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Bachelor's Thesis

The Transduction Hierarchy for Infinite Structures

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Bachelor-Thesis von Daniel Günzel

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Hiermit versichere ich, die vorliegende Bachelor-Thesis ohne Zuhilfenahme Dritter nur mit den angegebenen Quellen und Hilfsmitteln angefertigt zu haben. Alle Stellen, die den Quellen entnommen wurden, sind als solche gekennzeichnet. Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

Darmstadt, den 30. September 2010

(Daniel Günzel)

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Contents

1 Introduction

This work is a follow-up to the article "On the Monadic Second-Order Transduction Hierarchy" by Blumensath and Courcelle [1], in which classes of finite incidence structures are compared and linearly ordered with the help of transductions. The framework monadic second-order logic (MSO) will be also be used here. MSO has a great expressive power with the benefit of still having decidability for certain theories, for example the infinite binary tree and some linear orders have a decidable theory in MSO [6, 7].

The concept of a transduction is a transformation of relational structures by a list of formulae. It can be described as a generalisation of three kinds of operations: the definition of a relational structure inside another structure (like the definition of an interpretation in [8]), a copy operation creating multiple copies of the universe and extending the relations and the introduction of additional unary predicates. In general transductions are not single valued, they could be referred to as non-deterministic. If all elements of a class $\mathcal{A}$ can be obtained by a transduction from some element in a class $\mathcal{B}$, then one can introduce a preorder $\mathcal{A} \sqsubseteq \mathcal{B}$. In [1], one of the central techniques used by Blumensath and Courcelle is the concept of tree width, which is based on the tree decomposition of graphs. Their main result is a linear order with trees of height $1, 2, \ldots$ at the lower end, followed by the class of all paths, then all binary trees and at the far end of the hierarchy the class of all rectangular grids.

In this paper we want to consider one of the open problems mentioned in Blumensath and Courcelle [1, Section 9], namely the treatment of infinite structures. To avoid unnecessary confusion with the word "infinite", we restrict ourselves to countable structures throughout this paper. The techniques developed and presented in [1] can mostly be adapted to the new setting allowing infinite structures. For example, the concept of a transduction is not limited to finite structures. Unfortunately, the convenient and somewhat surprising result of a linear hierarchy is not achievable by extending the setting to infinite structures, simply because some classes become incomparable. Naturally one starts at the lower end of the hierarchy, where we have trees of different shapes and sizes. At the very bottom of the hierarchy we have the empty class, followed by the class having only the tree with one vertex as a member. The next level is still quite intuitive, we have the class of all finite trees of height two, namely the root having $1, 2, \ldots$ children. At this point linearity cannot be preserved, since we have the infinite tree of height two and all finite trees of height three, side by side. Neither is an infinite tree obtainable by any transduction from any finite tree, nor is an arbitrarily large finite tree of height three obtainable by a tree of lower height. The question how classes of paths and grids fit into the new hierarchy is left open for further analysis. Because our focus is largely on trees, we can find a more practical method of comparing two classes of trees than transductions.

Before we can produce any new results, we want to define all necessary notions and refer to a few concrete statements of the article by Blumensath and Courcelle. In Chapter 3 we show that we can use n-embeddings instead of transductions. There is an n-embedding from a class $\mathcal{A}$ into a class $\mathcal{B}$ if there exists $n < \omega$ such that for all members $\mathfrak{A} \in \mathcal{A}$ there is a structure $\mathfrak{B} \in \mathcal{B}$ and a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that the preimage of every vertex in $\mathfrak{B}$ has size of at most n. This gives us a much simpler criterion to show the relation between two classes. Note though, that for example the class of trees with at most n vertices, is n-embeddable into the class consisting of the tree having only the root.

For the classification of the different classes we introduce a few operations on classes of trees in Chapter 4. We can identify the roots of two trees ($\mathfrak{A} \oplus \mathfrak{B}$), simply consider the union of the classes ($\mathcal{A} \cup \mathcal{B}$) or extend trees at their leaves by elements from another class. These operations yield a toolkit of which we can assemble and name certain classes of trees. Much work will be required to show that between two classes, there can be no class in between in the hierarchy. This process is quite technical, but due to the introduction of n-embeddings still feasible. We would like to establish general patterns of the hierarchy, which occur repetitively as we will see in Theorem 5.10. The main target of this work is now to give a precise and complete picture of the lower part of the class hierarchy which is done in Chapter 5 and presented in a diagram where the "greatest" class is the class of infinite trees of height three.

2 Preliminaries

2.1 Basic Definitions

First we define the most important notions used in the following sections. Then we present a couple results of [1], which are required in some of the proofs. This includes the main theorem stating the linear order of classes we have discussed in the introduction. We will omit the proofs for these results and refer to [1] for a complete outline.

Definition 2.1.

1. For some set D, the set of all finite sequences of elements of D is denoted by $D^{<\omega}$. We define the *prefix relation* $\preceq$ on $D^{<\omega}$ by:

$$x \preceq y \;:\Leftrightarrow\; y = xz, \text{ for some } z \in D^{<\omega}.$$

2. A set T is called a *tree domain* if the following holds: $T \subseteq D^{<\omega}$ and T is prefix closed, i.e if the sequence $xy \in T$ then $x \in T$.

3. The tuple $\mathfrak{T} = (T, \preceq)$ is called *order tree* if T is a tree domain and $\preceq$ is the prefix relation. The empty sequence $\bullet$ is called *root*, $\preceq$-maximal elements are the *leaves*. The set of all leaves of $\mathfrak{T}$ is denoted by $L(\mathfrak{T})$. Elements having some successor are called *inner* elements.

4. A *successor tree* (T, edg) is also defined on a tree domain T, together with the binary relation $\langle x, y\rangle \in \text{edg}$ if $y = xd$ for some $d \in D$. If $\langle x, y\rangle \in \text{edg}$, we call x the *immediate predecessor* of y and analogously y an *immediate successor* of x.

5. For an order tree T, we call x a *predecessor* of y if we have $x \preceq y$ and analogously y a *successor* of x. In a successor tree x is a *predecessor* of y if there is a path $v_0, v_1, \ldots$ originating from the root where $x = v_i$ and $y = v_j$ for $i < j$. Conversely for $i > j$ we call x a successor of y.

6. The *level* of an element v is the number of its predecessors. We denote it by $|v|$. The *height* of a tree is the least ordinal α [5, Chapter 6.2] such that, for all vertices v, we have $|v| < \alpha$. Thus the empty tree has height zero, the height of the tree constructed in Example 2.2 below is 4.

7. We can expand trees of both types to a *coloured* tree by the addition of unary predicates $P_0, \ldots, P_{m-1}$. Every vertex is permitted to have from zero up to m colours. The class of m-coloured trees is denoted by $\mathbb{TREE}_m$.

8. The *infimum* of the vertices u and v is:

$$u \sqcap v := \text{the unique } x \in T \text{ such that } x \preceq u,\ x \preceq v \text{ and } \forall z \in T(z \preceq u \wedge z \preceq v \Rightarrow z \preceq x),$$

denoting the largest common prefix of u and v, or equivalently the maximal element of the intersection of the paths originating from the root to u or v respectively.

9. For a vertex v, we define the subtree $\mathfrak{T}_v$ rooted at v as the subtree of $\mathfrak{T}$ consisting of all vertices u with $v \leq u$.

Every vertex of the tree is represented by its unique path from the root. Hence each vertex has a unique immediate predecessor. It is quite simple (see Example 2.6 below) to obtain a successor tree from an order tree and vice versa, but they are not isomorphic, since in an order tree with root r, it holds for all vertices x, that $r \leq x$, but clearly $\langle r, x \rangle \in \mathrm{edg}$ is not true for all x in a successor tree.

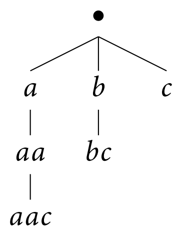
Example 2.2. Let $D := \{a, b, c\}$. Then the set of all (finite) sequences of elements of D is:

$$D^{<\omega} = \{\bullet, a, b, c, aa, ab, ac, ba, bb, bc, aaa, aab, \dots\}$$

A valid prefix closed tree domain $T \subset D^{<\omega}$ would be

$$T := \{\bullet, a, b, c, aa, bc, aac\},$$

producing the following tree:



Definition 2.3.

1. *Monadic second-order logic* (*MSO*) is an extension of first-order logic by adding set variables and the ability to quantify over those variables.

2. The *quantifier rank* $\mathrm{qr}(\varphi)$ of a MSO-formula φ is the nesting depth of quantifiers in φ. If no quantifiers occur in φ, it is called *quantifier-free*.

3. The *monadic theory of rank m* of a structure $\mathfrak{A}$ is the set of MSO-formulae φ with quantifier rank at most m such that $\mathfrak{A}$ is a model of φ.

$$\mathrm{MTh}_m(\mathfrak{A}) := \{\varphi \in \mathrm{MSO} \mid \mathfrak{A} \models \varphi, \mathrm{qr}(\varphi) \leq m\}.$$

For a tuple $\bar{a} \in A$, $\mathrm{MTh}_m(\mathfrak{A}, \bar{a})$ denotes the theory of the expansion $(\mathfrak{A}, \bar{a})$ and is called the *type* of $\bar{a}$.

If not explicitly stated otherwise, we assume all formulae to be MSO-formulae and to be normalised in the following way: all first-order variables are replaced by set-variables, followed by the statement that their size is one. For example $\exists x(x \leq c)$ is normalised to $\exists X(|X| = 1 \wedge X \leq c)$ where

$$|X| = 1 := \forall Y \forall Z(Y \subset X \wedge Z \subset X \Rightarrow Z = Y).$$

By formally eliminating first-order variables concepts like the quantifier rank become more straightforward, since there is no need to distinguish between first- and second-order quantification. For better readability we allow ourselves to write down formulae consisting of first-order variables, while referring to its normalised variant.

The next important concept is the transduction. For instance, it can be used to transform order trees into successor trees and vice versa. It is an operation on a relational structure $\mathfrak{A}$, producing a new structure $\tau(\mathfrak{A})$, or more precisely, a set of new structures. A transduction can be seen as a composition of three basic operations.

Definition 2.4.

1. The k-fold copy-operation maps a structure $\mathfrak{A}$ to

$$\mathrm{copy}_k(\mathfrak{A}) = \langle \mathfrak{A} \oplus \cdots \oplus \mathfrak{A}, \sim, P_0, \ldots, P_{k-1} \rangle .$$

   Elements of $\tau(\mathfrak{A})$ are of the form $\langle a, i \rangle$, for $a \in A, i < k$. The unary predicates P_i form a partition into the k components of $\tau(\mathfrak{A})$, since they contain all vertices of the i-th component.

$$P_i := \{\langle a, i \rangle \mid a \in A\} \quad \text{and} \quad \langle a, i \rangle \sim \langle b, j \rangle :\Leftrightarrow a = b$$

2. The *expansion* operator $\exp_m$ maps $\mathfrak{A}$ to the set of all possible expansions [4, Chapter 3.4] by m unary predicates $Q_0, \ldots, Q_{m-1} \subseteq A$. $\exp_0$ is the identity. Note that the operator cannot be seen as a function in the strict sense, since it is many-valued in general.

3. A *basic transduction* τ_0 is an operation on $\mathfrak{A}$ described by a list of formulae

$$\langle \chi, \delta(x), \varphi_0(\bar{x}), \ldots, \varphi_{s-1}(\bar{x}) \rangle$$

   called the *definition scheme* of τ_0. If a structure satisfies the formula χ, i.e. $\mathfrak{A} \models \chi$, then τ_0 produces a new structure:

$$\tau_0(\mathfrak{A}) := \langle D, R_0, \ldots, R_{s-1} \rangle$$

   where the new domain D is obtained by all elements $a \in A$ satisfying $\delta(a)$:

$$D := \{a \in A \mid \mathfrak{A} \models \delta(a)\}$$

   and R_i are the relations defined by the formulae $\varphi_i(\bar{x})$:

$$R_i := \left\{\bar{a} \in D^{\mathrm{ar}(R_i)} \mid \mathfrak{A} \models \varphi_i(\bar{a})\right\} .$$

   Here, $\mathrm{ar}(R_i)$ denotes the arity of the relation R_i. However, if $\mathfrak{A}$ does not satisfy χ, the image of $\mathfrak{A}$ under τ_0 is not defined.

4. A *transduction* τ is a composition of the three operations defined above, each of them is permitted to be the identity: $\tau = \tau_0 \circ \exp_m \circ \mathrm{copy}_k$. If copy_k is the identity, i.e. if $k = 1$, τ is called *noncopying*, otherwise it is referred to as *k-copying*.

Lemma 2.5. *The transitive closure of a relation R can be defined in MSO by the formula*

$$\varphi(u,v) := \forall X(Xu \wedge \forall x \forall y(Xx \wedge Rxy \rightarrow Xy) \rightarrow Xv)$$

or in words, every set which contains u and is closed under the relation R also contains v. We will use $(u,v) \in R^*$ as an abbreviation for the above MSO-formula.

Proof. See [2, Section 1.3.1] □

Example 2.6. We want to construct a noncopying transduction τ, such that the image of a successor tree is an order tree with the same height and shape. We use the abbreviation $\mathrm{edg}^*(x,y)$ for $\langle x,y\rangle \in \mathrm{edg}^*$. First define

$$\chi := \forall x \forall y\left((\mathrm{edg}^*(x,y) \wedge \mathrm{edg}^*(y,x)) \Rightarrow x = y\right)$$

ruling out all structures containing a cycle. For all trees $\mathfrak{T}$ it holds: $\mathfrak{T} \vDash \chi$. Since we want all of the tree domain to be kept, we put

$$\delta(x) := (x = x),$$

then all vertices satisfy this statement trivially: $\mathfrak{T} \vDash \delta(x)$. Next we define the new order relation by

$$\varphi(x,y) := (x,y) \in \mathrm{edg}^*.$$

The basic transduction τ with definition scheme $\langle \chi, \delta(x), \varphi(x,y)\rangle$ is now transforming successor trees into order trees.

In the following we will often consider the incidence structure $\mathfrak{A}_{\mathrm{in}}$ instead of the structure $\mathfrak{A}$ itself. The same concept is extended to classes, i.e. for a class $\mathcal{A}$ we consider the incidence structures of all its members, denoted by $\mathcal{A}_{\mathrm{in}}$.

2.2 Known Results on the Transduction Hierarchy

Definition 2.7. Let $\mathfrak{A} = \langle A, R_0, \ldots, R_{m-1}\rangle$ be a relational structure, r be the maximal arity of the relations R_i. We define the *incidence* structure $\mathfrak{A}_{\mathrm{in}}$:

$$\mathfrak{A}_{\mathrm{in}} := \langle A \uplus E, P_{R_0}, \ldots, P_{R_{m-1}}, \mathrm{in}_0, \ldots, \mathrm{in}_{r-1}\rangle .$$

It can be seen as a representation of $\mathfrak{A}$, where the domain is extended by new elements, one for each tuple in a relation.

$$E := R_0 \cup \cdots \cup R_{m-1}.$$

The new relations are unary and binary. The former are used to distinguish the type of relation, the latter to describe the membership of the elements of the regular domain to each other:

$$\begin{aligned} P_{R_i} &:= \{\bar{c} \in E \mid \bar{c} \in R_i\}, \\ \mathrm{in}_i &:= \{(a,\bar{c}) \in A \times E \mid |\bar{c}| > i \text{ and } a = c_i\}. \end{aligned}$$

The class of all incidence structure with signature Σ is $\mathbb{STR}_{\mathrm{in}}[\Sigma]$.

Remark 2.8. There exist transductions such that $\tau(\mathfrak{A}_{\mathrm{in}}) = \mathfrak{A}$ and, under certain restrictions, (compare [1, Section 3]), also in the opposite direction $\sigma(\mathfrak{A}) = \mathfrak{A}_{\mathrm{in}}$.

Definition 2.9. Let $\mathfrak{A} = (A, R_0, \dots, R_{n-1})$ be a structure. Then:

1. The *Gaifman graph* of $\mathfrak{A}$ is the undirected graph

$$\mathrm{Gf}(\mathfrak{A}) := \langle A, \mathrm{edg} \rangle$$

with the same domain A and with the edge relation containing all pairs of elements being contained in a tuple of any R_i:

$$\mathrm{edg} := \{(u, v) \mid u \neq v \text{ and there is some } \bar{c} \in R_i \text{ with } u, v \in \bar{c}\}.$$

2. A *strict tree decomposition* of $\mathfrak{A}$ is a pair (T, D) where T is a tree domain (sometimes referred to as the *index tree*) and $D = (U_v)_{v \in T}$ is a family of possibly empty subsets of A such that:
   - For every $a \in A$ it holds that $\{v \in T \mid a \in U_v\}$ is nonempty and connected.
   - For every tuple $\bar{c} \in R_i$, there is some index $v \in T$ with $\bar{c} \subseteq U_v$.
   - If u is a predecessor of v in T, then $U_v \setminus U_u \neq \emptyset$.
   - If $z \in T$ is not the root, the subgraph of $\mathrm{Gf}(\mathfrak{A})$ induced by the subset $\bigcup_{v \in T_z} U_v \setminus \bigcup_{v \in T \setminus T_z} U_v$ is connected.

3. The *height* of a tree decomposition (T, D) is the height of T, its *width* is defined as

$$\mathrm{wd}(T, D) := \sup_{v \in T} (|U_v| - 1) .$$

4. The *n-depth tree-width* $\mathrm{twd}_n(\mathfrak{A})$ of $\mathfrak{A}$ is the minimal width of a tree decomposition of $\mathfrak{A}$, whose index tree T has height at most n.

Definition 2.10.

1. A *minor* of a graph is a subgraph obtained by first deleting some vertices and edges and then contracting some of the remaining edges. Every graph is its own minor [3, Section 1.7]. If $\mathcal{A}$ is a class of graphs, $\mathrm{Min}(\mathcal{A})$ denotes the class of all minors of graphs in $\mathcal{A}$.

2. A *hypergraph* is a pair of two disjoint sets (V, E), where V is the set of vertices and $E \subseteq \wp(V)$ the set of hyperedges, i.e. edges that can connect any number of vertices.

3. Let $\mathcal{A}$ be a class of hypergraphs. We denote by $\mathrm{STD}_n(\mathcal{A})$ the class of all successor trees of height at most n, that are an index tree of some strict tree decomposition of a member of $\mathcal{A}$.

Lemma 2.11. *[1, Section 5] There exists a transduction τ with $\tau(\mathfrak{G}_{in}) = \mathrm{Min}(\mathfrak{G})$, for every graph $\mathfrak{G}$.*

Proof. Any minor $\mathfrak{H}$ of $\mathfrak{G}$ can be obtained by deleting some vertices, edges and then contracting some edges. We can encode $\mathfrak{H}$ with the help of four sets: the vertices being deleted, the edges being deleted, the edges being contracted and the vertices which are kept, i.e. one vertex of each contracted subgraph. With those sets as parameters it is possible to define $\mathfrak{H}$ inside of $\mathfrak{G}_{\mathrm{in}}$ by MSO-formulae. □

Theorem 2.12 (Excluded Path Theorem [1]). *For each path $\mathfrak{P}$, there exist numbers $k, n < \omega$ such that*

$$\mathfrak{P} \notin \mathrm{Min}(\mathfrak{G}) \quad \textit{implies} \quad \mathrm{twd}_n(\mathfrak{G}) < k, \qquad \textit{for every graph } \mathfrak{G}.$$

Proof. See [1]. □

Lemma 2.13 (Lemma 5.2 [1]). *For every signature Σ and every number $k < \omega$, there exists a transduction $\tau_k : \mathbb{TREE}_0 \to \mathbb{STR}_{\mathrm{in}}[\Sigma]$ that maps an order-tree $\mathfrak{T}$ to the class of all incidence structures $\mathfrak{A}_{\mathrm{in}}$ such that the corresponding Σ-structure $\mathfrak{A}$ has a tree decomposition of width at most k with underlying tree $\mathfrak{T}$.*

Proof. See [1]. □

Theorem 2.14 (Theorem 5.4 [1]). *For each constant $n < \omega$, there exists a transduction τ_n mapping a graph $\mathfrak{G}$ to the class of all (underlying trees of) strict tree decompositions of $\mathfrak{G}$ of height at most n.*

Proof. See [1]. □

Definition 2.15 (Definition 6.3 [1]). We consider the following subclasses of $\mathbb{STR}[\{\mathrm{edg}\}]$. (All trees below are considered to be successor-trees.)

1. $\mathcal{T}_n := \{m^{<n} \mid m < \omega\}$ is the set of all complete m-ary trees of height n.
2. $\mathcal{T}_{\mathrm{bin}}$ is the class of all binary trees.
3. $\mathcal{T}_\omega$ is the class of all trees.
4. $\mathcal{P}$ is the class of all paths.
5. $\mathcal{G}$ is the class of all rectangular grids.

Definition 2.16.

1. Let $\mathcal{A}, \mathcal{B}$ be classes of structures with common signature. We define $\mathcal{A} \sqsubseteq \mathcal{B}$ if there exists a transduction τ such that for all members $\mathfrak{A} \in \mathcal{A}$ we have $\mathfrak{A}_{\mathrm{in}} \in \tau(\mathfrak{B}_{\mathrm{in}})$ for some $\mathfrak{B} \in \mathcal{B}$.
2. $\mathcal{A} \equiv \mathcal{B}$ if $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqsubseteq \mathcal{A}$.
3. $\mathcal{A} \sqsubset \mathcal{B}$ if $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \not\sqsubseteq \mathcal{A}$.
4. $\mathcal{A} \lhd \mathcal{B}$ if $\mathcal{A} \sqsubset \mathcal{B}$ and there is no class $\mathcal{C}$ with $\mathcal{A} \sqsubset \mathcal{C} \sqsubset \mathcal{B}$.

Theorem 2.17 (Theorem 6.4 [1]). *We have the following hierarchy:*

$$\varnothing \lhd \mathcal{T}_0 \lhd \mathcal{T}_1 \lhd \dots \lhd \mathcal{T}_n \lhd \dots \sqsubset \mathcal{P} \lhd \mathcal{T}_\omega \equiv \mathcal{T}_{bin} \lhd \mathcal{G}.$$

For every signature Σ, every class $\mathcal{C} \subseteq \mathbb{STR}[\Sigma]$ is $\equiv$-equivalent to some class in this hierarchy.

Proof. See [1]. □

3 From Transductions to n-Embeddings

In this chapter we want to find an alternative for transductions which is capable of comparing two classes of trees. This is possible because we limit ourselves to classes of trees with bounded height. The main result will be Theorem 3.7 stating that whenever all elements of a class $\mathcal{A}$ can be described via some transduction inside an element of $\mathcal{B}$, there exists an n-embedding from the structure in $\mathcal{A}$ to the structure in $\mathcal{B}$. The other direction is also true: whenever we have an n-embedding from $\mathfrak{A}$ to $\mathfrak{B}$, we can find a transduction describing $\mathfrak{A}$ inside $\mathfrak{B}$. Here, we need to mimic the n-embedding via some transduction and then finding suitable relations and unary predicates. For the first claim, more work will be done, which is not surprising at all, since a transduction is a far more general object than an n-embedding. The plan is as follows: for a given transduction τ one creates a rearrangement σ of τ, whose "inverse function" is the desired n-embedding.

At the end of the chapter, in Proposition 3.8, we prove a nice result on classes of hypergraphs. Suppose the class of all finite paths is not transducible in a class of hypergraphs $\mathcal{A}$, i.e. $\mathcal{P} \not\sqsubseteq \mathcal{A}$. This implies that there exists a finite path which cannot be described by any transduction evaluated at an element of $\mathcal{A}$. Then $\mathcal{A}$ is equivalent in the sense of transductions to the class of all successor trees of height at most n that are an index tree of some strict tree decomposition of a member of $\mathcal{A}$. This means that we can express a class of hypergraphs having no trees as members in terms of a class of trees of bounded height under certain restrictions.

Proposition 3.1. *For every transduction τ and every constant $d < \omega$, there exists a transduction σ with the following properties:*

- $\sigma(\mathfrak{T}) \cong \tau(\mathfrak{T})$*, for all structures* $\mathfrak{T}$*.*
- *If $\mathfrak{T}$ is an order tree such that $\tau(\mathfrak{T})$ is a graph where every vertex has indegree at most d, then for every pair of vertices $\langle u, i\rangle$ and $\langle v, k\rangle$ of $\sigma(\mathfrak{T})$ that are connected by an edge we have $u \leq v$ or $v \leq u$ in $\mathfrak{T}$.*

Before we are able to prove the proposition, we need some preparational lemmata.

Definition 3.2. Let τ be a m-copying transduction, a constant $n < \omega$ and $\{\psi_i(x, y))_{i<m}\}$ a family of formulae. Then we define a new $(n + nm)$-copying transduction σ, called a *rearrangement* of τ induced by $\{\psi_i(x, y))_{i<m}\}$ as follows:

Let P_i^l, for $i < n$, $l < m$, be the new parameters of σ. If a given input structure $\mathfrak{T}$ does not satisfy the following conditions, we set $\sigma(\mathfrak{T}) := \tau(\mathfrak{T})$:

- $\mathfrak{T} = (T, \leq)$ is an order-tree.

- $\psi_i(x, y)$ defines a function $f_i : T \to T$ such that $|f_i^{-1}(w)| \leq n$, for all $w \in T$.
- For every $w \in T$ and $i < m$, each element of $f_i^{-1}(w)$ belongs to exactly one of the sets $P_i^0, \ldots, P_i^{n-1}$.

If $\mathfrak{T}$ satisfies these conditions, we set $\sigma(\mathfrak{T}) := \pi(\tau(\mathfrak{T}))$, where π is the isomorphism mapping $\langle u, i \rangle \in \tau(\mathfrak{T})$ to

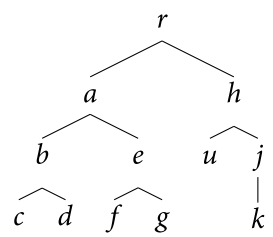
$$\pi(\langle u, i \rangle) = \begin{cases} \langle u, i \rangle & \text{if } f_i(u) = u \\ \langle f_i(u), m + l \rangle & \text{if } f_i(u) \neq u \text{ and } l < m \text{ is the unique index such that } u \in P_i^l. \end{cases}$$

Let τ be an m-copying transduction and $\mathfrak{T} = (T, \leq)$ an order tree and let $(\varphi_{ik}(x, y))_{i,k<m}$ be the formulae defining the edge relation in $\tau(\mathfrak{T})$. For $i < m$ and $u \in T$, we define the functions

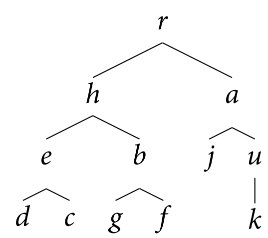
$$f_i(u) := \bigsqcap \{ u \sqcap v \mid u \nleq v, v \nleq u \text{ and } \mathfrak{T} \vDash \varphi_{ik}(u, v) \text{ for some } k < m \} .$$

Note that the image could possibly be empty for some u, in this case we set $f_i(u) := u$. Clearly the functions are MSO-definable on order trees. For a better understanding of the function f_i, consider Example 3.3 below.

Example 3.3. Consider the following order tree $\mathfrak{T}$, the vertices are labelled in a different way in comparison to Example 2.2 for better readability:



For simplicity reasons, we assume that τ is noncopying and that $\tau(\mathfrak{T})$ is an order tree, where the immediate successors of each vertex having exactly two successors are interchanged. Then $\tau(\mathfrak{T})$ looks as follows:



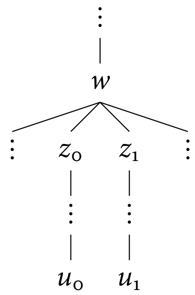
Now we want to compute $f_0(u)$: we know from the definition, that we need to consider all candidates for v which are not $\leq$-related to u but whose image is an immediate successor of $\tau(u)$. The only candidate for this is k, whose infimum with respect to u, is h. Hence $f_0(u) = h$.

One could describe the working principle of the functions as follows: for each new relation R_i only vertices v which are successors of u in the new produced structure are considered. Of those vertices, the ones being in $\leq$-relation with u in $\mathfrak{T}$ are withdrawn and then the infimum of the remaining vertices is given as result. If $f_i(u) = u$, that could mean u has no successor in $\tau(\mathfrak{T})$ or that they are all in $\leq$-relation with u.

Lemma 3.4. *Let τ be an m-copying transduction and $\mathfrak{T} = (T, \leq)$ an order tree such that $\tau(\mathfrak{T})$ is a graph where every vertex has indegree at most d. Let $\varphi_{ik}(x, y)$ be the formulae defining the edge relation of $\tau(\mathfrak{T})$. Let $w \in T$ and let r be the maximal quantifier rank of the φ_{ik}. For every type p of quantifier rank r, there are at most $d(d+1)$ vertices $u \geq w$ such that*

$$f_i(u) = w \quad \text{and} \quad \mathrm{MTh}_r(\mathfrak{T}_z, u) = p,$$

where z is the immediate successor of w with $z \leq u$.



Proof. For a contradiction suppose that they are at least $d(d+1)+1$ vertices $u_0, \ldots, u_{d(d+1)}$ with the same image under f_i. We distinguish two cases:

1. There is an immediate successor z of w such that T_z contains more than d of these vertices, say $u_0, \ldots, u_d \in T_z$. Since we have that $f_i(u_0) = w$, there is by the definition of f_i some $v \in T \smallsetminus T_z$ and some $k < m$ such that $\mathfrak{T} \vDash \varphi_{ik}(u_0, v)$. By assumption it holds that $\mathrm{MTh}_r(\mathfrak{T}_z, u_j) = \mathrm{MTh}_r(\mathfrak{T}_z, u_0)$ for $j \leq d$. Therefore, $\mathfrak{T} \vDash \varphi_{ik}(u_0, v)$ implies $\mathfrak{T} \vDash \varphi_{ik}(u_j, v)$ for $0 \leq j \leq d$. Hence, the vertex $\langle v, k\rangle$ of $\tau(\mathfrak{T})$ has indegree at least $d+1$. A contradiction.

2. For every immediate successor z of w the subtree T_z contains at most d of these vertices. Then there are distinct immediate successors $z_0, \ldots, z_{d+1}$ and vertices $u_{l_0}, \ldots, u_{l_{d+1}}$ with $z_j \leq u_{l_j}$. For simplicity, assume that $l_j = j$, for all j. As in case 1, there is a vertex $v \in T \smallsetminus T_{z_0}$ and $k < m$ such that $\mathfrak{T} \vDash \varphi_{ik}(u_0, v)$. There is at most one index j with $v \in T_{z_j}$. W.l.o.g. assume that $j = d+1$. For $1 \leq j \leq d$ it follows that

$$\mathrm{MTh}_r(\mathfrak{T}_{z_0}, u_0) = \mathrm{MTh}_r(\mathfrak{T}_{z_j}, u_j) \;\Rightarrow\; \mathfrak{T} \vDash \varphi_{ik}(u_j, v),$$

leading to the same contradiction as above.

$\square$

Now we are able to prove the proposition.

Proof of Proposition 3.1. It follows from Lemma 3.4, that there is a uniform bound for the size of the preimage $f_i^{-1}(w)$ for all tree vertices $w \in f_i(T)$, this bound is $d(d+1)$ times the number of types p with quantifier rank r.

Let σ be the rearrangement of τ induced by the functions $(f_i)_i$ defined above. We claim that σ has the desired properties. Let u, v be vertices such that in $\sigma(\mathfrak{T})$ there is an edge between $\langle u, i\rangle$ and $\langle v, k\rangle$. Distinguish four cases:

$i, k < m$: Both $\langle u, i\rangle$ and $\langle v, k\rangle$ were not moved from their position in $\tau(\mathfrak{T})$. Hence u, v are fixed points of f_i and f_k respectively and thus either $u \leq v$ or $v \leq u$ holds in $\mathfrak{T}$.

$i < m, k \geq m$: The vertex $\langle v, k\rangle$ was moved into a new copy of $\mathfrak{T}$, whereas $\langle u, i\rangle$ is still at its original position. There is some $x \neq v$ such that $f_{k-m}(x) = v$ and $\mathfrak{T} \vDash \varphi_{i(k-m)}(u, x)$. Since $f_i(u) = u$ we have $x \leq u$ or $u \leq x$. Hence, $v \prec x$ implies $v \leq u$ or $u \leq v$.

$i \geq m, k < m$: The vertex $\langle u, i\rangle$ was moved into a new copy of $\mathfrak{T}$, whereas $\langle v, k\rangle$ is still at its original position. There is some $x \neq u$ such that $f_{i-m}(x) = u$ and $\mathfrak{T} \vDash \varphi_{(i-m)k}(x, v)$. By definition of f_{i-m} it follows that $x \leq v$, $v \leq x$ or $x \sqcap v \geq u$. Since $u \prec x$ this implies that $u \leq v$ or $v \leq u$.

$i, k \geq m$: Both vertices were moved to new copies of $\mathfrak{T}$. There are vertices $x \neq u$ and $y \neq v$ such that $f_{i-m}(x) = u$, $f_{k-m}(y) = v$ and $\mathfrak{T} \vDash \varphi_{(i-m)(k-m)}(x, y)$. Note that $y \leq x$ or $u = f_{i-m}(x) \leq y$. In both cases $v \leq y$ implies that $u \leq v$ or $v \leq u$.

$\square$

Lemma 3.5. *For every transduction τ and every number $n < \omega$ there exists a transduction σ with the following property: We have $\sigma(\mathfrak{T}) \cong \tau(\mathfrak{T})$, for every structure $\mathfrak{T}$. Furthermore, if $\mathfrak{T} = (T, \leq)$ and $\tau(\mathfrak{T})$ are both order-trees and $\tau(\mathfrak{T})$ is of height of at most n then,*

$$\langle u, i\rangle \leq \langle v, k\rangle \text{ in } \sigma(\mathfrak{T}) \Rightarrow u \leq v \text{ in } \mathfrak{T}. \tag{3.1}$$

Proof. Suppose that τ is m-copying and let $\varphi_{ik}(u, v)$ be the formulae defining the order relation $\leq$ between the vertices $\langle u, i\rangle$ and $\langle v, k\rangle$ in $\tau(\mathfrak{T})$. Note that in an order tree of height n, each vertex has at most $n+1$ predecessors. Hence, using Proposition 3.1 with $d := n+1$, we may assume that, for every pair of vertices $\langle u, i\rangle \leq \langle v, k\rangle \in \tau(\mathfrak{T})$ we have $u \leq v$ or $v \leq u$ in $\mathfrak{T}$. For $i < m$ and $u \in T$, we define

$$f_i(u) := \bigsqcap \{ v \leq u \mid \mathfrak{T} \vDash \varphi_{ik}(u, v) \text{ for some } k < m \}.$$

These functions give the infimum of all vertices $\geq \langle u, i\rangle$ in $\tau(\mathfrak{T})$ which are $\prec u$ in $\mathfrak{T}$.

If $f_i(u) = v$, we have that $\langle u, i\rangle \leq \langle v, k\rangle$ for some $k < m$. We can find a bound for the size of the preimage $|f_i^{-1}(v)|$ as follows: by assumption τ is m-copying and the resulting tree has height at most n, implying that the path from the root to $\langle v, k\rangle$ has at most $m(n+1)$ distinct vertices. So we can bound the preimage by this number: $|f_i^{-1}(v)| \leq m(n+1)$. The rearrangement σ of τ induced by $(f_i)_i$ has the desired properties. $\square$

Definition 3.6.

1. Let $\mathfrak{S} = (S, \le)$ and $\mathfrak{T} = (T, \le)$ be order trees. We define an *n-embedding* to be a homomorphism $h : \mathfrak{S} \to \mathfrak{T}$ such that $|h^{-1}(u)| \le n$ for all $u \in T$.

2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be trees. We write $\mathfrak{A} \hookrightarrow_n \mathfrak{B}$ if there exists an n-embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

3. Let $\mathcal{A}, \mathcal{B}$ be classes of trees. Define $\mathcal{A} \le_n \mathcal{B}$ if for all $\mathfrak{A} \in \mathcal{A}$ there exists some $\mathfrak{B} \in \mathcal{B}$ such that $\mathfrak{A} \hookrightarrow_n \mathfrak{B}$.

Theorem 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be classes of order trees such that the height of trees in $\mathcal{A}$ and $\mathcal{B}$ is bounded by $m < \omega$. Then $\mathcal{A} \sqsubseteq \mathcal{B}$ if and only if $\mathcal{A} \le_n \mathcal{B}$ for some number $n < \omega$.*

Proof. ($\Leftarrow$) We construct a transduction τ such that, for every $\mathfrak{B} \in \mathcal{B}$,

$$\{\mathfrak{A} \mid \mathfrak{A} \hookrightarrow_n \mathfrak{B}\} \subseteq \tau(\mathfrak{B}).$$

Given $\mathfrak{B} \in \mathcal{B}$, we encode an n-embedding $h : \mathfrak{A} \hookrightarrow_n \mathfrak{B}$ as follows: for each vertex $v \in B$, we fix an enumeration $q_0(v), q_1(v), \dots$ of $h^{-1}(v)$. Furthermore, we denote by $\mathrm{pre}_l(v)$ the l-th predecessor of v, i.e. the unique vertex $u \in B$ such that there exists a path of length l from u to v. We use the following unary predicates to encode h:

$$\begin{aligned} P_k &:= \{v \in B \mid |h^{-1}(v)| = k\}, \\ R_{ikl} &:= \{v \in B \mid \text{there exist an edge in } \mathfrak{A} \text{ from } q_i(\mathrm{pre}_l(v)) \text{ to } q_k(v)\}. \end{aligned}$$

We can recover $\mathfrak{A}$ from these predicates by the transduction τ defined by the formulae

$$\begin{aligned} \delta_i(x) &:= \bigvee_{k \ge i} P_k(x), \\ \varphi_{ik}(x, y) &:= \bigvee_{l < m} \left(R_{ikl}(y) \wedge x = \mathrm{pre}_l(y)\right). \end{aligned}$$

($\Rightarrow$) Let τ be a transduction such that we have $\mathfrak{A} \in \tau(\mathfrak{B})$ and the height of $\mathfrak{A}$ and $\mathfrak{B}$ is at most m. We construct a transduction σ as in Lemma 3.5. Then we have $\mathfrak{A} \in \sigma(\mathfrak{B})$. Suppose then σ is n-copying. Then the map sending $\langle u, k \rangle$ to u is an n-embedding. □

Proposition 3.8. *Let $\mathcal{A}$ be a class of hypergraphs, $\mathcal{P}$ the class of all finite paths. If $\mathcal{P} \not\sqsubseteq \mathcal{A}$ then $\mathcal{A} \equiv \mathrm{STD}_n(\mathcal{A})$ for some $n < \omega$.*

Proof. $\mathcal{P} \not\sqsubseteq \mathcal{A}$ implies that for every transduction τ there is some path $\mathfrak{P} \in \mathcal{P}$ such that $\mathfrak{P} \notin \tau(\mathfrak{A})$. By Lemma 2.11 we know that $\mathfrak{P}$ is hence not contained in $\mathrm{Min}(\mathcal{A})$. The Excluded Path Theorem implies that $\mathrm{twd}_n(\mathcal{A}) < \infty$, for some n. Thus we can conclude with the help of Lemma 2.13 and Theorem 2.14 that $\mathcal{A} \equiv \mathrm{STD}_n(\mathcal{A})$. □

4 Classes of Trees with Bounded Height

From now on, all classes of trees considered have bounded height. In this chapter we define some operations on classes of trees which are a foundation for building all the classes we need to have a complete lower part of the transduction hierarchy. We also want to see, what the properties of our newly defined operations are, starting with associativity, commutativity and so on. Furthermore, we define a class $\mathcal{A}$ to be directed, which means that $\mathcal{A} \oplus \mathcal{A} \sqsubseteq \mathcal{A}$ or in words: the operation $\oplus$ applied to any two elements of $\mathcal{A}$ produces only structures which can be described via transduction in terms of some $\mathfrak{A} \in \mathcal{A}$, or roughly speaking $\oplus$ gives us nothing new.

At the end of the chapter we achieve a nice result telling us something of the structure which classes form together with the relation $\sqsubseteq$. Theorem 4.22 states that the class relation $\sqsubseteq$ forms a distributive join-semilattice on the $\equiv$-equivalence classes of all structures. The join of two classes, or more precisely, two representatives of equivalence classes, is simply their union.

Definition 4.1. Let $\mathcal{A}, \mathcal{B}$ be classes of trees.

1. For trees $\mathfrak{A}$ and $\mathfrak{B}$ we define $\mathfrak{A} \oplus \mathfrak{B}$ to be the tree obtained by identifying the roots of $\mathfrak{A}, \mathfrak{B}$, i.e. one merges the two roots into one vertex which then has the combined immediate successors from both former roots. For $\mathfrak{B} = \varnothing$, we set $\mathfrak{A} \oplus \varnothing := \mathfrak{A}$. We define

$$\mathcal{A} \oplus \mathcal{B} := \{\mathfrak{A} \oplus \mathfrak{B} \mid \mathfrak{A} \in \mathcal{A},\ \mathfrak{B} \in \mathcal{B}\}.$$

2. Let $\mathfrak{A}$ be a tree, $L = L(\mathfrak{A})$ and $(\mathfrak{B}_v)_{v \in L}$ a sequence of trees. $\mathfrak{A} \leftarrow (\mathfrak{B}_v)_{v \in L}$ denotes the tree obtained by replacing each leaf $v \in L$ of $\mathfrak{A}$ by the tree $\mathfrak{B}_v$. If the sequence is constant, i.e. for all $v \in L$ we have $\mathfrak{B}_v = \mathfrak{B}_0$ then the resulting tree is denoted by $\mathfrak{A} \leftarrowtail \mathfrak{B}_0$.

3. We define $\mathcal{A} \cdot \mathcal{B}$ to be the set of all trees of the form $\mathfrak{A} \leftarrow (\mathfrak{B}_v)_{v \in L}$ for $\mathfrak{A} \in \mathcal{A}$ and a sequence $(\mathfrak{B}_v)_{v \in L}$ in $\mathcal{B}$ where L is the set of leaves of $\mathfrak{A}$, i.e.

$$\mathcal{A} \cdot \mathcal{B} := \{\mathfrak{A} \leftarrow (\mathfrak{B}_v)_{v \in L} \mid \mathfrak{A} \in \mathcal{A}, \mathfrak{B}_v \in \mathcal{B}\}.$$

4. Define $\mathcal{A} : \mathcal{B}$ to be the set of all trees of $\mathcal{A} \cdot \mathcal{B}$ where the sequence $(\mathfrak{B}_v)_{v \in L}$ is constant, i.e.

$$\mathcal{A} : \mathcal{B} := \{\mathfrak{A} \leftarrowtail \mathfrak{B} \mid \mathfrak{A} \in \mathcal{A}, \mathfrak{B} \in \mathcal{B}\}.$$

5. We say the class $\mathcal{A}$ is *directed*, if $\mathcal{A} \oplus \mathcal{A} \sqsubseteq \mathcal{A}$.

From now on we will mostly deal with $\mathcal{A} \leq_n \mathcal{B}$, because n-embeddings are easier to handle than transductions. However, the results will be stated in the form $\mathcal{A} \sqsubseteq \mathcal{B}$ which is permissible by Theorem 3.7.

Lemma 4.2. *For classes $\mathcal{A}, \mathcal{B}$ we have that $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{A} \sqsubseteq \mathcal{B}$. If for every $\mathfrak{A} \in \mathcal{A}$ there is some $\mathfrak{B} \in \mathcal{B}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ it also follows that $\mathcal{A} \sqsubseteq \mathcal{B}$.*

Proof. For the first implication we simply use the identity map for a 1-embedding, for the second statement the inclusion map 1-embeds every $\mathfrak{A} \in \mathcal{A}$ into some $\mathfrak{B} \in \mathcal{B}$. □

Definition 4.3.

1. For $0 < m \leq \omega$ we denote complete m-ary trees of height 2 by $[\![m]\!] := m^{<2}$, with the special case $[\![0]\!] := \bullet$.
2. We denote the product of such trees by $[\![m_0, ..., m_{n-1}]\!] := [\![m_0]\!] \twoheadleftarrow \ldots \twoheadleftarrow [\![m_{n-1}]\!]$ and thus we have $[\![m^n]\!] = m^{<(n+1)}$.
3. We denote the class of finite trees with height two by $\Phi := \{[\![m]\!] \mid m < \omega\}$.
4. The class containing the countably infinite tree with height two is denoted by $\Omega := \{[\![\omega]\!]\}$.
5. We denote the tree having one vertex, i.e. only the root, by $\bullet$. The class consisting only of this tree is denoted by 1.
6. For all $n < \omega$ we define $\Phi^n := \Phi \cdot \ldots \cdot \Phi$, where $\Phi^0 := 1$. Ω^n is defined analogously.

Example 4.4. The class Ω^n contains only one element, which is denoted by $[\![\omega^n]\!]$. The class Φ^n contains elements like $[\![k^n]\!]$, for all $k < \omega$, but also other trees.

4.1 Laws for the Operations $\{\oplus, \cup, \cdot, :\}$

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be classes of trees.

Lemma 4.5. *The operations $\cup$ and $\oplus$ are associative and commutative. The neutral element of $\cup$ is the empty class $\emptyset$, the neutral element of $\oplus$ is 1. Both operations are monotone with respect to $\sqsubseteq$, i.e. we have for $\star \in \{\cup, \oplus\}$*

$$\mathcal{A} \star (\mathcal{B} \star \mathcal{C}) = (\mathcal{A} \star \mathcal{B}) \star \mathcal{C},$$

$$\mathcal{A} \star \mathcal{B} = \mathcal{B} \star \mathcal{A} \tag{4.1}$$

and

$$\mathcal{A} \sqsubseteq \mathcal{B} \text{ implies } \mathcal{C} \star \mathcal{A} \sqsubseteq \mathcal{C} \star \mathcal{B} \text{ and } \mathcal{A} \star \mathcal{C} \sqsubseteq \mathcal{B} \star \mathcal{C}.$$

Proof. Commutativity and the neutral element being the empty class are obvious. Associativity and monotonicity of $\oplus$ follow directly from Definition 4.1, for $\cup$ it is again obvious. □

Lemma 4.6. *The operation $\cdot$ is associative and monotone in the second argument with respect to $\sqsubseteq$. If $\mathcal{A} \sqsubseteq \mathcal{B}$ then $\mathcal{C} \cdot \mathcal{A} \sqsubseteq \mathcal{C} \cdot \mathcal{B}$ (but not in general $\mathcal{A} \cdot \mathcal{C} \sqsubseteq \mathcal{B} \cdot \mathcal{C}$). The neutral element is 1.*

Proof. Consider classes $\mathcal{A}, \mathcal{B}$ with $\mathcal{A} \leq_n \mathcal{B}$. Let $\mathfrak{C} \leftarrow (\mathfrak{A}_i)_{i \in L(\mathfrak{C})}$ be an element of $\mathcal{C} \cdot \mathcal{A}$. We want to find an element of $\mathcal{C} \cdot \mathcal{B}$ into which $\mathfrak{C} \leftarrow (\mathfrak{A}_i)_{i \in L(\mathfrak{C})}$ can be n-embedded. For each subtree $\mathfrak{A}_i$ we know that there is some $\mathfrak{B}_i \in \mathcal{B}$ such that $\mathfrak{A}_i \hookrightarrow_n \mathfrak{B}_i$. We construct the desired n-embedding of $\mathfrak{C} \leftarrow (\mathfrak{A}_i)_{i \in L(\mathfrak{A})}$ into $\mathfrak{C} \leftarrow (\mathfrak{B}_i)_{i \in L(B)}$as an identity for the common initial segment extended by these n-embeddings for the additional subtrees $\mathfrak{A}_i$ into the corresponding $\mathfrak{B}_i$.

For associativity let $\mathcal{D}_0 := (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}$ and $\mathcal{D}_1 := \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$ and let $\mathfrak{D} \in \mathcal{D}_0$. Then $\mathfrak{D}$ is of the form

$$\left(\mathfrak{A} \leftarrow (\mathfrak{B}_i)_{i \in L(\mathfrak{A})}\right) \leftarrow (\mathfrak{C}_j)_{j \in L(\mathfrak{A} \leftarrow \mathfrak{B}_i)}$$

for some $\mathfrak{A} \in \mathcal{A}$, $\mathfrak{B}_i \in \mathcal{B}$ and $\mathfrak{C}_j \in \mathcal{C}$. Now we want to show that this tree is a member of $\mathcal{D}_1$. Its initial segment is $\mathfrak{A}$. We define a sequence of elements of $\mathcal{B} \cdot \mathcal{C}$ by first taking the initial segments to be $\mathfrak{B}_i$ and extend them by the sequences $(\mathfrak{C}_j)_{j \in L(\mathfrak{B}_i)}$, i.e. $\mathfrak{B}_i \leftarrow (\mathfrak{C}_j)_{j \in L(\mathfrak{B}_i)}$, such that each $\mathfrak{B}_i$ is extended by the same tree as above. Since

$$L\left(\mathfrak{A} \leftarrow (\mathfrak{B}_i)_{i \in L(\mathfrak{A})}\right) = \bigcup_{i \in L(\mathfrak{A})} L(\mathfrak{B}_i)$$

we have equality:

$$\mathfrak{D} = \mathfrak{A} \leftarrow \left(\mathfrak{B}_i \leftarrow (\mathfrak{C}_j)_{j \in L(\mathfrak{B}_i)}\right)_{i \in L(\mathfrak{A})}$$

and $\mathfrak{A} \leftarrow \left(\mathfrak{B}_i \leftarrow (\mathfrak{C}_j)_{j \in L(\mathfrak{B}_i)}\right)_{i \in L(\mathfrak{A})}$ is an element of $\mathcal{D}_1$. The converse direction is analogous.

Since $\mathfrak{A} \twoheadleftarrow \bullet = \mathfrak{A}$ we have $\mathcal{A} \cdot 1 = \mathcal{A}$. □

Lemma 4.7. *The operation $:$ is monotone with respect to $\sqsubseteq$, i.e. if we have $\mathcal{A} \sqsubseteq \mathcal{B}$ then $\mathcal{C} : \mathcal{A} \sqsubseteq \mathcal{C} : \mathcal{B}$ and $\mathcal{A} : \mathcal{C} \sqsubseteq \mathcal{B} : \mathcal{C}$. Furthermore it is associative and has 1 as neutral element.*

Proof. The first statement of monotonicity is proven analogously to Lemma 4.6. For the second statement let $\mathfrak{A} \twoheadleftarrow \mathfrak{C} \in \mathcal{A} : \mathcal{C}$. Now choose some $\mathfrak{B} \in \mathcal{B}$ such that $\mathfrak{A} \hookrightarrow_n \mathfrak{B}$. Let h be the corresponding n-embedding. For all leaves $v \in L(\mathfrak{B})$ we have that $|h^{-1}(v)| \leq n$. Hence we can extend h such that it maps $\mathfrak{C}$ to itself in such a way that it is a valid n-embedding $(\mathfrak{A} \twoheadleftarrow \mathfrak{C}) \hookrightarrow_n (\mathfrak{B} \twoheadleftarrow \mathfrak{C})$.

The proof for associativity is analogous to the proof of Lemma 4.6, as well as the neutral element being 1. □

Lemma 4.8. *$\mathcal{A} \cup \mathcal{B} \sqsubseteq \mathcal{A} \oplus \mathcal{B} \sqsubseteq \mathcal{A} : \mathcal{B} \sqsubseteq \mathcal{A} \cdot \mathcal{B}$, for nonempty classes $\mathcal{A}, \mathcal{B}$ and $\mathcal{A} \neq \{\varnothing\}$.*

Proof. We proof each statement separately:

1. Let $\mathfrak{A} \in \mathcal{A} \cup \mathcal{B}$. W.l.o.g. we assume $\mathfrak{A} \in \mathcal{A}$. Since $\mathcal{B}$ is nonempty, there is some $\mathfrak{B} \in \mathcal{B}$. By Lemma 4.2, we have $\mathfrak{A} \hookrightarrow_1 \mathfrak{A} \oplus \mathfrak{B}$ implying the claim.

2. Let $\mathfrak{A} \oplus \mathfrak{B} \in \mathcal{A} \oplus \mathcal{B}$ for some elements $\mathfrak{A} \in \mathcal{A}$ and $\mathfrak{B} \in \mathcal{B}$ with $\mathfrak{A} \neq \varnothing$. We consider $\mathfrak{T} := \mathfrak{A} \twoheadleftarrow \mathfrak{B} \in \mathcal{A} : \mathcal{B}$. Now we know that $\mathfrak{A} \hookrightarrow_n \mathfrak{T}$. $\mathfrak{B}$ can be 1-embedded to one of the copies of itself in $\mathfrak{T}$. Hence we have that $(\mathfrak{A} \oplus \mathfrak{B}) \hookrightarrow_n (\mathfrak{A} \twoheadleftarrow \mathfrak{B})$. Suppose that $\varnothing \in \mathcal{A}$ and let $\mathfrak{B} \in \mathcal{B}$. Then $\mathfrak{B} = \varnothing \oplus \mathfrak{B} \in \mathcal{A} \oplus \mathcal{B}$ and $\mathfrak{B} \hookrightarrow_1 \mathfrak{A} \twoheadleftarrow \mathfrak{B}$ for any $\mathfrak{A} \in \mathcal{A}$ with $\mathfrak{A} \neq \varnothing$.

3. We have that $\mathcal{A} : \mathcal{B} \subseteq \mathcal{A} \cdot \mathcal{B}$, so we simply apply Lemma 4.2.

□

Lemma 4.9. $\sup\{\mathcal{A},\mathcal{B}\} = \mathcal{A} \cup \mathcal{B}$.

Proof. Obviously it holds that $\mathcal{A},\mathcal{B} \sqsubseteq \mathcal{A} \cup \mathcal{B}$. Now let $\mathcal{C}$ be a class such that $\mathcal{A},\mathcal{B} \sqsubseteq \mathcal{C}$. Hence there is some $n < \omega$ such that every member of $\mathcal{A}$ and $\mathcal{B}$ is n-embeddable into some element of $\mathcal{C}$ implying that $\mathcal{A} \cup \mathcal{B} \sqsubseteq \mathcal{C}$. □

Lemma 4.10. *$\mathcal{A} \cup \mathcal{B} \equiv \mathcal{A} \oplus \mathcal{B}$ implies that $\mathcal{A} \sqsubseteq \mathcal{B}$ or $\mathcal{B} \sqsubseteq \mathcal{A}$.*

Proof. Assume that $\mathcal{A} \oplus \mathcal{B} \leq_n \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \not\sqsubseteq \mathcal{B}$. Since $\mathcal{A} \not\sqsubseteq \mathcal{B}$ there is at least one $\mathfrak{A}_0 \in \mathfrak{A}$ such that $\mathfrak{A}_0 \not\hookrightarrow_n \mathfrak{B}$ for every $\mathfrak{B} \in \mathcal{B}$. $\mathcal{A} \oplus \mathcal{B} \leq_n \mathcal{A} \cup \mathcal{B}$ implies that, for every $\mathfrak{B} \in \mathcal{B}$, there is some $\mathfrak{C} \in \mathcal{A} \cup \mathcal{B}$ such that $\mathfrak{A}_0 \oplus \mathfrak{B} \hookrightarrow_n \mathfrak{C}$. Thus, $\mathfrak{A}_0 \hookrightarrow_n \mathfrak{C}$ and $\mathfrak{B} \hookrightarrow_n \mathfrak{C}$. Since the former implies that $\mathfrak{C} \notin \mathcal{B}$ it follows that $\mathfrak{C} \in \mathcal{A}$. Hence, for every $\mathfrak{B} \in \mathcal{B}$ there is some $\mathfrak{C} \in \mathcal{A}$ with $\mathfrak{B} \hookrightarrow_n \mathfrak{C}$. Thus we can conclude that $\mathcal{B} \sqsubseteq \mathcal{A}$. □

Lemma 4.11. *$\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \equiv \mathcal{A} \cup \mathcal{B}$ if and only if $\mathcal{C} \sqsubseteq \mathcal{A} \cup \mathcal{B}$.*

Proof. ($\Leftarrow$) Suppose that $\mathcal{C} \leq_n \mathcal{A} \cup \mathcal{B}$. For every $\mathfrak{C} \in \mathcal{C}$ there is some $\mathfrak{D} \in \mathcal{A} \cup \mathcal{B}$ with $\mathfrak{C} \hookrightarrow_n \mathfrak{D}$. Set

$$\mathcal{C}_0 := \{\mathfrak{C} \in \mathcal{C} \mid \exists \mathfrak{A} \in \mathcal{A} \text{ with } \mathfrak{C} \hookrightarrow_n \mathfrak{A}\},$$
$$\mathcal{C}_1 := \{\mathfrak{C} \in \mathcal{C} \mid \exists \mathfrak{B} \in \mathcal{B} \text{ with } \mathfrak{C} \hookrightarrow_n \mathfrak{B}\}.$$

Then $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$ and $\mathcal{C}_0 \leq_n \mathcal{A}, \mathcal{C}_1 \leq_n \mathcal{B}$. By monotonicity of $\cup$, this implies $\mathcal{C} \sqsubseteq \mathcal{A} \cup \mathcal{B}$. Again by monotonicity, it follows that $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \sqsubseteq \mathcal{A} \cup \mathcal{B}$.

($\Rightarrow$) $\mathcal{C} \sqsubseteq \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \sqsubseteq \mathcal{A} \cup \mathcal{B}$. □

Lemma 4.12. *If $\mathcal{A}$ is nonempty $\mathcal{A} \oplus \mathcal{B} \equiv \mathcal{A}$ implies $\mathcal{B} \sqsubseteq \mathcal{A}$. The converse is true if $\mathcal{A}$ is directed.*

Proof. ($\Rightarrow$) We have $\mathcal{B} \sqsubseteq \mathcal{A} \oplus \mathcal{B} \sqsubseteq \mathcal{A}$.

($\Leftarrow$) If $\mathcal{A}$ is directed and $\mathcal{B} \sqsubseteq \mathcal{A}$ then $\mathcal{A} \oplus \mathcal{B} \sqsubseteq \mathcal{A} \oplus \mathcal{A} \equiv \mathcal{A}$. □

Lemma 4.13.

1. $\mathcal{A} \oplus (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \oplus \mathcal{B}) \cup (\mathcal{A} \oplus \mathcal{C})$.
2. $\mathcal{A} \cup (\mathcal{B} \oplus \mathcal{C}) \sqsubseteq (\mathcal{A} \cup \mathcal{B}) \oplus (\mathcal{A} \cup \mathcal{C})$.
3. $(\mathcal{A} \cup \mathcal{B}) \cdot \mathcal{C} = (\mathcal{A} \cdot \mathcal{C}) \cup (\mathcal{B} \cdot \mathcal{C})$.
4. $(\mathcal{A} \cdot \mathcal{B}) \cup (\mathcal{A} \cdot \mathcal{C}) \sqsubseteq \mathcal{A} \cdot (\mathcal{B} \cup \mathcal{C})$.
5. $(\mathcal{A} \cup \mathcal{B}) : \mathcal{C} = (\mathcal{A} : \mathcal{C}) \cup (\mathcal{B} : \mathcal{C})$.
6. $(\mathcal{A} : \mathcal{B}) \cup (\mathcal{A} : \mathcal{C}) = \mathcal{A} : (\mathcal{B} \cup \mathcal{C})$.

Proof. All statements follow directly from the definition and Lemma 4.2. □

We shall denote the disjoint union by $\sqcup$.

Lemma 4.14. *$(\mathcal{A} \oplus \mathcal{B}) \cdot \mathcal{C} = (\mathcal{A} \cdot \mathcal{C}) \oplus (\mathcal{B} \cdot \mathcal{C})$ if $\mathcal{A}$ and $\mathcal{B}$ do not contain $\bullet$.*

Proof. ($\subseteq$) Let $(\mathfrak{A}\oplus\mathfrak{B}) \leftarrow (\mathfrak{C}_i)_{i\in L(\mathfrak{A}\oplus\mathfrak{B})} \in (\mathcal{A}\oplus\mathcal{B})\cdot\mathcal{C}$. If $\mathfrak{A}, \mathfrak{B} \neq \bullet$, then $L(\mathfrak{A}\oplus\mathfrak{B}) = L(\mathfrak{A}) \sqcup L(\mathfrak{B})$. We split the sequence $(\mathfrak{C}_i)_{i\in L(\mathfrak{A}\oplus\mathfrak{B})}$ into two disjoint parts $(\mathfrak{C}_j)_{j\in L(\mathfrak{A})}$ and $(\mathfrak{C}_k)_{k\in L(\mathfrak{B})}$. Then

$$(\mathfrak{A}\oplus\mathfrak{B}) \leftarrow (\mathfrak{C}_i)_{i\in L(\mathfrak{A}\oplus\mathfrak{B})} = \mathfrak{A} \leftarrow (\mathfrak{C}_j)_{j\in L(\mathfrak{A})} \oplus \mathfrak{B} \leftarrow (\mathfrak{C}_k)_{k\in L(\mathfrak{B})}.$$

($\supseteq$) Here the procedure is similar, this time we join the two sequences extending arbitrary $\mathfrak{A}$ and $\mathfrak{B}$ to show that it is a member of $(\mathcal{A}\oplus\mathcal{B})\cdot\mathcal{C}$. □

Lemma 4.15. *$(\mathcal{A}\oplus\mathcal{B}):\mathcal{C} \sqsubseteq (\mathcal{A}:\mathcal{C})\oplus(\mathcal{B}:\mathcal{C})$ if $\mathcal{A}$ and $\mathcal{B}$ do not contain $\bullet$. If $\mathcal{C}$ is directed also $(\mathcal{A}\oplus\mathcal{B}):\mathcal{C} \sqsupseteq (\mathcal{A}:\mathcal{C})\oplus(\mathcal{B}:\mathcal{C})$ holds.*

Proof. The first statement is proven analogously to Lemma 4.14, the only difference is that all sequences are constant.

Let $\mathcal{C}$ be directed and let $\mathfrak{A}\in\mathcal{A}$, $\mathfrak{B}\in\mathcal{B}$ and $\mathfrak{C}_0, \mathfrak{C}_1, \in \mathcal{C}$. Consider the element $(\mathfrak{A} \leftarrowtail \mathfrak{C}_0)\oplus(\mathfrak{B} \leftarrowtail \mathfrak{C}_1)$. Now find some element $\mathfrak{C}\in\mathcal{C}$ such that $\mathfrak{C}_0\oplus\mathfrak{C}_1 \hookrightarrow_n \mathfrak{C}$. We use this n-embedding to show that $(\mathfrak{A} \leftarrowtail \mathfrak{C}_0)\oplus(\mathfrak{B} \leftarrowtail \mathfrak{C}_1) \hookrightarrow_n (\mathfrak{A}\oplus\mathfrak{B}) \leftarrowtail \mathfrak{C}$. □

Lemma 4.16. *If the classes $\mathcal{A}$ and $\mathcal{B}$ are directed, then $\mathcal{A}\oplus\mathcal{B}$ is directed.*

Proof. Let $\mathfrak{A}_0\oplus\mathfrak{B}_0$ and $\mathfrak{A}_1\oplus\mathfrak{B}_1 \in \mathcal{A}\oplus\mathcal{B}$. By assumption there exists $\mathfrak{A}_2\in\mathcal{A}$ and $\mathfrak{B}_2\in\mathcal{B}$ such that $\mathfrak{A}_0\oplus\mathfrak{A}_1 \hookrightarrow_n \mathfrak{A}_2$ and $\mathfrak{B}_0\oplus\mathfrak{B}_1 \hookrightarrow_n \mathfrak{B}_2$ implying that $\mathfrak{A}_0\oplus\mathfrak{B}_0\oplus\mathfrak{A}_1\oplus\mathfrak{B}_1 \hookrightarrow_n \mathfrak{A}_2\oplus\mathfrak{B}_2$. □

Lemma 4.17. *If the classes $\mathcal{C}$ and $\mathcal{D}$ are directed, we have that $\mathcal{A}\cup\mathcal{B} \sqsubseteq \mathcal{C}\cup\mathcal{D}$ implies $\mathcal{A}\oplus\mathcal{B} \sqsubseteq \mathcal{C}\oplus\mathcal{D}$.*

Proof. $\mathcal{A}\cup\mathcal{B} \sqsubseteq \mathcal{C}\cup\mathcal{D}$ and Lemma 4.16 imply

$$\mathcal{A}\oplus\mathcal{B} \sqsubseteq (\mathcal{C}\cup\mathcal{D})\oplus(\mathcal{C}\cup\mathcal{D}) \sqsubseteq (\mathcal{C}\oplus\mathcal{C})\cup(\mathcal{C}\oplus\mathcal{D})\cup(\mathcal{D}\oplus\mathcal{D}) \sqsubseteq \mathcal{C}\oplus\mathcal{D}.$$

□

Lemma 4.18. *If the class $\mathcal{C}$ is directed and we have $\mathcal{A}\sqsubseteq\mathcal{C}$ and $\mathcal{B}\sqsubseteq\mathcal{C}$ then $\mathcal{A}\oplus\mathcal{B}\sqsubseteq\mathcal{C}$ holds.*

Proof. $\mathcal{A},\mathcal{B}\sqsubseteq\mathcal{C}$ implies $\mathcal{A}\oplus\mathcal{B}\sqsubseteq\mathcal{C}\oplus\mathcal{C}\equiv\mathcal{C}$. □

Corollary 4.19. Let $\mathcal{A},\mathcal{B}$ classes. For all directed classes $\mathcal{C}$ such that $\mathcal{A}\cup\mathcal{B}\sqsubseteq\mathcal{C}\sqsubseteq\mathcal{A}\oplus\mathcal{B}$ it follows that $\mathcal{C}\equiv\mathcal{A}\oplus\mathcal{B}$.

Proof. The statement follows directly from Lemma 4.18. □

Corollary 4.20. If $\mathcal{A}\cup\mathcal{B}$ is directed we have $\mathcal{A}\sqsubseteq\mathcal{B}$ or $\mathcal{B}\sqsubseteq\mathcal{A}$ and thus $\mathcal{A}\cup\mathcal{B}\equiv\mathcal{B}$ or $\mathcal{A}\cup\mathcal{B}\equiv\mathcal{A}$.

Proof. By Corollary 4.19 we have $\mathcal{A}\oplus\mathcal{B}\equiv\mathcal{A}\cup\mathcal{B}$. Hence the claim follows by Lemma 4.10. □

Definition 4.21.

1. A *join-semilattice* L is a partially ordered set, such that for every non-empty finite subset there exists a least upper bound. If a and b are elements of L, we denote their supremum (or join) by $a \vee b$.

2. A join-semilattice L is *distributive* if the following holds: for all $a, b, c \in L$ such that $c \leq a\vee b$ there exist elements $a_0, b_0 \in L$ with $a_0 \leq a$ and $b_0 \leq b$ such that $c = a_0 \vee b_0$.

Theorem 4.22. *The relation $\sqsubseteq$ defines the structure of a distributive join-semilattice with zero on the $\equiv$-equivalence classes of all structures.*

Proof. The join of two classes is their union, (see Lemma 4.9). The empty class is the zero element of this join. That $\sqsubseteq$ defines a partial ordering on its equivalence classes is obvious. It remains to show distributivity: let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be classes such that $\mathcal{C} \sqsubseteq \mathcal{A} \cup \mathcal{B}$. We divide the class $\mathcal{C}$ such that $\mathcal{C} = \mathcal{C}_{\mathcal{A}} \cup \mathcal{C}_B$ with $\mathcal{C}_A \sqsubseteq \mathcal{A}$ and $\mathcal{C}_B \sqsubseteq \mathcal{B}$, as in the proof of Lemma 4.11. $\square$

5 Hierarchy of Classes of Trees with Bounded Height

Now we have arrived at the hierarchy of classes of trees. The pattern for each statement will always be the same: first we show $\mathcal{A} \sqsubseteq \mathcal{B}$ for some classes $\mathcal{A}, \mathcal{B}$ which is mostly done by applying a lemma of the fourth chapter. Then we want to have $\mathcal{B} \not\sqsubseteq \mathcal{A}$ since otherwise we would end up with $\mathcal{A} \equiv \mathcal{B}$ instead of a further step in the hierarchy. The hardest part of most of the proofs is to show that the hierarchy is complete, i.e. there is no class $\mathcal{C}$ in between $\mathcal{A} \lhd \mathcal{B}$, which is not $\equiv$-equivalent to $\mathcal{A}$ or $\mathcal{B}$. This is done by a case distinction. We found two new classes in the process. As already announced in the introduction, we will have a theorem at the end comprising all propositions of the forthcoming chapter into one diagram.

Lemma 5.1. $\Phi \sqsubseteq \Omega$.

Proof. For every $[\![m]\!] \in \Phi$ we have that $[\![m]\!] \subset [\![\omega]\!]$. Hence we can apply Lemma 4.2. □

Lemma 5.2. *For every $n < \omega$ the following statements hold:*

1. $\Phi^n \sqsubseteq \Phi^{n+1}$.
2. $\Phi^n \sqsubseteq \Phi^n \cup \Omega$.
3. $\Phi^n \cup \Omega \sqsubseteq \Phi^{n+1} \cup \Omega$.

Proof.

1. $1 \sqsubseteq \Phi$ implies $\Phi^n = \Phi^n \cdot 1 \sqsubseteq \Phi^n \cdot \Phi = \Phi^{n+1}$.
2. $\varnothing \sqsubseteq \Omega$ implies $\Phi^n \cup \varnothing \sqsubseteq \Phi^n \cup \Omega$.
3. By (1) $\Phi^n \sqsubseteq \Phi^{n+1}$ implies $\Phi^n \cup \Omega \sqsubseteq \Phi^{n+1} \cup \Omega$.

□

Lemma 5.3. *The classes Φ^n and Ω^n are directed, for all $n < \omega$.*

Proof. Let $\mathfrak{A}, \mathfrak{B} \in \Phi^n$. There exists $m < \omega$ such that every vertex in $\mathfrak{A}$ or $\mathfrak{B}$ has at most m successors. Thus $\mathfrak{A}, \mathfrak{B} \subseteq [\![m^n]\!]$ and hence $\mathfrak{A} \oplus \mathfrak{B} \subseteq [\![(2m)^n]\!]$ proves the claim

For Ω^n the proof is even easier, since for all $n < \omega$ we have that $[\![\omega^n]\!] \oplus [\![\omega^n]\!] = [\![\omega^n]\!]$. □

Lemma 5.4. *If a class $\mathcal{C}$ contains at least one infinite tree, then $\Omega \sqsubseteq \mathcal{C}$.*

Proof. Let $\mathfrak{C} \in \mathcal{C}$ be infinite. Since the root of $\mathfrak{C}$ has infinitely many successors we can embed $[\![\omega]\!]$ into $\mathfrak{C}$ by mapping its root to the root of $\mathfrak{C}$. □

Corollary 5.5. The $\sqsubseteq$-minimal class of infinite structures is the class Ω.

Proof. The claim follows directly from Lemma 5.4. $\square$

Lemma 5.6. *For $n < \omega$ we have $\Phi^{n+1} \not\sqsubseteq \Omega^n$.*

Proof. Assume $\Phi^{n+1} \sqsubseteq \Omega^n$. Then there exists $k < \omega$ such that $\Phi^{n+1} \leq_k \Omega^n$, hence for all $m < \omega$ we have $[\![m^{n+1}]\!] \hookrightarrow_k [\![\omega^n]\!]$. For $m = k$, we find an $l < \omega$ such that $[\![k^{n+1}]\!] \hookrightarrow_k [\![l^n]\!]$. We construct a sequence $v_0, \dots, v_{n+1}$ of vertices of $[\![k^{n+1}]\!]$ such that v_i is mapped to a vertex of level at least i. For $i = n + 1$, this leads to the desired contradiction.

We start with $v_0 = \bullet$. For the inductive step, suppose that we have already defined $v_0, \dots, v_i$. Let w_i be the image of v_i and let $u_0, \dots, u_{k-1}$ be the immediate successors of v_i. By assumption $|w_i| \geq i$. There are at most $k - 1$ successors u_j that are mapped to w_i. Hence, there is some $j < k$ such that u_j is a proper successor of v_i. In particular, $|u_j| \geq i + 1$. We set $v_{i+1} := u_j$. $\square$

Proposition 5.7. $\Phi^n \lhd \Phi^{n+1}$.

Proof. We have $\Phi^n \sqsubseteq \Phi^{n+1}$ from Lemma 5.2. The strictness follows directly by Lemma 5.6 which implies $\Phi^{n+1} \not\sqsubseteq \Phi^n$.

Assume there exists a class $\mathcal{C}$ with $\Phi^n \sqsubseteq \mathcal{C} \sqsubseteq \Phi^{n+1}$. If $\mathcal{C}$ contains an infinite element, we would have by Lemma 5.4 that $\Omega \sqsubseteq \Phi^{n+1}$ which is a contradiction. Hence all trees of $\mathcal{C}$ are finite and Theorem 2.17 gives that $\mathcal{C} \equiv \Phi^n$ or $\mathcal{C} \equiv \Phi^{n+1}$. $\square$

Proposition 5.8. *If $1 \leq n < \omega$ then $\Phi^n \lhd \Phi^n \cup \Omega$ and $\Phi^n \sqsubset \mathcal{C} \sqsubseteq \Omega^n$ for some class $\mathcal{C}$ implies $\Phi^n \cup \Omega \sqsubseteq \mathcal{C} \sqsubseteq \Omega^n$.*

Proof. If $\mathcal{C}$ contains only finite trees, then Theorem 2.17 implies $\Phi^{n+1} \sqsubseteq \mathcal{C}$ contradicting Lemma 5.6. Thus $\mathcal{C}$ contains an infinite element, so we have by Lemma 5.4 that $\Omega \sqsubseteq \mathcal{C}$ yielding the assertion. $\square$

Proposition 5.9. *For all $n < \omega$ it holds: $\Phi^n \cup \Omega \lhd \Phi^{n+1} \cup \Omega$.*

Proof. The $\sqsubseteq$-relation is implied by Proposition 5.7. Strictness follows by Lemma 5.6.

Let $\mathcal{C}$ be a class with $\Phi^n \cup \Omega \sqsubseteq \mathcal{C} \sqsubseteq \Phi^{n+1} \cup \Omega$. Suppose that $\mathcal{C} \leq_k \Phi^{n+1} \cup \Omega$ and define

$$\begin{aligned} \mathcal{C}_\infty &:= \{\mathfrak{A} \in \mathcal{C} \mid \mathfrak{A} \hookrightarrow_k [\![\omega]\!]\}, \\ \mathcal{C}_{\text{fin}} &:= \mathcal{C} \setminus \mathcal{C}_\infty. \end{aligned}$$

Then $\mathcal{C}_\infty \leq_k \Omega$ and $\mathcal{C}_{\text{fin}} \leq_k \Phi^{n+1}$. Furthermore, $\Omega \sqsubseteq \mathcal{C}$ implies that $\mathcal{C}$ contains an infinite tree. This tree must be in $\mathcal{C}_\infty$. Hence, $\mathcal{C}_\infty \neq \varnothing$ and $\Omega \sqsubseteq \mathcal{C}_\infty$. Furthermore $\mathcal{C}_{\text{fin}} \sqsubseteq \Phi^{n+1}$ implies that $\mathcal{C}_{\text{fin}}$ only contains finite trees. By Theorem 2.17 it follows that $\mathcal{C}_{\text{fin}} \sqsubseteq \Phi^n$ or $\mathcal{C}_{\text{fin}} \equiv \Phi^{n+1}$. Consequently, we have $\mathcal{C} = \mathcal{C}_\infty \cup \mathcal{C}_{\text{fin}} \sqsubseteq \Omega \cup \Phi^n$ or $\mathcal{C} = \mathcal{C}_\infty \cup \mathcal{C}_{\text{fin}} \equiv \Omega \cup \Phi^{n+1}$. $\square$

Theorem 5.10. *We get for all $n < \omega$ the following hierarchy where the edges denote the $\lhd$-relation:*

$$\begin{tikzcd}[column sep=small] & \Phi^{n+1} \cup \Omega \arrow[dl, dash, "(5.8)"'] \arrow[dr, dash, "(5.9)"] & \\ \Phi^{n+1} \arrow[dr, dash, "(5.7)"'] & & \Phi^{n} \cup \Omega \arrow[dl, dash, "(5.8)"] \\ & \Phi^{n} & \end{tikzcd}$$

Proof. The hierarchy follows directly from the propositions the labels of the edges are referring to. ☐

Note that for $n = 1$ we have $\Phi \cup \Omega \equiv \Omega$, which follows directly from Lemma 4.2.

Proposition 5.11. *For any number $n < \omega$ the following holds: $(\Omega \oplus \Phi^n) \cup \Phi^{n+1} \lhd \Omega \oplus \Phi^{n+1}$.*

Proof. We have $(\Omega \oplus \Phi^n) \cup \Phi^{n+1} \sqsubseteq \Omega \oplus \Phi^{n+1}$ since $\Phi^n \sqsubseteq \Phi^{n+1}$ implies $\Omega \oplus \Phi^n \sqsubseteq \Omega \oplus \Phi^{n+1}$ and $1 \sqsubseteq \Omega$ implies $\Phi^{n+1} = 1 \cdot \Phi^{n+1} \sqsubseteq \Omega \oplus \Phi^{n+1}$. For the strictness claim assume $\Omega \oplus \Phi^{n+1} \sqsubseteq (\Omega \oplus \Phi^n) \cup \Phi^{n+1}$. Hence, $\Phi^{n+1} \sqsubseteq \Omega \oplus \Phi^{n+1} \sqsubseteq \Omega \oplus \Phi^n \sqsubseteq \Omega^n$ in contradiction to Lemma 5.6.

Assume there is a class $\mathcal{C}$ with $(\Omega \oplus \Phi^n) \cup \Phi^{n+1} \sqsubseteq \mathcal{C} \sqsubseteq \Omega \oplus \Phi^{n+1}$. We divide $\mathcal{C}$ into its infinite and finite parts: $\mathcal{C} = \mathcal{C}_\infty \sqcup \mathcal{C}_{\text{fin}}$. Since $\mathcal{C}_\infty$ contains an infinite element we have by Corollary 5.4 that $\Omega \sqsubseteq \mathcal{C}_\infty$.

Suppose that $\mathcal{C}_{\text{fin}} \leq_l \Omega \oplus \Phi^{n+1}$. For all $\mathfrak{C}_0 \in \mathcal{C}_{\text{fin}}$ we have some $\mathfrak{A}_0 \in \Omega \oplus \Phi^{n+1}$ with $\psi_{\mathfrak{C}_0} : \mathfrak{C}_0 \hookrightarrow_l \mathfrak{A}_0$. Obviously $\psi_{\mathfrak{C}_0}(\mathfrak{C}_0)$ is finite and has height at most $n+1$. Thus there is a vertex in $\psi_{\mathfrak{C}_0}(\mathfrak{C}_0)$ having a maximal number of successors, say $m_{\mathfrak{C}_0} < \omega$ many. Hence we can l-embed $\mathfrak{C}_0$ into $[\![m^n_{\mathfrak{C}_0}]\!]$ and we get $\mathcal{C}_{\text{fin}} \sqsubseteq \Phi^{n+1}$.

Now choose $k < \omega$ such that for all $\mathfrak{C} \in \mathcal{C}_\infty$ there is a k-embedding $\varphi_{\mathfrak{C}} : \mathfrak{C} \hookrightarrow_k [\![\omega]\!] \oplus \mathfrak{A}$ for some $\mathfrak{A} \in \Phi^{n+1}$. Set

$$\mathcal{D} := \{\mathfrak{D}_{\mathfrak{C}} \mid \mathfrak{D}_{\mathfrak{C}} = \varphi_{\mathfrak{C}}^{-1}(\mathfrak{A}) \text{ for } \mathfrak{C} \in \mathcal{C}_\infty \text{ and } \varphi_{\mathfrak{C}} : \mathfrak{C} \hookrightarrow_k [\![\omega]\!] \oplus \mathfrak{A}\}.$$

Then by definition $\mathcal{D}$ contains only finite elements and we have $\mathcal{D} \leq_k \Phi^{n+1}$ and also $\mathcal{D} \leq_1 \mathcal{C}_\infty$. We distinguish two cases which is sufficient by Proposition 5.7:

$\mathcal{D} \equiv \Phi^{n+1}$: Then we have $\Phi^{n+1} \leq_m \mathcal{C}_\infty$, for some $m < \omega$. Since $[\![\omega]\!] \hookrightarrow_1 \mathfrak{C}$, for all $\mathfrak{C} \in \mathcal{C}_\infty$, it follows that $\Omega \oplus \Phi^{n+1} \leq_{m+1} \mathcal{C}$..

$\mathcal{D} \sqsubseteq \Phi^n$: Then $\mathcal{C}_\infty \sqsubseteq \Omega \oplus \Phi^n$ and further we have $\mathcal{C}_\infty \sqsubseteq (\Omega \oplus \Phi^n) \cup \Phi^{n+1}$. Together with $\mathcal{C}_{\text{fin}} \sqsubseteq \Phi^{n+1}$ we conclude that $\mathcal{C} \sqsubseteq (\Omega \oplus \Phi^n) \cup \Phi^{n+1}$.

☐

Proposition 5.12. $\Omega \oplus \Phi^2 \lhd \Omega : \Phi$.

Proof. First we show that $\Omega \oplus \Phi^2 \sqsubseteq \Omega : \Phi$. Let $[\![\omega]\!] \oplus \mathfrak{A} \in \Omega \oplus \Phi^2$. Then there exists some $m < \omega$ such that $\mathfrak{A} \sqsubseteq [\![m^2]\!]$. Since $[\![\omega]\!] \oplus [\![m^2]\!] \subset [\![\omega, m]\!]$ it follows that $\mathfrak{A} \hookrightarrow_1 [\![\omega]\!] \oplus [\![m^2]\!] \hookrightarrow_1 [\![\omega, m]\!]$.

For the strictness claim, assume $\Omega : \Phi \leq_n \Omega \oplus \Phi^2$. Then for all $m < \omega$ there exists $l < \omega$ such that $[\![\omega, m]\!] \hookrightarrow_n [\![\omega]\!] \oplus [\![l^2]\!]$. The tree $[\![\omega, m]\!]$ has infinitely many inner vertices, whereas $[\![\omega]\!] \oplus [\![l^2]\!]$ has only finitely, precisely l, inner vertices. If $m > n$ we have the following situation: All but finitely many inner vertices of $[\![\omega, m]\!]$ are mapped to vertices of level 1 by an n-embedding. In particular there is an inner vertex v that is mapped to a leaf x of level 1. Then all successors of v are also mapped to x. Hence, the preimage of x contains at least $m + 1 > n$ vertices. A contradiction.

Finally, suppose that there is a class $\Omega \oplus \Phi^2 \sqsubseteq \mathcal{C} \sqsubseteq \Omega : \Phi$. For every $\mathfrak{C} \in \mathcal{C}$, we fix an n-embedding $h_{\mathfrak{C}} : \mathfrak{C} \hookrightarrow_n \mathfrak{T}_{\mathfrak{C}}$ into some tree $\mathfrak{T}_{\mathfrak{C}} \in \Omega : \Phi$. We can choose $h_{\mathfrak{C}}$ such that it maps the root of $\mathfrak{C}$ to the root of $\mathfrak{T}_{\mathfrak{C}}$. Furthermore, we can modify $h_{\mathfrak{C}}$ (by rearranging the vertices that are mapped to vertices of level 1) such that it satisfies the following conditions:

(a) There are no vertices $x, y \in C$ with $x \nleq y$ and $y \nleq x$ such that $h_{\mathfrak{C}}(x) = h_{\mathfrak{C}}(y)$ and $|h_{\mathfrak{C}}(x)| = 1$.

(b) For every vertex $x \in C$ with $|h_{\mathfrak{C}}(x)| = 2$, there is some $y < x$ with $|h_{\mathfrak{C}}(y)| = 1$.

Let $\mathfrak{S}_{\mathfrak{C}} \subseteq \mathfrak{T}_{\mathfrak{C}}$ be the image of $h_{\mathfrak{C}}$. We distinguish two cases:

1. First, suppose that there is some constant $k < \omega$, such that every tree $\mathfrak{S}_{\mathfrak{C}}$ has only finitely many vertices on level 1 with more than k successors. Then each $\mathfrak{S}_{\mathfrak{C}}$ can be $(k+1)$-embedded into some tree from $\Omega \oplus \Phi^2$. Hence, $\mathcal{C} \leq_n \{\mathfrak{S}_{\mathfrak{C}} \mid \mathfrak{C} \in \mathcal{C}\} \leq_{k+1} \Omega \oplus \Phi^2$ implies $\mathcal{C} \leq_{n(k+1)} \Omega \oplus \Phi^2$.

2. It remains to consider the case that, for every $k < \omega$, there is some tree $\mathfrak{S}_{\mathfrak{C}}$ with infinitely many vertices on level 1 with more than k successors. Let $\mathfrak{T}_k = [\![\omega, k]\!] \in \Omega : \Phi$. We select $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{S}_{\mathfrak{C}}$ has infinitely many vertices $v_0, v_1, \dots$ on level 1 with at least k successors. For each $n < \omega$, choose distinct successors $u_{n1}, \dots, u_{nk}$ of v_n. Let $x_n := \min\{h_{\mathfrak{C}}^{-1}(v_n)\}$ and choose vertices $y_{nk} \in h_{\mathfrak{C}}^{-1}(u_{nk})$. By the choice of $h_{\mathfrak{C}}$, we have $x_n \leq y_{nk}$ for all k. Hence, the subtree of $\mathfrak{C}$ consisting of the root and the vertices $x_n, y_{n1}, \dots, y_{nk}$, for all $n < \omega$, is isomorphic to $\mathfrak{T}_k$. Therefore, $\mathfrak{T}_k \hookrightarrow_1 \mathfrak{C}$. It follows that $\Omega : \Phi \leq_1 \mathcal{C}$.

$\square$

Before we prove the next $\triangleleft$-relation, we remind ourselves that $\Phi : \Omega = \Phi \cdot \Omega$, since Ω contains only $[\![\omega]\!]$. In such a case, we will always use the :-relation, i.e. $\Phi : \Omega$ in this example.

Lemma 5.13. *Let $\mathcal{C}$ be a class of trees. If, for every $k < \omega$ there exists $\mathfrak{C}_k \in \mathcal{C}$ with k distinct vertices $v_1, \dots, v_k$ having infinitely many successors, then $\Phi : \Omega \sqsubseteq \mathcal{C}$.*

Proof. For each $m < k$ it holds that $[\![m, \omega]\!] \hookrightarrow_1 \mathfrak{C}_k$ and thus we have $\Phi : \Omega \sqsubseteq \mathcal{C}$. $\square$

Lemma 5.14. *Let $\mathcal{C}$ be a class of trees with $\mathcal{C} \sqsubseteq \Phi : \Omega$. If there is some $k < \omega$ such that every tree $\mathfrak{C} \in \mathcal{C}$ has at most k vertices with infinitely many successors then $\mathcal{C} \sqsubseteq \Phi^2 \oplus \Omega$.*

Proof. Suppose that $\mathcal{C} \leq_n \Phi : \Omega$ and let $\mathfrak{C} \in \mathcal{C}$. There is some $m < \omega$ such that $\mathfrak{C} \hookrightarrow_n [\![m, \omega]\!]$. Let $\mathfrak{S} \subseteq [\![m, \omega]\!]$ be the image of this n-embedding. By assumption on $\mathfrak{C}$, $\mathfrak{S}$ contains at most k vertices on level 1 with infinitely many successors. Let $\mathfrak{S}_\infty \subseteq \mathfrak{S}$ be the set of these vertices together with their successors and the root. Then $\mathfrak{S}_\infty \hookrightarrow_{k+1} [\![\omega]\!]$. Let $\mathfrak{S}_{\text{fin}} \subseteq \mathfrak{S}$ be the tree consisting of all vertices of $\mathfrak{S} \setminus \mathfrak{S}_\infty$ and the root. Then $\mathfrak{S}_{\text{fin}}$ is a finite tree of height at most 2. Hence, there is some $l < \omega$ such that $\mathfrak{S}_{\text{fin}} \subseteq [\![l^2]\!]$. It follows that

$$\mathfrak{C} \hookrightarrow_n \mathfrak{S} \hookrightarrow_{k+1} [\![\omega]\!] \oplus [\![l^2]\!].$$

$\square$

Lemma 5.15. $\Phi : \Omega \not\sqsubseteq \Omega \cdot \Phi$.

Proof. Suppose that $\Phi : \Omega \leq_n \Omega \cdot \Phi$. Then there exist numbers $m_i < \omega$, for $i < \omega$, such that $[\![n, \omega]\!] \hookrightarrow_n [\![\omega]\!] \leftarrow ([\![m_i]\!])_{i<\omega}$. There is at least one vertex v of level 1 that is not mapped to the root by this n-embedding. Let u be its image. We obtain a contradiction since v has infinitely many successors, while u has only finitely many. □

Proposition 5.16. $\Omega \oplus \Phi^2 \lhd \Phi : \Omega$.

Proof. We first show that $\Omega \oplus \Phi^2 \sqsubseteq \Phi : \Omega$ holds. Let $[\![\omega]\!] \oplus \mathfrak{A} \in \Omega \oplus \Phi^2$, then there exists $m < \omega$ such that $\mathfrak{A} \subseteq [\![m^2]\!]$. Now we observe that $[\![\omega]\!] \oplus [\![m^2]\!] \hookrightarrow_1 [\![m+1, \omega]\!]$. Hence $[\![\omega]\!] \oplus \mathfrak{A} \hookrightarrow_1 [\![m+1, \omega]\!]$. Strictness follows directly from Lemma 5.15 since $\Omega \oplus \Phi^2 \sqsubseteq \Omega \cdot \Phi$.

Now we assume that there is some class $\mathcal{C}$ with $\Omega \oplus \Phi^2 \sqsubseteq \mathcal{C} \sqsubseteq \Phi : \Omega$. Suppose that $\mathcal{C} \not\equiv \Phi : \Omega$. By Lemma 5.13 it follows that there is some $k < \omega$ such that $\mathcal{C}$ does not contain a tree with k vertices each having infinitely many successors. Hence, it follows by Lemma 5.14 that $\mathcal{C} \sqsubseteq \Omega \oplus \Phi^2$. □

Proposition 5.17. $\Omega : \Phi \lhd \Omega : \Phi \cup \Phi : \Omega$.

Proof. The $\sqsubseteq$-relation holds trivially by Lemma 4.2. For strictness assume that $\Omega : \Phi \cup \Phi : \Omega \sqsubseteq \Omega : \Phi$. Then, $\Phi : \Omega \sqsubseteq \Omega : \Phi \sqsubseteq \Omega \cdot \Phi$. This contradicts Lemma 5.15.

Now let $\mathcal{C}$ be a class with $\Omega : \Phi \sqsubseteq \mathcal{C} \sqsubseteq \Omega : \Phi \cup \Phi : \Omega$. Distinguish two cases:

1. For all $k < \omega$ there is a tree $\mathfrak{C}_k \in \mathcal{C}$ with at least k distinct vertices having infinitely many successors. By Lemma 5.13 we have $\Phi : \Omega \sqsubseteq \mathcal{C}$. Since $\Omega : \Phi \sqsubseteq \mathcal{C}$ it follows that $\Phi : \Omega \cup \Omega : \Phi \sqsubseteq \mathcal{C}$.

2. There is a number $k < \omega$ such that all $\mathfrak{C} \in \mathcal{C}$ have at most k vertices with infinitely many successors. Suppose that $C \leq_n \Omega : \Phi \cup \Phi : \Omega$. Let

$$\mathcal{C}_0 := \{\mathfrak{A} \in \mathcal{C} \mid \mathfrak{A} \not\hookrightarrow_n \mathfrak{B} \text{ for all } \mathfrak{B} \in \Omega : \Phi\}.$$

Then $\mathcal{C}_0 \leq_n \Phi : \Omega$ and $\mathcal{C} \setminus \mathcal{C}_0 \leq_n \Omega : \Phi$. Then by Lemma 5.14 we see that $\mathcal{C}_0 \sqsubseteq \Phi^2 \oplus \Omega \sqsubseteq \Omega : \Phi$. Hence, $\mathcal{C} \sqsubseteq \Omega : \Phi$.

□

Proposition 5.18. $\Phi : \Omega \lhd \Omega : \Phi \cup \Phi : \Omega$.

Proof. The $\sqsubseteq$-relation is obvious by Lemma 4.2. For its strictness, suppose $\Omega : \Phi \leq_n \Phi : \Omega$. Then for each $m < \omega$ there exists $k < \omega$ such that $[\![\omega, m]\!] \hookrightarrow_n [\![k, \omega]\!]$. Since $[\![k, \omega]\!]$ has only finitely many level 1 vertices, infinitely many vertices of $[\![\omega, m]\!]$ in level 1 have to be mapped to level 2 vertices. Their m successors have to be mapped to the same vertex, for $m > n$ this is a contradiction.

Let $\mathcal{C}$ be a class with $\Phi : \Omega \sqsubseteq \mathcal{C} \sqsubseteq \Omega : \Phi \cup \Phi : \Omega$. Distinguish two cases:

1. For all $k < \omega$ there is a tree $\mathfrak{C}_k \in \mathcal{C}$ with infinitely many distinct vertices having at least k successors. Then we have for $m < k$ that $[\![\omega, m]\!] \hookrightarrow_1 \mathfrak{C}_k$, and thus $\Omega : \Phi \sqsubseteq \mathcal{C}$. Since $\Phi : \Omega \sqsubseteq \mathcal{C}$ it follows that $\Phi : \Omega \cup \Omega : \Phi \sqsubseteq \mathcal{C}$.

2. There is a number $k < \omega$ such that all $\mathfrak{C} \in \mathcal{C}$ have only finitely many vertices, say $n_{\mathfrak{C}} < \omega$, with more than k successors. We can 1-embed the subtrees of those vertices into $[\![n_{\mathfrak{C}}, \omega]\!]$. All other vertices have less than k successors, thus their subtrees can be $(k+1)$-embedded into $[\![\omega]\!]$. Hence, $\mathfrak{C} \hookrightarrow_{k+2} [\![n_{\mathfrak{C}}, \omega]\!]$ and we have $\mathcal{C} \sqsubseteq \Phi : \Omega$.

□

Definition 5.19. A tree $\mathfrak{A}$ contains a *strictly increasing sequence* $(v_n)_{n<\omega}$, if there are distinct vertices $v_1, v_2, \ldots$ such that for all $i < \omega$ the vertex v_i has at least i successors.

Lemma 5.20. *Let $\mathcal{C}$ be a class of trees. If there is some $\mathfrak{C} \in \mathcal{C}$ containing a strictly increasing sequence $(v_n)_{n<\omega}$ then $\Omega \cdot \Phi \leq_1 \mathcal{C}$.*

Proof. Let $\mathfrak{A} \in \Omega \cdot \Phi$ and enumerate its level 1 vertices $w_1, w_2, \ldots$. Let k_i be the number of successors of w_i. For each i, let l_i be the least index such that $l_i \geq k_i$ and $l_i > l_{i-1}$. We can 1-embed the subtree rooted at w_i into the subtree rooted at v_{l_i}. In this way we obtain $\mathfrak{A} \hookrightarrow_1 \mathfrak{C}$ implying $\Omega \cdot \Phi \leq_1 \mathcal{C}$. □

Corollary 5.21. Let $\mathcal{C}$ be a class of trees. If for all $\mathfrak{C} \in \mathcal{C}$ all level 1 vertices have finitely many successors, we have $\mathcal{C} \leq_1 \Omega \cdot \Phi$.

Proof. Let $\mathfrak{C} \in \mathcal{C}$ and enumerate its level 1 vertices $w_1, w_2, \ldots$. Let k_i be the number of successors of w_i. Then $\mathfrak{C} \hookrightarrow_1 [\![\omega]\!] \leftarrow ([\![k_i]\!])_{i<\omega}$. □

Lemma 5.22. *For all $k, n < \omega$,*

$$[\![\omega]\!] \leftarrow ([\![m]\!])_{m<\omega} \not\hookrightarrow_n [\![\omega, k]\!] \oplus [\![k, \omega]\!].$$

Proof. Suppose that there exists an n-embedding $[\![\omega]\!] \leftarrow ([\![m]\!])_{m<\omega} \hookrightarrow_n [\![\omega, k]\!] \oplus [\![k, \omega]\!]$. Since $[\![\omega, k]\!] \oplus [\![k, \omega]\!]$ has infinitely many level 1 vertices, we can assume that level 1 vertices are mapped to an equal level. Thus at most kn level 1 vertices are mapped into a subtree of $[\![k, \omega]\!]$. The remaining level 1 vertices are mapped to a vertex of $[\![\omega, k]\!]$. Hence, all but finitely many of the subtrees $[\![m]\!]$ are mapped to copies of $[\![k]\!]$. This leads to a contradiction for $m > nk$. □

Lemma 5.23. *Let $\mathcal{C} \sqsubseteq \Omega \cdot \Phi$ be a class such that, for every $\mathfrak{C} \in \mathcal{C}$, there is a finite number $n_{\mathfrak{C}} < \omega$ such that every vertex of $\mathfrak{C}$ has either infinitely many successors or at most $n_{\mathfrak{C}}$ successors. Then $\mathcal{C} \sqsubseteq \Omega : \Phi$.*

Proof. Let $\mathfrak{C} \in \mathcal{C}$ and $h : \mathfrak{C} \hookrightarrow_k [\![\omega]\!] \leftarrow ([\![m_i]\!])_{i<\omega}$ be a k-embedding. We may assume that h maps vertices of level 1 to vertices of the same level, since there are infinitely many vertices at this level. Let $\mathfrak{C}_{\text{fin}} \subseteq \mathfrak{C}$ be the set of all vertices of $\mathfrak{C}$ with at most $n_{\mathfrak{C}}$ successors and let $\mathfrak{C}_\infty$ be the set of all vertices with infinitely many successors. Then $\mathfrak{C} = \mathfrak{C}_{\text{fin}} \cup \mathfrak{C}_\infty$. Note that $h(\mathfrak{C}_{\text{fin}})$ is isomorphic to a subforest of $[\![\omega, n_{\mathfrak{C}}]\!]$, while $h(\mathfrak{C}_\infty) \subseteq [\![\omega]\!] \leftarrow ([\![m_i]\!])_{i<\omega}$ implies that h maps every vertex of $\mathfrak{C}_\infty$ to the root. It follows that $h(\mathfrak{C})$ is isomorphic to a subtree of $[\![\omega, n_{\mathfrak{C}}]\!]$. Hence, $\mathfrak{C} \hookrightarrow_k [\![\omega, n_{\mathfrak{C}}]\!]$. □

Proposition 5.24. $\Omega : \Phi \lhd \Omega \cdot \Phi$.

Proof. Lemma 4.2 implies $\Omega : \Phi \sqsubseteq \Omega \cdot \Phi$. Strictness is implied by Lemma 5.22.

Let $\mathcal{C}$ be a class with $\Omega : \Phi \sqsubseteq \mathcal{C} \sqsubseteq \Omega \cdot \Phi$. Distinguish the following cases:

1. There is some $\mathfrak{C} \in \mathcal{C}$ containing a strictly increasing sequence $(v_n)_{n<\omega}$. Then Lemma 5.20 implies $\Omega \cdot \Phi \leq_1 \mathcal{C}$.

2. For all $\mathfrak{C} \in \mathcal{C}$ there is some $n_{\mathfrak{C}} < \omega$ such that every vertex has either less than $n_{\mathfrak{C}}$ or infinitely many successors. Lemma 5.23 implies that $\mathcal{C} \sqsubseteq \Omega : \Phi$.

□

Proposition 5.25. $\Omega \cdot \Phi \lhd \Omega \cdot \Phi \cup \Phi : \Omega$.

Proof. Since $\Omega \cdot \Phi \subset \Omega \cdot \Phi \cup \Phi : \Omega$, the $\sqsubseteq$-relation is trivially true. For its strictness, suppose $\Omega \cdot \Phi \cup \Phi : \Omega \sqsubseteq \Omega \cdot \Phi$ implying $\Phi : \Omega \sqsubseteq \Omega \cdot \Phi$, which is a contradiction to Lemma 5.15.

Now let $\mathcal{C}$ be a class such that $\Omega \cdot \Phi \sqsubseteq \mathcal{C} \sqsubseteq \Omega \cdot \Phi \cup \Phi : \Omega$. Distinguish two cases:

1. For all $k < \omega$ there is a tree $\mathfrak{C}_k \in \mathcal{C}$ with at least k distinct vertices having infinitely many successors. By Lemma 5.13 we have $\Phi : \Omega \sqsubseteq \mathcal{C}$. Since $\Omega \cdot \Phi \sqsubseteq \mathcal{C}$ it follows that $\Omega \cdot \Phi \cup \Phi : \Omega \sqsubseteq \mathcal{C}$.

2. There is a number $k < \omega$ such that all $\mathfrak{C} \in \mathcal{C}$ have at most k vertices with infinitely many successors. Suppose that $C \leq_n \Omega \cdot \Phi \cup \Phi : \Omega$. Let

$$\mathcal{C}_0 := \{\mathfrak{A} \in \mathcal{C} \mid \mathfrak{A} \not\hookrightarrow_n \mathfrak{B} \text{ for all } \mathfrak{B} \in \Omega \cdot \Phi\}.$$

   Then $\mathcal{C}_0 \leq_n \Phi : \Omega$ and $\mathcal{C} \setminus \mathcal{C}_0 \leq_n \Omega \cdot \Phi$.

   By Lemma 5.14 we have $\mathcal{C}_0 \sqsubseteq \Omega \oplus \Phi^2 \sqsubseteq \Omega \cdot \Phi$. Hence, $\mathcal{C} \sqsubseteq \Omega \cdot \Phi$.

□

Proposition 5.26. $\Omega : \Phi \cup \Phi : \Omega \lhd \Omega \cdot \Phi \cup \Phi : \Omega$.

Proof. The $\sqsubseteq$-relation follows directly from Proposition 5.24. For the strictness, assume $\Omega \cdot \Phi \cup \Phi : \Omega \leq_n \Omega : \Phi \cup \Phi : \Omega$ implying directly $\Omega \cdot \Phi \leq_n \Omega : \Phi \cup \Phi : \Omega$. Let $\mathfrak{A} := [\![\omega]\!] \leftarrow ([\![m]\!])_{m<\omega}$. From Lemma 5.22 we know that for all $k, n < \omega$ we have $\mathfrak{A} \not\hookrightarrow_n [\![\omega, k]\!]$. Thus there is a number $m < \omega$ such that $\mathfrak{A} \hookrightarrow_n [\![m, \omega]\!]$. Since $\mathfrak{A}$ has infinitely many level 1 vertices, almost all of them have to be mapped to the leaves of $[\![m, \omega]\!]$. Because the number of successors a vertex of level 1 has is not bounded in $\mathfrak{A}$, we have a contradiction.

Let $\mathcal{C}$ be a class with $\Omega : \Phi \cup \Phi : \Omega \sqsubseteq \mathcal{C} \leq_n \Omega \cdot \Phi \cup \Phi : \Omega$. Distinguish two cases:

1. $\mathcal{C}$ contains a tree with a strictly increasing sequence. By Lemma 5.20 we have $\Omega \cdot \Phi \sqsubseteq \mathcal{C}$. Hence $\Omega : \Phi \cup \Omega \cdot \Phi \sqsubseteq \mathcal{C}$.

2. $\mathcal{C}$ does not contain a tree with a strictly increasing sequence. Hence, for each $\mathfrak{C} \in \mathcal{C}$ there is a constant $n_{\mathfrak{C}} < \omega$ such that all vertices have either less than $n_{\mathfrak{C}}$ or infinitely many successors. We split the class $\mathcal{C}$ as follows:

$$\mathcal{C}_0 := \{\mathfrak{C} \in \mathcal{C} \mid \mathfrak{C} \hookrightarrow_n [\![m, \omega]\!] \text{ for } m < \omega\},$$
$$\mathcal{C}_1 := \mathcal{C} \setminus \mathcal{C}_0.$$

   Then $\mathcal{C}_0 \sqsubseteq \Phi : \Omega$ and $\mathcal{C}_1 \sqsubseteq \Omega \cdot \Phi$. Since Lemma 5.23 implies that $\mathcal{C}_1 \sqsubseteq \Omega : \Phi$, it follows that $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \sqsubseteq \Phi : \Omega \cup \Omega : \Phi$.

$\square$

Lemma 5.27. *Let $\mathcal{C}$ be a class with $\mathcal{C} \leq_n \Omega : \Phi \oplus \Phi : \Omega$. If there is a constant $k < \omega$ such that every $\mathfrak{C} \in \mathcal{C}$ has at most one of the following properties, then $\mathcal{C} \sqsubseteq \Omega : \Phi \cup \Phi : \Omega$ holds:*

(a) *There are infinitely many vertices having more than k successors.*

(b) *There are more than k vertices having infinitely many successors.*

Proof. First we note, that for each $\mathfrak{C} \in \mathcal{C}$ there is some $n_{\mathfrak{C}} < \omega$ such that $n_{\mathfrak{C}}$ vertices have infinitely many successors. Otherwise both (a) and (b) would be fulfilled which is not permitted. Fix an n-embedding $h : \mathfrak{C} \hookrightarrow_n [\![\omega, m]\!] \oplus [\![m, \omega]\!]$.

Suppose that $\mathfrak{C}$ does not satisfy (b). Since there are at most k vertices with infinitely many successors, we may assume that $n \geq k$ and that h maps these vertices to the root of $[\![\omega, m]\!]$. Now we show that there is a constant $p_{\mathfrak{C}} < \omega$ such that all other vertices have at most $p_{\mathfrak{C}}$ successors. This implies that $\mathfrak{C} \hookrightarrow_n [\![\omega, m + p_{\mathfrak{C}}]\!]$. Assume the contrary, i.e. for each $l < \omega$ there is a vertex having at least l successors, but only finitely many. Thus some strictly increasing sequence $(v_i)_{i<\omega}$ would be included in $\mathfrak{C}$. By Lemma 5.22 only finitely many vertices v_i can be mapped by h into $[\![\omega, m]\!]$. Furthermore, h can only map v_i to a vertex of $[\![m, \omega]\!]$ of level 2 for $i \leq n$. Hence, all v_i with $i > n$ have to be mapped to a vertex of level at most 1. Since there are only finitely many such vertices, we obtain a contradiction..

Finally, suppose that $\mathfrak{C}$ does not satisfy (a), say $\mathfrak{C}$ has only $n_{\mathfrak{C}} < \omega$ vertices with more than k successors. We split $\mathfrak{C}$ as follows: $\mathfrak{C}_0$ contains all vertices of $\mathfrak{C}$ having more than k successors. Then we have that $h(\mathfrak{C}_0) \hookrightarrow_1 [\![n_{\mathfrak{C}}, \omega]\!]$ and thus $\mathfrak{C}_0 \hookrightarrow_n [\![n_{\mathfrak{C}}, \omega]\!]$. Let $\mathfrak{C}_1$ contain all remaining vertices of $\mathfrak{C}$. Then it holds $h(\mathfrak{C}_1) \hookrightarrow_{k+1} [\![\omega]\!]$ which implies $\mathfrak{C}_1 \hookrightarrow_{k+1} [\![1, \omega]\!]$. Altogether we get that $\mathfrak{C} \hookrightarrow_{n+k+1} [\![n_{\mathfrak{C}} + 1, \omega]\!]$.

If we combine cases (a) and (b) we conclude that $\mathcal{C} \sqsubseteq \Omega : \Phi \cup \Phi : \Omega$. $\square$

Proposition 5.28. $\Omega : \Phi \cup \Phi : \Omega \lhd \Omega : \Phi \oplus \Phi : \Omega$.

Proof. We have $\Omega : \Phi \cup \Phi : \Omega \sqsubseteq \Omega : \Phi \oplus \Phi : \Omega$ by Lemma 4.8. Propositions 5.17 and 5.18 imply $\Omega : \Phi \not\sqsubseteq \Phi : \Omega$ and $\Phi : \Omega \not\sqsubseteq \Omega : \Phi$. Hence, strictness follows by Lemma 4.10.

Let $\mathcal{C}$ be a class with $\Omega : \Phi \cup \Phi : \Omega \leq_n \mathcal{C} \leq_n \Omega : \Phi \oplus \Phi : \Omega$. Distinguish two cases:

1. For all $k < \omega$ there is some $\mathfrak{C}_k \in \mathcal{C}$ with at least k vertices having infinitely many successors and infinitely many vertices having at least k successors. Then for $m < k$ we have $[\![\omega, m]\!] \oplus [\![m, \omega]\!] \hookrightarrow_1 \mathfrak{C}_k$ and thus $\Omega : \Phi \oplus \Phi : \Omega \leq_1 \mathcal{C}$.

2. There is a constant $k < \omega$ such that every $\mathfrak{C} \in \mathcal{C}$ has at most one of the following properties:
   (a) There are infinitely many vertices having more than k successors.
   (b) There are more than k vertices having infinitely many successors.

   Lemma 5.27 implies directly $\mathcal{C} \sqsubseteq \Omega : \Phi \cup \Phi : \Omega$.

$\square$

Lemma 5.29. $\Phi : \Omega \not\sqsubseteq \Omega \cdot \Phi$

Proof. Assume $\Phi : \Omega \leq_n \Omega \cdot \Phi$. Then we have $[\![n+1, \omega]\!] \hookrightarrow_n \mathfrak{A}$ for some $\mathfrak{A} \in \Omega \cdot \Phi$. All members of $\Omega \cdot \Phi$ have the property, that only the root has infinitely many successors. Since $[\![n+1, \omega]\!]$ has $n+1$ level 1 vertices having infinitely many successors, at least one of them, call it v_0, cannot be mapped to the root, thus it is mapped to a level 1 or 2 vertex. In both cases infinitely many successors of v_0 have to be mapped to the same vertex. A contradiction. □

Proposition 5.30. $\Omega \cdot \Phi \cup \Phi : \Omega \lhd \Omega \cdot \Phi \cup (\Omega : \Phi \oplus \Phi : \Omega)$.

Proof. The $\sqsubseteq$-relation follows immediately by Lemma 4.2. $\Omega : \Phi \oplus \Phi : \Omega \not\sqsubseteq \Phi : \Omega$ is implied by Proposition 5.28 and $\Omega : \Phi \oplus \Phi : \Omega \not\sqsubseteq \Omega \cdot \Phi$ is implied by Lemma 5.29.

Let $\mathcal{C}$ be a class with $\Omega \cdot \Phi \cup \Phi : \Omega \leq_n \mathcal{C} \leq_n \Omega \cdot \Phi \cup (\Omega : \Phi \oplus \Phi : \Omega)$. We divide $\mathcal{C}$ into:

$$\begin{aligned} \mathcal{C}_0 &:= \{\mathfrak{C} \in \mathcal{C} \mid \mathfrak{C} \hookrightarrow_n \mathfrak{A} \text{ for some } \mathfrak{A} \in \Omega \cdot \Phi\}, \\ \mathcal{C}_1 &:= \mathcal{C} \setminus \mathcal{C}_0. \end{aligned}$$

Then $\mathcal{C}_0 \sqsubseteq \Omega \cdot \Phi$ and $\mathcal{C}_1 \sqsubseteq \Omega : \Phi \oplus \Phi : \Omega$. Distinguish the following cases:

1. There is a constant $k < \omega$ such that every $\mathfrak{C} \in \mathcal{C}_1$ has at most one of the following properties:
   (a) There are infinitely many vertices having more than k successors.
   (b) There are more than k vertices having infinitely many successors.

   Lemma 5.27 implies directly $\mathcal{C}_1 \sqsubseteq \Omega \cdot \Phi \cup \Phi : \Omega$ and thus we have $\mathcal{C} \sqsubseteq \Omega \cdot \Phi \cup \Phi : \Omega$.

2. For all $k < \omega$ there is some $\mathfrak{C}_k \in \mathcal{C}_1$ with at least k vertices having infinitely many successors and infinitely many vertices having at least k successors. Then for $m < k$ we have $[\![\omega, m]\!] \oplus [\![m, \omega]\!] \hookrightarrow_2 \mathfrak{C}_k$ and thus $\Omega : \Phi \oplus \Phi : \Omega \leq_2 \mathcal{C}_1$. Since by hypothesis $\Omega \cdot \Phi \sqsubseteq \mathcal{C}$, we have $\Omega \cdot \Phi \cup (\Omega : \Phi \oplus \Phi : \Omega) \sqsubseteq \mathcal{C}$.

□

Proposition 5.31. $\Omega \cdot \Phi \cup (\Omega : \Phi \oplus \Phi : \Omega) \lhd \Omega \cdot \Phi \oplus \Phi : \Omega$.

Proof. The $\sqsubseteq$-relation follows immediately by Lemma 4.2. Strictness follows by Lemma 4.10.

Now let $\mathcal{C}$ be a class with $\Omega \cdot \Phi \cup (\Omega : \Phi \oplus \Phi : \Omega) \sqsubseteq \mathcal{C} \sqsubseteq \Omega \cdot \Phi \oplus \Phi : \Omega$. Distinguish two cases:

1. For each number $k < \omega$ there is $\mathfrak{C}_k \in \mathcal{C}$ with at least k vertices having infinitely many successors and containing a strictly increasing sequence $(v_n)_{n<\omega}$. Let $\mathfrak{A} \in \Omega \cdot \Phi$ and $m < \omega$. $\mathfrak{A} \oplus [\![m, \omega]\!]$ is 2-embeddable into $\mathfrak{C}_k$ for $k > m$ since there are m vertices with infinitely many successors in $\mathfrak{C}_k$ and any sequence of vertices with finitely many successors can also be 1-embedded as seen in Lemma 5.20. Thus we have $\Omega \cdot \Phi \oplus \Phi : \Omega \leq_2 \mathcal{C}$.

2. There is some $k < \omega$ such that all elements $\mathfrak{C} \in \mathcal{C}$ that contain a strictly increasing sequence have at most k vertices with infinitely many successors. Let $\mathfrak{C} \in \mathcal{C}$ and $h : \mathfrak{C} \hookrightarrow_n \mathfrak{A} \oplus [\![m, \omega]\!]$ for some $m < \omega, \mathfrak{A} \in \Omega \cdot \Phi$ and $\mathfrak{S} := h(\mathfrak{C})$. First, assume that $\mathfrak{C}$ contains a strictly increasing sequence. By Corollary 5.21 there is some $\mathfrak{B} \in \Omega \cdot \Phi$ such that $\mathfrak{S} \cap \mathfrak{A} \hookrightarrow_1 \mathfrak{B}$. Furthermore, we have $\mathfrak{S} \setminus \mathfrak{A} \hookrightarrow_k [\![\omega]\!]$. Therefore, $\mathfrak{S} \hookrightarrow_{k+1} \mathfrak{B}$, which implies $\mathfrak{C} \hookrightarrow_{n(k+1)} \mathfrak{B}$.

   It remains to consider the case that $\mathfrak{C}$ does not contain a strictly increasing sequence. Let $\mathfrak{C}_\infty \subseteq \mathfrak{C}$ consist of those vertices mapped by h to $[\![m, \omega]\!]$ and let $\mathfrak{C}_{\text{fin}}$ consist of those vertices that are mapped by h to $\mathfrak{A}$. Then $\mathfrak{C} = \mathfrak{C}_{\text{fin}} \oplus \mathfrak{C}_\infty$. Furthermore, $\mathfrak{C}_{\text{fin}}$ does not contain an increasing sequence. Hence, there is a number $l < \omega$ such that every vertex of $\mathfrak{C}_{\text{fin}}$ with finitely many successors has at most l successors. It follows that $h(\mathfrak{C}_{\text{fin}})$ is isomorphic to a subtree of $[\![\omega, l]\!]$. Therefore, $h(\mathfrak{C}) \hookrightarrow_1 [\![\omega, l]\!] \oplus [\![m, \omega]\!]$, which implies that $\mathfrak{C} \hookrightarrow_n [\![\omega, l]\!] \oplus [\![m, \omega]\!]$.

□

Proposition 5.32. $\Omega : \Phi \oplus \Phi : \Omega \lhd \Omega \cdot \Phi \cup (\Omega : \Phi \oplus \Phi : \Omega)$.

Proof. The $\sqsubseteq$-relation follows from Lemma 4.2. Strictness is implied by Lemma 5.22.

Now let $\mathcal{C}$ be a class with $\Omega : \Phi \oplus \Phi : \Omega \leq_n \mathcal{C} \leq_n \Omega \cdot \Phi \cup (\Omega : \Phi \oplus \Phi : \Omega)$. We split $\mathcal{C}$ into:

$$\mathcal{C}_0 := \{\mathfrak{C} \in \mathcal{C} \mid \mathfrak{C} \hookrightarrow_n \mathfrak{A} \text{ for some } \mathfrak{A} \in \Omega : \Phi \oplus \Phi : \Omega\},$$
$$\mathcal{C}_1 := \mathcal{C} \setminus \mathcal{C}_0.$$

Then $\mathcal{C}_0 \sqsubseteq \Omega : \Phi \oplus \Phi : \Omega$ and $\mathcal{C}_1 \sqsubseteq \Omega \cdot \Phi$. Distinguish the following cases:

1. $\mathcal{C}_1$ contains a tree with a strictly increasing sequence. By Lemma 5.20 we have $\Omega \cdot \Phi \sqsubseteq \mathcal{C}_1$. Hence $\Omega \cdot \Phi \cup (\Omega : \Phi \oplus \Phi : \Omega) \sqsubseteq \mathcal{C}$.

2. $\mathcal{C}_1$ does not contain a tree with a strictly increasing sequence. Hence, for each $\mathfrak{C} \in \mathcal{C}_1$ there is a constant $n_{\mathfrak{C}} < \omega$ such that all vertices have either less than $n_{\mathfrak{C}}$ or infinitely many successors. Then Lemma 5.23 yields $\mathcal{C}_1 \sqsubseteq \Omega : \Phi$ and thus $\mathcal{C} \sqsubseteq \Omega : \Phi \oplus \Phi : \Omega$.

□

Proposition 5.33. $\Omega \cdot \Phi \oplus \Phi : \Omega \lhd \Omega^2$.

Proof. The $\sqsubseteq$-relation follows immediately by Lemma 4.2. For the strictness property, suppose $\Omega^2 \leq_n \Omega \cdot \Phi \oplus \Phi : \Omega$. Then there is $m < \omega$ and $\mathfrak{A} \in \Omega \cdot \Phi$ such that $[\![\omega^2]\!] \hookrightarrow_n \mathfrak{A} \oplus [\![m, \omega]\!]$. Every vertex of $[\![\omega^2]\!]$ of level 1 is mapped to a vertex with infinitely many successors. Since $\mathfrak{A} \oplus [\![m, \omega]\!]$ has only finitely many such vertices, there are infinitely many level 1 vertices that are mapped to the same vertex. A contradiction.

Now let $\mathcal{C}$ be a class such that $\Omega \cdot \Phi \oplus \Phi : \Omega \sqsubseteq \mathcal{C} \sqsubseteq \Omega^2$. Distinguish two cases:

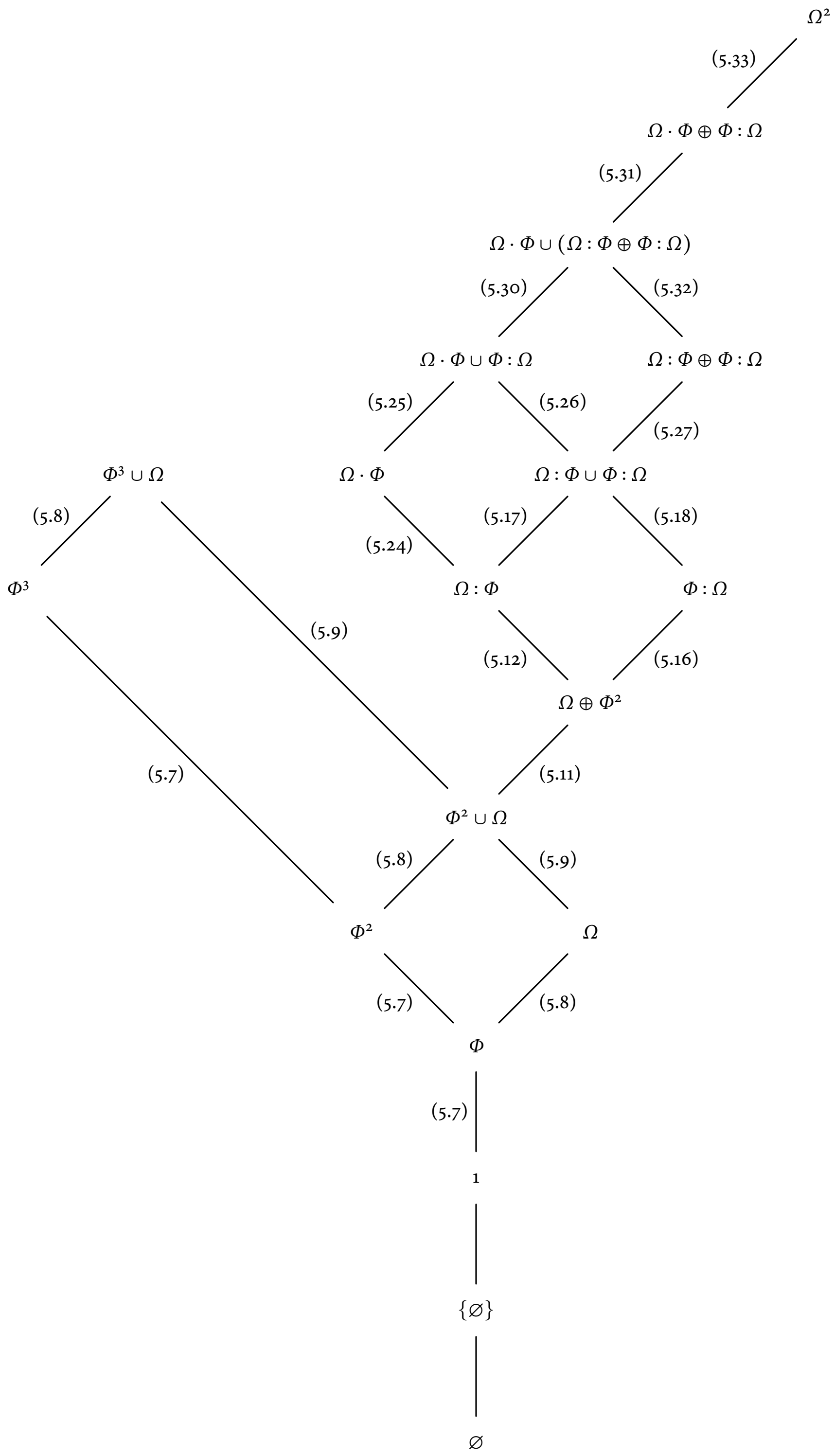
1. There is some $\mathfrak{C} \in \mathcal{C}$ with infinitely many vertices having infinitely many successors. Then we have $[\![\omega^2]\!] \hookrightarrow_1 \mathfrak{C}$ and thus $\Omega^2 \leq_1 \mathcal{C}$.

2. All elements $\mathfrak{C} \in \mathcal{C}$ have finitely many vertices, say $n_{\mathfrak{C}} < \omega$, having infinitely many successors. Let $\mathfrak{C} \in \mathcal{C}$ and $h : \mathfrak{C} \hookrightarrow_n [\![\omega^2]\!]$ and $\mathfrak{S} := h(\mathfrak{C})$. By assumption: $\mathfrak{S}$ is isomorphic to a subtree of $\mathfrak{A} \oplus [\![n_{\mathfrak{C}}, \omega]\!]$, for some $\mathfrak{A} \in \Omega \cdot \Phi$. Hence, $\mathfrak{C} \hookrightarrow_n \mathfrak{A} \oplus [\![n_{\mathfrak{C}}, \omega]\!]$. Therefore, $\mathcal{C} \leq_n \Omega \cdot \Phi \oplus \Phi : \Omega$.

□

Now we can state the main theorem of this paper compressing the results of the current chapter into a picture showing the lower end of the transduction hierarchy of the classes of trees.

Theorem 5.34. *We have the following hierarchy where the edges denote the $\lhd$-relation:*



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