Locality and modular Ehrenfeucht-Fraïssé games

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12th March 2009

We study ways to simplify Ehrenfeucht-Fraïssé games. In particular, we consider decompositions of a structure and their effect on Ehrenfeucht-Fraïssé games. We investigate notions of locality and we present a generalisation of the theorem of Gaifman to linearly ordered structures.

Keywords. Ehrenfeucht-Fraïssé games, locality, first-order logic

1 Introduction

Ehrenfeucht-Fraïssé games provide a versatile tool for the investigation of logics and their expressive power. When compared to other methods these games have the advantage that they can easily be adapted to many different logics and that they also work well in the context of finite structures. Unfortunately, in nontrivial applications the complexity of playing these games quickly becomes unmanageable. Therefore, it was suggested by Fagin, Stockmeyer, and Vardi [8] to create a library of tools that can be used to simplify Ehrenfeucht-Fraïssé games.

For first-order logic, progress in this direction has been made by the theorems of Hanf [10] and Gaifman [9], and by the more recent results of Libkin et.al. [11, 1]. Games for existential monadic second-order logic were investigated by Fagin et.al. and Schwentick [8, 13, 7, 2].

A shortcoming of most of these results is that they can only be used on sparse structures, i.e., structures where the relations contain few tuples (a notable exception being [13]). The reason for this is the notion of locality the statements are based on. In non-sparse structures all elements are in the vicinity of each other. Hence, if we are interested in results for non-sparse structures we have to adopt a different notion of locality.

In this paper we look at notions of locality and their effect on Ehrenfeucht-Fraïssé games. We try to isolate special cases where games can be simplified. Besides sparse structures such cases turn out to be structures with a hierarchical decomposition and linearly ordered structures. In the first part we present several simple ideas to simplify games on non-sparse structures. The second part consists of a generalisation of the theorem of Gaifman that also gives meaningful results for certain structures that are non-sparse.

2 Preliminaries

Let us recall some basic definitions and fix our notation. Let $[n] := \{0, \dots, n-1\}$. We tacitly identify tuples $\bar{a} = a_0 \dots a_{n-1} \in A^n$ with functions $[n] \to A$ and frequently we write $\bar{a}$ for the set $\{a_0, \dots, a_{n-1}\}$. This allows us to write $\bar{a} \subseteq \bar{b}$ or $\bar{a} = \bar{b}|_I$ for $I \subseteq [n]$. We denote the empty tuple by $\langle \rangle$.

We assume that the reader is familiar with basic concepts of first-order logic (see e.g. [6] for definitions and notation). We will only consider purely relational structures and we work with *infinitary* first-order logic throughout. The relation defined by a formula φ in a structure $\mathfrak{A}$ is denoted by $\varphi^{\mathfrak{A}}$.

Two Σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *m-equivalent*, in symbols $\mathfrak{A} \equiv_m \mathfrak{B}$, if they satisfy the same infinitary first-order sentences of quantifier rank at most m. We denote the quantifier rank of φ by $\mathrm{qr}(\varphi)$.

The m round *Ehrenfeucht-Fraïssé game* $\mathrm{EF}_m(\mathfrak{A}, \mathfrak{B})$ between two Σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is played as follows. There are two players *Spoiler* and *Duplicator* who make moves in turn. In every round Spoiler selects either some element $a \in A$ or an element $b \in B$. Duplicator replies with an element of the other structure. Let $\bar{a} = a_0 \dots a_{m-1} \in A^m$ and $\bar{b} = b_0 \dots b_{m-1} \in B^m$ be the elements selected during the m rounds. Duplicator wins the play if and only if the mapping $p =$

$\{(a_0, b_0), \dots, (a_{m-1}, b_{m-1})\}$ is a partial isomorphism, that is, an isomorphism between the substructures induced by $\bar{a}$ and $\bar{b}$, respectively. To simplify notation we will denote such mappings p by $\bar{a} \mapsto \bar{b}$.

A more algebraic way to look at Ehrenfeucht-Fraïssé games is via *back-and-forth systems*. Such a system consists of a sequence $(J_k)_{i \le m}$ of sets of partial isomorphisms with the following properties:

- *Forth property.* For every $\bar{a} \mapsto \bar{b} \in J_{k+1}$ and all $c \in A$, there exists an element $d \in B$ such that $\bar{a}c \mapsto \bar{b}d \in J_k$.

- *Back property.* For every $\bar{a} \mapsto \bar{b} \in J_{k+1}$ and all $d \in B$, there exists an element $c \in A$ such that $\bar{a}c \mapsto \bar{b}d \in J_k$.

Theorem 2.1 (Ehrenfeucht-Fraïssé). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures. The following statements are equivalent:*

(a) $\mathfrak{A} \equiv_m \mathfrak{B}$.

(b) *Duplicator has a winning strategy for* $\mathrm{EF}_m(\mathfrak{A}, \mathfrak{B})$.

(c) *There exists a back-and-forth system $(J_k)_{k \le m}$ with $J_m \neq \varnothing$.*

If the signature Σ is finite then we can replace (a) by m-equivalence with respect to finitary first-order logic. The reason why we consider infinitary logic is that, for some constructions below, we need to introduce infinite signatures.

3 DECOMPOSING STRUCTURES

The first thing that comes to mind when tasked with simplifying an Ehrenfeucht-Fraïssé game is trying to break it down into simpler games. One way of doing so consists in decomposing the structures in question into several parts on which one can play separately. Therefore, we will study operations f on structures for which there exists a function $g : \omega \to \omega$ such that

$$\mathfrak{A}_i \equiv_{g(m)} \mathfrak{B}_i, \text{ for all } i, \quad \text{implies} \quad f(\mathfrak{A}_0, \dots, \mathfrak{A}_{n-1}) \equiv_m f(\mathfrak{B}_0, \dots, \mathfrak{B}_{n-1}).$$

Let us recall several well-known instances of such operations. The canonical example consists of disjoint unions.

Lemma 3.1. *If $\mathfrak{A}_0 \equiv_m \mathfrak{B}_0$ and $\mathfrak{A}_1 \equiv_m \mathfrak{B}_1$ then $\mathfrak{A}_0 \uplus \mathfrak{A}_1 \equiv_m \mathfrak{B}_0 \uplus \mathfrak{B}_1$.*

For a proof, note that if Duplicator has strategies to win the games $\mathrm{EF}_m(\mathfrak{A}_0, \mathfrak{B}_0)$ and $\mathrm{EF}_m(\mathfrak{A}_1, \mathfrak{B}_1)$ then she can compose them to win the game $\mathrm{EF}_m(\mathfrak{A}, \mathfrak{B})$. The key reason why this is possible is that one can select elements of one component without knowledge of which elements of the other component have been chosen. For unions that are not disjoint the situation is more complex since the component games are not independent. We will return to this more general case below.

An analogous result holds for direct products, although it will not be used in this article.

Lemma 3.2. *If $\mathfrak{A}_0 \equiv_m \mathfrak{B}_0$ and $\mathfrak{A}_1 \equiv_m \mathfrak{B}_1$ then $\mathfrak{A}_0 \times \mathfrak{A}_1 \equiv_m \mathfrak{B}_0 \times \mathfrak{B}_1$.*

In fact, this result and its version for disjoint unions can be generalised to infinitely many operands. (There are even stronger generalisations possible.)

Theorem 3.3 (Feferman-Vaught). *If $\mathfrak{A}_i \equiv_m \mathfrak{B}_i$, for all $i \in I$, then*

$$\biguplus_{i \in I} \mathfrak{A}_i \equiv_m \biguplus_{i \in I} \mathfrak{B}_i \quad \text{and} \quad \prod_{i \in I} \mathfrak{A}_i \equiv_m \prod_{i \in I} \mathfrak{B}_i.$$

A third important class of operations that are compatible with first-order theories are first-order interpretations.

Definition 3.4. Let Σ and $\Gamma = \{R_0, \dots, R_s\}$ be signatures. A *k-dimensional first-order interpretation* (from Σ to Γ) is a list

$$\mathcal{I} = \big\langle \delta(\bar{x}), \varepsilon(\bar{x}, \bar{y}), \varphi_{R_0}(\bar{x}_0, \dots, \bar{x}_{n_0-1}), \dots, \varphi_{R_s}(\bar{x}_0, \dots, \bar{x}_{n_s-1}) \big\rangle$$

of infinitary first-order formulae over the signature Σ where each of the tuples $\bar{x}, \bar{y}, \bar{x}_i$ above has length k. Such an interpretation defines the following operation on structures. It maps a Σ-structure $\mathfrak{A}$ to the Γ-structure

$$\mathcal{I}(\mathfrak{A}) := (\delta^{\mathfrak{A}}, \varphi_{R_0}^{\mathfrak{A}}, \dots, \varphi_{R_s}^{\mathfrak{A}})/\varepsilon^{\mathfrak{A}}.$$

That is, the universe of $\mathcal{I}(\mathfrak{A})$ consists of all tuples $\bar{a} \in A^k$ satisfying δ and the relations are defined by the formulae φ_{R_i}. Additionally, we factorise by the relation defined by ε. Of course, this only works if $\varepsilon^{\mathfrak{A}}$ is a congruence of the structure $(\delta^{\mathfrak{A}}, \varphi_{R_0}^{\mathfrak{A}}, \dots, \varphi_{R_s}^{\mathfrak{A}})$. If this is not the case then $\mathcal{I}(\mathfrak{A})$ remains undefined.

Lemma 3.5. *Let $\mathcal{I}$ be a k-dimensional first-order interpretation where each formula has quantifier rank at most r. For every formula φ over Γ, there exists a formula $\varphi^{\mathcal{I}}$ over Σ of quantifier rank $\mathrm{qr}(\varphi^I) \le k \cdot \mathrm{qr}(\varphi) + r$ such that, for all structures $\mathfrak{A}$ where $\mathcal{I}(\mathfrak{A})$ is defined, we have*

$$\mathcal{I}(\mathfrak{A}) \vDash \varphi \quad \text{iff} \quad \mathfrak{A} \vDash \varphi^{\mathcal{I}}.$$

Corollary 3.6. *Let $\mathcal{I}$ be a k-dimensional first-order interpretation where each formula has quantifier rank at most r. If $\mathfrak{A} \equiv_{km+r} \mathfrak{B}$ then $\mathcal{I}(\mathfrak{A}) \equiv_m \mathcal{I}(\mathfrak{B})$, provided these are defined.*

One way to simplify proofs based on Ehrenfeucht-Fraïssé games with the help of interpretations consists in replacing the structures $\mathfrak{A}$ and $\mathfrak{B}$ by more convenient structures $\mathfrak{A}^+$ and $\mathfrak{B}^+$ such that $\mathfrak{A} = \mathcal{I}(\mathfrak{A}^+)$ and $\mathfrak{B} = \mathcal{I}(\mathfrak{B}^+)$, for some interpretation $\mathcal{I}$. Of course, these new structures cannot be really simpler than the original ones since we can recover the latter from the former. But they might be more convenient to play on.

For example, suppose that we want to prove that spoiler wins the m round game between $\mathfrak{A}$ and $\mathfrak{B}$. We might simplify his task by replacing these structures by expansions $\mathfrak{A}^+$ and $\mathfrak{B}^+$ with some additional, definable relations that make certain information directly available which, in the original game, spoiler would need several steps to check. For instance, one could add the immediate successor relation to a partial order. Then the player can check immediately whether two elements are immediate successors. Otherwise, he would need an additional move to select an element in between. If the definitions of the new relations have quantifier rank k then, by Corollary 3.6, a proof that spoiler can win $\mathrm{EF}_{m-k}(\mathfrak{A}^+, \mathfrak{B}^+)$ implies that he can also win the original game $\mathrm{EF}_m(\mathfrak{A}, \mathfrak{B})$.

Example. Consider finite linear orders $\mathfrak{A} = (A, <, \perp, \top)$ and $\mathfrak{B} = (B, <, \perp, \top)$ with constants for the least and greatest element. (Formally, we regard $\perp$ and $\top$ as unary predicates to remain in our purely relational framework.) If $|A|, |B| > 2^m$ then $\mathfrak{A} \equiv_m \mathfrak{B}$ (see, e.g., [6] Example 1.3.5). There exists an interpretation $\mathcal{I}$ of quantifier rank 1 that defines the relation

$$E := \{(\top, \perp)\} \cup \{(a, b) \mid b \text{ is the immediate successor of } a\}.$$

Thus, $\mathcal{I}(\mathfrak{A})$ is a cycle of length $|A|$. By Corollary 3.6, we have $\mathcal{I}(\mathfrak{A}) \equiv_{m-1} \mathcal{I}(\mathfrak{B})$. It follows that, if $\mathfrak{C}$ and $\mathfrak{D}$ are cycles of length greater than 2^m then $\mathfrak{C} \equiv_{m-1} \mathfrak{D}$.

To obtain more substantial simplifications we can combine interpretations with other operations like disjoint unions. In the remainder of this section we will consider partitions of a structure that do not correspond to a disjoint union. We would like to apply the above techniques to this case.

Suppose that we have a partition $A_0 \uplus A_1$ of A and let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be the corresponding substructures of $\mathfrak{A}$. We would like to find an operation f such that $f(\mathfrak{A}_0, \mathfrak{A}_1) = \mathfrak{A}$. It turns out that using the substructures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ directly is

not sufficient. We will use certain expansions $\mathfrak{A}_0^+$ and $\mathfrak{A}_1^+$ instead. The operations f we will consider consist of a disjoint union followed by a one-dimensional quantifier-free interpretation. By the lemmas above it follows that, if $\mathfrak{A}$ and $\mathfrak{B}$ are structures that can be written as $\mathfrak{A} = f(\mathfrak{A}_0^+, \mathfrak{A}_1^+)$ and $\mathfrak{B} = f(\mathfrak{B}_0^+, \mathfrak{B}_1^+)$, for the same operation f, then

$$\mathfrak{A}_0^+ \equiv_m \mathfrak{B}_0^+ \quad \text{and} \quad \mathfrak{A}_1^+ \equiv_m \mathfrak{B}_1^+ \quad \text{implies} \quad \mathfrak{A} \equiv_m \mathfrak{B}.$$

In order to recover the structure $\mathfrak{A}$ from its substructures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ we have to know which tuples $\bar{a}_0 \subseteq A_0$ and $\bar{a}_1 \subseteq A_1$ are connected by a relation. In the expansion $\mathfrak{A}_i^+$ we therefore colour all tuples by information about those tuples in the other component it is connected with.

Definition 3.7. Let $\mathfrak{A}$ be a structure and let r be the maximal arity of a relation of $\mathfrak{A}$. For $1 \le n < r$, let C_n be a set of *colours*. A $\bar{C}$-*colouring* of $\mathfrak{A}$ is a function χ that maps every tuple $\bar{a} \in A^n$ with $1 \le n < r$ to a colour $\chi(\bar{a}) \in C_n$. By $(\mathfrak{A}, \chi)$ we denote the expansion of $\mathfrak{A}$ by relations $R_c := \chi^{-1}(c)$, for every $c \in \bigcup_n C_n$.

The following theorem is an immediate consequence of Lemma 3.1 and Corollary 3.6.

Theorem 3.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures with partitions $A = A_0 \uplus A_1$ and $B = B_0 \uplus B_1$. Suppose that there exists a quantifier-free interpretation $\mathcal{I}$ and colourings χ_i of A_i and η_i of B_i such that*

$$\mathfrak{A} = \mathcal{I}\big((\mathfrak{A}_0, \chi_0) \uplus (\mathfrak{A}_1, \chi_1)\big) \quad \text{and} \quad \mathfrak{B} = \mathcal{I}\big((\mathfrak{B}_0, \eta_0) \uplus (\mathfrak{B}_1, \eta_1)\big).$$

If $(\mathfrak{A}_0, \chi_0) \equiv_m (\mathfrak{B}_0, \eta_0)$ and $(\mathfrak{A}_1, \chi_1) \equiv_m (\mathfrak{B}_1, \eta_1)$ then we have $\mathfrak{A} \equiv_m \mathfrak{B}$.

In order to use this theorem we have to find suitable colourings and interpretations. Let $A = A_0 \uplus A_1$ be a partition of $\mathfrak{A}$. We start by defining colourings χ_i of $\mathfrak{A}_i$ such that,

$$\mathfrak{A} = \mathcal{I}\big((\mathfrak{A}_0, \chi_0) \uplus (\mathfrak{A}_1, \chi_1)\big),$$

for some quantifier-free interpretation $\mathcal{I}$. There is a canonical choice for such colours. We can colour a tuple $\bar{a}$ by its *external type* as defined below.

Definition 3.9. Let $\mathfrak{A}$ be a Σ-structure and $X, U \subseteq A$.
(a) A formula $\varphi(\bar{x})$ is a *literal* if it is either atomic or the negation of an atomic formula. If, in addition, every variable x_i really appears in φ then we call $\varphi(\bar{x})$ a *strict literal*.

(b) The *atomic type* of a tuple $\bar{a} \subseteq A$ over a set $U \subseteq A$ of parameters is the set

$$\mathrm{atp}(\bar{a}/U) := \left\{ \varphi(\bar{x}) \mid \varphi \text{ a literal with parameters in } U, \ \mathfrak{A} \vDash \varphi(\bar{a}) \right\}.$$

For $U = \varnothing$, we just write $\mathrm{atp}(\bar{a})$. The *external type* of $\bar{a}$ is the set

$$\mathrm{etp}(\bar{a}/U) := \mathrm{atp}(\bar{a}/U) \smallsetminus \mathrm{atp}(\bar{a}).$$

(c) For $\bar{a}, \bar{b} \subseteq A$, we define the *type equivalence* relation

$$\bar{a} \simeq_U \bar{b} \quad : \text{iff} \quad \mathrm{etp}(\bar{a}/U) = \mathrm{etp}(\bar{b}/U).$$

(d) We denote the set of all external n-types over U realised in X by

$$I_n(X/U) := X^n/{\simeq_U}.$$

The union over all n is

$$I(X/U) := I_1(X/U) \cup \cdots \cup I_{r-1}(X/U),$$

where r is the maximal arity of relations of $\mathfrak{A}$. For $U = A \smallsetminus X$, we introduce the shorthands

$$I_n(X) := I_n(X/A \smallsetminus X) \quad \text{and} \quad I(X) := I(X/A \smallsetminus X).$$

(e) Set $\Sigma_{\mathfrak{L}}(X) := \Sigma \cup \left\{ R_\tau \mid \tau \in I(X) \right\}$. The *localisation* of $\mathfrak{A}$ to X is the $\Sigma_{\mathfrak{L}}(X)$-structure

$$\mathfrak{L}(X) := (\mathfrak{A}|_X, \chi)$$

where χ is the $I(X)$-colouring with $\chi(\bar{a}) := \mathrm{etp}(\bar{a}/A \smallsetminus X)$.

External types were introduced in [3, 4] generalising work of Courcelle [5]. See also [12] for similar techniques.

Example. Let $(A, \leq, \bar{P})$ be a linear order with unary predicates $\bar{P}$. For every convex subset $C \subseteq A$, we have

$$\bar{a} \simeq_{A \smallsetminus C} \bar{b} \quad \text{for all } \bar{a}, \bar{b} \in C^n.$$

When labelling tuples by their external type we can recover the original structure from its substructures with the help of a disjoint union and a quantifier-free interpretation.

Lemma 3.10. *Let $\mathfrak{A}$ be a Σ-structure and $X \subseteq A$. There exists a one-dimensional quantifier-free interpretation $\mathcal{I}$ such that*

$$\mathfrak{A} \cong \mathcal{I}(\mathfrak{L}(X) \cup \mathfrak{L}(A \smallsetminus X)).$$

If we are given two structures $\mathfrak{A}$ and $\mathfrak{B}$ and partitions $A_0 \uplus A_1 = A$ and $B_0 \uplus B_1 = B$ then it follows that there are interpretations $\mathcal{I}$ and $\mathcal{J}$ that reconstruct $\mathfrak{A}$ and $\mathfrak{B}$ from the respective localisations. But, in order to apply Theorem 3.8 we furthermore require that $\mathcal{I} = \mathcal{J}$. Note that, with our current definitions, this is never the case for the trivial reason that the sets of colours used by $\mathfrak{L}(A_i)$ and $\mathfrak{L}(B_i)$ are disjoint. Hence, we have to unify them by finding a suitable bijection mapping colours of $\mathfrak{L}(A_i)$ to those of $\mathfrak{L}(B_i)$. In order to be able to use the same interpretation for both structures we cannot use an arbitrary bijection between $I(A_i)$ and $I(B_i)$. We need one that respects the relations between tuples of these types. The following definition formalises this idea.

Definition 3.11. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures with partitions $A = X_0 \uplus \cdots \uplus X_{m-1}$ and $B = Y_0 \uplus \cdots \uplus Y_{m-1}$.

(a) A tuple $\bar{a}$ is *scattered* if there are at least two indices i with $\bar{a} \cap X_i \neq \varnothing$.

(b) Two tuples $\bar{a} \in A^n$ and $\bar{b} \in B^n$ are *congruent* if we have

$$a_i \in X_k \quad \text{iff} \quad b_i \in Y_k, \quad \text{for all } i < n \text{ and } k < m.$$

(c) For a sequence

$$g_i^n : I_n(X_i) \to I_n(Y_i), \quad \text{for } i < m \text{ and } 0 < n < r,$$

of bijections, we write

$$\bar{g} : \bar{X} \approx \bar{Y}$$

if, whenever $\bar{a}^i \subseteq X_i$ and $\bar{b}^i \subseteq Y_i$, $i < m$, are tuples such that
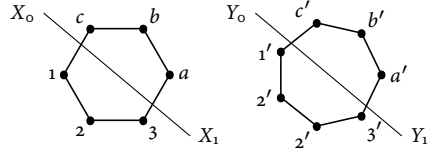
- $\bar{a}^0 \dots \bar{a}^{m-1}$ and $\bar{b}^0 \dots \bar{b}^{m-1}$ are congruent and scattered, and
- $g_i^{|\bar{a}^i|}\left(\mathrm{etp}(\bar{a}^i/A \smallsetminus X_i)\right) = \mathrm{etp}(\bar{b}^i/B \smallsetminus Y_i)$, for all i,

then we have

$$\mathfrak{A} \vDash \varphi(\bar{a}^0, \dots, \bar{a}^{m-1}) \quad \text{iff} \quad \mathfrak{B} \vDash \varphi(\bar{b}^0, \dots, \bar{b}^{m-1}),$$

for every *strict* literal $\varphi(\bar{x}^0, \dots, \bar{x}^{m-1})$.

Example. Consider the following partitions of circles $\mathfrak{C}_6$ and $\mathfrak{C}_7$ where the labels represent the external types:

Then we have

$$g_0^1 g_1^1 : X_0 X_1 \approx Y_0 Y_1 \,,$$

where $g_0^1 : X_0 \to Y_0$ and $g_1^1 : X_1 \to Y_1$ are the functions $x \mapsto x'$.

Remark. (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures with partitions $\bar{X}$ and $\bar{Y}$. We denote the substructure of $\mathfrak{A}$ induced by X_i by $\mathfrak{A}_i$ and the substructure of $\mathfrak{B}$ induced by Y_i by $\mathfrak{B}_i$. There exist functions $\bar{g}$ with $\bar{g} : \bar{X} \approx \bar{Y}$ if and only if, there exist colourings χ_i of $\mathfrak{A}_i$ and η_i of $\mathfrak{B}_i$ and a one-dimensional quantifier-free interpretation $\mathcal{I}$ such that

$$\mathfrak{A} = \mathcal{I}((\mathfrak{A}_0, \chi_0) \uplus \cdots \uplus (\mathfrak{A}_{m-1}, \chi_{m-1}))$$

and $\quad \mathfrak{B} = \mathcal{I}((\mathfrak{B}_0, \eta_0) \uplus \cdots \uplus (\mathfrak{B}_{m-1}, \eta_{m-1}))\,.$

(b) Suppose that $\bar{g} : \bar{X} \approx \bar{Y}$. If $\bar{a}^i \subseteq X_i$ and $\bar{b}^i \subseteq Y_i$ are tuples such that, for all i,

$$\mathrm{atp}(\bar{a}^i) = \mathrm{atp}(\bar{b}^i) \quad \text{and} \quad g_i^{|\bar{a}^i|}[\bar{a}^i] = [\bar{b}^i]$$

then we have $\mathrm{atp}(\bar{a}^0 \dots \bar{a}^{m-1}) = \mathrm{atp}(\bar{b}^0 \dots \bar{b}^{m-1})\,.$

Definition 3.12. Let Σ and Γ be signatures and $g : \Sigma \to \Gamma$ an arity preserving bijection. If $\mathfrak{A}$ is a Σ-structure and $\mathfrak{B}$ a Γ-structure then we write $\mathfrak{A} \equiv_m^g \mathfrak{B}$ if we have

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad \mathfrak{B} \vDash \varphi^g, \quad \text{for all } \varphi \text{ of quantifier rank at most } m,$$

where φ^g is the formula obtained from φ by replacing every relation symbol R by $g(R)$.

Lemma 3.13. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures, $X_0, \dots, X_{m-1} \subseteq A$ and $Y_0, \dots, Y_{m-1} \subseteq B$ sequences of disjoint subsets, and $\bar{g} : \bar{X} \approx \bar{Y}$. If $g_i : \Sigma_{\mathfrak{L}}(X_i) \to \Sigma_{\mathfrak{L}}(Y_i)$, $i < m$, are the corresponding bijections between the signatures then*

$$(*) \quad \mathfrak{L}(X_i), \bar{a}^i \equiv_0^{g_i} \mathfrak{L}(Y_i), \bar{b}^i, \quad \text{for all } i < m \,,$$

implies $\mathrm{atp}(\bar{a}^0 \dots \bar{a}^{m-1}) = \mathrm{atp}(\bar{b}^0 \dots \bar{b}^{m-1})$.

Proof. Note that $(*)$ implies

$$|\bar{a}^i| = |\bar{b}^i|, \quad \mathrm{atp}(\bar{a}^i) = \mathrm{atp}(\bar{b}^i), \quad \text{and} \quad g_i^{|\bar{a}^i|}[\bar{a}^i] = [\bar{b}^i]\,.$$

If there is at most one index i with $|\bar{a}^i| > 0$ then we are done. Otherwise, the claim follows from $\bar{g} : \bar{X} \approx \bar{Y}$ and the remark above. $\qquad \square$

With the help of Lemma 3.10 we can rewrite Theorem 3.8 in the following form.

Theorem 3.14. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures with partitions $X_0 \uplus X_1 = A$ and $Y_0 \uplus Y_1 = B$ of their universes. Suppose that $\bar{g} : X_0 X_1 \approx Y_0 Y_1$ and let $g_i : \Sigma_{\mathfrak{L}}(X_i) \to \Sigma_{\mathfrak{L}}(Y_i)$, $i < 2$, be the corresponding bijections between the signatures. If*

$$\mathfrak{L}(X_0) \equiv_m^{g_0} \mathfrak{L}(Y_0) \quad \text{and} \quad \mathfrak{L}(X_1) \equiv_m^{g_1} \mathfrak{L}(Y_1)$$

then we have $\mathfrak{A} \equiv_m \mathfrak{B}$.

Of course, whether we archive a simplification this way largely depends on the existence of suitable partitions of the given structures, preferably with few external types between the components.

Example. Let $\mathfrak{A} = (A, E, <)$ and $\mathfrak{B} = (B, E, <)$ be undirected graphs equipped with an additional linear order. Suppose that $A = X_0 \uplus X_1$ and $B = Y_0 \uplus Y_1$ where every element of X_0 and Y_0 is less than all elements of, respectively, X_1 and Y_1. Set $\mathfrak{A}_i := \mathfrak{A}|_{X_i}$ and $\mathfrak{B}_i := \mathfrak{B}|_{Y_i}$. Let $\bar{a}^i$ be an enumeration of all elements of $\mathfrak{A}_i$ that are adjacent to some element of $\mathfrak{A}_{1-i}$, and let $\bar{b}^i$ be the elements of $\mathfrak{B}_i$ adjacent to some element of $\mathfrak{B}_{1-i}$.

Note that all tuples (of a given arity) disjoint from $\bar{a}^0, \bar{a}^1, \bar{b}^0, \bar{b}^1$ have the same external type. Consequently, if we have

$$\mathrm{atp}(\bar{a}^0 \bar{a}^1) = \mathrm{atp}(\bar{b}^0 \bar{b}^1)\,,$$

then we can find functions $\bar{g} : X_0 X_1 \approx Y_0 Y_1$. Hence, the conditions

$$(\mathfrak{A}_0, \bar{a}^0) \equiv_m (\mathfrak{B}_0, \bar{b}^0)$$
$$(\mathfrak{A}_1, \bar{a}^1) \equiv_m (\mathfrak{B}_1, \bar{b}^1)$$
$$\mathrm{atp}(\bar{a}^0 \bar{a}^1) = \mathrm{atp}(\bar{b}^0 \bar{b}^1)$$

imply that $\mathfrak{A} \equiv_m \mathfrak{B}$.

4 CONTRACTING STRUCTURES

In many applications the systems under consideration have a hierarchical structure. For instance, when designing a circuit diagram one usually assembles it in a modular way by using several predefined units. Usually, these units in turn consist of subunits which, again, might be built up from even simpler parts. When playing Ehrenfeucht-Fraïssé games on such structures one would like to take this hierarchy into account, e.g., by playing on various levels of abstraction where all units of a lower level are considered as black-boxes without internal structure. To do so we introduce an operation on structures that contracts a unit to a single point. After contracting all subunits we can play the game on the remaining structure.

Definition 4.1. Let $\mathfrak{A}$ be a Σ-structure and $X_0, \dots, X_{n-1} \subseteq A$ a sequence of disjoint subsets such that, for all $i, k < n$, there are functions

$$\bar{g}_{ik} : X_i, (A \setminus X_i) \approx X_k, (A \setminus X_k).$$

(a) The $\bar{X}$-*contraction* $\mathfrak{C}(\mathfrak{A}, \bar{X})$ of $\mathfrak{A}$ is obtained by replacing each set X_i by a single element x_i and adding auxiliary relations P_τ that encode how the remaining elements were connected to those in X_i. Formally, we define the universe of the contraction as

$$C := (A \setminus \bigcup_i X_i) \cup \{x_0, \dots, x_{n-1}\},$$

and the relations are

$$R^{\mathfrak{C}(\mathfrak{A}, \bar{X})} := R^{\mathfrak{A}}|_C, \qquad\qquad \text{for } R \in \Sigma,$$
$$P_\tau^{\mathfrak{C}(\mathfrak{A}, \bar{X})} := \left\{ (\bar{a}, x_i) \mid \mathrm{etp}(\bar{a}/X_i) = g_{0i}(\tau) \right\}, \quad \text{for } \tau \in I(A \setminus X_0 / X_0),$$

and $\quad Q^{\mathfrak{C}(\mathfrak{A}, \bar{X})} := \{x_0, \dots, x_{n-1}\}.$

(b) Recall that a tuple $\bar{a} \in A^m$ is *scattered* if there are at least 2 indices $i < n$ with $\bar{a} \cap X_i \neq \varnothing$. For such a tuple $\bar{a}$, we define the *local type* of $\bar{a}$ as

$$\mathrm{ltp}(\bar{a}) := \left\langle \sim, F, (\tau_p)_p \right\rangle,$$

where

$$s \sim t \quad :\text{iff} \quad a_s \in X_i \Leftrightarrow a_t \in X_i, \quad \text{for all } i < n,$$
$$\tau_p := \mathrm{atp}(\bar{a}|_p), \quad \text{for } p \in [m]/\sim,$$
$$F := \left\{ s < m \mid a_s \in A \setminus (X_0 \cup \cdots \cup X_{n-1}) \right\}.$$

(c) We call $\mathfrak{A}$ *globally uniform w.r.t.* $\bar{X}$ if, for all $m < \omega$ and all scattered tuples $\bar{a}, \bar{b} \in A^m$,

$$\mathrm{ltp}(\bar{a}) = \mathrm{ltp}(\bar{b}) \quad \text{implies} \quad \mathrm{atp}(\bar{a}) = \mathrm{atp}(\bar{b}).$$

(d) Suppose that $\mathfrak{A}$ is globally uniform w.r.t. $\bar{X}$ and let $m < \omega$. The *global m-type* of $\mathfrak{A}, \bar{X}$ is the set

$$\mathrm{gtp}_m(\mathfrak{A}, \bar{X}) := \left\{ \left\langle \mathrm{ltp}(\bar{a}), \mathrm{atp}(\bar{a}) \right\rangle \mid \bar{a} \in A^m \text{ is scattered} \right\}.$$

Theorem 4.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures and suppose that $X_0, \dots, X_{m-1} \subseteq A$ and $Y_0, \dots, Y_{n-1} \subseteq B$ are subsets such that $\mathfrak{A}$ is globally uniform w.r.t. $\bar{X}$ and $\mathfrak{B}$ is globally uniform w.r.t. $\bar{Y}$. If we have*

$$\mathrm{gtp}_m(\mathfrak{A}, \bar{X}) = \mathrm{gtp}_m(\mathfrak{B}, \bar{Y}), \quad \text{for all } m \leq k,$$
$$\mathfrak{C}(\mathfrak{A}, \bar{X}) \equiv_k \mathfrak{C}(\mathfrak{B}, \bar{Y}) \quad \text{and} \quad \mathfrak{L}(X_i) \equiv_k \mathfrak{L}(Y_j), \quad \text{for all } i, j,$$

then it follows that $\mathfrak{A} \equiv_k \mathfrak{B}$.

Proof. Let $g : \mathfrak{A} \to \mathfrak{C}(\mathfrak{A}, \bar{X})$ and $h : \mathfrak{B} \to \mathfrak{C}(\mathfrak{B}, \bar{Y})$ be the contraction maps with $g(X_i) = \{x_i\}$. We call a map $\bar{a} \mapsto \bar{b}$ *l-good,* if $\bar{a} \in A^{k-l}$ and $\bar{b} \in B^{k-l}$,

$$\mathfrak{C}(\mathfrak{A}, \bar{X}), g(\bar{a}) \equiv_l \mathfrak{C}(\mathfrak{B}, \bar{Y}), h(\bar{b}),$$

and, for all indices $i < m$ and $j < n$ such that $I := \{ s \mid a_s \in X_i \} \neq \varnothing$ and $\bar{b}|_I \subseteq Y_j$, we have

$$\mathfrak{L}(X_i), \bar{a}|_I \equiv_l \mathfrak{L}(Y_j), \bar{b}|_I.$$

Let J_l be the set of all l-good maps. We claim that $(J_l)_l : \mathfrak{A} \equiv_k \mathfrak{B}$.

We have $\langle\rangle \mapsto \langle\rangle \in J_k \neq \varnothing$. To check the forth property assume that $\bar{a} \mapsto \bar{b} \in J_l$ and $c \in A$.

If $c \in A \smallsetminus (X_0 \cup \cdots \cup X_{m-1})$ then

$$\mathfrak{C}(\mathfrak{A}, \bar{X}), g(\bar{a}) \equiv_l \mathfrak{C}(\mathfrak{B}, \bar{Y}), h(\bar{b})$$

implies that there is some element $d \in B$ such that

$$\mathfrak{C}(\mathfrak{A}, \bar{X}), g(\bar{a})c \equiv_{l-1} \mathfrak{C}(\mathfrak{B}, \bar{Y}), h(\bar{b})d.$$

Hence, $\bar{a}c \mapsto \bar{b}d$ is $(l-1)$-good.

Suppose that $c \in X_i$. If $\bar{a} \cap X_i \neq \varnothing$ then, for suitable j and I, we have

$$\mathfrak{L}(X_i), \bar{a}|_I \equiv_l \mathfrak{L}(Y_j), \bar{b}|_I,$$

which implies that we can find an element $d \in Y_j$ with

$$\mathfrak{L}(X_i), \bar{a}|_I c \equiv_{l-1} \mathfrak{L}(Y_j), \bar{b}|_I d.$$

Hence, $\bar{a}c \mapsto \bar{b}d$ is $(l-1)$-good. Similarly, if $\bar{a} \cap X_i = \varnothing$ then

$$\mathfrak{C}(\mathfrak{A}, \bar{X}), g(\bar{a}) \equiv_l \mathfrak{C}(\mathfrak{B}, \bar{Y}), h(\bar{b})$$

implies that there is some index $j < n$ such that

$$\mathfrak{C}(\mathfrak{A}, \bar{X}), g(\bar{a})x_i \equiv_{l-1} \mathfrak{C}(\mathfrak{B}, \bar{Y}), h(\bar{b})y_j.$$

In particular, $\bar{b} \cap Y_j = \varnothing$. Let d be an element such that

$$\mathfrak{L}(X_i), c \equiv_{l-1} \mathfrak{L}(Y_j), d.$$

Then $\bar{a}c \mapsto \bar{b}d$ is $(l-1)$-good.

It remains to prove that every $\bar{a} \mapsto \bar{b} \in J_0$ is a partial isomorphism. Fix a subtuple $\bar{a}' \subseteq \bar{a}$ and let $\bar{b}' \subseteq \bar{b}$ be the corresponding subtuple of $\bar{b}$. If $\bar{a}'$ is scattered then we have $\mathrm{ltp}(\bar{a}') = \mathrm{ltp}(\bar{b}')$ and it follows that $\bar{a}'$ and $\bar{b}'$ satisfy the same strict literals.

If $\bar{a}'$ is not scattered then $\bar{a}' = \bar{a}_0 \cup \bar{a}_1$ and $\bar{b}' = \bar{b}_0 \cup \bar{b}_1$ where $\bar{a}_0 \subseteq X_i$ and $\bar{b}_0 \subseteq Y_j$, for some i, j. Hence, $\mathfrak{L}(X_i), \bar{a}_0 \equiv_0 \mathfrak{L}(Y_j), \bar{b}_0$ implies that $\mathrm{etp}(\bar{a}_0/A \smallsetminus X_i) = \mathrm{etp}(\bar{b}_0/B \smallsetminus Y_j)$ and it again follows that $\bar{a}'$ and $\bar{b}'$ satisfy the same strict literals. $\qquad \square$

Example. Let $\mathfrak{A}_n$ be the graph consisting of a cycle of length n to every point of which is attached a path of length n as follows:
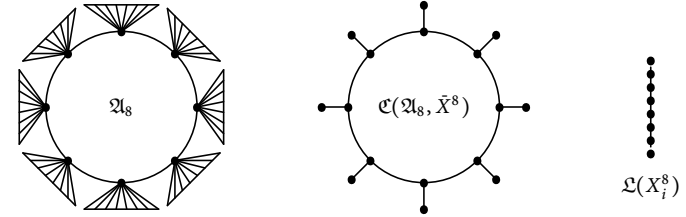

Let $X_0^n, \ldots, X_{n-1}^n$ be the sets of vertices of the attached paths. Then $\mathfrak{L}(X_i^n)$ is a path of length n and $\mathfrak{C}(\mathfrak{A}_n, \bar{X}^n)$ is a cycle of length n with additional edges attached at every vertex. If $m, n \geq 2^k$ then we have

$$\mathfrak{C}(\mathfrak{A}_m, \bar{X}^m) \equiv_k \mathfrak{C}(\mathfrak{A}_n, \bar{X}^n) \quad \text{and} \quad \mathfrak{L}(X_i^m) \equiv_k \mathfrak{L}(X_j^n)$$

which implies that $\mathfrak{A}_m \equiv_k \mathfrak{A}_n$.

5 GLUING STRUCTURES

In the preceding sections we have considered decompositions of a structure into disjoint parts. Now we study decompositions into two parts that overlap. Suppose that we have a subset $Z \subseteq A$ and some notion of distance between elements of $\mathfrak{A}$. For $i < n$, let X_i be the set of all elements whose distance from Z is i, and let X_n contain the remaining elements. We are interested in the decomposition of $\mathfrak{A}$ into the sets $X_0 \cup \cdots \cup X_{n-1}$ and $X_1 \cup \cdots \cup X_n$. This situation arose in [13]. We present a slightly generalised version of those results rephrased to fit our terminology.
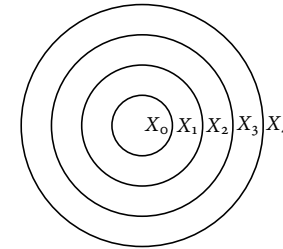
Definition 5.1. Let $\mathfrak{A}$ be a Σ-structure and $X_0 \uplus \cdots \uplus X_n = A$ a partition of its universe. We set $X_{<k} := X_0 \cup \cdots \cup X_{k-1}$ and similarly for $X_{>k}$ etc.

(a) The *inner part* of $\mathfrak{A}$ is

$$\mathfrak{J}(\bar{X}) := \left(\mathfrak{A}\big|_{A \smallsetminus X_n}, X_0, \ldots, X_{n-1}, (P_\tau^1)_\tau, \ldots, (P_\tau^{n-1})_\tau\right)$$

where

$$P_\tau^i := \left\{ \bar{a} \subseteq X_{<i} \mid \operatorname{etp}(\bar{a}/X_{>i}) = \tau \right\}.$$

Analogously, we define the *outer part* of $\mathfrak{A}$ by $\mathfrak{O}(X_0 \ldots X_n) := \mathfrak{J}(X_n \ldots X_0)$.

(b) Let $\mathfrak{B}$ be another Σ-structure and $Y_0 \uplus \cdots \uplus Y_n = B$ a partition of its universe. For bijections

$$g_i^k : I_k(X_{<i}/X_{>i}) \to I_k(Y_{<i}/Y_{>i}), \qquad 0 < i < n,\ k < r,$$
$$h_i^k : I_k(X_{>i}/X_{<i}) \to I_k(Y_{>i}/Y_{<i}), \qquad 0 < i < n,\ k < r,$$

we write

$$\bar{g}, \bar{h} : \bar{X} \approx \bar{Y}$$

if, whenever there is an index $0 < l < n$ and tuples $\bar{a} \subseteq X_{<l}$, $\bar{c} \subseteq X_{>l}$, $\bar{b} \subseteq Y_{<l}$, and $\bar{d} \subseteq Y_{>l}$ such that

- $|\bar{a}| = |\bar{b}| > 0$ and $|\bar{c}| = |\bar{d}| > 0$,
- $g_i^{|\bar{a}|}[\bar{a}] = [\bar{b}]$ and $h_i^{|\bar{c}|}[\bar{c}] = [\bar{d}]$

then we have

$$\mathfrak{A} \vDash \varphi(\bar{a}, \bar{c}) \quad \text{iff} \quad \mathfrak{B} \vDash \varphi(\bar{b}, \bar{d}), \quad \text{for every strict literal } \varphi(\bar{x}, \bar{y}).$$

If the distance between X_0 and X_n is large enough then we can play the game separately on $\mathfrak{J}(\bar{X})$ and $\mathfrak{O}(\bar{X})$.

Theorem 5.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures and $X_0 \uplus \cdots \uplus X_n = A$ and $Y_0 \uplus \cdots \uplus Y_n = B$ partitions of their universes. Suppose that m is a number such that $2^m \leq n$. If there are bijections $\bar{g}$ and $\bar{h}$ such that*

$$\bar{g}, \bar{h} : \bar{X} \approx \bar{Y}, \quad \mathfrak{J}(\bar{X}) \equiv_m^g \mathfrak{J}(\bar{Y}), \quad \text{and} \quad \mathfrak{O}(\bar{X}) \equiv_m^h \mathfrak{O}(\bar{Y}),$$

then we have $\mathfrak{A} \equiv_m \mathfrak{B}$.

Proof. For a set $Z \subseteq A$, we set

$$\mu(Z) := \max\left(\{-1\} \cup \{ k \mid Z \cap X_k \neq \varnothing \}\right),$$
$$\text{and} \quad \nu(Z) := \min\left(\{n+1\} \cup \{ k \mid Z \cap X_k \neq \varnothing \}\right).$$

We partition every tuple $\bar{a} \subseteq A$ as $\bar{a} = \lambda(\bar{a}) \cup \rho(\bar{a})$ with *left part* $\lambda(\bar{a}) \subseteq A \smallsetminus X_n$ and *right part* $\rho(\bar{a}) \subseteq A \smallsetminus X_0$ by induction on $|\bar{a}|$. We set $\lambda(\langle\rangle) := \langle\rangle$ and $\rho(\langle\rangle) := \langle\rangle$. For nonempty tuples $\bar{a}c$, we consider two cases. Suppose that $c \in X_l$. If c is nearer to the left part of $\bar{a}$ than to its right part, that is, if

$$l \leq \tfrac{1}{2}\left(\nu(\rho(\bar{a})) + \mu(\lambda(\bar{a}))\right),$$

then we add c to the left part, i.e., we set

$$\lambda(\bar{a}c) := \lambda(\bar{a})c \quad \text{and} \quad \rho(\bar{a}c) := \rho(\bar{a}).$$

Otherwise, we define

$$\lambda(\bar{a}c) := \lambda(\bar{a}) \quad \text{and} \quad \rho(\bar{a}c) := \rho(\bar{a})c.$$

For $\bar{b} \subseteq B$, we define $\lambda(\bar{b})$ and $\rho(\bar{b})$ analogously. Note that, if $k := |\bar{a}| \leq m$ then

$$\nu(\rho(\bar{a})) - \mu(\lambda(\bar{a})) \geq (n+2)2^{-k} > n2^{-m} \geq 1.$$

Define

$$J_k := \Big\{ \bar{a} \mapsto \bar{b} \mid \mathfrak{J}(\bar{X}), \lambda(\bar{a}) \equiv_k^g \mathfrak{J}(\bar{Y}), \lambda(\bar{b}) \text{ and}$$
$$\mathfrak{O}(\bar{X}), \rho(\bar{a}) \equiv_k^h \mathfrak{O}(\bar{Y}), \rho(\bar{b}) \Big\}.$$

We claim that $(J_k)_{k<m} : \mathfrak{A} \equiv_m \mathfrak{B}$. By definition, we have $\langle\rangle \mapsto \langle\rangle \in J_m \neq \varnothing$.

To check the forth property, let $\bar{a} \mapsto \bar{b} \in J_k$ and $c \in A$. By symmetry, we may assume that $\lambda(\bar{a}c) \neq \lambda(\bar{a})$ and $\rho(\bar{a}c) = \rho(\bar{a})$. Since $\mathfrak{J}(\bar{X}), \lambda(\bar{a}) \equiv_k^g \mathfrak{J}(\bar{Y}), \lambda(\bar{b})$ there is some element $d \in B \smallsetminus Y_n$ such that

$$\mathfrak{J}(\bar{X}), \lambda(\bar{a})c \equiv_{k-1}^g \mathfrak{J}(\bar{Y}), \lambda(\bar{b})d.$$

Consequently, we have $\bar{a}c \mapsto \bar{b}d \in J_{k-1}$.

It remains to show that every $\bar{a} \mapsto \bar{b} \in J_0$ is a partial isomorphism. Suppose that $\mathfrak{A} \vDash \varphi(\bar{a})$, for some literal φ. If $\lambda(\bar{a}) = \langle\rangle$ then

$$\mathfrak{O}(\bar{X}), \rho(\bar{a}) \equiv_0^h \mathfrak{O}(\bar{Y}), \rho(\bar{b})$$

implies $\mathfrak{B} \vDash \varphi(\bar{b})$. In a similar way it follows that $\rho(\bar{a}) = \langle\rangle$ implies $\mathfrak{B} \vDash \varphi(\bar{b})$.

Therefore, we may assume that $\lambda(\bar{a})$ and $\rho(\bar{a})$ are both nonempty. There exists some index $l < n$ such that $\lambda(\bar{a}) \subseteq X_{<l}$ and $\rho(\bar{a}) \subseteq X_{>l}$. Since

$$\mathfrak{J}(\bar{X}), \lambda(\bar{a}) \equiv_0^g \mathfrak{J}(\bar{Y}), \lambda(\bar{b})$$

we have

$$g\big(\mathrm{etp}(\lambda(\bar{a})/X_{>l})\big) = \mathrm{etp}(\lambda(\bar{b})/Y_{>l}) .$$

Again it follows that $\mathfrak{B} \vDash \varphi(\bar{b})$. $\qquad\square$

Example. Let $\mathfrak{A} = (A, <, \bar{R})$ be a finite linearly ordered structure. We define the distance between two elements $a, b \in A$ by

$$d(a, b) := \{ c \in A \mid a < c \le b \text{ or } b < c \le a \} .$$

Suppose that the relations R_l are local in the sense that there is a number k such that, for every tuple $\bar{a} \in R_l$, we have $d(a_i, a_j) \le k$, for all i, j.

Let $C \subseteq A$ be a subset that is convex with respect to $<$. If we are given a second structure $\mathfrak{B}$ with a convex subset $D \subseteq B$ such that $\mathfrak{B}|_{B \smallsetminus D} \cong \mathfrak{A}|_{A \smallsetminus C}$ then we can apply the above machinery by defining

$$X_i := \{ a \in A \mid k(i-1) < d(a, c) \le ki \text{ for some } c \in C \} ,$$
$$Y_i := \{ b \in B \mid k(i-1) < d(b, c) \le ki \text{ for some } c \in D \} .$$

$\mathfrak{A}$	X_3	X_2	X_1	C	X_1	X_2	X_3
$\mathfrak{B}$	Y_3	Y_2	Y_1	D	Y_1	Y_2	Y_3

Since $\mathfrak{O}(\bar{X}) \cong \mathfrak{O}(\bar{Y})$ we only have to prove that $\mathfrak{J}(\bar{X}) \equiv_m^h \mathfrak{J}(\bar{Y})$.

6 The theorem of Gaifman for globally uniform structures

For sparse structures the theorem of Gaifman provides a powerful method for proving expressibility results. Unfortunately, if the structures in question are non-sparse then the statement of the theorem becomes trivial. Nevertheless there are examples of successful arguments using Ehrenfeucht-Fraïssé games on non-sparse structures like linear orderings or Presburger Arithmetic. Furthermore, these arguments seem also to be based on a notion of locality. Therefore, there is hope to generalise Gaifman's theorem to cover these cases. For linear orders, we will present such a generalisation in the next section.

In order to obtain a meaningful generalisation of the theorem of Gaifman we need to consider other metrics. Hence, we start in this section with defining a quite general notion of a metric. For every element a of our structure, we assume that we are given some set $N_k(a)$ which we interpret as the set of all elements whose distance to a is at most k. In order for these sets $N_k(a)$ to induce a reasonable notion of distance we require them to satisfy some simple axioms.

Definition 6.1. Let $\mathfrak{A}$ be a structure.

(a) Let $N_k(a) \subseteq A$, for $a \in A$ and $k < \omega$, be a family of sets. We call $N = (N_k(a))_{a,k}$ a *system of neighbourhoods* if, for all $a \in A$ and every $k < \omega$, the following conditions are satisfied:

- $a \in N_0(a)$
- $N_k(a) \subseteq N_{k+1}(a)$
- There is an increasing function $\zeta : \omega \to \omega$ such that, for all $a, b \in A$,

$$b \in N_k(a) \quad \text{implies} \quad N_k(a) \subseteq N_{\zeta(k)}(b) .$$

For $X \subseteq A$, we set $N_k(X) := \bigcup_{a \in X} N_k(a)$.

(b) Let N be a system of neighbourhoods. A subset $X \subseteq A$ is *k-scattered* (w.r.t. N) if $a \notin N_k(b)$, for all $a, b \in X$ with $a \ne b$.

Let us collect some basic properties of systems of neighbourhoods.

Lemma 6.2. *Let N be a system of neighbourhoods.*

(a) *If $b \in N_k(a)$ then $N_{\zeta(k)}(b) \subseteq N_{\zeta^2(k)}(a)$.*

(b) *If $N_k(b) \nsubseteq N_{\zeta^2(k)}(a)$ then $N_k(a) \cap N_k(b) = \varnothing$.*

(c) *$N_{\zeta(k)}(N_k(a)) \subseteq N_{\zeta^2(k)}(a)$.*

(d) *If X is $\zeta(k)$-scattered then*

$$|X \cap N_k(c)| \le 1 , \qquad \text{for all } c \in A .$$

Proof. (a) By definition, $b \in N_k(a)$ implies $N_k(a) \subseteq N_{\zeta(k)}(b)$. In particular, $a \in N_{\zeta(k)}(b)$ which in turn implies $N_{\zeta(k)}(b) \subseteq N_{\zeta^2(k)}(a)$.

(b) Suppose that $c \in N_k(a) \cap N_k(b) \neq \varnothing$. Then $c \in N_k(a)$ implies $N_{\zeta(k)}(c) \subseteq N_{\zeta^2(k)}(a)$, and $c \in N_k(b)$ implies $N_k(b) \subseteq N_{\zeta(k)}(c)$. It follows that $N_k(b) \subseteq N_{\zeta^2(k)}(a)$.

(c) If $b \in N_k(a)$ and $c \in N_{\zeta(k)}(b)$ then it follows by (a) that

$$c \in N_{\zeta(k)}(b) \subseteq N_{\zeta^2(k)}(a).$$

(d) Let $a, b \in X$, $a \neq b$. If $a \in N_k(c)$ then $N_k(c) \subseteq N_{\zeta(k)}(a)$. Therefore, $b \notin N_{\zeta(k)}(a)$ implies $b \notin N_k(c)$. $\qquad\square$

In case of the usual Gaifman metric the distance between two elements is first-order definable. For general metrics this does not need to be the case. Therefore, we add new relations encoding the distances.

Definition 6.3. Let $\mathfrak{A}$ be a structure and N a system of neighbourhoods. For $\bar{a} \subseteq A$ we set

$$\mathfrak{N}_k(\bar{a}) := (\mathfrak{A}|_{N_k(\bar{a})}, (D_i)_{i<k}, \bar{a})$$

where $D_i := \{ (b,c) \mid c \in N_i(b) \}$.

Intuitively the reason why the theorem of Gaifman holds is that elements that are far away cannot be distinguished by atomic formulae. In order to generalise the theorem to other notions of distance we have to require the same property.

Definition 6.4. Let $\mathfrak{A}$ be a structure with system of neighbourhoods N.

(a) We call $\mathfrak{A}$ *globally uniform w.r.t.* N if, whenever $\bar{a}, \bar{a}', \bar{b}, \bar{b}'$ are tuples such that

- $|\bar{a}| = |\bar{a}'|$ and $|\bar{b}| = |\bar{b}'|$,
- $\bar{b} \cap N_0(\bar{a}) = \varnothing$ and $\bar{b}' \cap N_0(\bar{a}') = \varnothing$,
- $\mathrm{atp}(\bar{a}) = \mathrm{atp}(\bar{a}')$ and $\mathrm{atp}(\bar{b}) = \mathrm{atp}(\bar{b}')$,

then we have

$$\mathrm{atp}(\bar{a}\bar{b}) = \mathrm{atp}(\bar{a}'\bar{b}').$$

(b) If $\mathfrak{A}$ is globally uniform and $k, m < \omega$, then we define the *global type* of $\mathfrak{A}$ as

$$\mathrm{gtp}_{k,m}(\mathfrak{A}) :=$$
$$\left\{ \langle \mathrm{atp}(\bar{a}), \mathrm{atp}(\bar{b}), \mathrm{atp}(\bar{a}\bar{b}) \rangle \,\middle|\, \bar{a} \in A^k,\ \bar{b} \in A^m,\ \bar{b} \cap N_0(\bar{a}) = \varnothing \right\}.$$

The next lemma shows that globally uniform structures satisfy our requirement. Far away elements are indistinguishable.

Lemma 6.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be globally uniform structures such that*

$$\mathrm{gtp}_{k,m}(\mathfrak{A}) = \mathrm{gtp}_{k,m}(\mathfrak{B}), \qquad \text{for all } k, m < \omega.$$

Let $\bar{a}_0, \bar{a}_1 \subseteq A$ and $\bar{b}_0, \bar{b}_1 \subseteq B$ be tuples such that

$$N_r(\bar{b}_i) \cap N_{\zeta^2(r)}(\bar{a}_i) = \varnothing, \qquad \text{for both } i.$$

If we have

$$\mathfrak{N}_r(\bar{a}_0) \equiv_n \mathfrak{N}_r(\bar{b}_0) \quad \text{and} \quad \mathfrak{N}_r(\bar{a}_1) \equiv_n \mathfrak{N}_r(\bar{b}_1)$$

then $\mathfrak{N}_r(\bar{a}_0\bar{a}_1) \equiv_n \mathfrak{N}_r(\bar{b}_0\bar{b}_1)$.

Proof. For a tuple $\bar{c} \subseteq A$, we set $\bar{c}_i := \bar{c} \cap N_r(\bar{a}_i)$, and analogously for tuples $\bar{d} \subseteq B$. We claim that

$$(J_k)_{k \leq n} : \mathfrak{N}_r(\bar{a}_0\bar{a}_1) \equiv_n \mathfrak{N}_r(\bar{b}_0\bar{b}_1),$$

where

$$J_k := \left\{ \bar{c} \mapsto \bar{d} \,\middle|\, |\bar{c}| = |\bar{d}| = m - k,\ \mathfrak{N}_r(\bar{a}_i), \bar{c}_i \equiv_k \mathfrak{N}_r(\bar{b}_i), \bar{d}_i,\ \text{for both } i \right\}.$$

By definition, we have $\langle\rangle \mapsto \langle\rangle \in J_n \neq \varnothing$ and the back-and-forth property is easily verified. Hence, we only need to show that every $\bar{c} \mapsto \bar{d} \in J_0$ is a partial isomorphism.

By definition of J_0, we have $\mathrm{atp}(\bar{c}_0) = \mathrm{atp}(\bar{d}_0)$ and $\mathrm{atp}(\bar{c}_1) = \mathrm{atp}(\bar{d}_1)$. Furthermore, by Lemma 6.2 (c), $N_r(\bar{b}_i) \cap N_{\zeta^2(r)}(\bar{a}_i) = \varnothing$ implies that

$$N_r(\bar{b}_i) \cap N_0(N_r(\bar{a}_i)) = \varnothing.$$

Hence, we have $\bar{c}_1 \cap N_0(\bar{c}_0) = \varnothing$ and $\bar{d}_1 \cap N_0(\bar{d}_0) = \varnothing$. Since $\mathfrak{A}$ and $\mathfrak{B}$ are globally uniform and their global types coincide it follows that $\mathrm{atp}(\bar{c}_0\bar{c}_1) = \mathrm{atp}(\bar{d}_0\bar{d}_1)$. Consequently, $\bar{c} \mapsto \bar{d}$ is a partial isomorphism. $\qquad\square$

After these preparations we can prove an analogue of the theorem of Gaifman for globally uniform systems of neighbourhoods.

Definition 6.6. (a) A sentence φ is *basic local* if it is of the form

$$\exists \bar{x}\big(\text{`}\bar{x} \text{ is } r\text{-scattered'} \wedge \bigwedge_i \psi^{(N_r(x_i))}(x_i)\big),$$

where $\psi^{(N_r(x))}(x)$ denotes the relativisation of ψ to $N_r(x)$.

(b) A sentence φ is *basic global* if it is of the form

$$\exists \bar{x} \exists \bar{y}\big(\bar{y} \cap N_0(\bar{x}) = \varnothing \wedge \psi(\bar{x}, \bar{y})\big)$$

where ψ is quantifier-free.

Lemma 6.7. *If $\mathfrak{A}$ and $\mathfrak{B}$ are globally uniform structures that satisfy the same basic global sentences then we have*

$$\mathrm{gtp}_{k,m}(\mathfrak{A}) = \mathrm{gtp}_{k,m}(\mathfrak{B}), \qquad \textit{for all } k, m < \omega.$$

Theorem 6.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be globally uniform structures such that*

$$\mathrm{gtp}_{k,m}(\mathfrak{A}) = \mathrm{gtp}_{k,m}(\mathfrak{B}), \qquad \textit{for all } k, m < \omega.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same basic local sentences then $\mathfrak{A} \equiv_\omega \mathfrak{B}$.

Proof. It is sufficient to show that $\mathfrak{A} \equiv_m \mathfrak{B}$, for all $m < \omega$. Fix $m < \omega$ and set $\rho(k) := \zeta^k(0)$. We define

$$J_k := \big\{ \bar{a} \mapsto \bar{b} \mid \bar{a} \in A^{m-k}, \bar{b} \in B^{m-k}, \mathfrak{N}_{\rho(8k)}(\bar{a}) \equiv_{mk+1} \mathfrak{N}_{\rho(8k)}(\bar{b}) \big\}.$$

We claim that $(J_k)_{k \le m}$ is a back-and-forth system for $\mathfrak{A}$ and $\mathfrak{B}$. Since $\langle\rangle \mapsto \langle\rangle \in J_m$ it then follows that $\mathfrak{A} \equiv_m \mathfrak{B}$. Clearly, every $\bar{a} \mapsto \bar{b} \in J_0$ is a partial isomorphism. Therefore, we only need to prove the back-and-forth property.

By symmetry it is sufficient to consider the forth property. Let $\bar{a} \mapsto \bar{b} \in J_k$ and $c \in A$. We distinguish two cases.

First, suppose that $N_{\rho(8k-6)}(c) \subseteq N_{\rho(8k)}(\bar{a})$. Since

$$\mathfrak{N}_{\rho(8k)}(\bar{a}) \equiv_{mk+1} \mathfrak{N}_{\rho(8k)}(\bar{b})$$

we can find some $d \in N_{\rho(8k)}(\bar{b})$ such that

$$\mathfrak{N}_{\rho(8k)}(\bar{a}), c \equiv_{mk} \mathfrak{N}_{\rho(8k)}(\bar{b}), d.$$

It follows that

$$\mathfrak{N}_{\rho(8(k-1))}(\bar{a}c) \equiv_{mk} \mathfrak{N}_{\rho(8(k-1))}(\bar{b}d)$$

which implies that $\bar{a}c \mapsto \bar{b}d \in J_{k-1}$.

It remains to consider the case that $N_{\rho(8k-6)}(c) \nsubseteq N_{\rho(8k)}(\bar{a})$. Then we have

$$N_{\rho(8k-6)}(c) \cap N_{\rho(8k-6)}(\bar{a}) = \varnothing,$$

by Lemma 6.2 (b). If we find an element $d \in B$ such that

$$N_{\rho(8k-8)}(d) \cap N_{\rho(8k-6)}(\bar{b}) = \varnothing$$

and $\quad \mathfrak{N}_{\rho(8k-8)}(c) \equiv_{m(k-1)+1} \mathfrak{N}_{\rho(8k-8)}(d)$

then it follows by Lemma 6.5 that

$$\mathfrak{N}_{\rho(8(k-1))}(\bar{a}c) \equiv_{m(k-1)+1} \mathfrak{N}_{\rho(8(k-1))}(\bar{b}d).$$

Consequently, $\bar{a}c \mapsto \bar{b}d \in J_{k-1}$ and we are done.

In order to find a suitable element d let $\psi(x)$ be a formula such that

$$(\mathfrak{B}, (D_i)_{i<\rho(8k)}) \vDash \psi(d) \quad \text{iff} \quad \mathfrak{N}_{\rho(8k-8)}(c) \equiv_{m(k-1)+1} \mathfrak{N}_{\rho(8k-8)}(d),$$

and set

$$\vartheta_n(x_0, \ldots, x_{n-1}) := \text{`}\bar{x} \text{ is } (\rho(8k-3))\text{-scattered'} \wedge \bigwedge_{i<n} \psi(x_i),$$

and $\quad \chi_\lambda(\bar{y}) := \exists x_0 \cdots \exists x_{\lambda-1}\big(\bigwedge_{i<\lambda} N_{\rho(8k-6)}(x_i) \subseteq N_{\rho(8k-4)}(\bar{y}) \wedge \vartheta_\lambda(\bar{x})\big).$

Note that the quantifier rank of ψ is bounded by $m(k-1)+1$ and that of χ_λ by $\lambda + m(k-1) + 1$.

Let κ be the maximal finite cardinal such that

$$(\mathfrak{A}, (D_i)_{i<\rho(8k)}) \vDash \exists x_0 \cdots \exists x_{\kappa-1} \vartheta_\kappa(\bar{x})$$

(if no such cardinal exists we set $\kappa := \omega$), and let λ be the maximal finite cardinal such that

$$\mathfrak{N}_{\rho(8k)}(\bar{a}) \vDash \chi_\lambda(\bar{a}).$$

Note that $\lambda \leq |\bar{a}| = m - k$, by Lemma 6.2 (d).

Let $\bar{e}$ be some $\rho(8k - 3)$-scattered sequence of length λ with $N_{\rho(8k-6)}(e_i) \subseteq N_{\rho(8k-4)}(\bar{a})$ such that every e_i satisfies ψ. We claim that $\bar{e}c$ is $\rho(8k-3)$-scattered.

If $e_i \in N_{\rho(8k-3)}(c)$, for some i, then $N_{\rho(8k-3)}(c) \subseteq N_{\rho(8k-2)}(e_i)$. But there is some index l such that $e_i \in N_{\rho(8k-3)}(a_l)$ which implies

$$N_{\rho(8k-2)}(e_i) \subseteq N_{\rho(8k-1)}(a_l).$$

Hence, we have $N_{\rho(8k-3)}(c) \subseteq N_{\rho(8k-1)}(a_l)$ in contradiction to our assumption on c.

Similarly, if $c \in N_{\rho(8k-3)}(e_i)$, for some i, then $e_i \in N_{\rho(8k-4)}(a_l)$, for some l, implies $N_{\rho(8k-3)}(e_i) \subseteq N_{\rho(8k-2)}(a_l)$, and it follows that $c \in N_{\rho(8k-2)}(a_l)$. Therefore, we have $N_{\rho(8k-1)}(c) \subseteq N_{\rho(8k)}(a_l)$ which again contradicts our assumption on c.

We have shown that $\bar{e}c$ is a $\rho(8k-3)$-scattered sequence every element of which satisfies ψ. This implies $\kappa \geq \lambda + 1$.

Since $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same basic local sentences it follows that

$$(\mathfrak{B}, (D_i)_{i<\rho(8k)}) \vDash \exists x_0 \cdots \exists x_\lambda \vartheta_{\lambda+1}(\bar{x}).$$

By definition of J_k we further have

$$\mathfrak{N}_{\rho(8k)}(\bar{b}) \vDash \chi_\lambda(\bar{b}) \wedge \neg\chi_{\lambda+1}(\bar{b}),$$

since the quantifier rank of this formula is bounded by

$$\lambda + 1 + m(k-1) + 1 \leq m - k + 1 + m(k-1) + 1 \leq mk + 1.$$

Let $\bar{e} \in B^{\lambda+1}$ be a sequence satisfying $\vartheta_{\lambda+1}$. There must be some index i such that $N_{\rho(8k-6)}(e_i) \nsubseteq N_{\rho(8k-4)}(\bar{b})$. By Lemma 6.2 (b), it follows that

$$N_{\rho(8k-6)}(e_i) \cap N_{\rho(8k-6)}(\bar{b}) = \varnothing.$$

Thus, we have found an element $d := e_i$ such that

$$N_{\rho(8k-8)}(d) \cap N_{\rho(8k-6)}(\bar{b}) = \varnothing$$

and $\quad \mathfrak{N}_{\rho(8(k-1))}(c) \equiv_{m(k-1)+1} \mathfrak{N}_{\rho(8(k-1))}(d). \qquad \square$

Corollary 6.9. *On the class of globally uniform structures every first-order sentence is equivalent to a boolean combination of basic local and basic global sentences.*

7 The theorem of Gaifman for linearly Σ-uniform structures

The requirement of globally uniformity is a rather strong one. Essentially it only covers structures that can be obtained from a sparse structure by an interpretation. In particular, linear orders are not globally uniform. In order to extend the theorem of Gaifman to linearly ordered structures we therefore try to weaken our assumptions by considering structures that are globally uniform only with respect to some relations.

Definition 7.1. Let $\mathfrak{A}$ be a $(\Sigma \uplus \Xi)$-structure where Ξ contains a binary relation symbol $\leq \in \Xi$.

(a) A system of neighbourhoods N for $\mathfrak{A}$ is *linear* (w.r.t. $\leq$) if $\leq^{\mathfrak{A}}$ is a linear preorder on A such that every set $N_k(a)$ is convex w.r.t. $\leq$.

(b) Let N be a linear system of neighbourhoods for $\mathfrak{A}$. For $a, b \in A$ and $k < \omega$, we define

$$H_k(a,b) := \bigcup \{ N_k(c) \mid a \leq c \leq b \text{ or } b \leq c \leq a \}.$$

(c) We call $\mathfrak{A}$ *linearly Σ-uniform w.r.t. N* if

- N is a linear system of neighbourhoods,
- the Σ-reduct $\mathfrak{A}|_\Sigma$ is globally uniform w.r.t. N, and
- there is a number β such that, for all $n < \omega$, every pair $a, a' \in A$, and all tuples $\bar{b}, \bar{c} \in (H_0(a, a'))^n$,

$$\text{atp}(\bar{b}) = \text{atp}(\bar{c}) \quad \text{implies}$$
$$\text{atp}(\bar{b}/A \smallsetminus H_\beta(a, a')) = \text{atp}(\bar{c}/A \smallsetminus H_\beta(a, a')).$$

(All types are with respect to the full signature $\Sigma \cup \Xi$.)

Suppose that $\mathfrak{A}$ is a $(\Sigma \uplus \Xi)$-structure such that $\mathfrak{A}|_\Sigma$ is globally uniform. We try to simplify the Ehrenfeucht-Fraïssé game by removing all relations in Σ and playing on the resulting reduct. Of course, we have to somehow take into account the relations we deleted. We do so by labelling the elements of A by the type of their neighbourhoods. Thus, the simplification consists in replacing the relations of Σ by unary predicates.

Definition 7.2. Let $\mathfrak{A}$ be a $(\Sigma \cup \Xi)$-structures with linear systems of neighbourhoods. For $\bar{a} \subseteq A$ and $b, c \in A$, we define

$$\mathfrak{N}_k(\bar{a}) := \left(\mathfrak{A}|_{N_k(\bar{a})},\ (D_i)_{i<k},\ \bar{a} \right),$$

$$\mathfrak{H}_k(b, c) := \left(\mathfrak{A}|_{H_k(b,c)},\ (D_i)_{i<k} \right),$$

$$\text{and} \quad \mathfrak{L}_{m,k}(\bar{a}) := \left(\mathfrak{A}|_\Xi,\ (D_i)_{i\leq k},\ (P_\tau)_\tau,\ \bar{a} \right),$$

where

$$D_i := \left\{ (b, c) \mid c \in N_i(b) \right\},$$

$$P_\tau := \left\{ b \in A \mid \mathrm{Th}_m(\mathfrak{N}_k(b)) = \tau \right\},$$

and $\mathrm{Th}_m(\mathfrak{N})$ denotes the infinitary first-order theory of $\mathfrak{N}$ of quantifier rank m. We also set $\mathfrak{L}_{m,k}(\mathfrak{A}) := \mathfrak{L}_{m,k}(\langle\rangle)$.

We will reduce the game on two structures $\mathfrak{A}$ and $\mathfrak{B}$ to a game on the corresponding structures $\mathfrak{L}_{m,k}(\mathfrak{A})$ and $\mathfrak{L}_{m,k}(\mathfrak{B})$. Note that the classical theorem of Gaifman can be seen as a reduction of $\mathrm{EF}_m(\mathfrak{A}, \mathfrak{B})$ to a game between two structures $(A, \bar{P})$ and $(B, \bar{P})$ with only unary predicates $\bar{P}$.

Definition 7.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $(\Sigma \cup \Xi)$-structures with linear systems of neighbourhoods and let $\leq\, \in \Xi$ be the corresponding preorder.
(a) Two tuples $\bar{a} \in A^n$ and $\bar{b} \in B^n$ are m-congruent if

$$a_i \in N_k(a_l) \Leftrightarrow b_i \in N_k(b_l), \quad \text{for all } i, l < n \text{ and } k \leq m - \max\{i, l\}.$$

(b) Set $\rho(n) := \zeta^{2n}(\beta)$ and let $m < \omega$ and $\bar{a} \in A^{m-k}$. We define a partition of $\bar{a}$ into several intervals $H_k(a_i, a_l)$ as follows.
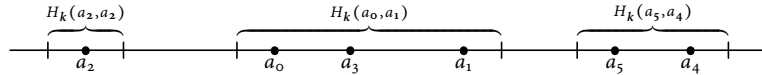
The partition is induced by the following equivalence relation $\sim\, \subseteq [m-k]^2$. For $l < n < m - k$, we define by induction on n

$$l \sim n \quad :\text{iff} \quad \text{there are } i, j < n \text{ with } i \sim j \sim l \text{ such that}$$
$$N_{\rho(m-n-1)}(a_n) \subseteq H_{\rho(m-n)}(a_i, a_j).$$

Further, we set

$$\sigma(n) := \min\{i \mid i \sim n\} \quad \text{and} \quad S := \{n \mid \sigma(n) = n\},$$

and we define functions μ and ν by

$$\mu(n) \sim n \sim \nu(n) \quad \text{and} \quad a_{\mu(n)} \leq a_n \leq a_{\nu(n)},$$

for every $n < m - k$.

Example. For the tuple $\bar{a} \in A^5$ in the above diagram, we have $S = \{0, 2, 4\}$,

$$\sigma^{-1}(0) = \{0, 1, 3\}, \quad \sigma^{-1}(2) = \{2\}, \quad \sigma^{-1}(4) = \{4, 5\},$$

$$\mu(0) = \mu(1) = \mu(3) = 0 \quad \text{and} \quad \nu(0) = \nu(1) = \nu(3) = 1,$$
$$\mu(2) = 2 \qquad\qquad\qquad \text{and} \quad \nu(2) = 2,$$
$$\mu(4) = \mu(5) = 5 \qquad\quad \text{and} \quad \nu(4) = \nu(5) = 4.$$

Remark. (a) For all n, we have

$$N_k(a_n) \subseteq H_k(a_{\mu(n)}, a_{\nu(n)}) \subseteq N_{m-\sigma(n)}(a_{\sigma(n)}).$$

(b) If $\bar{a}$ and $\bar{b}$ are m-congruent then both tuples lead to the same σ, S, μ, and ν.
(c) Note that each $\sim$-class is of the form $\sigma^{-1}(n)$, for some $n \in S$.

Theorem 7.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $(\Xi \cup \Sigma)$-structures with linearly Σ-uniform systems of neighbourhoods.*

$$\mathfrak{L}_{m-1, \rho(m-1)}(\mathfrak{A}) \equiv_{2m} \mathfrak{L}_{m-1, \rho(m-1)}(\mathfrak{B}) \quad \text{implies} \quad \mathfrak{A} \equiv_m \mathfrak{B}.$$

Proof. To simplify notation we set

$$H_k[\bar{a}; n] := H_k(a_{\mu(n)}, a_{\nu(n)}) \quad \text{and} \quad \mathfrak{H}_k[\bar{a}; n] := \mathfrak{H}_k(a_{\mu(n)}, a_{\nu(n)}).$$

Let $\bar{a} \in A^{m-k}$ and $\bar{b} \in B^{m-k}$. We call the map $\bar{a} \mapsto \bar{b}$ *good* if

- $\bar{a}$ and $\bar{b}$ are m-congruent,
- $\mathfrak{H}_{\rho(k)}[\bar{a}; n], \bar{a}|_{\sigma^{-1}(n)} \equiv_k \mathfrak{H}_{\rho(k)}[\bar{b}; n], \bar{b}|_{\sigma^{-1}(n)}$, for all $n \in S$,
- $\mathfrak{L}_{k-1, \rho(k-1)}(\bar{a}|_S) \equiv_{m+k} \mathfrak{L}_{k-1, \rho(k-1)}(\bar{b}|_S)$.

Let

$$J_k := \{ \bar{a} \mapsto \bar{b} \mid \bar{a} \in A^{m-k}, \bar{b} \in B^{m-k}, \bar{a} \mapsto \bar{b} \text{ is good} \} .$$

We claim that $(J_k)_k : \mathfrak{A} \equiv_m \mathfrak{B}$. By assumption we have $\langle \rangle \mapsto \langle \rangle \in J_m$. For the forth property, suppose that $\bar{a} \mapsto \bar{b} \in J_k$ and $c \in A$. We consider two cases.

First, suppose that $N_{\rho(k-1)}(c) \subseteq H_{\rho(k)}[\bar{a}; n]$, for some $n \in S$.

$$\mathfrak{H}_{\rho(k)}[\bar{a}; n], \bar{a}|_{\sigma^{-1}(n)} \equiv_k \mathfrak{H}_{\rho(k)}[\bar{b}; n], \bar{b}|_{\sigma^{-1}(n)}$$

implies that there is some $d \in H_{\rho(k)}[\bar{b}; n]$ such that

$$\mathfrak{H}_{\rho(k)}[\bar{a}; n], \bar{a}|_{\sigma^{-1}(n)} c \equiv_{k-1} \mathfrak{H}_{\rho(k)}[\bar{b}; n], \bar{b}|_{\sigma^{-1}(n)} d .$$

It follows that

$$\mathfrak{H}_{\rho(k-1)}(x, x'), \bar{a}|_{\sigma^{-1}(n)} c \equiv_{k-1} \mathfrak{H}_{\rho(k-1)}(y, y'), \bar{b}|_{\sigma^{-1}(n)} d ,$$

where $x := \min \{a_{\mu(n)}, c\}$ and $x' := \max \{a_{\nu(n)}, c\}$ and similarly for y and y'. Consequently, we have $\bar{a}c \mapsto \bar{b}d \in J_{k-1}$.

It remains to consider the case that $N_{\rho(k-1)}(c) \nsubseteq H_{\rho(k)}[\bar{a}; n]$, for all $n \in S$. Then we have $N_{\rho(k-1)}(c) \cap H_{\rho(k-1)}[\bar{a}; n] = \varnothing$, for all $n \in S$, and

$$\mathfrak{L}_{k-1, \rho(k-1)}(\bar{a}|_S) \equiv_{m+k} \mathfrak{L}_{k-1, \rho(k-1)}(\bar{b}|_S)$$

implies that there is some $d \in B$ such that

$$\mathfrak{L}_{k-1, \rho(k-1)}(\bar{a}|_S), c \equiv_{m+k-1} \mathfrak{L}_{k-1, \rho(k-1)}(\bar{b}|_S), d .$$

Therefore, we have

$$\mathfrak{L}_{k-2, \rho(k-2)}(\bar{a}|_S c) \equiv_{m+k-1} \mathfrak{L}_{k-2, \rho(k-2)}(\bar{b}|_S d) ,$$

and $\bar{a}c \mapsto \bar{b}d \in J_{k-1}$.

Finally, we have to show that every $\bar{a} \mapsto \bar{b} \in J_0$ is a partial isomorphism. Suppose that $\mathfrak{A} \vDash \varphi(\bar{c})$ where $\bar{c} \subseteq \bar{a}$ and φ is a literal. If there is some $n \in S$ such that $\bar{c} \subseteq H_\beta[\bar{a}; n]$ then

$$\mathfrak{H}_\beta[\bar{a}; n], \bar{a}|_{\sigma^{-1}(n)} \equiv_0 \mathfrak{H}_\beta[\bar{b}; n], \bar{b}|_{\sigma^{-1}(n)}$$

implies that $\mathfrak{B} \vDash \varphi(\bar{d})$ where $\bar{d} \subseteq \bar{b}$ is the corresponding subtuple of $\bar{b}$. Hence, we may assume that $\bar{c} = \bar{c}_0 \ldots \bar{c}_l$ where $l > 0$ and there are distinct numbers $n_0, \ldots, n_l \in S$ such that

$$\bar{c}_i \subseteq H_0[\bar{a}; n_i] , \qquad \text{for } i \leq l .$$

Since $\mathfrak{A}$ is linearly Σ-uniform we have

$$\mathfrak{A} \vDash \varphi(\bar{c}'_0, \ldots, \bar{c}'_l) \quad \text{for all tuples } \bar{c}'_i \subseteq H_0[\bar{a}; n_i] \text{ with } \mathrm{atp}(\bar{c}'_i) = \mathrm{atp}(\bar{c}_i) .$$

Let $\alpha_i(\bar{x}, y) := \bigwedge \mathrm{atp}(\bar{c}_i a_{n_i})$ be the quantifier-free formula describing the atomic type of the tuple $\bar{c}_i a_{n_i}$, and set

$$\beta_i(\bar{x}) := \alpha_i(\bar{x}, a_{n_i}) \wedge \bar{x} \subseteq N_{\rho(m-n_i)}(a_{n_i})$$
$$\wedge \bigwedge_{k \in S \smallsetminus \{n_i\}} N_\beta(\bar{x}) \cap N_{\rho(m-k)}(a_k) = \varnothing .$$

Then we have

$$\mathfrak{A} \vDash \forall \bar{x}_0 \cdots \forall \bar{x}_l \Big(\bigwedge_{i \leq l} \beta_i(\bar{x}_i) \to \varphi(\bar{x}_0, \ldots, \bar{x}_l) \Big) .$$

Since $\mathfrak{L}_{0, \beta}(\bar{a}|_S) \equiv_m \mathfrak{L}_{0, \beta}(\bar{b}|_S)$ it follows that

$$\mathfrak{B} \vDash \forall \bar{x}_0 \cdots \forall \bar{x}_l \Big(\bigwedge_{i \leq l} \beta_i(\bar{x}_i) \to \varphi(\bar{x}_0, \ldots, \bar{x}_l) \Big) .$$

Consequently, we have $\mathfrak{B} \vDash \varphi(\bar{d}_0, \ldots, \bar{d}_l)$ where $\bar{d}_i \subseteq \bar{b}$ is the subtuple of $\bar{b}$ corresponding to $\bar{c}_i$. $\qquad \square$

Example. Set $\Xi := \{\leq, E\}$ and $\Sigma := \bar{R}$. We consider two linearly ordered structures $\mathfrak{A} = (A, \leq, E, \bar{R})$ and $\mathfrak{B} = (B, \leq, E, \bar{R})$ where E is the successor relation of $\leq$. We define the distance $d(a, b)$ between elements $a, b \in A$ as their Gaifman distance in the reduct $\mathfrak{A}|_{E, \bar{R}}$, i.e., we ignore $\leq$. Let $N_r(a)$ be the r-neighbourhood of a with respect to this distance. The structures $\mathfrak{L}_{m, n}(\bar{a})$ are labelled linear orders where the colour of an element denotes the type of its r-neighbourhood. Hence, we have reduced the game on $\mathfrak{A}$ and $\mathfrak{B}$ to a simpler game on labelled linear orders.

We conclude this section with a more substantial application. Let $\mathfrak{R}$ be a ring. We consider a chain complex $(\mathfrak{M}_\bullet, d_\bullet)$ of $\mathfrak{R}$-modules $\mathfrak{M}_n = \langle M_n, +, (\lambda_r)_{r \in R} \rangle$,

$$\cdots \to \mathfrak{M}_2 \to \mathfrak{M}_1 \to \mathfrak{M}_0 \to 0$$

that is, a sequence of homomorphisms $d_n : \mathfrak{M}_n \to \mathfrak{M}_{n-1}$ between modules with $d_n \circ d_{n+1} = 0$. We encode such a complex as a structure

$$\mathfrak{C}(\mathfrak{M}_\bullet, d_\bullet) = \langle C, +, (\lambda_r)_{r \in R}, d, \leq \rangle$$

where

- $C = \biguplus_n M_n$ is the disjoint union of the universes M_n,
- $+$ is the union of the (graphs of the) addition operations on each M_n,
- λ_r is the union of the scalar multiplication operations on each M_n,
- d is the union of the graphs of the d_n, and
- the preorder $\leq$ is defined by

$$a \leq b \quad :\text{iff} \quad a \in M_i \text{ and } b \in M_k \text{ for } i \leq k.$$

Recall that the n-th homology group of $(\mathfrak{M}_\bullet, d_\bullet)$ is

$$H_n(\mathfrak{M}_\bullet, d_\bullet) := \operatorname{kern} d_n / \operatorname{rng} d_{n+1}.$$

We will prove that there does not exist a first-order formula φ that holds in a structure of the form $\mathfrak{C}(\mathfrak{M}_\bullet, d_\bullet)$ if and only if there exists a maximal index $n < \omega$ with $H_n(\mathfrak{M}_\bullet, d_\bullet) \neq 0$ and this index n is even.

For a contradiction, suppose that φ is a sentence with the desired properties. Let $\mathfrak{R} = \mathbb{Q}$ and $\mathfrak{M}_n := \mathbb{Q}^\omega$, for all n. We define a function $d : \mathbb{Q}^\omega \to \mathbb{Q}^\omega$ by

$$d(a_i)_{i < \omega} := (b_i)_{i < \omega} \quad \text{where} \quad b_i := \begin{cases} a_{i+1} & \text{if } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\operatorname{kern}(d) = \operatorname{rng}(d)$ and the sequence

$$\cdots \xrightarrow{d} \mathfrak{M}_2 \xrightarrow{d} \mathfrak{M}_1 \xrightarrow{d} \mathfrak{M}_0$$

is exact. Furthermore, let $d' : \mathbb{Q}^\omega \to \mathbb{Q}^\omega$ be the constant map with value 0.

Let m be the quantifier rank of φ, set $r := \rho(m-1)$, and fix a number $l > 2r + 2^{2m}$. We define two complexes $(\mathfrak{M}_\bullet, d_\bullet^0)$ and $(\mathfrak{M}_\bullet, d_\bullet^1)$ by setting

$$d_n^i := \begin{cases} d' & \text{if } n = 0 \text{ or } n = l + i, \\ d & \text{otherwise,} \end{cases} \quad \text{for } i < 2.$$

$$\cdots \xrightarrow{d} \mathfrak{M}_{l+i} \xrightarrow{d'} \mathfrak{M}_l \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{M}_2 \xrightarrow{d} \mathfrak{M}_1 \xrightarrow{d} \mathfrak{M}_0 \xrightarrow{d'} 0$$

It follows that

$$H_n(\mathfrak{M}_\bullet, d_\bullet^i) = \begin{cases} \mathbb{Q}^\omega & \text{if } n = 0 \text{ or } n = l + i, \\ 0 & \text{otherwise.} \end{cases}$$

By assumption on φ we therefore have

$$\mathfrak{C}(\mathfrak{M}_\bullet, d_\bullet^0) \vDash \varphi \quad \text{iff} \quad \mathfrak{C}(\mathfrak{M}_\bullet, d_\bullet^1) \nvDash \varphi.$$

In order to apply Theorem 7.4, we partition the signature into $\Xi := \{\leq\}$ and $\Sigma := \{+, (\lambda_r)_r, d\}$. We define a system of neighbourhoods N by setting

$$N_r(a) := \bigcup \{ M_i \mid n - r \leq i \leq n + r \},$$

where n is the index such that $a \in M_n$. With these definitions a structure of the form $\mathfrak{C}(\mathfrak{M}_\bullet, d_\bullet)$ becomes linearly Ξ-uniform.

Let a be an element of the first complex and b an element of the second one. Suppose that $a \in M_i$ and $b \in M_j$. If

- $i = j < l - r$ or
- $i = j - 1 > r$ or
- $r < i < l - r$ and $r < j < l + 1 - r$

then we have

$$\mathfrak{H}_r(a) \cong \mathfrak{H}_r(b).$$

It follows that $\mathfrak{L}_{m-1,r}(\mathfrak{C}(\mathfrak{M}_\bullet, d_\bullet^0))$ consists of a coloured linear preorder where in the middle part there are only two colours: the zero elements of each M_i have one colour and all other elements have the second colour. Furthermore, we obtain the structure $\mathfrak{L}_{m-1,r}(\mathfrak{C}(\mathfrak{M}_\bullet, d_\bullet^1))$ from the first one by inserting a copy of $\mathfrak{M}_i$ that is also coloured this way. Since the middle part consists of more than $2r + 2^{2m} - 2r \geq 2^{2m}$ copies of $\mathfrak{M}_i$ it follows that Duplicator has a winning strategy for the $2m$-round Ehrenfeucht-Fraïssé game between these structures. By Theorem 7.4, it follows that

$$\mathfrak{C}(\mathfrak{M}_\bullet, d_\bullet^0) \equiv_m \mathfrak{C}(\mathfrak{M}_\bullet, d_\bullet^1).$$

A contradiction.

8 Conclusion

We have investigated tools to simplify Ehrenfeucht-Fraïssé games on non-sparse structures. In the first part of the paper we have presented several simple ways to decompose a game on two structures into games on certain substructures. Technically the main idea behind these constructions was the colouring of tuples by their external type.

In the second part of the paper we have tried to generalise the theorem of Gaifman to non-sparse structures. In particular, we aimed at covering well-known examples from the literature which successfully employed locality-based Ehrenfeucht-Fraïssé arguments. By introducing the notions of global uniformity and linear uniformity we were able to do so for the case of linearly ordered structures. We conclude this article by mentioning two important cases which still remain open.

Open Problem. *Extend the theorem of Gaifman such that it covers*

(a) *trees (with ordering),*

(b) *Presburger Arithmetic and algebraically closed fields.*

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