# A Model Theoretic Characterisation of Clique Width

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**Abstract**

We generalise the concept of clique width to structures of arbitrary signature and cardinality. We present characterisations of clique width in terms of decompositions of a structure and via interpretations in trees. Several model-theoretic properties of clique width are investigated including VC-dimension and preservation of finite clique width under elementary extensions and compactness.

*Key words:* clique width, monadic second-order logic, model theory

MSC: 03C50, 03C85, 03C98, 05C75

## 1 Introduction

In the last decades several measures for the complexity of graphs have been defined and investigated. The most prominent one is the tree width which appears in the work of Robertson and Seymour [1] on graph minors and which also plays an important role in recent developments of graph algorithms. When studying non-sparse graphs and their monadic second-order properties the measure of choice seems to be the clique width defined by Courcelle and Olariu [2].

Although no hard evidence could be obtained so far, various partial results suggest that the property of having a finite clique width constitutes the dividing line between simple and complicated monadic theories. On the one hand every structure of finite clique width can be interpreted in the binary tree and, therefore, has a simple monadic theory. On the other hand, every structure with an MSO-definable pairing function is of infinite clique width. For graphs, the converse also holds. Answering a conjecture of Seese [3], Courcelle and Oum [4] have shown that every class of finite undirected graphs with unbounded clique width has an undecidable $(\mathrm{MSO}+\mathrm{C}_2)$-theory, where $\mathrm{MSO}+\mathrm{C}_2$ denotes the extension of monadic second-order logic by counting quantifiers modulo 2. Unfortunately, the case of arbitrary structures remains open.

The aim of this article is to show that clique width is a meaningful and natural concept not only in graph theory but also from a model-theoretic point of view. We generalise the definition of clique width to structures of arbitrary signature and cardinality and show that the resulting measure which we call *partition width* has natural model-theoretic properties such as preservation of finite partition width under elementary extensions and compactness.

The outline of the article is as follows: The next section is meant to fix notation and recall basic results.

In Section 3 we introduce infinite terms denoting relational structures of arbitrary cardinality. The main problem we will be dealing with is to equip these terms with a well-defined semantics. We prove that every structure denoted by such a term can be interpreted in some tree.

In the following section we define a certain kind of decomposition of a structure. The important parameter of such a decomposition is the number of atomic types realised in a component. This number, called partition width, will be our generalisation of the clique width of a graph. We conclude the section by proving a tight relationship between these decompositions and the terms defined in the previous section.

Section 5 contains technical results about the number of atomic types which are needed in Section 6 to prove that a structure has finite partition width if and only if it can be interpreted in some tree.

In the two final sections we turn to model-theoretic questions. In Section 7 we prove that the partition width of a structure is finite iff the width of its finite substructures is bounded; we give a kind of compactness theorem for structures of a given partition width; and we show that finiteness of the partition width is preserved under elementary extensions.

In Section 8 it is shown that structures with finite partition width do not contain an MSO-definable pairing function. In particular, they do not have the independence property.

## 2 Preliminaries

**Logic.** Let us recall some basic definitions and fix our notation. Let $[n] := \{0, \dots, n-1\}$. We tacitly identify tuples $\bar{a} = a_0 \dots a_{n-1} \in A^n$ with functions $[n] \to A$ and frequently we write $\bar{a}$ for the set $\{a_0, \dots, a_{n-1}\}$. This allows us to write $\bar{a} \subseteq \bar{b}$ or $\bar{a} = \bar{b}|_I$ for $I \subseteq [n]$. The complement of a set $X$ is denoted by $\overline{X}$. Recall that the $\alpha$-fold iterated exponentiation $\beth_\alpha(\kappa)$ is defined by

$$\beth_0(\kappa) = \kappa \quad \text{and} \quad \beth_\alpha(\kappa) = \sup \left\{ 2^{\beth_\beta(\kappa)} \;\middle|\; \beta < \alpha \right\}.$$

We will use this notation also for finite $\kappa$.

W.l.o.g. we will only consider relational structures $\mathfrak{A} = (A, R_0, R_1, \dots)$ in this article. The set of relation symbols $\{R_0, R_1, \dots\}$ is called the *signature* of $\mathfrak{A}$. When speaking of the arity of a structure or a signature we mean the supremum of the arities of its relations.

MSO, monadic second-order logic, extends first-order logic FO by quantification over sets. In places where the exact definition matters – say when considering the quantifier rank of a formula – we will use a variant of MSO without first-order variables where the atomic formulae are of the form $Y = Z$, $Y \subseteq Z$, and $RX_0 \dots X_{n-1}$, for set variables $X_i$, $Y$, $Z$ and relations $R$. Using slightly nonstandard semantics we say that an atom of the form $R\bar{X}$ holds if there are elements $a_i \in X_i$ such that $\bar{a} \in R$. Note that we do not require the $X_i$ to be singletons. Obviously, each MSO-formula can be brought into this form.

By $\mathrm{FO}_k$ and $\mathrm{MSO}_k$ we denote the fragments of the respective logic that consists of those formulae with quantifier rank at most $k$.

A formula $\varphi(\bar{x})$ where each free variable is first-order defines on a given structure $\mathfrak{A}$ the relation $\varphi^{\mathfrak{A}} := \{\, \bar{a} \mid \mathfrak{A} \models \varphi(\bar{a}) \,\}$.

**Definition 1** *Let $\mathfrak{A} = (A, R_0, R_1, \dots)$ and $\mathfrak{B}$ be relational structures. A (one-dimensional)* MSO-interpretation *of $\mathfrak{A}$ in $\mathfrak{B}$ is a sequence*

$$\mathcal{I} = \left\langle \delta(x),\ \varepsilon(x, y),\ \varphi_{R_0}(\bar{x}),\ \varphi_{R_1}(\bar{x}), \dots \right\rangle$$

*of* MSO*-formulae such that*

$$\mathfrak{A} \cong \mathcal{I}(\mathfrak{B}) := \left( \delta^{\mathfrak{B}},\ \varphi_{R_0}^{\mathfrak{B}},\ \varphi_{R_1}^{\mathfrak{B}}, \dots \right) \big/ \varepsilon^{\mathfrak{B}}.$$

*To make this expression well-defined we require that $\varepsilon^{\mathfrak{B}}$ is a congruence of the structure $(\delta^{\mathfrak{B}},\ \varphi_{R_0}^{\mathfrak{B}},\ \varphi_{R_1}^{\mathfrak{B}}, \dots)$. We denote the fact that $\mathcal{I}$ is an* MSO*-interpretation of $\mathfrak{A}$ in $\mathfrak{B}$ by $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{MSO}} \mathfrak{B}$.*

*The epimorphism $(\delta^{\mathfrak{B}},\ \varphi_{R_0}^{\mathfrak{B}},\ \varphi_{R_1}^{\mathfrak{B}}, \dots) \to \mathfrak{A}$ is also denoted by $\mathcal{I}$.*

If $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{MSO}} \mathfrak{B}$ then every formula $\varphi$ over the signature of $\mathfrak{A}$ can be translated to a formula $\varphi^{\mathcal{I}}$ over the signature of $\mathfrak{B}$ by replacing every relation symbol $R$ by its definition $\varphi_R$, replacing every $=$ by $\varepsilon$, and by relativising every quantifier to $\delta$ where set quantifiers are further relativised to sets closed under $\varepsilon$.

**Lemma 2** *If $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{MSO}} \mathfrak{B}$ then*

$$\mathfrak{A} \models \varphi(\mathcal{I}(\bar{b})) \quad \text{iff} \quad \mathfrak{B} \models \varphi^{\mathcal{I}}(\bar{b}) \qquad \textit{for all } \varphi \in \mathrm{MSO} \textit{ and } \bar{b} \subseteq \delta^{\mathfrak{B}}.$$

We will make use of Ramsey's theorem. Recall that $n \to (m)^d_p$ asserts that every colouring of $[n]^d$ with $p$ colours contains a homogeneous subset of size $m$. In order to avoid clumsy descriptions we define

$$R(m)^d_p := \min\left\{ n \;\middle|\; n \to (m)^d_p \right\}.$$

**Trees.** Let $\kappa$ be a cardinal and $\alpha$ an ordinal. By $\kappa^{<\alpha}$ we denote the set of all functions $\beta \to \kappa$ for $\beta < \alpha$. We write $x \preceq y$ for $x, y \in \kappa^{<\alpha}$ if $x$ is a prefix of $y$. The longest common prefix of $x$ and $y$ is denoted by $x \sqcap y$.

A *tree* is a partial order $(T, \preceq)$ where the universe $T \subseteq \kappa^{<\alpha}$ is closed under prefixes. Sometimes, we also add the successor functions $\mathrm{suc}_i(x) := xi$ for $i < \kappa$. Labelled trees are either represented as structures $(T, \preceq, (P_i)_{i \in \Lambda})$ with additional unary predicates $P_i$ for each label $i \in \Lambda$, or as functions $t : T \to \Lambda$.

**Graph grammars.** The notion of clique width arose in the study of graph grammars. We present two kinds of such grammars: VR-grammars as considered by Courcelle [5] and NLC-grammars studied by Wanke [6].

Let $C$ be a set of colours. Consider the following operations on $C$-coloured undirected graphs:

- $a$ denotes the trivial graph whose single vertex is coloured $a$;
- $\mathfrak{G}_0 + \mathfrak{G}_1$ is the disjoint union of $\mathfrak{G}_0$ and $\mathfrak{G}_1$;
- the recolouring $\varrho_\beta(\mathfrak{G})$ with $\beta : C \to C$ changes each colour $a$ to $\beta(a)$;
- $\alpha_{a,b}(\mathfrak{G})$ adds edges from all $a$-coloured vertices to every vertex of colour $b$;
- $\mathfrak{G}_0 \oplus_S \mathfrak{G}_1$ with $S \subseteq C \times C$ denotes the disjoint union of $\mathfrak{G}_0$ and $\mathfrak{G}_1$ where $a$-coloured vertices of $\mathfrak{G}_0$ are connected by an edge to $b$-coloured vertices of $\mathfrak{G}_1$ iff $(a, b) \in S$.

A *VR-term* is a term consisting of the operations $a$, $+$, $\varrho_\beta$, and $\alpha_{a,b}$, while *NLC-terms* are built up from $a$, $\varrho_\beta$, and $\oplus_S$.

**Definition 3** *The* clique width *of a graph $\mathfrak{G}$ is the minimal size of a set $C$ of colours such that there is a VR-term denoting $\mathfrak{G}$ which uses only colours from $C$. The* NLC-width *is defined analogously using NLC-terms.*

The following observation by Johansson [7] shows that these two measures are nearly the same.

**Lemma 4** *Let $k$ be the clique width of a graph $\mathfrak{G}$ and $m$ its NLC-width. Then $m \leq k \leq 2m$.*

The characterisation we aim to generalise is the following result of Courcelle [5] relating clique width with interpretations in the binary tree.

**Theorem 5** *A countable graph $\mathfrak{G} = (V, E)$ has finite clique width if and only if $\mathfrak{G} \leq_{\mathrm{MSO}} (2^{<\omega}, \preceq, P)$ for some unary predicate $P \subseteq 2^{<\omega}$.*

## 3 Infinite terms

We start by generalising NLC-terms to infinite terms describing relational structures of arbitrary cardinality. One approach, chosen by Grohe and Turan [8], consists in colouring the elements of the structure as for VR-terms above and generalising the operation $\alpha_{a,b}$ to tuples of length more than two. We choose a different route by colouring all tuples of elements instead of just singletons (see also [9,10]). That way we obtain a larger class of structures that still shares most properties of the class of graphs denoted by VR-terms. In particular, we are able to derive an analog of Theorem 5. An example of a structure which can be described by the terms defined below, but not by the terms introduced by Grohe and Turan, is $(\mathbb{Q}, R)$ where $R := \{ (a, b, c) \mid a < b < c \}$.

**Definition 6** *A* graded set of colours *is a set $C$ that is partitioned into finite nonempty sets $C_n$, $n < \omega$. Colours $c \in C_n$ are said to be of* arity $n$.

*A $C$-*colouring *of a structure $\mathfrak{M}$ is a function $\chi$ mapping every $n$-tuple $\bar{a} \in M^n$ to some colour $\chi(\bar{a}) \in C_n$. The empty tuple is also coloured. We call the pair $(\mathfrak{M}, \chi)$ a $C$-*coloured structure.

Analogously to the NLC-composition $\oplus_S$ we define two operators $\sum^\Theta$ and $\bigcup^\Theta$ to compose a family of $C$-coloured structures $(\mathfrak{M}_i, \chi_i)$, $i < \alpha$, one for ordered families and one for unordered ones. In both cases the resulting structure will consist of the union of the $\mathfrak{M}_i$. Additionally, we will update the colouring and add new tuples to the relations of $\mathfrak{M}$. If $\bar{a}$ is a tuple of $\mathfrak{M}$ then the colours of its parts $\bar{a} \cap M_i$, for $i < \alpha$, will determine both, its new colour and whether we add $\bar{a}$ to a relation $R$. We record this information in an *update instruction* $(n, \alpha, \bar{I}, \bar{c}, d, S)$ where $I_i := \{ k \mid a_k \in M_i \}$ is the partition of $\bar{a}$ induced by the union, $c_i := \chi_i(\bar{a}|_{I_i})$ is the colour of the tuple $\bar{a} \cap M_i$, $d$ is the new colour of $\bar{a}$, and $S$ contains all relation symbols to which $\bar{a}$ is added.

**Definition 7** *Let $\tau$ be a signature and $C$ a graded set of colours.*

(a) *An* update instruction *is a tuple $(n, \alpha, \bar{I}, \bar{c}, d, S)$ where*

- *$n < \omega$ is a natural number and $\alpha$ is an ordinal;*
- *$\bar{I}$ is a partition $\dot\bigcup_{i<\alpha} I_i = [n]$ of $[n]$ into $\alpha$ classes (of which all but finitely many are empty);*
- *$\bar{c} \in C^\alpha$ is a sequence of $\alpha$ colours such that the arity of $c_i$ is $|I_i|$ (which implies that the sum of their arities is $n$);*
- *$d \in C_n$ is a colour of arity $n$; and*
- *$S \subseteq \tau$ is a set of $n$-ary relation symbols.*

*The number $n$ is called the* arity *of the instruction.*

(b) *An* ordered $\alpha$-update *is a set $\Theta$ of update instructions that contains exactly one instruction $(n, \alpha, \bar{I}, \bar{c}, d, S)$, for all values of $n$, $\bar{I}$, and $\bar{c}$. Each such set $\Theta$ induces a family of functions*

$$\Theta_n(\bar{I}; \bar{c}) = (d, S) \quad :\text{iff} \quad (n, \alpha, \bar{I}, \bar{c}, d, S) \in \Theta\,.$$

(c) *A* symmetric update *is a set $\Theta$ of update instructions with the following properties:*

- *$\Theta$ contains exactly one instruction $(n, s, \bar{I}, \bar{c}, d, S)$ for all $n < \omega$, every $s \leq n$, all partitions $\bar{I} = I_0 \dot\cup \cdots \dot\cup I_{s-1}$ where each of the $I_i$ is nonempty, and all appropriate $\bar{c} \in C^s$.*
- *For all permutations $\sigma \in S_s$ we have*

$$\begin{array}{ll} & \big(n,\ s,\ \langle I_{\sigma 0}, \dots, I_{\sigma(s-1)}\rangle,\ \langle c_{\sigma 0}, \dots, c_{\sigma(s-1)}\rangle,\ d,\ S\big) \in \Theta \\ \textit{iff} & \big(n,\ s,\ \langle I_0, \dots, I_{s-1}\rangle,\ \langle c_0, \dots, c_{s-1}\rangle,\ d,\ S\big) \in \Theta\,. \end{array}$$

*The family of functions induced by $\Theta$ is*

$$\Theta_n^s(\bar{I}; \bar{c}) = (d, S) \quad :\text{iff} \quad (n, s, \bar{I}, \bar{c}, d, S) \in \Theta\,.$$

We use ordered updates to define a sum operation $\sum^\Theta$ where the ordering of the structures matters, whereas symmetric updates are used to define an operation $\bigcup^\Theta$ that is invariant under permutations of its arguments. For every symmetric sum there exists an equivalent ordered one, while the converse only holds if we are allowed to use more colours. (Basically, we need to colour each structure with a different copy of the colours.) Below we will use ordered sums only for finitely many arguments.

**Definition 8** *Let $(\mathfrak{M}_i, \chi_i)$, $i < \kappa$, be a sequence of $C$-coloured structures.*

(a) *Let $\Theta$ be an ordered $\kappa$-update. The* ordered sum

$$\sum_{i<\kappa}^{\Theta} (\mathfrak{M}_i, \chi_i)$$

*of $(\mathfrak{M}_i, \chi_i)$, $i < \kappa$, with respect to $\Theta$ is the structure $(\mathfrak{N}, \eta)$ obtained from the disjoint union of the $\mathfrak{M}_i$ by the following operation:*

*For every $n$-tuple $\bar{a} \in N^n$, $n < \omega$, if*

$$\Theta_n(\bar{I}; \bar{c}) = (d, S)$$

*where*

$$I_i := \{\, k < n \mid a_k \in M_i \,\} \quad \textit{and} \quad c_i := \chi_i(\bar{a}|_{I_i}) \quad \textit{for } i < \kappa\,,$$

*then we add $\bar{a}$ to all relations $R \in S$ and set the new colour to $\eta(\bar{a}) := d$.*

(b) *Let $\Theta$ be a symmetric update. The* symmetric sum

$$\bigcup_{i<\kappa}^{\Theta} (\mathfrak{M}_i, \chi_i)$$

*of $(\mathfrak{M}_i, \chi_i)$, $i < \kappa$, with respect to $\Theta$ is the structure $(\mathfrak{N}, \eta)$ obtained from the disjoint union of the $\mathfrak{M}_i$ by the following operation:*

*For every $n$-tuple $\bar{a} \in N^n$, $n < \omega$, containing elements from $M_{j_0}, \dots, M_{j_{s-1}}$, if*

$$\Theta_n^s(\bar{I}; \bar{c}) = (d, S)$$

*where*

$$I_i := \{\, k < n \mid a_k \in M_{j_i} \,\} \quad \textit{and} \quad c_i := \chi_i(\bar{a}|_{I_i}) \quad \textit{for } i < s\,,$$

*then we add $\bar{a}$ to all relations $R \in S$ and set the new colour to $\eta(\bar{a}) := d$.*

*Note that this definition does not depend on the ordering of $j_0, \dots, j_{s-1}$ since $\Theta$ is invariant under permutations.*

(c) *For every sequence of colours $c_n \in C_n$, $n < \omega$, let $\bar{c}$ denote the $C$-coloured structure $(\mathfrak{D}, \zeta)$ with universe $D := [1]$ and empty relations $R := \emptyset$ where the only $n$-tuple is coloured with $c_n$.*

**Example 9** *Consider three structures with universes $\{x, x'\}$, $\{y, y'\}$, and $\{z, z'\}$,* *and colouring*

$$\begin{aligned}
\chi(\langle\rangle) &= e\,, \\
\chi(x) &= a\,, & \chi(y) &= b\,, & \chi(z) &= c\,, \\
\chi(x') &= b\,, & \chi(y') &= c\,, & \chi(z') &= a\,, \\
\chi(x,x') &= d\,, & \chi(y,y') &= d\,, & \chi(z,z') &= f\,, \\
\chi(x',x) &= d\,, & \chi(y',y) &= f\,, & \chi(z',z) &= f\,.
\end{aligned}$$

(a) *Let $\Theta$ be a symmetric update. The following examples show how the new colour and relations of a tuple are determined.*

$$\begin{aligned}
(x,y):&\quad \Theta_2^2\big(\{0\},\{1\};\ a,b\big) \\
(y,x):&\quad \Theta_2^2\big(\{0\},\{1\};\ b,a\big) = \Theta_2^2\big(\{1\},\{0\};\ a,b\big) \\
(y',y):&\quad \Theta_2^1\big(\{0,1\};\ f\big) \\
(y,x,y'):&\quad \Theta_3^2\big(\{1\},\{0,2\};\ a,d\big) \\
(y,z,x):&\quad \Theta_3^3\big(\{1\},\{2\},\{0\};\ c,a,b\big)
\end{aligned}$$

(b) *For an ordered* 3*-update $\Theta$ we have:*

$$\begin{aligned}
(x,y):&\quad \Theta_2\big(\{0\},\{1\},\emptyset;\ a,b,e\big) \\
(y,x):&\quad \Theta_2\big(\{1\},\{0\},\emptyset;\ a,b,e\big) \\
(y',y):&\quad \Theta_2\big(\emptyset,\{0,1\},\emptyset;\ e,f,e\big) \\
(y,x,y'):&\quad \Theta_3\big(\{1\},\{0,2\},\emptyset;\ a,d,e\big) \\
(y,z,x):&\quad \Theta_3\big(\{2\},\{0\},\{1\};\ a,b,c\big)
\end{aligned}$$

Having decided on the operations we can start building terms. Since we want to support uncountable structures we consider terms as infinitely branching trees of ordinal height.

**Definition 10** (a) *For a graded set of colours $C$ and a signature $\tau$, let $\Upsilon^{<}_{C,\tau}$ be the signature consisting of all operations of the form $\bar{c}$ and $\sum^{\Theta}$ with colours from $C$ and relation symbols from $\tau$. Similarly, $\Upsilon_{C,\tau}$ consists of $\bar{c}$ and $\bigcup^{\Theta}$.*

(b) *Let $\Upsilon$ be a signature. A* $\Upsilon$-term *is a tree $T \subseteq \kappa^{<\alpha}$ labelled with symbols from $\Upsilon$ such that the number of successors of a node equals the arity of the symbol labelling it.*

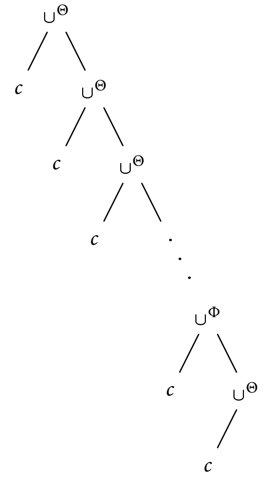


Figure 1. The term $T_{\omega+2}$

**Example 11** *Let* $C_1 = \{a, b, c\}$, $C_n = \{1\}$, *for* $n \neq 1$, *and*

$$\begin{aligned}\Theta := \big\{&(1,\ 1,\ \langle\{0\}\rangle,\ \langle a\rangle,\ b,\ \emptyset)\,,\\ &(1,\ 1,\ \langle\{0\}\rangle,\ \langle b\rangle,\ b,\ \emptyset)\,,\\ &(1,\ 1,\ \langle\{0\}\rangle,\ \langle c\rangle,\ a,\ \emptyset)\,,\\ &(2,\ 2,\ \langle\{0\},\{1\}\rangle,\ \langle c,a\rangle,\ 1,\ \{\mathrm{suc},\leq\})\,,\\ &(2,\ 2,\ \langle\{1\},\{0\}\rangle,\ \langle a,c\rangle,\ 1,\ \{\mathrm{suc},\leq\})\,,\\ &(2,\ 2,\ \langle\{0\},\{1\}\rangle,\ \langle c,b\rangle,\ 1,\ \{\leq\})\,,\\ &(2,\ 2,\ \langle\{1\},\{0\}\rangle,\ \langle b,c\rangle,\ 1,\ \{\leq\})\,,\\ &\dots\big\}\end{aligned}$$

*(where we left out the irrelevant entries). Let* $\Phi$ *be the update obtained from* $\Theta$ *by replacing the instruction* $(1,\ 1,\ \langle\{0\}\rangle,\ \langle c\rangle,\ a,\ \emptyset)$ *by* $(1,\ 1,\ \langle\{0\}\rangle,\ \langle c\rangle,\ b,\ \emptyset)$. *For each ordinal* $\alpha$, *we can define a term* $T_\alpha$ *denoting the structure* $(\alpha, \mathrm{suc}, \leq)$ *where the colour of the first element is* $a$ *and the other elements are coloured by* $b$. *(A formal definition of the value of a term can be found below.) For* $\beta < \alpha$, *we set*

$$T_\alpha(0^\beta) := \begin{cases} \cup^\Theta & \textit{if } \beta \textit{ is a successor}, \\ \cup^\Phi & \textit{if } \beta \textit{ is a limit}, \end{cases} \qquad \textit{and} \qquad T_\alpha(0^\beta 1) := c\,.$$

*For instance,*

$$T_4 = c \cup^\Theta (c \cup^\Theta (c \cup^\Theta \bigcup\nolimits^\Theta \{c\}))\,.$$

When trying to evaluate an infinite term $T \subseteq \kappa^{<\alpha}$ for $\alpha > \omega$ in a bottom-up fashion, we face the difficulty that, after having obtained the value of a subterm whose root is at a limit depth, we have to propagate this value to its predecessors. To do so, we start at the predecessor in question and trace the value back until we reach the already evaluated subterm.

**Definition 12** *Fix a relation $\leq$ well-ordering each colour set $C_n$ such that colours of different arities are incomparable.*

(1) *For sequences of colours $(c_i)_{i<\alpha}$ and $(d_i)_{i<\alpha}$ we define the ordering componentwise.*

$$\begin{aligned} & (c_i)_i \leq (d_i)_i && :\text{iff} \quad c_i \leq d_i \text{ for all } i < \alpha\,, \\ \text{and} \quad & (c_i)_i < (d_i)_i && :\text{iff} \quad (c_i)_i \leq (d_i)_i \text{ and } (d_i)_i \not\leq (c_i)_i\,. \end{aligned}$$

(2) *Let $T$ be a term, $v \in T$ a node, and $\alpha := |v|$. A* colour trace *to $v$ is a sequence $(c_i)_{i<\alpha+1}$ of colours of the same arity which satisfies the following conditions:*

(a) *If $\alpha = \beta+1$ is a successor then $(c_i)_{i<\beta+1}$ is a colour trace to the predecessor $u$ of $v$ and the operation at $u$ changes the colour of tuples from $c_{\beta+1}$ to $c_\beta$.*
(b) *If $\alpha$ is a limit then each subsequence $(c_i)_{i<\beta+1}$ for $\beta < \alpha$ is a colour trace to the corresponding prefix of $v$, and $c_\alpha$ is the minimal colour $c$ such that the set $\{\, \beta < \alpha \mid c_\beta = c \,\}$ is unbounded below $\alpha$.*

**Example 13** *For the terms $T_\alpha$ in the previous example, the colour traces are of the form $bb \dots bbac$, $bb \dots bba$, or $bb \dots bb$.*

With these notions we can define a subclass of terms to which we can assign a value. Basically, we call a term $T$ *well-formed* if its value $\mathrm{val}(T)$ (which we introduce below) is well-defined.

**Definition 14** *A term $T$ is* well-formed *if the following conditions are satisfied:*

(1) *For each $v \in T$, the set of colour traces to $v$ is linearly ordered by $\leq$.*
(2) *For every leaf $v$ labelled $\bar{c}$ and all arities $n$ there exists a colour trace $(d_i)_{i<\alpha+1}$ to $v$ with $d_\alpha = c_n$.*
(3) *For all finite sequences of vertices $v^k$, $k < m$, and all colour traces $(c_i^k)_i$ to $v^k$, there exists a colour trace $(d_i)_{i<\alpha+1}$ to $u := v^0 \sqcap \dots \sqcap v^{m-1}$ such that $d_\alpha$ is the result of the operation at $u$ applied to the colours $c_{\alpha+1}^k$.*

**Lemma 15** *Let $T$ be a well-formed term. For every $v \in T$ and all colours $c \in C$ there is at most one colour trace $(c_\beta)_{\beta<\alpha+1}$ to $v$ with $c_\alpha = c$.*

**PROOF.** Let $(c_\beta)_{\beta<\alpha+1}$ and $(d_\beta)_{\beta<\alpha+1}$ be colour traces to $v$ with $c_\alpha = d_\alpha$. We prove by induction on $\alpha$ that $(c_\beta)_\beta = (d_\beta)_\beta$. The case $\alpha = 0$ is trivial.

If $\alpha = \beta + 1$ is a successor ordinal then the operation at $v$ maps $c_\alpha = d_\alpha$ to $c_\beta = d_\beta$ and the claim follows by induction hypothesis.

Suppose that $\alpha$ is a limit and that $(c_\beta)_\beta \neq (d_\beta)_\beta$. By symmetry, we may assume that $(c_\beta)_\beta < (d_\beta)_\beta$. By definition, the set

$$S := \{\, \beta < \alpha \mid d_\beta = d_\alpha \,\}$$

is unbounded below $\alpha$. Let $e$ be the minimal colour such that the subset $S' := \{\, \beta \in S \mid c_\beta = e \,\}$ is also unbounded. Such a colour exists since there are only finitely many colours of the given arity.

By definition of a colour trace we have $e \geq c_\alpha$. Since $c_\beta \leq d_\beta$ for all $\beta < \alpha$ it follows that $e = c_\beta \leq d_\beta = d_\alpha = c_\alpha$ for $\beta \in S'$. Consequently, $c_\beta = d_\beta$ for all $\beta \in S'$. Since $S'$ is unbounded the induction hypothesis implies that $c_\beta = d_\beta$ for all $\beta < \alpha$. Contradiction. $\square$

**Definition 16** *Let $T \subseteq \kappa^{<\alpha}$ be a well-formed term and $L \subseteq T$ the set of its leaves.*

(a) *To every tuple $\bar{a} \in L^n$ we associate a colour trace $\chi(\bar{a})$ by induction on $n$. If $a_0 = \dots = a_{n-1}$ and the node $a_0$ is labelled by $\bar{d}$ then $\chi(\bar{a}) := (c_\beta)_{\beta<\alpha+1}$ is the (unique) colour trace to $a_0$ that ends in $c_\alpha = d_n$.*

*Otherwise, let $v := \bigsqcap \bar{a}$. There is a partition $I_0 \uplus \dots \uplus I_{s-1} = [n]$ of the indices such that*

- *$v \prec a_i \sqcap a_k$ if $i$ and $k$ belong to the same class $I_l$, and*
- *$v = a_i \sqcap a_k$ for $i$ and $k$ belonging to different classes.*

*The node $v$ is labelled by either $\sum^\Theta$ or $\bigcup^\Theta$ for some update $\Theta$. Let $(c^i_\beta)_\beta := \chi(\bar{a}|_{I_i})$, for $i < s$, and let $\alpha := |v|$. We either have*

$$(d, S) = \Theta_n\Big(\bar{I}'; (\hat{c}^i_\alpha)_{i<\kappa}\Big) \quad \text{or} \quad (d, S) = \Theta^s_n\Big(\bar{I}; c^0_\alpha, \dots, c^{s-1}_\alpha\Big)\,,$$

*where $(\hat{c}^i_\alpha)_{i<\kappa}$ is the sequence of length $\kappa$ obtained from $c^0_\alpha, \dots, c^{s-1}_\alpha$ by inserting the colour of the empty tuple at the appropriate places. We let $\chi(\bar{a}) := (c_\beta)_{\beta<\alpha+1}$ be the (unique) colour trace to $v$ with $c_\alpha = d$.*

(b) *The* value $\mathrm{val}(T)$ *of $T$ is the structure whose universe $M := L$ consists of all leaves of $T$. A tuple $\bar{a} \in M^n$ with associated colour trace $\chi(\bar{a}) = (c_\beta)_{\beta<\alpha+1}$ belongs to a relation $R$ iff there is some node $v \preceq \bigsqcap \bar{a}$ labelled by an operation $\sum^\Theta$ or $\bigcup^\Theta$ that adds tuples coloured $c_{|v|}$ to $R$.*

In the following we will tacitly assume that all terms are well-formed.

What structures can be the value of a $\Upsilon_{C,\tau}$-term? If $\mathfrak{M}$ is a finite structure with $|M^n| \leq |C_n|$, for all $n < \omega$, then, by assigning different colours to each tuple $\bar{a} \subseteq M$, we can easily construct a term denoting $\mathfrak{M}$.

But, if $M$ is infinite, this does not need to be the case. In the next lemma we prove that every structure denoted by an $\Upsilon_{C,\tau}$-term $T$ can be interpreted in some tree, namely, the term $T$ itself. The converse is shown in Section 6.

One remaining technicality we have to deal with is to fix an encoding of terms as structures. In order to allow infinite signatures we encode a $\Upsilon$-term $T \subseteq \kappa^{<\alpha}$ as a structure $(T, \preceq, \bar{P})$ with universe $T$, prefix ordering $\preceq$, and unary predicates $\bar{P}$ coding the functions in $\Upsilon$. Each operator is encoded by several predicates:

$$\begin{aligned} P_d &:= \{\, v \in T \mid v \text{ is labelled by some } \bar{c} \text{ with } d \in \bar{c} \,\}\,, \\ P_{(n,\alpha,\bar{I},\bar{c},d,R)} &:= \{\, v \in T \mid v \text{ is labelled by } \textstyle\sum^\Theta \text{ or } \bigcup^\Theta \text{ for some } \Theta \\ &\qquad\qquad \text{containing } (n, \alpha, \bar{I}, \bar{c}, d, S) \text{ with } R \in S \,\}\,. \end{aligned}$$

**Proposition 17** *For all signatures $\tau$ and every set $C$ of colours there are* MSO*-interpretations $\mathcal{V}$ and $\mathcal{V}_k^<$, $k < \aleph_0$, such that*

$$\begin{aligned} &\mathcal{V}_k^< : \mathrm{val}(T) \leq_{\mathrm{MSO}} (T, \preceq, \bar{P}, (\mathrm{suc}_i)_{i<k}) \quad \textit{for all } \Upsilon_{C,\tau}^<\textit{-terms } T \subseteq k^{<\alpha}, \\ \textit{and } &\mathcal{V} : \mathrm{val}(T) \leq_{\mathrm{MSO}} (T, \preceq, \bar{P}) \qquad\qquad\quad \textit{for all } \Upsilon_{C,\tau}\textit{-terms } T \subseteq \kappa^{<\alpha}. \end{aligned}$$

*If the arity of $\tau$ is bounded then there even exist* $\mathrm{MSO}_m$*-interpretations for some $m$.*

**PROOF.** The universe of $\mathrm{val}(T)$ consists of the set of leaves of $T$, which is definable. The above definition of the relations of $\mathrm{val}(T)$ can be translated immediately into MSO once we have shown how to encode colour traces. If colour traces $(c_i)_{i<\alpha} \in C_n^\alpha$ to some node $v \in T$ are represented by sets $(X_d)_{d\in C_n}$ such that $u \preceq v$ belongs to $X_d$ iff $c_{|u|} = d$, then there is an MSO-formula which expresses that the sequence of colours encoded in some tuple $\bar{X}$ is indeed a colour trace.

The quantifier rank of these formulae depends only on $|C_n|$ and the arity of the relations involved. □

## 4 Partition refinements

Our goal is to obtain a characterisation of the class of structures denoted by some term similar to Theorem 5. As an intermediate step before proving the converse of Proposition 17 we show that the structure denoted by a term can be decomposed in a certain way, and that, vice versa, every such decomposition yields a term.

If we ignore the colours, $\Upsilon_{C,\tau}$-terms consist purely of disjoint unions. Thus, when traversing a term $T$ from the root to its leaves we observe a progression of decompositions of the structure denoted by $T$. This process is captured by the following definition.

**Definition 18** (a) *A* partial $\kappa^{<\alpha}$-partition refinement *of a structure $\mathfrak{M}$ is a family $(U_v)_{v\in T}$ of nonempty subsets $U_v \subseteq M$ indexed by a tree $T \subseteq \kappa^{<\alpha}$ such that the following conditions are satisfied:*

(1) *$U_\varepsilon = M$ and for every $a \in M$ there is some leaf $v \in T$ with $a \in U_v$.*
(2) *Each $U_v$ is the disjoint union of its successors $U_{v\beta}$, $v\beta \in T$, $\beta < \kappa$.*
(3) *If $|v|$ is a limit ordinal then $U_v = \bigcap_{u \prec v} U_u$.*

*The* granularity *of a partial partition refinement $(U_v)_v$ is the supremum of the cardinalities $|U_v|$ of its leaves $v$.*

(b) *A* $\kappa^{<\alpha}$-partition refinement *is a partial $\kappa^{<\alpha}$-partition refinement of granularity* 1.

It turns out that it is not necessary to explicitly add information about the colouring to a partition refinement. Instead, the colours can directly be read off from a given partition refinement since the colour of a tuple corresponds to its type as explained below. As the colours are only needed to connect tuples $\bar{a} \subseteq U_v$ in some component $U_v$ with tuples $\bar{b} \subseteq \overline{U_v}$ in the complement we define a notion of type consisting only of formulae containing both, a free variable and some parameter.

**Definition 19** *Let $\mathfrak{M}$ be a structure, $\bar{a} \in M^n$, and $U \subseteq M$. Let $\Delta \subseteq$ FO. The* $\Delta$-type *of $\bar{a}$ over $U$ is the set*

$$\mathrm{tp}_\Delta(\bar{a}/U) := \{\, \varphi(\bar{x};\bar{c}) \mid \mathfrak{M} \models \varphi(\bar{a};\bar{c}),\ \varphi \in \Delta,\ \bar{c} \subseteq U \,\}.$$

*The* external $\Delta$-type *of $\bar{a}$ over $U$ is defined by*

$$\mathrm{etp}_\Delta(\bar{a}/U) := \{\, \varphi(\bar{x};\bar{c}) \in \mathrm{tp}_\Delta(\bar{a}/U) \mid \begin{array}{l}\textit{every atom of } \varphi \textit{ contains a variable}\\ \textit{and some parameter } c \in U \,\}.\end{array}$$

*We denote the set of all $\Delta$-types over $U$ with $n$ free variables by $S^n_\Delta(U)$ and its subset of external types by $ES^n_\Delta(U)$. In case $\Delta = \mathrm{FO}_k$ we simply write $\mathrm{tp}_k(\bar{a}/U)$ and $S^n_k(U)$.*

*For sets $\bar{A} \subseteq \mathcal{P}(M)$ and monadic formulae $\Delta \subseteq$ MSO we also define the* monadic $\Delta$-type *of* $\bar{A}$ *over* $U$ *and its external variant by*

$$\begin{aligned}\operatorname{mtp}_\Delta(\bar{A}/U) &:= \{\, \varphi(\bar{X};\bar{C}) \mid \mathfrak{M} \models \varphi(\bar{A},\bar{C}),\ \varphi \in \Delta,\ \bar{C} \subseteq \mathcal{P}(U)\,\} \\ \operatorname{emtp}_\Delta(\bar{A}/U) &:= \{\ \varphi(\bar{X};\bar{C}) \in \operatorname{mtp}_\Delta(\bar{A}/U) \mid \textit{every atom of } \varphi \textit{ contains} \\ &\qquad\qquad \textit{a variable and some parameter } C \subseteq U\,\}.\end{aligned}$$

*The set of all monadic* $\Delta$*-types over* $U$ *with* $n$ *free variables is denoted by* $MS^n_\Delta(U)$.

**Definition 20** *Let* $\mathfrak{M}$ *be a structure and* $U \subseteq M$. *For tuples* $\bar{a}$, $\bar{b} \subseteq M$ *we define*

$$\begin{aligned}\bar{a} \approx^\Delta_U \bar{b} \quad &: \text{iff} \quad \operatorname{tp}_\Delta(\bar{a}/U) = \operatorname{tp}_\Delta(\bar{b}/U)\,, \\ \bar{a} \simeq^\Delta_U \bar{b} \quad &: \text{iff} \quad \operatorname{etp}_\Delta(\bar{a}/U) = \operatorname{etp}_\Delta(\bar{b}/U)\,.\end{aligned}$$

*For sets* $\bar{A}$, $\bar{B} \subseteq \mathcal{P}(M)$ *we reuse the these symbols and write*

$$\begin{aligned}\bar{A} \approx^\Delta_U \bar{B} \quad &: \text{iff} \quad \operatorname{mtp}_\Delta(\bar{A}/U) = \operatorname{mtp}_\Delta(\bar{B}/U)\,, \\ \bar{A} \simeq^\Delta_U \bar{B} \quad &: \text{iff} \quad \operatorname{emtp}_\Delta(\bar{A}/U) = \operatorname{emtp}_\Delta(\bar{B}/U)\,.\end{aligned}$$

*The* [external] [monadic] $\Delta$-type index *of a set* $X$ *over* $U$ *is*

$$\begin{aligned}\operatorname{ti}^n_\Delta(X/U) &:= |X^n/{\approx^\Delta_U}|\,, \qquad & \operatorname{mti}^n_\Delta(X/U) &:= |\mathcal{P}(X)^n/{\approx^\Delta_U}|\,, \\ \operatorname{eti}^n_\Delta(X/U) &:= |X^n/{\simeq^\Delta_U}|\,, \qquad & \operatorname{emti}^n_\Delta(X/U) &:= |\mathcal{P}(X)^n/{\simeq^\Delta_U}|\,.\end{aligned}$$

*Again, in case* $\Delta = \mathrm{FO}_k$ *we simply write* $\approx^k_U$, $\operatorname{ti}^n_k(X/U)$, *and so on.*

**Remark 21** *Note that, for undirected graphs, the relations* $\simeq^0_U$ *coincides with the relation* $\sim_{\overline{U}}$ *defined by Courcelle in [11].*

For the most part we will concentrate on atomic external types $\operatorname{etp}_0(\bar{a}/U)$ and the corresponding index $\operatorname{eti}^n_0(X/U)$.

**Example 22** *Consider the binary tree* $\mathfrak{T} = (2^{\leq\omega}, \preceq)$ *and fix a vertex* $w \in 2^{\leq\omega}$. *If* $v \in {\uparrow}w := \{\, v \in 2^{\leq\omega} \mid w \preceq v \,\}$ *then*

$$\begin{aligned}&u \preceq v \qquad && \textit{for all } u \in {\downarrow}w := \{\, v \in 2^{\leq\omega} \mid v \prec w \,\}\,, \\ \textit{and }\ &u \npreceq v \qquad && \textit{for all } u \in 2^{\leq\omega} \setminus ({\uparrow}w \cup {\downarrow}w)\,.\end{aligned}$$

*Hence* $\operatorname{eti}^1_0({\uparrow}w/\overline{{\uparrow}w}) = 1$ *since only one external atomic type over* $2^{\leq\omega} \setminus {\uparrow}w$ *is realised in* ${\uparrow}w$. *On the other hand,* $\operatorname{eti}^1_0(\overline{{\uparrow}w}/{\uparrow}w) = 2$ *because there are two external atomic types over* ${\uparrow}w$ *realised in* $2^{\leq\omega} \setminus {\uparrow}w$.

Below it will be shown that, when colouring a component $U_v$ of a partition refinement, we can take as colours the classes of the relation $\simeq^0_{\overline{U_v}}$, i.e., the atomic external types over the complement of $U_v$. Therefore, the number of $n$-ary colours we need equals $\operatorname{eti}^n_0(U_v/\overline{U_v})$.

**Definition 23** (1) *The $n$-ary* partition width *of a partition refinement $(U_v)_{v\in T}$ is the number*

$$\operatorname{pwd}_n(U_v)_{v\in T} := \sup\Big\{\operatorname{eti}_0^n(U_v/\overline{U_v}) \;\Big|\; v \in T\Big\},$$

*and the $n$-ary* symmetric partition width *is*

$$\operatorname{spwd}_n(U_v)_{v\in T} := \sup\Big\{\operatorname{eti}_0^n\Big(\textstyle\bigcup_{i\in I} U_{vi} \Big/ \overline{\bigcup_{i\in I} U_{vi}}\Big) \;\Big|\; v \in T,\ I \subseteq \kappa\Big\}.$$

(2) *The $n$-ary* partition width $\operatorname{pwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *of a structure $\mathfrak{M}$ is defined inductively as follows:* $\operatorname{pwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is the minimal cardinal $\lambda$ such that there exists a $\kappa^{<\alpha}$-partition refinement $(U_v)_v$ with*

$$\operatorname{pwd}_n(U_v)_v = \lambda \quad \text{and} \quad \operatorname{pwd}_i(U_v)_v = \operatorname{pwd}_i(\mathfrak{M}, \kappa^{<\alpha}) \quad \text{for } i < n.$$

*If $\kappa^{<\alpha} = 2^{<|M|^+}$ we omit the second parameter and simply write* $\operatorname{pwd}_n \mathfrak{M}$. $\mathfrak{M}$ *is said to be of* finite partition width *if* $\operatorname{pwd}_n \mathfrak{M}$ *is finite for all $n < \omega$.*

*The $n$-ary* symmetric partition width $\operatorname{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *of $\mathfrak{M}$ is defined analogously. We set* $\operatorname{spwd}_n \mathfrak{M} := \operatorname{spwd}_n\Big(\mathfrak{M}, |M|^{<|M|^+}\Big)$.

(3) *The* monadic [symmetric] partition widths $\operatorname{mpwd}_n$ *and* $\operatorname{smpwd}_n$ *of a partition refinement or a structure are defined similarly by replacing* $\operatorname{eti}_0^n$ *by* $\operatorname{emti}_0^n$.

**Remark 24** (1) *Obviously, we have* $\operatorname{pwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \operatorname{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$.

(2) *In each partition refinement $(U_v)_{v\in T}$ we can remove all nodes $v \in T$ with exactly one successor. In that way we can transform any $\kappa^{<\alpha}$-partition refinement of a structure of cardinality $\lambda$ into a $\kappa^{<\lambda^+}$-partition refinement.*

(3) *It is not clear whether there always exists a partition refinement $(U_v)_v$ such that* $\operatorname{pwd}_n \mathfrak{M} = \operatorname{pwd}_n(U_v)_v$ *for all $n$.*

**Lemma 25** *Every linear order $\mathfrak{M} = (M, \leq)$ has a $2^{<|M|^+}$-partition refinement $(U_v)_{v\in T}$ of monadic partition width* $\operatorname{mpwd}_n(U_v)_v = 1$ *where every $U_v$ forms an interval of $\mathfrak{M}$.*

**PROOF.** We define $U_v$ by induction on $|v|$. Let $U_\varepsilon := M$. Given an interval $U_v$ containing at least two different elements, we pick some $a \in U_v$ that is not the least element of $U_v$ and set

$$U_{v0} := \{\, b \in U_v \mid b < a \,\} \quad \text{and} \quad U_{v1} := \{\, b \in U_v \mid b \geq a \,\}.$$

Finally, if $|v|$ is a limit ordinal, we set $U_v := \bigcap_{u \preceq v} U_u$. □

**Lemma 26** *For the tree $\mathfrak{T} := (\beta^{<\alpha}, \preceq)$ we have*

$$\operatorname{smpwd}_n(\mathfrak{T}, \beta^{<2\alpha}) = 1 \quad \textit{and} \quad \operatorname{smpwd}_n(\mathfrak{T}, 2^{<(\beta+2)\alpha}) = 1.$$

**PROOF.** We define a $\beta^{<2\alpha}$-partition refinement $(U_v)_v$ by induction on $v$. Set $U_\varepsilon := \beta^{<\alpha}$. Suppose that $U_v$ is already defined and of the form $\uparrow w := \{\, x \in \beta^{<\alpha} \mid w \preceq x \,\}$ for some $w$. We define

$$U_{v0} := \{w\}\,, \quad U_{v1} = U_v \setminus \{w\}\,, \quad U_{v1i} := \uparrow wi \quad \text{for } i < \beta\,.$$

Then we have $\operatorname{emti}_0^n(U_v/\overline{U_v}) = 1$ for all $v$, as desired.

The second claim is proved analogously. If $U_v = \uparrow w$ is already defined, we set

$$U_{v0} := \{w\}\,, \quad U_{v11^\gamma} := \bigcup_{i \geq \gamma} \uparrow wi\,, \quad U_{v11^\gamma 0} := \uparrow w\gamma \quad \text{for } \gamma < \beta\,.$$

□

We promised above that we will show how one can use types to define a canonical colouring. For the symmetric case we first need a technical lemma which relates infinite symmetric sums and symmetric partition width.

We say that a disjoint union $\bigcup_i X_i$ *induces* the equivalence relation

$$a \sim b \quad :\text{iff} \quad \text{there is some } i \text{ with } a, b \in X_i\,.$$

When considering an $n$-tuple $\bar{a}$, this relation induces a partition $I_0 \cup\!\!\!\!\cdot\; \cdots \cup\!\!\!\!\cdot\; I_s = [n]$ of the indices such that $a_i \sim a_k$iff$i, k \in I_l$ for some $l$.

We call a tuple $\bar{a} \subseteq \bigcup_i X_i \cup U$ *fragmented* if the induced partition consists of at least two classes. Further, we say that a colouring $\chi$ of a set $X$ is *compatible* with the equivalence relation $\simeq_U^0$ if

$$\chi(\bar{a}) = \chi(\bar{b}) \quad \text{iff} \quad \bar{a} \simeq_U^0 \bar{b} \qquad \text{for all } \bar{a}, \bar{b} \subseteq X\,.$$

**Proposition 27** *Let $\mathfrak{M}$ be a structure of arity $r < \omega$, $Y := \bigcup_{i<\kappa} X_i \subseteq M$ a disjoint union, and $U \subseteq M$ disjoint from $Y$. For $I \subseteq \kappa$, define $X_I := \bigcup_{i \in I} X_i$ and $U_I := U \cup (Y \setminus X_I)$. Let $\sim$ be the equivalence relation induced by the union $\bigcup_i X_i$. Consider the following statements:*

(1) *There is a bound $\bar{w} \in \omega^\omega$ with $w_n \leq w_{n+1}$ such that*

$$\operatorname{eti}_0^n(X_I/U_I) \leq w_n \qquad \textit{for all } n < \omega \textit{ and } I \subseteq \kappa\,.$$

(2) *There exists a set of colours $C$ and $C$-colourings $\eta$ of $Y$ and $\chi_i$ of $X_i$ compatible with, respectively, $\simeq^0_U$ and $\simeq^0_{U_{\{i\}}}$ such that*

$$(\mathfrak{M}|_Y, \eta) = \bigcup_{i<\kappa}^{\Theta} (\mathfrak{M}|_{X_i}, \chi_i) \qquad \text{for suitable } \Theta\,.$$

*The following implications hold:*

(2) $\Rightarrow$ (1) *with* $w_n \leq n^{n+1}(c_n)^n$ *where* $c_n := \max_{i\leq n}|C_i|$.
(1) $\Rightarrow$ (2) *with* $|C_n| < (w_n(r-n)+1)R(K_n)^2_{r^{3r+1}}$ *where*

$$K_n := w_n(rw_r)^r + R(w_n + 2(r-n)+2)^3_8\,.$$

**PROOF.** (2) $\Rightarrow$ (1) Define $\chi(\bar a) := \chi_i(\bar a)$ for $\bar a \subseteq X_i$, $i<\kappa$. Let $I \subseteq \kappa$ and $\bar a$, $\bar a' \in (X_I)^n$. We claim that, if $\sim$ induces the same partition $J_0 \cupdot \dots \cupdot J_s = [n]$ of the indices of $\bar a$ and $\bar a'$ and if $\chi(\bar a|_{J_i}) = \chi(\bar a'|_{J_i})$ for all $i \leq s$, then $\bar a \simeq^0_{U_I} \bar a'$.

First suppose that $\mathfrak{M} \models \varphi(\bar a; \bar b)$ for some atomic formula $\varphi$ and parameters $\bar b \subseteq Y \setminus X_I$. Then $\bigcup^\Theta$ adds all tuples of colour $\eta(\bar a\bar b) = \eta(\bar a'\bar b)$ to the corresponding relation. Hence, $\mathfrak{M} \models \varphi(\bar a'; \bar b)$.

It remains to consider the case $\mathfrak{M} \models \varphi(\bar a; \bar b, \bar c)$ where $\bar b \subseteq Y \setminus X_I$ and $\bar c \subseteq U$. $\eta(\bar a\bar b) = \eta(\bar a'\bar b)$ implies $\bar a\bar b \simeq^0_U \bar a'\bar b$. Thus, $\mathfrak{M} \models \varphi(\bar a'; \bar b, \bar c)$.

Setting $c_n := \max_{i\leq n}|C_i|$ it follows that

$$\begin{aligned} w_n &\leq \sum \Big\{ |C_{|J_0|}| \cdot \dots \cdot |C_{|J_{s-1}|}| \;\Big|\; J_0 \cupdot \dots \cupdot J_{s-1} = [n],\ s \leq n \Big\} \\ &\leq \sum_{s\leq n} s^n (c_n)^s \leq n^{n+1}(c_n)^n\,. \end{aligned}$$

(1) $\Rightarrow$ (2) We call a sequence $(f_n)_{n\leq r}$ of functions

$$f_n : \bigcup_{\alpha<\kappa} X^n_\alpha \to C_n$$

a *valid colouring* iff

$$(\mathfrak{M}|_Y, \eta) = \bigcup_{\alpha<\kappa}^{\Theta} (\mathfrak{M}|_{X_\alpha}, \chi_\alpha)$$

for some $\Theta$ where $\chi_\alpha$ is the colouring of $X_\alpha$ induced by $(f_n)_n$. This condition is equivalent to the following one: $(f_n)_n$ is valid if and only if, for all tuples $\bar a$, $\bar b \in Y^n$ such that $\sim$ induces the same partition $J_0 \cup \dots \cup J_s$ of their indices, $f_{|J_i|}(\bar a|_{J_i}) = f_{|J_i|}(\bar b|_{J_i})$, $i \leq s$, and for every atomic formula $\varphi(\bar x; \bar d)$ with parameters $\bar{d} \subseteq U$ such that $\bar{a}\bar{d}$ and $\bar{b}\bar{d}$ are fragmented, we have

$$\mathfrak{M} \models \varphi(\bar{a}; \bar{d}) \leftrightarrow \varphi(\bar{b}; \bar{d}) .$$

Fix $(f_n)_n$. For $\bar{a}_0 \in X^n_\alpha$ and $\bar{b}_0 \in X^n_\beta$, we write $\bar{a}_0 \rightharpoondown \bar{b}_0$ if there are tuples $\bar{a}_1 \subseteq Y \setminus X_\alpha$ and $\bar{b}_1 \subseteq Y \setminus X_\beta$ such that

- $\sim$ induces the same partition $J_0 \cup \cdots \cup J_s$ of their indices,
- $f_{|J_i|}(\bar{a}_1|_{J_i}) = f_{|J_i|}(\bar{b}_1|_{J_i})$, for $i \leq s$, and
- for some atomic formula $\varphi(\bar{x}, \bar{y}; \bar{d})$ with parameters $\bar{d} \subseteq U$ such that $\bar{a}_0\bar{a}_1\bar{d}$ and $\bar{b}_0\bar{b}_1\bar{d}$ are fragmented, we have

$$\mathfrak{M} \models \varphi(\bar{a}_0, \bar{a}_1; \bar{d}) \leftrightarrow \neg\varphi(\bar{b}_0, \bar{b}_1; \bar{d}) .$$

We will call such tuples $\bar{a}_1$ and $\bar{b}_1$ *witnesses* of the fact that $\bar{a}_0 \rightharpoondown \bar{b}_0$.

By the above remark, it follows that $(f_n)_n$ is a valid colouring if and only if $\bar{a} \rightharpoondown \bar{b}$ implies $f_n(\bar{a}) \neq f_n(\bar{b})$ for all $\bar{a}$ and $\bar{b}$.

Let $(f_n)_n$ be a valid colouring such that $C_n := \operatorname{rng} f_n$ is of minimal size. Suppose that

$$m := |C_n| \geq (w_n(r-n)+1)R(K_n)^2_{r^{3r+1}} .$$

We fix an arbitrary ordering of each $C_n$ and we order colourings pointwise:

$$(f_n)_n \leq (g_n)_n \quad :\text{iff} \quad f_n(\bar{a}) \leq g_n(\bar{a}) \quad \text{for all } n \leq r,\ \bar{a} \in \bigcup_\alpha X^n_\alpha .$$

W.l.o.g. we may assume that $(f_n)_n$ is minimal w.r.t. this ordering. It follows that, for all $\bar{a} \in \bigcup_\alpha X^n_\alpha$ and every colour $c \in C_n$ with $c < f_n(\bar{a})$, there exists some tuple $\bar{b} \in f_n^{-1}(c)$ with $\bar{a} \rightharpoondown \bar{b}$ since, otherwise, the sequence $(g_n)_n$ defined by

$$g_n(\bar{x}) := \begin{cases} c & \text{if } \bar{x} = \bar{a} , \\ f_n(\bar{x}) & \text{otherwise} , \end{cases}$$

and $g_i := f_i$ for $i \neq n$, would be a strictly smaller valid colouring.

Further, it follows that $|\operatorname{rng} f_n|_{X^n_\alpha}| \leq w_n$ for all $\alpha < \kappa$ since, if $\bar{a} \simeq^0_{U_{\{\alpha\}}} \bar{b}$ and $f_n(\bar{a}) < f_n(\bar{b})$, then we could change the colour of $\bar{b}$ to $f_n(\bar{a})$ and the colouring would still be valid.

(A) Fix a decreasing enumeration $c_0 > \cdots > c_{m-1}$ of $C_n$. We construct a sequence $(\bar{a}^i)_i$ such that $\bar{a}^i \rightharpoondown \bar{a}^k$ for $i \neq k$. By induction on $i$, we define

- an increasing sequence of indices $s_i \in [m]$;

- a decreasing sequence of sets $H_i \subseteq [m]$;
- sets $I_{it} \subseteq \kappa$, for $s_i < t < m$; and
- tuples $\bar{a}^i \in f_n^{-1}(c_{s_i}) \cap X^n_{I_{i-1,s_i}}$

such that

- $\bar{b} \rightharpoondown \bar{a}^i$ for all $\bar{b} \in f_n^{-1}(c_t) \cap X^n_{I_{it}}$, $s_i < t < m$, and
- $f_n^{-1}(c_t) \cap X^n_{I_{it}} \neq \emptyset$ for all $t \in H_i$.

Let $H_{-1} := [m]$ and $I_{-1,t} := \kappa$. For every $i$, we perform the following steps. If $H_{i-1} = \emptyset$ we stop. Otherwise, let $s_i := \min H_{i-1}$ and choose an arbitrary tuple $\bar{a}^i \in f_n^{-1}(c_{s_i}) \cap X^n_{I_{i-1,s_i}}$, say $\bar{a}^i \in X^n_\alpha$. Since $I_{i-1,s_i} \subseteq I_{ks_i}$ for $k < i$ and by induction hypothesis, we have $\bar{a}^i \rightharpoondown \bar{a}^k$, for every $k < i$, as desired.

To define $I_{it}$, $s_i < t < m$, fix some $\bar{b}_0 \in f_n^{-1}(c_t)$ such that $\bar{b}_0 \rightharpoondown \bar{a}^i$, say, $\bar{b}_0 \in X^n_\beta$. By definition, there exist an atomic formula $\varphi(\bar{x}, \bar{y}; \bar{d})$ with parameters $\bar{d} \subseteq U$ and tuples $\bar{a}_1$ and $\bar{b}_1$ such that $\bar{a}^i\bar{a}_1\bar{d}$ and $\bar{b}_0\bar{b}_1\bar{d}$ are fragmented, $\sim$ induces the same partition $J_0 \cup \cdots \cup J_s$ of the indices of $\bar{a}_1$ and $\bar{b}_1$, $f_{|J_l|}(\bar{a}_1|_{J_l}) = f_{|J_l|}(\bar{b}_1|_{J_l})$, for $l \leq s$, and we have

$$\mathfrak{M} \models \varphi(\bar{a}^i, \bar{a}_1; \bar{d}) \leftrightarrow \neg\varphi(\bar{b}_0, \bar{b}_1; \bar{d}) .$$

Let $J \subseteq \kappa$ be the minimal set such that $\bar{b}_1 \subseteq X_J$. If $\bar{b}' \in f_n^{-1}(c_t) \cap X^n_{\kappa \setminus J}$ then

$$\mathfrak{M} \models \varphi(\bar{b}', \bar{b}_1; \bar{d}) \leftrightarrow \varphi(\bar{b}_0, \bar{b}_1; \bar{d})$$

since $(f_n)_n$ is a valid colouring. This implies $\bar{b}' \rightharpoondown \bar{a}^i$. Therefore, we can set $I_{it} := I_{i-1,t} \setminus J$. We conclude the construction by setting

$$H_i := \{\, t \in H_{i-1} \setminus \{s_i\} \mid f_n^{-1}(c_t) \cap X^n_{I_{it}} \neq \emptyset \,\} .$$

The sequence $(\bar{a}^i)_{i<m_1}$ obtained this way satisfies $\bar{a}^i \rightharpoondown \bar{a}^k$ for $i \neq k$. It remains to determine its length $m_1$. We have

$$\begin{aligned} |H_i| &\geq |H_{i-1}| - w_n|J| - 1 \\ &\geq |H_{-1}| - (i+1)(w_n(r-n)+1) \\ &= m - (i+1)(w_n(r-n)+1) . \end{aligned}$$

We can define $\bar{a}^i$ provided $H_{i-1} \neq \emptyset$. This is the case if

$$i < \frac{m}{w_n(r-n)+1} .$$

Consequently,

$$m_1 \geq \frac{m}{w_n(r-n)+1} \geq R(K_n)^2_{r^{3r+1}} .$$

(B) Denote the index $\alpha$ such that $\bar{a}^i \in X^n_\alpha$ by $\alpha_i$. For all $i < k$, we fix tuples $\bar{b}^{ik} \subseteq X_{\kappa \setminus \{\alpha_i\}}$ and $\bar{b}^{ki} \subseteq X_{\kappa \setminus \{\alpha_k\}}$ witnessing the fact that $\bar{a}^i \rightharpoondown \bar{a}^k$, that is,

$$\mathfrak{M} \models \varphi(\bar{a}^i, \bar{b}^{ik}; \bar{d}) \leftrightarrow \neg\varphi(\bar{a}^k, \bar{b}^{ki}; \bar{d})$$

for some atomic formula $\varphi(\bar{x}, \bar{y}; \bar{d})$. Let $J_0 \cup \cdots \cup J_s$ be the partition of the indices of $\bar{b}^{ik}$ (or of $\bar{b}^{ki}$) induced by $\sim$. Set

$$\bar{b}^{ik}_l := \bar{b}^{ik}|_{J_l}\,, \qquad \bar{b}^{ki}_l := \bar{b}^{ki}|_{J_l}\,,$$

and let $\beta^{ik}_l, \beta^{ki}_l < \kappa$ be the indices such that $\bar{b}^{ik}_l \subseteq X_{\beta^{ik}_l}$ and $\bar{b}^{ki}_l \subseteq X_{\beta^{ki}_l}$. Assume that we have chosen $\bar{b}^{ik}$ and $\bar{b}^{ki}$ such that the set

$$N := \{\, l \mid \bar{b}^{ik}_l = \bar{b}^{ki}_l \,\}$$

is maximal.

It follows that, for each $l \notin N$, we either have $\beta^{ik}_l = \alpha_k$ or there exists some index $\sigma(l) \neq l$ such that $\beta^{ik}_l = \beta^{ki}_{\sigma(l)}$. Otherwise, we could replace $\bar{b}^{ki}_l$ by $\bar{b}^{ik}_l$ and the resulting pair of tuples would still witness $\bar{a}^i \rightharpoondown \bar{a}^k$ in contradiction to the maximality of $N$.

Let $\sigma_{ik} : [s+1] \setminus N \to ([s+1] \setminus N) \cup \{*\}$ be the function such that

$$\beta^{ik}_l = \begin{cases} \alpha_k & \text{if } \sigma_{ik}(l) = *\,, \\ \beta^{ki}_{\sigma_{ik}(l)} & \text{otherwise}\,, \end{cases}$$

and define $\sigma_{ki}$ analogously. The maximality of $N$ further implies that there exists no sequence $l_0, \ldots, l_t$ of indices such that $\sigma_{ik}(l_j) = l_{j+1}$, for $j < t$, and $\sigma_{ik}(l_t) = l_0$ since, otherwise, we could simultaneously replace each $\bar{b}^{ki}_{l_j}$ by $\bar{b}^{ik}_{l_j}$ and again obtain witnesses for $\bar{a}^i \rightharpoondown \bar{a}^k$ with strictly larger $N$.

It follows that $\beta^{ik}_l \in \{\alpha_k, \beta^{ki}_0, \ldots, \beta^{ki}_s\}$, for every $l \notin N$, and there is some number $j$ such that $\sigma^j_{ik}(l) = *$, i.e., $\beta^{ik}_{\sigma^{j-1}_{ik}(l)} = \alpha_k$.

For each pair $i < k$ of indices we record

- the partition $J_0 \cup \cdots \cup J_s$ of the indices of $\bar{b}^{ik}$ induced by $\sim$,
- the size $|N|$ of the set $N$ defined above, and
- the functions $\sigma_{ik}$ and $\sigma_{ki}$.

There exists a subset $I \subseteq \kappa$ of size

$$\begin{aligned} |I| &\geq m_2 := \max\{\, k \mid m_1 \to (k)^2_{r^{3r+1}} \,\} \\ &\geq K_n = w_n(rw_r)^r + R(w_n + 2(r-n) + 2)^3_8 \end{aligned}$$

such that all pairs $i, k \in I$ with $i < k$ are coloured in the same way. W.l.o.g. we may assume that $I = [m_2]$.

(c) First, consider the case that $N = [s+1]$ for all $i, k \in I$. Let $B_{ik} \subseteq \kappa$ be the smallest set of indices such that $\bar{b}^{ik} = \bar{b}^{ki} \subseteq X_{B_{ik}}$. Clearly, $B_{ik} = B_{ki}$. Also note that, by definition of $\bar{b}^{ik}$ and $\bar{b}^{ki}$, we have $\alpha_i, \alpha_k \notin B_{ik}$. For each set $\{i, k, l\}$ of indices $i < k < l$, we record which of the following conditions hold:

$$\alpha_i \in B_{kl}\,, \quad \alpha_k \in B_{il}\,, \quad \alpha_l \in B_{ik}\,.$$

There exists a subset $I' \subseteq [m_2]$ of size

$$|I'| \geq m_3 := \max\{\, k \mid m_2 \to (k)^3_{2^3} \,\} \geq w_n + 2(r-n) + 2$$

such that all triples $i, k, l \in I'$ are coloured in the same way. W.l.o.g. we may assume that $I' = [m_3]$.

First we consider the case that $\alpha_l \in B_{ik}$ for all $i < k < l < m_3$. Then $\alpha_i \in B_{01}$, for $1 < i < m_3$. Furthermore, for $0 < i < k$, we have $\alpha_i \notin B_{0i}$ and $\alpha_k \in B_{0i} \setminus B_{0k}$ which implies that $\alpha_i \neq \alpha_k$. Hence,

$$m_3 \leq |B_{01}| + 2 \leq r - n + 2\,.$$

Contradiction. Analogously, if $\alpha_i \in B_{kl}$ or $\alpha_k \in B_{il}$, for $i < k < l$, then we obtain, respectively,

$$m_3 \leq |B_{m_3-2,m_3-1}| + 2 \quad \text{and} \quad m_3 \leq |B_{0,m_3-1}| + 2\,,$$

which lead to similar contradictions.

The only remaining case is that none of the above conditions holds, that is, we have $\alpha_i \notin B_{kl}$ for all pairwise distinct sets of indices $i, k, l$. Let $H := \{\, \alpha_i \mid i < m_3 \,\}$. $\bar{b}^{ik} \subseteq U_H$ implies $\bar{a}^i \not\approx^0_{U_H} \bar{a}^k$, for all $i \neq k$. Consequently, we have

$$\mathrm{eti}^n_0(X_H/U_H) \geq m_3 > w_n.$$

Contradiction.

(d) It remains to consider the case that $[s+1] \setminus N \neq \emptyset$. Let $l_0 \in \sigma^{-1}_{10}(*)$, i.e., $\beta^{ki}_{l_0} = \alpha_i$, for all $i < k$, and define $l_{j+1} := \sigma_{01}(l_j)$. Let $l_0, \dots, l_t$ be the sequence of indices obtained in this way where $l_t = *$. Note that, for $i < k$ and $j < t-1$, we have $\beta^{ik}_{l_j} = \beta^{ki}_{\sigma_{01}(l_j)} = \beta^{ki}_{l_{j+1}}$. For notational convenience, we also set $\beta^{ki}_* := \beta^{ik}_{l_{t-1}} = \alpha_k$.

By induction on $j \leq t$, we construct a decreasing sequence of subsets $I_j \subseteq I$

of size

$$|I_j| \geq (|I| - 1)/(rw_r)^j$$

such that

$$\beta^{i0}_{l_j} = \beta^{k0}_{l_j} \quad \text{and} \quad f_{|J_{l_{j-1}}|}(\bar{b}^{0i}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{0k}_{l_{j-1}}) \quad \text{for all } i, k \in I_j \,.$$

For all indices $i, k \in I_t$ it follows that $\alpha_i = \beta^{i0}_* = \beta^{k0}_* = \alpha_k$. Since each tuple $\bar{a}^i$ has a different colour it further follows that $|I_t| \leq w_n$ which implies that

$$w_n \geq |I_t| \geq (|I| - 1)/(rw_r)^t > w_n \,.$$

Contradiction.

(E) We still have to construct the sets $I_j$. Let $I_0 := I \setminus \{0\}$. Since $\beta^{i0}_{l_0} = \alpha_0 = \beta^{k0}_{l_0}$ our claim holds for $j = 0$. Suppose that $I_0, \dots, I_{j-1}$ are already defined. Since $\beta^{i0}_{l_{j-1}} = \beta^{k0}_{l_{j-1}}$, for $i, k \in I_{j-1}$, there exists a subset $I'_j \subseteq I_{j-1}$ of size

$$|I'_j| \geq |I_{j-1}|/w_{|J_{l_{j-1}}|} \geq |I_{j-1}|/w_r$$

such that $f_{|J_{l_{j-1}}|}(\bar{b}^{i0}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{k0}_{l_{j-1}})$ for all $i, k \in I'_j$. It follows that

$$c := f_{|J_{l_{j-1}}|}(\bar{b}^{0i}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{i0}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{k0}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{0k}_{l_{j-1}}) \,,$$

and, by the remarks in (B), we have $f^{-1}_{|J_{l_{j-1}}|}(c) \subseteq X_{\{\alpha_0, \beta^{0i}_0, \dots, \beta^{0i}_s\}}$. Therefore, there exists a subset $I_j \subseteq I'_j$ of size

$$|I_j| \geq |I'_j|/(s+2) \geq |I_{j-1}|/(rw_r) \geq (|I| - 1)/(rw_r)^j$$

such that $\beta^{0i}_{l_{j-1}} = \beta^{0k}_{l_{j-1}}$ for all $i, k \in I_j$. It follows that

$$\beta^{i0}_{l_j} = \beta^{0i}_{l_{j-1}} = \beta^{0k}_{l_{j-1}} = \beta^{k0}_{l_j}$$

as desired. $\square$

After these somewhat lengthy preparations we are finally able to prove that every structure denoted by a term has finite partition width and, conversely, every structure with finite partition width is denoted by a term.

**Proposition 28** *Let $C$ be a graded set of colours, $\tau$ a signature, and $n < \omega$.*

(1) $\mathrm{pwd}_n(\mathrm{val}(T), \kappa^{<\alpha}) < \aleph_0$ *for all $\Upsilon^{<}_{C,\tau}$-terms $T \subseteq \kappa^{<\alpha}$.*
(2) $\mathrm{spwd}_n(\mathrm{val}(T), \kappa^{<\alpha}) < \aleph_0$ *for every $\Upsilon_{C,\tau}$-term $T \subseteq \kappa^{<\alpha}$.*

**PROOF.** (1) Consider the subterm $T_v$ of $T$ with root $v \in T$ and let $U_v$ be the universe of $\mathrm{val}(T_v)$. We claim that $(U_v)_{v\in T}$ is the desired partition refinement.

Suppose that $\bar{a}$, $\bar{b} \in U_v^n$ are tuples such that, for all $I \subseteq [n]$, the subtuples $\bar{a}|_I$ and $\bar{b}|_I$ have the same colour at node $v$. Let $\varphi(\bar{x}, \bar{c})$ be an atomic formula with parameters $\bar{c} \subseteq \overline{U_v}$. If $\mathrm{val}(T) \models \varphi(\bar{a}, \bar{c})$ then there exists a node $u \prec v$ such that $\bar{a}$, $\bar{c} \subseteq U_u$ and the operation $\sum^\Theta$ at $u$ adds all tuples with the colour of $(\bar{a}\bar{c})|_I$ to the relation in $\varphi$ where $I$ is the set of those indices that actually appear in $\varphi$. Since $(\bar{b}\bar{c})|_I$ has the same colour it follows that also $\mathrm{val}(T) \models \varphi(\bar{b}, \bar{c})$. Consequently, we have $\bar{a} \simeq^0_{U_v} \bar{b}$.

(2) Define $(U_v)_v$ as above. By the preceding proposition, we have

$$\mathrm{eti}_0^n\Big(\bigcup_{i\in I} U_{vi} \Big/ \overline{\bigcup_{i\in I} U_{vi}}\Big) \leq n^{n+1}\Big(\max_{i\leq n} |C_n|\Big)^n .$$

□

**Remark 29** *Note that, for* $n = 1$*, the proof above implies* $\mathrm{pwd}_1(\mathrm{val}(T), \kappa^{<\alpha}) \leq |C_1|$.

**Proposition 30** *Let $\mathfrak{M}$ be a $\tau$-structure.*

(1) *Let $k < \aleph_0$. For every $k^{<\alpha}$-partition refinement $(U_v)_{v\in S}$ of $\mathfrak{M}$ of finite partition width, there exists a $\Upsilon^<_{C,\tau}$-term $T \subseteq k^{<\alpha}$ denoting $\mathfrak{M}$ where $C$ is a set of colours with $|C_n| \leq \mathrm{pwd}_n(U_v)_v$ for $n < \omega$.*

(2) *If the arity of $\mathfrak{M}$ is finite and there exists a $\kappa^{<\alpha}$-partition refinement $(U_v)_{v\in S}$ of $\mathfrak{M}$ such that $\mathrm{spwd}_n(U_v)_v < \aleph_0$ for all $n$, then there is a $\Upsilon_{C,\tau}$-term $T \subseteq \kappa^{<\alpha}$ denoting $\mathfrak{M}$ for some set of colours $C$.*

**PROOF.** (1) Let $w_n := \mathrm{pwd}_n(U_v)_v$. Let $T := S \cup \{\, w0 \mid w \text{ leaf of } S \,\}$ be the tree obtained from $S$ by adding to every leaf of $S$ a new vertex as successor. We construct a $\Upsilon^<_{C,\tau}$-term with domain $T$ such that, for every $v \in S$, the subterm $T_v := \{\, w \in T \mid w \succeq v \,\}$ will evaluate to the substructure $\mathfrak{M}|_{U_v}$ of $\mathfrak{M}$ induced by $U_v$.

In a first step, each such component $U_v$ will be coloured by a different set $C^v$ of colours with $|C^v_n| \leq w_n$. To obtain a single set of colours $C$ we then define injective functions $\mu^v_n : C^v_n \to [w_n]$ and identify colours $c \in C^u_n$ and $d \in C^v_n$ iff $\mu^u_n(c) = \mu^v_n(d)$.

Colour each tuple $\bar{a} \subseteq U_v$ by its external type $\mathrm{etp}_0(\bar{a}/\overline{U_v})$. If $\bar{a}_i \subseteq U_{vi}$, for $i < k$, then the type $\mathrm{etp}_0(\bar{a}_0 \dots \bar{a}_{k-1}/\overline{U_v})$ is uniquely determined by $\mathrm{etp}_0(\bar{a}_i/\overline{U_{vi}})$ for $i < k$. Hence, these colourings $\chi_v$ enable us to express $U_v$ as the ordered sum of the $U_{vi}$

$$(\mathfrak{M}|_{U_v}, \chi_v) = \sum_{i<k}^{\Theta_v} (\mathfrak{M}|_{U_{vi}}, \chi_{vi})$$

for a suitable set $\Theta_v$.

For non-leaves $v \in S$, we define the labelling of $T$ by $T(v) := \sum^{\Theta_v}$. Then we have $T_v = \sum_{i<k}^{\Theta_v} T_{vi}$.

For leaves $v \in S$ with $U_v = \{a\}$ we set $T(v) := \sum^{\Theta}$ and $T(v0) := \bar{c}$, i.e., $T_v = \sum^{\Theta} \bar{c}$, where $c_n := \mathrm{etp}_0(a^n/M \setminus \{a\})$ and

$$\Theta := \{\, (n, 1, [n], c_n, c_n, S_n) \mid n < \omega \,\}$$

with $S_n := \{\, R \mid a^n \in R \,\}$.

It remains to define the functions $\mu_n^v : C_n^v \to [w_n]$ such that the resulting term $T := T_\varepsilon$ is well-formed. For $v \in T$, we denote by $v_\beta \preceq v$ the prefix of $v$ of length $|v_\beta| = \beta$ and, for each type $p \in C_n^v$ over $\overline{U_v}$, we denote by $p_\beta$ its restriction to $\overline{U_{v_\beta}}$.

For $T$ to be well-formed it is sufficient to define $\mu_n^v$ such that

- for each $p \in C_n^v$, the sequence $(\mu_n^{v_\beta}(p_\beta))_{\beta<|v|+1}$ forms a colour trace to $v$;
- the colour traces to $v$ are linearly ordered.

We define $\mu_n^v$ by induction on $|v|$. Let $\mu_n^\varepsilon$ be an arbitrary injective function $C_n^\varepsilon \to [w_n]$. (Note that $|C_n^\varepsilon| = 1$ since there is only one external type over the empty set.) Suppose that $\mu_n^u$ is already defined for all $|u| < \alpha$ and let $|v| = \alpha$.

First, consider the case that $\alpha = \beta + 1$ is a successor. Set $u := v_\beta$ and let $<$ be the ordering on $C_n^u$ induced by the function $\mu_n^u$. We order $C_n^v$ in the following way. If $p_\beta < p'_\beta$, for $p$, $p' \in C_n^v$, then we set $p < p'$ and, if $p_\beta = p'_\beta$, then we choose an arbitrary ordering between them. Finally, let $\mu_n^v$ be some injective order preserving function $C_n^v \to [w_n]$.

It remains to consider limit ordinals $\alpha$. Let $p \in C_n^v$ and let $c$ be the minimal number such that the set $\{\, \beta < \alpha \mid \mu_n^{v_\beta}(p_\beta) = c \,\}$ is unbounded. We set $\mu_n^v(p) := c$.

With these definitions, $(\mu_n^{v_\beta} p_\beta)_\beta$ satisfies both conditions on a colour trace, and we have ensured that all colour traces to some node $v$ are linearly ordered.

(2) In the symmetric case the proof is analogous except that, according to the above proposition, we have to use a suitable refinement of the colouring given

by the external types. This poses no problem since the number of additional $n$-ary colours only depends on the arity of $\mathfrak{M}$ and $\mathrm{spwd}_i(U_v)_v$, for $i < \omega$, so the bound $\sup\{\, |C_i^v| \mid v \in S \,\}$ remains finite. $\square$

We claimed above that partition width generalises the notion of clique-width or NLC-width. This is justified by the following lemma.

**Lemma 31** *Let $\mathfrak{G} = (V, E)$ be a countable undirected graph of NLC-width $k$.*

$$\mathrm{pwd}_1(\mathfrak{G}, 2^{<\omega}) \leq k \leq \mathrm{cwd}\,\mathfrak{G} \leq 2 \cdot \mathrm{pwd}_1(\mathfrak{G}, 2^{<\omega})\,.$$

**PROOF.** The first inequality follows since VR- and NLC-operations can be expressed by suitable $\Upsilon^{<}_{C,\tau}$-terms using the same set of colours. For the last inequality, fix a $\Upsilon^{<}_{C,\tau}$-term $T$ denoting $\mathfrak{G}$ with $n := |C_1|$ colours of arity 1. We construct a VR-term using colours $[2n]$.

For $w \in 2^{<\omega}$, let $T_w$ be the subterm of $T$ with root $w$ and let $U_w$ be the universe of $\mathrm{val}(T_w)$. For every injective mapping $\varphi$ of the atomic external 1-types over $\overline{U_w}$ realised in $U_w$ into the set $[2n]$, we will construct a VR-term $t_w^\varphi$ that denotes $\mathrm{val}(T_w)$ such that the colouring of elements $a \in U_w$ is the one induced by $\varphi$.

If $w$ is a leaf with $U_w = \{a\}$ then we set

$$t_w^\varphi := \varphi\Big(\mathrm{etp}_0(a/V \setminus \{a\})\Big)\,.$$

Otherwise, $T_w = T_{w0} +^{\Theta} T_{w1}$, and we set

$$t_w^\varphi := \varrho_\beta \mathrm{add}(t_{w0}^{\psi_0} + t_{w1}^{\psi_1})$$

where $\psi_0$ and $\psi_1$ are mappings with disjoint ranges, $\beta$ maps the colours induced by $\psi_0$ and $\psi_1$ to the ones required by $\varphi$, and add is a sequence of operations $\alpha_{a,b}$ adding all the necessary edges. $\square$

## 5 The type equivalence

Before proceeding we need to collect some basic properties of type indices. In the following lemmas let $\mathfrak{M}$ be a fixed relational structure.

Recall that, when speaking of the quantifier rank of monadic second-order formulae, we consider the variant of MSO without first-order variables where the atomic formulae are of the form $X \subseteq Y$ and $R\bar{X}$, where the latter means that there exist some elements $a_i \in X_i$ such that $\bar{a} \in R$.

The first lemma summarises some immediate relations between the various kinds of type indices.

**Lemma 32** *Let $X, U \subseteq M$ and $\bar{a}, \bar{b} \in M^n$.*

(1) *If $m \leq n$ and $\Gamma \subseteq \Delta$ then $\mathrm{ti}^m_\Gamma(X/U) \leq \mathrm{ti}^n_\Delta(X/U)$ and analogously for the external and monadic case.*
(2) $\mathrm{eti}^n_0(X/U) \quad \leq \mathrm{ti}^n_0(X/U) \quad \leq |S^n_0(\emptyset)| \cdot \mathrm{eti}^n_0(X/U)\,,$

$$\mathrm{emti}^n_0(X/U) \leq \mathrm{mti}^n_0(X/U) \leq |MS^n_0(\emptyset)| \cdot \mathrm{emti}^n_0(X/U)\,.$$
(3) $a_0 \dots a_{n-1} \approx^0_U b_0 \dots b_{n-1}$iff$\{a_0\} \dots \{a_{n-1}\} \approx^0_U \{b_0\} \dots \{b_{n-1}\}$.
(4) *If the arity of $\mathfrak{M}$ is bounded by $r$ then*

$$\mathrm{eti}^n_0(X/U) \leq \left(\mathrm{eti}^{r-1}_0(X/U)\right)^{2^n}.$$

**PROOF.** (1) $\bar{a} \approx^\Delta_U \bar{b}$ implies $\bar{a}|_I \approx^\Gamma_U \bar{b}|_I$ for all sets of indices $I$.

(2) $\bar{a} \approx^0_U \bar{b}$ iff $\bar{a} \simeq^0_U \bar{b}$ and $\mathrm{tp}_0(\bar{a}) = \mathrm{tp}_0(\bar{b})$.

(3) For singletons $X_i = \{a_i\}$ we have $R\bar{X}$ iff $R\bar{a}$.

(4) Let $\bar{a}$, $\bar{b} \in X^n$ such that $\bar{a}|_I \simeq^0_U \bar{b}|_I$ for all $I \subseteq [n]$ of size $|I| < r$. If $\bar{a} \not\simeq^0_U \bar{b}$ then there is some atomic formula $\varphi(\bar{x}; \bar{c})$ with $\bar{c} \subseteq U$ such that

$$\mathfrak{M} \models \varphi(\bar{a}; \bar{c}) \leftrightarrow \neg\varphi(\bar{b}; \bar{c})\,.$$

Let $I \subseteq [n]$ be the set of those indices $i$ such that the variable $x_i$ appears in $\varphi$. Then $|I| < r$ and $\bar{a}|_I \not\simeq^0_U \bar{b}|_I$. Contradiction.

Since there are

$$\sum_{i=0}^{r-1} \binom{n}{i} \leq \sum_{i<n} \binom{n}{i} = 2^n$$

subsets of $[n]$ of size less than $r$ the claim follows. $\square$

Frequently, one would like to compute the type index of a boolean combination of sets from their respective type indices. For arbitrary structures this is only possible in special cases and even then quite complicated. For instance, we can construct a structure $\mathfrak{M}$ such that $\mathrm{pwd}_n \mathfrak{M} \geq \aleph_0$, for all $n$, but there exists a single element $v \in M$ such that $\mathrm{pwd}_n \mathfrak{M}|_{M \setminus v} = 1$ for all $n < \omega$:

Let $(\mathbb{Z} \times \mathbb{Z}, E)$ be the infinite grid, and let $v$ be a new vertex. We can set $\mathfrak{M} := (M, R)$ where

$$M := \mathbb{Z} \times \mathbb{Z} \cup \{v\}$$

and $\quad R := \{ (a, b, v) \mid (a, b) \in E \}$ .

Nevertheless, some results can be obtained.

**Lemma 33** *Let $X$, $Y \subseteq M$ and $n < \omega$.*

$$\begin{aligned} \operatorname{ti}^n_\Delta(X \cup Y / \overline{X \cup Y}) &\leq \sum_{i \leq n} \binom{n}{i} \operatorname{ti}^i_\Delta(X/\overline{X}) \operatorname{ti}^{n-i}_\Delta(Y \setminus X / \overline{Y \setminus X}) \\ &\leq 2^n \operatorname{ti}^n_\Delta(X/\overline{X}) \operatorname{ti}^n_\Delta(Y \setminus X / \overline{Y \setminus X}) \,. \end{aligned}$$

*The same holds for* $\operatorname{eti}^n_\Delta$.

**PROOF.** The second inequality holds by Lemma 32 (1). To prove the first one, let $i < n$, $\bar{a}, \bar{a}' \in X^i$, and $\bar{b}, \bar{b}' \in (Y \setminus X)^{n-i}$. Set $U := \overline{X \cup Y}$. We claim that

$$\bar{a} \approx^\Delta_{U \cup \bar{b}'} \bar{a}' \text{ and } \bar{b} \approx^\Delta_{U \cup \bar{a}} \bar{b}' \quad \text{implies} \quad \bar{a}\bar{b} \approx^\Delta_U \bar{a}\bar{b}' \approx^\Delta_U \bar{a}'\bar{b}' \,.$$

Suppose for a contradiction that $\bar{a}\bar{b} \not\approx^0_U \bar{a}'\bar{b}'$. There exists some formula $\varphi(\bar{x}, \bar{y}; \bar{c}) \in \Delta$ with parameters $\bar{c} \subseteq U$ such that

$$\mathfrak{M} \models \varphi(\bar{a}, \bar{b}; \bar{c}) \leftrightarrow \neg\varphi(\bar{a}', \bar{b}'; \bar{c}) \,.$$

But $\bar{b} \approx^\Delta_{U \cup \bar{a}} \bar{b}'$ implies that

$$\mathfrak{M} \models \varphi(\bar{a}, \bar{b}; \bar{c}) \leftrightarrow \varphi(\bar{a}, \bar{b}'; \bar{c}) \,,$$

and $\bar{a} \approx^\Delta_{U \cup \bar{b}'} \bar{a}'$ implies that

$$\mathfrak{M} \models \varphi(\bar{a}, \bar{b}'; \bar{c}) \leftrightarrow \varphi(\bar{a}', \bar{b}'; \bar{c}) \,.$$

Contradiction. The result follows since there are $\binom{n}{i}$ possible ways to shuffle an $i$-tuple and an $(n - i)$-tuple. $\square$

**Lemma 34** *Let $\mathfrak{M}$ be a relational structure, $X, U \subseteq M$. Let $m$ be the number of relations of arity greater than 1 and let $r$ be the supremum of their arities.*

$$\operatorname{eti}^n_0(U/X) \leq 2^{m(n+1)^r \operatorname{eti}^{r-1}_0(X/U)} \,.$$

**PROOF.** Let $\bar{a}, \bar{a}' \in U^n$. We have $\bar{a} \simeq^0_X \bar{a}'$ iff

$$\mathfrak{M} \models \varphi(\bar{a}, \bar{b}) \leftrightarrow \varphi(\bar{a}', \bar{b})$$

for all $\bar{b} \subseteq X$ and for all atomic formulae $\varphi(\bar{x}, \bar{y})$ containing at least one $x_i$ and one $y_j$. Obviously, we only need to consider tuples $\bar{b}$ of less than $r$ elements. Also note that, if $\bar{b} \simeq^0_U \bar{b}'$, then $\mathfrak{M} \models \varphi(\bar{a}, \bar{b})$ iff $\mathfrak{M} \models \varphi(\bar{a}, \bar{b}')$. Hence, it is sufficient to take one representative of each $\simeq^0_U$-class. Finally, if $\varphi'(\bar{x}, \bar{y})$ is obtained from $\varphi(\bar{x}, \bar{y})$ by a permutation of $\bar{y}$, then $\mathfrak{M} \models \varphi(\bar{a}, \bar{b})$ iff $\mathfrak{M} \models \varphi'(\bar{a}, \bar{b}')$ where $\bar{b}'$ is the corresponding permutation of $\bar{b}$. Thus, we can ignore the ordering of the variables $\bar{y}$. The claim follows since there are at most $m(n+1)^r$ atomic formulae with variables $\bar{x}\bar{y}$ and the number $\simeq^0_U$-classes is $\mathrm{eti}^{r-1}_0(X/U)$. □

In the definition of partition width we only considered atomic formulae. This is no restriction as the type indices of formulae of higher quantifier rank are bounded by the quantifier-free ones.

**Lemma 35** *Let $\mathfrak{M}$ be a structure, $X \subseteq M$, and $n, k < \omega$.*

(1) $\mathrm{eti}^n_k(X/\overline{X}) \leq \beth_k(\mathrm{eti}^{n+k}_0(X/\overline{X}))$.
(2) $\mathrm{ti}^n_k(X/\overline{X}) \leq \beth_k(\mathrm{ti}^{n+k}_0(X/\overline{X}))$.
(3) $\mathrm{mti}^n_k(X/\overline{X}) \leq \beth_k(\mathrm{mti}^{n+k}_0(X/\overline{X}))$.
(4) $\mathrm{emti}^n_k(X/\overline{X}) \leq \beth_k(\mathrm{emti}^{n+k}_0(X/\overline{X}))$.

**PROOF.** Since the proofs are very similar we only show a strong version of (3). Let $\Delta(k)$ be the fragment of *infinitary* monadic second-order logic consisting of all formulae of quantifier rank at most $k$. We prove that $\mathrm{mti}^n_{\Delta(k+1)}(X/\overline{X}) \leq 2^{\mathrm{mti}^{n+1}_{\Delta(k)}(X/\overline{X})}$.

For $\bar{A}, \bar{A}' \in \mathcal{P}(X)^n$ we have

$$\bar{A} \approx^{\Delta(k+1)}_X \bar{A}' \text{ iff for all } B \text{ there is some } B' \text{ with } \bar{A}B \approx^{\Delta(k)}_X \bar{A}'B' \\ \text{and vice versa}\,.$$

Since $\bar{A}B \approx^{\Delta(k)}_X \bar{A}'B'$ iff $\bar{A}(B \cap X) \approx^{\Delta(k)}_X \bar{A}'(B' \cap X)$ and $B \setminus X = B' \setminus X$, we only need to consider sets $B \subseteq X$. Defining

$$e(\bar{A}) := \left\{ [\bar{A}B] \in \mathcal{P}(X)^{n+1}/{\approx^{\Delta(k)}_X} \;\middle|\; B \subseteq X \right\}$$

we obtain $\bar{A} \approx^{\Delta(k+1)}_X \bar{A}'$ iff $e(\bar{A}) = e(\bar{A}')$. It follows that

$$\begin{aligned} \mathrm{mti}^n_{\Delta(k+1)}(X/\overline{X}) &= \left|\mathcal{P}(X)^n \;/\; {\approx^{\Delta(k+1)}_X}\right| \\ &\leq \left|\mathcal{P}\big(\mathcal{P}(X)^{n+1} \;/\; {\approx^{\Delta(k)}_X}\big)\right| = 2^{\mathrm{mti}^{n+1}_{\Delta(k)}(X/\overline{X})}\,. \end{aligned}$$

□

The next result shows that having finite partition width is a finitary condition. This is the reason for the various compactness properties of Section 7.

**Lemma 36** *Let $X, U \subseteq M$, $\Delta \subseteq \mathrm{FO}$, and $n < \omega$.*

(1) *Let $\bar{a}, \bar{b} \subseteq M$. If $\bar{a} \not\approx^{\Delta}_{U} \bar{b}$ then there is a finite subset $U_0 \subseteq U$ and a single formula $\varphi \in \Delta$ such that $\bar{a} \not\approx^{\varphi}_{U_0} \bar{b}$. The same holds for $\simeq^{\Delta}_{U}$.*
(2) *If $\mathrm{ti}^n_\Delta(X/U)$ is finite then there are finite subsets $U_0 \subseteq U$ and $\Delta_0 \subseteq \Delta$ such that $X^n/{\approx^{\Delta}_{U}} = X^n/{\approx^{\Delta_0}_{U_0}}$. The same holds for $\mathrm{eti}^n_\Delta$ and $\simeq^{\Delta}_{U}$.*
(3) *If $\mathrm{eti}^n_\Delta(X/U)$ is finite then*

$$\mathrm{eti}^n_\Delta(X/U) = \sup\{\, \mathrm{eti}^n_{\Delta_0}(X/U) \mid \Delta_0 \subseteq \Delta \textit{ finite} \,\}.$$

(4) *If $\mathrm{ti}^n_\Delta(X/U)$ is finite then the relation $\approx^{\Delta}_{U}$ is $\mathcal{B}(\Delta)$-definable on $X^n$. ($\mathcal{B}(\Delta)$ is the boolean closure of $\Delta$.)*

**PROOF.** (1) If $\bar{a} \not\approx^{\Delta}_{U} \bar{b}$ then there is some formula $\varphi(\bar{x}, \bar{c}) \in \Delta$ with $\bar{c} \subseteq U$ such that $\mathfrak{M} \models \varphi(\bar{a}, \bar{c}) \leftrightarrow \neg\varphi(\bar{b}, \bar{c})$. Setting $U_0 := \bar{c}$ we obtain $\bar{a} \not\approx^{\varphi}_{U_0} \bar{b}$.

(2) According to (1) there are finite sets $U_{[\bar{a}][\bar{b}]}$ and formulae $\varphi_{[\bar{a}][\bar{b}]}$, for each pair of distinct classes $[\bar{a}], [\bar{b}] \in X^n/{\approx^{\Delta}_{U}}$, such that $\bar{a} \not\approx^{\varphi_{[\bar{a}][\bar{b}]}}_{U_{[\bar{a}][\bar{b}]}} \bar{b}$. Setting $U_0 := \bigcup_{[\bar{a}] \neq [\bar{b}]} U_{[\bar{a}][\bar{b}]}$ and $\Delta_0 := \{\, \varphi_{[\bar{a}][\bar{b}]} \mid [\bar{a}] \neq [\bar{b}] \,\}$ we obtain

$$\bar{a} \approx^{\Delta}_{U} \bar{b} \quad \text{iff} \quad \bar{a} \approx^{\Delta_0}_{U_0} \bar{b} \qquad \text{for all } \bar{a}, \bar{b} \in X^n\,.$$

(3) immediately follows from (2).

(4) For each pair $[\bar{a}], [\bar{b}] \in X^n/{\approx^{\Delta}_{U}}$ of distinct classes we fix a $\Delta$-formula $\varphi_{[\bar{a}][\bar{b}]}(\bar{x}, \bar{y})$ and parameters $\bar{c}_{[\bar{a}][\bar{b}]}$ such that

$$\mathfrak{M} \models \varphi_{[\bar{a}][\bar{b}]}(\bar{a}, \bar{c}_{[\bar{a}][\bar{b}]}) \leftrightarrow \neg\varphi_{[\bar{a}][\bar{b}]}(\bar{b}, \bar{c}_{[\bar{a}][\bar{b}]})\,.$$

Then we have $\bar{a} \approx^{\Delta}_{U} \bar{a}'$ iff

$$\mathfrak{M} \models \bigwedge_{[\bar{b}] \neq [\bar{b}']} \Big(\varphi_{[\bar{b}][\bar{b}']}(\bar{a}, c_{[\bar{b}][\bar{b}']}) \leftrightarrow \varphi_{[\bar{b}][\bar{b}']}(\bar{a}', c_{[\bar{b}][\bar{b}']})\Big)\,.$$

□

**Lemma 37** *Let $\bar{w} \in \omega^\omega$. Let $(X_v)_{v \in I}$ be an increasing chain of sets $X_v$ (i.e., $u \leq v$ implies $X_u \subseteq X_v$) indexed by an arbitrary linear order $(I, \leq)$ such that $\mathrm{eti}^n_0(X_v/\overline{X_v}) \leq w_n$ for all $n < \omega$.*

$$\mathrm{eti}^n_0\Big(\bigcup_{v \in I} X_v \Big/ \overline{\bigcup_{v \in I} X_v}\Big) \leq w_n \quad \textit{and} \quad \mathrm{eti}^n_0\Big(\bigcap_{v \in I} X_v \Big/ \overline{\bigcap_{v \in I} X_v}\Big) \leq w_n\,.$$

**PROOF.** For the first claim, let $W := \bigcup_{v \in I} X_v$. Suppose there are $w_n + 1$ tuples $\bar{a}_i \in W^n$, $i \leq w_n$, such that $\bar{a}_i \not\approx^0_{\overline{W}} \bar{a}_k$ for $i \neq k$. There exists some $v \in I$ with $\bar{a}_i \subseteq X_v$ for all $i \leq w_n$. Hence,

$$\mathrm{eti}^n_0(X_v/\overline{X_v}) \geq \mathrm{eti}^n_0(X_v/\overline{W}) \geq w_n + 1\,.$$

Contradiction.

To prove the second bound, set $W := \bigcap_{v \in I} X_v$. Suppose there are $w_n + 1$ tuples $\bar{a}_i \in W^n$, $i \leq w_n$, such that $\bar{a}_i \not\approx^0_{\overline{W}} \bar{a}_k$ for $i \neq k$. By the preceding lemma, there exist finite sets $U_{ik} \subseteq \overline{W}$, $i \neq k$, such that $\bar{a}_i \not\approx^0_{U_{ik}} \bar{a}_k$ for $i \neq k$. Since $U := \bigcup_{i \neq k} U_{ik}$ is finite there is some $v \in I$ with $U \subseteq \overline{X_v}$. As $\bar{a}_i \subseteq X_v$ for all $i \leq w_n$ it follows that

$$\mathrm{eti}^n_0(X_v/\overline{X_v}) \geq \mathrm{eti}^n_0(X_v/U) \geq w_n + 1\,.$$

Contradiction. $\square$

Finally, we note that adding unary predicates does not change the partition width since $\mathrm{etp}_\Delta(\bar{a}/U)$ does not contain formulae of the form $Px_i$, and $\mathrm{emtp}_\Delta(\bar{A}/U)$ no formulae $PX_i$.

**Lemma 38** *Let $X, U \subseteq M$. $\mathrm{eti}^\alpha_\Delta(X/U)$ and $\mathrm{emti}^\alpha_\Delta(X/U)$ do not change if we add arbitrarily many unary predicates to $\mathfrak{M}$.*

## 6 Interpretations

Now we are ready to give a characterisation of the class of structures of finite partition width in terms of interpretations in trees. One direction was already presented in Proposition 17. For the other one, we show that finiteness of partition width is preserved by interpretations.

**Proposition 39** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be structures of finite signature and $\mathcal{I} : \mathfrak{M} \leq_{\mathrm{MSO}_k} \mathfrak{N}$. If $\mathrm{mpwd}_n(\mathfrak{N}, \kappa^{<\alpha})$ is finite for all $n < \omega$ then so is $\mathrm{mpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$. The same holds for $\mathrm{smpwd}_n(\mathfrak{N}, \kappa^{<\alpha})$.*

**PROOF.** Let $(U_v)_v$ be a partition refinement of $\mathfrak{N}$ of finite width. We claim that the partition refinement $(\mathcal{I}(U_v))_v$ of $\mathfrak{M}$ also has a finite width. By Lemmas 32 (2) and 35 it is sufficient to prove that, for all $\bar{A}$, $\bar{B} \subseteq \mathcal{P}(N)$, $U \subseteq N$, and $n < \omega$,

$$\bar{A} \approx^{n+k}_U \bar{B} \quad \text{implies} \quad \mathcal{I}(\bar{A}) \approx^n_{\mathcal{I}(U)} \mathcal{I}(\bar{B})\,.$$

Suppose $\mathcal{I}(\bar{A}) \not\approx^n_{\mathcal{I}(U)} \mathcal{I}(\bar{B})$. There exists an $\mathrm{MSO}_n$-formula $\varphi(\bar{x}, \bar{C})$ with parameters $\bar{C} \subseteq \mathcal{P}(\mathcal{I}(U))$ such that

$$\mathfrak{M} \models \varphi(\mathcal{I}(\bar{A}), \bar{C}) \wedge \neg\varphi(\mathcal{I}(\bar{B}), \bar{C}) .$$

Choose $\bar{D} \subseteq \mathcal{P}(U)$ such that $\bar{C} = \mathcal{I}(\bar{D})$. Then

$$\mathfrak{N} \models \varphi^{\mathcal{I}}(\bar{A}, \bar{D}) \wedge \neg\varphi^{\mathcal{I}}(\bar{B}, \bar{D}) .$$

Since $\varphi^{\mathcal{I}} \in \mathrm{MSO}_{n+k}$ we have $\bar{A} \not\approx^{n+k}_U \bar{B}$. $\square$

**Proposition 40** *If* $\mathfrak{M} \leq_{\mathrm{MSO}_k} (\kappa^{<\alpha}, \preceq, \bar{P})$ *for finitely many unary predicates* $\bar{P}$ *and some* $k < \omega$*, then* $\mathrm{smpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is finite for all* $n < \omega$*.*

The following theorem summarises the various characterisations we have obtained so far.

**Theorem 41** *Let* $\mathfrak{M}$ *be a structure of finite signature.*

(a) *For each tree* $\kappa^{<\alpha}$ *the following statements are equivalent:*
  (1) $\mathrm{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is finite for all* $n < \omega$*.*
  (2) $\mathrm{smpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is finite for all* $n < \omega$*.*
  (3) $\mathfrak{M} = \mathrm{val}(T)$ *for some* $\Upsilon_{C,\tau}$*-term* $T \subseteq \kappa^{<\alpha}$*.*
  (4) $\mathfrak{M} \leq_{\mathrm{MSO}_n} (\kappa^{<\alpha}, \preceq, \bar{P})$ *for finitely many unary predicates* $\bar{P}$ *and some* $n < \omega$*.*
(b) *If* $\kappa < \aleph_0$ *is finite then the following statements are equivalent to those above:*
  (5) $\mathrm{pwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is finite for all* $n < \omega$*.*
  (6) $\mathrm{mpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is finite for all* $n < \omega$*.*
  (7) $\mathfrak{M} = \mathrm{val}(T)$ *for some* $\Upsilon^{<}_{C,\tau}$*-term* $T \subseteq \kappa^{<\alpha}$*.*
  (8) $\mathfrak{M} \leq_{\mathrm{MSO}_n} (\kappa^{<\alpha}, \preceq, (\mathrm{suc}_i)_{i<\kappa}, \bar{P})$ *for finitely many unary predicates* $\bar{P}$ *and some* $n < \omega$*.*

**PROOF.** (1) $\Rightarrow$ (3) Since the arity of $\mathfrak{M}$ is bounded Lemma 32 (4) implies that there exists a partition refinement $(U_v)_v$ of $\mathfrak{M}$ such that $\mathrm{spwd}_n(U_v)_v$ is finite for all $n < \omega$. Consequently, the claim follows from Proposition 30.

(3) $\Rightarrow$ (4) $\Rightarrow$ (2) follows by Propositions 17 and 40.

(2) $\Rightarrow$ (1) $\mathrm{spwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \mathrm{smpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$.

Analogously, (5) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (6) follows from, respectively, Propositions 30, 17, and 40, together with the fact that $\mathrm{pwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \mathrm{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$.

(6) $\Rightarrow$ (5) is trivial.

(1) $\Rightarrow$ (5) also follows from $\operatorname{pwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \operatorname{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$.

(8) $\Rightarrow$ (4) If $\kappa$ is finite then $(\kappa^{<\alpha}, \preceq, (\mathrm{suc}_i)_{i<\kappa}, \bar{P}) \leq_{\mathrm{MSO}_1} (\kappa^{<\alpha}, \preceq, \bar{P}, \bar{Q})$ where $Q_i := \operatorname{rng} \mathrm{suc}_i$ since we can define

$$\mathrm{suc}_i(x, y) \quad :\text{iff} \quad x \prec y \wedge Q_i y \wedge \neg\exists z(x \prec z \prec y).$$

$\square$

## 7 Coding and compactness

In the final two sections we are going to show that the concept of partition width is a natural one from a model-theoretic point of view. We prove that elementary extensions preserve finiteness of partition width and present a compactness theorem for structures of finite partition width. In Section 8 it is shown that structures of finite partition width do not have the independence property.

We will restrict our attention to binary trees $2^{<\alpha}$. This can be done without loss of generality since $(\alpha^{<\beta}, \preceq) \leq_{\mathrm{FO}} (2^{<\alpha\beta}, \preceq, P)$ for a suitable unary predicate $P$. We start with a simple observation.

**Lemma 42** *If $\mathfrak{M} \subseteq \mathfrak{N}$ then $\operatorname{pwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \operatorname{pwd}_n(\mathfrak{N}, \kappa^{<\alpha})$ for all $n < \omega$.*

**PROOF.** Each partition refinement $(U_v)_{v\in T}$ of $\mathfrak{N}$ induces the partition refinement $(U_v \cap M)_{v\in T}$ of $\mathfrak{M}$ which has the width $\operatorname{pwd}_n(U_v \cap M)_v \leq \operatorname{pwd}_n(U_v)_v$. $\square$

In order to compute the partition width of structures constructed by model-theoretic means we need to code partition refinements by relations.

**Definition 43** (a) *Let $(U_v)_{v\in T}$ be a family of sets $U_v \subseteq M$ indexed by a partial order $(T, \preceq)$. A pair $(U, \sqsubseteq)$ of relations $U \subseteq M^{1+n}$ and $\sqsubseteq \subseteq M^{2n}$* code $(U_v)_{v\in T}$ *if there exists an isomorphism*

$$f : (D, \sqsubseteq) \cong (T, \preceq)\,,$$

*where $D := \{\, \bar{a} \in M^n \mid \bar{a} \sqsubseteq \bar{a} \,\}$, such that*

$$U := \{\, (a, \bar{b}) \in M \times D \mid a \in U_{f(\bar{b})} \,\}\,,$$

*and $\bar{a} \sqsubseteq \bar{b}$ implies $\bar{a}, \bar{b} \in D$.*

(b) *We call a partition refinement* $(U_v)_{v\in T}$ *of* $\mathfrak{M}$ reduced *if all non-leaves of* $T$ *have at least two immediate successors. If* $(U_v)_{v\in T}$ *is reduced we can define a* canonical coding *of* $(U_v)_v$ *in the following way. For each* $v \in T$ *choose leaves* $u_0, u_1 \in T$ *with* $v = u_0 \sqcap u_1$ *and set* $h(v) := (a_0, a_1)$ *where* $U_{u_i} = \{a_i\}$, $i < 2$. *Let* $D := \operatorname{rng} h$. *We define*

$$\bar{a} \sqsubseteq \bar{b} \quad \text{:iff} \quad \bar{a}, \bar{b} \in D \text{ and } h^{-1}(\bar{a}) \preceq h^{-1}(\bar{b}),$$

$$U := \{\, (c, \bar{a}) \mid \bar{a} \in D,\ c \in U_{h^{-1}(\bar{a})} \,\}.$$

**Remark 44** *Note that not every partition refinement* $(U_v)_{v\in T}$ *of a structure* $\mathfrak{M}$ *can be coded, since we might have* $|T| > |M^n|$ *for all* $n < \omega$. *But we can always obtain a codable partition refinement by removing some vertices* $v \in T$ *with exactly one immediate successor. The same holds for non-standard partition refinements which will be defined below.*

The fact that a relation $U$ codes some partition refinement can be expressed in first-order logic, with the sole exception that it is not possible to state that the components are arranged in a tree. Therefore, we consider partition refinements indexed by non-standard trees.

**Definition 45** *Let* $T^{\kappa}_{\text{tree}}$ *be the theory of all trees* $(S, \preceq)$ *where* $S \subseteq \kappa^{<\omega}$ *is prefix-closed.*

**Definition 46** *A* non-standard $\kappa^{<\omega}$-partition refinement *of a structure* $\mathfrak{M}$ *is a family* $(U_v)_{v\in T}$ *of subsets* $U_v \subseteq M$ *indexed by a model* $T$ *of* $T^{\kappa}_{\text{tree}}$ *satisfying the following conditions:*

(1) *For all* $a \in M$ *there exists some* $v \in T$ *with* $U_v = \{a\}$.
(2) *If* $u \preceq v$, *for* $u, v \in T$, *then* $U_u \supseteq U_v$.
(3) *If* $u, v \in T$ *are incomparable then* $U_u \cap U_v = \emptyset$.

*Note that we do not require the* $U_v$ *to be nonempty.*

*The widths* $\operatorname{pwd}_n(U_v)_v$ *and* $\operatorname{spwd}_n(U_v)_v$ *of* $(U_v)_v$ *are defined in the same way as for standard partition refinements.*

*For a structure* $\mathfrak{M}$ *we define the* non-standard [symmetric] partition width $\operatorname{pwd}^{\text{ns}}_n \mathfrak{M}$ *[*$\operatorname{spwd}^{\text{ns}}_n \mathfrak{M}$*] of* $\mathfrak{M}$ *as the minimal partition width of a non-standard* $2^{<\omega}$*-[*$\aleph_0^{<\omega}$*-]partition refinement of* $\mathfrak{M}$.

**Lemma 47** *If* $(U_v)_{v\in T}$ *is a non-standard* $\kappa^{<\omega}$*-partition refinement of* $\mathfrak{M}$ *and* $C \subseteq M$ *then* $(U_v \cap C)_{v\in T}$ *is a non-standard* $\kappa^{<\omega}$*-partition refinement of* $\mathfrak{M}|_C$ *of width*

$$\operatorname{pwd}_n(U_v \cap C)_{v\in T} \le \operatorname{pwd}_n(U_v)_v \qquad \text{for all } n < \omega.$$

**Corollary 48** *If* $\mathfrak{M} \subseteq \mathfrak{N}$ *then* $\operatorname{pwd}^{\text{ns}}_n(\mathfrak{M}, \kappa^{<\alpha}) \le \operatorname{pwd}^{\text{ns}}_n(\mathfrak{N}, \kappa^{<\alpha})$ *for all* $n < \omega$.

**Lemma 49** *Let $\mathfrak{M}$ be a $\tau$-structure and $(U, \sqsubseteq)$ a pair of additional relation symbols. For each $\kappa \leq \aleph_0$, there exists an* FO*-theory $T^\kappa_{\mathrm{pr}}$ such that $(\mathfrak{M}, U, \sqsubseteq) \models T^\kappa_{\mathrm{pr}}$ if and only if $(U, \sqsubseteq)$ codes a non-standard $\kappa^{<\omega}$-partition refinement of $\mathfrak{M}$.*

**PROOF.** Let $\Psi$ be the theory obtained from $T^\kappa_{\mathrm{tree}}$ by replacing every occurrence of $\preceq$ by $\sqsubseteq$ and relativising every formula to the set $D := \{\, \bar{a} \mid \bar{a} \sqsubseteq \bar{a} \,\}$. Further, let $\Phi$ consist of the following formulae which express the properties of a non-standard partition refinement:

$$\forall x \exists \bar{y} \forall z (U z \bar{y} \leftrightarrow z = x)$$
$$\forall \bar{y} \forall \bar{z} (\bar{y} \sqsubseteq \bar{z} \rightarrow \forall x (U x \bar{z} \rightarrow U x \bar{y}))$$
$$\forall \bar{y} \forall \bar{z} (\bar{y} \not\sqsubseteq \bar{z} \wedge \bar{z} \not\sqsubseteq \bar{y} \rightarrow \neg \exists x (U x \bar{y} \wedge U x \bar{z}))$$
$$\forall \bar{x} \forall \bar{y} (\bar{x} \sqsubseteq \bar{y} \rightarrow \bar{x} \sqsubseteq \bar{x} \wedge \bar{y} \sqsubseteq \bar{y})$$
$$\forall x \forall y (U x \bar{y} \rightarrow \bar{y} \sqsubseteq \bar{y})$$

Let $T^\kappa_{\mathrm{pr}} := \Phi \cup \Psi$. We claim that $(\mathfrak{M}, U, \sqsubseteq) \models T^\kappa_{\mathrm{pr}}$ iff $(U, \sqsubseteq)$ codes a non-standard $\kappa^{<\omega}$-partition refinement of $\mathfrak{M}$.

$(\Leftarrow)$ is obvious. For $(\Rightarrow)$, suppose that $(\mathfrak{M}, U, \sqsubseteq) \models T^\kappa_{\mathrm{pr}}$. We define

$$T := \{\, \bar{a} \in M^n \mid \bar{a} \sqsubseteq \bar{a} \,\},$$
$$\text{and } U_{\bar{a}} := \{\, b \in M \mid (b, \bar{a}) \in U \,\}, \qquad \text{for } \bar{a} \in T.$$

Then $(T, \sqsubseteq) \models T^\kappa_{\mathrm{tree}}$, $\bar{a} \sqsubseteq \bar{b}$ implies $\bar{a}, \bar{b} \in D$, and $(U_{\bar{a}})_{\bar{a} \in T}$ forms the desired non-standard $\kappa^{<\omega}$-partition refinement coded by $(U, \sqsubseteq)$. □

**Lemma 50** *Let $\mathfrak{M}$ be a $\tau$-structure and $(U, \sqsubseteq)$ a pair of additional relation symbols.*

(1) *For every sequence $\bar{w} \in \omega^\omega$ there is a set of sentences $\Pi^2_{\bar{w}} \subseteq$ FO such that $(\mathfrak{M}, U, \sqsubseteq) \models \Pi^2_{\bar{w}}$ if and only if $(U, \sqsubseteq)$ codes a non-standard $2^{<\omega}$-partition refinement $(U_v)_v$ of $\mathfrak{M}$ with $\mathrm{pwd}_n(U_v)_v \leq w_n$ for all $n < \omega$.*

(2) *For every sequence $\bar{w} \in \omega^\omega$ there is a set of sentences $\Pi^\omega_{\bar{w}} \subseteq$ FO such that $(\mathfrak{M}, U, \sqsubseteq) \models \Pi^\omega_{\bar{w}}$ if and only if $(U, \sqsubseteq)$ codes a non-standard $\aleph_0^{<\omega}$-partition refinement $(U_v)_v$ of $\mathfrak{M}$ with $\mathrm{spwd}_n(U_v)_v \leq w_n$ for all $n < \omega$.*

**PROOF.** (1) Since $(\mathfrak{M}, U, \sqsubseteq) \models T^2_{\mathrm{pr}}$ iff $(U, \sqsubseteq)$ codes a non-standard $2^{<\omega}$-partition refinement of $\mathfrak{M}$, it remains to express that the partition width is bounded.

According to Lemma 36 (3) it is sufficient to do so for all finite subsets $\tau_0 \subseteq \tau$. We construct formulae $\varphi^{\tau_0}_{n,m}$ expressing that the $n$-ary partition width of the

$\tau_0$-reduct is at most $m$. Then we can set

$$\Pi^2_{\bar{w}} := T^2_{\mathrm{pr}} \cup \{ \varphi^{\tau_0}_{n,w_n} \mid n < \omega,\ \tau_0 \subseteq \tau \text{ finite} \} .$$

Let $r$ be the maximal arity of relations in $\tau_0$. For $\bar{a}$, $\bar{b} \in X$, we have

$$\bar{a} \simeq^0_X \bar{b} \quad \text{iff} \quad \bar{a} \simeq^0_{\bar{c}} \bar{b} \quad \text{for all } \bar{c} \in \overline{X}^r .$$

Consequently, we can express that $\bar{x} \simeq^0_X \bar{y}$ by the formula

$$\psi(\bar{x}, \bar{y}; X) := \Big(\forall \bar{z}. \bigwedge_{i<r} \neg X z_i\Big)\Big[\mathrm{etp}_{\tau_0}(\bar{x}/\bar{z}) = \mathrm{etp}_{\tau_0}(\bar{y}/\bar{z})\Big]$$

where $\bar{z}$ is an $r$-tuple. Finally, we set

$$\begin{aligned} \varphi^{\tau_0}_{n,m} := \Big(\forall \bar{y}. \bar{y} \sqsubseteq \bar{y}\Big)\Big(\exists \bar{x}^0 \dots \bar{x}^{m-1}. \bigwedge_{i<n, j<m} U x^j_i \bar{y}\Big) \\ \Big(\forall \bar{x}'. \bigwedge_{i<n} U x'_i \bar{y}\Big) \bigvee_{j<m} \psi(\bar{x}', \bar{x}^j; U\_\bar{y}) \end{aligned}$$

where the $\bar{x}^j$, $\bar{x}'$, and $\bar{y}$ are $n$-tuples, and $U\_\bar{y}$ indicates that every atom $Xz$ in $\psi$ should be replaced by $Uz\bar{y}$.

(2) As above we construct formulae $\varphi^{\tau_0}_{n,m}$ expressing that the $n$-ary symmetric partition width of the $\tau_0$-reduct is at most $m$, and set

$$\Pi^\omega_{\bar{w}} := T^{\aleph_0}_{\mathrm{pr}} \cup \{ \varphi^{\tau_0}_{n,w_n} \mid n < \omega,\ \tau_0 \subseteq \tau \text{ finite} \} .$$

Let $r$ be the maximal arity of relations in $\tau_0$. The formula

$$\eta(\bar{y}_0, \bar{y}_1) := \bar{y}_0 \sqsubset \bar{y}_1 \wedge \neg \exists \bar{z}(\bar{y}_0 \sqsubset \bar{z} \sqsubset \bar{y}_1)$$

defines the successor relation of the partial order $\sqsubseteq$. For tuples $\bar{x}^0, \dots, \bar{x}^m$ contained in $U\_\bar{y}$ the formula

$$\vartheta(z; \bar{y}, \bar{x}^0, \dots, \bar{x}^m) := \forall \bar{y}'\Big(\Big(\eta(\bar{y}, \bar{y}') \wedge Uz\bar{y}'\Big) \to \neg \bigwedge_{i<n, j\leq m} U x^j_i \bar{y}'\Big)$$

states that the element $z$ is not a member of any component $U\_\bar{y}'$ containing some of the $\bar{x}^j$.

We have to express that there is no sequence $\bar{a}^0, \dots, \bar{a}^m$ of $m+1$ tuples of pairwise distinct types over all components that do not contain any of the $\bar{a}^i$.

This can be done by defining

$$\varphi^{\tau_0}_{n,m} := \forall \bar{y} \neg \Big( \exists \bar{x}^0 \dots \bar{x}^m . \bigwedge_{i<n, j<m} U x^j_i \bar{y} \Big)$$
$$\bigwedge_{j \neq k} \Big( \exists \bar{z} . \bigwedge_{i<r} \vartheta(z_i; \bar{y}, \bar{x}^0, \dots, \bar{x}^m) \Big)$$
$$\Big[ \mathrm{etp}_0(\bar{x}^j / \bar{z}) \neq \mathrm{etp}_0(\bar{x}^k / \bar{z}) \Big] .$$

□

Having established our main tool we first apply it to show that the non-standard partition width of a structure is determined by the non-standard partition widths of its finite substructures. This generalises the analogous result for the clique width of countable graphs by Courcelle [11].

**Proposition 51** *Let $\mathfrak{M}$ be a relational structure and $\bar{w} \in \omega^\omega$.*

(1) $\mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M} \leq w_n$, *for all $n < \omega$, if and only if all finite substructures of $\mathfrak{M}$ have a non-standard $2^{<\omega}$-partition refinement of width at most $\bar{w}$.*
(2) $\mathrm{spwd}^{\mathrm{ns}}_n \mathfrak{M} \leq w_n$, *for all $n < \omega$, if and only if all finite substructures of $\mathfrak{M}$ have a non-standard $\aleph_0^{<\omega}$-partition refinement of width at most $\bar{w}$.*

**PROOF.** One direction immediately follows from Corollary 48. For the other one, set $\Phi := \Delta \cup \Pi$ where $\Delta$ is the atomic diagram of $\mathfrak{M}$ and $\Pi$ is either $\Pi^2_{\bar{w}}$ or $\Pi^\omega_{\bar{w}}$.

If $\Phi$ has a model $(\mathfrak{N}, U, \sqsubseteq)$ then there is a non-standard partition refinement $(U_v)_v$ of $\mathfrak{N}$ of width $\bar{w}$. The restriction $(U_v \cap M)_v$ of $(U_v)_v$ to $M$ yields the desired refinement of $\mathfrak{M}$.

To prove that $\Phi$ is consistent let $\Phi_0 \subseteq \Phi$ be finite. Then there is a finite set $A \subseteq M$ such that $\Phi_0 \subseteq \Delta_0 \cup \Pi$ where $\Delta_0$ is the atomic diagram of $\mathfrak{M}|_A$. Let $(U_v)_v$ be a reduced partition refinement of $\mathfrak{M}|_A$ of width $\bar{w}$, and let $(U, \sqsubseteq)$ be relations coding it. Then $(\mathfrak{M}|_A, U, \sqsubseteq) \models \Phi_0$. □

Of course, we are interested in a standard partition refinement. Unfortunately, the width of a non-standard partition refinement may increase when we transform it into a standard one.

**Example 52 (Courcelle [11])** *Let $\mathfrak{G}$ be the graph with universe $V := [2] \times \omega$ and edge relation*

$$E := \{ (\langle b, k \rangle, \langle 1, n \rangle) \mid k < n,\ b < 2 \} .$$

*Then* $\mathrm{pwd}_1 \mathfrak{G}_0 = \mathrm{pwd}_1^{\mathrm{ns}} \mathfrak{G}_0 = \mathrm{pwd}_1^{\mathrm{ns}} \mathfrak{G} = 1$ *for every finite induced subgraph* $\mathfrak{G}_0 \subseteq \mathfrak{G}$ *but* $\mathrm{pwd}_1 \mathfrak{G} = 2$.

*To compute* $\mathrm{pwd}_1 \mathfrak{G}_0$ *and* $\mathrm{pwd}_1^{\mathrm{ns}} \mathfrak{G}_0$ *it is sufficient to consider the case that* $\mathfrak{G}_0 = \mathfrak{G}|_{[2]\times[n]}$. *A partition refinement of width* $1$ *is given by* $(U_v)_{v\in T}$ *where* $T := 0^{<2n}1^{<2}$ *and*

$$\begin{aligned}
U_{0^{2k}} &:= [2] \times [n-k]\,, \\
U_{0^{2k}1} &:= \{\langle 0, n-k-1\rangle\}\,, \\
U_{0^{2k}0} &:= [2] \times [n-k-1] \cup \{\langle 1, n-k-1\rangle\}\,, \\
U_{0^{2k}01} &:= \{\langle 1, n-k-1\rangle\}\,.
\end{aligned}$$

*For* $\mathrm{pwd}_1^{\mathrm{ns}} \mathfrak{G}$ *we use as index structure the tree* $T$ *of all sequences* $w : I \to [2]$ *where* $I$ *is a prefix of* $\omega + \zeta$. *Then we can define analogously*

$$\begin{aligned}
U_{0^n} &:= [2] \times \omega\,, \qquad \text{for } n < \omega\,, \\
U_{0^{\omega+\omega^*-2k}} &:= [2] \times [k]\,, \\
U_{0^{\omega+\omega^*-2k}1} &:= \{\langle 0, k-1\rangle\}\,, \\
U_{0^{\omega+\omega^*-2k}0} &:= [2] \times [k-1] \cup \{\langle 1, k-1\rangle\}\,, \\
U_{0^{\omega+\omega^*-2k}01} &:= \{\langle 1, k-1\rangle\}\,, \\
\text{and} \qquad U_v &:= \emptyset\,, \qquad \text{for all other indices } v\,.
\end{aligned}$$

*Suppose that there exists a partition refinement* $(U_v)_v$ *of* $\mathfrak{G}$ *of width* $1$. *By symmetry, we may assume that* $U_0 \cap [b] \times \omega$ *is infinite for some* $b < 2$.

*If* $\langle b, n\rangle \in U_0$ *and* $k > n$ *then* $\langle 1-b, k\rangle \notin U_1$ *since there exists some* $n' > k$ *with* $\langle b, n'\rangle \in U_0$ *and* $\langle b, n\rangle \not\approx^0_{\langle 1-b,k\rangle} \langle b, n'\rangle$. *Similarly,* $\langle b, k\rangle \notin U_1$ *for* $k > n$ *since* $\langle b, n\rangle \not\approx^0_{\langle b,k\rangle} \langle b, n'\rangle$ *for all* $n' > k$. *Hence,* $U_1 \subseteq [2] \times [m]$ *for some* $m < \omega$.

*Fix some element* $\langle c, k\rangle \in U_1$. *There are elements* $\langle 0, n_0\rangle, \langle 1, n_1\rangle \in U_0$ *with* $n_0, n_1 > k$. *But* $\langle 0, n_0\rangle \not\approx^0_{\langle c,k\rangle} \langle 1, n_1\rangle$ *contradicts our assumption that* $\mathrm{eti}^1_0(U_0/U_1) = 1$.

**Proposition 53** *Let* $\mathfrak{M}$ *be a structure with* $m$ *relations of arity greater than* $1$ *and let* $r$ *be the maximum of their arities.*

(1) *If* $(U_v)_v$ *is a non-standard* $2^{<\omega}$*-partition refinement* $(U_v)_v$ *of* $\mathfrak{M}$ *of width* $w_n := \mathrm{pwd}_n(U_v)_v$ *then*

$$\mathrm{pwd}_n \mathfrak{M} \leq 2^{m(n+1)^r w_{r-1} 2^{mr^r w_{r-1}+r-1}}.$$

(2) *If* $(U_v)_v$ *is a non-standard* $\aleph_0^{<\omega}$*-partition refinement* $(U_v)_v$ *of* $\mathfrak{M}$ *of width*

$w_n := \operatorname{spwd}_n(U_v)_v$ *then*

$$\operatorname{spwd}_n \mathfrak{M} \leq 2^{m(n+1)^r w_{r-1} 2^{mr^r w_{r-1}+r-1}}.$$

**PROOF.** Since both cases are similar we only prove (1). Let $(U_v)_{v \in T}$ be a non-standard $2^{<\omega}$-partition refinement of $\mathfrak{M}$. By induction on $\alpha$, we define

- a strictly decreasing sequence $T_\alpha \subseteq T$ of subsets of $T$;
- an increasing sequence of trees $S_\alpha$; and
- a partial partition refinement $(V_v)_{v \in S_\alpha}$

such that $u \in T_\alpha$ and $u \preceq v$ imply $v \in T_\alpha$ and we can partition $T_\alpha$ into sets $T_\alpha^\beta$ satisfying the following conditions:

- $u, v \in T_\alpha$ belong to the same component $T_\alpha^\beta$ iff $u \sqcap v \in T_\alpha$.
- For every maximal path $C \subseteq S_\alpha$ such that $W := \bigcap_{v \in C} V_v$ contains at least 2 elements, there exists some $\beta$ with $\bigcup_{v \in T_\alpha^\beta} U_v = W$ and, vice versa, for every component $T_\alpha^\beta$ there exists such a chain $C \subseteq S_\alpha$.

Intuitively, $S_\alpha$ is the part of $T$ we have already converted and $T_\alpha$ is the part that still has to be transformed into a standard refinement.

Let $S_0$ be the standard part of $T$, set $T_0 := T \setminus S_0$, and let $V_v := U_v$ for $v \in S_0$. If $\alpha$ is a limit we set $S_\alpha := \bigcup_{\beta < \alpha} S_\beta$ and $T_\alpha := \bigcap_{\beta < \alpha} T_\beta$.

Suppose that $\alpha = \beta + 1$. Fix a maximal chain $C \subseteq S_\beta$ such that $W := \bigcap_{v \in C} V_v$ contains at least 2 elements. If such a chain does not exist then $(V_v)_{v \in S_\beta}$ is already a partition refinement of $\mathfrak{M}$ (after adding some singletons as leaves if necessary) and we are done.

If there is some $v_0 \in T_\beta$ such that $U_v = W$ then let $T'$ consists of all $u \in T_\beta$ with $v_0 \preceq u$. We add the standard part of $T'$ to $S_\beta$ above $C$ and remove from $T_\beta$ this part and all other elements $v$ with $v \sqcap v_0 \in T_\beta$ (the elements below $v_0$). Set $V_u := U_u$ for the new elements $u \in S_{\beta+1} \setminus S_\beta$.

If such a vertex $v_0$ does not exist, let $T' \subseteq T_\beta$ be the set of all $v \in T_\beta$ such that $U_v \subseteq W$. Then, by assumption, $\bigcup_{v \in T'} U_v = W$. Fix a maximal chain $I \subseteq T'$. Note that, for every $v \in T'$ and all $u \in I$ we have $u \sqcap v \in I$. Since $I$ is a linear order there exists a partition refinement $(H_v)_{v \in F}$ of $(I, \preceq)$ of width 1 where each component is some interval $H_v \subseteq I$. We add the tree $F$ to $S_\beta$ above $C$, define

$$V_v := \bigcup_{w \in H_v} U_w \setminus \bigcup \{ U_w \mid w \in I,\ w > u \text{ for all } u \in H_v \},$$

for $v \in F$, and set $T_{\beta+1} := T_\beta \setminus I$.

Since $T_\alpha \supset T_\beta$ for $\alpha < \beta$, the construction must stop after at most $|T|^+$ steps with some partition refinement $(V_v)_{v \in S}$.

The components $V_v$ are of the form $X$ or $X \setminus Y$ where $X$ and $Y$ are either components $U_w$, for some $w \in T$, or of the form $\bigcup_{w \in C} U_w$, for some chain $C \subseteq T$. By Lemma 37, we have $\mathrm{eti}_0^n(X/\overline{X}) \leq w_n$ in both cases. It follows, by Lemmas 33 and 34, that

$$\begin{aligned}\mathrm{eti}_0^n(Y \cup \overline{X}/X \setminus Y) &\leq 2^n \mathrm{eti}_0^n(Y/\overline{Y}) 2^{m(n+1)^r \mathrm{eti}_0^{r-1}(X/\overline{X})} \\ &\leq 2^n w_n 2^{m(n+1)^r w_{r-1}},\end{aligned}$$

where $m$ is the number of relations of arity greater than 1, and $r$ is the maximum of their arities. Therefore,

$$\mathrm{eti}_0^n(X \setminus Y/Y \cup \overline{X}) \leq 2^{m(n+1)^r w_{r-1} 2^{mr^r w_{r-1}+r-1}}.$$

□

**Corollary 54** (1) *If there exists a sequence $\bar{w} \in \omega^\omega$ such that* $\mathrm{pwd}_n \mathfrak{A} \leq w_n$*, $n < \omega$, for every finite substructure $\mathfrak{A} \subseteq \mathfrak{M}$ then* $\mathrm{pwd}_n \mathfrak{M} \leq \aleph_0$ *for $n < \omega$.*

(2) *If there exists a sequence $\bar{w} \in \omega^\omega$ such that* $\mathrm{spwd}_n \mathfrak{A} \leq w_n$*, $n < \omega$, for every finite substructure $\mathfrak{A} \subseteq \mathfrak{M}$ then* $\mathrm{spwd}_n \mathfrak{M} \leq \aleph_0$ *for $n < \omega$.*

A direct consequence of Proposition 51 is the fact that having a finite partition width is a property of first-order theories.

**Theorem 55** *If $\mathfrak{M}$ is of finite non-standard partition width and $\mathfrak{M} \equiv_{\mathrm{FO}} \mathfrak{N}$ then*

$$\mathrm{pwd}_n^{\mathrm{ns}} \mathfrak{M} = \mathrm{pwd}_n^{\mathrm{ns}} \mathfrak{N} \quad \textit{and} \quad \mathrm{spwd}_n^{\mathrm{ns}} \mathfrak{M} = \mathrm{spwd}_n^{\mathrm{ns}} \mathfrak{N}$$

*for all $n < \omega$.*

**PROOF.** Let $w_i := \mathrm{pwd}_i \mathfrak{M}$, for $i < \omega$. W.l.o.g. assume that the signature is finite. Since there are only finitely many structures of size $n$ there exists an FO-formula $\psi_{i,k}^n(x_0, \dots, x_{n-1})$ stating that $\mathrm{pwd}_i \mathfrak{M}|_{\bar{x}} \leq k$. $\mathfrak{M} \models \forall \bar{x} \psi_{i,w_i}^n(\bar{x})$ implies $\mathfrak{N} \models \forall \bar{x} \psi_{i,w_i}^n(\bar{x})$. By Proposition 51 it follows that $\mathrm{pwd}_n^{\mathrm{ns}} \mathfrak{N} \leq \mathrm{pwd}_n^{\mathrm{ns}} \mathfrak{M}$ for $n < \omega$. The claim follows by symmetry.

In the same way we can show that the non-standard symmetric partition widths are equal. □

**Corollary 56** *If $\mathfrak{M} \equiv_{\mathrm{FO}} \mathfrak{N}$ and $\mathfrak{M}$ is of finite [symmetric] partition width then so is $\mathfrak{N}$.*

For the non-standard partition width we are able to prove that for every structure $\mathfrak{M}$ such that $\mathrm{pwd}_n^{\mathrm{ns}}\,\mathfrak{M}$ is finite there exists a non-standard partition refinement of exactly this width.

**Proposition 57** *Let $\mathfrak{M}$ be a structure.*

(1) *There exists a non-standard $2^{<\omega}$-partition refinement $(U_v)_v$ of partition width $\mathrm{pwd}_n(U_v)_v = \mathrm{pwd}_n^{\mathrm{ns}}\,\mathfrak{M}$ for all $n < \omega$.*
(2) *There exists a non-standard $\aleph_0^{<\omega}$-partition refinement $(U_v)_v$ of partition width $\mathrm{spwd}_n(U_v)_v = \mathrm{spwd}_n^{\mathrm{ns}}\,\mathfrak{M}$ for all $n < \omega$.*

**PROOF.** Since the proofs are nearly identical, we prove only (1). Let $w_n := \mathrm{pwd}_n^{\mathrm{ns}}\,\mathfrak{M}$, and let $\Delta$ be the atomic diagram of $\mathfrak{M}$. If $(\mathfrak{N}, U, \sqsubseteq) \models \Phi := \Delta \cup \Pi^2_{\bar{w}}$ then $\mathfrak{M} \subseteq \mathfrak{N}$ and $(U, \sqsubseteq)$ codes a non-standard partition refinement of $\mathfrak{N}$ of width $\bar{w}$ which induces one of $\mathfrak{M}$ of the same width.

To show that $\Phi$ is consistent let $\Phi_0 \subseteq \Phi$ be finite. There exists some $k < \omega$ such that $\Phi_0$ does not contain any formula of the form $\varphi^{\tau_0}_{n,m}$ for $n \geq k$. Let $(U, \sqsubseteq)$ code a non-standard partition refinement $(U_v)_v$ of $\mathfrak{M}$ such that

$$\mathrm{pwd}_n(U_v)_v = \mathrm{pwd}_n^{\mathrm{ns}}\,\mathfrak{M} \qquad \text{for all } n < k\,.$$

Then $(\mathfrak{M}, U, \sqsubseteq) \models \Phi_0$. $\square$

Consider an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ of $\mathfrak{M}$. Every non-standard partition refinement $(U_v)_{v\in T}$ of $\mathfrak{N}$ induces a corresponding refinement $(U_v \cap M)_{v\in T}$ of $\mathfrak{M}$, that is, each partition refinement of $\mathfrak{N}$ can be obtained by extending one of $\mathfrak{M}$. The following proposition states the converse: every non-standard partition refinement of $\mathfrak{M}$ can be extended to one of $\mathfrak{N}$.

**Proposition 58** *Let $(U_v)_{v\in T}$ be a non-standard $2^{<\omega}$-partition refinement of $\mathfrak{M}$. For every $\mathfrak{N} \succeq \mathfrak{M}$ there exists an elementary extension $S \succeq T$ and a non-standard $2^{<\omega}$-partition refinement $(V_v)_{v\in S}$ of $\mathfrak{N}$ of the same width such that $V_{h(v)} \supseteq U_v$ for all $v \in T$ where $h : T \to S$ is the corresponding elementary embedding.*

**PROOF.** W.l.o.g. we may assume that $|M| \geq \aleph_0$. Set $w_n := \mathrm{pwd}_n(U_v)_v$. Let $(U, \sqsubseteq)$ be relations coding $(U_v)_v$. Let $\Delta_{\mathfrak{N}}$ be the elementary diagram of $\mathfrak{N}$, $\Xi$ the elementary diagram of $(M, U, \sqsubseteq)$, and set

$$\Gamma := \{\, Pa \mid a \in N \,\}\,.$$

By a straightforward modification of $\Pi^2_{\bar w}$ we obtain a set of formulae expressing that $(U \cap (P \times M^n), \sqsubseteq)$ codes a non-standard partition refinement of $P$. Let $\Pi^P$ be this set.

We have to show that $\Psi := \Xi \cup \Gamma \cup \Pi^P \cup \Delta_{\mathfrak{N}}$ has a model $(\mathfrak{N}', P, V, \sqsubseteq')$. Then there exists an elementary embedding $h : (T, \sqsubseteq) \preceq (S, \sqsubseteq')$ where

$$S := \{ \bar a \in (N')^n \mid \bar a \sqsubseteq \bar a \},$$

and $(V_{\bar a})_{\bar a \in S}$ with $V_{\bar a} := \{ b \in N \mid (b, \bar a) \in V \}$ is a non-standard partition refinement of $\mathfrak{N}$ with $U_v \subseteq V_{h(v)}$.

Let $\Psi_0 \subseteq \Psi$ be a finite subset. Then $\Psi_0 \subseteq \Xi_0 \cup \Gamma_0 \cup \Pi^P \cup \Delta_0$ for some finite sets $\Xi_0 \subseteq \Xi$, $\Gamma_0 \subseteq \Gamma$, and $\Delta_0 \subseteq \Delta_{\mathfrak{N}}$. Let $A \subseteq N$ be the finite set of elements mentioned in $\Xi_0 \cup \Gamma_0 \cup \Delta_0$, and set $M_0 := A \cap M$, $N_0 := A \setminus M$. Let $\bar a$ be an enumeration of $N_0$. There exists a tuple $\bar b \subseteq M$ such that $\mathrm{tp}(\bar b / M_0) = \mathrm{tp}(\bar a / M_0)$. Then $(\mathfrak{M}, M_0 \cup \bar b, U, \sqsubseteq) \models \Psi_0$. □

We conclude this section with the proof of a compactness theorem for structures of finite non-standard partition width.

**Theorem 59 (Compactness)** *Let $\bar w \in \omega^\omega$. A set $\Phi \subseteq \mathrm{FO}$ of sentences has a model $\mathfrak{M}$ with $\mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M} \leq w_n$ for $n < \omega$ if and only if every finite subset $\Phi_0 \subseteq \Phi$ has such a model. The same holds for $\mathrm{spwd}^{\mathrm{ns}}_n \mathfrak{M}$.*

**PROOF.** $\Phi$ has a model $\mathfrak{M}$ of width $\mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M} \leq w_n$ if and only if $\Phi \cup \Pi^2_{\bar w}$ is consistent. Since all finite subsets of $\Phi \cup \Pi^2_{\bar w}$ are consistent, so is the whole set. □

**Corollary 60** *A set $\Phi \subseteq \mathrm{FO}$ of sentences has a model of finite partition width if and only if there exists a sequence $\bar w \in \omega^\omega$ such that every finite subset $\Phi_0 \subseteq \Phi$ has a model $\mathfrak{M}$ with $\mathrm{pwd}_n \mathfrak{M} \leq w_n$ for $n < \omega$. The same holds for the symmetric partition width.*

## 8 Pairing functions and the independence property

Baldwin and Shelah argue in [12] that monadic second-order theories in which a pairing function can be defined are hopelessly complicated and then proceed to classify the other ones. They show that the models of every stable theory without definable pairing function can be decomposed in a tree-like fashion and that these theories can be interpreted in the theory of a suitable class of trees. Extended to include unstable theories a finitary version of their results would answer the analogue of the conjecture of Seese [3] for partition width. It is quite easy to show that the existence of a pairing function implies an infinite partition width while a proof of the converse seems to be quite involved requiring an adaptation of the excluded grid theorem of Robertson and Seymour [13].

Recently, a slightly weaker form of the conjecture of Seese has been proved by Courcelle and Oum [4]. Let us denote by $\mathrm{MSO} + \mathrm{C}_2$ the extension of monadic second-order logic by quantifiers "The number of elements $x$ such that ... is finite and even." The *$m \times n$ grid* is the undirected graph $(V, E)$ with $V = [m] \times [n]$ and

$$E := \{ ((i,k),(j,l)) \in V \times V \mid |i-j| + |k-l| = 1 \} .$$

**Theorem 61 (Courcelle and Oum)** *Let $\mathcal{K}$ be a class of finite undirected graphs. If the clique width of the graphs in $\mathcal{K}$ is unbounded then there exists an $(\mathrm{MSO} + \mathrm{C}_2)$-interpretation $\mathcal{I}$ such that $\mathcal{I}(\mathcal{K})$ is the class of all finite grids.*

Note that this result only applies to graphs. Furthermore, it seems that for the case of arbitrary structures a fundamentaly different proof is required.

**Definition 62** *A structure $\mathfrak{M}$* admits MSO-coding *if there exists an* MSO-*formula $\varphi(x, y, z; \bar{X})$ such that, for each natural number $n < \omega$, there are sets $A$, $B$, $C \subseteq M$ of size $|A| = |B| = n$ such that, for suitable monadic parameters $\bar{P}$, $\varphi(x, y, z; \bar{P})$ defines a bijection $A \times B \to C$.*

**Lemma 63** *Let $\mathfrak{M}$ be a structure and $n < \aleph_0$. The following statements are equivalent:*

(1) *There exists an* MSO-*formula $\chi(x, y, z)$ with monadic parameters that defines a bijection $A \times B \to C$ for sets of size $|A| = |B| = n$.*
(2) *There exists an* MSO-*formula $\vartheta(x, y)$ with monadic parameters that defines an $n \times n$ grid.*
(3) *There exist* MSO-*formulae $\varphi(x, y)$ and $\psi(x, y)$ each of which defines an equivalence relation with $n$ classes such that every class of the first one intersects each class of the other one.*

**PROOF.** (1) $\Rightarrow$ (3) Let $f : A \times B \to C$ be the given bijection. We can define two equivalence relations on $C$ by setting

$$\varphi(x, y) := \exists u \exists v \exists z (f(u, z) = x \wedge f(v, z) = y) ,$$
$$\text{and } \psi(x, y) := \exists u \exists v \exists z (f(z, u) = x \wedge f(z, v) = y) .$$

(2) $\Rightarrow$ (1) Fix $n < \aleph_0$ and $C \cong n \times n$ as above. Let $A := n \times \{0\} \subseteq C$ and

$B := \{0\} \times n \subseteq C$. We claim that the function $f : A \times B \to C$ defined by $f((i,0),(0,k)) := (i,k)$ is MSO-definable.

With the help of the parameters

$$\begin{aligned} & H_m := \{\, (i,k) \mid i \cong m \pmod 3 \,\} \subseteq C \\ \text{and}\quad & V_m := \{\, (i,k) \mid k \cong m \pmod 3 \,\} \subseteq C\,, \end{aligned}$$

for $m < 3$, we can define the successor relations

$$\begin{aligned} & S_0 := \{\, ((i,k),(i+1,k)) \mid i < n-1,\ k < n \,\} \\ \text{and}\quad & S_1 := \{\, ((i,k),(i,k+1)) \mid i < n,\ k < n-1 \,\}\,. \end{aligned}$$

Then the desired coding function can be defined by

$$f(x,y) = z \quad \text{iff} \quad (x,z) \in (S_1)^* \text{ and } (y,z) \in (S_0)^*\,.$$

(3) $\Rightarrow$ (2) Let $\sim_0$ and $\sim_1$ be the two equivalences. Fix elements $a_{ik}$, $i,k < n$, such that

$$a_{ik} \sim_0 a_{ml} \text{ iff } i = m \qquad \text{and} \qquad a_{ik} \sim_1 a_{ml} \text{ iff } k = l\,.$$

With the help of the parameters

$$P := \{\, a_{ii} \mid i < n \,\} \quad \text{and} \quad Q := \{\, a_{i(i+1)} \mid i+1 < n \,\}\,,$$

we define the relations

$$\begin{aligned} & S_0 := \{\, (a_{ik}, a_{(i+1)k}) \mid i,k < n \,\}\,, \\ & S_1 := \{\, (a_{ik}, a_{i(k+1)}) \mid i,k < n \,\}\,, \end{aligned}$$

by setting

$$\begin{aligned} & S_0xy := x \sim_1 y \wedge \exists u \exists v (Qu \wedge Pv \wedge x \sim_0 u \wedge y \sim_0 v \wedge u \sim_1 v)\,, \\ \text{and}\quad & S_1xy := x \sim_0 y \wedge \exists u \exists v (Pu \wedge Qv \wedge x \sim_1 u \wedge y \sim_1 v \wedge u \sim_0 v)\,. \end{aligned}$$

□

**Remark 64** *Note that the translation in the preceding lemma is uniform, that is, given $\chi(x,y,z;\bar{Z})$ we can construct a formula $\vartheta(x,y;\bar{Z})$ such that, whenever $\bar{P}$ are parameters such that $\chi(x,y,z;\bar{P})$ defines a bijection $A \times B \to C$ with $|A| = |B| = n$, then we can find parameters $\bar{Q}$ such that $\vartheta(x,y;\bar{Q})$ defines an $n \times n$ grid. Analogous statements hold for the other directions.*

It follows that structures admitting MSO-coding are complicated. In particular, it follows from the following theorem that their MSO-theory is undecidable.

**Theorem 65 (Seese [14])** *The* MSO*-theory of the class of all finite grids is undecidable.*

An easy proof consists in coding domino problems (see [15]). Together with Lemma 63 this theorem implies the following result.

**Theorem 66** *If* $\mathfrak{M}$ *is a structure that admits* MSO*-coding then the* MSO*-theory of* $\mathfrak{M}$ *is undecidable.*

We conjecture that the property of admitting MSO-coding is equivalent to an infinite partition width.

**Conjecture 67** *A structure* $\mathfrak{M}$ *with finite signature has finite partition width if and only if it does not admit* MSO*-coding.*

Note that this conjecture fails if we allow infinite signatures. Consider $\mathfrak{M} = (\omega \times \omega, (E_n)_{n<\omega})$ where

$$E_n := \Big\{ (\langle i, k\rangle,\ \langle j, l\rangle) \ \Big|\ |i-j| + |k-l| = 1,\ i, j, k, l < n \Big\}.$$

Then, $\mathrm{pwd}_1 \mathfrak{M} = \aleph_0$. On the other hand, the MSO-theory of $\mathfrak{M}$ is decidable since each formula contains only finitely many relation symbols and every finite reduct of $\mathfrak{M}$ is the disjoint union of a finite structure and an infinite set.

Since all structures admitting MSO-coding have an undecidable MSO-theory a proof of this conjecture would settle the conjecture of Seese that every class of finite graphs with decidable MSO-theory has finite clique width. The following lemma deals with the easy direction.

We call a function $f : A \times B \to C$ *cancellative* if $f(a, b) = f(a', b)$ implies $a = a'$ and $f(a, b) = f(a, b')$ implies $b = b'$.

**Proposition 68** *Let* $\mathfrak{M}$ *be a* $\tau$*-structure. If there are unary predicates* $\bar{P}$ *and an* $\mathrm{MSO}_k$*-formula* $\varphi(x, y, z; \bar{P})$ *defining a cancellative function* $f : A \times B \to C$ *then* $|A| \leq K$ *or* $|B| \leq K$ *where*

$$K := 3 \cdot \beth_k(N_{k+2}\, \mathrm{mpwd}_{k+2}\, \mathfrak{M}) \quad \text{and} \quad N_k := |MS_0^k(\emptyset)|$$

*where* $MS_0^k$ *is taken with respect to the signature* $\tau \cup \bar{P}$*.*

**PROOF.** Let $f : A \times B \to C$ be the given function. Fix a partition refinement $(U_v)_{v \in T}$ of $\mathfrak{M}$ such that $\mathrm{mpwd}_{k+2}(U_v)_v$ is minimal and define

$$w_n := \sup \{ \mathrm{mti}_k^n(U_v/\overline{U_v}) \mid v \in T \}.$$

By Lemmas 32 (2) and 35 we have

$$w_2 \leq \beth_k(N_{2+k}\,\mathrm{mpwd}_{2+k}\,\mathfrak{M}) = K/3\,.$$

Suppose, for a contradiction, that $m := |A| = |B| > 3w_2$.

We claim that there exists some vertex $v \in T$ such that

$$\begin{aligned} &\tfrac{1}{3}m \leq |U_v \cap A| \leq \tfrac{2}{3}m \quad \text{and} \quad |B \setminus U_v| > w_2\,, \\ \text{or} \quad &\tfrac{1}{3}m \leq |U_v \cap B| \leq \tfrac{2}{3}m \quad \text{and} \quad |A \setminus U_v| > w_2\,. \end{aligned}$$

Let $v_0$ be some vertex with $\frac{1}{3}m \leq |U_{v_0} \cap B| \leq \frac{2}{3}m$. If $|A \setminus U_{v_0}| \leq w_2$ then there exists some $v \succeq v_0$ such that

$$\tfrac{1}{3}m \leq |U_v \cap A| \leq \tfrac{2}{3}m$$
and $|B \setminus U_v| \geq |B \setminus U_{v_0}| \geq m/3 > w_2\,.$

Thus, by symmetry we may assume that there exists some $v \in T$ satisfying the first condition.

There are at most $w_2$ elements $b \in B \setminus U_v$ such that $f(a,b) = c$ for some $a \in U_v \cap A$, $c \in U_v \cap C$. Otherwise, there would be tuples $f(a,b) = c$ and $f(a',b') = c'$ with $b \neq b'$ and $\{a\}\{c\} \approx^k_{\overline{U_u}} \{a'\}\{c'\}$. Then, $f(a',b') = c'$ would imply

$$f(a,b') = c = f(a,b)\,,$$

and by cancellation, we would have $b = b'$ in contradiction to our assumption.

Since $|B \setminus U_v| > w_2$ it follows that there exists some $b \in B \setminus U_v$ such that $f(a,b) \in \overline{U_v}$ for all $a \in U_v \cap A$.

Furthermore, since $|U_v \cap A| \geq m/3 > w_2$ there are two different elements $a, a' \in U_v \cap A$ such that $a \approx^k_{\overline{U_v}} a'$. This implies $f(a,b) = c$ iff $f(a',b) = c$ for all $c \in \overline{U_v}$. Contradiction. □

**Corollary 69** *If $\mathfrak{M}$ admits* MSO*-coding then* $\mathrm{pwd}_n\,\mathfrak{M} \geq \aleph_0$ *for some $n$.*

**Corollary 70** *A group has finite partition width if and only if it is finite.*

Proposition 68 can be used to link the concept of partition width with the model theoretic notion of VC-dimension or, equivalently, the independence property.

**Definition 71** *Let $T$ be a first-order theory. An* FO*-formula $\varphi(\bar{x}, \bar{y})$ has the* independence property *(w.r.t. $T$) if there exists a model $\mathfrak{M}$ of $T$ containing* *sequences* $(\bar{a}_I)_{I\subseteq\omega}$ *and* $(\bar{b}_i)_{i<\omega}$ *such that*

$$\mathfrak{M} \models \varphi(\bar{a}_I, \bar{b}_i) \quad \text{iff} \quad i \in I\,.$$

*We say that a structure* $\mathfrak{M}$ *has the independence property if there exists a formula* $\varphi$ *that has the independence property w.r.t.* $\mathrm{Th}(\mathfrak{M})$*. If* $\bar{a}_I$ *and* $\bar{b}_i$ *are singletons we say that* $\mathfrak{M}$ *has the* independence property on singletons*.*

In [12] it is shown that these two notions coincide if we allow monadic parameters.

**Lemma 72** *Let* $\mathfrak{M}$ *have the independence property. There exists an elementary extension* $\mathfrak{N} \succeq \mathfrak{M}$ *and unary predicates* $\bar{P}$ *such that* $(\mathfrak{N}, \bar{P})$ *has the independence property on singletons.*

It immediately follows that the independence property implies MSO-coding.

**Lemma 73** *Let* $\mathfrak{M}$ *have the independence property on singletons. There exists an elementary extension* $\mathfrak{N} \succeq \mathfrak{M}$ *that admits* FO*-coding.*

**PROOF.** Choose an elementary extension $\mathfrak{N}$ that contains sequences $(a_I)_{I\subseteq\omega}$ and $(b_i)_{i\in\omega}$ such that, for some formula $\varphi(x, y)$, we have

$$\mathfrak{N} \models \varphi(a_I, b_i) \quad \text{iff} \quad i \in I\,.$$

Fix disjoint infinite sets $X,\ Y \subseteq B := \{\, b_i \mid i < \omega \,\}$, and define a function $f : X \times Y \to M$ by $f(b_i, b_j) := a_{\{i,j\}}$. For $x \in X$, $y \in Y$, and $z \in Z := f(X, Y)$ we have

$$f(x, y) = z \quad \text{iff} \quad \mathfrak{M} \models \varphi(z, x) \wedge \varphi(z, y)\,.$$

Hence, $f$ is an FO-definable bijection $X \times Y \to Z$. $\square$

Together with the results above it follows that no structure with the independence property has finite partition width. This slightly extends a result of Parigot [16] who showed that trees do not have the independence property.

**Proposition 74** *If* $\mathfrak{M}$ *is a structure with the independence property then* $\mathrm{pwd}_n\, \mathfrak{M} \geq \aleph_0$ *for some* $n$*.*

**PROOF.** If $\mathfrak{M}$ has the independence property then there exists an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ and unary predicates $\bar{P}$ such that $(\mathfrak{N}, \bar{P})$ has the independence property on singletons. Hence, there exists an elementary extension

$(\mathfrak{N}', \bar{P}')$ which admits FO-coding. If $\mathfrak{M}$ where of finite partition width, then so would be $\mathfrak{N}$, $(\mathfrak{N}, \bar{P})$, and $(\mathfrak{N}', \bar{P}')$. The latter contradicts Corollary 69. $\square$

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