MASARYK UNIVERSITY
FACULTY OF INFORMATICS

Non-well-founded Conway games

BACHELOR'S THESIS

Martin Blahynka

Brno, Spring 2020

Masaryk University
Faculty of Informatics

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Bachelor's Thesis

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Declaration

Hereby I declare that this paper is my original authorial work, which I have worked out on my own. All sources, references, and literature used or excerpted during elaboration of this work are properly cited and listed in complete reference to the due source.

Martin Blahynka

Advisor: Dr. rer. nat. Achim Blumensath

Acknowledgements

I am very grateful to my thesis supervisor Dr. rer. nat. Achim Blumensath for introducing me to the topic and helping me with the development of this thesis.

Abstract

Conway games are a type of partisan combinatorial game between two players. We give an exposition of (possibly non-well-founded) Conway games based on a simple graph model. We consider every infinite play to be a draw. In this formalism we prove determinacy of Conway games. We similarly formalise positional games – another type of game – and show how the two types of games can be reduced to each other.

Keywords

Conway games, winning regions, determinacy, positional games.

Contents

Introduction

A Conway game is a combinatorial game between two players. The players alternate in making a move and when a player whose turn it is cannot make a move, he loses. It is a partisan game – in every position the possible moves of each player can be different.

Conway games were first introduced by John Horton Conway in his book *On numbers and games* [1]. He studied well-founded games – that is, games in which there can be no infinite plays. In this thesis, however, we allow the games to be non-well-founded. The possibility of an infinite play gives rise to a new result – we consider every infinite play to be a draw.

Conway games can be used to model many abstract strategy board games, such as chess or go, where the players alternate in making a move. Fundamental rules of these board games often allow for an infinite play by repetition of positions. For instance, in chess an infinite play is considered to be a draw and some additional rules have been introduced to stop a play that would go indefinitely. Non-well-founded Conway games where every infinite play is a draw are a good mathematical model for such board games.

Admittedly, many abstract strategy board games are too complex; optimal strategies cannot be found in a reasonable time by analysing a corresponding combinatorial game, and machine learning algorithms are used instead to approximate optimal strategies.

Among other possible applications, combinatorial games can be used for model checking for various logics. The basic idea is to reduce the question of whether a logical formula is true in a given structure to the question of whether there exists a winning strategy for one of the players in an associated combinatorial game. For instance, a model-checking game for a first-order logic problem is played by two players V and F. V tries to prove that the formula is true in the given structure, her moves correspond to choosing values for existentially quantified variables, whereas F chooses values for universally quantified variables and tries to prove that the formula is not true [2].

One of the ways to extend the theory of Conway games to non-well-founded Conway games is by coalgebraic methods [3]. In this thesis, however, we aim to give a simpler exposition of (possibly non-well-founded) Conway games based on a graph model.

In the first chapter we introduce Conway games as oriented graphs and provide basic definitions. The players move between vertices along edges.

The graphs can be arbitrary, we do not require them to be finite nor acyclic; some graphs therefore correspond to non-well-founded games.

In the second chapter we introduce winning and losing regions (that is, the sets of positions in which players can win, respectively cannot avoid to lose). Then we prove a few auxiliary theorems needed in the third chapter.

In the third chapter we prove determinacy of Conway games – for every Conway game exactly one of the following three statements holds: (i) the starting player has a winning strategy, (ii) the other player has a winning strategy, (iii) both players have a non-losing strategy.

In the fourth chapter we introduce another type of game between two players – positional games. They are also defined in terms of a simple graph model.

In the fifth chapter we connect the two types of games. We define reductions of Conway games to positional games and vice-versa.

1 Conway games as graphs

In this chapter we introduce Conway games. We define basic terms and show a few examples of simple Conway games.

1.1 Basic definitions

There are two players in Conway games playing against each other. Formally, players are numbers.

Definition 1.1.1 (Players)**.** A *player* p is a number from $\{1, 2\}$.

Notation. Let p be a player. We will write $\overline{p}$ for the other player.

Definition 1.1.2 (Conway game)**.** A *Conway game* is a tuple (V, E_1, E_2, q, z) where V is a set of *positions*, $E_1, E_2 \subseteq V \times V$ are sets of *moves of Player 1*, respectively *moves of Player 2*, $q \in V$ is *the starting position*, and $z \in \{1, 2\}$ is *the starting player*.

A game represents an initial setup. Given a game we can define a play of the game. A play is a sequence of positions. It starts in the starting position, the starting player makes the first move, then both players alternate in making a move (the possible moves of a player p are determined by the current position and by E_p). If the play is finite, then the player whose turn it is in the last position has no move to make (that is, a play cannot end in a situation in which a further development is possible).

Now we give a formal definition.

Definition 1.1.3 (Play of a Conway game)**.** Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game.

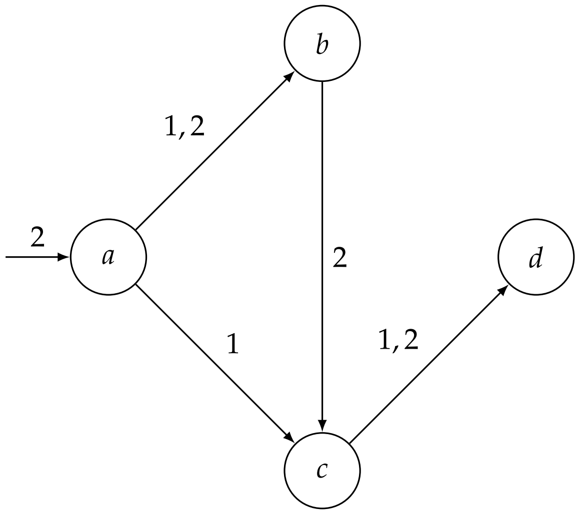
A *play* of $\mathcal{G}$ is any sequence of positions $(p_i)_{i=1}^{n}$ where $n \in \mathbb{N}^+ \cup \{\omega\}$ [1], such that all the following statements hold.

- $p_1 = q$.
- for every odd $m \in \mathbb{N}$ such that $m < n$ we have $(p_m, p_{m+1}) \in E_z$.
- for every even $m \in \mathbb{N}^+$ such that $m < n$ we have $(p_m, p_{m+1}) \in E_{\overline{z}}$.
- if $n \in \mathbb{N}$, then there is no $v \in V$ such that $(p_n, v) \in E_{((n+z) \bmod 2)+1}$.

1. $\mathbb{N}^+$ denotes the set of all natural numbers except zero (whereas $\mathbb{N}$ does include zero), ω the least infinite ordinal.

Notation. Given a player p, when we say that it is p's turn in a position p_i in the given play $P = (p_i)_{i=1}^{n}$, we mean that it is p's turn in the i-th position of P (that is, $i \equiv z + \overline{p} \pmod 2$ where z is the starting player). It can be any player's turn in p_i when it appears in j-th position of P for some $j \neq i$.

Figure 1.1: A simple well-founded Conway game $\mathcal{G}$



Example. The game in Figure 1.1 is formally the following tuple.

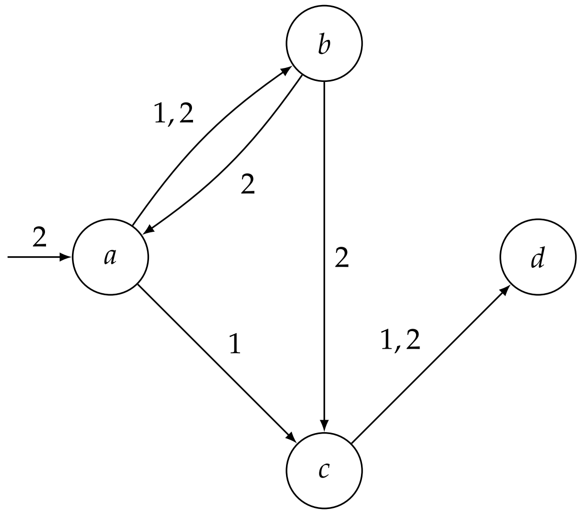
$$\mathcal{G} = (\{a, b, c, d\}, \{(a, b), (a, c), (c, d)\}, \{(a, b), (b, c), (c, d)\}, a, 1)$$

The vertices in the graph depict the positions. The edges depict the moves. The information of "to which player does a move belong" is conveyed by the labels of the edges. When both players have the same move, we draw them a common edge. We mark the starting vertex by an arrow pointing to it with no beginning vertex. It is labeled by the player who does not start (one can imagine that this player makes the zeroth move to the starting vertex).

The possible plays of $\mathcal{G}$ are $P = (a, b, c, d)$ and $P' = (a, c, d)$.

P' is a losing play for Player 1, as it is 1's turn in the end. However, she has a winning strategy. If she moves from a to b in her first move, she will win – the winning play is P. The terms *strategy*, *winning play*, etc. are defined later in this chapter. We took the liberty of using them here, before defining them, as they are intuitively intelligible.

Figure 1.2: A simple non-well-founded Conway game $\mathcal{G}'$



Example. In Figure 1.2 we added the edge (b, a) for Player 2 to Figure 1.1. The cycle induces the possibility of an infinite play of the game $\mathcal{G}'$. Now there are infinitely many possible plays, among which there is $P = (ab)^{\omega}$. P is a draw because it is infinite. In terms of strategies, both players have a non-losing strategy, no player has a winning strategy.

Definition 1.1.4 (Positional strategy)**.** Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game and p a player.

A *positional strategy* of p for $\mathcal{G}$ is a partial function $S : V \to V$ such that for every position v on which S is defined we have $(v, S(v)) \in E_p$, and for every position v on which S is not defined, there is no position w such that $(v, w) \in E_p$.

Notation. We will abbreviate the term *positional strategy* and use the term *strategy* instead.

Note. A strategy is a recipe for how to play. As defined in Definition 1.1.3, no play ends when a further development is still possible. This is why we demand in the definition of a strategy that players cannot just give up (make no move in a position in which they can make a move).

Notation. When the Conway game $\mathcal{G}$ is apparent from the context, we will only say "a strategy of p", instead of "a strategy of p for $\mathcal{G}$".

Note. Our terminology is not standard. A strategy usually means a recipe describing where to move depending on the position *and the history* (that is, the sequence of already visited positions). We will call such a strategy a *general strategy*[2]. This concept is especially important when the winning condition is specified in such a way that some infinite plays are winning for one of the players (in which case one of the players might have, for instance, a winning general strategy but no winning positional strategy). In our case, however, positional strategies are "sufficiently strong". That is, regardless of whether we define strategies to take into account the history or not, the existence of winning and non-losing strategies will be the same for every Conway game. An argument for this is given later in the proof of Theorem 3.2.2; it is a consequence of determinacy of Conway games.

Definition 1.1.5 (Play conforming to a strategy)**.** Let $\mathcal{G}$ be a Conway game, p a player, S a strategy of p, and $P = (p_n)_{i=1}^{n}$ a play of $\mathcal{G}$.

We say that P conforms to S iff for every position p_i in P in which it is p's turn we have $i = n$ or $S(p_i) = p_{i+1}$.

Definition 1.1.6 (Play conforming to strategies of both players)**.** Let $\mathcal{G}$ be a Conway game, $P = (p_n)_{i=1}^{n}$ a play of $\mathcal{G}$, S a strategy of Player 1, and S' a strategy of Player 2.

Then $S \circ S' = S' \circ S$ is the unique play conforming to both S and S'.

Note. It is easy to prove by induction that, given a strategy for each player, there exists a play conforming to both strategies and that it is unique. The first position in it is q. The even positions are uniquely determined by the strategy of the starting player and by the previous position. The odd positions except for the first one are uniquely determined by the strategy of the other player and by the previous position. Therefore, there cannot be several such plays. The requirements of the two strategies do not overlap. The strategy of the starting player determines only the even positions, the strategy of the other player determines only the odd positions except for the first one, which is q. Therefore, there exists such a play.

2. One of the ways of defining both terms is to first define a general strategy, and then define a positional strategy as a special case of general strategy [3].

Definition 1.1.7 (Draw, losing and winning play)**.** Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game, p a player, and $P = (p_i)_{i=1}^{n}$ a play of $\mathcal{G}$.

We say that P is a *draw* iff $n = \omega$. P is *winning* for p iff $n \in \mathbb{N}$ and $p \equiv (n+z) \pmod 2$. P is losing for p iff $n \in \mathbb{N}$ and P is not winning for p.

Note. Note that for a game $\mathcal{G}$ and a play P we can decide the result of P solely from its length. If P is infinite, the result is a draw. If the length of P is even, the starting player wins (because it is the other player whose turn it is in the last position). If the length of p is odd, the player who does not start wins.

Note. Clearly, a play is winning for one of the players iff it is losing for the other one.

Definition 1.1.8 (Winning and non-losing strategy)**.** Let $\mathcal{G}$ be a Conway game, p a player, and S a strategy of p.

We say that S is *non-losing* (*winning*) iff every play of $\mathcal{G}$ conforming to S is non-losing (winning) for p.

2 Winning and losing regions of Conway games

2.1 Winning and losing regions

Notation. When the game $\mathcal{G} = (V, E_1, E_2, q, z)$ is apparent from the context, we will use the following term for a more comfortable expression. The *game when a player p starts in a position v* means the game (V, E_1, E_2, v, p).

For a given Conway game, the winning (losing) region of a player p is the set of those positions v in which p has a winning (has no non-losing) strategy when p starts in v. By determinacy (Theorem 3.1.2), the losing region of p is the set of those positions v in which $\overline{p}$ has a winning strategy when p starts in v.

Positions in which a player p has a winning strategy are therefore in two sets – in the winning region of p (p can win in these positions when he starts) and in the losing region of $\overline{p}$ (p can win in these positions when $\overline{p}$ starts). Similarly, positions in which a player has no non-losing strategy are in two sets.

Definition 2.1.1 (Winning and losing regions of a Conway game)**.** Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game and p a player. We define

$$W_{\mathcal{G}}^{(p)} = \{v \in V \mid p \text{ has a winning strategy for } (V, E_1, E_2, v, p)\},$$
$$L_{\mathcal{G}}^{(p)} = \{v \in V \mid \ p \text{ has no non-losing strategy for } (V, E_1, E_2, v, p)\}.$$

We call $W_{\mathcal{G}}^{(p)}$ ($L_{\mathcal{G}}^{(p)}$) the winning (losing) region of p.

The definition of the regions is not constructive. In the following section we show how it is possible to construct the regions.

2.2 Constructive winning and losing regions

We will define a constructive version of the regions below which provides a powerful insight for studying the existence of winning and non-losing strategies. In particular, it will help us prove determinacy in the next chapter.

Notation. For a Conway game $\mathcal{G} = (V, E_1, E_2, q, z)$ and a player p, we will omit the subscript $\mathcal{G}$ in the notation of the regions. Instead of $L_{\mathcal{G}}^{(p)}$, $W_{\mathcal{G}}^{(p)}$ we will write $L^{(p)}$, $W^{(p)}$. We will use the subscript for defining new sets of positions. (If $\mathcal{G}$ was not clear from the context, we would need two subscripts.)

Definition 2.2.1 (Constructive regions)**.** Let p be a player and $n \in \mathbb{ON}$ [1]. We define

$$W_n^{(p)} = \{v \in V \mid \text{there exist}^2\ m < n \text{ and } w \in L_m^{(\overline{p})} \text{ such that } (v,w) \in E_p\},$$
$$L_n^{(p)} = \{v \in V \mid \text{for every } w \in V \text{ such that } (v,w) \in E_p \text{ we have } w \in W_n^{(\overline{p})}\}.$$

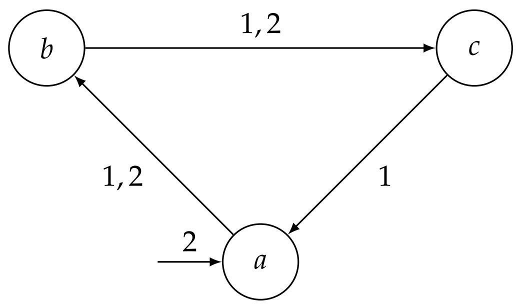
Note. Note that $W_0^{(p)}$ is the empty set because there is no ordinal $m < 0$.

$L_0^{(p)}$ is the set of those positions in which p has no move (otherwise there would have to be some $w \in W_0^{(\overline{p})}$, but $W_0^{(\overline{p})}$ is empty).

For $n \in \mathbb{N}^+$, $W_n^{(p)}$ is the set of those positions in which p can win when he starts, making no more than n moves. $L_n^{(p)}$ is the set of those positions in which p is losing when he starts and cannot make more than n moves when $\overline{p}$ plays to win as fast as possible.

Note. Trivially, for every Conway game $\mathcal{G}$, all $i, j \in \mathbb{ON}$ such that $i < j$, and every player p we have $W_i^{(p)} \subseteq W_j^{(p)}$ and $L_i^{(p)} \subseteq L_j^{(p)}$.

Figure 2.1: A simple Conway game



1. $\mathbb{ON}$ denotes the class of all ordinal numbers.
2. In similar cases we do not mention the obvious domain of m, which is $\mathbb{ON}$.

Example. We look at the winning and losing constructive regions of the game in Figure 2.1.

$$L_n^{(1)} = \emptyset, \quad W_n^{(2)} = \emptyset \qquad \text{for every } n \in \mathbb{ON},$$
$$W_0^{(1)} = \emptyset, \quad L_0^{(2)} = \{c\}, \quad W_1^{(1)} = \{b\}, \quad L_1^{(2)} = \{c, a\}, \quad W_2^{(1)} = \{b, c\},$$
$$L_n^{(2)} = W_{n+1}^{(1)} = \{a, b, c\} \qquad \text{for every } n \in \mathbb{ON} \text{ such that } n > 1.$$

We can also see that $W^{(1)} = L^{(2)} = \{a, b, c\}$ and $W^{(2)} = L^{(1)} = \emptyset$ by analysing the game in terms of the existence of winning and non-losing strategies. In this simple game, every player has only one strategy. The strategies are:

$S_1 = \{(a, b), (b, c), (c, a)\}$ for Player 1,

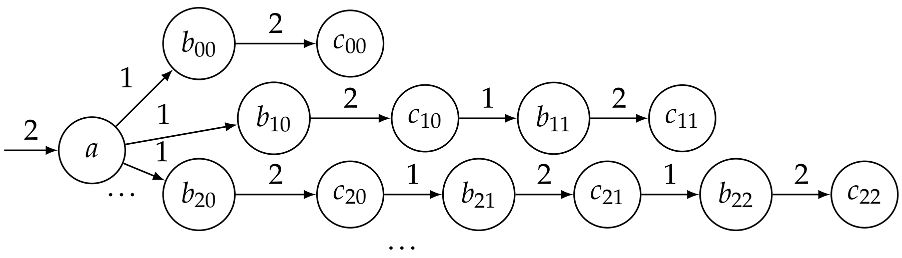
$S_2 = \{(a, b), (b, c)\}$ for Player 2.

For every player p and every position v, there is only one play conforming to S_p when p starts in v, and it is winning for 1 and losing for 2.

We have seen an example in which $W^{(p)} = \bigcup_{i=0}^{\infty} W_i^{(p)}$ and $L^{(p)} = \bigcup_{i=0}^{\infty} L_i^{(p)}$ for every player p. Are the sets always equal?

For finite graphs, they indeed are. But our games may have any ordinal number of positions. A union over natural numbers does not suffice.

Figure 2.2: An infinite game



Example. The starting position a in the game in Figure 2.2 is such that for every $n \in \mathbb{N}$ we have $a \notin L_n^{(1)}$ because Player 1 has a move to some $w \notin W_n^{(2)}$ (in fact, almost all the moves are such, one of them being (a, b_{n0})). However, every strategy of 1 is such that there is a play conforming to it that is losing for 1 (2 even cannot go wrong as there is always only one play conforming to the strategy of 1), therefore $a \in L^{(1)}$.

2.3 Ranks

In this last section of this chapter we introduce ranks of positions. For instance, whenever we know that $v \in W_n^{(p)}$, we also know that $v \in W_m^{(p)}$ for all $m > n$ but can only guess whether $v \in W_m^{(p)}$ for some $m < n$. In such a case we are missing an important information – what is the least m such that $v \in W_m^{(p)}$ (in other words, when did v first appear in the ascending sequence of constructive winning regions of p).

Definition 2.3.1 (Ranks)**.** Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game, p a player, and $n \in \mathbb{ON}$. We define

$$\widehat{W}_n^{(p)} = W_n^{(p)} - \bigcup_{i<n} W_i^{(p)}, \qquad \widehat{L}_n^{(p)} = L_n^{(p)} - \bigcup_{i<n} L_i^{(p)}.$$

We say that a position $v \in V$ *has a winning* (*losing*) *rank* n *for* p iff $v \in \widehat{W}_n^{(p)}$ ($v \in \widehat{L}_n^{(p)}$).

Note. Note that for every player p, every position has at most one winning rank and one losing rank for p.

Also note that $\bigcup_i W_i^{(p)} = \bigcup_i \widehat{W}_i^{(p)}$ and $\bigcup_i L_i^{(p)} = \bigcup_i \widehat{L}_i^{(p)}$ [3].

Note. For every limit ordinal n and every player p we have $\widehat{W}_n^{(p)} = \emptyset$. (Because when $v \in W_n^{(p)}$, then there exists $m < n$ such that p has a move from v to $L_m^{(\overline{p})}$, therefore $v \in W_{m+1}^{(p)}$. And when n is a limit ordinal, then $m + 1 < n$, therefore $v \notin \widehat{W}_n^{(p)}$.) However, $\widehat{L}_n^{(p)}$ can be non-empty, as we have seen in Figure 2.2.

We conclude the chapter by a few theorems. Corollary 2.3.3 is useful for the proof of determinacy in the next chapter.

Theorem 2.3.1. *For every Conway game $\mathcal{G}$, every player p, and every $n \in \mathbb{ON}$ such that $\widehat{L}_n^{(p)} = \emptyset$:* $\quad \widehat{L}_m^{(p)} = \widehat{W}_m^{(\overline{p})} = \emptyset \qquad$ *for every $m > n$.*

Proof. Let $\mathcal{G}$ be a Conway game, p a player, and let $n \in \mathbb{ON}$ be such that $\widehat{L}_n^{(p)} = \emptyset$.

We have $\widehat{W}_m^{(\overline{p})} = \emptyset$ for every limit ordinal m. We can prove the remaining cases by induction.

3. By this notation we always mean that i ranges over all ordinals.

Let $m > n$ be a successor ordinal and let $v \in W_m^{(\overline{p})}$. Then there is some $w \in L_k^{(p)}$ for some $k < m$, such that $\overline{p}$ has the move (v, w). Because all the sets $\widehat{L}_i^{(p)}$ for $n \leq i < m$ are empty by inductive hypothesis, every such w is in $L_k^{(p)}$ for some $k < n$. Therefore, $v \in W_l^{(\overline{p})}$ for some $l \leq n$. And because $n < m$, we have $v \notin \widehat{W}_m^{(\overline{p})}$. Therefore, $\widehat{W}_m^{(\overline{p})} = \emptyset$.

Let $m > n$ be a successor or limit ordinal. Let $v \in L_m^{(p)}$. That is, all the moves of p in v are to $W_k^{(\overline{p})}$ for some $k \leq m$. Because (by inductive hypothesis) $\widehat{W}_i^{(\overline{p})} = \emptyset$ for $n < i \leq m$, there is some $k \leq n$ such that all the moves of p in v are to $W_k^{(\overline{p})}$. That is, $v \in L_k^{(p)}$ for some $k \leq n$. And because $n < m$, we have $v \notin \widehat{L}_m^{(p)}$. Therefore, $\widehat{L}_m^{(p)} = \emptyset$. $\square$

Theorem 2.3.2. *For every Conway game $\mathcal{G} = (V, E_1, E_2, q, z)$, there is $n \in \mathbb{ON}$ such that for every player p and every $m > n$:* $\widehat{W}_m^{(p)} = \widehat{L}_m^{(p)} = \emptyset$.

Proof. Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game and p a player. Because every $v \in V$ appears in at most one $\widehat{L}_i^{(p)}$, the number of non-empty sets $\widehat{L}_i^{(p)}$ is limited, it is (by Dirichlet's principle) smaller than the least cardinal greater than $|V|$; let us denote this ordinal α. From this fact and from Theorem 2.3.1 it follows that $\widehat{L}_\alpha^{(p)} = \emptyset$ (at least one of the sets $\widehat{L}_i^{(p)}$ for $0 \leq i \leq \alpha$ is empty, and therefore by Theorem 2.3.1 we have $\widehat{L}_\alpha^{(p)} = \emptyset$). Then (again by Theorem 2.3.1) for every ordinal $m > \alpha$, $\widehat{L}_m^{(p)} = \widehat{W}_m^{(\overline{p})} = \emptyset$. Therefore, we have $\widehat{W}_m^{(1)} = \widehat{W}_m^{(2)} = \widehat{L}_m^{(1)} = \widehat{L}_m^{(2)} = \emptyset$ for all ordinals $m > \alpha$. $\square$

Corollary 2.3.3. *For every Conway game $\mathcal{G} = (V, E_1, E_2, q, z)$, every player p, and every position $v \notin \bigcup_i L_i^{(p)}$: p has a move from v to some $w \notin \bigcup_i W_i^{(\overline{p})}$.*

Proof. This is because if every move of p in v is to $\bigcup_i W_i^{(\overline{p})}$, then (by Theorem 2.3.2) every move of p in v is to $W_n^{(\overline{p})}$ for some $n \in \mathbb{ON}$, and therefore v is in $L_n^{(p)}$. $\square$

3 Determinacy of Conway games

3.1 Road to determinacy

We now aim to prove determinacy of Conway games. That is, that for every Conway game exactly one of the following 3 statements holds: 1 has a winning strategy; 2 has a winning strategy; both players have a non-losing strategy. If we knew that the existence of a winning strategy of one player is equivalent to the non-existence of a non-losing strategy of the other player, determinacy would be trivial (see the proof of Theorem 3.1.2). However, we do not know this yet. It could be sometimes the case, for instance, that for every strategy S of Player 1, Player 2 has a counter-strategy S' such that $S \circ S'$ is non-losing for 2 (hence 1 does not have a winning strategy) but S' depends on S and 2 has no non-losing strategy that would work against all the strategies of 1. In such a case Conway games would not be determined. In this section we show a way to proving determinacy.

We now prove that the existence of a winning strategy of a player p is equivalent the non-existence of a non-losing strategy of $\overline{p}$. We also connect the constructive regions $W_i^{(p)}$, $L_i^{(p)}$ with the regions $W^{(p)}$, $L^{(p)}$.

Theorem 3.1.1. *For every Conway game $\mathcal{G} = (V, E_1, E_2, q, z)$, every $v \in V$, and every player p, the following three statements are equivalent.*

(I_W) $v \in W^{(p)}$.

(II_W) $v \in \bigcup_i W_i^{(p)}$.

(III_W) *$\overline{p}$ has no non-losing strategy when p starts in v.*

Similarly, the following three statements are equivalent.

(I_L) $v \in L^{(p)}$.

(II_L) $v \in \bigcup_i L_i^{(p)}$.

(III_L) *$\overline{p}$ has a winning strategy when p starts in v.*

Note. We need the equivalence of (I) and (III) for the proof of determinacy; (II) helps us prove this equivalence.

Proof. Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game, $v \in V$ a position, and p a player. We first prove the equivalence of (I_W), (II_W) and (III_W).

$(I_W) \Rightarrow (III_W)$

Clearly, when p has a winning strategy S, $\overline{p}$ has no non-losing strategy, since for every strategy S' of $\overline{p}$, one of the plays conforming to S' is $S \circ S'$, which is losing for $\overline{p}$.

$(III_W) \Rightarrow (II_W)$ (We prove the contrapositive.)

Consider $v \notin \bigcup_i W_i^{(p)}$. Let S_N be a strategy of $\overline{p}$ such that for every $w \notin \bigcup_i L_i^{(\overline{p})}$, $S_N(w) = x$ for some $x \notin \bigcup_i W_i^{(p)}$ (such x always exists, see Corollary 2.3.3). We can define the rest of S_N arbitrarily.

We can easily prove by induction that every play of $\mathcal{G}$ when p starts outside of $\bigcup_i W_i^{(p)}$ conforming to S_N is such that when it is p's turn in some w, then $w \notin \bigcup_i W_i^{(p)}$, and when it is $\overline{p}$'s turn in some w, then $w \notin \bigcup_i L_i^{(\overline{p})}$.

Whenever p makes a move (w, x) in some $w \notin \bigcup_i W_i^{(p)}$, then $x \notin \bigcup_i L_i^{(\overline{p})}$ (if x were in $L_n^{(\overline{p})}$ for some $n \in \mathbb{ON}$, w would be in $W_{n+1}^{(p)}$).

Whenever $\overline{p}$ makes a move $(w, S_N(w))$ in some $w \notin \bigcup_i L_i^{(\overline{p})}$, then $S_N(w) \notin \bigcup_i W_i^{(p)}$ (from the definition of S_N).

In particular, when p starts in v, no play conforming to S_N contains a position $p_i \in \bigcup_i L_i^{(\overline{p})}$ such that it is $\overline{p}$'s turn in p_i. Outside of $\bigcup_i L_i^{(\overline{p})}$, $\overline{p}$ always has a move and makes a move according to S_N. S_N is therefore non-losing for $\overline{p}$ when p starts in v.

$(II_W) \Rightarrow (I_W)$

Let $v \in \bigcup_i W_i^{(p)}$. Then p has a winning strategy S_W for $\mathcal{G}$ when p starts in v defined as follows.

For every $w \in \bigcup_i W_i^{(p)}$, let $n \in \mathbb{ON}$ be such that $w \in \widehat{W}_n^{(p)}$ (hence n is a successor ordinal). Because $w \in W_n^{(p)}$, p has a move from w to some $x \in L_{n-1}^{(\overline{p})}$. Because $w \notin W_{n-1}^{(p)}$, this x is not in $L_{n-2}^{(\overline{p})}$. We can therefore put $S_W(w) = x$ for some $x \in \widehat{L}_{n-1}^{(\overline{p})}$. We can define the rest of S_W arbitrarily.

S_W of p is winning when p starts in $\bigcup_i W_i^{(p)}$ since in any play conforming to S_W, the losing rank for $\overline{p}$ of the positions in which it is $\overline{p}$'s turn decreases (every move of $\overline{p}$ in $\widehat{L}_n^{(\overline{p})}$ is to $\widehat{W}_m^{(p)}$ for some successor ordinal $m \leq n$, then p makes a move according to S_W to $\widehat{L}_{m-1}^{(\overline{p})}$, therefore to $\widehat{L}_k^{(\overline{p})}$ for some $k < n$) and every decreasing sequence of ordinals is finite and reaches 0, at which point it will be $\overline{p}$'s turn in a position in which $\overline{p}$ has no move. In particular, S_W of p is winning when p starts in v.

We can prove the equivalence of the other 3 statements similarly.

$(I_L) \Rightarrow (II_L)$ (We prove the contrapositive.)

Consider $v \notin \bigcup_i L_i^{(p)}$. Then p has a non-losing strategy when p starts in v that we have already described in the proof of $(III_W) \Rightarrow (II_W)$ (we called it S_N). p avoids $\bigcup_i W_i^{(\overline{p})}$. No matter the moves of $\overline{p}$, it will never be p's turn in $\bigcup_i L_i^{(p)}$.

$(II_L) \Rightarrow (III_L)$

Let $v \in \bigcup_i L_i^{(p)}$. Then $\overline{p}$ has a winning strategy that we have already described in the proof of $(II_W) \Rightarrow (I_W)$ (we called it S_W). $\overline{p}$ always decreases the rank (makes a move from $\widehat{W}_i^{(\overline{p})}$ to $\widehat{L}_{i-1}^{(p)}$), it will eventually be p's turn in $L_0^{(p)}$.

$(III_L) \Rightarrow (I_L)$

Clearly, no strategy of p is non-losing, since p always loses against the winning strategy of $\overline{p}$. $\square$

Theorem 3.1.2 (Determinacy of Conway games)**.** *For every Conway game $\mathcal{G}$, exactly one of the following statements holds.*

(i) *Player 1 has a winning strategy.*

(ii) *Player 2 has a winning strategy.*

(iii) *Both players have a non-losing strategy.*

Proof. Let $\mathcal{G}$ be a Conway game.

We can prove determinacy as an easy consequence of the results above.

Suppose neither of the players has a winning strategy. By the implications $(III_W) \Rightarrow (I_W)$ and $(I_L) \Rightarrow (III_L)$ of Theorem 3.1.1, both players have a non-losing strategy. Therefore, at least one of the statements $(i), (ii), (iii)$ holds.

Suppose at least two of the statements hold. Therefore, at least one of the players has a winning strategy. Let p be a player with a winning strategy. By the implications $(I_W) \Rightarrow (III_W)$ and $(III_L) \Rightarrow (I_L)$ of Theorem 3.1.1, $\overline{p}$ has no non-losing strategy. Therefore, $\overline{p}$ does not have any winning strategy, hence one of the statements $(i), (ii)$ does not hold. Also (iii) does not hold. This contradicts the supposition that at least two of the statements hold. Therefore, at most one of $(i), (ii), (iii)$ holds. $\square$

3.2 Some consequences of determinacy

One of the consequences of determinacy is Theorem 3.2.1. It will be useful for the proofs of Theorem 5.1.1 and Theorem 5.2.2 in the last chapter.

Theorem 3.2.1. *For every Conway game $\mathcal{G}$ and every player p: p has a non-losing (winning) strategy iff there is a strategy S of p such that for every strategy S' of $\overline{p}$, $S \circ S'$ is non-losing (winning) for p.*

Proof. Let $\mathcal{G}$ be a Conway game and p a player.

When S is a non-losing (winning) strategy of p, that is, all plays conforming to S are non-losing (winning) for p, then in particular the conforming plays $S \circ S'$ for some strategy S' of $\overline{p}$ are non-losing (winning) for p.

Let S be a strategy of p such that for every strategy S' of $\overline{p}$, $S \circ S'$ is non-losing (winning) for p. Therefore, for every strategy S' of $\overline{p}$, there is a conforming play to S' – the play $S \circ S'$ – that is not winning (is losing) for $\overline{p}$. Therefore, $\overline{p}$ has no winning (non-losing) strategy. By determinacy, p has a non-losing (winning) strategy. □

We conclude this chapter by proving the proposition mentioned in the first chapter where we claimed that (positional) strategies are "sufficiently strong" with respect to the existence of winning and non-losing strategies. Recall that we call strategies taking into account the history (that is, making a move based on the current position and all the already visited positions) *general strategies*. To avoid an unnecessary confusion, in the rest of this section we do not abbreviate the term *positional strategy*.

Theorem 3.2.2. *For every Conway game $\mathcal{G}$ and every player p: there is a winning (non-losing) general strategy*[1] *of p iff there is a winning (non-losing) positional strategy of p.*

Proof. Let $\mathcal{G}$ be a Conway game and p a player.

Clearly, when p has a winning (non-losing) positional strategy, then p also has a winning (non-losing) general strategy – the one mirroring the positional strategy, disregarding the history.

If p does not have a winning positional strategy, then $\overline{p}$ has a non-losing positional strategy S by determinacy. Therefore, p does not have a winning general strategy (for any general strategy S' of p, the play conforming to both S and S'[2] is not winning for p).

1. Analogously to the definition of a winning (non-losing) positional strategy, a winning (non-losing) general strategy of a player p is a general strategy such that every play conforming to it is winning (non-losing) for p.
2. We have not written $S \circ S'$ because this is strictly speaking not defined (one of the strategies is positional, the other one is general).

Analogously, if p does not have a non-losing positional strategy, then $\overline{p}$ has a winning positional strategy. Therefore, p does not have a non-losing general strategy. $\square$

Note. The same way we distinguish between positional strategies and general strategies, we can distinguish between positional determinacy and general determinacy. We have proved positional determinacy – Theorem 3.1.2 and abbreviated it to determinacy, the same way we have abbreviated positional strategies to strategies.

Because of Theorem 3.2.2, general determinacy does indeed hold as well.

Note. For every position v, a player playing by a positional strategy makes the same move every time he is in v. Therefore, a positional strategy does not formalize the notion of a person playing the game (spontaneously deciding where to move whenever it is their turn). On the other hand, a general strategy does. Every play of a Conway game is equal to $S \circ S'$ for some general strategies S of Player 1 and S' of Player 2. This is because every time it is 1's turn in the given play $(p_i)_{i=1}^{n}$, the history is unique. We can define the general strategy S of 1 such that whenever it is 1's turn in some p_i (for some $i < n$), S gives p_{i+1} on the position p_i and the history $p_1, p_2, ..., p_{i-1}$; S on other arguments can be arbitrary. We can define S' of 2 analogously.

For the same reason, a general strategy S of p is winning (non-losing) iff for every general strategy S' of $\overline{p}$, $S \circ S'$ is winning (non-losing) for p.

4 Positional games

A similar type of game to Conway games are positional games. In this chapter we briefly introduce them. In the fifth chapter, we show how it is possible to reduce Conway games to positional games and vice-versa.

4.1 Basic definitions

As in Conway games, in positional games there are two players 1 and 2 playing against each other, moving between positions. In positional games, however, every position belongs to one of the players, whose turn it is every time a play gets to this position.

The following definitions are similar to their counterparts for Conway games and therefore do not require explanations.

Definition 4.1.1 (Positional game)**.** A *positional game* is a tuple (V_1, V_2, E, q) where V_1, V_2 are disjoint sets of *positions of Player 1*, respectively *positions of Player 2*, $E \subseteq (V_1 \cup V_2) \times (V_1 \cup V_2)$ is a set of *moves*, $q \in V_1 \cup V_2$ is *the starting position*. We call a player p *the starting player* iff $q \in V_p$.

Definition 4.1.2 (Play of a positional game)**.** Let $\mathcal{G} = (V_1, V_2, E, q)$ be a positional game.

A *play* of $\mathcal{G}$ is any sequence of positions $(p_i)_{i=1}^{n}$ where $n \in \mathbb{N}^+ \cup \{\omega\}$, such that all the following statements hold.

- $p_1 = q$.
- for every $i \in \mathbb{N}^+$ such that $i < n$ we have $(p_i, p_{i+1}) \in E$.
- if $n \in \mathbb{N}$, then there is no $v \in V_1 \cup V_2$ such that $(p_n, v) \in E$.

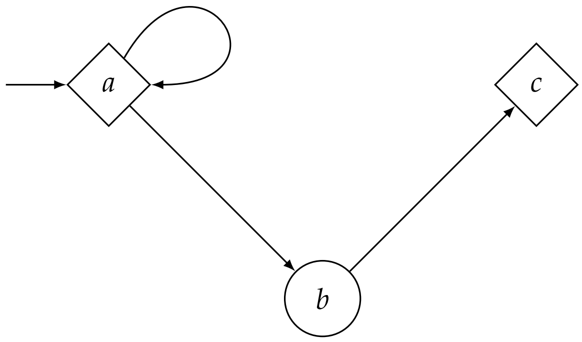
Definition 4.1.3 (Positional strategy)**.** Let $\mathcal{G} = (V_1, V_2, E, q)$ be a positional game and p a player.

A positional strategy of p for $\mathcal{G}$ is a partial function $S : V_p \to V_1 \cup V_2$ such that for every position $v \in V_p$ on which S is defined we have $(v, S(v)) \in E$, and for every position $v \in V_p$ on which S is not defined, there is no position w such that $(v, w) \in E$.

Note. The word *positional* in the term *positional strategy* does not come from the term *positional game* but from the fact that every move is determined by the position alone (disregarding the history), analogously to the terminology for Conway games.

Notation. We will abbreviate the term *positional strategy* to *strategy*, the same way we do for Conway games.

Figure 4.1: A simple positional game



Example. Positions are depicted by vertices, moves by edges. In positional games we only have one set of moves, hence no need to label the edges. On the other hand, we have two sets of positions – we depict the positions of Player 1 by diamonds and the positions of Player 2 by circles. We mark the starting position by an extra arrow.

The set of all possible plays of the game in Figure 4.1 can be described as $\{a^n bc \mid n \in \mathbb{N}\} \cup \{a^\omega\}$. Player 1 has a non-losing strategy – he loops in a and never lets the other player move.

Definition 4.1.4 (Play conforming to a strategy)**.** Let $\mathcal{G}$ be a positional game, p a player, S a strategy of p, and $P = (p_n)_{i=1}^{n}$ a play of $\mathcal{G}$.

We say that P conforms to S iff for every position p_i in P such that $p_i \in V_p$ we have $i = n$ or $S(p_i) = p_{i+1}$.

Definition 4.1.5 (Play conforming to strategies of both players)**.** Let $\mathcal{G}$ be a positional game, $P = (p_n)_{i=1}^{n}$ a play of $\mathcal{G}$, S a strategy of Player 1, and S' a strategy of Player 2.

Then $S \circ S' = S' \circ S$ is the unique play conforming to both S and S'.

Definition 4.1.6 (Draw, losing and winning play)**.** Let $\mathcal{G} = (V_1, V_2, E, q)$ be a positional game, p a player, and $P = (p_i)_{i=1}^{n}$ a play of $\mathcal{G}$.

We say that P is a *draw* iff $n = \omega$. P is *winning* for p iff $n \in \mathbb{N}$ and $p_n \in V_{\overline{p}}$. P is losing for p iff $n \in \mathbb{N}$ and P is not winning for p.

Definition 4.1.7 (Winning and non-losing strategy)**.** Let $\mathcal{G}$ be a positional game, p a player, and S a strategy of p.

We say that S is *non-losing* (*winning*) iff every play of $\mathcal{G}$ conforming to S is non-losing (winning) for p.

4.2 Regions and determinacy

Definition 4.2.1 (Winning and losing regions of a positional game)**.** Let $\mathcal{G} = (V_1, V_2, E, q)$ be a positional game and p a player. We define

$$\begin{aligned} W^{(p)} &= \{v \in V_p \mid p \text{ has a winning strategy for } (V_1, V_2, E, v)\}, \\ L^{(p)} &= \{v \in V_p \mid p \text{ has no non-losing strategy for } (V_1, V_2, E, v)\}. \end{aligned}$$

Note. The constructive version of the regions of positional games can be constructed in a similar fashion to Conway games. For every $n \in \mathbb{ON}$ we can define

$$\begin{aligned} W_n^{(p)} &= \{v \in V_p \mid \text{ there exists } m < n \text{ and a position } w \in L_m^{(\overline{p})} \cup W_m^{(p)} \\ &\quad \text{such that } (v, w) \in E\}, \\ L_n^{(p)} &= \{v \in V_p \mid \text{ for every } w \text{ such that } (v, w) \in E : \text{there is } m < n \\ &\quad \text{such that } w \in W_m^{(\overline{p})} \cup L_m^{(p)}\}. \end{aligned}$$

However, we do not need a definition of constructive regions of positional games anywhere in this thesis.

Notation. We will again use the terminology "*a game in which p starts in v*" as we did for Conway games. When we use this terminology for positional games, we have to make sure that $v \in V_p$, because p cannot start in $v \in V_{\overline{p}}$.

It is a well-known fact that positional games, too, are determined. See, e.g., [4].

Theorem 4.2.1 (Determinacy of positional games)**.** *For every positional game $\mathcal{G}$, exactly one of the following statements holds.*

(*i*) *Player 1 has a winning strategy.*

(*ii*) *Player 2 has a winning strategy.*

(*iii*) *Both players have a non-losing strategy.*

Theorem 4.2.2 is an analogy of Theorem 3.2.1 for positional games and can be proved analogously; it is a consequence of determinacy of positional games.

Theorem 4.2.2. *For every positional game $\mathcal{G}$ and every player p: p has a non-losing (winning) strategy iff there is a strategy S of p such that for every strategy S' of $\overline{p}$, $S \circ S'$ is non-losing (winning) for p.*

5 Connecting Conway games and positional games

We now describe reductions of Conway games to positional games and vice-versa. In both cases we require that the reduction preserves the existence of winning and non-losing strategies. That is, to every Conway (positional) game $\mathcal{G}$, we assign a positional (Conway) game $R(\mathcal{G})$ such that for every player p, p has a winning strategy for $\mathcal{G}$ iff p has a winning strategy for $R(\mathcal{G})$, and p has a non-losing strategy for $\mathcal{G}$ iff p has a non-losing strategy for $R(\mathcal{G})$.

We might sometimes be interested not only in the existence of winning and non-losing strategies when the starting player starts in the starting position but also in their existence in any position when any of the players starts (to answer questions like "What would happen if the game started in another position?", "What if the other player starts?", "What positions should a player avoid?"). For this reason we choose reductions such that there is a simple correspondence between the winning and losing regions of the original game and those of the reduced game.

When we have such reductions, we can, given a game of one kind, reduce it to a game of the other kind and study the reduced game, giving us answers about the original game.

5.1 Reducing Conway games to positional games

We now define a reduction R of Conway games to positional games.

We split every original position v in two – $v_1 \in V_1$ and $v_2 \in V_2$, one for each player. The position v_p in the reduced game corresponds to the position v when it is p's turn in the original game.

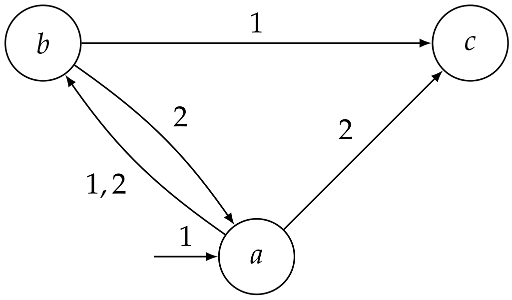
Because the players alternate in moves in every Conway game $\mathcal{G}$ and we want $R(\mathcal{G})$ to mirror $\mathcal{G}$, every move in the reduced game is from V_p to $V_{\overline{p}}$ for one of the players p. Therefore, the players alternate in making moves in the positional game $R(\mathcal{G})$ as well.

Definition 5.1.1 (Reduction R of Conway games to positional games)**.** Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game. We define $R(\mathcal{G}) = (V_1, V_2, E, (q, z))$ where

$$V_i = V \times \{i\} \qquad \text{for every } i \in \{1, 2\},$$
$$E = \{((l, 1), (r, 2)) \mid (l, r) \in E_1\} \cup \{((r, 2), (l, 1)) \mid (r, l) \in E_2\}.$$

Notation. For a player p we will write v_p instead of (v, p) for better readability.

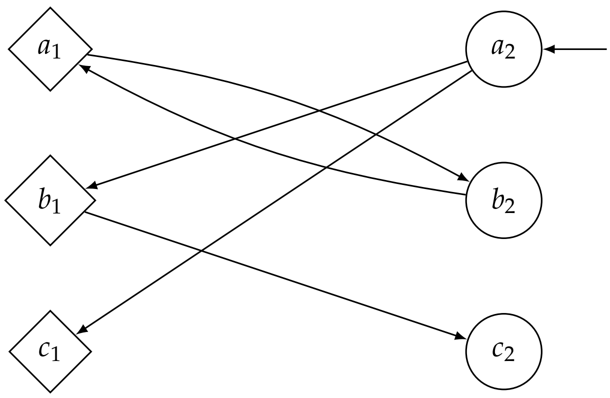
Figure 5.1: A Conway game $\mathcal{G}$



$$
\begin{aligned}
W^{(1)}_{\mathcal{G}} &= \{b\},\\
W^{(2)}_{\mathcal{G}} &= \{a\},\\
L^{(1)}_{\mathcal{G}} &= \{c\},\\
L^{(2)}_{\mathcal{G}} &= \{c\}.
\end{aligned}
$$

Example. The game $\mathcal{G}$ in Figure 5.1 is reduced to $R(\mathcal{G})$ in Figure 5.2.

Figure 5.2: The positional game $R(\mathcal{G})$



$$
\begin{aligned}
W^{(1)}_{R(\mathcal{G})} &= \{b_1\},\\
W^{(2)}_{R(\mathcal{G})} &= \{a_2\},\\
L^{(1)}_{R(\mathcal{G})} &= \{c_1\},\\
L^{(2)}_{R(\mathcal{G})} &= \{c_2\}.
\end{aligned}
$$

Notation. Given two or more games of one kind (Conway or positional), we will say that the games *have the same arena* iff they have the same sets of positions and moves.

Note. Note that games with the same arena have the same strategies. Given a strategy S of p for $\mathcal{G}$, we consider S to also be a strategy of p for any other game with the arena of $\mathcal{G}$.

We aim to prove that the reduction R preserves the existence of winning and non-losing strategies. We also aim to show how the regions of a Conway game $\mathcal{G}$ correspond to the regions of $R(\mathcal{G})$. Theorem 5.1.1 is instrumental.

Theorem 5.1.1. *For every Conway game $\mathcal{G} = (V, E_1, E_2, q, z)$, every player p, and every position $v \in V$:*
$v \in W^{(p)}_{\mathcal{G}}$ *iff* $v_p \in W^{(p)}_{R(\mathcal{G})}$,
$v \in L^{(p)}_{\mathcal{G}}$ *iff* $v_p \in L^{(p)}_{R(\mathcal{G})}$.

Proof. Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game.

We first describe an obvious bijection f between strategies for $\mathcal{G}$ and strategies for $R(\mathcal{G})$. For every strategy S of a player p for $\mathcal{G}$, the strategy $f(S)$ is a strategy of p for $R(\mathcal{G})$ defined on $v_p \in V_p$ iff S is defined on v, in which case it gives $f(S)(v_p) = S(v)_{\overline{p}}$.

We now prove that $v \in W^{(p)}_{\mathcal{G}}$ iff $v_p \in W^{(p)}_{R(\mathcal{G})}$, and $v \notin L^{(p)}_{\mathcal{G}}$ iff $v_p \notin L^{(p)}_{R(\mathcal{G})}$.

Let $v \in W^{(p)}_{\mathcal{G}}$ ($v \notin L^{(p)}_{\mathcal{G}}$), that is, p has a winning (non-losing) strategy for $\mathcal{G}$ when p starts in v. Then, by Theorem 3.2.1[1], there is a strategy S of p for $\mathcal{G}$ such that for every strategy S' of $\overline{p}$ for $\mathcal{G}$, $S \circ S'$ is winning (non-losing) for p when p starts in v. Let S be such a strategy.

Now, let S' be any strategy of $\overline{p}$ for $R(\mathcal{G})$. Because f is surjective, $S' = f(T)$ for some strategy T of $\overline{p}$ for $\mathcal{G}$. Let T be such a strategy.

The plays $S \circ T$ of $\mathcal{G}'$ when p starts in v and $f(S) \circ f(T)$ of $R(\mathcal{G}')$ when p starts in v_p have the same starting player, same length and in both plays the players alternate in making a move[2]. And because $S \circ T$ is winning (non-losing) for p when p starts in v, $f(S) \circ f(T) = f(S) \circ S'$ is also winning (non-losing) for p when p starts in v_p.

S' was an arbitrary strategy of $\overline{p}$ for $R(\mathcal{G})$. Therefore, $f(S)$ is such that for every strategy S' of $\overline{p}$ for $R(\mathcal{G})$, $f(S) \circ S'$ is winning (non-losing) for p when p starts in v_p.

By Theorem 4.2.2, p has a winning (non-losing) strategy for $R(\mathcal{G})$ when p starts in v_p.

Analogously we can go in the opposite direction and for a winning (non-losing) strategy S of p for $R(\mathcal{G})$ prove that the strategy $f^{-1}(S)$ of p is also winning (non-losing).

□

1. This is the trivial part of the theorem. It is still the same strategy.
2. It is easy to prove by induction that $f(S) \circ f(T) = (p_i, ((p+i) \bmod 2) + 1)_{i=1}^{n}$ where $(p_i)_{i=1}^{n} = S \circ T$.

There are two important consequences of Theorem 5.1.1. Corollary 5.1.2 states that R preserves the existence of winning and non-losing strategies. Corollary 5.1.3 gives the correspondence between the regions of a Conway game $\mathcal{G}$ and the regions of the positional game $R(\mathcal{G})$.

Corollary 5.1.2. *For every Conway game $\mathcal{G}$ and every player p: p has a winning (non-losing) strategy for $\mathcal{G}$ iff p has a winning (non-losing) strategy for $R(\mathcal{G})$.*

Proof. Theorem 5.1.1 shows that the reduction R preserves the existence of winning and non-losing strategies for the starting player. Then, by determinacy of both positional and Conway games, R also preserves the existence of winning and non-losing strategies of the other player. $\square$

Corollary 5.1.3. *For every Conway game $\mathcal{G}$ and every player p:*
$W^{(p)}_{R(\mathcal{G})} = W^{(p)}_{\mathcal{G}} \times \{p\}$,
$L^{(p)}_{R(\mathcal{G})} = L^{(p)}_{\mathcal{G}} \times \{p\}$.

Proof. Let $\mathcal{G} = (V, E_1, E_2, q, z)$ be a Conway game, $R(\mathcal{G}) = (V_1, V_2, E, q_z)$ the corresponding positional game, and p a player.

Let $w \in W^{(p)}_{R(\mathcal{G})}$. From Definition 4.2.1, $w \in V_p$, therefore $w = (v, p)$ for some $v \in V$. By Theorem 5.1.1, $v \in W^{(p)}_{\mathcal{G}}$. Hence $(v, p) \in W^{(p)}_{\mathcal{G}} \times \{p\}$.

Let $(v, p) \in W^{(p)}_{\mathcal{G}} \times \{p\}$. By Theorem 5.1.1, $(v, p) \in W^{(p)}_{R(\mathcal{G})}$.

The proof of the other equality is analogous. It is a consequence of the other equivalence in Theorem 5.1.1.

$\square$

When knowing the regions of $R(\mathcal{G})$, we can easily recover the regions of $\mathcal{G}$ by forgetting the second components of the positions in the regions of $R(\mathcal{G})$.

5.2 Reducing positional games to Conway games

Now we describe a reduction of positional games to Conway games. It is similar to the reduction of Conway games to positional games in that it splits every position of the original game in two. We call one of them *the main position* and the other one *the auxiliary position*. We need auxiliary positions in the Conway game because in the original positional game there can be a play in which one player makes several moves in succession. A move from a player p's position x to p's position y corresponds, in the Conway game, to a move from the main position of x to the auxiliary position of y. There is only one possible move for the other player in the auxiliary position of y – to the main position of y. Therefore, when p chooses to go from x to y, she will have guaranteed that she will make her next move in y, as is the case in the positional game.

Notation. In the following definition we describe that for every position x in the original game, we have the main position (x, M) and the auxiliary position (x, A) in the reduced game. We will write x_M, x_A instead of (x, M), (x, A) for better readability[3].

Note. In the following definition, it might be convenient for the reader to read m as 'me' and o as 'opponent' (viewed from the perspective of p).

Definition 5.2.1 (Reduction R of positional games to Conway games)**.** Let $\mathcal{G} = (V_1, V_2, E, q)$ be a positional game. We define $R(\mathcal{G}) = (V, E_1, E_2, q_M, z)$ where

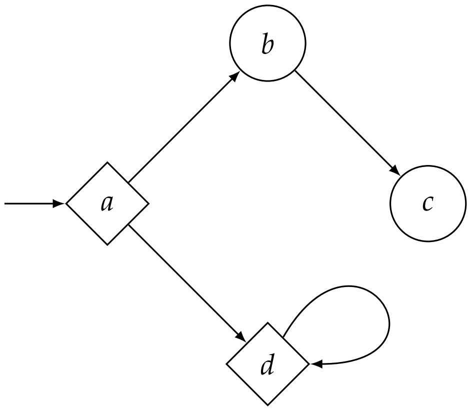
$V = (V_1 \cup V_2) \times \{M, A\}$,

$z \in \{1, 2\}$ is such that $q \in V_z$,

for every player p:

$$E_p = \{(m_M, o_M) \mid (m, o) \in E \text{ and } m \in V_p \text{ and } o \in V_{\overline{p}}\} \cup \{(m_M, {m'}_A) \mid (m, m') \in E \text{ and } m, m' \in V_p\} \cup \{(o_A, o_M) \mid o \in V_{\overline{p}}\}.$$

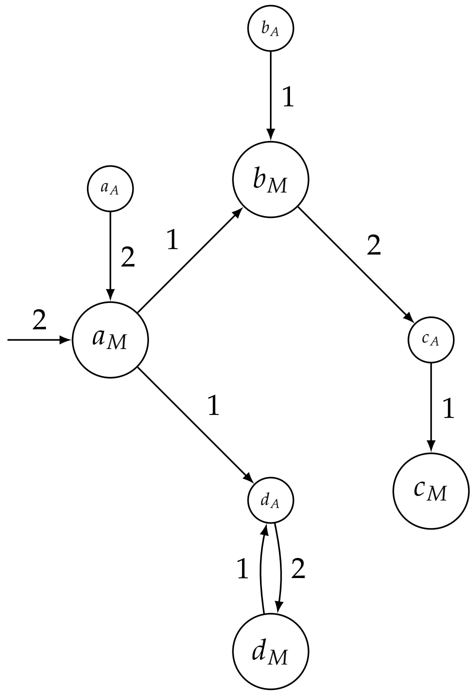
Figure 5.3: A positional game $\mathcal{G}$



$$W_{\mathcal{G}}^{(1)} = \{a\},$$
$$L_{\mathcal{G}}^{(2)} = \{b, c\},$$
$$W_{\mathcal{G}}^{(2)} = L_{\mathcal{G}}^{(1)} = \emptyset.$$

3. M and A are elementary symbols, not variables.

Figure 5.4: The Conway game $R(\mathcal{G})$



$$W^{(1)}_{R(\mathcal{G})} = \{a_M, c_A, b_A\},$$

$$W^{(2)}_{R(\mathcal{G})} = \emptyset,$$

$$L^{(1)}_{R(\mathcal{G})} = \{a_A, b_M, d_A, c_M\},$$

$$L^{(2)}_{R(\mathcal{G})} = \{b_M, c_M, a_A, b_A, c_A, a_M, d_M\}.$$

Example. The game in Figure 5.3 is reduced to the game in Figure 5.4.

Now we turn our attention to winning and losing regions.

In every auxiliary position v_A in $R(\mathcal{G})$, one player has no move, therefore v_A belongs to their losing region. The other player p has exactly one move in v_A – to v_M. Therefore, $v_A \in W^{(p)}_{R(\mathcal{G})}$ iff $v_M \in L^{(\overline{p})}_{R(\mathcal{G})}$, and $v_A \in L^{(p)}_{R(\mathcal{G})}$ iff $v_M \in W^{(\overline{p})}_{R(\mathcal{G})}$.

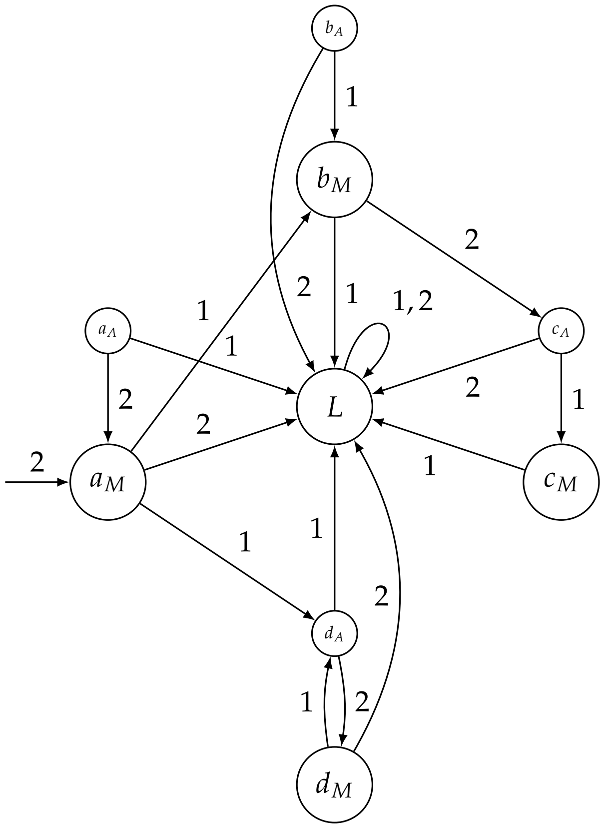
In every main position v_M, one player has no move, therefore v_M belongs to their losing region. The other player can have any number of moves in v_M, corresponding to the moves of this player in v in $\mathcal{G}$.

There is one problem with regards to the regions of $R(\mathcal{G})$. They do not correspond nicely to the regions of $\mathcal{G}$ (see the example above).

For a position $v \in V_p$ in $\mathcal{G}$, the plays of $R(\mathcal{G})$ when p starts in v_M or $\overline{p}$ starts in v_A (and has to make a move to v_M) correspond to the plays of $\mathcal{G}$ when p starts in v. However, the plays of $R(\mathcal{G})$ when p starts in v_A or when $\overline{p}$ starts in v_M do not correspond to any of the plays of any of the games with the arena of $\mathcal{G}$ because $\mathcal{G}$ is a positional game and a player p cannot start in $V_{\overline{p}}$. These plays add positions to the regions of $R(\mathcal{G})$ that are not relevant to what the regions of $\mathcal{G}$ are.

We therefore define a new reduction r such that all the plays of games with the arena of $r(\mathcal{G})$ that do not correspond to any of the plays of any of the games with the arena of $\mathcal{G}$ are draws, while all the plays that do correspond remain the same as in $R(\mathcal{G})$. This simplifies the regions of the reduced game while they still give us all the information about the regions of $\mathcal{G}$.

Figure 5.5: The Conway game $r(\mathcal{G})$



$$W^{(1)}_{r(\mathcal{G})} = \{a_M, b_A, c_A\},$$

$$W^{(2)}_{r(\mathcal{G})} = \emptyset,$$

$$L^{(1)}_{r(\mathcal{G})} = \emptyset,$$

$$L^{(2)}_{r(\mathcal{G})} = \{b_M, c_M, a_A\}.$$

The game $r(\mathcal{G})$ extends $R(\mathcal{G})$ – it has an additional position and additional moves related to this position.

$r(\mathcal{G})$ contains a position L (for "LOOP") in which both players have to loop (make the move (L, L)). For every position $v \in V_p$ in the original positional game, $\overline{p}$ has a move from v_M to L and p has a move from v_A to L.

Definition 5.2.2 (Reduction r of positional games to Conway games)**.** Let $\mathcal{G} = (V_1, V_2, E, q)$ be a positional game. We define $r(\mathcal{G}) = (V, E_1, E_2, q_M, z)$ where

$V = ((V_1 \cup V_2) \times \{M, A\}) \cup \{L\}$,

$z \in \{1, 2\}$ is such that $q \in V_z$,

for every player p:

$$\begin{aligned} E_p = &\{(m_M, o_M) \mid (m, o) \in E \text{ and } m \in V_p \text{ and } o \in V_{\overline{p}}\} \cup \\ &\{(m_M, {m'}_A) \mid (m, m') \in E \text{ and } m, m' \in V_p\} \cup \{(o_A, o_M) \mid o \in V_{\overline{p}}\} \cup \\ &\{(L, L)\} \cup \{(o_M, L) \mid o \in V_{\overline{p}}\} \cup \{(m_A, L) \mid m \in V_p\}. \end{aligned}$$

Example. Figure 5.5 depicts the game $r(\mathcal{G})$ for the positional game $\mathcal{G}$ in Figure 5.3.

When we are interested only in main positions, the regions of $r(\mathcal{G})$ are ${W'}^{(1)}_{r(\mathcal{G})} = \{a_M\}, \quad {W'}^{(2)}_{r(\mathcal{G})} = {L'}^{(1)}_{r(\mathcal{G})} = \emptyset, \quad {L'}^{(2)}_{r(\mathcal{G})} = \{b_M, c_M\}$.

We now aim to prove how, in general, the regions of a Conway game $\mathcal{G}$ correspond to the regions of $r(\mathcal{G})$. We start with Lemma 5.2.1.

Lemma 5.2.1. *For every positional game $\mathcal{G} = (V_1, V_2, E, q)$, there is a bijection f between strategies for $\mathcal{G}$ and strategies for $r(\mathcal{G})$ such that for every player p, every strategy S of p, every strategy S' of $\overline{p}$, and every $v \in V_p$, the following statements hold.*

(i) *$f(S)$ is a strategy of p and $f(S')$ is a strategy of $\overline{p}$.*

Let $(p_i)_{i=1}^n$ denote the play $S \circ S'$ of $\mathcal{G}$ when p starts in v and $(q_i)_{i=1}^m$ denote the subsequence of all main positions in $f(S) \circ f(S')$ of $r(\mathcal{G})$ when p starts in v_M.

(ii) $(q_i)_{i=1}^m = (p_{iM})_{i=1}^n$ [4].

(iii) *for every finite $i \leq n$, it is p's turn in p_i iff it is p's turn in q_i.*

Proof. Let $\mathcal{G}$ be a positional game.

We first describe the bijection f. Let S be a strategy of a player p for $\mathcal{G}$. For every $w \in V_{\overline{p}}$, p has exactly one move in the positions w_M and w_A in $r(\mathcal{G})$. For every $w \in V_p$, p has exactly one move in w_A. Therefore, the strategies

4. $p_{iM} = (p_i, M)$

of p for $r(\mathcal{G})$ differ only on positions w_M for $w \in V_p$. In every such w_M, p has either the move $(w_M, S(w)_M)$ (when $S(w) \in V_{\overline{p}}$) – in which case we put $f(S)(w) = S(w)_M$, or p has the move $(w_M, S(w)_A)$ (when $S(w) \in V_p$) – in which case we put $f(S)(w) = S(w)_A$. f is clearly a bijection.

It is easy to prove (ii) and (iii) by induction. p starts in v in $S \circ S'$ (respectively in v_M in $f(S) \circ f(S')$). Whenever it is a player's turn twice in succession in some p_i and p_{i+1}, there are p_{iM}, $p_{(i+1)A}$, $p_{(i+1)M}$ in $f(S) \circ f(S')$. Whenever p_i and p_{i+1} do not belong to the same player, $f(S) \circ f(S')$ contains p_{iM}, $p_{(i+1)M}$. In both cases $q_{i+1} = p_{(i+1)M}$ and it is p's turn in p_{i+1} iff it is p's turn in q_{i+1}. $\square$

Theorem 5.2.2. *For every player p, every positional game $\mathcal{G}$, and every position $v \in V_p$:*

$v \in W^{(p)}_{\mathcal{G}}$ iff $v_M \in W^{(p)}_{r(\mathcal{G})}$,

$v \in L^{(p)}_{\mathcal{G}}$ iff $v_M \in L^{(p)}_{r(\mathcal{G})}$.

Proof. We prove the first equivalence.

Let $\mathcal{G} = (V_1, V_2, E, q)$ be a positional game, p a player, $v \in V_p$ a position.

Suppose that p has a winning strategy for $\mathcal{G}$ when p starts in v. Then let S be a strategy of p for $\mathcal{G}$ such that for every strategy S' of $\overline{p}$ for $\mathcal{G}$, $S \circ S'$ when p starts in v is winning for p (such S exists by Theorem 4.2.2). That is, S is such that for every strategy S' of $\overline{p}$ for $\mathcal{G}$, $S \circ S'$ when p starts in v ends when it is $\overline{p}$'s turn.

Now let S' be any strategy of $\overline{p}$ for $r(\mathcal{G})$. Because f is surjective, $S' = f(T)$ for some strategy T of $\overline{p}$ for $\mathcal{G}$. Let T be such a strategy. Let $w \in V_{\overline{p}}$ be the last position of $S \circ T$ when p starts in v. By Lemma 5.2.1, it is $\overline{p}$'s turn in the last main position w_M of $f(S) \circ f(T) = f(S) \circ S'$. Because $\overline{p}$ has no move in w, $\overline{p}$ has no move in w_M (by the definition of r). Therefore, w_M is the last position of $f(S) \circ S'$ when p starts in v_M. Therefore, $f(S) \circ S'$ is losing for $\overline{p}$ when p starts in v_M. S' was an arbitrary strategy of $\overline{p}$ for $r(\mathcal{G})$; therefore, by Theorem 3.2.1, p has a winning strategy for $r(\mathcal{G})$ when p starts in v_M.

An analogous argument works in the other direction as well. When a strategy S for $r(\mathcal{G})$ is winning, $f^{-1}(S)$ for $\mathcal{G}$ is winning.

The other equivalence (or rather the contrapositives) can be proved similarly (using Lemma 5.2.1 to claim that when $S \circ T$ is infinite or ends when it is p's in w, then $f(S) \circ f(T)$ is infinite or ends when it is p's turn in w_M). $\square$

A direct consequence of Theorem 5.2.2 and determinacy is the fact that r preserves the existence of winning and losing strategies (Corollary 5.2.3).

Corollary 5.2.3. *For every positional game $\mathcal{G}$ and every player p: p has a winning (non-losing) strategy for $\mathcal{G}$ iff p has a winning (non-losing) strategy for $r(\mathcal{G})$.*

Proof. Theorem 5.2.2 shows that the reduction r preserves the existence of winning and non-losing strategies for the starting player. Then, by determinacy of both positional and Conway games, r also preserves the existence of winning and non-losing strategies of the other player. □

Theorem 5.2.4 is the last result we present. It describes the correspondence between the regions of a positional game $\mathcal{G}$ and the regions of the Conway game $r(\mathcal{G})$.

Theorem 5.2.4. *For every positional game $\mathcal{G}$ and every player p:*
$W^{(p)}_{r(\mathcal{G})} = \{v_M \mid v \in W^{(p)}_{\mathcal{G}}\} \cup \{v_A \mid v \in L^{(\overline{p})}_{\mathcal{G}}\}$,
$L^{(p)}_{r(\mathcal{G})} = \{v_M \mid v \in L^{(p)}_{\mathcal{G}}\} \cup \{v_A \mid v \in W^{(\overline{p})}_{\mathcal{G}}\}$.

Proof. We prove the first equality.

Let $\mathcal{G}$ be a positional game and p a player. p clearly has no winning strategy when p starts in L in $r(\mathcal{G})$.

Every other position in $r(\mathcal{G})$ is either v_M or v_A where either $v \in V_p$ or $v \in V_{\overline{p}}$. We look at all four cases.

Let $v \in V_p$.

p has only one move in v_A – to L. Therefore, $v_A \notin W^{(p)}_{r(\mathcal{G})}$. v_A is not in $\{v_M \mid v \in W^{(p)}_{\mathcal{G}}\} \cup \{v_A \mid v \in L^{(\overline{p})}_{\mathcal{G}}\}$ either.

By Theorem 5.2.2, $v_M \in W^{(p)}_{r(\mathcal{G})}$ iff $v \in W^{(p)}_{\mathcal{G}}$. v_M is either in both sets or in neither of them.

Let $v \in V_{\overline{p}}$.

p has only one move in v_M – to L. Such v_M is in neither of the sets.

p has only one move in v_A – to v_M. Therefore, p has a winning strategy when p starts in v_A iff p has a winning strategy when $\overline{p}$ starts in v_M iff (by determinacy) $\overline{p}$ has no non-losing strategy when $\overline{p}$ starts in v_M . That is, $v_A \in W^{(p)}_{r(\mathcal{G})}$ iff $v_M \in L^{(\overline{p})}_{r(\mathcal{G})}$. By Theorem 5.2.2, $v_M \in L^{(\overline{p})}_{r(\mathcal{G})}$ iff $v \in L^{(\overline{p})}_{\mathcal{G}}$. Hence $v_A \in W^{(p)}_{r(\mathcal{G})}$ iff $v \in L^{(\overline{p})}_{\mathcal{G}}$. v_A is either in both sets or in neither of them.

Therefore, the sets in the first equation are equal.

The other equality can be proved analogously. The most interesting case is the one with v_A when $v \in V_{\overline{p}}$. As in the last case of the proof of the other equality, we need the fact that p has only one move in v_A – to v_M, determinacy, and Theorem 5.2.2. □

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