Lemma 3.9 (c) to obtain similar bounds $k(C)$. Setting $k := \max\{\, k(C) \mid C \in \mathcal{D} \,\}$ we obtain the desired bound on the size of $R_{\bar{a}}$. Hence, it remains to prove the claims.

(1) Fix $\bar{a} \in B$ and suppose that $b_0 < \dots < b_{k-1}$ is an enumeration of $R_{\bar{a}}$. Since B is R-normal we can find open intervals $I_0, \dots, I_{k-1}$ with $b_i \in I_i$ and continuous functions $s_0, \dots, s_{k-1} \in \mathrm{Cn}(B)$ such that

$$R \cap (B \times I_i) = s_i\,, \quad \text{for all } i < k\,.$$

We claim that $s_0, \dots, s_{k-1}$ is a local enumeration of R over B.

First, let us show that $s_0 < \dots < s_{k-1}$. For a contradiction, suppose that $s_i \nless s_{i+1}$. Since s_i and s_{i+1} are continuous this implies that there is some point $\bar{c} \in B$ with $s_i(\bar{c}) = s_{i+1}(\bar{c})$. In particular, $s_{i+1}(\bar{c}) \in I_i$. As s_{i+1} is continuous, there is a neighbourhood $U \subseteq B$ of $\bar{c}$ such that $s_{i+1}[U] \subseteq I_i$.

Since $R \cap (B \times I_i) = s_i$, it follows that $s_{i+1} \upharpoonright U = s_i \upharpoonright U$. Thus, the set $\{ \bar{c} \in B \mid s_i(\bar{c}) = s_{i+1}(\bar{c}) \}$ is open. Since

$$\{ \bar{c} \in B \mid s_i(\bar{c}) < s_{i+1}(\bar{c}) \} \quad \text{and} \quad \{ \bar{c} \in B \mid s_i(\bar{c}) > s_{i+1}(\bar{c}) \}$$

are also open and B is definably connected it follows that $s_i = s_{i+1}$. But $s_i(\bar{a}) < s_{i+1}(\bar{a})$. A contradiction.

It remains to prove that $R \cap (B \times M) = s_0 \cup \dots \cup s_{k-1}$. Let $\bar{b}c \in R \cap (B \times M)$. There exists a continuous function $f \in \mathrm{Cn}(B)$ with $f(\bar{b}) = c$ and $f \subseteq R$. In particular, $\langle \bar{a}, f(\bar{a}) \rangle \in R$. Hence, there is some index $i < k$ such that $f(\bar{a}) = b_i = s_i(\bar{a})$. As above, it follows that $f = s_i$.

(2) If S is empty there is nothing to do. Hence, we may assume that there is some $\bar{a} \in S$. Let $k := |R_{\bar{a}}|$. By (1), the set $\{ \bar{b} \in S \mid |R_{\bar{b}}| = k \}$ is clopen in S. This implies that $|R_{\bar{b}}| = k$, for all $\bar{b} \in S$. Consequently, we can find functions $s_0 < \dots < s_{k-1}$ such that

$$R_{\bar{b}} = \{ s_0(\bar{b}), \dots, s_{k-1}(\bar{b}) \}, \quad \text{for } \bar{b} \in S .$$

It follows from (1) that each s_i is continuous.

(3) Let $B \subseteq C$ be a box. We will show that B contains an R-normal point. Suppose that $B = B_0 \times I$, for a box $B_0 \subseteq M^{n-2}$ and an open interval $I \subseteq M$. For $\bar{a} \in B_0$, we define

$$R(\bar{a}) := \{ \langle b, c \rangle \mid b \in I \text{ and } \bar{a}bc \in R \} .$$

Then $R(\bar{a})$ is finite over M. By Corollary 3.7 it follows that the set

$$\{ c \in M \mid c \text{ is not } R(\bar{a})\text{-normal} \}$$

is finite. Consequently, the set

$$S_B(R) := \{ \langle \bar{a}, b \rangle \in B \mid b \text{ is not } R(\bar{a})\text{-normal} \}$$

has empty interior. By inductive hypothesis, we can find a cell decomposition $\mathcal{D}$ of M^{n-1} partitioning B and $S_B(R)$. Let $C \in \mathcal{D}$ be an open cell with $C \subseteq B$. Then $C \cap S_B(R) = \varnothing$. Replacing B by a box contained in C we may assume that $S_B(R) = \varnothing$. We can apply (2) to $R(\bar{a})$ to find numbers $k(\bar{a}) < \omega$, for $\bar{a} \in B_0$, such that $|R_{\bar{a}b}| = k(\bar{a})$, for all $b \in I$.

We claim that there exists a bound k with $k(\bar{a}) \leq k$, for all $\bar{a}$. Fix $c \in I$ and define

$$R^c := \{ \langle \bar{a}, b \rangle \mid \langle \bar{a}, c, b \rangle \in R \} .$$

This set is finite over M^{n-2}. By inductive hypothesis, there exists a number m such that $|R^c_{\bar{a}}| < m$, for all $\bar{a} \in B_0$. Since $R^c_{\bar{a}} = R_{\bar{a}c}$ it follows that $|R_{\bar{a}c}| < m$. Consequently, we have $k(\bar{a}) \leq m$, for all $\bar{a} \in B_0$, which implies that $|R_{\bar{a}b}| < m$, for all $\bar{a}b \in B$, as desired.

We still have to find an R-normal element in B. For $k < m$, set

$$B_k := \{ \bar{a} \in B \mid |R_{\bar{a}}| = k \} ,$$

and let $s^k_0, \dots, s^k_{k-1} : B_k \to M$ be a local enumeration of $R_{\bar{a}}$ over B_k. By inductive hypothesis, we can find a cell decomposition $\mathcal{D}$ partitioning each set B_k such that, for every $C \in \mathcal{D}$, all restrictions $s^k_i \upharpoonright C$ are continuous. Since B is open and partitioned by $\mathcal{D}$ there exists an open cell $C \in \mathcal{D}$ with $C \subseteq B$. Fix k such that $C \subseteq B_k$. The functions $s^k_0, \dots, s^k_{k-1}$ are continuous on C. Consequently, each point of C is R-normal.

We prove (a) next. Let $R_0, \dots, R_{t-1} \subseteq M^n$ be parameter-definable and set

$$B := \partial_{n-1} R_0 \cup \dots \cup \partial_{n-1} R_{t-1} ,$$

where

$$\partial_{n-1} R := \{ \bar{a}b \in M^n \mid b \in \partial R_{\bar{a}} \} .$$

Note that B is finite over M^{n-1}. By (c), it follows that there is some bound $m < \omega$ such that $|B_{\bar{a}}| < m$, for all $\bar{a} \in M^{n-1}$. For $k < m$, let

$$B^k := \{ \bar{a} \mid |B_{\bar{a}}| = k \} ,$$

and let $s^k_1, \dots, s^k_k : B^k \to M$ be a local enumeration of $B_{\bar a}$ over B^k. We set $s^k_0 := -\infty$ and $s^k_{k+1} := \infty$. Finally, let

$$C_{lki} := \{ \bar a \in B^k \mid s^k_i(\bar a) \in (R_l)_{\bar a} \},$$
$$D_{lki} := \{ \bar a \in B^k \mid (s^k_i(\bar a), s^k_{i+1}(\bar a)) \subseteq (R_l)_{\bar a} \},$$

for $l < t$ and $0 \leq i \leq k \leq m$. By inductive hypothesis, there exists a cell decomposition $\mathcal{C}_0$ of M^{n-1} simultaneously partitioning the sets B^k, C_{lki}, and D_{lki}. By (b) we can choose a suitable refinement $\mathcal{C}$ of $\mathcal{C}_0$ such that, for every $C \in \mathcal{C}$ with $C \subseteq B^k$, the functions $s^k_1 \restriction C, \dots, s^k_k \restriction C$ are continuous.

For $C \in \mathcal{C}$ with $C \subseteq B^k$, we define a partition of $C \times M$ by

$$\mathcal{D}_C := \{ (s^k_i \restriction C, s^k_{i+1} \restriction C) \mid 0 \leq i < m \} \cup \{ s^k_i \restriction C \mid 0 < i < m \}.$$

The union $\mathcal{D} := \bigcup_{C \in \mathcal{C}} \mathcal{D}_C$ is the desired cell decomposition of M^n.

It remains to prove (b). Let $f : S \to M$ be parameter-definable with domain $S \subseteq M^n$. By (a), it is sufficient to show that we can find a partition $S = R_0 \cup \dots \cup R_{k-1}$ where each R_i is a parameter-definable set such that $f \restriction R_i$ is continuous. First, we can use (a) to partition S into finitely many cells. To find the desired partition of S it is sufficient to consider each of these cells separately. Hence, we may assume that S is a single cell.

If S is not open then we can use the definable homeomorphism $p : S \to p[S] \subseteq M^{\dim S}$ from Lemma 3.9 (c). By inductive hypothesis, we know that the image $p[S]$ can be partitioned into parameter-definable subsets $C_0, \dots, C_{k-1}$ such that all restrictions $(f \circ p^{-1}) \restriction C_i$ are continuous. Consequently, we can set $R_i := p^{-1}[C_i]$ to obtain the desired partition of S.

It remains to consider the case that S is an open cell. We call a point $\langle \bar a, b \rangle \in S$ *regular* if there exists a box $B \subseteq M^{n-1}$ and an open interval $I \subseteq M$ such that

(1) $\langle \bar a, b \rangle \in B \times I \subseteq S$,

(2) for every $\bar{c} \in B$, the function $f(\bar{c}, \cdot)$ is continuous and monotone on I,

(3) the function $f(\,\cdot\,, b)$ is continuous at $\bar{a}$.

Let $S_{\text{reg}} \subseteq S$ be the set of all regular points. Note that S_{reg} is parameter-definable.

First, we prove that S_{reg} is dense in S. Let $B \subseteq M^{n-1}$ be a box and $I = (c, d) \subseteq M$ an interval such that $B \times I \subseteq S$. We have to show that $(B \times I) \cap S_{\text{reg}} \neq \varnothing$. By the Monotonicity Theorem, we can find, for every $\bar{a} \in B$, a greatest element $\lambda(\bar{a}) \in (c, d]$ such that the function $f(\bar{a}, \cdot)$ is continuous and monotone on $(c, \lambda(\bar{a}))$. Since $\lambda : B \to M$ is parameter-definable we can use the inductive hypothesis to find a box $C_0 \subseteq B$ such that $\lambda \upharpoonright C_0$ is continuous. Fix elements $c < e < b < d$. We can find a cell $C_1 \subseteq C_0$ such that $\lambda(\bar{a}) \geq b$, for all $\bar{a} \in C_1$. By inductive hypothesis, there is a cell $C_2 \subseteq C_1$ such that $f(\,\cdot\,, e)$ is continuous on C_2. It follows that every point of $C_2 \times \{e\}$ is regular. Hence, $C_2 \times \{e\} \subseteq (B \times I) \cap S_{\text{reg}} \neq \varnothing$, as desired.

By (a), we obtain a cell decomposition $\mathcal{D}$ partitioning both S and S_{reg}. We claim that $f \upharpoonright C$ is continuous, for every $C \in \mathcal{D}$ with $C \subseteq S$. Since S_{reg} is dense in S we have $S_{\text{reg}} \cap C \neq \varnothing$, for such a cell C. This implies that $C \subseteq S_{\text{reg}}$. Consequently, for each $\bar{a}b \in C$, the function $f(\,\cdot\,, b)$ is continuous at $\bar{a}$. It follows that C can be written as a union of boxes $B \times I$ that, for every $\langle \bar{a}, b \rangle \in B \times I$, satisfy conditions (1)–(3) above. Consequently, we can use Lemma 1.3 to conclude that f is continuous on each box $B \times I$. This implies that f is continuous on C. $\square$

The Cell Decomposition Theorem has a number of important corollaries.

Proposition 3.12. *Let $R \subseteq M^m$ be a nonempty parameter-definable relation. Then R has only finitely many definably connected components. These components form a partition of R and each of them is clopen in R.*

Proof. Let $\mathcal{D}$ be a cell decomposition partitioning R and set

$$\mathcal{D}_0 := \{\, C \in \mathcal{D} \mid C \subseteq R \,\}.$$

Let $\mathcal{C}$ be a maximal subset of $\mathcal{D}_0$ such that $C := \bigcup \mathcal{C}$ is definably connected. We claim that every definably connected subset $S \subseteq R$ with $C \cap S \neq \emptyset$ is contained in C.

Let $\mathcal{D}_S := \{ D \in \mathcal{D}_0 \mid D \cap S \neq \emptyset \}$. Then $S \subseteq \bigcup \mathcal{D}_S$. Since every cell is definably connected it follows that $\bigcup \mathcal{D}_S$ is definably connected. Furthermore, we have $C \cap \bigcup \mathcal{D}_S \supseteq C \cap S \neq \emptyset$. Hence, $C \cup \bigcup \mathcal{D}_S$ is also definably connected. By choice of $\mathcal{C}$ it follows that $\mathcal{D}_S \subseteq \mathcal{C}$. Hence, $S \subseteq \bigcup \mathcal{D}_S \subseteq C$, as desired.

We have shown that C is a definably connected component of R. It follows that we can partition R into definably connected components of the form $\bigcup \mathcal{C}$, for $\mathcal{C} \subseteq \mathcal{D}_0$. Since $\mathcal{D}_0$ is finite there are only finitely many such components.

Finally, note that the closure of a definably connected subset of R is also definably connected. Therefore, each definably connected component of R is closed in R. Since its complement is a finite union of closed sets it follows that each component is also open. $\square$

Proposition 3.13. *Let $\mathfrak{M}$ be o-minimal and let $\pi : M^{m+n} \to M^m$ be the projection to the first m coordinates.*

(a) *For every cell $C \subseteq M^{m+n}$ and every point $\bar{a} \in \pi(C)$, the fibre $C_{\bar{a}}$ is a cell in M^n.*

(b) *For every cell decomposition $\mathcal{D}$ of M^{m+n} and every $\bar{a} \in M^m$, we obtain a cell decomposition*

$$\mathcal{D}_{\bar{a}} := \{ C_{\bar{a}} \mid C \in \mathcal{D},\ \bar{a} \in \pi(C) \}$$

of M^n.

Proof. (a) For $n = 1$, the fibre $C_{\bar{a}}$ is either a single point of an open interval. Hence, it is a cell. Suppose we have proved the claim already for $n - 1$ and let $C \subseteq M^{m+n}$. For $f \in \mathrm{Cn}(D)$, let $f_{\bar{a}} \in \mathrm{Cn}(D_{\bar{a}})$ be the function defined by $f_{\bar{a}}(x) := f(\bar{a}, x)$.

If C is the graph of a function $f \in \mathrm{Cn}(D)$ then $C_{\bar{a}}$ is the graph of $f_{\bar{a}}$. Similarly, if $C = (f, g)$, for $f, g \in \mathrm{Cn}_\infty(D)$, then $C_{\bar{a}} = (f_{\bar{a}}, g_{\bar{a}})$. Hence, $C_{\bar{a}}$ is again a cell.

(b) Clearly, $\mathcal{D}_{\bar{a}}$ is a finite partition of M^n. Therefore, the claim follows by (a). $\square$

Corollary 3.14. *Let $R \subseteq M^m \times M^n$ be parameter-definable.*

(a) *There exists a number $k < \omega$ such that, for every $\bar{a} \in M^m$, the fibre $R_{\bar{a}} \subseteq M^n$ has a partition into at most k cells. In particular, each fibre $R_{\bar{a}}$ has at most k definably connected components.*

(b) *There exists a number $k < \omega$ such that, for every $\bar{a} \in M^m$, the fibre $R_{\bar{a}} \subseteq M^n$ has at most k isolated points. In particular, the size of every finite fibre $R_{\bar{a}}$ is bounded by k.*

Proof. (a) Let $\mathcal{D}$ be a cell decomposition of M^{m+n} partitioning R. For every $\bar{a} \in M^m$, the induced cell decomposition $\mathcal{D}_{\bar{a}}$ of M^n partitions R and it contains at most $|\mathcal{D}|$ cells. Hence, we can set $k := |\mathcal{D}|$.

(b) follows immediately from (a). $\square$

Corollary 3.15. *Every o-minimal theory is graduated and, hence, admits elimination of $\exists^{\aleph_0}$.*

Proof. This follows by Theorem D1.2.15. $\square$

An important consequence of the Cell Decomposition Theorem is the fact that whether a structure is o-minimal only depends on its first-order theory.

Theorem 3.16. *Let $\mathfrak{M}$ be an o-minimal structure. If $\mathfrak{N} \equiv \mathfrak{M}$ then $\mathfrak{N}$ is also o-minimal.*

Proof. Let $\varphi(x; \bar{y})$ be a first-order formula. We have to show that, for every choice of paramters $\bar{a} \subseteq N$, the set $\varphi(x; \bar{a})^{\mathfrak{N}}$ can be written as a finite union of intervals.

For $n < \omega$, let ψ_n be the first-order sentence stating that there are elements $\bar{a}$ such that $\varphi(x; \bar{a})$ is not a union of at most n intervals. By Theorem 3.11, there exists a number $m < \omega$ such that $\mathfrak{M} \nvDash \psi_m$. Hence, $\mathfrak{N} \nvDash \psi_m$ and every set of the form $\varphi(x; \bar{a})^{\mathfrak{N}}$ with $\bar{a} \subseteq N$ can be written as a union of at most m intervals. $\square$

Recommended Literature

Set theory

M. D. Potter, *Sets. An Introduction,* Oxford University Press 1990.

A. Lévy, *Basic Set Theory,* Springer 1979, Dover 2002.

K. Kunen, *Set Theory. An Introduction to Independence Proofs,* North-Holland 1983.

T. J. Jech, *Set Theory,* 3rd ed., Springer 2003.

Algebra and Category Theory

G. M. Bergman, *An Invitation to General Algebra and Universal Constructions,* 2nd ed., Springer 2015.

P. M. Cohn, *Universal Algebra,* 2nd ed., Springer 1981.

P. M. Cohn, *Basic Algebra,* Springer 2003.

S. Lang, *Algebra,* 3rd ed., Springer 2002.

F. Borceux, *Handbook of Categorical Algebra,* Cambridge University Press 1994.

S. MacLane, *Categories for the Working Mathematician,* 2nd ed., Springer 1998.

J. Adámek, J. Rosický, and M. Vitale, *Algebraic Theories,* Cambridge University Press 2011.

J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories,* Cambridge University Press 1994.

Topology and lattice theory

R. Engelking, *General Topology,* 2nd ed., Heldermann 1989.

C.-A. Faure, A. Frölicher, *Modern Projective Geometry,* Kluwer 2000.

P. T. Johnstone, *Stone Spaces,* Cambridge University Press 1982.

G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous Lattices and Domains,* Cambridge University Press 2003.

Model theory

K. Tent and M. Ziegler, *A Course in Model Theory,* Cambridge University Press 2012.

W. Hodges, *Model Theory,* Cambridge University Press 1993.

B. Poizat, *A Course in Model Theory,* Springer 2000.

C. C. Chang and H. J. Keisler, *Model Theory,* 3rd ed., North-Holland 1990.

General model theory

J. Barwise and S. Feferman, eds., *Model-Theoretic Logics,* Springer 1985.

J. T. Baldwin, *Categoricity,* AMS 2010.

R. Diaconescu, *Institution-Independent Model Theory,* Birkhäuser 2008.

H.-D. Ebbinghaus and J. Flum, *Finite Model Theory,* Springer 1995.

Stability theory

S. Buechler, *Essential Stability Theory,* Springer 1996.

E. Casanovas, *Simple Theories and Hyperimaginaries,* Cambridge University Press 2011.

A. Pillay, *Geometric Stability Theory,* Oxford Science Publications 1996.

F. O. Wagner, *Simple Theories,* Kluwer Academic Publishers 2000.

S. Shelah, *Classification Theory,* 2nd ed., North-Holland 1990.

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A	a	$\mathfrak{A}$	$\mathfrak{a}$	N	n	$\mathfrak{N}$	$\mathfrak{n}$
B	b	$\mathfrak{B}$	$\mathfrak{b}$	O	o	$\mathfrak{O}$	$\mathfrak{o}$
C	c	$\mathfrak{C}$	$\mathfrak{c}$	P	p	$\mathfrak{P}$	$\mathfrak{p}$
D	d	$\mathfrak{D}$	$\mathfrak{d}$	Q	q	$\mathfrak{Q}$	$\mathfrak{q}$
E	e	$\mathfrak{E}$	$\mathfrak{e}$	R	r	$\mathfrak{R}$	$\mathfrak{r}$
F	f	$\mathfrak{F}$	$\mathfrak{f}$	S	s	$\mathfrak{S}$	ſ $\mathfrak{s}$
G	g	$\mathfrak{G}$	$\mathfrak{g}$	T	t	$\mathfrak{T}$	$\mathfrak{t}$
H	h	$\mathfrak{H}$	$\mathfrak{h}$	U	u	$\mathfrak{U}$	$\mathfrak{u}$
I	i	$\mathfrak{I}$	$\mathfrak{i}$	V	v	$\mathfrak{V}$	$\mathfrak{v}$
J	j	$\mathfrak{J}$	$\mathfrak{j}$	W	w	$\mathfrak{W}$	$\mathfrak{w}$
K	k	$\mathfrak{K}$	$\mathfrak{k}$	X	x	$\mathfrak{X}$	$\mathfrak{x}$
L	l	$\mathfrak{L}$	$\mathfrak{l}$	Y	y	$\mathfrak{Y}$	$\mathfrak{y}$
M	m	$\mathfrak{M}$	$\mathfrak{m}$	Z	z	$\mathfrak{Z}$	$\mathfrak{z}$

The Greek alphabet					
A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ε	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	ϑ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega