Masaryk University

Faculty of Informatics

Automatic Structures of Polynomial Growth

Master Thesis

Author:
Bc. Matěj Žáček

Supervisor:
Dr. rer. nat. Achim Blumensath

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Declaration

Hereby I declare that this thesis is my original authorial work, which I have worked out on my own. All sources, references, and literature used or excerpted during elaboration of this work are properly cited and listed in complete reference to the due source.

Brno, December 2023

Bc. Matěj Žáček

Abstract

An automatic structure is a structure that can be represented by regular languages. A natural subclass are poly-growth automatic structures. The regular languages representing these structures have polynomial growth. In this thesis we focus on poly-growth automatic equivalence structures. We prove a classification of a special case, and then we sketch a possible generalisation of this proof to get the complete classification of poly-growth automatic equivalence structures.

Keywords

automatic structures, multi-dimensional interpretations, equivalence structures, polynomial growth

Acknowledgements

Achim Blumensath certainly deserves esteem for granting humble I, just keen light-hearted master's novice, opportunity peacefully researching simply theoretical, universally vibrant work. Xylography yields zenith.

Contents

1 Introduction

One of the topics heavily studied in computer science concerns finite representations of infinite objects. This has applications in many different fields from database theory and knowledge representation to formal verification and computational group theory. *Model theory* approaches this problem via investigating logical properties of structures.

However, not all finite representations are useful, for example

$$\text{“}K := \{i \in \mathbb{N} \mid \text{turing machine with index } i \text{ halts on input } i\}\text{”}$$

is a finite representation of an infinite set, but we do not have any way of computing this set.

Algorithmic model theory aims to restrict model theory such that the model checking problem is decidable. This approach gives results that are more useful in practice.

Algorithmic model theory studies structures with finite representation (e.g. by an axiomatisation in some logic, by an algorithm, by a collection of automata, ...), with requirement for *effective semantics*, i.e. for relevant logics L, given $\varphi \in L$ and a representation of a structure $\mathfrak{A}$, it should be decidable whether $\mathfrak{A} \models \varphi$.

These are just minimal required conditions, and depending on a context we might want to add for example some closure properties to be able to argue more easily, or require effective query evaluation for our theory to be more useful in practice.

One of the methods currently studied ([Hod76; KN94; BG00]) are *automatic structures*. Informally, a structure $\mathfrak{A}$ is automatic if its universe and relations can be recognised by a multi-headed automaton.

As an example, we can intuitively see that $\langle \mathbb{N}, + \rangle$ is automatic. If we represent each number in binary with the least significant bit first, we can construct a multi-head automaton that keeps track of the carry bit during addition. On the other hand, it was proven that $\langle \mathbb{Q}, + \rangle$ is not automatic [Tsa11].

One of the advantages of this approach is that it can be equivalently defined via model-theoretic interpretations [BG00] so we are able to tackle given problems by different techniques.

This concept was first researched in algorithmic group theory (see [BE92]), where they realised that many computational problems about certain fundamental groups are related to automata theory. Their definition is however slightly different than the one we use for structures. Every automatic group (in the group theoretic sense) is an automatic structure, but not every group that is an automatic structure is an automatic group (for example the *discrete Heisenberg group*, see [Blub]).

This notion was then refined for structures by Khoussainov and Nerode [KN94], and their theory has been developed by Blumensath in [BG00]. It has been shown there that the first-order theory of automatic structures is decidable. However in it was also shown that the transition graph of any Turing machine is automatic, therefore if we extend first-order logic with reachability or transitive closure, we obtain an undecidable theory.

However, it seems that the classification of automatic structures is a challenging problem. It is therefore only natural to study natural subclasses of automatic structures. The first natural subclass that comes to mind are unary automatic structures, i.e. structures with a representation over a unary alphabet, for example $\langle \mathbb{N}, \leq \rangle$. We can represent each $n \in \mathbb{N}$ as the word 0^n, and we can simply check which word is longer. For many types of structures (equivalence structures, linear orders, ...) there exist complete classifications of unary automatic structures (see [Blu99]), but this subclass seems too simple to be interesting.

It was shown in [Szi+92] that for any regular language the number of words of shorter than n is either bounded by some polynomial (called a language with polynomial growth), or this number is asymptotically exponential. Therefore another potentially interesting natural subclass are automatic structures where the universe of the representation is a language with polynomial growth. We call these *automatic structures with polynomial growth*. We will give an exact definition in chapter 2.

Note that every unary automatic structure has polynomial growth, but it does not hold the other way. For example $\langle \mathbb{N}^2, \sim \rangle$ where $(a, b) \sim (a', b')$ iff $a = a'$ has polynomial growth but it is not unary.

This subclass has been researched briefly by Bárány in [Bár07] and Huschenbett in [Hus16]. Our thesis aims to present notions necessary to properly outline these results in chapter 2. Then in chapter 3 we will prove a complete classification of equivalence structures over 0^*1^* and discuss a possible modification of our proof so that it would work for arbitrary $0^*1^* \dots (n-1)^*$, which would result in the complete classification of automatic equivalence structures with polynomial growth. Lastly, in chapter 4 we will summarise our results.

Shortly before submitting this thesis, we discovered a paper [GK20] by Ganardi and Khoussainov containing a more general version of our results. Let us note that our proofs have been obtained independently and use very different techniques.

2 Preliminaries

2.1 Convolution

As we explained in the previous chapter, we would like to define automatic structures via being recognised by multi-head automata. However it will be easier to do arguments with regular languages. For this, we define a convolution of tuples of words and a convolution of finitary relations over words. Intuitively, the convolution transforms a k-tuple of words over the alphabet Σ into a word over Σ^k. Then, a relation R is recognised by a multi-head automaton if the convolution of R (written $R^\otimes$) is recognised by a single-head automaton, i.e. $R^\otimes$ is a regular language.

We will have to deal with tuples of words where not all the words will have the same length. For this we will have to introduce a blank symbol $\square$.

Definition 2.1. (a) For an alphabet Σ, we denote by $\Sigma_\square := \Sigma \cup \{\square\}$ the extension by a new blank symbol $\square \notin \Sigma$.

(b) Let $s_0, \dots, s_{k-1}$ be words over the alphabet Σ, which we regard as functions $s_i : \mathrm{dom}(s_i) \to \Sigma$, where $\mathrm{dom}(s_i) = \{0, 1, \dots, \mathrm{len}(s_i) - 1\}$. The *convolution* $s_0 \otimes \cdots \otimes s_{k-1}$ is the word:

$$s_0 \otimes \cdots \otimes s_{k-1} : \bigcup_{i<k} \mathrm{dom}(s_i) \to \Sigma_\square^k$$

over the alphabet $\Sigma_\square^k$ where the labeling is given by

$$(s_0 \otimes \cdots \otimes s_{k-1})(x) := \begin{bmatrix} s_0(x) \\ s_1(x) \\ \vdots \\ s_{k-1}(x) \end{bmatrix},$$

with the convention that $s_i(x) = \square$ if $x \notin \mathrm{dom}(s_i)$. We also use the shorthand

$$\overline{s}^\otimes = s_0 \otimes \cdots \otimes s_{k-1}$$

(c) Let $R \subseteq (\Sigma^*)^k$ be a k-ary relation. Then

$$R^\otimes := \{\overline{s}^\otimes \mid \overline{s} \in R\} \subseteq (\Sigma_\square^k)^*.$$

We will call R regular if $R^\otimes$ is regular.

$$\begin{bmatrix} a \\ b \\ a \end{bmatrix} \begin{bmatrix} b \\ a \\ a \end{bmatrix} \begin{bmatrix} b \\ \square \\ a \end{bmatrix} \begin{bmatrix} a \\ \square \\ c \end{bmatrix} \begin{bmatrix} a \\ \square \\ c \end{bmatrix} \begin{bmatrix} c \\ \square \\ b \end{bmatrix} \begin{bmatrix} \square \\ \square \\ a \end{bmatrix} \begin{bmatrix} \square \\ \square \\ b \end{bmatrix} \begin{bmatrix} \square \\ \square \\ a \end{bmatrix}$$

Figure 1: $abbaac \otimes ba \otimes aaaccbaba$

In the figure 1 you can see an example of a convolution of words. Now we will show some examples of regular relations by giving regular expressions that generate that language.

Example. Let $\Gamma = \{a_0, \ldots, a_{n-1}\}$ be a finite alphabet, and let $<$ be a strict linear ordering on Γ.

(a) The *equality relation* $=$ over Γ^* is given by

$$\left(\begin{bmatrix}a_0\\a_0\end{bmatrix} + \begin{bmatrix}a_1\\a_1\end{bmatrix} + \cdots + \begin{bmatrix}a_{n-1}\\a_{n-1}\end{bmatrix}\right)^* = \left(\sum_{x\in\Gamma}\begin{bmatrix}x\\x\end{bmatrix}\right)^* = \mathrm{ID}$$

(b) The symmetric relation $\bowtie$ where all words are only in relation with ε is given by

$$\left(\sum_{x\in\Gamma}\begin{bmatrix}\square\\x\end{bmatrix}\right)^* + \left(\sum_{x\in\Gamma}\begin{bmatrix}x\\\square\end{bmatrix}\right)^* = \mathrm{END}$$

(c) The *equal length relation* $=_{\mathrm{len}} := \{(u,v) \mid |u| = |v|\}$ is given by

$$\left(\sum_{x,y\in\Gamma}\begin{bmatrix}x\\y\end{bmatrix}\right)^* = \mathrm{LEN}$$

(d) The *total relation* $\Gamma^* \times \Gamma^*$ is given by

$$\left(\sum_{x,y\in\Gamma}\begin{bmatrix}x\\y\end{bmatrix}\right)^* \cdot \left[\left(\sum_{x\in\Gamma}\begin{bmatrix}\square\\x\end{bmatrix}\right)^* + \left(\sum_{x\in\Gamma}\begin{bmatrix}x\\\square\end{bmatrix}\right)^*\right] = \mathrm{LEN}\cdot\mathrm{END}$$

(e) The *prefix order* $\preceq$ is given by

$$\left(\sum_{x\in\Gamma}\begin{bmatrix}x\\x\end{bmatrix}\right)^* \cdot \left(\sum_{x\in\Gamma}\begin{bmatrix}\square\\x\end{bmatrix}\right)^* = \mathrm{PO}$$

(f) The *lexicographic ordering*

$$u \leq_{\mathrm{lex}} v \text{ :iff } u \preceq v \text{ or } (u = wau' \text{ and } v = wbv' \text{ for some } u', v' \in \Gamma^*)$$

is given by

$$\mathrm{PO} + \mathrm{ID}\cdot\left(\sum_{\substack{x,y\in\Gamma'\\x<y}}\begin{bmatrix}x\\y\end{bmatrix}\right)\cdot\mathrm{LEN}\cdot\mathrm{END}$$

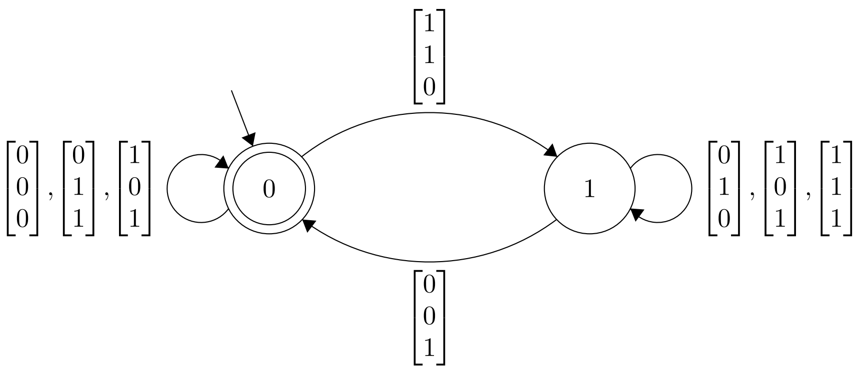
(g) The *length-lexicographic ordering*

$$u \leq_{\mathrm{llex}} v : \text{iff } |u| < |v| \text{ or } (|u| = |v| \text{ and } u \leq_{\mathrm{lex}} v)$$

is given by

$$\mathrm{LEN}\cdot\left(\sum_{x\in\Gamma}\begin{bmatrix}\square\\x\end{bmatrix}\right)^+ + \mathrm{ID}\cdot\left(\sum_{\substack{x,y\in\Gamma'\\x<y}}\begin{bmatrix}x\\y\end{bmatrix}\right)\cdot\mathrm{LEN} + \mathrm{ID}$$

(h) Addition $+$ on natural numbers represented in binary with least significant bit first is regular. For this, we will present an automaton accepting $+$:



We have only given the part of the automaton that does not mention the blank symbols $\square$. The full automaton can be obtained by adding more copies of the automaton above where some labels 0 are replaced by $\square$.

To help us construct regular relations, we have the following closure properties.

Definition 2.2. A homomorphism $\varphi : \Sigma^* \to \Gamma^*$ is *uniform* if

$$|\varphi(a)| = |\varphi(b)|, \text{ for all } a, b \in \Sigma$$

Lemma 2.3. *The class of regular relations over Σ^* is closed under:*

(a) boolean operations,

(b) direct products,

(c) projections,

(d) inverse uniform homomorphisms,

(e) uniform homomorphisms.

Proof. See e.g. [Blub]. $\square$

2.2 Automatic structures

To be complete, first we have to define *relational structures*:

Definition 2.4. (a) A *relational signature* Γ is a set of relational symbols, each of which has fixed (finite) arity.

(b) Let Γ be a relational signature. A (relational) Γ-*structure* $\mathfrak{A}$ consists of a set A called the *universe* of $\mathfrak{A}$, and for each relational symbol $R \in \Gamma$ with arity n an n-ary relation $R^{\mathfrak{A}}$.

Formally we can define a relational structure to be a pair $\langle A, \sigma \rangle$, where A is the universe and σ maps relational symbols from Γ to the relation it denotes. Note that we do not have requirements on the size of Γ, but in this thesis we will be working only with finite relational signatures, and we will write them simply as a tuple $\langle A, R_0^{\mathfrak{A}}, R_1^{\mathfrak{A}}, \ldots, R_j^{\mathfrak{A}} \rangle$. Many $(n+1)$-ary relational symbols will represent n-ary functions and we will use the standard functional notation when working with them (i.e. we will write $a + b = c$ instead of $(a, b, c) \in +$).

The main topic of our thesis are *equivalence structures*:

Definition 2.5. Let $\langle A, E\rangle$ be a relational structure, where E has arity 2. We call $\langle A, E\rangle$ an *equivalence structure* if E is an equivalence relation on A.

Since the first-order theory of regular languages is decidable (see [KN94]), encoding structures using regular languages is a reasonable approach for constructing models with effective semantics. This motivates the definition of *automatic structures*:

Definition 2.6. Let $\mathfrak{A}$ be a Γ-structure for some finite relational signature Γ.

(a) An *automatic presentation* of $\mathfrak{A}$ (over the finite alphabet Σ) is a surjective partial function $\pi : \Sigma^* \to A$ such that the languages

$$\begin{aligned} &L_\delta := \mathrm{dom}(\pi),\\ &L_= := \{u \otimes v \mid \pi(u) = \pi(v)\},\\ &L_R := \{\overline{u}^{\otimes} \mid (\pi(u_0), \ldots, \pi(u_{n_R-1})) \in R\} \text{ for all } R \in \Gamma, \end{aligned}$$

are all regular.

(b) The structure $\mathfrak{A}$ is *automatic* if it has an automatic presentation.

(c) If we fix a presentation π of $\mathfrak{A}$, for $a \in A$ we will denote the length of the shortest word representing a by $||a||$, i.e. $||a|| = \min\{|u| \mid \pi(u) = a\}$. We will say that a has the norm $||a||$.

Usually we will only write the sets $\langle L_\delta, L_=, (L_R)_{R\in\Sigma}\rangle$, leaving π implicit. Note that π can be recovered from $\langle L_\delta, L_=, (L_R)_{R\in\Sigma}\rangle$ (up to isomorphism) since

$$\mathfrak{A} \cong \langle L_\delta, (L_R)_{R\in\Sigma}\rangle / L_=.$$

Moreover, from the definition it is clear that only structures with countable size are automatic, therefore in the rest of the paper whenever we say *infinite number*, we mean countable infinity.

Example. (a) Given $m > 1$, the structure $\langle \omega, \leq, m|\cdot\rangle$ is automatic. We will encode each number n by the word 0^n:

- $L_\delta = 0^*$,
- $L_= = \begin{bmatrix}0\\0\end{bmatrix}^*$,
- $L_\leq = \begin{bmatrix}0\\0\end{bmatrix}^* \cdot \begin{bmatrix}\square\\0\end{bmatrix}^*$
- $L_{m|\cdot} = (0^m)^*$

(b) The structure $\langle \omega, \leq, +\rangle$ is automatic. We will encode each number n in binary with the least significant bit first, i.e. $\pi(x \cdot w) = x + 2\pi(w)$ with $\pi(\varepsilon) = 0$.

- $L_\delta = (0+1)^*$,
- $L_= = \left(\begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}\right)^* \cdot \left(\begin{bmatrix}\square\\0\end{bmatrix}^* + \begin{bmatrix}0\\\square\end{bmatrix}^*\right)$,
- $L_\leq = L_= + \left(\sum_{x,y\in\{0,1\}} \begin{bmatrix}x\\y\end{bmatrix}\right)^* \cdot \begin{bmatrix}0\\1\end{bmatrix} \cdot L_=$
- L_+: we have shown in the example (h) in the section 2.1 that this language is regular.

(c) For each n, the ordinal $\langle \omega^n, \leq \rangle$ is automatic. Let $\pi((n-1)^{x_{n-1}} \dots 1^{x_1} 0^{x_0}) := \omega^{n-1} \cdot x_{n-1} + \dots + \omega \cdot x_1 + x_0$. Then:

- $L_\delta = (n-1)^* \dots 1^* 0^*$,
- $L_= = \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}^* \cdot \dots \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}^* \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}^*$,
- $L_\leq$ is lexicographic ordering $\leq_{\text{lex}}$ (from the example (f) in the section 2.1) restricted to $(n-1)^* \dots 1^* 0^*$

(d) The structure $\langle T, \text{root}, (\text{succ}_i)_{i<n} \rangle$ is automatic, where T is the complete n-ary rooted tree of depth ω. You can find the picture of this tree in the figure 2. The alphabet of our encoding will be $\Gamma = \{0, 1, \dots, n-1\}$ and we will encode each node t by the path from the root node, i.e. if t_p is the predecessor of t, t is the i-th successor of t_p, and $\pi(e_p) = t_p$, then $\pi(e_p \cdot i) = t$.

- $L_\delta = \Gamma^*$,
- $L_= = \left(\sum_{i<n} \begin{bmatrix} i \\ i \end{bmatrix} \right)^*$,
- $L_{\text{root}} = \{\varepsilon\}$
- $L_{\text{succ}_j} = L_= \cdot \begin{bmatrix} \Box \\ j \end{bmatrix}$

(e) The structure $\mathcal{P}_{\text{fin}}\langle \omega, \leq \rangle := \langle \mathcal{P}_{\text{fin}}(\omega), \subseteq, \leq' \rangle$ (where $\leq'$ compares only single element sets) is automatic. We will encode each subset by its characteristic function (for example $\pi(1100110) = \{0, 1, 4, 5\}$):

- $L_\delta = (0+1)^*$,
- $L_= = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^* \cdot \left(\begin{bmatrix} \Box \\ 0 \end{bmatrix}^* + \begin{bmatrix} 0 \\ \Box \end{bmatrix}^* \right)$,
- $L_\subseteq = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^* \cdot \left(\begin{bmatrix} \Box \\ 0 \end{bmatrix}^* + \begin{bmatrix} 0 \\ \Box \end{bmatrix}^* \right)$,
- $L_{\leq'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^* \cdot \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}^* \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}^* \left(\begin{bmatrix} \Box \\ 0 \end{bmatrix}^* + \begin{bmatrix} 0 \\ \Box \end{bmatrix}^* \right)$.

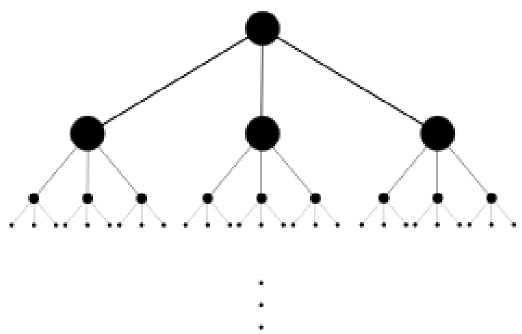


Figure 2: A picture of the tree structure from the example (d) for $n = 3$.

2.3 Interpretations

Often when working with structures, we use some representation of its elements – for example when working with complex numbers we often use cartesian coordinates over $\mathbb{R}^2$. The definition of a *k-dimensional interpretation* formalizes this notion:

Definition 2.7. Let Σ and Γ be relational signatures.

(a) Given an FO[Σ]-formula φ and a Σ-structure $\mathfrak{A}$, we will denote

$$\varphi^{\mathfrak{A}} := \{\overline{x} \mid \mathfrak{A} \models \varphi(\overline{x})\}$$

(b) A *k-dimensional first-order interpretation* (from signature Σ to Γ) is given by a list of FO[Σ]-formulae

$$\tau = \langle \delta(\overline{x}), \varepsilon(\overline{x}, \overline{y}), (\varphi_R(\overline{x}_0, \ldots, \overline{x}_{n_R-1}))_{R\in\Gamma}\rangle$$

where $\overline{x}, \overline{y}, \overline{x}_i$ are k-tuples of variables and n_R is the arity of the relation R.

Given a Σ-structure $\mathfrak{A}$, it produces the Γ-structure

$$\tau(\mathfrak{A}) := \langle \delta^{\mathfrak{A}}, (\varphi_R^{\mathfrak{A}})_{R\in\Gamma}\rangle / \approx,$$

where $\approx \subseteq (A^k)^2$ is the equivalence relation generated by the relation $\varepsilon^{\mathfrak{A}}$. If we do not want to specify the number k, we speak of a *multi-dimensional* interpretation

(c) We write $\mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$ if there exists a multi-dimensional interpretation τ such that $\tau(\mathfrak{B}) \cong \mathfrak{A}$.

Since this definition can seem complicated, we will give an example of a 2-dimensional interpretation. Note that neither of the mentioned structures are automatic.

Example. We will show that $\langle \mathbb{C}, +\rangle \leq_{\mathrm{FO}} \langle \mathbb{R}, +\rangle$ via 2-dimensional interpretation. Intuitively we will represent every complex number by its real and imaginary part.

- $\delta((x_0, x_1)) := x_0 = x_0$ (or any other tautology)
- $\varepsilon((x_0, x_1), (y_0, y_1)) := x_0 = y_0 \wedge x_1 = y_1$
- $\varphi_+((x_0, x_1), (y_0, y_1), (r_0, r_1)) := x_0 + y_0 = r_0 \wedge x_1 + y_1 = r_1$

Then we can write down the according sets:

- $\delta^{\mathfrak{R}} = \mathbb{R}^2$
- $\approx = \{((r_0, r_1), (r_0, r_1)) \mid (r_0, r_1) \in \mathbb{R}^2\}$ (i.e. identity on $\mathbb{R}^2 \times \mathbb{R}^2$)
- $\varphi_+^{\mathfrak{R}} = \{((r_0, r_1), (q_0, q_1), (r_0 + q_0, r_1 + q_1)) \mid (r_0, r_1, q_0, q_1) \in \mathbb{R}^4\}$

Since $\approx$ is the identity, we just have to show that $\langle \mathbb{C}, +\rangle \cong \langle \mathbb{R}^2, \varphi_+^{\mathfrak{R}}\rangle$. We can choose the bijection $\psi((r_0, r_1)) := r_0 + ir_1$. It is trivial to show that it indeed is a bijection and that $\psi((r_0, r_1)) + \psi((q_0, q_1)) = \psi((r_0 + q_0, r_1 + q_1))$.

The notion of a multi-dimensional interpretation is interesting for us because the class of automatic structures is closed under them, formally:

Theorem 2.8 (Blumensath, Grädel[BG00])**.** *Let $\mathfrak{B}$ be automatic structure and $\mathfrak{A}$ be a structure. Then if $\mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$, then $\mathfrak{A}$ is automatic.*

As we mentioned in the introduction, there is an equivalent definition of automatic structures via multi-dimensional interpretations that is often more convenient for proving statements.

Theorem 2.9 (Colcombet[CL07]). *Let $\mathfrak{A}$ be a structure. $\mathfrak{A}$ is automatic if and only if $\mathfrak{A} \leq_{\mathrm{FO}} \mathcal{P}_{\mathrm{fin}}\langle \omega, \leq \rangle$.*

We can say that $\mathcal{P}_{\mathrm{fin}}\langle \omega, \leq \rangle$ is a complete structure with respect to first-order interpretations. Note that $\mathcal{P}_{\mathrm{fin}}\langle \omega, \leq \rangle$ is just one of the complete structures. More can be found in [BG00].

2.4 Unary automatic structures

Definition 2.10. A structure $\mathfrak{A}$ is a *unary automatic structure* if it has an automatic presentation over a unary alphabet.

Example. The following structures are unary automatic.

(a) Every finite structure is unary automatic.

(b) In the example (a) in the section 2.2 we have shown that given $m > 1$, the structure $\langle \omega, \leq, m|\cdot \rangle$ has an automatic presentation over a unary alphabet, therefore it is unary automatic.

(c) The ordinal $\langle \omega + \omega, \leq \rangle$ is unary automatic. Let $\pi(0^{2k}) := k$, and $\pi(0^{2k+1}) = \omega + k$. Then:

- $L_\delta = 0^*$,
- $L_= = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^*$,
- $L_\leq = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}^2\right)^* \cdot \begin{bmatrix} \square \\ 0 \end{bmatrix}^* + \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}^2\right)^* \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} \square \\ 0 \end{bmatrix}^2\right)^*$

(d) The structure $\langle S, \mathrm{root}, (\mathrm{succ}_i)_{i<n} \rangle$ is unary automatic, where S is the n-ary rooted tree, where the 0-th successor has n successors, and other successors have 0. You can find the picture of this tree in the figure 3. We will encode each node t according to the path from the root node in the following way: let t_p be the predecessor of t and $\pi(e_p) = t_p$. Then if t is the 0-th successor of t_p $\pi(e_p \cdot 0^n) = t$, otherwise if t is the i-th successor, then $\pi(e_p \cdot 0^i) = t$.

- $L_\delta = 0^*$,
- $L_= = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^*$,
- $L_{\mathrm{root}} = \{\varepsilon\}$
- $L_{\mathrm{succ}_0} = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}^n\right)^* \cdot \begin{bmatrix} \square \\ 0 \end{bmatrix}^n$
- $L_{\mathrm{succ}_{j>0}} = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}^n\right)^* \cdot \begin{bmatrix} \square \\ 0 \end{bmatrix}^j$

Unary automatic structures are the simplest natural subclass of automatic structures. It is closed under finite disjoint unions and one-dimensional interpretations, but it is not closed under direct products nor under multi-dimensional interpretations (see [Blu99]).

We can get an equivalent definition of unary automatic structures using one-dimensional interpretation:

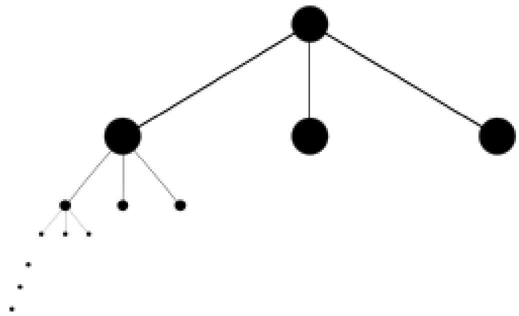


Figure 3: A picture of the tree structure from the example (d) for $n = 3$.

Theorem 2.11 (Blumensath[Blu99]). *A structure $\mathfrak{A}$ is unary automatic structure if and only if there exists $m > 1$ such that $\mathfrak{A} \leq_{\mathrm{FO}} \langle \omega, \leq, m|\cdot \rangle$ via a one-dimensional interpretation.*

From this characterisation we can easily see that a representation over a unary alphabet gives us only information about the length, which when sufficiently large can be tested only modulo some constant. Because of this simplicity, linear ordering, equivalence structures, and more have been completely characterised. We will present the result for equivalence structures:

Theorem 2.12 (Khoussainov,Rubin[KR99]). *Let $\langle A, \sim \rangle$ be an equivalence structure. Then the following statements are equivalent:*

i. *$\langle A, \sim \rangle$ is unary automatic,*

ii. *There exists a constant k such that $|\mathcal{C}| < k$ for all finite equivalence classes $\mathcal{C}$, and there is only finite number of infinite equivalence classes.*

2.5 Automatic structures with polynomial growth

The main goal of this thesis is to characterise automatic equivalence structures with *polynomial growth.* This class is richer than the one of unary structures, but its characterisation seems much simpler than the one for ordinary automatic structures.

Definition 2.13. A structure $\mathfrak{A}$ is an *automatic structure with polynomial growth* if it has an automatic presentation π such that there exists a polynomial $p(x)$ where for all $n \in \mathbb{N}$

$$|\{a \in A \mid ||a|| \leq n\}| \leq p(n)$$

We also say that $\mathfrak{A}$ has *polynomial growth*, or that $\mathfrak{A}$ is *poly-growth automatic.* Note that by [Bár07] we can without loss of generality assume that this presentation is injective, and in the rest of the paper we will do so.

Example. The following structures are poly-growth automatic:

(a) Every unary automatic structure has polynomial growth, in particular $\langle \omega, \leq, m|\cdot \rangle$

(b) In the example (c) in the section 2.2 we have shown that for all n, the ordinal $\langle \omega^n, \leq \rangle$ has an automatic presentation over $(n-1)^* \dots 1^* 0^*$. In the theorem 2.16 we will show that this implies that $\langle \omega^n, \leq \rangle$ is poly-growth automatic

(c) The equivalence structure $\langle \mathbb{N}^2, \sim \rangle$ is poly-growth automatic, where $\sim$ is equality on the first element. Let $\pi(0^a 1^b) = (a, b)$.

- $L_\delta = 0^*1^*$,
- $L_= = \begin{bmatrix}0\\0\end{bmatrix}^* \cdot \begin{bmatrix}1\\1\end{bmatrix}$,
- $L_\sim = \begin{bmatrix}0\\0\end{bmatrix}^* \cdot \begin{bmatrix}1\\1\end{bmatrix}^* \cdot \left(\begin{bmatrix}\square\\1\end{bmatrix}^* + \begin{bmatrix}1\\\square\end{bmatrix}^*\right)$

Since $L_\delta = 0^*1^*$, by the theorem 2.16 this is an automatic structure with polynomial growth.

(d) The structure $\langle R, \mathrm{root}, (\mathrm{succ}_i)_{i<n}\rangle$ is poly-growth automatic, where S is the n-ary rooted tree, where the $(n-1)$-th successor has n successors, and all other successors have 1. You can find the picture of this tree in the figure 4. We will use the same encoding as in the example (d) in the section 2.2, i.e. the alphabet of our encoding will be $\Gamma = \{0, 1, \ldots, n-1\}$ and we will encode each node t by the path from the root node. The only difference will be the domain of π.

- $L_\delta = (n-1)^* \cdot (\sum_{i<n-1} i) \cdot 0^* + (n-1)^*$,
- $L_= = \begin{bmatrix}n-1\\n-1\end{bmatrix}^* \cdot \left(\sum_{i<n-1} \begin{bmatrix}i\\i\end{bmatrix}\right) \cdot \begin{bmatrix}0\\0\end{bmatrix}^* + \begin{bmatrix}n-1\\n-1\end{bmatrix}^*$,
- $L_{\mathrm{root}} = \{\varepsilon\}$
- $L_{\mathrm{succ}_0} = L_= \cdot \begin{bmatrix}\square\\0\end{bmatrix}$
- $L_{\mathrm{succ}_{j>0}} = \begin{bmatrix}n-1\\n-1\end{bmatrix}^* \cdot \begin{bmatrix}\square\\j\end{bmatrix}$

We can see that $L_\delta \subseteq (n-1)^* \ldots 1^*0^*$, therefore by the theorem 2.16 this is an automatic structure with polynomial growth.

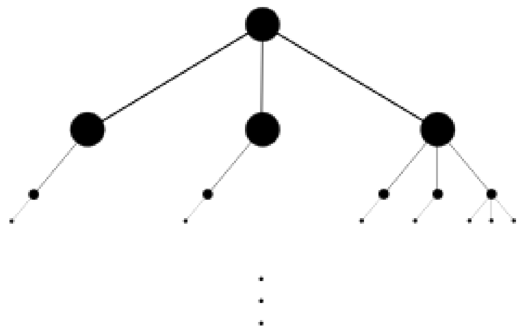


Figure 4: A picture of the tree structure from the example (d) for $n = 3$.

To show that this is a natural class of structures, Bárány has shown the following closure properties:

Theorem 2.14 (Bárány[Bár07])**.** *The class of automatic structures with polynomial growth is closed under disjoint union, direct product, and first-order interpretations.*

Similarly as for automatic structures, there is an equivalent definition using interpretations for automatic structures with polynomial growth.

Theorem 2.15 (Bárány[Bár07])**.** *Let $\mathfrak{A}$ be a structure. $\mathfrak{A}$ is automatic with polynomial growth if and only if there exists $m > 1$ such that $\mathfrak{A} \leq_{\mathrm{FO}} \langle \omega, \leq, m|\cdot \rangle$.*

The most useful tool for our thesis is the following result:

Theorem 2.16 (Bárány[Bár07]). *Let $\mathfrak{A}$ be a structure. $\mathfrak{A}$ is automatic with polynomial growth if and only if $\mathfrak{A}$ has an automatic presentation whose universe is a subset of $a_0^*a_1^* \dots a_{n-1}^*$ for some n and a_i's.*

If we could characterise equivalence structures with automatic presentation over $0^*1^* \dots (n-1)^*$ for arbitrary n, we would obtain a complete characterisation of equivalence structures with polynomial growth. In the next chapter we will give a proof of the characterisation of equivalence structures with automatic presentation over 0^*1^*, and then we will give a proof sketch for equivalence structures with automatic presentation over $0^*1^* \dots (n-1)^*$ for arbitrary n.

2.5.1 Known classifications

Finally the following complete classifications of automatic structures with polynomial growth are known:

Theorem 2.17 (Huschenbett[Hus16]). *An ordinal $\langle \alpha, \leq \rangle$ is poly-growth automatic if and only if $\alpha < \omega^\omega$.*

Theorem 2.18 (Blumensath[Blua]). *An abelian group is poly-growth automatic if and only if it is finite.*

In the next chapter we will give a complete classification of poly-growth automatic equivalence structures over 0^*1^*. This subclass is also closed under finite unions, formalized by the following lemma:

Lemma 2.19. *Let $\mathfrak{A}, \mathfrak{B}$ be automatic structures with injective presentations π_A, π_B over $0^*1^* \dots (n-1)^*$. Then $\mathfrak{A} \cup \mathfrak{B}$ has an injective presentation π over $0^*1^* \dots (n-1)^*$.*

Idea of proof: We will encode elements of $\mathfrak{A}$ by words such that they have even number of each letter, and elements of $\mathfrak{B}$ by words such that they have odd number of each letter.

Proof. Let $\mathfrak{A}, \mathfrak{B}$ be automatic structures with injective presentations π_A, π_B over $0^*1^* \dots (n-1)^*$. Let $\pi : 0^*1^* \dots (n-1)^* \to A \cup B$ given by:

$$\pi(0^{m_0}1^{m_1} \dots (n-1)^{m_{n-1}}) = \begin{cases} \pi_A(0^{\frac{m_0}{2}}1^{\frac{m_1}{2}} \dots (n-1)^{\frac{m_{n-1}}{2}}) & \text{if } 2 \mid m_i \text{ for all i} \\ \pi_B(0^{\frac{m_0-1}{2}}1^{\frac{m_1-1}{2}} \dots (n-1)^{\frac{m_{n-1}-1}{2}}) & \text{if } 2 \nmid m_i \text{ for all i} \\ \perp & \text{else} \end{cases}$$

Trivially π is injective, and $\mathrm{dom}(\pi)$ is regular. Now for every R from the relational signature, we know that we can separate $R^{\mathfrak{A}\cup\mathfrak{B}}$ into $R^{\mathfrak{A}} \cup R^{\mathfrak{B}}$. Then $L_R = L_{R,A} \cup L_{R,B}$, where $L_{R,X} := \{(u,v)^{\otimes} \mid (\pi(u), \pi(v)) \in R^{\mathfrak{X}}\}$.

Let ψ be a endomorphism given by

$$\psi(x) = x^2 \text{ for all } x \in \{0, 1, \dots, n-1\}.$$

Then $L_{R,A} = \psi^{n_R}(\{(u_0, u_1, \dots, u_{n_R-1}) \mid (\pi_A(u_0), \pi_A(u_1), \dots, \pi_A(u_{n_R-1}) \in R^{\mathfrak{A}}\})^{\otimes}$. Since π_A is an automatic presentation, and regular relations are closed under uniform homomorphisms by the theorem 2.3, $L_{R,A}$ is regular.

To prove that $L_{R,B}$, we have to introduce function $\eta : 0^*1^* \dots (n-1)^* \to 0^*1^* \dots (n-1)^*$:

$$\eta(0^{m_0}1^{m_1}\dots(n-1)^{m_{n-1}}) = 0^{m_0+1}1^{m_1+1}\dots(n-1)^{m_{n-1}+1}$$

Then $L_{R,B} = \eta^{n_R}(\psi^{n_R}(\{(u_0, u_1, \dots, u_{n_R-1}) \mid (\pi_B(u_0), \pi_B(u_1), \dots, \pi_B(u_{n_R-1}) \in R^{\mathfrak{B}}\}))^{\otimes}$. Similarly since π_B is an automatic presentation, regular relations are closed under uniform homomorphisms by the theorem 2.3, and trivially η keeps regularity, $L_{R,B}$ is regular.

We then have shown that π is an injective presentation of $\mathfrak{A} \cup \mathfrak{B}$ over $0^*1^*\dots(n-1)^*$. $\square$

3 Equivalence structures

In this chapter we will study poly-growth automatic equivalence structures. First we will prove two very helpful theorems; the first tells us that we can study equivalence structures with only finite equivalence classes and with only infinite ones separately. The second one is a stronger version of the pumping lemma, stating that if our language is a subset of a finite union of languages of the form $a_0^* a_1^* \dots a_{n-1}^*$ for some n and a_i's, we can pump every word (that has a long enough sub-word) by the same constant m, and we will pump only a part of the subword that contains only one type of letters.

3.0.1 Separation of equivalence structures

Theorem 3.1. *Let $\langle A, \sim \rangle$ be an equivalence structure with an injective automatic representation π with universe $L \subseteq \Sigma^*$. Then there exist structures $\langle A_{\text{fin}}, \sim_{\text{fin}} \rangle$ and $\langle A_{\text{inf}}, \sim_{\text{inf}} \rangle$ with injective automatic presentations $\pi_{\text{fin}}, \pi_{\text{inf}}$ over Σ^* such that:*

1. $\langle A_{\text{fin}}, \sim_{\text{fin}} \rangle \cup \langle A_{\text{inf}}, \sim_{\text{inf}} \rangle \cong \langle A, \sim \rangle$
2. *All equivalence classes of $\langle A_{\text{fin}}, \sim_{\text{fin}} \rangle$ are finite*
3. *All equivalence classes of $\langle A_{\text{inf}}, \sim_{\text{inf}} \rangle$ are infinite*
4. *The universe of these presentations are disjoint subsets of L.*

Idea of proof: Since $\leq_{\text{llex}}$ is regular, we can use this to construct a first-order formula $\varphi(x)$ that is satisfied if x is part of a finite equivalence class. This separates A into A_{fin} and A_{inf}.

Proof. Let $D := \pi^{-1}(A)$, and $E := \{(a,b)^{\otimes} \mid \pi(a) \sim \pi(b)\}$. First we will show that id_D is an automatic presentation of $\langle D, E, \leq_{\text{llex}} \rangle$. Trivially, $id_D^{-1}(D) = D$ is regular, so is $\{(a,b)^{\otimes} \mid (a,b) = (id_D(a), id_D(b)) \in E\}$. From the example (g) in the section 2.1 we also know that $\leq_{\text{llex}}$ is regular, so $\{(a,b)^{\otimes} \mid (a,b) = (id_D(a), id_D(b)) \in \leq_{\text{llex}}\}$ is regular.

Now let's consider the following formula:

$$\varphi(x) := \forall y.[E(x,y) \Rightarrow (\exists z. y \neq z \land y \leq_{\text{llex}} z \land E(x,z))]$$

It holds for all $x \in D$ that:

$$\langle D, E, \leq_{llex} \rangle \models \varphi(x) \iff x \textit{ is in an infinite equivalence class}$$

Let $D_{\text{fin}} := \{x \in D \mid \langle D, E, \leq_{\text{llex}} \rangle \not\models \varphi(x)\}$ and $D_{\text{inf}} := \{x \in D \mid \langle D, E, \leq_{\text{llex}} \rangle \models \varphi(x)\}$. Since φ is a first-order formula, both D_{fin} and D_{inf} are regular. Also trivially $E_{\text{fin}} := \{(a,b)^{\otimes} \mid (a,b)^{\otimes} \in E, a \in D_{\text{fin}}\}$ and $E_{\text{inf}} := \{(a,b)^{\otimes} \mid (a,b)^{\otimes} \in E, a \in D_{\text{inf}}\}$ are both regular. Now let $\pi_{\text{fin}} := \pi|_{D_{\text{fin}}}$ and $\pi_{\text{inf}} := \pi|_{D_{\text{inf}}}$.

Let $A_{\text{fin}} := \text{im}(\pi_{\text{fin}})$, $A_{\text{inf}} := \text{im}(\pi_{\text{inf}})$, $\sim_{\text{fin}} := \{(\pi_{\text{fin}}(x), \pi_{\text{fin}}(y)) \mid x, y \in D_{\text{fin}}, (x \otimes y) \in E\}$, and $\sim_{\text{inf}} := \{(\pi_{\text{inf}}(x), \pi_{\text{inf}}(y)) \mid x, y \in D_{\text{inf}}, (x \otimes y) \in E\}$. As we discussed earlier, all D_{fin}, D_{inf}, E_{fin}, and E_{inf} are all regular, so π_{fin} is an automatic presentation of $\langle A_{\text{fin}}, \sim_{\text{fin}} \rangle$ and π_{inf} is an automatic presentation of $\langle A_{\text{inf}}, \sim_{\text{inf}} \rangle$.

Moreover, from our construction it is trivial to see that both $\pi_{\text{fin}}, \pi_{\text{inf}}$ are injective, are over Σ^*, and all 4 points from our theorem hold.

□

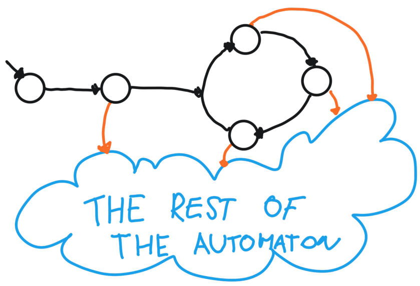
3.0.2 Pumping arguments

Definition 3.2. Let L be a regular language with $L \subseteq a_0^* \dots a_{l-1}^*$ for some distinct $a_0, \dots, a_{l-1}$. Then we say that L has rank l.

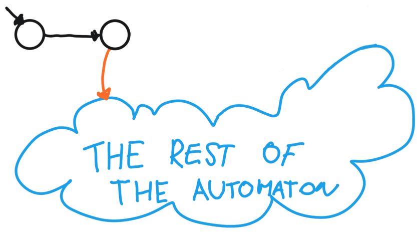
To prove our pumping lemma more easily, first we will prove that regular languages with rank l (for some l) can be decomposed into a finite union of concatenations of unary languages.

Lemma 3.3. *Let $L \subseteq a_0^* \dots a_{n-1}^*$ be a regular language, and all $a_0, \dots, a_{n-1}$ are distinct. Then there exists a finite set I and regular languages $A_{0,i}, \dots, A_{n-1,i}$ for all $i \in I$ such that $A_{j,i} \subseteq a_j^*$ and $L = \bigcup_{i \in I} A_{0,i}^* \dots A_{n-1,i}^*$.*

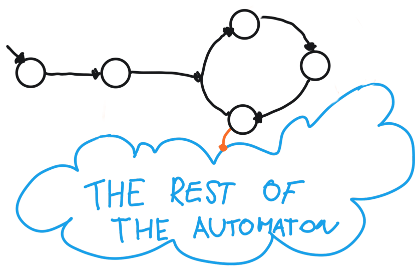
Idea of proof: We will prove this by induction on the rank n. The intuition behind the inductive step can be seen in the figure 5. We transform the original automaton $\mathcal{A}$ into a finite number of pairs of automata $\mathcal{K}_i, \mathcal{R}_i$, where the language accepted by $\mathcal{K}_i$ is a subset of a_0^*, the language of $\mathcal{R}_i$ is a subset of $a_1^* \dots a_{n-1}^*$, and $L(\mathcal{A}) = \bigcup_{i \in I} L(\mathcal{K}_i) \cdot L(\mathcal{R}_i)$.



(a) The original automaton



(b) The automaton accepting $L(\mathcal{K}_0) \cdot L(\mathcal{R}_0)$



(c) The automaton accepting $L(\mathcal{K}_3) \cdot L(\mathcal{R}_3)$

Figure 5: The intuition behind the inductive step in the proof of the lemma 3.3

Proof. We will prove this by induction on the rank n. If $n = 1$, our statement trivially holds. Now let $n > 1$, $L \subseteq a_0^* \dots a_{n-1}^*$ and assume that our lemma holds for all regular languages K satisfying our assumption with $n_k < n$.

First, let $A_{0,0} := L \cap a_0^*$. It is obvious that $A_{0,0}$ is regular since L is regular.

Now, since L is a regular language, there exists a deterministic automaton $\mathcal{A}$ with a single initial state q_0 such that $L(\mathcal{A}) = L$. Let $\mathcal{T}$ be the labeled transition system that we obtain from $\mathcal{A}$ by removing all edges that are not labeled by a_0 and removing all states that are not reachable from q_0. The set of labels for the states of $\mathcal{T}$ is equal to $\Sigma \setminus \{a_0\}$, and the labeling is given by:

$$l(q) := \{a_j \mid \exists q' \in \text{STATES}(\mathcal{A}).(q, a_j, q') \in \text{EDGES}(\mathcal{A})\}.$$

Now let $I' := \{(q, a_j) \mid q \in \text{STATES}(\mathcal{T}), a_j \in l(q)\}$. For every pair $(q, a_j) \in I'$ we will construct two automata $\mathcal{K}_{(q,a_j)}$, $\mathcal{R}_{(q,a_j)}$, and the languages they accept will be our regular languages $A_{0,(q,a_j)}$, $R_{(q,a_j)}$ that decompose $L \setminus A_{0,0}$.

Let $(q, a_j) \in I'$. Then we get $\mathcal{K}_{(q,a_j)}$ from $\mathcal{T}$ by setting q_0 (from the original automaton $\mathcal{A}$) as the initial state, and setting q as the only accepting state. Now $\mathcal{R}_{(q,a_j)}$ we construct from $\mathcal{A}$ by removing all edges labeled by a_i for all $i < j$, adding a new state q' with the same outgoing edges as q that are labeled by a_j (so that we don't accept ε and the first letter is a_j), setting q' as its initial state, and leaving the accepting states the same as in $\mathcal{A}$.

Let $A_{0,i} := L(\mathcal{K}_i)$, $R_i := L(\mathcal{R}_i)$ for every $i \in I'$. Now we will prove that $L = A_{0,0} \cdot \{\varepsilon\} \cup \bigcup_{i \in I} A_{0,i} \cdot R_i$.

Let $w \in L \cap A_{0,0}$. It is obvious that $w \in A_{0,0} \cdot \{\varepsilon\}$. Now let $w \in L \setminus A_{0,0}$. From that we can infer that w has to be of form $w = a_0^m \cdot a_j \cdot w'$ for some $m \in \mathbb{N}_0$, $j > 0$, and $w' \in (\Sigma \setminus \{a_0\})^*$. Then there exists an accepting run r on $\mathcal{A}$. Let q be the $(m+1)$-th state of this run (the state after we've read the a_0^m). It is obvious from our construction that $a_0^m \in A_{0,(q,a_j)}$ and that $a_j \cdot w' \in R_{(q,a_j)}$, therefore $w \in \bigcup_{i \in I} A_{0,i} \cdot R_i$. From this we get that $L \subseteq A_{0,0} \cdot \{\varepsilon\} \cup \bigcup_{i \in I} A_{0,i} \cdot R_i$.

Let $w \in A_{0,0} \cdot \{\varepsilon\}$. Since $A_{0,0} \cdot \{\varepsilon\} = A_{0,0}$ and $A_{0,0} \subseteq L$, $w \in L$. Now let $w \in \bigcup_{i \in I} A_{0,i} \cdot R_i$. Then there exists $(q, a_j) \in I'$ such that $w \in A_{0,(q,a_j)} \cdot R_{(q,a_j)}$, thus $w = w_k \cdot w_r$, where $w_k \in A_{0,(q,a_j)}$, $w_r \in R_{(q,a_j)}$. Then there exist accepting runs r_k, r_l in $\mathcal{K}_{(q,a_j)}$ and $\mathcal{R}_{(q,a_j)}$ respectively. From our construction we can transform $r_k \cdot r_l$ into run r in $\mathcal{A}$ that is accepting, thus $w \in L$. From this we get that $A_{0,0} \cdot \{\varepsilon\} \cup \bigcup_{i \in I'} A_{0,i} \cdot R_i \subseteq L$.

We've proven that $L = \bigcup_{i \in I \cup \{0\}} K_i \cdot R_i$, now we can apply our induction hypothesis I times to get that $L = \bigcup_{i \in J \cup \{0\}} A_{0,i} \cdot \bigcup_{r_i \in R_i} A_{1,r_i}^* \dots A_{n-1,r_i}^* = \bigcup_{i \in J'} A_{0,i}^* \dots A_{n-1,i}^*$, which we wanted to prove. □

Corollary 3.3.1. *Let $L \subseteq \bigcup_{j \in J} a_{0,j}^* \dots a_{l_j,j}^*$ be a regular language, where J is finite, and for all $j \in J$, $a_{0,j}, \dots, a_{l_j,j}$ are distinct. Then for each $j \in J$ there exist finite set I_j and regular languages $A_{0,i,j}, \dots, A_{l_j,i,j}$ for all $i \in I_j$ such that $A_{k,i,j} \subseteq a_{k,j}^*$ and $L = \bigcup_{j \in J, i \in I_j} A_{0,i,j}^* \dots A_{l_j,i,j}^*$.*

Proof. Apply the previous lemma for each $j \in J$ and take their union. □

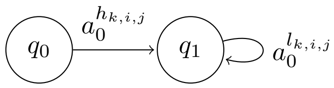
Theorem 3.4. *Let $L \subseteq \bigcup_{j \in J} a_{0,j}^* \dots a_{l_j,j}^*$ be a regular language, where J is finite, and for all $j \in J$, $a_{0,j}, \dots, a_{l_j,j}$ are distinct. Then there exist constants k, m such that for all $j \in J$, $o_0, \dots, o_{l_j} \in \mathbb{N}$, and for every subword w of $a_{0,j}^{o_0} \dots a_{l_j,j}^{o_{l_j}} \in L$ with $|w| > k$:*

$$\exists i \in \{0, \dots, l_j\}. a_{i,j} \in w \wedge \forall n \in \mathbb{N}. a_{0,j}^{o_0} \dots a_{i,j}^{o_i - m + nm} \dots a_{l_j,j}^{o_{l_j}} \in L$$

Idea of proof: With the previous lemma, we can construct an automaton where all the cycles have the same length, and there are no two cycles reachable from one another. The rest of this proof is similar to a proof of the standard pumping lemma.

Proof. From the corollary 3.3.1 we know that there exist a finite number of unary regular languages such that $L = \bigcup_{j\in J, i\in I_j} A^*_{0,i,j} \dots A^*_{l_j,i,j}$ with $A_{k,i,j} \subseteq a^*_{k,j}$. First, for each $A_{k,i,j}$ we construct an automat $\mathcal{A}_{k,i,j}$.

Since $A_{k,i,j}$ is regular, there exists a minimal deterministic automat $\mathcal{A}$ over the unary alphabet $\{a_{k,j}\}$ such that $L(\mathcal{A}'_{k,i,j}) = A_{k,i,j}$. This automaton has a pre-period of some length $h_{k,i,j} \geq 0$, and if $A_{k,i,j}$ is not finite, it ends with loop of length $l_{k,i,j} \geq 0$, represented by the following diagram:



Now let $m := \mathrm{LCM}(\{l_{k,i,j} |\ |A_{k,i,j}| = \omega\})$. We transform all $\mathcal{A}'_{k,i,j}$ with loops into $\mathcal{A}_{k,i,j}$ by extending the loop by a factor $\frac{m}{l_{k,i,j}}$, so that all the created automata will have a loop of length m. This will also multiply the number of accepting states on the loop by the same factor so that $L(\mathcal{A}_{k,i,j}) = L(\mathcal{A}'_{k,i,j})$. If $\mathcal{A}'_{k,i,j}$ has no loop, we will set $\mathcal{A}_{k,i,j} := \mathcal{A}'_{k,i,j}$.

For every language $A^*_{0,i,j} \dots A^*_{l_j,i,j}$ we will construct an automaton $\mathcal{A}_{j,i}$ from the automata for the unary languages by the standard construction for concatenation of languages, and finally we will construct nondeterministic automata $\mathcal{A}$ for L from $\mathcal{A}_{j,i}$ by the standard construction for finite union of languages. Let k be the number of states of $\mathcal{A}$.

Let $u := a_{0,j}^{o_0} \dots a_{l_j,j}^{o_{l_j}} \in L$ for some $j \in J$, $o_0, \dots, o_{l_j} \in \mathbb{N}$, and let w be a subword of u longer than k. Since $u \in L$, there exists an accepting run $(r_0, r_1, \dots, r_{|u|+1})$ in $\mathcal{A}$, and let $(r_p, ..., r_{p+|w|+1})$ be a subword of r be states visited when reading w. Since $|w| > k$, $p > k$ and therefore there must exist two indices $i < j$ such that $r_i = r_j$. Let i_M, j_M be such a pair of indices with i_M being minimal among those i's and j_M be minimal among j's such that $r_{i_M} = r_j$. Therefore $(r_{i_M}, r_{i_M+1}, \dots, r_{j_M})$ form a cycle, and because of minimality of j_M this cycle has length m. Moreover, from our construction of $\mathcal{A}$ we can see that this cycle corresponds to reading a^m for some $a \in w$.

From this we get that for every $n \in \mathbb{N}$, $r_0, \dots, (r_{i_M}, r_{i_M+1}, \dots, r_{j_M})^n, r_{j_M+1}, \dots, r_{|u|+1}$ is again accepting run, i.e. there exists $a_{i,j} \in w$ such that for all $n \in \mathbb{N}$:

$$a_{0,j}^{o_0} \dots a_{i,j}^{o_i - m + nm} \dots a_{l_j,j}^{o_{l_j}} \in L,$$

which we wanted to prove.

□

Note that in the rest of this chapter whenever we will claim an existence of constants k, m, in the proof it will always be the constants from this pumping lemma.

In the following corollary you can see how we will be using our pumping lemma. Often we will be able to prove equivalence statements about elements that are "*far enough*". The figure 6 illustrates this lemma. The two rows represent two words, each blue block represents a block of letters (the longer the block, the more letters there are), and the orange box highlights the part of the convolution of words we will be pumping. Note that we will be using similar figures in the rest of this thesis when explaining the intuition behind theorems that use the pumping lemma.

Corollary 3.4.1. *Let $\langle A, \sim\rangle$ be an equivalence structure with an injective automatic representation π over $0^*1^* \dots (n-1)^*$. Then there exist constants k, m such that for all $0^a \cdot u, 0^{a'} \cdot u' \in \pi^{-1}(A)$ with $a, a' > k$ it holds that:*

$$\pi(0^a \cdot u) \sim \pi(0^{a'} \cdot u') \Leftrightarrow \pi(0^{a+m} \cdot u) \sim \pi(0^{a'+m} \cdot u')$$

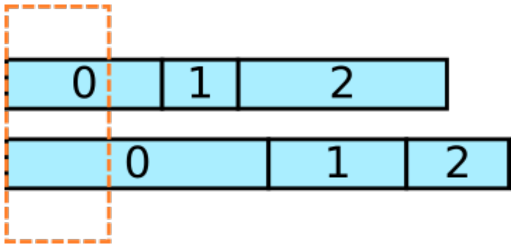


Figure 6: The intuition behind the corollary 3.4.1 for $n = 3$

Proof. Since $E := \{(x, y)^{\otimes} \mid \pi(x) \sim \pi(y)\}$ is a finite union of languages of some finite rank, we will be able to use the theorem 3.4. Choose k, m according to the theorem 3.4, and let $0^a \cdot u, 0^{a'} \cdot u' \in \pi^{-1}(A)$ with $a, a' > k$.

First assume that $\pi(0^a \cdot u) \sim \pi(0^{a'} \cdot u')$. Then $(0^a \cdot u) \otimes (0^{a'} \cdot u') = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^{k+1} \cdot v$ for some v. From the theorem 3.4 (choosing $n = 2$) we get that $\begin{bmatrix} 0 \\ 0 \end{bmatrix}^{k+1+m} \cdot v \in E$, which is equivalent to $\pi(0^{a+m} \cdot u) \sim \pi(0^{a'+m} \cdot u')$.

Similarly, for $(0^{a+m} \cdot u) \otimes (0^{a'+m} \cdot u') \in E$ we can choose $n = 0$ to get that $(0^a \cdot u) \otimes (0^{a'} \cdot u') \in E$, which is equivalent to $\pi(0^a \cdot u) \sim \pi(0^{a'} \cdot u')$. $\square$

From the theorem 3.1 we know we can study finite and infinite equivalence classes separately. The infinite classes are quite uninteresting – theorem 2.12 tells us that if the equivalence structure has a finite number of infinite equivalence classes, it is unary. The equivalence structure from the example (c) in the section 2.5 has an infinite number of infinite equivalence classes, and has representation over 0^*1^*. Because of this we will be interested only in studying the finite equivalence classes.

The first important observations then is that if we have a finite equivalence class $\mathcal{C}$, the words that represent elements of $\mathcal{C}$ have to have similar length, otherwise we could use the theorem 3.4 to obtain an infinite number of distinct elements that have to be in $\mathcal{C}$. The figure 7 shows this.

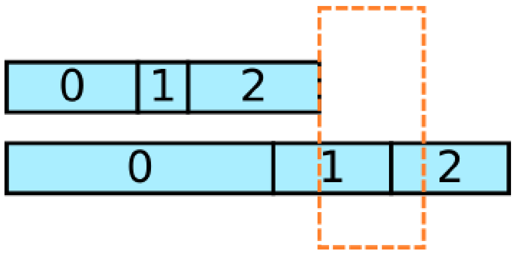


Figure 7: The intuition behind the theorem 3.5 for $n = 3$

Theorem 3.5. *Let $\langle A, \sim \rangle$ be an equivalence structure with an injective automatic representation π over $0^*1^* \dots (n-1)^*$. Then there exists a constant k such that for all finite equivalence classes $\mathcal{C} \in A/\sim$ and for all $a, b \in \pi^{-1}(\mathcal{C})$:*

$$|b| - |a| \leq k$$

Proof. Let $\langle A, \sim \rangle$ be an equivalence structure with an injective automatic representation π over $0^*1^* \dots (n-1)^*$, and choose k, m according to the theorem 3.4. Let $\mathcal{C} \in A/\sim$ be finite equivalence class, and let $a, b \in \pi^{-1}(\mathcal{C})$ such that $|a| < |b| - k$. Then $a \otimes b = u \cdot v$ for some u, v such that:

$$v \in (\sum_{i=0}^{n-1} \begin{bmatrix} \square \\ i \end{bmatrix})^{k+1}$$

Since $E := \{(x, y)^{\otimes} \mid \pi(x) \sim \pi(y)\}$ is a finite union of languages of some finite rank, and $|v| > k$, from the theorem 3.4 there exist x, y, z with $|y| > 0$ (more specifically $|y| = m$) and $v = xyz$ such that for all $n \in \mathbb{N}$, $uxy^n z \in E$. From the definition of E, for each n there exists a_n, b_n such that $a_n \otimes b_n = uxy^n z$. But since v had only blanks at the zeroth position, $a_n = a$ for all $n \in \mathbb{N}$. Moreover, since π is injective, and for all $i \neq j$, $b_i \neq b_j$, we obtain that $|\mathcal{C}| \geq \omega$, contradicting the finiteness of $\mathcal{C}$. □

The direct consequence of this is an upper bound on the size of an equivalence class depending on the length of the shortest word representing an element that class.

Corollary 3.5.1. *Let $\langle A, \sim \rangle$ be an equivalence structure with an injective automatic representation π over $0^*1^* \dots (n-1)^*$. Then there exists a constant k such that for all finite equivalence classes $\mathcal{C} \in A/\sim$:*

$$|\mathcal{C}| \leq \sum_{i=||\mathcal{C}||}^{||\mathcal{C}||+k} \binom{i+n-1}{n-1},$$

where $||\mathcal{C}|| = \min\{||c|| \mid c \in \mathcal{C}\}$. In particular, for $n = 1$ we get that $|\mathcal{C}| \leq k+1$, and for $n = 2$ we get $|\mathcal{C}| \leq (k+1) \cdot ||\mathcal{C}|| + \frac{(k+2)\cdot(k+1)}{2}$.

Proof. Choose k according to the theorem 3.5. For each c, we know that the number of words of length c is $\binom{c+n-1}{n-1}$. From the theorem 3.5 we know that we only can have words of length from $||\mathcal{C}||$ to $||\mathcal{C}|| + k$, so we get that:

$$|\mathcal{C}| \leq \sum_{i=||\mathcal{C}||}^{||\mathcal{C}||+k} \binom{i+n-1}{n-1},$$

□

3.1 Unary equivalence structures

We have given the complete characterisation of this class in the theorem 2.12. Here we will present an equivalent characterisation that mirrors the one we will give for the equivalence structures over 0^*1^*, and which we will propose for the general case. Intuitively if our equivalence structure has a representation over $0^*1^* \dots (n-1)^*$, we can represent "*big enough*" classes uniquely with a pair $i \in \mathbb{N}$ and polynomial $p(x)$ of degree $(n-1)$, i.e. in the unary case with constant polynomials.

Theorem 3.6. *Let $\langle A, \sim \rangle$ be an equivalence structure. Then the following statements are equivalent:*

i. *$\langle A, \sim \rangle$ has an injective automatic representation π over 0^**

ii. *There exists a constant k such that for all finite equivalence classes $\mathcal{C}$, $|\mathcal{C}| < k$, and there is only finite number of infinite equivalence classes.*

iii. *There is only a finite number of infinite equivalence classes and there exist a constant s, and a finite index set I and for each $i \in I$, a number $c_i \in \mathbb{N}$ such that there exists a bijection f between $I \times \mathbb{N}$ and all finite equivalence classes $\mathcal{C}$ with $|\mathcal{C}| > s$. Additionally, it holds that $|f(i, n)| = c_i$.*

Proof. Let $\langle A, \sim \rangle$ be an equivalence structure with only finite number of infinite equivalence classes. First we will show that *ii* implies *iii*, and then that *iii* implies *ii*.

Let k be such that all for all finite equivalence classes $\mathcal{C}$, $|\mathcal{C}| < k$. Then we can choose $s := k$ and $I := \emptyset$. Since there are no finite equivalence classes bigger than s, the bijection between two empty sets satisfies *iii*.

For the other implication, assume that *iii* holds. If I is empty, we can choose $k := s + 1$. Otherwise choose $k := \max\{c_i \mid i \in I\} + 1$. □

3.2 Equivalence structures over 0^*1^*

To obtain the classification of equivalence relations over 0^*1^*, first we will show there is some internal structure within equivalence classes in the lemma 3.9, and then we will show that this structure is copied among different equivalence classes in the lemma 3.10. From this we will create an enumeration of all equivalence classes, from which we can effectively compute their sizes (corollary 3.10.1).

In this section, we will write (a, b) instead of 0^a1^b. Moreover, if we have $\langle A, \sim \rangle$ with some fixed automatic representation π, we will write $(a, b) \sim (a', b')$ instead of $\pi((a, b)) \sim \pi((a', b'))$.

First we will prove useful consequences of our pumping lemma. Note that for the last four points we will be using the pumping argument twice, illustrated in the figure 8.

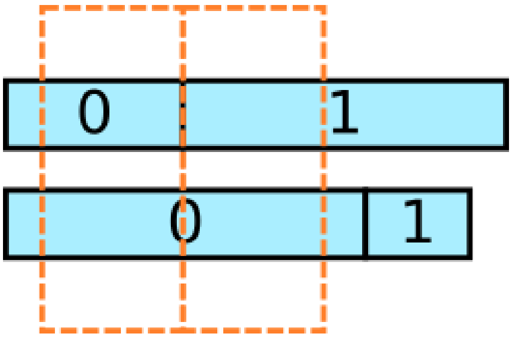


Figure 8: The intuition behind iii. from the theorem 3.7

In figure 9 you can see different visualisation of this theorem. To illustrate our arguments, we will represent each word as a point on a two dimensional plane. The horizontal axis corresponds to the number of zeros, the vertical axis corresponds to the number of ones, and two points have the same colour if they are part of the same equivalence class. The diagonal lines enclose the regions where points from the same class can be (as limited by the theorem 3.5).

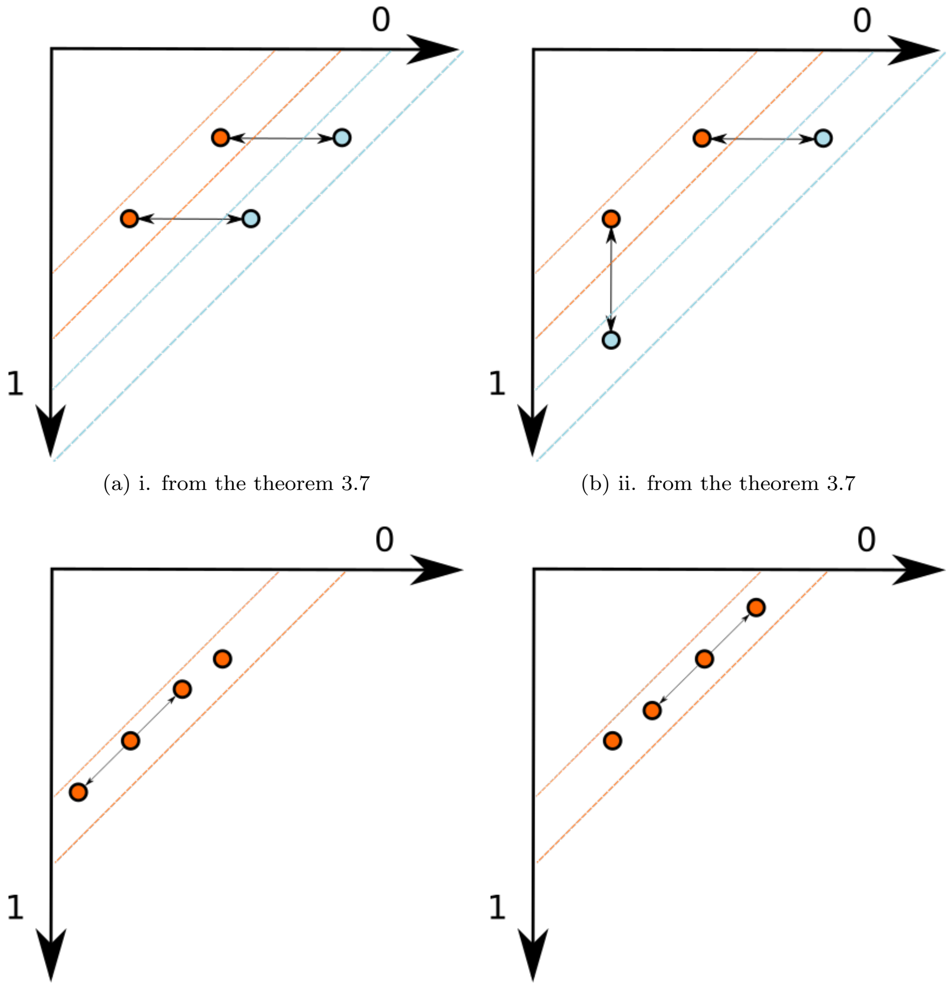
Theorem 3.7. *Let $\langle A, \sim \rangle$ be an equivalence structure with an injective automatic representation π over 0^*1^*. Then there exist constants k, m such that for all $(a_L, b_L), (a_R, b_R) \in \pi^{-1}(A)$:*

i. $(a_L, a_R > k) \Rightarrow ((a_L, b_L) \sim (a_R, b_R) \Leftrightarrow (a_L + m, b_L) \sim (a_R + m, b_R))$

ii. $(a_R - a_L > k) \wedge (b_L > k) \Rightarrow ((a_L, b_L) \sim (a_R, b_R) \Leftrightarrow (a_L, b_L + m) \sim (a_R + m, b_R))$

iii. $[(a_R - a_L > k) \wedge (a_L > k) \wedge (b_L > k) \wedge (a_L, b_L) \sim (a_R, b_R)] \Rightarrow ((a_L + m, b_L - m) \sim (a_R, b_R))$

iv. $[(a_R - a_L > k) \wedge (a_L > k) \wedge (b_L > k) \wedge (a_L, b_L) \sim (a_R, b_R)] \Rightarrow ((a_L - m, b_L + m) \sim (a_R, b_R))$



(a) i. from the theorem 3.7

(b) ii. from the theorem 3.7

(c) iii. and iv. from the theorem 3.7

(d) v. and vi. from the theorem 3.7

Figure 9: The visualisation of the theorem 3.7

Moreover, for all finite equivalence classes $\mathcal{C} \in A/\sim$ with $\pi((a_L, b_L)), \pi((a_R, b_R)) \in \mathcal{C}$:

v. $[(a_R - a_L > k) \wedge (b_L > k) \wedge (b_R > k)] \Rightarrow ((a_L, b_L) \sim (a_R + m, b_R - m))$

vi. $[(a_R - a_L > k) \wedge (b_L > k) \wedge (b_R > k)] \Rightarrow ((a_L, b_L) \sim (a_R - m, b_R + m))$

Proof. Since $E := \{(x, y)^{\otimes} \mid \pi(x) \sim \pi(y)\}$ is a finite union of languages of some finite rank, we will be able to use the theorem 3.4. Choose k, m according to the theorem 3.4, and let $(a, b), (a', b') \in \pi^{-1}(A)$.

i. : This follows directly from the corollary 3.4.1.

ii. : Let's assume that $a_R - a_L > k$ and $b_L > k$. Then for some v, v':

$$(a_L, b_L) \otimes (a_R, b_R) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^{a_L} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\min(a_R - a_L, b_L)} \cdot v' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^{a_L} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{k+1} \cdot v$$

If we assume $(a_L, b_L) \otimes (a_R, b_R) \in E$, then from the theorem 3.4 (choosing $n = 2$) we get that:

$$\begin{bmatrix}0\\0\end{bmatrix}^{a_L} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{k+1+m} \cdot v = (a_L, b_L + m) \otimes (a_R + m, b_R) \in E$$

which is equivalent to $(a_L, b_L + m) \sim (a_R + m, b_R)$. Similarly for $(a_L, b_L + m) \sim (a_R+m, b_R)$ we can choose $n = 0$ to get that $(a_L, b_L) \otimes (a_R, b_R) \in E$, which is equivalent to $(a_L, b_L) \sim (a_R, b_R)$.

iii. : Lets assume $(a_R - a_L > k)$, $(a_L > k)$, and $(b_L > k)$. Then for some v, v':

$$(a_L, b_L) \otimes (a_R, b_R) = \begin{bmatrix}0\\0\end{bmatrix}^{a_L} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{\min(a_R - a_L, b_L)} \cdot v' = \begin{bmatrix}0\\0\end{bmatrix}^{a_L} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{k+1} \cdot v$$

Since $a_L > k$, and $k + 1 > k$, we can use the theorem 3.4 (choosing first $n = 2$ and then $n = 0$) to get that:

$$\begin{bmatrix}0\\0\end{bmatrix}^{a_L + m} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{k+1-m} \cdot v \in E,$$

which is equivalent to $(a_L + m, b_L - m) \sim (a_R, b_R)$.

iv. : Same proof as for *iii.*, choosing first $n = 0$ and then $n = 2$.

Now let $\mathcal{C} \in A/\sim$ be finite equivalence class, and let $C = \pi^{-1}(\mathcal{C})$, and assume $(a_L, b_L), (a_R, b_R) \in C$.

v. : Lets assume $a_R - a_L > k$, $b_L > k$, and $b_R > k$. We know that $a_R - a_L \le b_L$, otherwise $a_L + b_L < a_R + b_R - k$, thus by the theorem 3.5 C would be infinite. First assume that $a_L + b_L \ge a_R + b_R$. Then:

$$(a_L, b_L) \otimes (a_R, b_R) = \begin{bmatrix}0\\0\end{bmatrix}^{a_L} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{a_R - a_L} \cdot \begin{bmatrix}1\\1\end{bmatrix}^{b_R} \cdot \begin{bmatrix}1\\\square\end{bmatrix}^{b_L - b_R - (a_R - a_L)},$$

Since $b_R > k$, we can use the theorem 3.4 (choosing $n = 0$) to get that:

$$\begin{bmatrix}0\\0\end{bmatrix}^{a_L} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{a_R - a_L} \cdot \begin{bmatrix}1\\1\end{bmatrix}^{b_R - m} \cdot \begin{bmatrix}1\\\square\end{bmatrix}^{b_L - b_R - (a_R - a_L)} \in E,$$

Moreover since $a_R - a_L > k$, we can use the theorem 3.4 (choosing $n = 2$) to get that:

$$\begin{bmatrix}0\\0\end{bmatrix}^{a_L} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{a_R - a_L + m} \cdot \begin{bmatrix}1\\1\end{bmatrix}^{b_R - m} \cdot \begin{bmatrix}1\\\square\end{bmatrix}^{b_L - b_R - (a_R - a_L)} \in E,$$

which is equivalent to $(a_L, b_L) \sim (a_R + m, b_R - m)$.

Now assume that $a_L + b_L < a_R + b_R$. Then:

$$(a_L, b_L) \otimes (a_R, b_R) = \begin{bmatrix}0\\0\end{bmatrix}^{a_L} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{a_R - a_L} \cdot \begin{bmatrix}1\\1\end{bmatrix}^{b_L - (a_R - a_L)} \cdot \begin{bmatrix}\square\\1\end{bmatrix}^{b_R - b_L + a_R - a_L}$$

Since $b_R > k$, we know that subword $\begin{bmatrix}1\\1\end{bmatrix}^{b_L - (a_R - a_L)} \cdot \begin{bmatrix}\square\\1\end{bmatrix}^{b_R - b_L + a_R - a_L}$ is longer than k. From the theorem 3.4, we know that we can apply it on either $\begin{bmatrix}1\\1\end{bmatrix}^{b_L - (a_R - a_L)}$ or $\begin{bmatrix}\square\\1\end{bmatrix}^{b_R - b_L + a_R - a_L}$. If we applied it on the latter, we would obtain that $(a_L, b_L) \sim (a_R, b_L + km)$ for all $k \in \mathbb{N}$, contradicting the finiteness of C. Therefore we know that we can apply it on the former; choosing $n = 0$ we get:

$$\begin{bmatrix}0\\0\end{bmatrix}^{a_L} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{a_R-a_L} \cdot \begin{bmatrix}1\\1\end{bmatrix}^{b_L-(a_R-a_L)-m} \cdot \begin{bmatrix}\square\\1\end{bmatrix}^{b_R-b_L+a_R-a_L} \in E$$

Also we can again use the theorem 3.4 (choosing $n = 2$) to get that

$$\begin{bmatrix}0\\0\end{bmatrix}^{a_L} \cdot \begin{bmatrix}1\\0\end{bmatrix}^{a_R-a_L+m} \cdot \begin{bmatrix}1\\1\end{bmatrix}^{b_L-(a_R-a_L)-m} \cdot \begin{bmatrix}\square\\1\end{bmatrix}^{b_R-b_L+a_R-a_L} \in E,$$

which is equivalent to $(a_L, b_L) \sim (a_R + m, b_R - m)$, proving v.

$vi.$: Almost the same proof as for $v.$, first choosing $n = 2$ and then choosing $n = 0$. $\square$

The first two points of the previous theorem intuitively tell us there is some repeating structure between different equivalence classes. The rest of the points on the other hand show that there is some repeating structure within an equivalence class.

To be able to more easily argue in the rest of the proofs, we will prove that if the equivalence class is "*big enough*", for each point in some bounds there is an element of that equivalence class that is closer than some constant. This is illustrated in the figure 10. Note that the diagonal lines correspond to $||\mathcal{C}||$ and $||\mathcal{C}|| + k$, the vertical and horizontal line correspond to k number of zeros and ones, and the blue vertical lines correspond to the intervals $i \pm m$.

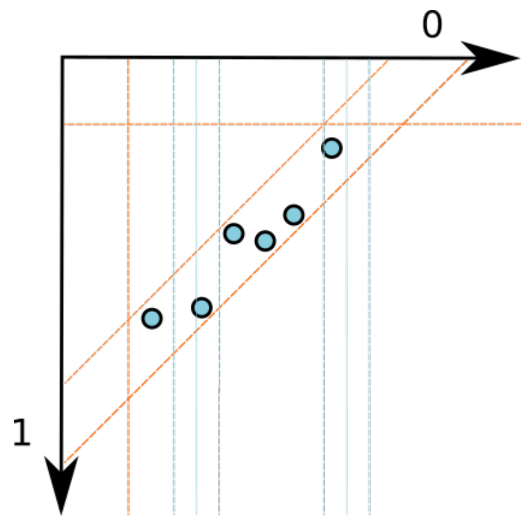


Figure 10: The intuition behind the lemma 3.8

Lemma 3.8. *Let $\langle A, \sim\rangle$ be an equivalence structure with an injective automatic representation π over 0^*1^*. Then there exist constants k, m such that for all finite equivalence classes $\mathcal{C} \in A/\sim$ with $|\mathcal{C}| > 4(k+1)^2$:*

$$\forall i \in \mathbb{N}.(k+m < i < ||\mathcal{C}|| - k - m) \Rightarrow (\exists (a,b) \in \pi^{-1}(\mathcal{C}).|i-a| < m \wedge (k < a < ||\mathcal{C}|| - k)).$$

Proof. Choose k, m according to the theorem 3.7. Let $\mathcal{C}$ be a finite equivalence class bigger than $4(k+1)^2$. From the theorem 3.5 we know that there are at least $4(k+1)^2 - 2(k+1)^2 = 2(k+1)^2$ elements of $\pi^{-1}(\mathcal{C})$ such that they have more than k zeros and k ones. Lets denote this subset $\mathcal{C}'$.

Let I be set of indices i, such that $k + m < i < ||\mathcal{C}|| - k - m$ and there is no $(a, b) \in \mathcal{C}'$ such that $|i - a| < m$, and assume that I is nonempty. Let $i_L := \min(I)$, and $i_R := \max(I)$.

If $i_L = k + m + 1$, there are no $(a', b') \in \mathcal{C}'$ with $a < i_L$. Lets take $(a_L, b_L) \in \mathcal{C}'$ such that for all other elements of $(a', b') \in \mathcal{C}'$ holds that $a_L \leq a'$ and $a_L = a' \Rightarrow b_L \geq b'$. Since for each $a \in \mathbb{N}$, there are at most $k + 1$ elements from $\mathcal{C}'$ that have a zeros, there must exist $(a_R, b_R) \in \mathcal{C}'$ such that $a_R - a_L > k$. From definition of $\mathcal{C}'$ we know that $b_L > k$. Moreover, since $a_L > i_L$, we know that $a_L > k$. Now we can use iv. from the theorem 3.7 to get that $(a_L - m, b_L + m) \in \mathcal{C}'$, obtaining the contradiction to the minimality of (a_L, b_L).

Similarly, if $i_R = ||\mathcal{C}|| - k - m - 1$, we can take $(a_R, b_R) \in \mathcal{C}'$ such that for all other elements of $(a', b') \in \mathcal{C}'$ holds that $a_R \geq a'$ and $a_R = a' \Rightarrow b_R \geq b'$, and get that there must exist $(a_L, b_L) \in \mathcal{C}'$ such that $a_R - a_L > k$, $b_L > k$, and $b_R > k$. Now we can use v. from the theorem 3.7 to get that $(a_R + m, b_R - m) \in \mathcal{C}'$, obtaining the contradiction to the maximality of (a_R, b_R).

Now assume that $i_L \neq k + m + 1$ and $i_R \neq ||\mathcal{C}|| - k - m - 1$. Let $E_L = \{(a', b') \in \mathcal{C}' \mid i_L - m > a' > k\}$. Similarly let $E_R = \{(a', b') \in \mathcal{C}' \mid i_L + m < a' < k\}$. Note that $E_L \cup E_R = \mathcal{C}'$. From $i_L \neq k + m + 1$ we get there exists $(a', b') \in \mathcal{C}'$ with $i_L - m > a' > k$, so E_L is not empty. Similarly from $i_R \neq ||\mathcal{C}|| - k - m - 1$ we get that there exists $(a', b') \in \mathcal{C}'$ with $i_L + m \leq i_R + m < a' < k$, so E_R is also not empty. Since $|\mathcal{C}'| = 2(k + 1)^2$, either E_L or E_R has to have at least $(k + 1)^2$ elements.

First, lets assume that $|E_L| \geq (k + 1)^2$, and take $(a_R, b_R) \in E_R$ such that for all $(a', b') \in E_R$, $a_R \leq a' \wedge (a_R = a' \Rightarrow b_R \geq b')$. Since $|E_L| \geq k^2$, there must exist $(a_L, b_L) \in E_L$ such that $i_L - a_L > k$, thus $a_R - a_L > k$. From definition of E_R we also know that $b_R > k$, and with iv. from the theorem 3.7 we get that $(a_R - m, b_R + m) \in \mathcal{C}'$. Since (a_R, b_R) is minimal, $(a_R - m, b_R + m) \notin E_R$. But $(a_R - m, b_R + m)$ cannot be an element of E_L, since for all $(a', b') \in E_L$, $a' < i_L - m$, and with $a_R > i_L + m$ we get that $a' < a_R - 2m$, obtaining a contradiction.

Finally, lets assume that $|E_R| \geq (k + 1)^2$, and take $(a_L, b_L) \in E_L$ such that for all $(a', b') \in E_L$, $a_L \geq a'$ and $a_L = a' \Rightarrow b_L \geq b'$. Similarly as in the previous paragraph we can get $(a_R, b_R) \in E_R$ such that $a_R - i_L > k$, thus $a_R - a_L > k$. From the definition of E_L we also know that $b_L > k$, and with iv. from the theorem 3.7 we get that $(a_L + m, b_L - m) \in C'$. Since (a_L, b_L) is maximal, $(a_L + m, b_L - m) \notin E_L$. But $(a_L + m, b_L - m)$ cannot be an element of E_R, since for all $(a', b') \in E_R$, $a' > i_L + m$, and with $a_L < i_L - m$ we get that $a' > a_L + 2m$, obtaining a contradiction.

We have discussed all the cases and obtaining contradictions in each of them, therefore I has to be an empty set, proving our theorem. □

The two corollaries of this are lower and upper bounds on $|\mathcal{C}|$ given by terms of $||\mathcal{C}||$, and vice versa.

Corollary 3.8.1. *Let $\langle A, \sim \rangle$ be an equivalence structure with an injective automatic representation π over 0^*1^*. Then there exist constants k, m such that for all finite equivalence classes $\mathcal{C} \in A/\sim$ with $|\mathcal{C}| > 4(k + 1)^2$:*

$$\frac{||\mathcal{C}|| - 2k - 1}{2m} \leq |\mathcal{C}| \leq (k + 1) \cdot ||\mathcal{C}|| + \frac{(k + 2) \cdot (k + 1)}{2}$$

Proof. Choose k, m according to the lemma 3.8. From the lemma 3.8 we get that there have to be at least $\frac{||\mathcal{C}|| - 2k - 1}{2m}$ elements, and from the corollary 3.5.1 we get the upper bound. □

Corollary 3.8.2. *Let $\langle A, \sim \rangle$ be an equivalence structure with an injective automatic representation π over 0^*1^*. Then there exist constants k, m such that for all finite equivalence classes $\mathcal{C} \in A/\sim$ with $|\mathcal{C}| > 4(k+1)^2$:*

$$\frac{|\mathcal{C}|}{k+1} - \frac{k}{2} - 1 \leq ||\mathcal{C}|| \leq 2m|\mathcal{C}| + 2k + 1$$

Proof. Trivial consequence of 3.8.1. □

Now we will show that "*big enough*" classes have some internal structure. The structure is given as solutions to a finite number of equations of the following structure (for a fixed m):

$$\begin{aligned} x_0, x_1 >& \ k \\ x_0 + x_1 =& \ s \\ x_0 \equiv_m& \ c \end{aligned}$$

You can see the visual intuition of this lemma in the figure 11.

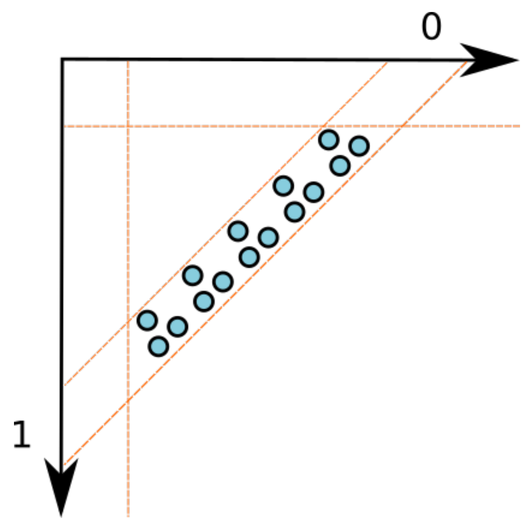


Figure 11: The intuition behind the lemma 3.9

Lemma 3.9. *Let $\langle A, \sim \rangle$ be an equivalence structure without infinite classes. Then there exist constants k, m such that for all $\mathcal{C} \in A/\sim$ with $|\mathcal{C}| > 13(k+1)^2$, there exist sets of constants $P_0, \ldots, P_k \subseteq \{0, \ldots, m-1\}$ such that:*

$$\forall (a,b) \in \mathbb{N}^2 . \forall i \in \{0, \ldots, k\} . ((a > k \wedge b > k \wedge a + b = ||\mathcal{C}|| + i)$$
$$\Rightarrow ((\exists p \in P_i . a \equiv_m p) \Leftrightarrow (a,b) \in \pi^{-1}(\mathcal{C})))$$

Proof. Choose k, m according to the lemma 3.8, from the theorem 3.8.2 (and that $m \leq k$, $k \geq 1$) we get that:

$$4m + 5k + 2 \leq 12k - 1 = \frac{13(k+1)k}{k+1} - k - 1 \leq \frac{13(k+1)^2}{k+1} - \frac{k}{2} - 1 \leq ||\mathcal{C}||.$$

Let $\mathcal{C}$ be an equivalence class satisfying the assumptions, let $i \in \{0, \ldots, k\}$ and let $C'_i := \{(a,b) \in \pi^{-1}(\mathcal{C}) \mid a > k \wedge b > k \wedge a + b = ||\mathcal{C}|| + i\}$. Lets choose $P_i := \{a'$

mod $m \mid (a', b') \in C'_i\}$. It is trivial that for all $(a, b) \in C'_i$ there exists according $p \in P_i$. Note that P_i is empty if and only if C'_i is empty.

First, if P_i is empty, C'_i is empty, and the conclusion of our implication trivially holds.

Now let $p \in P_i$, and let $C_p := \{(a, b) \in C'_i \mid a \equiv_m p\}$ and $R_p := \{(a, b) \in \mathbb{N}^2 \mid a > k \wedge b > k \wedge a + b = ||\mathcal{C}|| + i \wedge a \equiv_m p\} \setminus C_p$. In the rest of the proof we will assume that any a_X, b_X will be greater than k since our lemma only argues about elements with this property. To obtain contradiction, assume that R_p is not empty. We know that there must exists $(a_E, b_E) \in R_p$ such that there exists $(a_C, b_C) \in C_p$ with $|a_C - a_E| = m$ (otherwise C_p would be empty).

First assume that $a_E + m = a_C$. If $a_C \leq b_C$, choose $(a_R, b_R) \in C'_i$ such that $a_R > ||\mathcal{C}|| - k - 2m$ (existence of such element is guaranteed by the lemma 3.8). Since $a_R - a_C > k$ (from $a_R - a_C \geq a_R - \frac{||\mathcal{C}||+i+1}{2} > \frac{||\mathcal{C}||-i-1}{2} - k - 2m \geq \frac{||\mathcal{C}||-k-1}{2} - k - 2m > 2m + 2k - k - 2m = k$), we can use iv. from the theorem 3.7 to get that $(a_C - m, b_C + m) = (a_E, b_E) \in C'_i$, therefore $(a_E, b_E) \in C_p$ obtaining contradiction.

If $a_C > b_C$, choose $(a_L, b_L) \in C'_i$ such that $a_L < k + 2m$ (existence of such element is guaranteed by the lemma 3.8). Since $a_C - a_L > k$ (from similar counting argument as in previous case), we can use vi. from the theorem 3.7 to get that $(a_C - m, b_C + m) = (a_E, b_E) \in C'_i$, therefore $(a_E, b_E) \in C_p$ obtaining contradiction.

Now assume that $a_E - m = a_C$. If $a_C \leq b_C$, choose $(a_R, b_R) \in C'_i$ such that $a_R > ||\mathcal{C}|| - k - 2m$ (existence of such element is guaranteed by the lemma 3.8). Since $a_R - a_C > k$ (again from the same counting argument), we can use iii. from the theorem 3.7 to get that $(a_C + m, b_C - m) = (a_E, b_E) \in C'_i$, therefore $(a_E, b_E) \in C_p$ obtaining contradiction.

Finally, if $a_C > b_C$, choose $(a_L, b_L) \in C'_i$ such that $a_L < k + 2m$ (existence of such element is guaranteed by the lemma 3.8). Since $a_C - a_L > k$ (from similar counting argument as in previous case), we can use v. from the theorem 3.7 to get that $(a_C + m, b_C - m) = (a_E, b_E) \in C'_i$, therefore $(a_E, b_E) \in C_p$ obtaining a contradiction.

We have discussed all possibilities, obtaining contradictions, therefore R_p has to be empty, proving our lemma. $\square$

A simple consequence of this lemma is that if two classes have two elements with the same length and same residual after dividing the first component by m, these two classes are the same.

Corollary 3.9.1. *Let $\langle A, \sim\rangle$ be an equivalence structure without infinite classes. Then there exist constants k, m such that for all $\mathcal{C}, \mathcal{C}' \in \mathcal{A}/\sim$ with $|\mathcal{C}| > 13(k + 1)^2$ and $\mathcal{C}'$ of arbitrary size:*

$$[\exists(a, b) \in \pi^{-1}(\mathcal{C}), (a', b') \in \pi^{-1}(\mathcal{C}').(a, a', b, b' > k \wedge a + b = a' + b' \wedge a \equiv_m a')] \Rightarrow \mathcal{C} = \mathcal{C}'$$

Proof. From the lemma 3.9 we get that $(a, b) \sim (a'b')$, by transitivity we get $\mathcal{C} = \mathcal{C}'$. $\square$

For a given class $\mathcal{C}$ and $i \in \{0, \ldots, k\}$, we will denote the set of constants from the lemma 3.9 by $P_{\mathcal{C},i}$ and we will set $\overline{P}_{\mathcal{C}} := (P_{\mathcal{C},0}, P_{\mathcal{C},1}, \ldots, P_{\mathcal{C},k})$. For a given class $\mathcal{C}$, let $\mathcal{C}_M := \{(a, b) \in \mathcal{C} \mid a > k \wedge b > k\}$. For a given length l, we will denote the set of all classes $\mathcal{C}$ with $||\mathcal{C}_M|| = l$ by $\mathbb{C}_l$.

Now we will show there is a unique correspondence between equivalence classes according to their sets $P_{\mathcal{C},i}$ and length $||\mathcal{C}_M||$ of the shortest element with $a, b > k$. You can see the visual intuition of this lemma in the figure 12.

Lemma 3.10. *Let $\langle A, \sim\rangle$ be an equivalence structure without infinite classes. Then there exist constants k, m such that for all $i \in \mathbb{N}$, and all $\overline{P} \in \mathcal{P}(\{0, \ldots, m - 1\})^{k+1}$:*

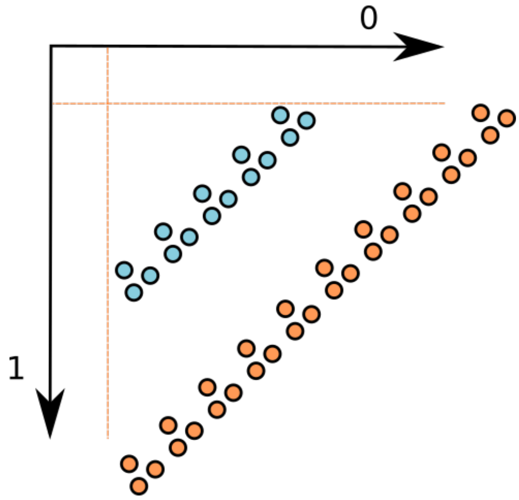


Figure 12: The intuition behind the lemma 3.10

$$(\exists \mathcal{C} \in \mathbb{C}_i.|C| > 13(k+1)^2 \wedge \overline{P}_{\mathcal{C}} = \overline{P}) \Rightarrow \exists \mathcal{C}' \in \mathcal{C}_{i+m}.\overline{P}_{\mathcal{C}'} = \overline{P} \tag{1}$$

and

$$(\exists \mathcal{C}' \in \mathbb{C}_{i+m}.|\mathcal{C}'| > 13(k+1)^2 \wedge \overline{P}_{\mathcal{C}'} = \overline{P}) \Rightarrow \exists \mathcal{C} \in \mathbb{C}_i.\overline{P}_{\mathcal{C}} = \overline{P} \tag{2}$$

Proof. Choose k, m according to the lemma 3.9, let $i \in \mathbb{N}$, and let $P_l \subseteq \{0, \dots, m-1\}$. First we will prove (1). Let's assume that there exists $\mathcal{C} \in \mathbb{C}_i$ with $|\mathcal{C}| > 13(k+1)^2$ and $P_{\mathcal{C},l} = P_l$ for each l. Fix some l, and for each $p \in P_l$, choose $(m \cdot a_p + p, b_p) \in \mathcal{C}$ such that $k < m \cdot a_p + p < k + 2m$ and $m \cdot a_p + p, b_p = i + l$, and $(a_c, b_c) \in \mathcal{C}$ such that $k + 2m < a_c$ (we know these exist, because $|\mathcal{C}| > 2k + 4m$). For each $p \in P$, we can use i. from the theorem 3.7 to get that $(m \cdot a_p + p + m, b_p) = (m \cdot (a_p + 1) + p, b_p) \sim (a_c + m, b_c)$. Lets call this equivalence class $\mathcal{C}'$.

If $\mathcal{C}' \in \mathbb{C}_j$ for some $j < i + m$, there would exist an element $(a_X, b_X) \in \mathcal{C}'$ with $a_X + b_X = j$ and $a_X > k, b_X > k$. By the lemma 3.9 we can then find $(a'_X, b'_X) \in \mathcal{C}'$ with $a'_X + b'_X = j$ and $a'_X > k + m, b'_X > k + m$ (and with $a_X \equiv_m a'_X$). Then we can find an element $(a_Y, b_Y) \in \mathcal{C}'$ such that $a_Y > k + m$ and $(a_Y - m, b_Y) \in \mathcal{C}$. Then we can use i. from the theorem 3.7 to get that $(a'_X - m, b'_X) \in \mathcal{C}$. Since $a'_X - m + b'_X = j - m < i$, and $a'_X - m > k, b'_X > k$, this would mean that $\mathcal{C} \notin \mathbb{C}_i$, getting contradiction. Moreover, since $\mathcal{C} \in \mathbb{C}_i$, $P_{C,i}$ is not empty and therefore there exists $(a, b) \in \mathcal{C}'$ with $a + b = i + m$, obtaining that $\mathcal{C}' \in \mathbb{C}_{i+m}$.

Now with the lemma 3.9 we can see that $P_{\mathcal{C},l} \subseteq P_{\mathcal{C}',l}$ for all l.

Now let $p' \in P_{\mathcal{C}',l}$ for some l. There exist $(m \cdot a' + p', b') \in \mathcal{C}'$ such that $k + m < m \cdot a' + p' < k + 2m$. Since $(m \cdot a' + p', b') \sim (a_c + m, b_c)$, we can again use i. from the theorem 3.7 to get that $(m \cdot a' + p' - m, b') = (m \cdot (a'_p - 1) + p) \sim (a_c, b_c)$. With the lemma 3.9 we get that $P_{\mathcal{C}',l} \subseteq P_{\mathcal{C},l}$, obtaining $P_{\mathcal{C}',l} = P_{\mathcal{C},l}$ for all l.

The proof for (2) is the same, because of the size we can always find elements $(a, b) \in \mathcal{C}'$ with $a > k + m, b > k + m$ for any given length and residual class. □

From the corollary 3.9.1 we know that the corresponding $\mathcal{C}'$ is unique. We will denote it by $\alpha(\mathcal{C})$. Also for every "*big enough*" $\mathcal{C}'$, there is exactly one $\mathcal{C}$ such that $\alpha(\mathcal{C}) = \mathcal{C}'$.

The last thing we need is a relation between $|\mathcal{C}|$ and $|\alpha(\mathcal{C})|$. We can easily show that $|\mathcal{C} \setminus \mathcal{C}_M| = |\alpha(\mathcal{C}) \setminus \alpha(\mathcal{C})_M|$, so we will only have to determine the size of $\alpha(\mathcal{C})_M$. This will be achieved by a simple combinatorial calculation.

Corollary 3.10.1. *Let $\langle A, \sim\rangle$ be an equivalence structure without infinite classes. Then there exist constants k, m such that for all $\mathcal{C} \in A/\sim$ with $|\mathcal{C}| > 13(k+1)^2$:*

$$\forall n \in \mathbb{N}. |\alpha^n(\mathcal{C})| = |\mathcal{C}| + n \cdot \sum_{j=0}^{k} |P_{\mathcal{C},j}|$$

Proof. For any equivalence class $\mathcal{C}$, we will use $\mathcal{C}_{R,j} = \{(a,b) \in \mathcal{C} \mid a \geq ||\mathcal{C}|| - k \wedge a + b = ||\mathcal{C}|| + j\}$, $\mathcal{C}_{L,j} = \{(a,b) \in \mathcal{C} \mid a \leq k \wedge a + b = ||\mathcal{C}|| + j\}$, and $\mathcal{C}_{M,j} = \mathcal{C}\{(a,b) \in \mathcal{C} \mid k < a < ||\mathcal{C}|| - k \wedge a + b = ||\mathcal{C}|| + j\}$. Note that $\mathcal{C} = \bigcup_{j=0}^{k} \mathcal{C}_{R,j} \cup \mathcal{C}_{M,j} \cup \mathcal{C}_{L,j}$.

Choose k, m according to the lemma 3.9, let $\mathcal{C} \in \mathbb{C}_i$ with $|\mathcal{C}| > 13(k+1)^2$, let $j \in \{0, \ldots, k\}$, and let $n \in \mathbb{N}$. From the lemma 3.8 we know there exist $(a_L, b_L), (a_R, b_R) \in \mathcal{C}_{M,j}$ such that $a_L < k + 2m$ and $a_R > ||\mathcal{C}|| - k - 2m$. For every $(a', b') \in \mathcal{C}_{L,j}$ we apply ii. from the theorem 3.7 with (a_R, b_R) n times to get that $(a', b') \sim (a_R, b_R) \Leftrightarrow (a', b' + nm) \sim (a_R + nm, b_R)$, obtaining $|\mathcal{C}_{L,j}| = |\alpha(\mathcal{C})_{L,j}|$. Similarly for every $(a', b') \in \mathcal{C}_{R,j}$ we use ii. from the theorem 3.7 with (a_L, b_L) n times to get that $(a', b') \sim (a_L, b_L) \Leftrightarrow (a' + nm, b') \sim (a_L, b_L + nm)$, obtaining $|\mathcal{C}_{R,j}| = |\alpha(\mathcal{C})_{R,j}|$.

Last, we can use i. from the theorem 3.7 to get that for every $(a', b') \in \mathcal{C}_{M,j}$, $(a' + nm, b' + nm) \sim (a_R + nm, b_R + nm)$, so we get that $|\alpha^n(\mathcal{C})_{M,j}| = |\mathcal{C}_{M,j}| + |\{(a,b) \in \alpha^n(\mathcal{C})_{M,j} \mid a \leq k + mn\}|$

Since $P_{\mathcal{C},j} = P_{\alpha^n(\mathcal{C}),j}$, and since we know that for every $p \in P_{\mathcal{C},j}$ there are n elements from $\{(a,b) \in \alpha^n(\mathcal{C})_{M,j} \mid a \leq k + mn\}$ with $a \equiv_m p$, we get that $|\{(a,b) \in \alpha^n(\mathcal{C})_{M,j} \mid a \leq k + mn\}| = n \cdot |P_{\mathcal{C},j}|$.

With this, we get that for each j, $|\alpha^n(\mathcal{C})_{R,j} \cup \alpha^n(\mathcal{C})_{M,j} \cup \alpha^n(\mathcal{C})_{L,j}| = |\mathcal{C}_{R,j} \cup \mathcal{C}_{M,j} \cup \mathcal{C}_{L,j}| + n \cdot |P_{\mathcal{C},j}|$. By summing these we get that:

$$|\alpha^n(\mathcal{C})| = |\mathcal{C}| + n \cdot \sum_{j=0}^{k} |P_{\mathcal{C},j}|$$

□

Finally, we have all the tools necessary to prove the classification theorem.

Theorem 3.11 (The classification of equivalence structures over 0^*1^*)**.** *Let $\langle A, \sim\rangle$ be an equivalence structure. Then the following statements are equivalent:*

i. $\langle A, \sim\rangle$ has an injective automatic representation π over 0^*1^*

ii. There exist a constant s, and a finite index set I and for each $i \in I$, a pair $(c_i, p_i) \in \mathbb{N}^2$ such that there exists bijection f between $I \times \mathbb{N}$ and all finite equivalence classes $\mathcal{C}$ with $|\mathcal{C}| > s$. Additionally, it holds that $|f(i,n)| = c_i + n \cdot p_i$.

Proof. Let $\langle A, \sim\rangle$ be an equivalence structure. First we will prove "⇒". Assume that $\langle A, \sim\rangle$ has an injective automatic representation π over 0^*1^*. By the theorem 3.1 there exist $\langle A_{\text{fin}}, \sim_{\text{fin}}\rangle$ and $\langle A_{\text{inf}}, \sim_{\text{inf}}\rangle$ with injective automatic presentations $\pi_{\text{fin}}, \pi_{\text{inf}}$ over Σ^* such that $\langle A_{\text{fin}}, \sim_{\text{fin}}\rangle \cup \langle A_{\text{inf}}, \sim_{\text{inf}}\rangle \cong \langle A, \sim\rangle$ and such that $\langle A_{\text{fin}}, \sim_{\text{fin}}\rangle$ has only finite equivalence classes, and $\langle A_{\text{inf}}, \sim_{\text{inf}}\rangle$ has only infinite equivalence classes. From now we will only consider $\langle A', \sim'\rangle := \langle A_{\text{fin}}, \sim_{\text{fin}}\rangle$.

Let k, m be constants from the previous theorems for $\langle A', \sim'\rangle$, choose $s := 13(k+1)^2$, and let $\mathbb{C} := \{\mathcal{C} \in A'/\sim' \mid |\mathcal{C}| > 13(k+1)^2\}$. Now let $\mathbb{B} := \mathbb{C} \setminus \alpha(\mathbb{C})$. In the lemma 3.10 we have shown that for every $\mathcal{C} \in \mathbb{C}$, there exists $\mathcal{C}' \in A'/\sim'$ such that $\mathcal{C} = \alpha(\mathcal{C}')$. With the fact that if $\mathcal{C} \in \mathbb{C}_i$ then $\alpha(\mathcal{C}) \in \mathbb{C}_{i+m}$ for some fixed m, we get that $\mathbb{B}$ is finite. Moreover we have shown that α is injective, therefore function $f : \mathbb{B} \times \mathbb{N} \to \mathbb{C}$ given by $f(\mathcal{B}, n) := \alpha^n(\mathcal{B})$ is a bijection.

For each $\mathcal{B} \in \mathbb{B}$, let $p_{\mathcal{B}} := \sum_{j=0}^{k} |P_{\mathcal{B},j}|$ and $c_b := |\mathcal{B}|$. From the corollary 3.10.1 we get that $|f(\mathcal{B}, n)| = |\alpha^n(\mathcal{B})| = |\mathcal{B}| + n \cdot \sum_{j=0}^{k} |P_{\mathcal{B},j}| = c_b + n \cdot p_b$, which we wanted to show.

For "$\Leftarrow$", we will first decompose $\langle A, \sim\rangle$ into $\langle A_{\text{fin}}, \sim_{\text{fin}}\rangle \cup \langle A_{\text{inf}}, \sim_{\text{inf}}\rangle$, where A_{fin} contains all the finite equivalence classes, and A_{inf} contains all the infinite classes.

If $A_{\text{inf}} = \emptyset$, then $\langle A_{\text{fin}}, \sim_{\text{fin}}\rangle \cong \langle A, \sim\rangle$. If $|A_{\text{inf}}/\sim_{\text{inf}}| = n \in \mathbb{N}$ then $\langle A_{\text{inf}}, \sim_{\text{inf}}\rangle \cong \langle \mathbb{N}, \equiv_n\rangle$, which has automatic representation over 0^*1^* by the example (b) from the section 2.4. If $|A_{\text{inf}}/\sim_{\text{inf}}| = \omega$, then $\langle A_{\text{inf}}, \sim_{\text{inf}}\rangle$ is isomorphic to the structure from the example (c) in the section 2.5, therefore has automatic representation over 0^*1^*.

Let $s \in \mathbb{N}$. Now we will decompose $\langle A_{\text{fin}}, \sim_{\text{fin}}\rangle$ into $\langle A_{\text{const}}, \sim_{\text{const}}\rangle \cup \langle A', \sim'\rangle$, where $\langle A_{\text{const}}, \sim_{\text{const}}\rangle$ contains all the equivalence classes smaller or equal to s. From the theorem 2.12 we know that $\langle A_{\text{const}}, \sim_{\text{const}}\rangle$ has automatic representation over 0^*, therefore it has automatic representation over 0^*1^*.

Now let I be finite, and for each $i \in I$ let $(c_i, p_i) \in \mathbb{N}^2$. Finally let f be a bijection between $I \times \mathbb{N}$ and all finite equivalence classes $\mathcal{C}$ with $|\mathcal{C}| > s$ with $|f(i, n)| = c_i + n \cdot p_i$. We will decompose $\langle A', \sim'\rangle = \bigcup_{i \in I} \langle A_i, \sim_i\rangle$, where $\langle A_i, \sim_i\rangle$ contains only classes in $\text{im}(f(i, -))$. Let $i \in I$, we will construct an equivalence relation $\langle B_i, \equiv_i\rangle$ such that $\langle A_i, \sim_i\rangle \cong \langle B_i, \equiv_i\rangle$.

Since all equivalence classes have size bigger than some s, $c > 0$. Now let $B_n := \{(a, b) \in \mathbb{N}^2 \mid a + b = n \cdot p_i + c_i - 1\}$ for all $n \in \mathbb{N}$. It is easy to see that $|B_n| = n \cdot p_i + c_i$, and consider $B_i = \bigcup_{n \in \mathbb{N}} B_n$, and $(a, b) \equiv_i (a', b') := a + b = a' + b'$. We will show that $\pi_i(0^a 1^b) := (a, b)$ is automatic presentation:

- π is trivially surjective
- $D := \text{dom}(\pi) = \{0^a 1^b \mid \exists n \in \mathbb{N}_0 . a + b = n \cdot p_i + c_i - 1\} = \{0^a 1^b \mid a + b \equiv_{p_i} c_i - 1\}$, which is easy to see that it is regular.
- π is injective, therefore $\{(x, y)^{\otimes} \mid \pi(x) = \pi(y)\}$ is trivially regular
- $\{(x, y)^{\otimes} \mid \pi(x) \equiv \pi(y)\} = \{(x, y)^{\otimes} \mid \pi(x) \equiv_i \pi(y)\} = \{(x, y)^{\otimes} \mid x, y \in D, |x| = |y|\}$, therefore it is trivially regular since D is regular.

Since $\langle A, \sim\rangle = \langle A_{\text{inf}}, \sim_{\text{inf}}\rangle \cup \langle A_{\text{const}}, \sim_{\text{const}}\rangle \cup \bigcup_{i \in I} \langle B_i, \equiv_i\rangle$, and I is finite, from the theorem 2.19 we get that there also exists automatic presentation over 0^*1^* for $\langle A, \sim\rangle$, which we wanted to show.

□

3.3 Notes on the general case

We think our approach could be used to solve the general case. First we will define a class of functions we assume would arise in the proof of the general case. We will give the explanation of our intuition in the next section.

For every $M, m > 0$, let $F_{M,m} : \mathbb{N} \to \mathbb{N}$ given by:

$$F_{M,m}(x) = \begin{cases} \binom{\frac{x}{M} + m}{m} & \text{if } M \mid x \text{ and } \frac{x}{M} \geq m \\ 0 & \text{else} \end{cases}$$

Let $\mathbb{F} := \{F_{M,m} \mid M, m > 0\}$ and let $\mathbb{P}$ be a closure of $\mathbb{F}$ under addition of functions and under substitution of x by $x - n$ for all $n \in \mathbb{N}$.

We would like to prove the following conjecture:

Conjecture 3.12. *Let $\langle A, \sim\rangle$ be an equivalence structure. Then the following statements are equivalent:*

i. $\langle A, \sim \rangle$ *is poly-growth automatic*

ii. There exist a constant s, and a finite index set I and for each $i \in I$, a function $p_i \in \mathbb{P}$ such that there exists bijection f between $I \times \mathbb{N}$ and all finite equivalence classes $\mathcal{C}$ with $|\mathcal{C}| > s$. Additionally, it holds that $|f(i,n)| = p_i(n)$.

We will discuss how the proof for the 0^*1^* case can be modified to prove the general case, and then we will discuss that *ii* implies *i*.

3.3.1 Modification of the proof

Let $\langle A, \sim \rangle$ be an automatic equivalence structure over $0^*1^* \dots (n-1)^*$ with only finite equivalence classes. We have shown in the theorem 3.5 that all equivalence classes lie in k consecutive layers, where each layer corresponds to the elements of the same length. Each of these layers (if the length is "*big enough*") can be decomposed into 2^m distinct regions R_φ, where $\varphi(\overline{a}) := \bigwedge \psi_i(a_i)$ and $\psi_i(x) \in \{x > k, x \leq k\}$.

If we are able to find pumping arguments similar to the theorem 3.7, then if a region R_φ has "*enough elements*" from the same class $\mathcal{C}$, we can show that the elements of $\mathcal{C}$ are distributed over the region (similar to 3.8). Then we are able to show that an element $\overline{a} \in R_\varphi$ is in $\mathcal{C}$ if and only if it is a solution to one of the following systems of equations (similar to 3.9):

$$\begin{aligned}
\sum_{i=0}^{l} x_i &= \quad s \\
x_0 &\equiv_m \quad c_0 \\
x_1 &\equiv_m \quad c_1 \\
&\vdots \\
x_{l-1} &\equiv_m c_{l-1}
\end{aligned}$$

This equation has an equivalent number of solutions as the following one (for some K):

$$\sum_{i=0}^{l} x_i = \frac{s-K}{m}$$

The number of solutions of this equation is $\binom{\frac{s-K}{m}+l}{l}$.

Finally to finish the proof for the general case, we have to find a similar correspondence α of "*big enough*" equivalence classes (similar to the corollary 3.10.1), which would finish our proof.

3.3.2 Equivalence structures definable by "combinatorial polynomials"

We give a sketch of the proof of the easy direction of the conjecture 3.12. First, we will show that for each $p \in \mathbb{P}$, there exists a poly-growth structure $\langle A_p, \sim_p \rangle$ such that sizes of the equivalence class C_i is equal to $p(i)$ for $i \in \mathbb{N}$. Then we will show that if an automatic structure $\langle A, \sim \rangle$ fulfills *ii* from the conjecture 3.12, it is isomorphic to a union of structures $\langle A_{\text{const}}, \sim_{\text{const}} \rangle$ (where all equivalence classes have size smaller than some constant), $\langle A_{\text{inf}}, \sim_{\text{inf}} \rangle$ (where all equivalence classes are infinite), and $\langle A_p, \sim_p \rangle$ for finitely many $p \in \mathbb{P}$.

Definition 3.13. Let $f : \mathbb{N} \to \mathbb{N}$. A *canonical structure of* f is the structure $\langle A_f, \sim_f \rangle$ such that if there exists an enumeration of $A_f / \sim_f \cup \{\emptyset\}$ such that $|C_i| = f(i)$ for all $i \in \mathbb{N}$, and for $i \neq j$, $C_i = C_j$ implies $C_i = \emptyset$.

Note that the canonical structure is not unique, but it is unique up to isomorphism.

Clearly, if $\langle A_p, \sim_p \rangle$ is poly-growth automatic, then if we obtain $\langle A, \sim \rangle$ from $\langle A_p, \sim_p \rangle$ by adding or removing equivalence classes smaller than some constant s, $\langle A, \sim \rangle$ will again be poly-growth automatic. This implies that if we prove that for all $p \in \mathbb{P}$ their canonical structure $\langle A_p, \sim_p \rangle$ is poly-growth automatic, we will prove that *ii* implies *i* from the conjecture 3.12.

Now we will show that for all $p \in \mathbb{P}$, $\langle A_p, \sim_p \rangle$ is poly-growth automatic. First we will show this for the base case $p \in \mathbb{F}$, then for the closure under the substitution of x by $x - n$, and last for the closure under the addition of functions.

Lemma 3.14. *Let $m, M > 0$, and let $F : \mathbb{N} \to \mathbb{N}$ given by:*

$$F(x) = \begin{cases} \binom{\frac{x}{M}+m-1}{m-1} & \text{if } M \mid x \\ 0 & \text{if } M \nmid x \end{cases}$$

Then there exists an automatic canonical structure $\langle A_F, \sim_F \rangle$ with $A_F \subseteq 0^* \dots (m-1)^*$ such that $\sim_F$ is realised as a restriction of $=_{\text{len}}$, and for the enumeration from the definition 3.13 it holds that elements from C_i have length i.

Sketch of proof: Choose $\langle (0^M)^*(1^M)^* \dots ((m-1)^M)^*, =_{\text{len}} \rangle$

Lemma 3.15. *Let $n > 0$, $f : \mathbb{N} \to \mathbb{N}$ with automatic canonical structure $\langle A_f, =_{\text{len}} \rangle$ of F, where $A_f \subseteq 0^*1^* \dots (m-1)^*$ for some m, and assume there exists $c \in \mathbb{N}$ such that for the canonical enumeration holds that elements of C_i have length $i + c$.*

Then there exists an automatic canonical structure $\langle A_{f(x-n)}, \sim \rangle$ of $f(x-n)$ with $A_{f(x-n)} \subseteq 0^* \dots (m-1)^*$ such that $\sim$ is realised as a restriction of $=_{\text{len}}$, and for the enumeration from the definition 3.13 it holds that elements from C_i have length $i + c$.

Sketch of proof: Choose $\langle 0^n \cdot A, =_{\text{len}} \rangle$.

Lemma 3.16. *Let $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$ with automatic canonical structures $\langle A_f, =_{\text{len}} \rangle$, $\langle A_g, =_{\text{len}} \rangle$, where $A_f, A_g \subseteq 0^*1^* \dots (m-1)^*$ for some m, and assume there exists $c_f, c_g \in \mathbb{N}$ such that for the canonical enumerations holds that elements of $C_i \in A_f/_{\text{len}}$ have length $i + c_f$, and elements of $C_i \in A_f/_{\text{len}}$ have length $i + c_g$.*

Then there exists an automatic canonical structure $\langle A, \sim \rangle$ of $f(x) + g(x)$ with $A \subseteq 0^* \dots (m-1)^*$ such that $\sim$ is realised as a restriction of $=_{\text{len}}$, and there exists c such that for the enumeration from the definition 3.13 it holds that elements from C_i have length $i + c$.

Sketch of proof: Let $\varphi : 0^*1^* \dots (m-1)^* \to 0^*1^* \dots (m+1)^*$ be a homomorphism given by $\varphi(x) = x + 2$, and $A'_f = \varphi(A_f)$, $A'_g = \varphi(A_g)$. Without loss of generality assume $c_f \geq c_g$ and let $c' = c_f - c_g$. Then choose $\langle 00 \cdot A'_f + 011^{c'} \cdot A'_g, =_{\text{len}} \rangle$.

Corollary 3.16.1. *For every $p \in \mathbb{P}$ there exist a poly-growth automatic structure $\langle A, \sim \rangle$, and an enumeration of equivalence classes C_i of A such that $p(n) = |C_n|$ for all $n \in \mathbb{N}$.*

Corollary 3.16.2. *Let $\langle A, \sim \rangle$ be an equivalence structure. If there exist a constant s, and a finite index set I and for each $i \in I$, a function $p_i \in \mathbb{P}$ such that there exists bijection f between $I \times \mathbb{N}$ and all finite equivalence classes $\mathcal{C}$ with $|\mathcal{C}| > s$ with $|f(i, n)| = p_i(n)$, then $\langle A, \sim \rangle$ is poly-growth automatic.*

Sketch of proof: First we will decompose $\langle A, \sim \rangle$ into $\langle A_{\mathrm{inf}}, \sim_{\mathrm{inf}} \rangle \cup \langle A_{\mathrm{const}}, \sim_{\mathrm{const}} \rangle \cup \bigcup_{i \in I} \langle A_{p_i}, \sim_{p_i} \rangle$, where:

- $\langle A_{\mathrm{const}}, \sim_{\mathrm{const}} \rangle$ contains all the equivalence classes with size smaller or equal to s
- $\langle A_{\mathrm{inf}}, \sim_{\mathrm{inf}} \rangle$ contains all the equivalence classes with infinite size
- $\langle A_{p_i}, \sim_{p_i} \rangle$ contains the equivalence classes from the image of $f(i, -)$

From the proof of the theorem 3.11 we know that $\langle A_{\mathrm{const}}, \sim_{\mathrm{const}} \rangle$ and $\langle A_{\mathrm{inf}}, \sim_{\mathrm{inf}} \rangle$ are poly-growth automatic. From the previous corollary we get that for all p_i there exists according poly-growth automatic structure that is isomorphic to $\langle A_{p_i}, \sim_{p_i} \rangle$. Since poly-growth automatic structures are closed under finite unions, $\langle A, \sim \rangle$ is poly-growth automatic.

4 Conclusion

We have defined the notion of automatic structures, unary automatic structures and automatic structures with polynomial growth. We have defined multi-dimensional interpretations, and used this to give an equivalent definition of automatic structures, unary automatic structures and automatic structures with polynomial growth. We have presented known results about automatic structures with polynomial growth, the most important one stating that structure is poly-growth automatic if and only if it has an automatic presentation over $a_0^* a_1^* \dots a_{n-1}^*$ for some n and a_i's.

We have shown that automatic structures with a presentation over $a_0^* a_1^* \dots a_{n-1}^*$ for some fixed n are closed under finite unions. We have then presented the lemmas necessary to prove the complete classification of equivalence structures with an automatic presentation over 0^*1^*.

Lastly we have presented a possible generalisation of our proof to obtain the complete classification of equivalence structures with an automatic presentation over $0^*1^* \dots (n-1)^*$ for arbitrary n, therefore a complete classification of poly-growth automatic equivalence structures.

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