The Expansion Problem for Infinite Trees

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We study Ramsey like theorems for infinite trees and similar combinatorial tools. As an application we consider the expansion problem for tree algebras.

1 Introduction

While the theory of formal languages of infinite trees is well-established by now, it is far less developed than other formal language theories. For instance, only in recent years work on an algebraic theory for such languages has begun [10, 4, 5, 11, 1, 12, 9, 6, 7, 3]. Further progress in this direction is currently hampered by our lack of understanding of the combinatorial properties of infinite trees. In particular, for many purposes the currently known Ramsey type theorems for trees are simply not strong enough. What would be needed instead are, for instance, analogues of Simon factorisation trees for infinite trees. Such Ramsey arguments are ubiquitous in the study of languages of infinite objects. For instance, in the theory of ω-words they appear in the original complementation proof for Büchi automata, when expanding a Wilke algebra to an ω-semigroup, or in the more recent work on distance automata and boundedness problems.

There are at least two persistent problems when trying to extend our repertoire of combinatorial tools to infinite trees. The first one concerns the step from arbitrary trees to regular ones: while many arguments only work if the considered trees are regular, there are currently no known algebraic methods of reducing a given tree to an equivalent regular one. For instance, in [10] all proofs work exclusively with regular trees, and only at the very end the authors transfer their results to arbitrary trees, which was possible in this particular case since the languages under consideration were regular and therefore uniquely determined by which regular trees they contain.

The second problem concerns trees that are highly-branching. Many of the known tools from the theory of ω-semigroups can be generalised to trees that are thin, i.e., that have only countably many infinite branches. But all attempts to extend them to (at least some) non-thin trees have failed so far. For example, in [11] the authors only consider languages of thin trees since their methods do not apply to non-thin ones. Later in [1], they then successfully adapted their approach to study unambiguous languages of non-thin trees, utilising the fact that trees in unambiguous languages are in a certain sense governed by their thin prefixes.

The only combinatorial methods know so far that work well even in light of the above issues are those based on automata and games since, in a certain sense, games provide a way to reduce a problem concerning the whole tree into one only involving a single branch. (In fact, one of the motivations for this paper stems from a wish to deeper understand how exactly this works, in particular during the translation of a formula into an automaton.) Unfortunately, there are many algebraic questions that resist to being phrased in automata-theoretic terms.

In this article we start by reviewing the existing techniques to study combinatorial properties of infinite trees, presenting them in the unifying language of tree algebras from [6]. We then take a look at several new approaches. We determine how far they carry and which the problems are that prevent us from continuing. Our contributions are mainly conceptual. We raise many open questions, but provide few answers. None of the results below are very deep and several remain partial. The main purpose of the article is to draw attention to a problem I consider central for further progress.

It seems that such progress will likely not come from abstract considerations but by solving concrete problems. Our focus will therefore be on a particular application, one we call the *Expansion Problem:* the problem of whether a given algebra whose product is defined only for some trees can be expanded to one whose product is defined everywhere, analogously to the expansion of a Wilke algebra (where the product is defined only for ultimately periodic words) to an ω-semigroup (where we can multiply arbitrary ω-words). This problem turns out to be a good test bed for the various approaches we consider. We solve it in some special cases, but none of our approaches is strong enough to solve the general case.

The overview of the article is as follows. We start in Section 2 with setting up the algebraic framework we will be working in. Section 3 contains a brief survey of the existing Ramsey Theorems for trees. The Expansion Problem is defined in Section 4, where we also recall some tools from [7] to prove uniqueness of expansions. The main technical part of the article are Sections 5 and 6, which contain two techniques to study expansions. The first one are so-called *evaluations,* which are a weak form of a Simon tree, the second one are *consistent labellings,* which are somewhat similar to automata. The final two sections (7 and 8) contain two applications. The first one recalls results of [1] about a characterisation of unambiguous languages in terms of consistent labellings, while the second one uses consistent labellings to define classes of tree algebras with unique expansions.

2 Tree algebras

We start with a brief introduction to the algebraic framework we will be working in. To make this article accessible to a wider audience, we have tried to keep the category-theoretic prerequisites at a minimum. A more detailed account can be found in [6, 7]. To model ranked trees, we work in a many-sorted setting where the sort of a tree represents the set of *variables* or *holes* appearing in it. Hence, we fix a countably infinite set X of variables and use the set $\Xi := \wp_{\mathrm{fin}}(X)$ of finite subsets of X as sorts. In addition, for technical reasons we equip the labels of our trees with an ordering. Hence, we will work with *partially ordered Ξ-sorted sets* which are families $A = (A_\xi)_{\xi \in \Xi}$ where each component A_ξ is a partial ordered. A function $f : A \to B$ between two such sets is then a family $f = (f_\xi)_{\xi \in \Xi}$ of monotone functions $f_\xi : A_\xi \to B_\xi$. In the following we will for simplicity use the term *set* for 'partially orderd Ξ-sorted set' and the term *function* for a function between such sets. Sometimes it is convenient to identify a set $A = (A_\xi)_{\xi \in \Xi}$ with its disjoint union $A = \sum_{\xi \in \Xi} A_\xi$. Then a function $f : A \to B$ corresponds to a sort-preserving and order-preserving function between the corresponding disjoint unions.

Given such a set A an *A-labelled tree* is a possibly infinite tree t where the vertices are labelled by elements of A and the edges by variables from X in such a way that a vertex with a label $a \in A_\xi$ of sort ξ as exactly one outgoing edge labelled by x, for every $x \in \xi$ (and no other edges). We identify such a tree with a function $t : \mathrm{dom}(t) \to A$, where $\mathrm{dom}(t)$ is the set of vertices of t. The root of t is denoted by $\langle\rangle$ (the empty tuple). If there is an x-labelled edge from a vertex u to v, we call v the *x-successor* of u.

Definition 2.1. Let A be a set.

(a) We set $\mathbb{T}^\times A := (\mathbb{T}^\times_\xi A)_{\xi \in \Xi}$ where $\mathbb{T}^\times_\xi A$ denotes the set of all $(A + \xi)$-labelled trees t (where the elements in ξ are assumed to have sort $\varnothing$) satisfying the following conditions.

- Every variable $x \in \xi$ appears at least once in t.
- The root of t is not labelled by a variable.

(b) For a function $f : A \to B$, we denote by $\mathbb{T}^\times f : \mathbb{T}^\times A \to \mathbb{T}^\times B$ the function applying f to every label of the given tree (leaving the variables unchanged).

(c) The *tree order* $\leq$ is the ordering on $\mathrm{dom}(t)$ is defined by

$$u \leq v \quad :\text{iff} \quad u \text{ lies on the path from the root to } v\,.$$

(d) For vertices u and $(v_x)_{x \in \xi}$ of a tree $t \in \mathbb{T}^\times A$, we set

$$[u, \bar{v}) := \{\, w \in \mathrm{dom}(t) \mid u \leq w \text{ and } v_x \nleq w\,,\ \text{for all } x \,\}\,.$$

We denote by $t[u, \bar{v})$ the restriction of $t : \mathrm{dom}(t) \to A$ to the set $[u, \bar{v})$ (where v_x is labelled by the variable x while all other vertices have the same label as in t). We call $t[u, \bar{v}) \in \mathbb{T}_\xi A$ the *factor* of t *between* u and $\bar{v}$. The set ξ is the *sort* of the factor. In the special case where $\xi = \langle\rangle$, we obtain the *subtree* of t *rooted* at u, which we usually denote by $t|_u$. ⌟

Remark. We can identify the vertices of a tree with words in X^*. With this convention the tree order $\leq$ is just the prefix ordering. This also explains our notation $\langle\rangle$ for the root. ⌟

We need the following two operations on trees.

Definition 2.2. Let A be a set.

(a) The *singleton operation* $\mathrm{sing} : A \to \mathbb{T}^\times A$ maps every letter $a \in A_\xi$, to the tree $\mathrm{sing}(a)$ consisting of the root with label a attached to which is one leaf with label x, for every $x \in \xi$.

(b) The *flattening operation* $\mathrm{flat} : \mathbb{T}^\times\mathbb{T}^\times A \to \mathbb{T}^\times A$ is a generalisation of term substitution. It takes a tree t labelled by trees $t(v) \in \mathbb{T}^\times A$ and combines them into a single tree as follows (see Figure 1).

- We take the disjoint union of all trees $t(v)$, for $v \in \mathrm{dom}(t)$,
- delete from each component $t(v)$ every vertex labelled by a variable $x \in X$, and
- redirect every edge of $t(v)$ leading to such a deleted vertex to the root of $t(u_x)$, where u_x is the x-successor of v in t. ⌟

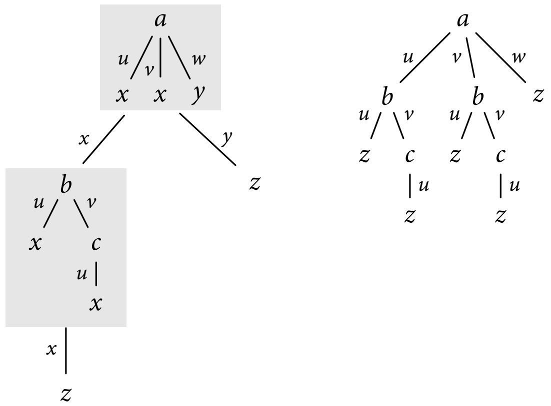


Figure 1: The flattening operation: t and $\mathrm{flat}(t)$

Remark. The triple $\langle \mathbb{T}^\times, \mathrm{flat}, \mathrm{sing}\rangle$ forms what is called a *monad* in category-theoretical language, which means it satisfies the following three equations.

$$\begin{aligned} \mathrm{flat} \circ \mathrm{sing} &= \mathrm{id}\,, \\ \mathrm{flat} \circ \mathbb{T}^\times \mathrm{sing} &= \mathrm{id}\,, \\ \mathrm{flat} \circ \mathbb{T}^\times \mathrm{flat} &= \mathrm{flat} \circ \mathrm{flat}\,. \end{aligned}$$

⌟

There are several special classes of trees we are interested in below.

Definition 2.3. (a) A *submonad* of $\mathbb{T}^\times$ is a functor $\mathbb{T}^\circ$ such that, $\mathbb{T}^\circ A \subseteq \mathbb{T}^\times A$, for every set A, and $\mathbb{T}^\circ A$ is closed under flat and sing, that is,

$$\begin{aligned} \mathrm{flat}(t) &\in \mathbb{T}^\circ A\,, && \text{for all } t \in \mathbb{T}^\circ\mathbb{T}^\circ A\,, \\ \mathrm{sing}(a) &\in \mathbb{T}^\circ A\,, && \text{for all } a \in A\,. \end{aligned}$$

We write $\mathbb{T}^\circ \subseteq \mathbb{T}^\times$ to denote this fact.

(b) We are particularly interested in the following submonads. $\mathbb{T}$ denotes the set of all *linear trees,* i.e., trees where each variable appears exactly once. $\mathbb{T}^{\mathrm{fin}}$ denotes the set of *finite linear trees,* $\mathbb{T}^{\mathrm{reg}}$ the set of *regular linear trees,* $\mathbb{T}^{\mathrm{thin}}$ the set of *thin linear trees,* i.e., trees with only countably many infinite branches, and $\mathbb{T}^{\mathrm{wilke}} := \mathbb{T}^{\mathrm{thin}} \cap \mathbb{T}^{\mathrm{reg}}$ the set of all trees that are thin and regular. The corresponding classes of non-linear trees are denoted by $\mathbb{T}^{\times\mathrm{fin}}$, $\mathbb{T}^{\times\mathrm{reg}}$, etc. ⌟

In algebraic language theory one uses algebras (usually finite ones) to recognise languages. In our setting these algebras take the following form.

Definition 2.4. Let $\mathbb{T}^\circ \subseteq \mathbb{T}^\times$.

(a) A $\mathbb{T}^\circ$-*algebra* $\mathfrak{A} = \langle A, \pi \rangle$ consists of a set A and a *product* $\pi : \mathbb{T}^\circ A \to A$ satisfying

$$\pi \circ \mathrm{sing} = \mathrm{id} \quad \text{and} \quad \pi \circ \mathbb{T}^\circ \pi = \pi \circ \mathrm{flat}\,.$$

(b) A $\mathbb{T}^\circ$-algebra is *finitary* if every domain A_ξ is finite. ⌟

Example. The following algebra $\mathfrak{A} = \langle A, \pi \rangle$ recognises the language K of all trees $t \in \mathbb{T}^\times_\varnothing\{a, b\}$ that contain at least one letter a. For each sort ξ, we use two elements $0_\xi, 1_\xi$. Hence,

$$A_\xi := \{0_\xi, 1_\xi\}\,.$$

The product is defined by

$$\pi(t) := \begin{cases} 1 & \text{if } t \text{ contains the label } 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $K = \varphi^{-1}(1_\varnothing)$, where $\varphi : \mathbb{T}\{a, b\} \to A$ is the morphism mapping a to 1 and b to 0. ⌟

A complication of the theory of infinite trees is the fact that some finitary $\mathbb{T}^\times$-algebras recognise non-regular languages [12]. For this reason we have to consider a smaller class of algebras.

Definition 2.5. Let $\mathbb{T}^\circ \subseteq \mathbb{T}^\times$. A $\mathbb{T}^\circ$-algebra $\mathfrak{A}$ is MSO-*definable* if there exists a finite set $C \subseteq A$ of generators with the following property: for every $a \in A$, there exists an MSO-formula φ_a such that

$$t \vDash \varphi_a \quad :\text{iff} \quad \pi(t) \geq a\,, \quad \text{for all } t \in \mathbb{T}^\circ A\,.$$ ⌟

Using this notion we obtain the following characterisation (for proofs see [6, 7]).

Theorem 2.6. *A* $\mathbb{T}^\times$*-algebra* $\mathfrak{A}$ *is* MSO*-definable if, and only if, every language recognised by* $\mathfrak{A}$ *is regular.*

The definition above is not very enlightening as it is basically just a restatement of the preceding theorem. Although a more algebraic characterisation has be found in [6], a simpler one would be appreciated. In particular, it would be nice to find a system of inequalities axiomatising the class of MSO-definable algebras.

Open Question. *Find a concrete description of a system of inequalities that axiomatises the class of* MSO-*definable* $\mathbb{T}^\times$*-algebras.*

By general arguments, we know that such a system of inequalities exists, although it might be infinite and the terms in the inequalities are in general profinite (see [7] for the details).

3 Partition theorems for trees

Let us start with a brief overview of the existing partition theorems for trees, followed by some remarks on how they might be extended and how they might not. The seminal partition theorem for trees is the one by Milliken.

Definition 3.1. Let $t \in \mathbb{T}A$ be a tree.

(a) For a vertex u of sort ξ and a variable $x \in \xi$, we define the relation

$$u \prec_x v \quad :\text{iff} \quad v \text{ belongs to the subtree } t|_w \text{ where } w \text{ is the } x\text{-successor of } u\,.$$

(b) A factor $[u, \bar{v})$ of t is *properly embedded* if $\bar{v} = (v_x)_{x \in \xi}$ and $u \prec_x v_x$, for all $x \in \xi$, where ξ is the sort of u. Similarly, we say that a set $P \subseteq \mathrm{dom}(t)$ is *properly embedded* if, for every vertex $u \in P$ that is not maximal in P and for every x-successor v_x of u, the subtree $t|_{v_x}$ has a unique minimal vertex that belongs to P.

(c) A *proper labelling* of t is a function λ that assigns a colour to every properly embedded factor of t.

(d) Let λ be a proper labelling of t. A set $H \subseteq \mathrm{dom}(t)$ is *homogeneous* (with respect to λ) if

$$\lambda([u, \bar{v})) = \lambda([u', \bar{v}'))\,, \quad \text{for all } u, u', v_x, v'_x \in H \text{ such that } [u, \bar{v}) \text{ and } [u', \bar{v}') \text{ are properly embedded.}$$

⌟

Theorem 3.2 (Milliken [16]). *Let $t \in \mathbb{T}_\varnothing A$ be an infinite tree without leaves and C a finitary set of colours. For every proper labelling λ of t with colours from C, there exists an infinite homogeneous set H that is properly embedded in t.*

Remark. The actual theorem by Milliken is stronger than the above version, but the extra strength is not relevant in our context. ⌟

The problem with the theorem of Milliken is that it does not give any information about factors $[u, \bar{v})$ whose end-points do not belong to the set H. For a stronger statement we need additional assumptions on the labelling. For instance, for labellings of finite words, there is the Factorisation Tree Theorem of Simon which states that, if the labelling is additive (i.e., the colours form a semigroup), we can recursively factorise the given word into homogeneous parts. This theorem has been adapted to trees by Colcombet [14] as follows.

Definition 3.3. Let $t \in \mathbb{T}A$ be a tree and $\mathfrak{S}$ an ω-semigroup.

(a) An $\mathfrak{S}$-*labelling* of t is a function λ that maps every edge of t to a semigroup element. Each such function can be extended to all non-empty (finite or infinite) paths $p = (v_i)_i$ by setting

$$\lambda(p) := \prod_i \lambda(v_i, v_{i+1}) .$$

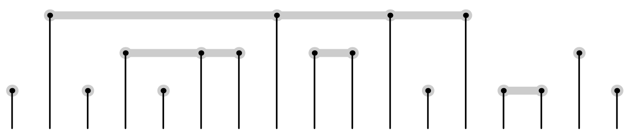
For $u < v$, we will also use the notation

$$\lambda(u, v) := \lambda(p) , \quad \text{where } p \text{ is the path from } u \text{ to } v .$$

(b) Given a function $\sigma : \mathrm{dom}(t) \to [k]$, we define a binary relation $\sqsubset_\sigma$ on $\mathrm{dom}(t)$ by

$$x \sqsubset_\sigma y \quad \text{iff} \quad x < y , \quad \sigma(x) = \sigma(y) , \quad \text{and}$$
$$\sigma(z) \le \sigma(x) , \quad \text{for all } x \le z \le y .$$

As usual, $\sqsubseteq_\sigma$ denotes the reflexive version of $\sqsubset_\sigma$.



(c) A *weak Ramseyan split* of an $\mathfrak{S}$-labelling λ is a function $\sigma : \mathrm{dom}(t) \to [k]$ such that

$$\lambda(x, y) = \lambda(x, y) \cdot \lambda(x', y') , \quad \text{for all } x \sqsubset_\sigma y \text{ and } x' \sqsubset_\sigma y'$$
$$\text{such that } y \sqsubseteq_\sigma y' \text{ or } y' \sqsubseteq_\sigma y . \quad ⌟$$

Theorem 3.4. *Let t be a tree and $\mathfrak{S}$ a finite ω-semigroup. Every $\mathfrak{S}$-labelling λ of t has a weak Ramseyan split. Furthermore, this split is* MSO*-definable.*

As an example of how to apply this theorem, let us mention the following result from [5] that can be used to turn arbitrary trees into regular ones.

Definition 3.5. Let $\mathfrak{S}$ be an ω-semigroup and λ an $\mathfrak{S}$-labelling of some tree t. We denote by $\lim \lambda$ the set

$$\lim \lambda := \{ \lambda(\beta) \mid \beta \text{ a branch of } t \} .$$

⌟

Theorem 3.6. *Let $\mathfrak{S}$ be a finite ω-semigroup. For every $\mathfrak{S}$-labelling λ of a tree t, there exists a regular tree t_0 and a regular $\mathfrak{S}$-labelling λ_0 of t_0 such that* $\lim \lambda_0 = \lim \lambda$.

The proof consists in fixing a weak Ramseyan split of λ and using it to replace certain subtrees of t by back-edges. The unravelling of the resulting graph is the desired regular tree t_0.

As a second application let us see how to use Ramseyan splits to evaluate products in a $\mathbb{T}$-algebra. Since every tree $t \in \mathbb{T}A$ can be factorised into a finite number of (I) trees of arity 1 and (II) singletons, we can use an edge-labelling to reduce every product $\pi(t)$ to a finite one. (Note that this does not work for trees in $\mathbb{T}^\times A$.) The details are as follows.

Definition 3.7. Let $\mathfrak{A}$ be a $\mathbb{T}$-algebra.

(a) The *canonical edge labelling* λ of a tree $t \in \mathbb{T}A$ is defined by

$$\lambda(u, v) := \pi(t[u, v)) , \quad \text{for } u < v .$$

(b) Let L be a logic. We say that $\mathfrak{A}$ is L-definable *with respect to* a certain type of labelling (of edges or vertices) if $\mathfrak{A}$ is finitary and, for every finite $C \subseteq A$ and every $a \in A$, there exists an L-formula φ such that

$$\langle t, \lambda \rangle \vDash \varphi \quad \text{iff} \quad \pi(t) \geq a , \quad \text{for all } t \in \mathbb{T}C ,$$

where λ is the labelling for t and $\langle t, \lambda \rangle := \langle \mathrm{dom}(t), \preceq, (\mathrm{suc}_x)_x, (P_c)_{c \in C}, (R_s)_{s \in S} \rangle$ denotes the usual encoding of t as a relational structure with additional relations R_s (unary or binary) for the labelling λ. ⌟

Proposition 3.8. *Every finitary $\mathbb{T}$-algebra is* FO*-definable with respect to the canonical edge labelling.*

Proof. Fix $a \in A_\xi$ and $C \subseteq A$. We define the desired formula φ by induction on $|\xi|$. First consider the case where $\xi = \varnothing$. The formula φ checks whether the given tree t has a leaf. If this is the case it picks one, say v, and then checks whether

$$\lambda(\langle\rangle, v) \cdot t(v) \geq a\,.$$

Otherwise, we fix, for every sort ζ used by some element of C, some variable $z \in \zeta$. If we start at the root of t and follow the successors labelled by one of these chosen variables, we obtain an infinite branch of t. This branch is FO-definable. The formula φ checks that the branch contains an infinite sequence $v_0 \prec v_1 \prec \dots$ of vertices such that

$$\lambda(v_i, v_k) = \lambda(v_0, v_1)\,, \quad \text{for all } i < k < \omega\,,$$
$$\lambda(\langle\rangle, v_0) \cdot \lambda(v_0, v_1)^\omega \geq a\,.$$

This can be done in first-order logic using a trick of Thomas [18].

Next suppose that $\xi = \{x\}$ is a singleton. Then the formula φ guesses the leaf v of the tree t labelled by x and checks that

$$\lambda(\langle\rangle, v) \geq a\,.$$

Finally, we consider the case where $|\xi| > 1$. Given a tree t, the formula guesses the longest common prefix w of all vertices labelled by some variable in ξ. Let w_0 be the predecessor of w and let $v_0, \dots, v_{n-1}$ be its successors. By inductive hypothesis, there exist formulae computing $\pi(t|_{v_i})$, for every i. Consequently, we can also compute

$$\pi(t) = \lambda(\langle\rangle, w_0) \cdot t(w)\big(\pi(t|_{v_0}), \dots, \pi(t|_{v_{n-1}})\big)\,. \qquad \square$$

Definition 3.9. Let L be a logic and let τ be an edge labelling of some tree $\mathfrak{T}$.

(a) We say that τ is L-definable if, for every $c \in \operatorname{rng}\tau$, there exists an L-formula $\varphi_c(x, y)$ such that

$$\tau(u, v) = c \quad \text{iff} \quad \mathfrak{T} \models \varphi_c(u, v)\,, \quad \text{for every edge } \langle u, v\rangle\,.$$

The definition for labellings of vertices is analogous.

(b) We say that the product π of an algebra $\mathfrak{A}$ is L-definable on a term $t \in \mathbb{T}A$ *in terms of* a labelling $\tau : \operatorname{dom}(t) \to C$ if, for every $a \in A$, there exists an L-formula $\psi_a(x, \bar{y})$ such that

$$\pi(t[u, \bar{v})) = a \quad \text{iff} \quad \langle t, \tau\rangle \models \psi_a(u, \bar{v})\,, \quad \text{for every factor } [u, \bar{v})\,. \qquad ⌟$$

Theorem 3.10. *Let $\mathfrak{A}$ be a finitary $\mathbb{T}$-algebra. For every finite set $C \subseteq A$, there exists a finite set S such that every $t \in \mathbb{T}C$ has a labelling $\sigma : \mathrm{dom}(t) \to S$ such that the product on t is* FO*-definable in terms of σ. Furthermore, if $\mathfrak{A}$ is* MSO*-definable, then so is σ.*

Proof. We have shown in Proposition 3.8 that $\mathfrak{A}$ is FO-definable with respect to the canonical edge labelling λ. Consequently, it is sufficient to find a vertex labelling σ such that λ is FO-definable in terms of σ.

By Theorem 3.4, there exists an MSO-definable weak Ramseyan split $\sigma_0 : \mathrm{dom}(t) \to [k]$ for λ. Unfortunately, λ does not need to be FO-definable in terms of σ_0. Therefore, we define an extended labelling

$$\sigma : \mathrm{dom}(t) \to [k] \times A_{\{x\}} \times A_{\{x\}}$$

as follows. For a vertex $v \in \mathrm{dom}(t)$, let $p(v)$ be the predecessor of v (if it exists) and let $q(v) \prec v$ be the maximal vertex (if it exists) such that $q(v) \sqsubset_{\sigma_0} v$. Fixing an arbitrary element $a_0 \in A_{\{x\}}$, we set

$$\sigma(v) := \begin{cases} \langle \sigma_0(v),\ \lambda(p(v), v),\ \lambda(q(v), v) \rangle & \text{if } q(v) \text{ is defined,} \\ \langle \sigma_0(v),\ \lambda(p(v), v),\ a_0 \rangle & \text{if } p(v) \text{ is defined, but } q(v) \\ & \quad \text{is not,} \\ \langle \sigma_0(v),\ a_0,\ a_0 \rangle & \text{otherwise.} \end{cases}$$

Note that, if $\mathfrak{A}$ is MSO-definable, then so is σ. Hence, it remains to prove that λ is FO-definable in $\langle t, \sigma \rangle$.

Consider two vertices $u \prec v$. We can compute $\lambda(u, v)$ from σ as follows. We start by defining the sequence $u_0 \prec \cdots \prec u_m \preceq v_n \prec \cdots \prec v_0$ where $u_0 = u$, $v_0 = v$, u_{i+1} is the minimal vertex bettween u_i and v such that $\sigma_0(u_{i+1}) > \sigma_0(u_i)$, and v_{i+1} is the maximal vertex bettween u and v_i such that $\sigma_0(v_{i+1}) > \sigma_0(v_i)$. Note that this sequence if FO-definable and its length is bounded in terms of k. Since

$$\begin{aligned} \lambda(u, v) = \lambda(u_0, u_1) \cdot \dots \cdot \lambda(u_{m-1}, u_m) \\ \cdot \lambda(u_m, v_n) \\ \cdot \lambda(v_n, v_{n-1}) \cdot \dots \cdot \lambda(v_1, v_0) \end{aligned}$$

(where we drop the factor $\lambda(u_m, v_n)$ in case $u_m = v_n$), we can evaluate $\lambda(u, v)$ in first-order logic, provided that we can define the values $\lambda(u_i, u_{i+1})$, $\lambda(u_m, v_n)$, and $\lambda(v_{i+1}, v_i)$.

Let us explain how to compute $\lambda(u_i, u_{i+1})$. The other two cases are analogous. To do so, let $m < k$ be a number and $u' < v'$ two vertices such that

$$\sigma_0(w) \leq m\,, \quad \text{for all } u' \leq w < v'\,.$$

By induction on m, we will construct an FO-formula computing $\lambda(u', v')$. Let w_0 be the minimal vertex $u' \leq w_0 < v'$ with $\sigma_0(w_0) = m$, let w_2 be the maximal one, and let w_1 be the minimal vertex $w_0 < w_1 \leq w_2$ with $\sigma_0(w_2) = m$. We distinguish several cases. If v' is the successor of u', we can read off $\lambda(u', v')$ from $\sigma(v')$. If w_0 does not exist, we can use the formula from the inductive hypothesis. If $w_0 = w_2$, we have

$$\lambda(u', v') = \lambda(u', w_0) \cdot \lambda(w_0, v')\,,$$

where both factors can be computed by inductive hypothesis. Finally, consider the case where $w_0 < w_2$. Then $w_0 \sqsubset_{\sigma_0} w_1 \sqsubseteq_{\sigma_0} w_2$ implies that

$$\begin{aligned}\lambda(u', v') &= \lambda(u', w_0) \cdot \lambda(w_0, w_2) \cdot \lambda(w_2, v') \\ &= \lambda(u', w_0) \cdot \lambda(w_0, w_1) \cdot \lambda(w_2, v')\,,\end{aligned}$$

where the middle factor can be read of from $\sigma(w_1)$ and the other two factors can be computed by inductive hypothesis. □

The problem with the above theorems is that they require access to the canonical edge-labelling λ and, in order to obtain this labelling, we need to be able to compute products $\pi([u, v)]$ where the factors $[u, v)$ are usually infinite. Unfortunately, in many applications we only know the products of *finite* factors. For instance, we cannot combine Theorems 3.6 and 3.10 to conclude that every tree $t \in \mathbb{T}A$ over a finitary $\mathbb{T}$-algebra $\mathfrak{A}$ can be replaced by a regular one with the same product since we do not know whether or not the regular labelling λ_0 constructed in Theorem 3.6 is a canonical edge-labelling. It seems that a good first step towards progress would be to prove a finitary version of Theorem 3.10 like the following one.

Conjecture. *Let $\mathfrak{A}$ be a finitary $\mathbb{T}^{\text{fin}}$-algebra. For every finite set $C \subseteq A$, there exists a finite set S such that every $t \in \mathbb{T}C$ has a labelling $\sigma : \text{dom}(t) \to S$ such that, for every* finite *factor $[u, \bar{v})$ of t,*

$$\pi(t[u, \bar{v}))\ \textit{can be computed from } \sigma \textit{ by an } \text{FO-}\textit{formula}\,.$$

We conclude this section with a counterexample showing that some natural ways to approach this conjecture (or a similar one) do not work. It turns out that in general, if we want to compute $\pi(t[u,\bar{v}))$, we need to know how the factor $[u,\bar{v})$ is embedded in the tree. Just looking at the values $\sigma(u)$ and $\sigma(v_i)$ provided by some labelling σ is not enough.

Definition 3.11. Let $t \in \mathbb{T}^{\times}A$ be a tree.

(a) For $u, v \in \mathrm{dom}(t)$, we denote by $u \sqcap v$ their infimum in the tree order.

(b) The *branching pattern* of a tuple $\bar{v}$ of vertices is the order tree consisting of the root of t and all vertices of the form $v_i \sqcap v_j$ where each edge $u \prec w$ is labelled by the variable x such that $\mathrm{suc}_x(u) \leq w$.

(c) The *branching type* of a factor $[u,\bar{v})$ of t is the isomorphism type of the branching patterns of $\bar{v}$ in the subtree $t|_u$. Alternatively, we can define the branching type as the atomic type of $\bar{v}$ in the structure $\langle t|_u, (\prec_x)_x, \sqcap, u\rangle$. ⌟

Example. We construct a finitary $\mathbb{T}$-algebra $\mathfrak{A}$ such that, for every $t \in \mathbb{T}A$ with factors $[u,\bar{v})$ and $[u',\bar{v}')$,

$$\pi(t[u,\bar{v})) = \pi(t[u',\bar{v}')) \quad \text{implies} \quad [u,\bar{v}) \text{ and } [u',\bar{v}') \text{ have the same branching type.}$$

For $\xi \in \Xi$, let A_ξ be the set of all branching types of trees of sort ξ. (Up to the labelling of the edges, there are only finitely many of such types. We may assume that every sort ξ appearing in t is a finite initial segment of ω.) Since we can compute the branching type of $\mathrm{flat}(t)$ from the patterns of $t(v)$, for $v \in \mathrm{dom}(t)$, there exists a function $\pi : \mathbb{T}A \to A$ such that $\pi \circ \mathbb{T}\tau = \tau \circ \pi$, where $\tau : \mathbb{T}\Sigma \to A$ is the function mapping each tree to its branching type. It follows that $\mathfrak{A} := \langle A, \pi\rangle$ is a finitary $\mathbb{T}$-algebra with the desired property.

Let $t \in \mathbb{T}\Sigma$ be an infinite binary tree and let λ be the labelling mapping a factor $[u,\bar{v})$ to its branching type. We claim that there is no function $\sigma : \mathrm{dom}(t) \to C$ with a finite codomain C such that the label $\lambda([u,\bar{v}))$ only depends on the values $\sigma(u)$ and $\sigma(v_x)$. We fix a vertex $u \in \mathrm{dom}(t)$ such that the set

$$C_0 := \{\, \sigma(v) \mid v \geq u \,\}$$

is minimal. Set $c := \sigma(u)$. Ay choice of u there are vertices $v_0, v_1 \in \sigma^{-1}(c)$ with $u \prec_0 v_0$ and $u \prec_1 v_1$. Similarly, we can find $v_1' \in \sigma^{-1}(c)$ with $v_0 \prec_1 v_1'$. Then $u \prec_0 v_1'$ implies that $\lambda([u, v_0 v_1)) \neq \lambda([u, v_0 v_1'))$, but all four vertices have the same colour c. ⌟

We conclude this section by briefly mentioning two alternative approaches.

Remark. When talking about partition theorems for trees, we also have to mention automata. Every automaton can be seen as a prescription producing labellings (runs) of trees. The advantage of automata is that they can be used even if we know very little about the underlying algebra. In particular, in cases where we can only evaluate finite trees. Their disadvantage is that runs are usually not unique and that every run only contains a limited amount of information. For instance, in general there is no automaton what allows us to evaluate every factor $\pi(t[u,\bar{v}])$ of a given tree t. ⌟

Remark. The proof of Theorem 3.4 is based on semigroup-theoretic methods, in particular, it makes heavy use of Green's relations. It looks plausible that, in order to prove a stronger partition theorem for trees, we have to develop a similar theory of Green's relations for tree algebras. As it turns out, it is rather straightforward to generalise these relations to the setting of monoidal categories where all hom-sets are finite. (We omit the details since the statements and proofs are virtually identical to those for semigroups.) Furthermore, every $\mathbb{T}^{\mathrm{fin}}$-algebra $\mathfrak{A}$ can be seen as such a category $\mathcal{A}$ where the objects are the sorts $\xi \in \Xi$ and the hom-sets $\mathcal{A}(\xi,\zeta)$ are given by $(A_\xi)^\zeta$. The question is therefore how to apply these results to the case at hand. The main problem with doing so seems to be that, in general, a finitary $\mathbb{T}^{\mathrm{fin}}$-algebra can have infinitely many J-classes. ⌟

4 Expansions and dense submonads

When looking for a strengthening of the results in the previous section it is always useful to have an application in mind that can serve as a test case and reality check. The following problem on expansions seems to be a good candidate for this purpose.

Definition 4.1. Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}^\times$ be submonads. We say that a $\mathbb{T}^0$-algebra $\mathfrak{A}_0$ is a *reduct* of a $\mathbb{T}^1$-algebra $\mathfrak{A}_1 = \langle A, \pi \rangle$ if $\mathfrak{A}_0 = \langle A, \pi_0 \rangle$ where $\pi_0 : \mathbb{T}^0 A \to A$ is the restriction of $\pi_1 : \mathbb{T}^1 A \to A$. In this case, we call $\mathfrak{A}_1$ a $\mathbb{T}^1$-expansion of $\mathfrak{A}_0$. ⌟

Expansion Problem. *Given monads $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}$, which $\mathbb{T}^0$-algebras have $\mathbb{T}^1$-expansions? And for which algebras are these expansions unique?*

We start our investigation with recalling some results from [7] that can be used to prove the uniqueness of expansions, if not their existence.

Definition 4.2. A submonad $\mathbb{T}^\circ \subseteq \mathbb{T}^1$ is *dense* in $\mathbb{T}^1$ over a class $\mathcal{C}$ of $\mathbb{T}^1$-algebras if, for all $\mathfrak{A} \in \mathcal{C}$, $C \subseteq A$, and $s \in \mathbb{T}^1 C$, there exists $s^\circ \in \mathbb{T}^\circ C$ with $\pi(s^\circ) = \pi(s)$. ⌟

We have shown in Lemma 4.13 (a) of [7] that denseness implies that expansions are unique (if they exist).

Proposition 4.3. *Let $\mathbb{T}^\circ \subseteq \mathbb{T}^1$ be dense over some class $\mathcal{C}$ that is closed under binary products. Then every $\mathbb{T}^\circ$-algebra $\mathfrak{A}$ has at most one $\mathbb{T}^1$-expansion that belongs to $\mathcal{C}$.*

The fact that a regular language is uniquely determined by which regular trees it contains, can be generalised to the following theorem (which is a consequence of Theorem 10.1 of [7]).

Theorem 4.4. $\mathbb{T}^{\times\mathrm{reg}} \subseteq \mathbb{T}^\times$ *is dense over the class of all* MSO*-definable* $\mathbb{T}^\times$*-algebras.*

We can strengthen this result to include existence.

Theorem 4.5. *Every* MSO*-definable* $\mathbb{T}^{\times\mathrm{reg}}$*-algebra can be expanded to a unique* MSO*-definable* $\mathbb{T}^\times$*-algebra.*

Proof. Uniqueness follows by Theorem 4.4 and Proposition 4.3. For existence, consider an MSO-definable $\mathbb{T}^{\times\mathrm{reg}}$-algebra $\mathfrak{A}$ and let $C \subseteq A$ be a finite set of generators. For $a \in A$, fix an MSO-formula φ_a defining the set $\pi^{-1}(a) \cap \mathbb{T}^{\times\mathrm{reg}} C$. We define the desired expansion $\mathfrak{A}_+ = \langle A, \pi_+ \rangle$ by as follows. As C generates $\mathfrak{A}$, there exists a function $\sigma : A \to \mathbb{T}^{\times\mathrm{reg}} C$ such that $\pi(\sigma(a)) = a$, for all $a \in A$. We choose σ such that $\sigma(c) = \mathrm{sing}(c)$, for $c \in C$. Let $\hat\sigma := \mathrm{flat} \circ \mathbb{T}^\times \sigma : \mathbb{T}^\times A \to \mathbb{T}^\times C$ be its extension to $\mathbb{T}^\times A$. We set

$$\pi_+(t) := a \quad :\text{iff} \quad \hat\sigma(t) \vDash \varphi_a\,, \quad \text{for } t \in \mathbb{T}^\times A\,.$$

We have to show that π_+ is well-defined, that it extends π, and that it satisfies the axioms of a $\mathbb{T}^\times$-algebra.

To see that π_+ is well-defined, we have to check that, for every tree $t \in \mathbb{T}^\times_\xi A$, there is exactly one element $a \in A_\xi$ with $t \vDash \varphi_a$. For a contradiction, suppose otherwise. Then there exists a tree $t \in \mathbb{T}^\times_\xi A$ such that

$$\hat\sigma(t) \vDash \bigwedge_{a \neq b} (\varphi_a \wedge \varphi_b) \vee \neg \bigvee_{a \in A_\xi} \varphi_a\,.$$

Since every non-empty MSO-definable tree language contains a regular tree, it follows that we can choose $t \in \mathbb{T}^{\times\mathrm{reg}}\mathbb{T}^{\times\mathrm{reg}} A$. By choice of the formulae φ_a this means that $\pi(t) = \pi(\hat\sigma(t))$ has either no value, or more than one. A contradiction.

Furthermore, π_+ extends π since, for $t \in \mathbb{T}^{\times\mathrm{reg}} A$, we have

$$\pi_+(t) = a \quad \text{iff} \quad \hat\sigma(t) \vDash \varphi_a \quad \text{iff} \quad \pi(\hat\sigma(t)) = a \quad \text{iff} \quad \pi(t) = a\,.$$

Hence, it remains to check the axioms of a $\mathbb{T}^\times$-algebra. First, note that we have

$$\pi_+ \circ \hat\sigma = \pi_+$$

since

$$\begin{aligned}
\hat\sigma \circ \hat\sigma &= \mathrm{flat} \circ \mathbb{T}^\times\sigma \circ \mathrm{flat} \circ \mathbb{T}^\times\sigma \\
&= \mathrm{flat} \circ \mathrm{flat} \circ \mathbb{T}^\times(\mathbb{T}^\times\sigma \circ \sigma) \\
&= \mathrm{flat} \circ \mathrm{flat} \circ \mathbb{T}^\times(\mathbb{T}^\times\mathrm{sing} \circ \sigma) \\
&= \mathrm{flat} \circ \mathbb{T}^\times\mathrm{sing} \circ \mathrm{flat} \circ \mathbb{T}^\times\sigma \\
&= \mathrm{flat} \circ \mathbb{T}^\times\sigma \\
&= \hat\sigma
\end{aligned}$$

implies that

$$\begin{aligned}
\pi_+(\hat\sigma(t)) = a \quad &\text{iff} \quad \hat\sigma(\hat\sigma(t)) \vDash \varphi_a \\
&\text{iff} \quad \hat\sigma(t) \vDash \varphi_a \qquad \text{iff} \quad \pi_+(t) = a\,.
\end{aligned}$$

For the unit law, it therefore follows that

$$(\pi_+ \circ \mathrm{sing}) = (\pi_+ \circ \hat\sigma \circ \mathrm{sing}) = (\pi_+ \circ \sigma) = (\pi \circ \sigma) = \mathrm{id}\,.$$

For the associative law, let $t \in \mathbb{T}^\times\mathbb{T}^\times A$. For $a \in A$, let ψ_a be a formula stating that

> for every factorisation of the given tree, there exists a labelling of the factors by elements of A such that
>
> – each factor with label b satisfies the formula φ_b and
>
> – the tree consisting of the guessed labels satisfies φ_a.

Since $\mathfrak{A}$ is associative, we have

$$s \vDash \psi_a \to \varphi_a\,, \qquad \text{for every } s \in \mathbb{T}^{\times\mathrm{reg}} C\,.$$

It follows that the same is true for every $s \in \mathbb{T}^\times C$. In particular, it holds for the tree $s := (\mathrm{flat} \circ \mathbb{T}^\times\hat\sigma)(t)$. Consequently, we have

$$(\pi_+ \circ \mathrm{flat} \circ \mathbb{T}^\times\hat\sigma)(t) = (\pi_+ \circ \mathbb{T}^\times\pi_+ \circ \mathbb{T}^\times\hat\sigma)(t)\,,$$

which implies that

$$
\begin{aligned}
(\pi_+ \circ \mathrm{flat})(t) &= (\pi_+ \circ \hat{\sigma} \circ \mathrm{flat})(t) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}^\times \sigma \circ \mathrm{flat})(t) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathrm{flat} \circ \mathbb{T}^\times \mathbb{T}^\times \sigma)(t) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}^\times \mathrm{flat} \circ \mathbb{T}^\times \mathbb{T}^\times \sigma)(t) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}^\times \hat{\sigma})(t) \\
&= (\pi_+ \circ \mathbb{T}^\times \pi_+ \circ \mathbb{T}^\times \hat{\sigma})(t) = (\pi_+ \circ \mathbb{T}^\times \pi_+)(t) \,. \qquad \square
\end{aligned}
$$

Note that our uniqueness result in Proposition 4.3 only concerns expansions in the given class $\mathcal{C}$. It is possible that there exist additional expansions outside of $\mathcal{C}$.

Example. In [12] Bojańczyk and Klin have presented an example of a finitary $\mathbb{T}$-algebra that is not MSO-definable. This algebra can be used to find an MSO-definable $\mathbb{T}^{\mathrm{reg}}$-algebra with several $\mathbb{T}$-expansions, one of them MSO-definable. (By the preceding theorem, there can only be one of the latter.) The construction of this $\mathbb{T}^{\mathrm{reg}}$-algebra $\mathfrak{A}^\circ = \langle A, \pi \rangle$ and two of its $\mathbb{T}$-expansions $\mathfrak{A}^{\mathrm{reg}} = \langle A, \pi_{\mathrm{reg}} \rangle$ (MSO-definable) and $\mathfrak{A}^{\mathrm{non}} = \langle A, \pi_{\mathrm{non}} \rangle$ (not MSO-definable) is as follows.

For $\mathfrak{A}^{\mathrm{non}}$, we take (a simplified version) of the algebra from [12]. Set $\Sigma := \{a, b\}$ where both elements have arity 2 and let Δ_ξ be the set of all finite trees in $\mathbb{T}_\xi \Sigma$. As Σ contains only binary elements, every leaf of a tree $t \in \Delta_\xi$ must be labelled by a variable. Hence, t has at most $|\xi|$ leaves and, therefore, at most $|\xi| - 1$ internal vertices. This implies that Δ_ξ is a finite set.

We call a tree $t \in \mathbb{T}\Sigma$ *antiregular* if not two subtrees of t are isomorphic. It is *densely antiregular* if every subtree of t has an antiregular subtree.

The domains of all three algebras are

$$
A_\xi := \Delta_\xi \uplus \{\bot, *\} \,, \quad \text{for } \xi \in \Xi \,,
$$

which we order such that $\bot$ is the least element and all other elements are incomparable. For $t \in \mathbb{T}_\xi A$, we define the product $\pi_{\mathrm{non}}(t)$ of $\mathfrak{A}^{\mathrm{non}}$ by the following case distinction.

- $\pi_{\mathrm{non}}(t) = \bot$ if t contains the label $\bot$.
- $\pi_{\mathrm{non}}(t) = \mathrm{flat}(t)$ if t is finite and $t \in \mathbb{T}_\xi \Delta$.
- $\pi_{\mathrm{non}}(t) = *$ if t is finite and t contains the label $*$.
- $\pi_{\mathrm{non}}(t) = *$ if t is infinite and every infinite subtree of t contains the label $*$.

- $\pi_{\text{non}}(t) = *$ if t is infinite and, for every infinite subtree s of t with $s \in \mathbb{T}\Delta$, the tree $\text{flat}(s)$ has a subtree without variables that is not antiregular.
- $\pi_{\text{non}}(t) = \bot$ if t is infinite and there exists an infinite subtree s of t such that $s \in \mathbb{T}\Delta$ and every subtree of $\text{flat}(s)$ without variables is densely antiregular.

The product $\pi_{\text{reg}}(t)$ of $\mathfrak{A}^{\text{reg}}$ is defined as follows.

- $\pi_{\text{reg}}(t) = \bot$ if t contains the label $\bot$.
- $\pi_{\text{reg}}(t) = \text{flat}(t)$ if t is finite and $t \in \mathbb{T}_{\xi}\Delta$.
- $\pi_{\text{reg}}(t) = *$ if t is finite and t contains the label $*$.
- $\pi_{\text{reg}}(t) = *$ if t is infinite.

Note that this product is MSO-definable and that the restrictions of π_{reg} and π_{non} to $\mathbb{T}^{\text{reg}}A$ coincide. Thus $\mathfrak{A}^{\text{reg}}$ and $\mathfrak{A}^{\text{non}}$ are $\mathbb{T}$-expansions of the same $\mathbb{T}^{\text{reg}}$-algebra. ⌟

To conclude this section, let me mention one of the main open problems concerning the relation between regular trees and arbitrary ones.

Open Question. *Does there exist a system of equations modulo which every tree is equivalent to a regular one? If so, does it have an explicit description?*

Having such an equational characterisation would be invaluable for applications, where proofs frequently require a reduction to regular trees. For instance, there exists an equational characterisation of bisimulation-invariant languages of *regular* trees [10], but so far nobody was able to generalise it to languages of arbitrary trees.

5 Evaluations

The notion of denseness seems to be only useful if we already know that the algebra in question has an expansion. To actually prove existence we need different techniques. We start with the following simple idea. We try to compute the product of a tree $t \in \mathbb{T}A$ inductively bottom-up using the given $\mathbb{T}^{\circ}$-product. That is, we factorise t into pieces that belong to $\mathbb{T}^{\circ}$, evaluate them, and then recursively evaluate the remaining tree. If we can show that,

- after a finite number of such steps, the tree t is reduced to a single vertex and
- that the final result does not depend on the choice of which factorisation to use in each step,

it follows that we can uniquely evaluate every tree in $\mathbb{T}A$ using only the $\mathbb{T}^{\circ}$-product. In particular, the product of $\mathfrak{A}$ can be uniquely extended to the set of all trees.

A well-known use of this technique is given by Simon's Factorisation Tree Theorem. Such a factorisation tree is just a hierarchical decomposition of a given semigroup-product using binary products and products of idempotents only. A second example of this approach was used in [13] to prove, in our terminology, that a certain inclusion between monads of countable linear orders is dense. The aim of the current section is to make this idea of using an inductive approach work for thin trees and to show that unfortunately it does not work for general ones.

The definition below is a bit more general than the above intuitive description. Suppose that we are given a $\mathbb{T}^{\circ}$-algebra, which we want to expand to a $\mathbb{T}^{1}$-algebra, and suppose that we have already found some set $S \supseteq \mathbb{T}^{\circ}A$ such that we can extend the product $\pi : \mathbb{T}^{\circ}A \to A$ to $\sigma : S \to A$. To extend σ further to a function $\mathbb{T}^{1}A \to A$, consider a tree $t \in \mathbb{T}^{1}A$. We choose a factorisation T of t, i.e., a tree $T \in \mathbb{T}^{1}\mathbb{T}^{1}A$ with $\mathrm{flat}(T) = t$, where we have already inductively assigned some value $\mathrm{val}(T(v))$ to each factor. If the reduced tree $\mathbb{T}^{1}\mathrm{val}(T)$ belongs to S, then we can set $\mathrm{val}(t) := \sigma(\mathbb{T}^{1}\mathrm{val}(T))$. The formal definition is as follows.

Definition 5.1. Let $\mathbb{T}^{\circ} \subseteq \mathbb{T}^{1} \subseteq \mathbb{T}$ be functors, $\mathfrak{A}$ a $\mathbb{T}^{\circ}$-algebra, and $\sigma : S \to A$ a function with domain $S \supseteq \mathbb{T}^{\circ}A$.

(a) For each ordinal α, we inductively define the set $\mathbb{E}_{\alpha}(\sigma, \mathbb{T}^{1})$ of *σ-evaluations* and two functions

$$\mathrm{term}_{\alpha} : \mathbb{E}_{\alpha}(\sigma, \mathbb{T}^{1}) \to \mathbb{T}^{1}A \quad \text{and} \quad \mathrm{val}_{\alpha} : \mathbb{E}_{\alpha}(\sigma, \mathbb{T}^{1}) \to A$$

by

$$\begin{aligned}
\mathbb{E}_{0}(\sigma, \mathbb{T}^{1}) &:= A\,, \\
\mathbb{E}_{\alpha+1}(\sigma, \mathbb{T}^{1}) &:= \mathbb{E}_{\alpha}(\sigma, \mathbb{T}^{1}) \cup \big\{\, \gamma \in \mathbb{T}^{1}\mathbb{E}_{\alpha}(\sigma, \mathbb{T}^{1}) \,\big|\, \mathbb{T}^{1}\mathrm{val}_{\alpha}(\gamma) \in S \,\big\}\,, \\
\mathbb{E}_{\delta}(\sigma, \mathbb{T}^{1}) &:= \bigcup_{\alpha<\delta} \mathbb{E}_{\alpha}(\sigma, \mathbb{T}^{1})\,, \qquad \text{for limit ordinals } \delta\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{term}_{0} &:= \mathrm{sing}\,, & \mathrm{val}_{0} &:= \mathrm{id}\,, \\
\mathrm{term}_{\alpha+1} &:= \mathrm{flat} \circ \mathbb{T}^{1}\mathrm{term}_{\alpha}\,, & \mathrm{val}_{\alpha+1} &:= \sigma \circ \mathbb{T}^{1}\mathrm{val}_{\alpha}\,, \\
\mathrm{term}_{\delta} &:= \bigcup_{\alpha<\delta} \mathrm{term}_{\alpha}\,. & \mathrm{val}_{\delta} &:= \bigcup_{\alpha<\delta} \mathrm{val}_{\alpha}\,.
\end{aligned}$$

Finally, we set

$$\mathbb{E}(\sigma, \mathbb{T}^1) := \bigcup_\alpha \mathbb{E}_\alpha(\sigma, \mathbb{T}^1), \quad \text{term} := \bigcup_\alpha \text{term}_\alpha, \quad \text{and} \quad \text{val} := \bigcup_\alpha \text{val}_\alpha.$$

(b) We call $\text{term}(\gamma)$ the *underlying term* of $\gamma \in \mathbb{E}_\alpha(\sigma, \mathbb{T}^1)$ and $\text{val}(\gamma)$ its *value.* If $t = \text{term}_\alpha(\gamma)$, we say that γ is a *σ-evaluation* of t.

(c) We say that the algebra $\mathfrak{A}$ *has σ-evaluations* if $\text{term} : \mathbb{E}(\sigma, \mathbb{T}^1) \to \mathbb{T}^1 A$ is surjective, and we say that it has *essentially unique σ-evaluations* if furthermore

$$\text{term}(\gamma) = \text{term}(\gamma') \quad \text{implies} \quad \text{val}(\gamma) = \text{val}(\gamma').$$

(d) In the special case where $\sigma = \pi$, we also write $\mathbb{E}(\mathfrak{A}, \mathbb{T}^1) := \mathbb{E}(\pi, \mathbb{T}^1)$ and we call the elements of $\mathbb{E}(\mathfrak{A}, \mathbb{T}^1)$ *simple $\mathbb{T}^0$-evaluations.* ⌟

Let us start by explaining how to use σ-evaluations to construct $\mathbb{T}^1$-expansions. The proof makes use of the following glueing operation for evaluations.

Lemma 5.2. *Let $\gamma \in \mathbb{T}^1\mathbb{E}(\sigma, \mathbb{T}^1)$ and let β be a σ-evaluation of the tree $t := \mathbb{T}^1\text{val}(\gamma)$. There exists a σ-evaluation $\beta|\gamma$ of the tree* $(\text{flat} \circ \mathbb{T}^1\text{term})(\gamma)$ *such that* $\text{val}(\beta|\gamma) = \text{val}(\beta)$.

Proof. We define $\beta|\gamma$ by induction on the ordinal α with $\beta \in \mathbb{E}_\alpha(\sigma, \mathbb{T}^1)$. If $\alpha = 0$, then $\beta = a \in A$ and $\gamma = \text{sing}(\gamma_0)$, for some $\gamma_0 \in \mathbb{E}(\sigma, \mathbb{T}^1)$ with $\text{val}(\gamma_0) = a$. Setting $\beta|\gamma := \gamma_0$ we obtain $\text{val}(\beta|\gamma) = \text{val}(\gamma_0) = a = \text{val}(\beta)$.

Suppose that $\beta \in \mathbb{E}_{\alpha+1}(\sigma, \mathbb{T}^1) \setminus \mathbb{E}_\alpha(\sigma, \mathbb{T}^1)$. Then $\beta \in \mathbb{T}^1\mathbb{E}_\alpha(\sigma, \mathbb{T}^1)$. Let $\gamma' \in \mathbb{T}^1\mathbb{T}^1\mathbb{E}(\sigma, \mathbb{T}^1)$ be the tree with the same domain as β where $\gamma'(v)$ is the restriction of γ to $\text{dom}(\text{term}(\beta(v))) \subseteq \text{dom}(\gamma)$. We choose for $\beta|\gamma$ the tree with the same domain as β that is given by

$$(\beta|\gamma)(v) := \beta(v)|\gamma'(v), \quad \text{for } v \in \text{dom}(s).$$

By inductive hypothesis, it follows that

$$\begin{aligned} \text{term}((\beta|\gamma)(v)) &= \text{term}(\beta(v)|\gamma'(v)) = (\text{flat} \circ \mathbb{T}^1\text{term})(\gamma'(v)), \\ \text{val}((\beta|\gamma)(v)) &= \text{val}(\beta(v)|\gamma'(v)) \quad = \text{val}(\beta(v)). \end{aligned}$$

Consequently, we have

$$\begin{aligned}
\mathrm{term}(\beta|\gamma) &= (\mathrm{flat} \circ \mathbb{T}^1\mathrm{term})(\beta|\gamma) \\
&= (\mathrm{flat} \circ \mathbb{T}^1(\mathrm{flat} \circ \mathbb{T}^1\mathrm{term}))(\gamma') \\
&= (\mathrm{flat} \circ \mathrm{flat} \circ \mathbb{T}^1\mathbb{T}^1\mathrm{term})(\gamma') \\
&= (\mathrm{flat} \circ \mathbb{T}^1\mathrm{term} \circ \mathrm{flat})(\gamma') \\
&= (\mathrm{flat} \circ \mathbb{T}^1\mathrm{term})(\gamma)\,, \\
\mathrm{val}(\beta|\gamma) &= (\sigma \circ \mathbb{T}^1\mathrm{val})(\beta|\gamma) \\
&= (\sigma \circ \mathbb{T}^1\mathrm{val})(\beta) = \mathrm{val}(\beta)\,.
\end{aligned}$$

Furthermore, note that $\beta|\gamma$ really is an evaluation since

$$\mathbb{T}^1\mathrm{val}(\beta|\gamma) = \mathbb{T}^1\mathrm{val}(\beta) \in \mathrm{dom}(\sigma)\,.$$

Finally, if $\beta \in \mathbb{E}_\delta(\sigma, \mathbb{T}^1)$, for some limit ordinal δ, then there is some $\alpha < \delta$ with $\beta \in \mathbb{E}_\alpha(\sigma, \mathbb{T}^1)$. Hence, the claim follows by inductive hypothesis. □

Theorem 5.3. *Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}$ be submonads. Every $\mathbb{T}^0$-algebra $\mathfrak{A}$ with essentially unique σ-evaluations has a $\mathbb{T}^1$-expansion $\langle A, \pi_+\rangle$ such that*

$$\pi_+ \circ \mathrm{term} = \mathrm{val}\,.$$

Proof. For $t \in \mathbb{T}^1 A$, we define

$$\pi_+(t) := \mathrm{val}(\gamma)\,, \quad \text{for some } \gamma \in \mathrm{term}^{-1}(t)\,.$$

As σ-evaluations are essentially unique, it does not matter which evaluation γ we choose. Hence, this function is well-defined. We claim that $\mathfrak{A}_+ := \langle A, \pi_+\rangle$ is the desired $\mathbb{T}^1$-expansion of $\mathfrak{A}$.

The equation $\pi_+ \circ \mathrm{term} = \mathrm{val}$ follows immediately from the definition of π_+. Since every tree $t \in \mathbb{T}^0 A = \mathbb{E}_1(\mathfrak{A}, \mathbb{T}^1)$ is its own evaluation, we further have

$$\pi_+(t) = \mathrm{val}_1(t) = \sigma(t) = \pi(t)\,, \quad \text{for } t \in \mathbb{T}^0 A\,.$$

Consequently, π_+ is an extension of π and it remains to check the axioms of a $\mathbb{T}^1$-algebras.

For the unit law, we have $\pi_+(\mathrm{sing}(a)) = \pi(\mathrm{sing}(a)) = a$ since $\mathrm{sing}(a) \in \mathbb{T}^0 A$. For associativity, let $t \in \mathbb{T}^1\mathbb{T}^1 A$. Then there exists a tree of σ-evaluations $\gamma \in \mathbb{T}^1\mathbb{E}(\mathfrak{A}, \mathbb{T}^1)$ such that $t = \mathbb{T}^1\mathrm{term}(\gamma)$. Furthermore, we can fix an evaluation

$\beta \in \mathbb{E}(\mathfrak{A}, \mathbb{T}^1)$ of the tree $\mathbb{T}^1\mathrm{val}(\gamma)$. Let $\beta|\gamma$ be the σ-evaluation from Lemma 5.2. Then

$$\begin{aligned}(\pi_+ \circ \mathrm{flat})(t) &= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}^1\mathrm{term}_n)(\gamma) \\ &= (\pi_+ \circ \mathrm{term})(\beta|\gamma) \\ &= \mathrm{val}(\beta|\gamma) \\ &= \mathrm{val}(\beta) \\ &= (\pi_+ \circ \mathrm{term})(\beta) \\ &= (\pi_+ \circ \mathbb{T}^1\mathrm{val}_n)(\gamma) \\ &= (\pi_+ \circ \mathbb{T}^1\pi_+ \circ \mathbb{T}^1\mathrm{term}_n)(\gamma) = (\pi_+ \circ \mathbb{T}^1\pi_+)(t)\,. \end{aligned}$$ □

The theorem tells us how to use evaluations to construct expansions. Next, let us see where the limits of this method are.

Proposition 5.4. *Let $\mathfrak{A} = \langle A, \pi\rangle$ be a $\mathbb{T}^0$-algebra with a $\mathbb{T}^1$-expansion $\mathfrak{A}_+ = \langle A, \pi_+\rangle$, and suppose that $\sigma : S \to A$ is the restriction of π_+ to some set $S \supseteq \mathbb{T}^0 A$. Then*

$$\mathrm{val}(\gamma) = (\pi_+ \circ \mathrm{term})(\gamma)\,, \quad \textit{for every } \sigma\textit{-evaluation } \gamma\,.$$

Proof. We prove that $\mathrm{val}_\alpha = \pi_+ \circ \mathrm{term}_\alpha$ by induction on α. For $\alpha = 0$, we have

$$\mathrm{val}_0(\gamma) = \gamma = \pi_+(\mathrm{sing}(\gamma)) = \pi_+(\mathrm{term}_0(\gamma))\,, \quad \text{for } \gamma \in \mathbb{E}_0(\sigma, \mathbb{T}^1) = A\,.$$

For the successor step, suppose that the equation holds for α and consider $\gamma \in \mathbb{E}_{\alpha+1}(\sigma, \mathbb{T}^1)$. Then

$$\begin{aligned}\mathrm{val}_{\alpha+1}(\gamma) &= \sigma(\mathbb{T}^1\mathrm{val}_\alpha(\gamma)) \\ &= \sigma(\mathbb{T}^1(\pi_+ \circ \mathrm{term}_\alpha)(\gamma)) \\ &= \pi_+(\mathbb{T}^1\pi_+(\mathbb{T}^1\mathrm{term}_\alpha(\gamma))) \\ &= \pi_+(\mathrm{flat}(\mathbb{T}^1\mathrm{term}_\alpha(\gamma))) \\ &= \pi_+(\mathrm{term}_{\alpha+1}(\gamma))\,. \end{aligned}$$

Finally, for a limit ordinal α, the claim follows immediately from the inductive hypothesis. □

Corollary 5.5. *Let $\mathfrak{A}$ be a $\mathbb{T}^0$-algebra.*

(a) *If $\mathfrak{A}$ has several $\mathbb{T}^1$-expansions, there exist trees $t \in \mathbb{T}^1 A$ without a simple $\mathbb{T}^0$-evaluation.*

(b) *If $\mathfrak{A}$ has simple $\mathbb{T}^0$-evaluations that are not essentially unique, it has no $\mathbb{T}^1$-expansion.*

Example. Before using these results to study thin trees, let us quickly recall the results of [13] about countable chains. We denote by $\mathbb{C}A$ the set of all countable A-labelled linear orders and by $\mathbb{C}^{\text{reg}}A \subseteq \mathbb{C}A$ the subset of all *regular* linear orders. By definition, a linear order is regular if it can be denoted by a finite term using the following operations: (I) constants for singletons, (II) binary ordered sums, (III) multiplication by ω and ω^{op} (ω with the reverse ordering), and (IV) dense shuffles. In [13] it is shown that every finite $\mathbb{C}^{\text{reg}}$-algebra has essentially unique $\mathbb{C}^{\text{reg}}$-evaluations. This fact can be used to prove the following results (for the proofs, see [13, 8, 7]).

- $\mathbb{C}^{\text{reg}} \subseteq \mathbb{C}$ is dense over the class of all finite $\mathbb{C}$-algebras.
- Every finite $\mathbb{C}^{\text{reg}}$-algebra has a unique $\mathbb{C}$-expansion.
- Every finite $\mathbb{C}$-algebra is MSO-definable.
- A language $K \subseteq \mathbb{C}\Sigma$ of countable chains is MSO-definable if, and only if it is recognised by some finite $\mathbb{C}$-algebra.
- Every MSO-definable language $K \subseteq \mathbb{C}\Sigma$ has a finite syntactic algebra. ⌟

5.1 Thin trees

As an application of simple evaluations we consider thin trees, where we can use the Theorem of Ramsey and other tools from semigroup-theory.

Proposition 5.6. *Every finitary $\mathbb{T}^{\text{wilke}}$-algebra has essentially unique simple $\mathbb{T}^{\text{wilke}}$-evaluations for trees in $\mathbb{T}^{\text{thin}}$.*

Proof. Let $\mathfrak{A}$ be a finitary $\mathbb{T}^{\text{wilke}}$-algebra and $t \in \mathbb{T}^{\text{thin}}A$ a thin tree. We construct the desired simple evaluation of t by induction on the Cantor-Bendixson rank α of t. By inductive hypothesis, every subtree $t|_v$ of rank less than α has a simple evaluation $\gamma_v \in \mathbb{E}(\mathfrak{A}, \mathbb{T}^{\text{thin}})$. Let s be tree obtained from t by replacing every such subtree $t|_v$ by $\text{val}(\gamma_v)$. It is sufficient to find a simple evaluation of s. Then we can use the glueing operation from Lemma 5.2 to construct the desired evaluation of t.

By construction, s has only finitely many infinite branches. We distinguish three cases.

(I) If s is finite, it is its own evaluation.

(II) Next, suppose that s has a single infinite branch. By the theorem of Ramsey, we can find a factorisation $s = p_0 p_1 p_2 \dots$ such that $\pi(p_i) = \pi(p_j)$, for all $i, j > 0$. As each factor p_i is finite, we obtain simple evaluations β_i of p_i by (I). The path $\rho := \pi(p_0), \pi(p_1), \pi(p_2), \dots$ is of the form ae^ω for $a := \pi(p_0)$ and $e := \pi(p_1)$. In particular, it is regular. Let β_* be the path $\beta_0, \beta_1, \beta_2, \dots$. Then

$$\mathbb{T}^{\text{thin}}\text{val}(\beta_*) = aeee\cdots \in \mathbb{T}^{\text{reg}}A \quad \text{and} \quad \text{term}(\beta_*) = p_0 p_1 p_2 \cdots = s\,.$$

Hence, β_* is the desired simple evaluation of s.

(III) Finally, suppose that s has at least two infinite branches. Then we can factorise s into a finite prefix and finitely many trees with a single infinite branch. By (I) and (II), each of these factors has a simple evaluation. Let β be the finite tree consisting of these evaluations. Then β is an simple evaluation of s. □

Corollary 5.7. *$\mathbb{T}^{\text{wilke}} \subseteq \mathbb{T}^{\text{thin}}$ is dense over the class of all finitary $\mathbb{T}^{\text{thin}}$-algebras.*

Corollary 5.8. *Every finitary $\mathbb{T}^{\text{wilke}}$-algebra has a unique $\mathbb{T}^{\text{thin}}$-expansion.*

It follows that the step from a $\mathbb{T}^{\text{wilke}}$-algebra to a $\mathbb{T}^{\text{thin}}$-expansion is fairly well understood. The inclusion $\mathbb{T}^{\text{fin}} \subseteq \mathbb{T}^{\text{wilke}}$ is slightly more complicated since expansions are no longer unique.

Proposition 5.9. *Let $\mathfrak{A} = \langle A, \pi \rangle$ be a finitary $\mathbb{T}^{\text{fin}}$-algebra. There exists a bijection between all $\mathbb{T}^{\text{wilke}}$-expansions of $\mathfrak{A}$ and all functions $-^\omega : A_1 \to A_0$ satisfying the axioms of a Wilke algebra.*

Proof. Clearly, every $\mathbb{T}^{\text{wilke}}$-expansion $\mathfrak{A}^+ = \langle A, \pi_+ \rangle$ of $\mathfrak{A}$ induces an ω-power operation by

$$a^\omega := \pi_+(aaa\dots)\,.$$

This operation satisfies the axioms of a Wilke algebra since π_+ is associative. It therefore remains to show that this correspondence is bijective.

Note that every tree $t \in \mathbb{T}^{\text{wilke}}A$ is the unravelling of a finite graph all of which strongly connected components are either singletons or induced cycles.

For injectivity, suppose that there are two expansions $\mathfrak{A}_0 = \langle A, \pi_0 \rangle$ and $\mathfrak{A}_1 = \langle A, \pi_1 \rangle$ of $\mathfrak{A}$ with the same associated ω-power. Let $t \in \mathbb{T}^{\text{wilke}}A$ be the unravelling of a graph $\mathfrak{G}$ with n strongly connected components. By induction on n, we prove that

$$\pi_0(t) = \pi_1(t)\,.$$

Let C be the strongly connected component of $\mathfrak{G}$ containing the root of t. For every vertex $v \notin C$, it follows by inductive hypothesis that

$$\pi_0(t|_v) = \pi_1(t|_v) .$$

Hence, replacing these subtrees by their product we may assume that $\mathfrak{G}$ has a single strongly connected component C. If C is a single vertex, we have $t = \operatorname{sing}(a)$ and

$$\pi_0(t) = a = \pi_1(t) .$$

Otherwise, C is a cycle and there exists a finite path p such that $t = p^\omega$. This implies that

$$\pi_0(t) = \pi(p)^\omega = \pi_1(t) .$$

For surjectivity, suppose that $-^\omega : A_1 \to A_0$ is an ω-power operation. We construct an expansion $\mathfrak{A}^+ = \langle A, \pi_+ \rangle$ of $\mathfrak{A}$ as follows. Let $t \in \mathbb{T}^{\text{wilke}} A$ be the unravelling of a graph $\mathfrak{G}$ with n strongly connected components. We define $\pi(t)$ by induction on n. Let C be the strongly connected component of $\mathfrak{G}$ containing the root of t. For every vertex $v \notin C$, we can compute $\pi(t|_v)$ by inductive hypothesis. Let t' be the tree obtained from t by replacing every such subtree by its product. If $t' = \operatorname{sing}(a)$, we set

$$\pi_+(t) := a .$$

Otherwise, there exists a finite path p such that $t = p^\omega$ and we set

$$\pi_+(t) := \pi(p)^\omega .$$

It remains to check that the function π_+ defined this way satisfies the axioms of a $\mathbb{T}^{\text{wilke}}$-algebra and that the associated ω-operation coincides with the given one.

We start with the latter. Let $a \in A_1$. By definition of π_+, there are numbers $m, n < \omega$ such that

$$\pi_+(aaa\dots) = \pi(a^m) \cdot \pi(a^n)^\omega = a^m \cdot (a^n)^\omega = a^\omega ,$$

as desired.

For the unit law, it follows directly by definition that

$$\pi_+(\operatorname{sing}(a)) = a .$$

Next, let us show that π_+ is well-defined. Let $\mathfrak{G}$ and $\mathfrak{H}$ be two finite graphs with the same unravelling t. We have to show that we obtain the same result when defining $\pi_+(t)$ in terms of $\mathfrak{G}$ and in terms of $\mathfrak{H}$. Since $\mathfrak{H}$ and $\mathfrak{G} \times \mathfrak{H}$ have the same unravelling, we may assume that $\mathfrak{G}$ is a quotient of $\mathfrak{H}$. We prove the claim by induction on the number of strongly connected components of $\mathfrak{H}$. Let C be the strongly connected component of $\mathfrak{H}$ containing the root. For $v \notin C$, let $\mathfrak{G}_v$ and $\mathfrak{H}_v$ the subgraphs of $\mathfrak{G}$ and $\mathfrak{H}$, respectively, reachable from the vertex v, and let $t|_v$ be their unravelling. By inductive definition, the values of $\pi(t|_v)$ unravelling defined using these two graphs coincide. Hence, replacing each such subgraph by its product, we may assume that $\mathfrak{H}$ has a single strongly connected component. Then so does $\mathfrak{G}$. If these components are singletons, the products are the same. Otherwise, there is some finite path p and numbers m, k such that $\mathfrak{G}$ is a cycle consisting of m copies of p, and $\mathfrak{H}$ consists of km copies. Setting $a := \pi(p)$, it follows that the product defined in terms of $\mathfrak{G}$ is equal to $(a^m)^\omega$, while the one defined via $\mathfrak{H}$ is $(a^{km})^\omega$. Since these values coincide, the claim follows.

Finally for associativity, fix $t \in \mathbb{T}^{\text{wilke}}\mathbb{T}^{\text{wilke}}A$ and let $\mathfrak{G}$ be a finite graph with unravelling t. For each vertex v of $\mathfrak{G}$, we fix a finite graph $\mathfrak{H}_v$ with unravelling $t(v)$. Then $\text{flat}(t)$ is the unravelling of the graph obtained from the disjoint union of all $\mathfrak{H}_v$ by adding edges according to $\mathfrak{G}$. We prove that

$$\pi_+(\text{flat}(t)) = \pi_+(\mathbb{T}^{\text{wilke}}\pi_+(t))$$

by induction on the number of strongly connected components of the graph for $\text{flat}(t)$. Let C be the strongly connected component of $\mathfrak{G}$ containing the root. For every vertex $v \notin C$, it follows by inductive hypothesis that

$$\pi_+(\text{flat}(t|_v)) = \pi_+(\mathbb{T}^{\text{wilke}}\pi_+(t|_v))\,.$$

Replacing such subtrees by their respective products, we may therefore assume that $\mathfrak{G}$ consists of a single strongly connected component C. If C is a singleton, we have $t = \text{sing}(s)$ and

$$\pi_+(\text{flat}(t)) = \pi_+(s) = \pi_+(\mathbb{T}^{\text{wilke}}\pi_+(t))\,.$$

Hence, suppose that C is a cycle. Each graph $\mathfrak{H}_v$ consists of a finite path p_v to which are possibly attached additional trees. By inductive hypothesis, associativity holds for these subtrees. Again, replacing each such subtree by its product, we may assume that $\mathfrak{H}_v$ is equal to p_v. Consequently, $\text{flat}(t)$ is a single infinite path consisting of the concatenation of all p_v, while $\mathbb{T}^{\text{wilke}}\pi_+(t)$ is the infinite path labelled by the products $\pi(p_v)$. The product of these two paths is the same. □

Corollary 5.10. *Let $\mathfrak{A} = \langle A, \pi \rangle$ be a finitary $\mathbb{T}^{\text{fin}}$-algebra. There exists a bijection between all $\mathbb{T}^{\text{thin}}$-expansions of $\mathfrak{A}$ and all functions $-^{\omega} : A_1 \to A_0$ satisfying the axioms of a Wilke algebra.*

Note that every $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A} = \langle A, \pi \rangle$ induces an associated ω-semigroup $\langle A_{\{x\}}, A_{\varnothing} \rangle$ (for some arbitrary variable x). Using this observation, we can rephrase the above corollary as follows.

Corollary 5.11. *Every $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A}$ is uniquely determined by* (I) *its $\mathbb{T}^{\text{fin}}$-reduct and* (II) *the associated ω-semigroup.*

5.2 Evaluations with merging

When we try to go beyond $\mathbb{T}^{\text{thin}}$ our machinery breaks down since we cannot use the results for semigroups anymore. The following counterexample shows that a naïve generalisation of our definitions does not work.

Lemma 5.12. *There exists an* MSO*-definable $\mathbb{T}^{\text{reg}}$-algebra $\mathfrak{A}$ and a tree $t \in \mathbb{T}A$ that has no simple $\mathbb{T}^{\text{reg}}$-evaluation.*

Proof. Let $\mathfrak{A}$ be the $\mathbb{T}^{\text{reg}}$-reduct of the Bojańczyk-Klin algebra from the example on page 17. Then the claim follows immediately from Corollary 5.5 (a). Nevertheless we give an explicit proof to see what exactly is going wrong. Set $\Delta := \mathbb{T}^{\text{fin}}\{a, b\}$ and recall that $\Delta \subseteq A$. We will prove by induction on α that

$$t \notin \operatorname{rng} \operatorname{term}_\alpha\,, \qquad \text{for all } t \in \mathbb{T}\Delta \text{ where every subtree has vertices of arbitrarily high arity.}$$

For a contradiction, suppose otherwise. Let α be the minimal ordinal such that there is some simple evaluation $\gamma \in \mathbb{E}_\alpha(\mathfrak{A}, \mathbb{T})$ where every subtree of $\operatorname{term}(\gamma)$ has vertices of arbitrarily high arity. If $\alpha = 0$, then $\operatorname{term}(\gamma) = \operatorname{sing}(a)$ in contradiction to our choice of γ. Hence, $\alpha = \beta + 1$, for some β. Fix $v \in \operatorname{dom}(\gamma)$. Note that every subtree s of $\operatorname{term}_\beta(\gamma(v))$ has a simple evaluation in $\mathbb{E}_\beta(\mathfrak{A}, \mathbb{T})$ which, by inductive hypothesis, means that s has a subtree where the arity of the vertices is bounded. We claim that this implies that $t_v := \operatorname{term}(\gamma(v))$ is finite. Suppose otherwise. Since t_v has only finitely many variables, it has some infinite subtree s without variables. But s is also a subtree of $\operatorname{term}(\gamma)$. By choice of γ this implies that the arities of the vertices of s are unbounded. A contradiction.

Hence, we have $\operatorname{term}(\gamma(v)) \in \mathbb{T}^{\text{fin}}\Delta$, which implies implies that $\operatorname{val}(\gamma(v)) = \pi(\operatorname{term}(\gamma(v))) = \operatorname{term}(\gamma(v))$. Furthermore, $\operatorname{term}(\gamma(v))$ being finite its arity is at least as high as the maximal arity of a vertex in $\mathrm{dom}(\gamma(v))$. It follows that, for every $n < \omega$, there is some $v \in \mathrm{dom}(\gamma)$ such that $\mathrm{val}(\gamma(v))$ has arity at least n. But $\gamma \in \mathbb{E}_{\alpha+1}(\mathfrak{A}, \mathbb{T})$ implies that $\mathbb{T}\mathrm{val}(\gamma) \in \mathbb{T}^{\mathrm{reg}}A$. In particular, $\mathbb{T}\mathrm{val}(\gamma)$ uses only finitely many different labels. This implies that their arity is bounded. A contradiction. □

A closer look at the above proof reveals two possible reasons making simple evaluations impossible. Firstly, our counterexample uses a tree with infinitely many different labels. It still might be possible that trees with only finitely many different labels always have simple evaluations. Secondly, we made essential use of the fact that every factor of an infinite binary tree has a subtree that is itself an infinite binary tree. To be able to use factorisations of trees into pieces that are significantly simpler, we will probably have to allow more general factors, which then necessarily have infinitely many variables. Unfortunately, it is hard to combine these two modifications since factors with infinitely many variables usually give rise to infinitely many different elements of the algebra. What seems to be missing is some technique that, given a tree with infinitely many different labels, allows us to bound their arity by merging different variables (e.g., replacing $a(x, y, z)$ by, say, $a(x, x, z)$).

This observation leads to the following attempt to allow for evaluations where variables are merged. To make our definitions precise we need a bit of terminology. First, as we want to identify variables, we need to work in $\mathbb{T}^\times$ instead of $\mathbb{T}$. We also need a set of labels telling us which variables to identify.

Definition 5.13. Let $\sigma : \zeta \to \xi$.

(a) For $t \in \mathbb{T}^\times_\zeta A$, we denote by ${}^\sigma t \in \mathbb{T}^\times_\xi A$ the tree obtained from t by replacing every variable z by $\sigma(z)$.

If $\mathbb{T}^\circ \subseteq \mathbb{T}^\times$ is closed under the operation ${}^\sigma-$ we can extend this operation to $\mathbb{T}^\circ$-algebras $\mathfrak{A}$ by setting

$$ {}^\sigma a := \pi({}^\sigma \mathrm{sing}(a)) \,, \quad \text{for } a \in A_\zeta \,. $$

(b) For a sort $\xi \in \Xi$, we set $\Gamma(\xi) := (\Gamma_\zeta(\xi))_{\zeta \in \Xi}$ where

$$ \Gamma_\zeta(\xi) := \{ \sigma \mid \sigma : \zeta \to \xi \} \,. $$

⌟

Given a tree $t \in \mathbb{T}^\times A$ we can choose some sort $\xi \in \Xi$ and functions $\sigma_v \in \Gamma(\xi)$, for every $v \in \mathrm{dom}(t)$, and then replace every label $t(v)$ by ${}^{\sigma_v} t(v)$. The problem is that the resulting tree is not well-formed since the sorts do not match anymore.

For instance, in the tree $a(b,c)$ with $a = a(z_0, z_1)$ we can replace z_0 and z_1 by the same variable x. This produces the label $a' := a(x,x)$ of arity $\{x\}$. Consequently, we need to produce a tree where the corresponding vertex has a single successor. Given the tree $a(b,c)$ the only obvious choices for such a tree would be $a'(b)$ or $a'(c)$. This idea can be generalised as follows.

Definition 5.14. Let $\mathfrak{A}$ be a $\mathbb{T}^0$-algebra where $\mathbb{T}^0 \subseteq \mathbb{T}^\times$ is closed under the operations ${}^\sigma-$, let $p : \Gamma(\xi) \times A \to \Gamma(\xi)$ and $q : \Gamma(\xi) \times A \to A$ be the two projections, and let $s \in \mathbb{T}^\times(\Gamma(\xi) \times A)$ be a tree. For $v \in \operatorname{dom}(s)$, we set $\sigma_v := p(s(v))$.

(a) A *choice function* for s is a family $\mu = (\mu_v)_{v \in \operatorname{dom}_0(s)}$ of functions $\mu_v : \operatorname{rng} \sigma_v \to \operatorname{dom} \sigma_v$ such that $\sigma_v \circ \mu_v = \mathrm{id}$.

(b) Given a choice function μ for s, we define the tree $s \parallel \mu \in \mathbb{T}^\times A$ as follows. For every $v \in \operatorname{dom}(s)$,

- we delete from s all subtrees $s|_u$ where u is a x-successor of v with $x \notin \operatorname{rng} \mu_v$, and
- for $x \in \operatorname{rng} \mu_v$, we change the x-successor of v to a $\sigma_v(x)$-successor.
- we replace every label $s(v) = \langle \sigma, a \rangle$ by ${}^\sigma a$. ⌟

We can produce well-formed trees $s \parallel \mu$ using a choice function μ. But which one do we take? The easiest case is if all choice functions produce the same result (cf. [17]), then it does not matter. (A more general construction will be presented further below.)

Definition 5.15. Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}^\times$ be functors and $\mathfrak{A}$ a $\mathbb{T}^0$-algebra.

(a) We say that a tree $s \in \mathbb{T}^\times(\Gamma(\xi) \times A)$ is *uniform* if

$$s \parallel \mu = s \parallel \mu', \quad \text{for all choice functions } \mu, \mu'.$$

(b) A *uniform* $\mathbb{T}^0$*-condensation* of $t \in \mathbb{T}^1 A$ is a uniform tree $s \in \mathbb{T}^1(\Gamma(\xi) \times A)$ such that

$$\mathbb{T}^1 q(s) = t \quad \text{and} \quad s \parallel \mu \in \mathbb{T}^0 A, \quad \text{for some/all } \mu.$$

(c) We set

$$\pi^{\mathrm{u}}(t) := \pi(s \parallel \mu), \quad \text{where } s \text{ is some uniform } \mathbb{T}^0\text{-condensation of } t \text{ and } \mu \text{ is an arbitrary choice function}.$$

If there is no condensation s, we let $\pi^{\mathrm{u}}(t)$ be undefined. We call π^{u}-evaluations $\mathbb{T}^0$*-evaluations with uniform merging,* and we denote the corresponding set by

$$\mathbb{E}^{\mathrm{u}}_\alpha(\mathfrak{A}, \mathbb{T}^1) := \mathbb{E}_\alpha(\pi^{\mathrm{u}}, \mathbb{T}^1).$$ ⌟

Remark. Note that, in general, the value of $\pi^{\mathrm{u}}(t)$ does depend on the condensation s we have chosen, but it is obviously independent from μ. In the following, whenever we use π^{u} we tacitly assume some fixed choice of condensation for every term t where such a condensation exists. ⌟

Example. $\mathbb{T}^{\times\mathrm{reg}}$-evaluations with uniform merging were introduced in [17] where they were used to derive decidability results for trees. To do so Puppis considers trees $t \in \mathbb{T}^{\times}X$ such that (in our terminology), for every MSO-definable $\mathbb{T}^{\times\mathrm{reg}}$-algebra $\mathfrak{A}$ and every function $\beta : X \to A$, the image $\mathbb{T}^{\times}\beta(t)$ has an evaluation $\gamma \in \mathbb{E}^{\mathrm{u}}_n(\mathfrak{A}, \mathbb{T}^{\times})$, for some fixed $n < \omega$ independent of β and $\mathfrak{A}$, and for some particular choice of condensations based on the runs of the automata recognising the product (see the proof of Proposition 5.19 below for a similar construction). Let us call such trees *reducible.*

By induction on n, we can transform every reducible tree t into a regular tree t_0 with the same value as t. Puppis considers reducible trees t where this transformation is computable using a particular algorithm. (We omit the details.) Let us call such trees *effectively reducible.* [17] contains the following results.

- Every regular tree is effectively reducible.
- More generally, every deterministic tree in the Caucal hierarchy is effectively reducible.
- The class of effectively reducible trees is closed under a number of natural operations.
- Every effectively reducible tree has a decidable MSO-theory. ⌟

The main technical result of [17] is the following recipe of how to evaluate products in an MSO-definable $\mathbb{T}^{\times}$-algebra using uniform evaluations.

Proposition 5.16 (Puppis). *Given* MSO*-definable $\mathbb{T}^{\times}$-algebra $\mathfrak{A}$ and a sort $\xi \in \Xi$, there exists an* MSO*-definable $\mathbb{T}^{\times}$-algebra $\mathfrak{B}$, a morphism $\rho : \mathbb{T}^{\times}A \to \mathfrak{B}$, and* MSO*-formulae φ_a, for $a \in A_\xi$, such that, given a tree $T \in \mathbb{T}^{\times}_{\xi}\mathbb{T}^{\times}A$ and a uniform $\mathbb{T}^{\times}$-condensation s of $\mathbb{T}^{\times}\rho(T)$, we have*

$$\pi(\mathrm{flat}(T)) = a \quad \text{iff} \quad s \parallel \mu \vDash \varphi_a\,, \quad \text{for all } a \in A_\xi \text{ and all choice functions } \mu\,.$$

The proof uses similar techniques as that of Proposition 5.19 below.

Since evaluations with uniform merging generalise simple evaluations, they allow us to decompose more trees. Unfortunately, there are still trees left without an evaluation. We can generalise our evaluations even further by not requiring that all choice functions lead to the same tree, but only to one that is 'equivalent'.

Definition 5.17. Let $s \in \mathbb{T}^\times(\Gamma(\xi) \times A)$ and let $\pi : \mathbb{T}^\circ A \to A$ be a function. We call s *π-consistent* if

$$\begin{aligned} &s|_v \parallel \mu \in \mathbb{T}^\circ A\,, && \text{for every choice function } \mu \text{ of } s|_v \text{ and each} \\ & && \text{vertex } v \in \mathrm{dom}(s)\,, \\ &\pi(s|_v \parallel \mu) = \pi(s|_v \parallel \mu')\,, && \text{for all choice functions } \mu, \mu' \text{ of } s|_v \text{ and each} \\ & && \text{vertex } v \in \mathrm{dom}_0(s)\,. \end{aligned}$$
⌟

Definition 5.18. Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}$ be functors, $\mathfrak{A} = \langle A, \pi \rangle$ a $\mathbb{T}^0$-algebra, and $t \in \mathbb{T}^1 A$.

(a) A *consistent* $\mathbb{T}^0$*-condensation* of t is a π-consistent tree $s \in \mathbb{T}^1(\Gamma(\xi) \times A)$ such that $\mathbb{T}^1 q(s) = t$.

(b) We set

$$\pi^{\mathrm{c}}(t) := \pi(s \parallel \mu)\,, \quad \text{where } s \text{ is some consistent } \mathbb{T}^0\text{-condensation of } t \text{ and } \mu \text{ is an arbitrary choice function}\,.$$

If there is no condensation s, we let $\pi^{\mathrm{c}}(t)$ be undefined. We call π^{c}-evaluations $\mathbb{T}^0$*-evaluations with consistent merging*, and we denote the corresponding set by

$$\mathbb{E}^{\mathrm{c}}_\alpha(\mathfrak{A}, \mathbb{T}^1) := \mathbb{E}_\alpha(\pi^{\mathrm{c}}, \mathbb{T}^1)\,.$$
⌟

Clearly, consistent merging generalises uniform merging. While $\mathbb{T}^{\times\mathrm{reg}}$-evaluations with uniform merging seem to exist only in special cases, our hope is that $\mathbb{T}^{\times\mathrm{reg}}$-evaluations with consistent merging always exist (at least for MSO-definable algebras). At the moment we are only able to prove the existence of $\mathbb{T}^\times$-evaluations. Note that this statement is non-trivial since trees can contain labels of arbitrarily high arity while every $\mathbb{T}^\times$-condensation produces a tree where these arities are bounded.

Proposition 5.19. *Let $\mathfrak{A}$ be an* MSO*-definable $\mathbb{T}^\times$-algebra. Every tree $t \in \mathbb{T}^\times A$ has a consistent $\mathbb{T}^\times$-condensation s such that*

$$\pi(t) = \pi(s \parallel \mu)\,, \quad \textit{for all choice functions } \mu\,.$$

Proof. Let $C \subseteq A$ be a finite set of generators and let $\vartheta : A \to \mathbb{T}^\times C$ be the function mapping every $a \in A$ to some tree $\vartheta(a)$ whose product is a. Given $t \in \mathbb{T}^\times_\xi A$, we consider the tree $T := \mathbb{T}^\times \vartheta(t)$. By choice of ϑ, it follows that $\mathbb{T}^\times \pi(T) = t$. Hence,

$$\pi(t) = \pi(\mathbb{T}^\times \pi(T)) = \pi(\mathrm{flat}(T)) .$$

Furthermore, for every $w \in \mathrm{dom}(T)$, there is some factor $[u, \bar{v})$ of $\mathrm{flat}(T)$ such that

$$T(w) = \mathrm{flat}(T)[u, \bar{v}) .$$

Let $\varphi : \mathrm{dom}(T) \to \mathrm{dom}(\mathrm{flat}(T))$ be the function mapping every vertex w to the root u of the corresponding factor.

We recall a few needed notions and facts from automata theory (for details, we refer the reader to [19, 15]). A non-deterministic tree automaton is a tuple $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega \rangle$ where Q is the set of states, $\Omega : Q \to \omega$ a priority function, and Δ the transition relation which consists of triples of the form $\langle p, a, (q_x)_{x \in \xi} \rangle$ for states $p, q_x \in Q$ and a letter $a \in \Sigma_\xi$. Let $\mathcal{A}$ be such an automaton evaluating products of trees in $\mathbb{T}^\times_\xi C$, that is, an automaton such that, for every $a \in A_\xi$, there is some state $q_a \in Q$ such that for all $t' \in \mathbb{T}^\times_\xi C$,

$$\pi(t') = a \quad \text{iff} \quad \mathcal{A} \text{ accepts } t' \text{ when starting in the state } q_a ,$$

and let G be the corresponding Automaton-Pathfinder game for the input tree $\mathrm{flat}(T)$. Recall that this is a parity game where the positions for player Automaton are the pairs in $\mathrm{dom}(\mathrm{flat}(T)) \times Q$ while the positions for player Pathfinder are those in $\mathrm{dom}(\mathrm{flat}(T)) \times \Delta$. In a position $\langle v, q \rangle$ Automaton chooses a transition $\langle p, a, \bar{r} \rangle \in \Delta$ with $p = q$ and $a = \mathrm{flat}(T)(v)$; Pathfinder replies with some x-successor u_x of v; and the game continues in the position $\langle u_x, r_x \rangle$. Automaton wins a play in this game if he either manoevres Pathfinder into a position where the latter cannot make a move, or if the play is infinite and the corresponding sequence $(q_i)_{i<\omega}$ of states from the Automaton positions satisfies the parity condition:

$$\liminf_{n<\omega} \Omega(q_n) \quad \text{is even.}$$

The salient fact about the Automaton-Pathfinder game G is that player Automaton has a winning strategy for G when starting in position $\langle v, q \rangle$ if, and only if, the automaton $\mathcal{A}$ accepts the subtree $\mathrm{flat}(T)|_v$ when starting in the state q. Since G is a parity game, there exists a single positional strategy τ that is winning for all these starting positions.

We use this strategy τ to define a labelling χ of $\mathrm{dom}(t)$ as follows. For a vertex v of t with parent u, we set

$$\chi(v) := \langle P, \kappa \rangle ,$$

where $P \subseteq Q$ is the set of states p such that $\mathcal{A}$ accepts the subtree $\mathrm{flat}(T)|_{\varphi(v)}$ when starting in state p, and $\kappa : Q \to \mathrm{rng}\,\Omega$ is the function defined as follows. For $q \in Q$, let π_q be the play of G that results from Automaton using the strategy τ when starting in position $\langle \varphi(u), q \rangle$ and when Pathfinder chooses the vertices on the path from u to v. (We do not care about Pathfinder's choices after the play has reached the vertex v.) Then κ maps each state $q \in Q$ to the least priority k seen in the prefix of π_q between the vertices u and v.

We can use the labelling χ to define the desired condensation s as follows. We use $\xi := \mathrm{rng}\,\chi$ for the variables (or rather, some set $\xi \in \Xi$ isomorphic to it). For a vertex $v \in \mathrm{dom}(t)$ of sort ζ_v, we set

$$s(v) := \langle \sigma_v, t(v) \rangle ,$$

where the function $\sigma_v : \zeta_v \to \xi$ is defined as

$$\sigma_v(z) := \chi(u_z) , \quad \text{where } u_z \text{ is the } z\text{-successor of } v .$$

We will prove below that

$$\pi(s|_v \parallel \mu|_v) = \pi(t|_v) , \quad \text{for all } v \in \mathrm{dom}(t) \text{ and every choice function } \mu .$$

This implies that s is consistent since

$$\pi(s|_v \parallel \mu|_v) = \pi(t|_v) = \pi(s|_v \parallel \mu'|_v) , \quad \text{for all choice functions } \mu, \mu' .$$

Hence, the lemma follows for $v = \varnothing$.

To prove the claim, fix a choice function μ for s and set $S := \mathbb{T}^\times \vartheta(s \parallel \mu)$. Replacing t, s, and μ by, respectively, $t|_v$, $s|_v$, and $\mu|_v$, we may assume without loss of generality that $v = \langle\rangle$. Note that

$$\pi(s \parallel \mu) = \pi(\mathbb{T}^\times \pi(S)) = \pi(\mathrm{flat}(S)) .$$

Hence, it is sufficient to show that

$$\pi(\mathrm{flat}(T)) = \pi(\mathrm{flat}(S)) ,$$

i.e., that $\mathcal{A}$ accepts $\text{flat}(T)$ when staring in some state q if, and only if, it accepts $\text{flat}(S)$ when starting in q.

Let H be the Automaton-Pathfinder game for the tree $\text{flat}(S)$ and let $\psi : \text{dom}(S) \to \text{dom}(\text{flat}(S))$ be the function mapping each vertex of S to the corresponding one of $\text{flat}(S)$ (cf. the definition of φ above). Note that $\text{flat}(S)$ can be obtained from $\text{flat}(T)$ by

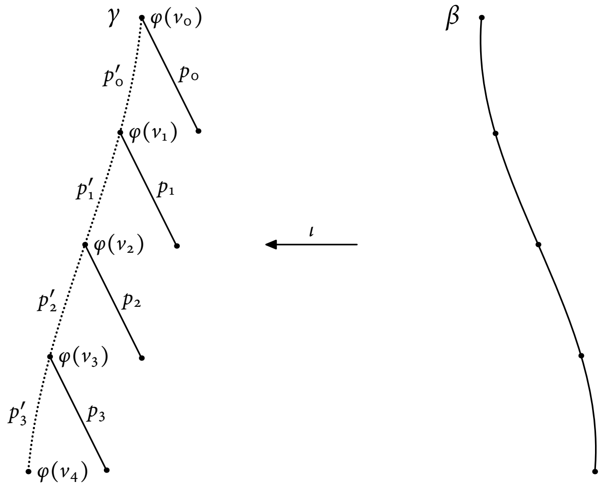
- for each $v \in \text{dom}(S)$ with successors $(u_x)_{x \in \xi}$, redirecting the edge leading to $\psi(u_x)$ to $\psi(u_y)$ where $y = \mu_v(\sigma_v(x))$,
- unravelling the resulting graph (thereby deleting unreachable vertices).

In particular, there exists a canonical function $\iota : \text{dom}(\text{flat}(S)) \to \text{dom}(\text{flat}(T))$ which restricts to a function $\text{dom}(s \parallel \mu) \to \text{dom}(t)$. We define a strategy ρ for Automaton in the game H by

$$\rho(\langle v, q\rangle) := \tau(\langle \iota(v), q\rangle) .$$

To prove the above statement it is sufficient to show that, for every branch β of $\text{flat}(S)$, there is a branch γ of $\text{flat}(T)$ such that the least priority seen infinitely often in the play of H following ρ along the branch β is the same as the least priority seen infinitely often in the play of G following τ along γ. The image $\iota[\beta]$ of a given branch β under ι consists of a sequence $p_0, p_1, p_2, \dots$ of disconnected paths with the following properties.

- For every $i < \omega$, there is some vertex $v_i \in \text{dom}(T)$ such that p_i corresponds to a path in $T(v_i)$ from the root to some variable.
- The vertices $v_0, v_1, \dots$ form a branch of T.



It follows that the starting points of the paths p_i lie on some branch γ of flat(T) which contains the vertices $\varphi(v_0), \varphi(v_1), \dots$. Let p'_i be the part of γ between $\varphi(v_i)$ and $\varphi(v_{i+1})$. By definition of s it follows that, for every i, the least priorities seen along p'_i and along p_i are the same. Let us call this priority k_i. By definition of ρ it follows that k_i is also the least priority seen in the part of β corresponding to p_i. Consequently, the least priorities seen infinitely often along β and along γ are both equal to $\liminf_i k_i$. □

Our hope is that a more elaborate version of the construction from the preceding proof can be used to construct a $\mathbb{T}^{\times\text{reg}}$-condensation instead of a $\mathbb{T}^{\times}$-one, or at least that we can iterate such a construction to obtain a $\mathbb{T}^{\times\text{reg}}$-evaluation.

Conjecture. *Let $\mathfrak{A}$ be an* MSO*-definable $\mathbb{T}^{\times\text{reg}}$-algebra. Then every tree $t \in \mathbb{T}A$ has a $\mathbb{T}^{\times\text{reg}}$-evaluation with consistent merging.*

6 Consistent labellings

As we have seen in the previous section we can construct expansions with the help of evaluations if the two monads in question are sufficiently well-behaved. What do we do if they are not? Let us turn to a second idea of how to prove that a $\mathbb{T}^{\circ}$-algebra $\mathfrak{A}$ has a $\mathbb{T}$-expansion: when we want to define the product of $t \in \mathbb{T}A$, we first annotate t with additional information that makes it easier to determine the value of the product. For instance, for each vertex v, we can guess the value $\pi(t|_v)$ of the corresponding subtree and the check that these guesses are correct.

Definition 6.1. Let $\mathbb{T}^\circ \subseteq \mathbb{T}$ be a submonad, $\mathfrak{A}^\circ = \langle A, \pi^\circ \rangle$ a $\mathbb{T}^\circ$-algebra, and $t \in \mathbb{T}_\xi A$.

(a) A *labelling* of t is a function $\sigma : \mathrm{dom}(t) \to A$ (not necessarily arity-preserving) such that, for every vertex v,

$$\sigma(v) \in A_\zeta \quad \text{iff} \quad \zeta \text{ is the set of variables appearing in } t|_v\,.$$

(b) A labelling $\sigma : \mathrm{dom}(t) \to A$ is *weakly* $\mathbb{T}^\circ$*-consistent* if, for every factor $[u, \bar{v})$ with $t[u, \bar{v}) \in \mathbb{T}^\circ A$,

$$\sigma(u) = \pi(t[u, \bar{v}))\big(\sigma(v_0), \sigma(v_1), \dots\big)\,. \qquad \lrcorner$$

We have noted in the previous section that using factors with only finitely many variables can cause pathological behaviour. For this reason, we introduce a more general notion of consistency where we also allow factors with infinitely many variables.

Definition 6.2. Let $\mathbb{T}^\circ \subseteq \mathbb{T}$ be a submonad, $\mathfrak{A}^\circ = \langle A, \pi^\circ \rangle$ a $\mathbb{T}^\circ$-algebra, and $t \in \mathbb{T}_\xi A$.

(a) Given a factor $[u, \bar{v})$ of t, possibly with infinitely many holes $\bar{v}$, we denote by $t[u, \bar{v})(a_0, a_1, \dots)$ the tree obtained from $t[u, \bar{v})$ by replacing each leaf labelled by a variable x_i by the tree $\mathrm{sing}(a_i)$.

(b) A labelling $\sigma : \mathrm{dom}(t) \to A$ is *strongly* $\mathbb{T}^\circ$*-consistent* if, for every factor $[u, \bar{v})$, possibly with infinitely many holes $\bar{v}$, with $t[u, \bar{v})(\sigma(v_0), \sigma(v_1), \dots) \in \mathbb{T}^\circ A$, we have

$$\sigma(u) = \pi\big(t[u, \bar{v})(\sigma(v_0), \sigma(v_1), \dots)\big)\,. \qquad \lrcorner$$

We start with the following easy observation.

Lemma 6.3. *Let $\mathfrak{A}$ be a finitary $\mathbb{T}^{\mathrm{fin}}$-algebra. Every tree $t \in \mathbb{T}A$ has a strongly $\mathbb{T}^{\mathrm{fin}}$-consistent labelling.*

Proof. We call a labelling σ of some tree t *locally consistent* if

$$\sigma(v) = t(v)\big(\sigma(u_0), \dots, \sigma(u_{n-1})\big)\,,$$

for every vertex v with successors $u_0, \dots, u_{n-1}$. Fix an increasing sequence $P_0 \subset P_1 \subset \dots \subset \mathrm{dom}(t)$ of finite prefixes of t with $\bigcup_i P_i = \mathrm{dom}(t)$, and let Λ_i be the set of all locally consistent labellings of P_i, for $i < \omega$. Then $\Lambda := \bigcup_i \Lambda_i$ ordered by $\subset$ forms a finitely-branching tree. By Kőnig's Lemma, there exists an infinite branch $\sigma_0 \subset \sigma_1 \subset \dots$. Let σ be its limit. Then σ is locally consistent.

It therefore, remains to prove that every locally consistent labelling of t is $\mathbb{T}^{\mathrm{fin}}$-consistent. Consider a finite factor H of t with root v and leaves $u_0, \dots, u_{m-1}$. By induction on $|H|$ it follows that

$$\sigma(v) = \pi\big((t \restriction H)(\sigma(u_0), \dots, \sigma(u_{m-1}))\big)\,. \qquad \square$$

Next, let us show how use consistent labellings to characterise possible $\mathbb{T}$-expansions of a given $\mathbb{T}^\circ$-algebra. We need the following additional property.

Definition 6.4. Let $\mathbb{T}^\circ \subseteq \mathbb{T}$.

(a) A *weak labelling scheme* for a $\mathbb{T}^\circ$-algebra $\mathfrak{A}$ is a function σ assigning to each tree $t \in \mathbb{T}A$ a weakly $\mathbb{T}^\circ$-consistent labelling $\sigma(t)$ of t. Similarly, a *strong labelling scheme* σ assigns to each tree $t \in \mathbb{T}A$ a strongly $\mathbb{T}^\circ$-consistent labelling $\sigma(t)$.

(b) A labelling scheme σ for $\mathfrak{A}$ is *associative* if, for every tree $T \in \mathbb{T}\mathbb{T}A$, we have

$$\sigma(t) = \sigma(\mathrm{flat}(T)) \circ \mu\,,$$

where $\mu : \mathrm{dom}(T) \to \mathrm{dom}(\mathrm{flat}(T))$ maps each vertex $v \in \mathrm{dom}(T)$ to the vertex of $\mathrm{flat}(T)$ corresponding to the root of $T(v)$, and $t \simeq_{\mathrm{sh}} T$ is the tree such that

$$t(v) := \sigma(T(v))(\langle\rangle)\,, \quad \text{for } v \in \mathrm{dom}(T)\,. \qquad \lrcorner$$

Example. There are algebras with several associative strong labelling schemes. Let $\mathfrak{A}$ be the $\mathbb{T}^{\mathrm{thin}}$-algebra with domains $A_\xi := [n]$, for some fixed number $n < \omega$, where the product is just the maximum

$$\pi(t) := \max\{\, t(v) \mid v \in \mathrm{dom}(t) \,\}\,.$$

For every $k < n$, we obtain an associative labelling scheme σ_k defined by

$$\sigma_k(t)(v) := \begin{cases} \pi(t|_v) & \text{if } t|_v \in \mathbb{T}^{\mathrm{thin}}A\,, \\ \max(\{k\} \cup \{\, t(u) \mid u \geq v \,\}\,, & \text{otherwise}\,. \end{cases} \qquad \lrcorner$$

There is a tight connection between $\mathbb{T}$-expansions and associative labelling schemes (weak or strong, it does not matter).

Proposition 6.5. *Let $\mathbb{T}^\circ \subseteq \mathbb{T}$ and let $\mathfrak{A}$ be a $\mathbb{T}^\circ$-algebra.*

(a) *Every associative weak labelling scheme for $\mathfrak{A}$ is strong.*

(b) *There exists a bijective correspondence between associative labelling schemes σ and $\mathbb{T}$-expansions of $\mathfrak{A}$.*

Proof. We define two mutually-inverse functions mapping (I) every associative weak labelling scheme to a $\mathbb{T}$-expansion of $\mathfrak{A}$ and (II) every such expansion to an associative strong labelling scheme.

(I) Given a weak scheme σ we define the corresponding function π_+ by

$$\pi_+(t) := \sigma(t)(\langle\rangle)\,, \quad \text{for } t \in \mathbb{T}A\,.$$

Then π_+ extends π since weak $\mathbb{T}^\circ$-consistency of σ implies that

$$\pi(t) = \sigma(t)(\langle\rangle) = \pi_+(t)\,, \quad \text{for } t \in \mathbb{T}^\circ A\,.$$

Hence, it remains to show that π_+ is associative. Fix $T \in \mathbb{T}\mathbb{T}A$. By the definition of associativity, we have

$$\sigma(\mathbb{T}\pi_+(T)) = \sigma(\text{flat}(T)) \circ \mu\,,$$

which in particular implies that

$$\begin{aligned}\pi_+(\mathbb{T}\pi_+(T)) &= \sigma(\mathbb{T}\pi_+(T))(\langle\rangle) \\ &= \sigma(\text{flat}(T))(\langle\rangle) = \pi_+(\text{flat}(T))\,.\end{aligned}$$

(II) Conversely, given a product $\pi_+ : \mathbb{T}A \to A$ we define a strong scheme σ by

$$\sigma(t)(v) := \pi_+(t|_v)\,, \quad \text{for } t \in \mathbb{T}A \text{ and } v \in \text{dom}(t)\,.$$

To show that this function σ is strongly $\mathbb{T}^\circ$-consistent, fix a factor $[u, \bar{v})$ of some tree $t \in \mathbb{T}A$. Then

$$\begin{aligned}\sigma(u) &= \pi_+(t|_u) \\ &= \pi_+\big(t[u,\bar{v})(\pi_+(t|_{v_0}), \pi_*(t|_{v_1}), \dots)\big) \\ &= \pi_+\big(t[u,\bar{v})(\sigma(v_0), \sigma(v_1), \dots)\big)\,,\end{aligned}$$

as desired. To show that σ is associative, let $T \in \mathbb{T}\mathbb{T}A$, $v \in \operatorname{dom}(T)$, and let t be the tree from the definition of associativity. Then

$$\begin{aligned}\sigma(t)(v) &= \pi_+(t|_v) \\ &= \pi_+(\mathbb{T}\pi_+(t)|_v) \\ &= \pi_+(\operatorname{flat}(T|_v)) \\ &= \pi_+(\operatorname{flat}(T)|_{\mu(v)}) = \sigma(\operatorname{flat}(T))(\mu(v)) .\end{aligned}$$

Finally, note that the mappings $\sigma \mapsto \pi_+$ and $\pi_+ \mapsto \sigma$ are clearly inverse to each other. □

In particular, if labellings are unique, so is the expansion. In fact, we do not need to assume associativity here.

Proposition 6.6. *Let $\mathbb{T}^\circ \subseteq \mathbb{T}$ and let $\mathfrak{A}$ be a $\mathbb{T}^\circ$-algebra satisfying at least one of the following two conditions.*

(i) *Every tree $t \in \mathbb{T}A$ has a unique weak $\mathbb{T}^\circ$-consistent labelling.*

(ii) *Every tree $t \in \mathbb{T}A$ has a unique strong $\mathbb{T}^\circ$-consistent labelling.*

Then $\mathfrak{A}$ has a unique $\mathbb{T}$-expansion.

Proof. Let σ be the unique labelling scheme (weak or strong). By Proposition 6.5, it is sufficient to prove that σ is associative. Hence, fix a tree $T \in \mathbb{T}\mathbb{T}A$ and let t and μ be as in the definition of associativity. We claim that the labelling

$$\tau := \sigma(\operatorname{flat}(T)) \circ \mu$$

is a $\mathbb{T}^\circ$-consistent labelling of t. Then uniqueness of labellings implies that

$$\sigma(t) = \tau = \sigma(\operatorname{flat}(T)) \circ \mu ,$$

as desired.

For the proof, fix a factor $[u, \bar{v})$ of t. We have to show that

$$\tau(u) = \pi\big(t[u, \bar{v})(\tau(v_0), \tau(v_1), \dots)\big) .$$

Since every tree $T(w)$ only contains finitely many variables, we can replace in $T(w)$ some subtrees $T(w)|_{w'}$ (without variables) by the corresponding constant $\sigma(T(w))(w')$. Let $P(w)$ be a finite tree obtained in this way from $T(w)$.

Using the consistency of $\sigma(P(w))$ and $\sigma(T(w))$, we can show by induction on w' (starting at the leaves) that

$$\sigma(P(w))(w') = \sigma(T(w))(w'), \quad \text{for all } w' \in \operatorname{dom}(P(w)).$$

Consequently,

$$\pi(P(w)) = \sigma(P(w))(\langle\rangle) = \sigma(T(w))(\langle\rangle) = t(w).$$

We regard the family $(P(w))_w$ as a tree $P \in \mathbb{T}\mathbb{T}A$ with domain

$$\operatorname{dom}(P) = [u, \bar{v}) \cup \{v_0, v_1, \dots\},$$

where, for the leaves v_i, we choose

$$P(v_i) := \tau(v_i).$$

Then the domain of P belongs to a $\mathbb{T}^\circ$-tree. Hence, we have $P \in \mathbb{T}^\circ\mathbb{T}^{\text{fin}}A \subseteq \mathbb{T}^\circ\mathbb{T}^\circ A$. This implies that $\operatorname{flat}(P) \in \mathbb{T}^\circ A$ and

$$\begin{aligned}\sigma(\operatorname{flat}(P))(\langle\rangle) = \pi(\operatorname{flat}(P)) &= \pi(\mathbb{T}^\circ\pi(P)) \\ &= \pi\big(t[u, \bar{v})(\tau(v_0), \tau(v_1), \dots)\big),\end{aligned}$$

where the last step follows by the fact that

$$\pi(P(w)) = \begin{cases} t(w) & \text{if } w \in [u, \bar{v}), \\ \tau(v_i) & \text{if } w = v_i. \end{cases}$$

Hence,

$$\begin{aligned}&\pi\big(t[u, \bar{v})(\tau(v_0), \tau(v_1), \dots)\big) \\ &= \sigma(\operatorname{flat}(P))(\langle\rangle) \\ &= \sigma(\operatorname{flat}(T))(\mu(u)) \\ &= \tau(u),\end{aligned}$$

where the second step follows by the uniqueness of labellings. □

As an application let us show how to use consistent labellings to prove that an algebra is definable

Proposition 6.7. *Every finitary* $\mathbb{T}^{\text{thin}}$*-algebra is* MSO*-definable.*

Proof. Let $\mathfrak{A}$ be a $\mathbb{T}^{\text{thin}}$-algebra and $t \in \mathbb{T}^{\text{thin}}A$. We construct a formula guessing the labelling $\lambda : \text{dom}(t) \to A$ induced by the product $\lambda(v) := \pi(t|_v)$ and then verifying the correctness of its guess by checking for each vertex v that

- $\lambda(v) = t(v)(\lambda(u_0), \dots, \lambda(u_{n-1}))$, where $u_0, \dots, u_{n-1}$ are the successors of v,
- $\lambda(v) = \pi(s_\beta)$, for every branch β starting at v, where s_β is the path obtained from $t|_v$ by replacing every vertex u not belonging to β by the constant $\lambda(u)$.

The latter condition can be expressed in MSO since this logic can evaluate products in ω-semigroups. By induction on the Cantor-Bendixson rank of t it follows that the above checks ensure that the guessed labelling coincides with the intended one. □

Corollary 6.8. *Let* $\mathfrak{A}$ *be a finitary* $\mathbb{T}$*-algebra where every tree* $t \in \mathbb{T}A$ *has exactly one* $\mathbb{T}^{\text{thin}}$*-consistent labelling. Then* $\mathfrak{A}$ *is* MSO*-definable.*

Proof. It follows by Proposition 6.7 that being $\mathbb{T}^{\text{thin}}$-consistent can be expressed in MSO. To evaluate a given product $\pi(t)$ in MSO, we can therefore guess a $\mathbb{T}^{\text{thin}}$-consistent labelling of t and take the label at the root. □

7 Unambiguous algebras

In Corollary 5.10, we have obtained a complete classification of all $\mathbb{T}^{\text{thin}}$-expansions of a $\mathbb{T}^{\text{fin}}$-algebra. In the present section, we use consistent labellings to study the inclusion $\mathbb{T}^{\text{thin}} \subseteq \mathbb{T}$. First let us remark that it not dense.

Lemma 7.1. *There exists a* $\mathbb{T}^{\text{thin}}$*-algebra* $\mathfrak{A}$ *with two different* MSO*-definable* $\mathbb{T}$*-expansions.*

Proof. Let A be the set with two elements $0_\xi, 1_\xi$ for every sort ξ. We consider two different products $\pi_0, \pi_1 : \mathbb{T}A \to A$ on this set. The first one is just the maximum operation:

$$\pi_0(t) := \max\{\, t(v) \mid v \in \text{dom}(t) \,\}, \quad \text{for } t \in \mathbb{T}A\,.$$

The second one is given by

$$\pi_1(t) := \begin{cases} 0 & \text{if } t \in \mathbb{T}^{\text{thin}}C\,, \\ 1 & \text{if } t \in \mathbb{T}C \setminus \mathbb{T}^{\text{thin}}C\,, \\ 1 & \text{if } t \notin \mathbb{T}C\,, \end{cases}$$

where $C \subseteq A$ is the subset consisting of the elements 0_ξ, $\xi \in \Xi$. Since there exists an MSO-formula expressing that a given tree is thin, both products are MSO-definable. Furthermore, π_0 is clearly associative. To show that so is π_1, fix a tree $t \in \mathbb{T}\mathbb{T}A$. We distinguish four cases.

- If $t(v) \notin \mathbb{T}C$, for some v, then $\pi_1(\mathbb{T}\pi_1(t)) = 1 = \pi_1(\text{flat}(t))$.
- If $t \in \mathbb{T}^{\text{thin}}\mathbb{T}^{\text{thin}}C$, then $\text{flat}(t) \in \mathbb{T}^{\text{thin}}C$ and $\pi_1(\mathbb{T}\pi_1(t)) = 0 = \pi_1(\text{flat}(t))$.
- If there is some $v \in \text{dom}(t)$ with $t(v) \notin \mathbb{T}^{\text{thin}}C$, we have $\text{flat}(t) \notin \mathbb{T}^{\text{thin}}C$ and $\pi_1(\mathbb{T}\pi_1(t)) = 1 = \pi_1(\text{flat}(t))$.
- Finally, suppose that $t \in \mathbb{T}\mathbb{T}^{\text{thin}}C \setminus \mathbb{T}^{\text{thin}}\mathbb{T}^{\text{thin}}C$. Then $\text{flat}(t) \in \mathbb{T}C \setminus \mathbb{T}^{\text{thin}}C$ and $\pi_1(\mathbb{T}\pi_1(t)) = 1 = \pi_1(\text{flat}(t))$. □

Consistent labellings have been used in [1] to study unambiguous tree languages. Let us give a brief overview over these results. The central notion is the following one.

Definition 7.2. Let $\mathbb{T}^\circ \subseteq \mathbb{T}$. A $\mathbb{T}^\circ$-algebra $\mathfrak{A}$ is *unambiguous* if every tree $t \in \mathbb{T}A$ has at most one strongly $\mathbb{T}^\circ$-consistent labelling. ⌟

Remarks. (a) For $\mathbb{T}^\circ = \mathbb{T}^{\text{thin}}$ these algebras were introduced in [1] under the name *prophetic thin algebras.*

(b) The fact that a given tree has a unique strongly $\mathbb{T}^{\text{thin}}$-consistent labelling is expressible in MSO. And so are the facts that every tree has a unique $\mathbb{T}^{\text{thin}}$-consistent labelling and that the corresponding labelling scheme is associative. ⌟

First, note that there exist $\mathbb{T}^{\text{thin}}$-algebras which are not unambiguous.

Example. Let $\mathfrak{A}$ be the $\mathbb{T}^{\text{thin}}$-algebra generated by the elements $0, 1$ (of arity 0), b_0, b_1, c_0, c_1 (of arity 1), and a (of arity 2) subject to the following equations.

$$\begin{aligned} b_i(j) &= j\,, & a(x,i) &= b_i(x)\,,\\ b_i b_j(x) &= b_{\max\{i,j\}}(x)\,, & a(i,x) &= c_i(x)\,,\\ c_i(j) &= i\,, & b_i^\omega &= 1-i\,,\\ c_i(b_j(x)) &= c_i(x)\,, & c_i^\omega &= i\,,\\ c_i(c_j(x)) &= c_i(x)\,, \end{aligned}$$

for $i, j \in \{0,1\}$. This algebra is not unambiguous since the (unique) tree $t \in \mathbb{T}\{a\}$ has several consistent labellings, including

$$\lambda(w) := |w|_1 \bmod 2 \quad \text{and} \quad \mu(w) := (|w|_1 + 1) \bmod 2\,.$$ ⌟

The connection between unambiguous $\mathbb{T}^{\text{thin}}$-algebras and unambiguous tree languages is given by the following theorem.

Definition 7.3. (a) A tree automaton is *unambiguous* if it has at most one accepting run on each given input tree.

(b) A language $K \subseteq \mathbb{T}_\xi\Sigma$ is called *bi-unambiguous* if both K and $\mathbb{T}_\xi\Sigma \setminus K$ are recognised by unambiguous automata. ⌟

Theorem 7.4 (Bilkowski, Skrzypczak [1]). *A language $K \subseteq \mathbb{T}_\xi\Sigma$ is bi-unambiguous if, and only if, it is recognised by a $\mathbb{T}^{\text{thin}}$-morphism $\varphi : \mathbb{T}\Sigma \to \mathfrak{A}$ to a finitary unambiguous $\mathbb{T}^{\text{thin}}$-algebra.*

Unfortunately, the existence of $\mathbb{T}^{\text{thin}}$-consistent labellings is still an open problem, one which turns out to be equivalent to the existence of the following kind of choice functions.

Definition 7.5. The *Thin Choice Conjecture* states that there does *not* exist an MSO-formula $\varphi(x; Z)$ such that, for every thin (unlabelled) tree t and every non-empty set $P \subseteq \text{dom}(t)$ of parameters, the formula $\varphi(x; P)$ defines a unique element of P. ⌟

Theorem 7.6 (Bilkowski, Skrzypczak [1]). *The following statements are equivalent.*

(1) *The Thin Choice Conjecture holds.*

(2) *All trees have strongly $\mathbb{T}^{\text{thin}}$-consistent labellings, for every finitary $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A}$.*

(3) *The unique tree in $\mathbb{T}_\varnothing\{a\}$ has a strongly $\mathbb{T}^{\text{thin}}$-consistent labelling, for every $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A}$ and every $a \in A$.*

(4) *For every morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ of $\mathbb{T}^{\text{thin}}$-algebras and every strongly $\mathbb{T}^{\text{thin}}$-consistent labelling β of some tree $t \in \mathbb{T}B$, there exists a strongly $\mathbb{T}^{\text{thin}}$-consistent labelling α with $\varphi \circ \alpha = \beta$.*

Theorem 7.7 (Bilkowski, Skrzypczak [1]). *Suppose that the Thin Choice Conjecture holds.*

(a) *Every finitary unambiguous $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A}$ has a unique $\mathbb{T}$-extension. Furthermore, this extension is* MSO*-definable.*

(b) *The class of unambiguous $\mathbb{T}^{\text{thin}}$-algebras forms a pseudo-variety.*

(c) *A language $K \subseteq \mathbb{T}\Sigma$ is bi-unambiguous if, and only if, (the $\mathbb{T}^{\text{thin}}$-reduct of) its syntactic algebra* $\text{Syn}(K)$ *is unambiguous.*

8 Branch-continuous algebras

In this final section, we take a look at a few other natural classes of $\mathbb{T}^{\text{thin}}$-algebras where unique $\mathbb{T}$-expansions exist. The simplest example consists of algebras that are constructed from an ω-semigroup as follows.

Definition 8.1. (a) Let $\mathfrak{S} = \langle S, S_\omega \rangle$ be an ω-semigroup. We denote by $\mathrm{TA}(\mathfrak{S})$ the $\mathbb{T}$-algebra $\langle A, \pi \rangle$ with domains

$$A_\xi := S_\omega + S \times \xi\,, \quad \text{for } \xi \in \Xi\,.$$

For elements $\langle a, x \rangle \in S \times \xi$, we will use the more suggestive notation $a(x)$. The product is defined as follows. Given $t \in \mathbb{T}_\xi A$, let $\beta = (v_i)_i$ be the path defined as follows. We start with the root v_0 of t. Having chosen v_i, we take a look at its label $t(v_i)$. If $t(v_i) = a_i(z_i) \in S \times \zeta_i$, we chose for v_{i+1} the z_i-successor of v_i. Otherwise, the path ends at v_i. Let $(a_i)_i$ be the corresponding sequence of labels. (If the path is finite, the last label a_n is either an element of S_ω or a variable.) We set

$$\pi(t) := \prod_i a_i\,.$$

Note that this product can be of one the following forms:

- an infinite product $a_0 \cdot a_1 \cdot \dots \in S_\omega$ with $a_i \in S$,
- a finite product $a_0 \cdot \dots \cdot a_n \in S_\omega$ with $a_0, \dots, a_{n-1} \in S$ and $a_n \in S_\omega$,
- a finite product $\langle a_0 \cdot \dots \cdot a_{n-1}, a_n \rangle \in S \times \xi$ with $a_0, \dots, a_{n-1} \in S$ and $a_n \in \xi$ is a variable.

(b) A $\mathbb{T}$-algebra $\mathfrak{A}$ is *semigroup-like* if it is isomorphic to $\mathrm{TA}(\mathfrak{S})$, for some ω-semigroup $\mathfrak{S}$. Similarly, for a subfunctor $\mathbb{T}^\circ \subseteq \mathbb{T}$, we call a $\mathbb{T}^\circ$-algebra $\mathfrak{A}$ *semigroup-like* if it is the $\mathbb{T}^\circ$-reduct of a semigroup-like $\mathbb{T}$-algebra. ⌟

Lemma 8.2. *Every semigroup-like $\mathbb{T}^{\text{thin}}$-algebra is unambiguous and has a unique $\mathbb{T}$-expansion. This expansion is again semigroup-like.*

Proof. Let $\mathfrak{A}$ be a semigroup-like $\mathbb{T}^{\text{thin}}$-algebra and let $\mathfrak{S}$ be the ω-semigroup such that $\mathfrak{A} \cong \mathrm{TA}(\mathfrak{S})|_{\mathbb{T}^{\text{thin}}}$. First, note that $\mathfrak{A}$ has a semigroup-like $\mathbb{T}$-expansion: the algebra $\mathrm{TA}(\mathfrak{S})$. Hence, we can use Proposition 6.5 (b) to find a strong labelling scheme for $\mathfrak{A}$. It therefore remains to prove that this labelling scheme is unique. Fix a tree $t \in \mathbb{T}A$ and let λ and μ be two strongly $\mathbb{T}^{\text{thin}}$-consistent labellings of t.

Given a vertex $v \in \mathrm{dom}(t)$, let β be the path starting at v that we construcgted in the definition of $\pi_+(t|_v)$. We choose a thin factor p of $t|_v$ containing this path. Then $\mathbb{T}^{\mathrm{thin}}$-consistency implies that $\lambda(v) = \pi(p) = \mu(v)$. Hence, $\lambda = \mu$. □

This lemma is hardly surprising, since the product of a semigroup-like algebra only depends on a single branch of the given tree. We can extend this result to more complicated classes of algebras as follows. So far, we have mostly ignored the fact that our algebras are ordered. The next two classes of examples on the other hand make essential use of the ordering. We start by introducing some notation concerning meets and joins.

Definition 8.3. Let A be a Ξ-sorted partially ordered set.

(a) For $X \subseteq A$, we set

$$\Uparrow X := \{\, a \in A \mid a \geq x \text{ for some } x \in X \,\}\,,$$
$$\Downarrow X := \{\, a \in A \mid a \geq x \text{ for some } x \in X \,\}\,.$$

(b) We define two functors $\mathbb{U}$ and $\mathbb{D}$ as follows. For sets A, we set

$$\mathbb{U}A := \{\, I \subseteq A \mid I \text{ is upwards closed} \,\}\,,$$
$$\mathbb{D}A := \{\, I \subseteq A \mid I \text{ is downwards closed} \,\}\,.$$

For functions $f : A \to B$, we define

$$\mathbb{U}f(I) := \Uparrow\{\, f(a) \mid a \in I \,\}\,,$$
$$\mathbb{D}f(I) := \Downarrow\{\, f(a) \mid a \in I \,\}\,.$$

(c) For $t \in \mathbb{T}A$ and $T \in \mathbb{T}\mathbb{U}A$ or $T \in \mathbb{T}\mathbb{D}A$, we write

$$t \in^{\mathbb{T}} T \quad :\text{iff} \quad t \text{ and } T \text{ have the same domain and if } t(v) \in T(v)\,, \quad \text{for all vertices } v\,.$$

(d) Let $C \subseteq A$. We denote by $\langle\!\langle C \rangle\!\rangle_{\mathrm{inf}}$ the closure of C under arbitrary meets and by $\langle\!\langle C \rangle\!\rangle_{\mathrm{sup}}$ its closure under joins. C is a set of *meet-generators* if $\langle\!\langle C \rangle\!\rangle_{\mathrm{inf}} = A$ and a set of *join-generators* if $\langle\!\langle C \rangle\!\rangle_{\mathrm{sup}} = A$. ⌟

The next, more interesting class of algebras we take a look at is the class of *deterministic* algebras, which was introduced in [6] to give an algebraic characterisation of the class of MSO-definable $\mathbb{T}$-algebras. Here, we are interested in the fact that their product is determined by its $\mathbb{T}^{\mathrm{thin}}$-reduct. The definition is as follows.

Definition 8.4. (a) We define $\mathrm{dist} : \mathbb{T}\mathbb{U}A \to \mathbb{U}\mathbb{T}A$ by

$$\mathrm{dist}(t) := \{\, s \in \mathbb{T}A \mid s \in^{\mathbb{T}} t \,\}\,.$$

(b) A function $g : \mathfrak{A} \to B$ from a $\mathbb{T}$-algebra $\mathfrak{A}$ to a completely ordered set B is *meet-distributive* if there exists a function $\sigma : \mathbb{T}\langle\!\langle \mathrm{rng}\, g \rangle\!\rangle_{\mathrm{inf}} \to B$ such that

$$\sigma \circ \mathbb{T}(\inf \circ \mathbb{U}g) = \inf \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}g\,.$$

A completely ordered $\mathbb{T}$-algebra $\mathfrak{A}$ is *meet-distributive* if the identity $\mathrm{id} : \mathfrak{A} \to \mathfrak{A}$ is meet-distributive. *Join-distributivity* is defined analogously.

(c) A $\mathbb{T}^{\mathrm{thin}}$-algebra $\mathfrak{A}$ is *deterministic* if it is meet-distributive and it has a semigroup-like subalgebra $\mathfrak{C}$ such that C forms a set of meet-generators of $\mathfrak{A}$ and the inclusion $C \to A$ is meet-distributive. ⌟

We start with two technical lemmas. The first one is trivial.

Lemma 8.5. *A completely ordered $\mathbb{T}$-algebra $\mathfrak{A}$ is meet-distributive if, and only if,*

$$\pi \circ \mathbb{T}\inf = \inf \circ \mathbb{U}\pi \circ \mathrm{dist}\,.$$

Proof. ($\Leftarrow$) In the definition of meet-distributivity, we can take $\sigma := \pi$.

($\Rightarrow$) Let σ be the function from the definition of meet-distributivity. Given a tree $t \in \mathbb{T}A$, let $T \in \mathbb{T}\mathbb{U}A$ be the tree with labels $T(v) = \{\, a \in A \mid a \geq t(v) \,\}$. Then

$$\sigma(t) = \sigma(\mathbb{T}\inf(T)) = \inf\{\, \pi(s) \mid s \in^{\mathbb{T}} T \,\} = \inf\{\pi(t)\} = \pi(t)\,.$$

Hence, we have

$$\begin{aligned} \pi \circ \mathbb{T}\inf &= \sigma \circ \mathbb{T}(\inf \circ \mathbb{U}\mathrm{id}) \\ &= \inf \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\mathrm{id} = \inf \circ \mathbb{U}\pi \circ \mathrm{dist}\,. \end{aligned}$$ □

Lemma 8.6. *Let $\varphi : \mathfrak{C} \to A$ be a meet-distributive function such that* $\mathrm{rng}\,\varphi$ *is a set of meet-generators of A. There exists a unique function $\sigma : \mathbb{T}A \to A$ such that $\langle A, \sigma\rangle$ is a meet-distributive $\mathbb{T}$-algebra and φ a morphism of $\mathbb{T}$-algebras.*

Proof. To make our proof more concise, we use some properties of the function $\mathrm{dist} : \mathbb{T}\mathbb{U} \Rightarrow \mathbb{U}\mathbb{T}$. We have shown in [3] that dist is what is called a *distributive law,* which means it is a natural transformation satisfying the equations

$$\mathrm{dist} \circ \mathrm{flat} = \mathbb{U}\mathrm{flat} \circ \mathrm{dist} \circ \mathbb{T}\mathrm{dist}\,, \qquad \mathrm{dist} \circ \mathrm{sing} = \mathbb{U}\mathrm{sing}\,,$$
$$\mathrm{dist} \circ \mathbb{T}\mathrm{union} = \mathrm{union} \circ \mathbb{U}\mathrm{dist} \circ \mathrm{dist}\,, \qquad \mathrm{dist} \circ \mathbb{T}\mathrm{pt} = \mathrm{pt}\,,$$

where union : $\mathbb{U}\mathbb{U}A \to \mathbb{U}$ maps a set of sets to its union and pt : $A \to \mathbb{U}A$ is defined by $\mathrm{pt}(a) := \Uparrow\{a\}$.

Let $\sigma : \mathbb{T}A \to A$ be the function from the definition of meet-distributivity. To see that $\langle A, \sigma \rangle$ is a $\mathbb{T}$-algebra, note that

$$\begin{aligned}
\sigma \circ \mathrm{sing} \circ (\inf \circ \mathbb{U}\varphi) &= \sigma \circ \mathbb{T}(\inf \circ \mathrm{pt}) \circ \mathrm{sing} \circ (\inf \circ \mathbb{U}\varphi) \\
&= \sigma \circ \mathbb{T}(\inf \circ \mathrm{pt}) \circ \mathbb{T}(\inf \circ \mathbb{U}\varphi) \circ \mathrm{sing} \\
&= \sigma \circ \mathbb{T}\inf \circ \mathbb{T}\mathbb{U}(\inf \circ \mathbb{U}\varphi) \circ \mathbb{T}\mathrm{pt} \circ \mathrm{sing} \\
&= \inf \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}(\inf \circ \mathbb{U}\varphi) \circ \mathbb{T}\mathrm{pt} \circ \mathrm{sing} \\
&= \inf \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathrm{sing} \circ \mathbb{U}(\inf \circ \mathbb{U}\varphi) \circ \mathrm{pt} \\
&= \inf \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathrm{sing} \circ \mathrm{pt} \circ (\inf \circ \mathbb{U}\varphi) \\
&= \inf \circ \mathbb{U}\pi \circ \mathbb{U}\mathrm{sing} \circ \mathrm{pt} \circ (\inf \circ \mathbb{U}\varphi) \\
&= \inf \circ \mathrm{pt} \circ (\inf \circ \mathbb{U}\varphi) \\
&= \inf \circ \mathbb{U}\varphi\,, \\
\sigma \circ \mathbb{T}\sigma \circ \mathbb{T}\mathbb{T}(\inf \circ \mathbb{U}\varphi) &= \sigma \circ \mathbb{T}(\inf \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\varphi) \\
&= \sigma \circ \mathbb{T}(\inf \circ \mathbb{U}\pi \circ \mathbb{U}\mathbb{T}\varphi \circ \mathrm{dist}) \\
&= \sigma \circ \mathbb{T}(\inf \circ \mathbb{U}\varphi \circ \mathbb{U}\pi \circ \mathrm{dist}) \\
&= \inf \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\varphi \circ \mathbb{T}(\mathbb{U}\pi \circ \mathrm{dist}) \\
&= \inf \circ \mathbb{U}\pi \circ \mathbb{U}\mathbb{T}(\varphi \circ \pi) \circ \mathrm{dist} \circ \mathbb{T}\mathrm{dist} \\
&= \inf \circ \mathbb{U}(\pi \circ \mathbb{T}(\pi \circ \mathbb{T}\varphi)) \circ \mathrm{dist} \circ \mathbb{T}\mathrm{dist} \\
&= \inf \circ \mathbb{U}(\pi \circ \mathrm{flat} \circ \mathbb{T}\mathbb{T}\varphi) \circ \mathrm{dist} \circ \mathbb{T}\mathrm{dist} \\
&= \inf \circ \mathbb{U}(\pi \circ \mathbb{T}\varphi \circ \mathrm{flat}) \circ \mathrm{dist} \circ \mathbb{T}\mathrm{dist} \\
&= \inf \circ \mathbb{U}(\pi \circ \mathbb{T}\varphi) \circ \mathrm{dist} \circ \mathrm{flat} \\
&= \inf \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\varphi \circ \mathrm{flat} \\
&= \inf \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathrm{flat} \circ \mathbb{T}\mathbb{T}\mathbb{U}\varphi \\
&= \sigma \circ \mathbb{T}\inf \circ \mathrm{flat} \circ \mathbb{T}\mathbb{T}\mathbb{U}\varphi \\
&= \sigma \circ \mathrm{flat} \circ \mathbb{T}\mathbb{T}(\inf \circ \mathbb{U}\varphi)\,.
\end{aligned}$$

Since $\inf \circ \mathbb{U}\varphi$ is surjective and $\mathbb{T}$ preserves surjectivity, it follows that

$$\sigma \circ \mathrm{sing} = \mathrm{id} \quad \text{and} \quad \sigma \circ \mathbb{T}\sigma = \sigma \circ \mathrm{flat}\,.$$

To see that $\mathfrak{A}$ is meet-distributive, note that

$$\begin{aligned}
&\sigma \circ \mathbb{T}(\mathrm{inf} \circ \mathbb{U}\mathrm{id}) \circ \mathbb{T}\mathbb{U}(\mathrm{inf} \circ \mathbb{U}\varphi) \\
&= \sigma \circ \mathbb{T}(\mathrm{inf} \circ \mathbb{U}\mathrm{inf} \circ \mathbb{U}\mathbb{U}\varphi) \\
&= \sigma \circ \mathbb{T}(\mathrm{inf} \circ \mathrm{union} \circ \mathbb{U}\mathbb{U}\varphi) \\
&= \sigma \circ \mathbb{T}(\mathrm{inf} \circ \mathbb{U}\varphi \circ \mathrm{union}) \\
&= \mathrm{inf} \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\varphi \circ \mathbb{T}\mathrm{union} \\
&= \mathrm{inf} \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}(\mathrm{union} \circ \mathbb{U}\mathbb{U}\varphi) \\
&= \mathrm{inf} \circ \mathbb{U}\pi \circ \mathrm{union} \circ \mathbb{U}\mathrm{dist} \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\mathbb{U}\varphi \\
&= \mathrm{inf} \circ \mathbb{U}\pi \circ \mathrm{union} \circ \mathbb{U}\mathrm{dist} \circ \mathbb{U}\mathbb{T}\mathbb{U}\varphi \circ \mathrm{dist} \\
&= \mathrm{inf} \circ \mathrm{union} \circ \mathbb{U}(\mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\varphi) \circ \mathrm{dist} \\
&= \mathrm{inf} \circ \mathbb{U}\mathrm{inf} \circ \mathbb{U}(\mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\varphi) \circ \mathrm{dist} \\
&= \mathrm{inf} \circ \mathbb{U}(\sigma \circ \mathbb{T}(\mathrm{inf} \circ \mathbb{U}\varphi)) \circ \mathrm{dist} \\
&= \mathrm{inf} \circ \mathbb{U}\sigma \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}(\mathrm{inf} \circ \mathbb{U}\varphi)) \\
&= \mathrm{inf} \circ \mathbb{U}\sigma \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\mathrm{id} \circ \mathbb{T}\mathbb{U}(\mathrm{inf} \circ \mathbb{U}\varphi)\,.
\end{aligned}$$

By surjectivity of $\mathbb{T}\mathbb{U}(\mathrm{inf} \circ \mathbb{U}\varphi)$, this implies that

$$\sigma \circ \mathbb{T}(\mathrm{inf} \circ \mathbb{U}\mathrm{id}) = \mathrm{inf} \circ \mathbb{U}\sigma \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\mathrm{id}\,.$$

To see that φ is a morphism of $\mathbb{T}$-algebras, note that

$$\begin{aligned}
\sigma \circ \mathbb{T}\varphi &= \sigma \circ \mathbb{T}(\mathrm{inf} \circ \mathrm{pt} \circ \varphi) \\
&= \sigma \circ \mathbb{T}(\mathrm{inf} \circ \mathbb{U}\varphi \circ \mathrm{pt}) \\
&= \mathrm{inf} \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\varphi \circ \mathbb{T}\mathrm{pt} \\
&= \mathrm{inf} \circ \mathbb{U}\pi \circ \mathbb{U}\mathbb{T}\varphi \circ \mathrm{dist} \circ \mathbb{T}\mathrm{pt} \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{dist} \circ \mathbb{T}\mathrm{pt} \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{pt} \\
&= \mathrm{inf} \circ \mathrm{pt} \circ \varphi \circ \pi \\
&= \varphi \circ \pi\,.
\end{aligned}$$

Finally, for uniqueness suppose that $\sigma' : \mathbb{T}A \to A$ is another function like this. Then we have

$$\sigma \circ \mathbb{T}(\mathrm{inf} \circ \mathbb{U}\varphi) = \mathrm{inf} \circ \mathbb{U}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathbb{U}\varphi = \sigma' \circ \mathbb{T}(\mathrm{inf} \circ \mathbb{U}\varphi)\,,$$

and the fact that $\mathbb{T}(\mathrm{inf} \circ \mathbb{U}\varphi)$ is surjective implies that $\sigma = \sigma'$. □

Theorem 8.7. *Every deterministic $\mathbb{T}^{\text{thin}}$-algebra has a unique meet-distributive $\mathbb{T}$-expansion.*

Proof. Let $\mathfrak{A} = \langle A, \pi \rangle$ be a deterministic $\mathbb{T}^{\text{thin}}$-algebra and let $\mathfrak{C} \subseteq \mathfrak{A}$ be the corresponding semigroup-like subalgebra. We can use Lemma 8.2 to find a unique $\mathbb{T}$-expansion $\mathfrak{C}_+$ of $\mathfrak{C}$, and Lemma 8.6 to find a unique meet-distributive algebra $\mathfrak{A}_+ = \langle A, \pi_+ \rangle$ with universe A that contains $\mathfrak{C}_+$ as a subalgebra.

It therefore remains to prove that $\mathfrak{A}$ is the $\mathbb{T}^{\text{thin}}$-reduct of this algebra $\mathfrak{A}_+$. Hence, let $t \in \mathbb{T}^{\text{thin}} A$ and fix a tree $T \in \mathbb{T}^{\text{thin}} \mathbb{U} C$ such that $t = \mathbb{T}^{\text{thin}} \inf(T)$. By meet-distributivity and the fact that the products π and π_+ agree on trees in $\mathbb{T}^{\text{thin}} C$, it follows that

$$\pi_+(t) = \inf \{ \pi_+(s) \mid s \in^{\mathbb{T}} T \} = \inf \{ \pi(s) \mid s \in^{\mathbb{T}} T \} = \pi(t) ,$$

as desired. □

Corollary 8.8. *Every deterministic $\mathbb{T}$-algebra is uniquely determined by its $\mathbb{T}^{\text{thin}}$-reduct.*

We can generalise deterministic algebras by also allowing joins. The resulting algebras are called *branch-continuous.* They were introduced in [2] as an algebraic analogue to tree automata.

Definition 8.9. A $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A}$ is *branch-continuous* if it is join-distributive and it has a deterministic subalgebra $\mathfrak{C}$ such that C forms a set of join-generators of $\mathfrak{A}$ and the inclusion $C \to A$ is join-distributive. ⌟

Using join-distributivity and meet-distributivity, one can show that a product $\pi(t)$ in a branch-continuous algebra can be computed by taking a join over meets over products along single branches of t (see [2] for details). In particular, a product of this form is MSO-definable. Together with the translation of automata into branch-continuous $\mathbb{T}$-algebras, this leads to the following two results from [2].

Proposition 8.10. *Every finitary branch-continuous $\mathbb{T}$-algebra is* MSO-*definable.*

Theorem 8.11. *A language $K \subseteq \mathbb{T}\Sigma$ is regular if, and only if, it is recognised by a morphism into a finitary branch-continuous $\mathbb{T}$-algebra.*

Thus branch-continuous algebras play a similar role as the MSO-definable ones. The reason we usually work with the latter is that the former do not form a pseudo-variety: the class of branch-continuous algebras is not closed under finitely-generated subalgebras.

Here, we are more interested in the fact that branch-continuous algebras have unique expansions.

Theorem 8.12. *Every branch-continuous $\mathbb{T}^{\text{thin}}$-algebra has a unique join-distributive $\mathbb{T}$-expansion.*

Proof. Let $\mathfrak{A}$ be a branch-continuous $\mathbb{T}^{\text{thin}}$-algebra and let $\mathfrak{C}$ be the corresponding deterministic subalgebra. By Theorem 8.7, $\mathfrak{C}$ has a unique meet-distributive $\mathbb{T}$-expansion $\mathfrak{C}_+$. By the dual version of Lemma 8.6, it follows that there exist a unique join-distributive $\mathbb{T}$-algebra $\mathfrak{A}_+$ extending $\mathfrak{C}_+$. By the same argument as in the proof of Theorem 8.7, it follows that $\mathfrak{A}^+$ is a $\mathbb{T}$-expansion of $\mathfrak{A}$. □

Corollary 8.13. *Every branch-continuous $\mathbb{T}$-algebra is uniquely determined by its $\mathbb{T}^{\text{thin}}$-reduct.*

9 Conclusion

We have presented several approaches to the expansion problem for tree algebras. In each cases, we could use the existing combinatorial theory for ω-semigroups to solve the problem for $\mathbb{T}^{\text{thin}}$-expansions, but we always hit a wall when considering the problem for expansions to non-thin trees.

In particular, the methods we developed seem to work well if there exists a unique expansion (or at least a unique expansion with a certain property, like a unique MSO-definable expansion, or a unique branch-continuous one), but there is currently no approach to prove the existence of several expansions.

Promising next steps towards further progress seem to include

- trying to generalise some of our existing tools to general trees; and/or
- finding counterexamples delineating the parameter space where such generalisations do not exist any more.

The most promising approach to do so appears to be to flesh out the theory of Green's relations for tree algebras, in the hope of it enabling the transfer of the proof Simon's Factorisation-Tree Theorem to this new setting. But this seems to be a very hard problem.

References

[1] M. Bilkowski and M. Skrzypczak, *Unambiguity and uniformization problems on infinite trees*, in Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2–5, 2013, Torino, Italy, vol. 23 of LIPIcs, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2013, pp. 81–100.

[2] A. Blumensath, *Branch-Continuous Tree Algebras.* arXiv:1807.04568, unpublished.

[3] ———, *The Power-Set Operation for Tree Algebras.* submitted.

[4] ———, *Recognisability for algebras of infinite trees*, Theoretical Computer Science, 412 (2011), pp. 3463–3486.

[5] ———, *An Algebraic Proof of Rabin's Tree Theorem*, Theoretical Computer Science, 478 (2013), pp. 1–21.

[6] ———, *Regular Tree Algebras*, Logical Methods in Computer Science, 16 (2020), pp. 16:1–16:25.

[7] ———, *Algebraic Language Theory for Eilenberg–Moore Algebras*, Logical Methods in Computer Science, 17 (2021), pp. 6:1–6:60.

[8] M. Bojańczyk, *Languages Recognises by Finite Semigroups and their generalisations to objects such as Trees and Graphs with an emphasis on definability in Monadic Second-Order Logic.* lecture notes, arXiv:2008.11635, 2020.

[9] M. Bojańczyk, F. Cavallari, T. Place, and M. Skrzypczak, *Regular tree languages in low levels of the Wadge Hierarchy*, Log. Methods Comput. Sci., 15 (2019), pp. 27:1–27:61.

[10] M. Bojańczyk and T. Idziaszek, *Algebra for Infinite Forests with an Application to the Temporal Logic EF*, in Proc. 20th International Conference on Concurrency Theory, CONCUR, LNCS 5710, 2009, pp. 131–145.

[11] M. Bojańczyk, T. Idziaszek, and M. Skrzypczak, *Regular languages of thin trees*, in Proc. 30th International Symposium on Theoretical Aspects of Computer Science, STACS 2013, 2013, pp. 562–573.

[12] M. Bojańczyk and B. Klin, *A non-regular language of infinite trees that is recognizable by a finite algebra*, Logical Methods in Computer Science, 15 (2019).

[13] O. Carton, T. Colcombet, and G. Puppis, *An Algebraic Approach to MSO-Definability on Countable linear Orderings*, J. Symb. Log., 83 (2018), pp. 1147–1189.

[14] T. Colcombet, *The Factorisation Forest Theorem*, in Handbook of Automata Theory, J.-É. Pin, ed., European Mathematical Society, 2021, pp. 653–693.

[15] C. Löding, *Automata on infinite trees*, in Handbook of Automata Theory, J.-É. Pin, ed., European Mathematical Society, 2021, pp. 265–302.

[16] K. R. Milliken, *A Ramsey theorem for trees*, Journal of Combinatorial Theory, Series A, 26 (1979), pp. 215–237.

[17] G. Puppis, *Automata for Branching and Layered Temporal Structures: An Investigation into Regularities of Infinite Transition Systems*, vol. 5955 of Lecture Notes in Computer Science, Springer, 2010.

[18] W. Thomas, *A combinatorial approach to the theory of ω-automata*, Information and Control, 48 (1981), pp. 261–283.

[19] ———, *Languages, Automata, and Logic*, in Handbook of Formal Languages, G. Rozenberg and A. Salomaa, eds., vol. 3, Springer, New York, 1997, pp. 389–455.