TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics

Bachelorthesis

Linear Rank-Width and Interpretations

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Abstract In this paper we prove the theorem stating that a class $\mathcal{G}$ of finite graphs has bounded linear rank-width if and only if there is an MSO interpretation τ and a class $\mathcal{I}$ of coloured linear orders such that $\mathcal{G} = \tau(\mathcal{I})$. For this proof we define linear rank- and clique-width and show that they are bounded over the same classes of graphs. We use compatible colourings and encodings to show the sufficiency of bounded linear rank-width and MSO theories for its necessity.

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Introduction

There are several notions of graph-widths whose origin is closely related with the analysis of graph algorithms.

Robertson and Seymour introduced path-width in the first article of their graph-minor project in 1983 using path-decompositions of finite graphs [9]. They defined tree-width in the project's third article the following year using tree-decompositions, such that path-width is the linearized variant of tree-width [10]. Courcelle and Engelfriet showed later the equivalence of bounded tree-width and the existence of a guarded second-order interpretation in a tree [4]. Since hierarchical decompositions of graphs are useful for defining graphs with good algorithmic properties, Courcelle and Olariu introduced clique-width to broaden the class of graphs for which for example NP complete problems have linear complexity [6]. They used another form of graph-decomposition which builds a graph using a defined set of operations. Oum and Seymour explored branch-width and its relation to clique-width and introduced rank-width in the process [8]. Linear rank-width was first introduced by Ganian and is a linearized variant of the rank-width, like path-width is of tree-width. It was introduced to obtain a class of graphs with better algorithmic properties than the classes of bounded rank- or tree-width [7].

The central theorem of this thesis states that a class $\mathcal{G}$ of finite graphs has bounded linear rank-width if and only if there is an MSO interpretation τ and a class $\mathcal{I}$ of coloured linear orders such that $\mathcal{G} = \tau(\mathcal{I})$. It is therefore analogue to the sentence of Courcelle and Engelfriet.

The first chapter of this thesis introduces some basic notions of Monadic Second-Order Logic and graphs. Here we will also define the used linear rank- and clique-width. We will use linear clique-width for the following proof since it is equivalent to linear rank-width with respect to boundedness.

The second chapter contains the proof of the main theorem. We will prove this theorem using compatible colourings and encodings of graphs for the sufficiency and MSO theories for the necessity of bounded linear rank-width.

Chapter 1

Definitions

In this chapter we will introduce Monadic Second-Order Logic and MSO interpretations. This chapter also deals with graphs and their orders. We will define linear rank- and clique-width and prove that one is bounded if and only if the other one is.

1.1 Basic definitions

I will first recall some basic definitions.

Definition 1.1. The logic used in this thesis is *Monadic Second-Order Logic (MSO)*. It extends First-Order Logic by set quantifiers.

Definition 1.2. An *MSO interpretation* $\tau : \Sigma \to \Gamma$, is a mapping between structures of relational signatures Σ and Γ. It is defined by a definition scheme $\langle \delta(x), (\varphi_R(\bar{x}))_{R\in\Gamma} \rangle$.

Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ be a Σ-structure. Then $\tau(\mathfrak{A}) := \langle \delta^{\mathfrak{A}}, (\varphi_R^{\mathfrak{A}})_{R\in\Gamma} \rangle$ is a Γ-structure whose universe is defined by $\delta^{\mathfrak{A}} := \{a \in A \mid \mathfrak{A} \models \delta(a)\}$ and whose relations are $\varphi_R^{\mathfrak{A}} := \{\bar{a} \mid \mathfrak{A} \models \varphi_R(\bar{a})\}$.

Definition 1.3. A *graph* $\mathfrak{G}$ is a structure $\langle V, E \rangle$ whose universe V contains the vertices and whose binary relation $E \subseteq V \times V$ defines the edges.

Definition 1.4. An *ordered graph* is a structure $\langle V, E, \leq \rangle$, where V is the universe, E the edge relation and $\leq$ a linear ordering on the vertices in V.

From here on, the symbol $\leq$ will always denote a linear ordering. For two sets S_1 and S_2 we define $S_1 \leq S_2 := (\forall x \in S_1)(\forall y \in S_2)[x \leq y]$.

Definition 1.5. We define a *partition* of the graph $\mathfrak{G} = \langle V, E \rangle$ as two non-empty sets of vertices X and X^c with $X \cap X^c = \emptyset$, $X \cup X^c = V$. A *cut* of an ordered graph $\langle V, E, \leq \rangle$ is a partition which also satisfies $X \leq X^c$.

We denote with x^+ the cut of a given graph $\mathfrak{G}$ with x as the biggest element of X, therefore the cut after x, and with x^- the cut with x as the smallest element of X^c, hence the cut before x.

The notation $\mathrm{pre}(x)$ ($\mathrm{suc}(x)$) denotes the predecessor (successor) of x. So the following are true:

$$(\mathrm{pre}(x) = a) \Leftrightarrow (a \leq x \wedge a \neq x \wedge \forall z[a \leq z \wedge z \leq x \rightarrow (z = a \vee z = x)]),$$

$$(\mathrm{suc}(x) = a) \Leftrightarrow (a \geq x \wedge a \neq x \wedge \forall z[a \geq z \wedge z \geq x \rightarrow (z = a \vee z = x)]).$$

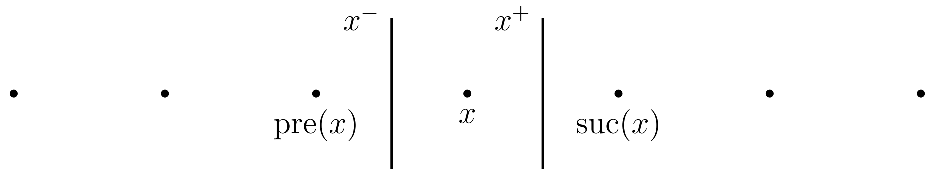

Figure 1.1: Cuts before and after x.

1.2 Notions of graph-width

In this section, we will introduce several definitions of linear rank- and clique-width and their correlation. We only discuss finite graphs, so we only need linear rank-width for finite graphs. The following definitions originate from [3] and [5].

Definition 1.6. A *linear decomposition* of a graph $\mathfrak{G}$ is a linear order $\leq$ on V.

Definition 1.7. Let $\mathfrak{G}$ be a graph. The *adjacency matrix* $M_{\mathfrak{G}}[X, X^c]$ of a partition (X, X^c) is the mapping $M : X \times X^c \rightarrow \{0, 1\}$ defined as

$$M(x, y) := \begin{cases} 1 \text{ if } (x, y) \in E \\ 0 \text{ if } (x, y) \notin E \end{cases} .$$

Definition 1.8. For a graph $\mathfrak{G} = \langle V, E\rangle$, the *rank-width* of a cut (X, X^c) is defined as $\mathrm{rwd}(\mathfrak{G}, X) := \mathrm{rk}(M_{\mathfrak{G}}[X, X^c])$ where $\mathrm{rk}(M)$ is the rank of M.
The *rank-width* of a linear decomposition $\leq$ is

$$\mathrm{rwd}(\mathfrak{G}, \leq) = \sup\{\mathrm{rwd}(\mathfrak{G}, X) \mid (X, X^c) \text{ is a cut of } \mathfrak{G} \text{ with decomposition } \leq\}$$

and the *linear rank-width* of $\mathfrak{G}$ is

$$\mathrm{lrwd}(\mathfrak{G}) = \min\{\mathrm{rwd}(\mathfrak{G}, \leq) \mid \leq \text{ is a decomposition of } \mathfrak{G}\}.$$

In the same way, we define the clique-width of a graph.

Definition 1.9. The *clique-width* $\mathrm{cwd}(\mathfrak{G}, X)$ of a graph $\mathfrak{G} = \langle V, E\rangle$ and a cut (X, X^c) is the number of pairwise distinct rows of $M_{\mathfrak{G}}[X, X^c]$.
For the decomposition $\leq$ we define the *clique-width* as

$$\mathrm{cwd}(\mathfrak{G}, \leq) = \sup\{\mathrm{cwd}(\mathfrak{G}, X), \mathrm{cwd}(\mathfrak{G}, X^c) \mid (X, X^c) \text{ is a cut of } \mathfrak{G} \text{ with decomposition } \leq\}.$$

The *linear clique-width* of $\mathfrak{G}$ is then defined as

$$\mathrm{lcwd}(\mathfrak{G}) = \min\{\mathrm{cwd}(\mathfrak{G}, \leq) \mid \leq \text{ is a decomposition}\}.$$

Lemma 1.10. *For every finite graph $\mathfrak{G} = \langle V, E\rangle$ we have $\mathrm{lcwd}(\mathfrak{G}) \leq |V|$.*

Proof. This is clear, since the adjacency matrix cannot have more rows than vertices. □

Lemma 1.11. *For every finite graph $\mathfrak{G} = \langle V, E\rangle$ we have $\mathrm{lrwd}(\mathfrak{G}) \leq \mathrm{lcwd}(\mathfrak{G}) \leq 2^{\mathrm{lrwd}(\mathfrak{G})}$.*

Proof. First we show that $\mathrm{lrwd}(\mathfrak{G}) \leq \mathrm{lcwd}(\mathfrak{G})$.

The rank of a matrix is defined as the number of linearly independent rows (or columns). Since linearly independent rows are in particular distinct, $\mathrm{rwd}(\mathfrak{G}, X) \leq \mathrm{cwd}(\mathfrak{G}, X)$ holds for every cut (X, X^c) of $\mathfrak{G}$. Let $\leq$ be a decomposition of G of minimal cwd. It follows that $\mathrm{lrwd}(\mathfrak{G}) \leq \mathrm{rwd}(\mathfrak{G}, \leq) \leq \mathrm{cwd}(\mathfrak{G}, \leq) = \mathrm{lcwd}(\mathfrak{G})$.

Now we show that $\mathrm{lcwd}(\mathfrak{G}) \leq 2^{\mathrm{lrwd}(\mathfrak{G})}$.

Let (X, X^c) be a cut and S be the set of linearly independent rows of $M_{\mathfrak{G}}[X, X^c]$. Hence, $\mathrm{rwd}(\mathfrak{G}, X) = |S|$. The maximum number of pairwise distinct rows of M is the number of linear combinations of elements of S. Since we are working in the field with two elements, there are at most $2^{|S|}$ of them. Therefore $\operatorname{cwd}(\mathfrak{G}, X) \leq 2^{\operatorname{rwd}(\mathfrak{G}, X)}$ holds for every cut of any partition $\leq$ and so we have $\operatorname{cwd}(\mathfrak{G}, \leq) \leq 2^{\operatorname{rwd}(\mathfrak{G}, \leq)}$. Let $\leq$ be a decomposition of minimal rwd, then $\operatorname{lcwd}(\mathfrak{G}) \leq \operatorname{cwd}(\mathfrak{G}, \leq) \leq 2^{\operatorname{rwd}(\mathfrak{G}, \leq)} = 2^{\operatorname{lrwd}(\mathfrak{G})}$. $\square$

Corollary 1.12. *The linear rank-width of a graph is finite if and only if its clique-width is finite.*

Corollary 1.13. *For a graph $\mathfrak{G}$ and a linear decomposition $\leq$ we have*

$$\operatorname{rwd}(\mathfrak{G}, \leq) \leq \operatorname{cwd}(\mathfrak{G}, \leq) \leq 2^{\operatorname{rwd}(\mathfrak{G}, \leq)}.$$

Chapter 2

Linear rank-width and MSO interpretations

This chapter deals with the relation between the linear rank-width of a graph and MSO interpretations. The following lemmas and theorems discuss graphs of bounded linear rank-width. Since we have shown that a graph's linear rank-width is bounded if and only if its linear clique-width is, we will use the more convenient linear clique-width for the proof of the following theorem.

Theorem 2.1. *Let $\mathcal{G}$ be a class of finite graphs. Then the following are equivalent:*

1. *$\mathcal{G}$ has bounded linear rank-width,*
2. *There is an MSO interpretation τ and a class $\mathcal{I}$ of coloured linear orders such that $\mathcal{G} = \tau(\mathcal{I})$.*

2.1 MSO interpretations of graphs with bounded width

Before proving the sufficiency of bounded linear rank-width, we will introduce compatible colourings and their encodings. A family of colourings $(\chi_C)_C$ of a graph $\mathfrak{G} = \langle V, E\rangle$ assigns for every cut C a colour $\chi_C(v) \in [k]$ to each node $v \in V$.

Definition 2.2. Let $\mathfrak{G} = \langle V, E\rangle$ be a graph. A family $(\chi_C)_C$ of colourings $\chi_C : V \to [k]$ for every cut C is *compatible* with a decomposition $\langle V, \leq\rangle$ if for all cuts $C = (X, X^c)$

and $D = (Y, Y^c)$ with $X \subset Y$ and all $a, a' \in X, b, b' \in X^c$

$$\chi_C(a) = \chi_C(a') \text{ and } \chi_C(b) = \chi_C(b') \Rightarrow (\langle a, b\rangle \in E \Leftrightarrow \langle a', b'\rangle \in E)$$

and

$$\chi_C(a) = \chi_C(a') \Rightarrow \chi_D(a) = \chi_D(a')$$

and for $b, b' \in Y^c$

$$\chi_D(b) = \chi_D(b') \Rightarrow \chi_C(b) = \chi_C(b').$$

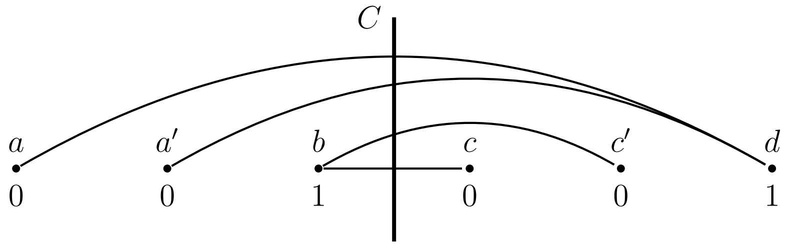

Figure 2.1: a and a' have the same colour 0, hence they have the same edges over C. The same holds for c and c'.

Lemma 2.3. $\mathrm{cwd}(\langle V, \leq\rangle) \leq k$ *if and only if there exists a compatible family of colourings* $(\chi_C)_C$ *with* k *colours.*

Proof. $(\Rightarrow)$: Let $\mathrm{cwd}(\langle V, \leq\rangle) \leq k$ and let $C = (X, X^c)$ be a cut of V. Then the corresponding adjacency matrix $M[C]$ has at most k different columns or rows. We now number the different rows and columns by elements of $[k]$ and let χ_C map a node x to the number of its row or column. We can use the same colours for the rows and columns, consequently χ_C is a colouring of the cut C using k colours. For a and a' in $X \in C$ and b and b' in $X^c \in C$ we get

$$\begin{aligned}
&\chi_C(a) = \chi_C(a') \text{ and } \chi_C(b) = \chi_C(b') \\
&\quad \Longrightarrow \text{ the rows for } a \text{ and } a' \text{ and the columns for } b \text{ and } b' \text{ in } M[C] \text{ are equal} \\
&\quad \Longrightarrow (\langle a, b\rangle \in E \Leftrightarrow \langle a', b'\rangle \in E).
\end{aligned}$$

For another cut $D = (Y, Y^c)$ wit $X \subset Y$ we get

$$\begin{aligned}
\chi_C(a) = \chi_C(a') &\Longrightarrow \text{ the rows for } a \text{ and } a' \text{ in } M[C] \text{ are equal} \\
&\Longrightarrow \text{ the rows for } a \text{ and } a' \text{ in } M[D] \text{ are equal} \\
&\Longrightarrow \chi_D(a) = \chi_D(a').
\end{aligned}$$

The reverse holds for the columns of M and $b, b' \in Y^c$. Consequently $(\chi_C)_C$ is a compatible colouring of $\mathfrak{G}$. Since we can use the same colours in different cuts, the number of colours needed is the maximum number of colours used in one cut. Therefore a compatible colouring $(\chi_C)_C$ with k colours exists whenever $\mathrm{cwd}(\langle V, \leq \rangle) \leq k$.

($\Leftarrow$): Let $(\chi_C)_C$ be a compatible colouring for $\langle V, \leq \rangle$ with k colours and let C be a cut. Nodes that are assigned the same colour by χ_C act in the same way with respect to the edges across the cut. Therefore the corresponding rows/columns of the adjacency matrix are the same. So, any adjacency matrix of the graph can have at most k different rows/columns and k is an upper bound for $\mathrm{cwd}(\langle V, \leq \rangle)$. $\square$

Now we introduce a single colouring function $\eta : V \to \Phi$ encoding a compatible colouring $(\chi_C)_C : V \to [k]$. This function η contains for every node $v \in V$ information about its own colour in the cuts v^- and v^+, changes of the other nodes' colours from the cut v^- to v^+ or v^+ to v^- and the set of connected colours over the cut v^+.

Definition 2.4. Let $(\chi_C)_C : V \to [k]$ be a family of colourings. We define the encoding $\eta(x) = \langle c^-, c^+, \overleftarrow{h}, \overrightarrow{h}, \gamma \rangle$ where:

- $c^- \coloneqq \chi_{x^-}(x)$ is the initial colouring of x in the cut left of x,
- $c^+ \coloneqq \chi_{x^+}(x)$ is the initial colouring of x in the cut right of x,
- the colour-change function $\overrightarrow{h}$ is defined by $\overrightarrow{h}(\chi_{x^-}(y)) \coloneqq \chi_{x^+}(y)$ for all $y \leq x$,
- the colour-change function $\overleftarrow{h}$ is defined by $\overleftarrow{h}(\chi_{x^+}(z)) \coloneqq \chi_{x^-}(z)$ for all $x \leq z$,
- the binary relation $\gamma \subseteq [k] \times [k]$ defines the connected colours in the cut x^+ defined by $\gamma \coloneqq \{\langle \chi_{x^+}(y), \chi_{x^+}(z) \rangle \mid \langle y, z \rangle \in E, y \leq x < z\}$.

For an encoding η and a vertex x we will denote the components of $\eta(x)$ by

$$\eta(x) = \langle c_x^-, c_x^+, \overleftarrow{h_x}, \overrightarrow{h_x}, \gamma_x \rangle.$$

By the second condition in the definition of a compatible colouring it is possible to find an encoding η for every compatible colouring $(\chi_C)_C$.

Definition 2.5. An encoding η is *valid* for a linear order $\langle V, \leq \rangle$ if for every $y, z \in V$ with $y < z$ it holds that the existence of an x with $y \leq x < z$ and

$$\langle \overrightarrow{h}_x(\overrightarrow{h}_{\mathrm{pre}(x)} \dots (\overrightarrow{h}_{\mathrm{suc}(y)}(c_y^+))), \overleftarrow{h}_{\mathrm{suc}(x)} \dots (\overleftarrow{h}_{\mathrm{pre}(z)}(c_z^-)) \rangle \in \gamma_x$$

implies the same holds for all x with $y \leq x < z$.

Lemma 2.6. *Let $\mathfrak{G} = \langle V, E\rangle$ be a graph and $(\chi_C)_C$ a compatible colouring of $\mathfrak{G}$. The encoding η of $(\chi_C)_C$ is valid.*

Proof. Let $y, x, z \in V$ with $y \leq x < z$ and

$$\langle \overrightarrow{h}_x(\overrightarrow{h}_{\mathrm{pre}(x)} \dots (\overrightarrow{h}_{\mathrm{suc}(y)}(c_y^+))), \overleftarrow{h}_{\mathrm{suc}(x)} \dots (\overleftarrow{h}_{\mathrm{pre}(z)}(c_z^-))\rangle \in \gamma_x.$$

For $v \in V$ with $y \leq v < z$ we have

$$\begin{aligned}
&\langle \overrightarrow{h}_x(\overrightarrow{h}_{\mathrm{pre}(x)} \dots (\overrightarrow{h}_{\mathrm{suc}(y)}(c_y^+))), \overleftarrow{h}_{\mathrm{suc}(x)} \dots (\overleftarrow{h}_{\mathrm{pre}(z)}(c_z^-))\rangle \in \gamma_x \\
&\implies \langle \chi_{x^+}(y), \chi_{x^+}(z)\rangle \in \gamma_x \\
&\implies \langle y, z\rangle \in E \\
&\implies \langle \chi_{v^+}(y), \chi_{v^+}(z)\rangle \in \gamma_v \\
&\implies \langle \overrightarrow{h}_v(\overrightarrow{h}_{\mathrm{pre}(v)} \dots (\overrightarrow{h}_{\mathrm{suc}(y)}(c_y^+))), \overleftarrow{h}_{\mathrm{suc}(v)} \dots (\overleftarrow{h}_{\mathrm{pre}(z)}(c_z^-))\rangle \in \gamma_v.
\end{aligned}$$

So η is valid. □

It is also possible to derive a compatible colouring from a valid encoding.

Lemma 2.7. *Let $I = \langle V, \leq\rangle$ be a linear order. For every valid encoding η there exist an edge relation E and a compatible family $(\chi_C)_C$ of colourings for $\langle V, E, \leq\rangle$ such that η is an encoding of $(\chi_C)_C$.*

Proof. Let η be a valid encoding. Suppose that $\eta(x) = \langle c_x^-, c_x^+, \overleftarrow{h}_x, \overrightarrow{h}_x, \gamma_x\rangle$.

We define $(\chi_C)_C$ as follows. Let $\chi_{x^+}(x) := c_x^+$ and $\chi_{x^-}(x) := c_x^-$. For $y < x$ we define

$$\chi_{x^+}(y) := \overrightarrow{h}_x(\overrightarrow{h}_{\mathrm{pre}(x)} \dots (\overrightarrow{h}_{\mathrm{suc}(y)}(c_y^+)))$$

and for $x < z$

$$\chi_{x^-}(z) := \overleftarrow{h}_x(\overleftarrow{h}_{\mathrm{suc}(x)} \dots (\overleftarrow{h}_{\mathrm{pre}(z)}(c_z^-))).$$

In this manner we can construct a colouring χ_C for every cut C. Finally, we set

$$E := \{\langle y, z\rangle \in V \times V \mid y < z \text{ and } \langle \chi_{x^+}(y), \chi_{x^+}(z)\rangle \in \gamma_x \text{ for some } y \leq x < z\}.$$

We will now show that $(\chi_C)_C$ is compatible.

Let $x, a, a', b, b' \in V$ with $a, a' \leq x < b, b'$, satisfying $\chi_{x^+}(a) = \chi_{x^+}(a')$ and $\chi_{x^+}(b) = \chi_{x^+}(b')$. Then

$$\begin{aligned}
&\langle a, b\rangle \in E \\
&\implies \exists a \leq z < b \text{ s.t. } \langle \chi_{z^+}(a), \chi_{z^+}(b)\rangle \in \gamma_z \\
&\overset{\eta \text{ is valid}}{\implies} \langle \chi_{x^+}(a), \chi_{x^+}(b)\rangle \in \gamma_x \\
&\implies \langle \chi_{x^+}(a'), \chi_{x^+}(b')\rangle \in \gamma_x \\
&\implies \langle a', b'\rangle \in E.
\end{aligned}$$

Let $a, a' \leq x < y \in V$, then

$$\begin{aligned}
&\chi_{x^+}(a) = \chi_{x^+}(a') \\
&\implies \chi_{y^+}(a) = \overrightarrow{h}_y(\overrightarrow{h}_{\mathrm{pre}(y)} \dots (\overrightarrow{h}_{\mathrm{suc}(x)}(\chi_{x^+}(a)))) \\
&\qquad\qquad = \overrightarrow{h}_y(\overrightarrow{h}_{\mathrm{pre}(y)} \dots (\overrightarrow{h}_{\mathrm{suc}(x)}(\chi_{x^+}(a')))) = \chi_{y^+}(a').
\end{aligned}$$

The same holds for $a, a' \geq x > y \in V$. Ergo $(\chi_C)_C$ is compatible. □

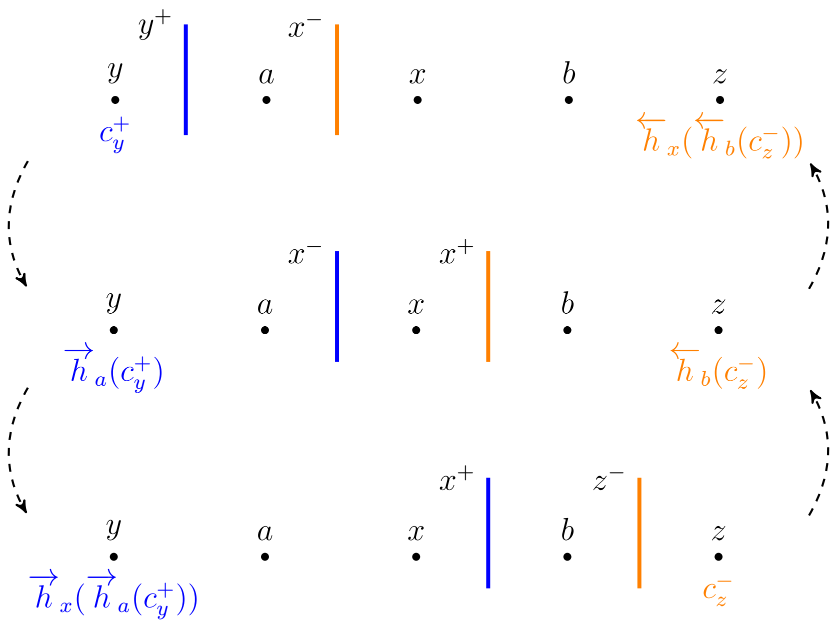

Figure 2.2: The downward and upward colourchanges of y and z.

Now we can use a compatible colouring and its encoding to find an interpretation τ yielding the graph $\mathfrak{G}$.

Lemma 2.8. *There exists an interpretation τ such that if $(\chi_C)_C$ is compatible with $\langle V, E, \leq\rangle$ and $\eta : V \to \Phi$ encodes $(\chi_C)_C$, then $\tau(\langle V, \leq, \eta\rangle) = \mathfrak{G}$.*

Proof. We will first introduce some MSO formulae.

$\varphi_d^<(x,y)$ determines for $x<y$ whether the colour $\chi_{x^+}(y)$ of y in the cut x^+ is d. The formula states that there are sets Z_c for each colour c such that Z_c contains exactly the nodes v such that y has colour c in the cut v^+.

$$\varphi_d^<(x,y) \coloneqq \exists (Z_c)_{c\in[k]}\Big[\bigwedge_{c\neq d} Z_c \cap Z_d = \emptyset \wedge Z_{c_y^-}(\mathrm{pre}(y)) \wedge \forall z. \bigwedge_{c\in[k]} (Z_c z \rightarrow Z_{\overleftarrow{h}_z(c)}(\mathrm{pre}(z))) \wedge Z_d x\Big]$$

$\varphi_d^>(x,y)$ determines in a similar way for $x>y$ whether the colour $\chi_{x^-}(y)$ of y in the cut before x is d.

$$\varphi_d^>(x,y) \coloneqq \exists (Z_c)_{c\in[k]}\Big[\bigwedge_{c\neq d} Z_c \cap Z_d = \emptyset \wedge Z_{c_y^+}(\mathrm{suc}(y)) \wedge \forall z. \bigwedge_{c\in[k]} (Z_c z \rightarrow Z_{\overrightarrow{h}_z(c)}(\mathrm{suc}(z))) \wedge Z_d x\Big]$$

$$\begin{array}{cccc} x & \mathrm{suc}(x) & \mathrm{pre}(y) & y \\ \bullet & \bullet & \bullet & \bullet \\ \overleftarrow{h}_{\mathrm{suc}(x)}(\overleftarrow{h}_{\mathrm{pre}(y)}(c_y^-)) = d & \overleftarrow{h}_{\mathrm{pre}(y)}(c_y^-) & c_y^- & \end{array}$$

Figure 2.3: Every node v is assigned the colour of y in the cut v^+.

$\psi(x,y)$ is a formula stating whether or not x and y are connected by an edge, that is $\langle x,y\rangle \in E$.

$$\psi(x,y) \coloneqq \bigvee_{\langle c,d\rangle\in\gamma_{x^+}} [x<y \wedge c_x^+ = c \wedge \varphi_d^<(x,y)] \vee \bigvee_{\langle c,d\rangle\in\gamma_{x^-}} [x>y \wedge c_x^- = d \wedge \varphi_c^>(x,y)].$$

We will now give the interpretation τ and show that $\tau(\langle V,\leq,\eta\rangle) = \mathfrak{G}$. Let $\mathfrak{I} = \langle V,\leq,\eta\rangle$ and $\mathfrak{G} = \langle V,E\rangle$. We define $\delta(V) \coloneqq \mathrm{true}$. Hence the universe is $\delta^{\mathfrak{I}} \coloneqq \{v\in V \mid \mathfrak{I} \models \delta(v)\} = V$. And we define $\varphi_E(x,y) \coloneqq \psi(x,y)$ where $\psi(x,y)$ is defined as above. As a result $\varphi_E^{\mathfrak{I}} \coloneqq \{\langle x,y\rangle \mid \langle x,y\rangle \in E\}$. Since η encodes a compatible colouring $(\chi_C)_C$ of $\mathfrak{G}$, the set of edges defined by $\varphi_E^{\mathfrak{I}}$ is the set of edges E of $\mathfrak{G}$ and therefore $\tau(\langle V,\leq,\eta\rangle) = \mathfrak{G}$. □

Now we can prove that bounded linear rank-width is sufficient for the existence of an interpretation.

Proof. Let $\mathcal{G}$ be a class of finite graphs with $\mathrm{lrwd}(\mathfrak{G}) \leq k$ for every $\mathfrak{G} \in \mathcal{G}$. Let τ be the interpretation from Lemma 2.5. There is a linear order $\leq$ for every $\mathfrak{G}$ such that $\mathrm{rwd}(\mathfrak{G},\leq) \leq k$ and in consequence $\mathrm{cwd}(\mathfrak{G},\leq) \leq 2^k$. With Lemma 2.4 follows the existence of a compatible colouring $(\chi_C)_C$ of V with 2^k colours. Let η be an encoding of $(\chi_C)_C$. Let $\mathcal{I}$ denote the class of these coloured linear orders. Then $\tau(\langle V, \leq, \eta \rangle) = \mathfrak{G}$, for every $\mathfrak{G} \in \mathcal{G}$ and thus $\mathcal{G} = \tau(\mathcal{I})$. $\square$

2.2 MSO interpretation implies finite linear rank-width

For proving the second direction of the theorem, we will use MSO theories. First we will recall some standard definitions and facts which can also be reviewed in eg. [1].

Definition 2.9. The *m-theory* of a Σ-structure $\mathfrak{A}$ is the set

$$\mathrm{Th}_m(\mathfrak{A}) := \{\varphi(\chi) \mid \mathrm{qr}(\varphi) \leq m, \mathfrak{A} \models \varphi\}$$

of all formulae of quantifier rank $\mathrm{qr}(\varphi)$ at most m satisfied by $\mathfrak{A}$.
Two Σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ are called *m-equivalent* if they have the same m-theory:

$$\mathfrak{A} \equiv_m \mathfrak{B} \Leftrightarrow \mathrm{Th}_m(\mathfrak{A}) = \mathrm{Th}_m(\mathfrak{B}).$$

We need to prove some properties of m-theories before using them as a compatible colouring. The following two lemmas and their proofs are taken from [1].

Lemma 2.10. *Let Σ be a finite signature and $m < \infty$. Up to logical equivalence, there are only finitely many m-theories over the signature Σ.*

Proof. First we will use induction to prove that up to logical equivalence there are only finitely many Σ-formulae of quantifier rank m with k parameters.

Start: $m = 0$

For $m = 0$ a formula is quantifier-free and can be written in disjunctive normal form. There are only finitely many atomic formulae and negated atomic formulae using k parameters, because Σ is finite, and thus the number of possible conjunctions of these is finite up to logical equivalence. It follows that there are only finitely many disjunctions of these conjunctions and so the number of formulae in disjunctive normal form is finite.

Step: $m > 0$

Hypothesis: The number of formulae with quantifier rank smaller than m is finite. Every formula of quantifier-rank at most m consists of boolean combinations of atomic formulae and formulae of the form $\exists x\psi$ or $\exists X\psi$, where ψ has quantifier-rank smaller than m. With the induction hypothesis, the number of such formulae is finite and therefore the number of possible disjunctive normal forms of the boolean combinations is finite, too.

Every Σ-theory of quantifier rank m with k parameters is a subset of the set S of Σ-formulae of quantifier rank m with k parameters. Since $|S| < \infty$, there are only $2^{|S|} < \infty$ possible pairwise distinct subsets, and consequently different Σ-theories. $\square$

Definition 2.11. Let $\mathfrak{A} = \langle A, \leq_{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle B, \leq_{\mathfrak{B}}\rangle$ be orders. Their ordered sum is $\mathfrak{A} \oplus \mathfrak{B} := \langle A \sqcup B, \leq_{\mathfrak{A}\oplus\mathfrak{B}}\rangle$ with $x \leq_{\mathfrak{A}\oplus\mathfrak{B}} y$ if and only if $(x, y \in A, x \leq_{\mathfrak{A}} y)$ or $(x, y \in B, x \leq_{\mathfrak{B}} y)$ or $(x \in A, y \in B)$.

Lemma 2.12. *For orders $\mathfrak{A} = \langle A, \leq_{\mathfrak{A}}\rangle$, $\mathfrak{A}' = \langle A', \leq_{\mathfrak{A}'}\rangle$, $\mathfrak{B} = \langle B, \leq_{\mathfrak{B}}\rangle$ and $\mathfrak{B}' = \langle B', \leq_{\mathfrak{B}'}\rangle$ the following holds:*

$$\mathfrak{A}, \bar{a} \equiv_m \mathfrak{A}', \bar{a}' \textit{ and } \mathfrak{B}, \bar{b} \equiv_m \mathfrak{B}', \bar{b}' \implies \mathfrak{A} \oplus \mathfrak{B}, \bar{a}\bar{b} \equiv_m \mathfrak{A}' \oplus \mathfrak{B}', \bar{a}'\bar{b}'.$$

Proof. We will use induction over m.

Start: $m = 0$

Every quantifier-free formula is a boolean combination of atomic formulae. By symmetry, it is sufficient to prove that

$$\mathfrak{A} \oplus \mathfrak{B} \models \varphi(\bar{a}, \bar{b}) \text{ implies } \mathfrak{A}' \oplus \mathfrak{B}' \models \varphi(\bar{a}', \bar{b}'),$$

for every atomic formula $\varphi(\bar{x}, \bar{y})$.

Let $\varphi = x \leq y$ and $\mathfrak{A} \oplus \mathfrak{B} \models \varphi(c, d)$, then either $c = a_i, d = a_j$ and $\mathfrak{A} \models a_i \leq a_j$, $c = b_i, d = b_j$ and $\mathfrak{B} \models b_i \leq b_j$ or $c = a_i, d = b_j$ holds. In the first case we have

$$\begin{aligned} &\mathfrak{A} \models a_i \leq a_j \text{ with } \mathfrak{A}, \bar{a} \equiv_0 \mathfrak{A}', \bar{a}' \\ \implies &\mathfrak{A}' \models a_i' \leq a_j' \\ \implies &\mathfrak{A}' \oplus \mathfrak{B}' \models a_i' \leq a_j' \\ \implies &\mathfrak{A}' \oplus \mathfrak{B}' \models \varphi(\bar{a}', \bar{b}'). \end{aligned}$$

The second case can be shown in the same way. For $c = a_i, d = b_j$ we trivially have

$$\mathfrak{A}' \oplus \mathfrak{B}' \models a_i' \leq b_j'.$$

Step: $m > 0$

Hypothesis: The claim holds for formulae of quantifier rank smaller m.
Suppose $\mathfrak{A}, \bar{a} \equiv_m \mathfrak{A}', \bar{a}'$ and $\mathfrak{B}, \bar{b} \equiv_m \mathfrak{B}', \bar{b}'$. We want to show that $\mathfrak{A} \oplus \mathfrak{B}, \bar{a}\bar{b} \equiv_m \mathfrak{A}' \oplus \mathfrak{B}', \bar{a}'\bar{b}'$. It is sufficient to prove that for every parameter c (first-order or monadic second-order) of $\mathfrak{A} \oplus \mathfrak{B}$, there is a parameter c' of $\mathfrak{A}' \oplus \mathfrak{B}'$ such that $\mathfrak{A} \oplus \mathfrak{B}, \bar{a}\bar{b}c \equiv_{m-1} \mathfrak{A}' \oplus \mathfrak{B}', \bar{a}'\bar{b}'c'$. If c is a first-order parameter we may assume by symmetry that $c \in A$. Since $\mathfrak{A}, \bar{a} \equiv_m \mathfrak{A}', \bar{a}'$, we can find a $c' \in A'$ with $\mathfrak{A}, \bar{a}c \equiv_{m-1} \mathfrak{A}', \bar{a}'c'$. Since $\mathfrak{B}, \bar{b} \equiv_{m-1} \mathfrak{B}', \bar{b}'$ it follows by induction hypothesis that $\mathfrak{A} \oplus \mathfrak{B}, \bar{a}\bar{b}c \equiv_{m-1} \mathfrak{A}' \oplus \mathfrak{B}', \bar{a}'\bar{b}'c'$.

If c is a set parameter, we will split c into its A and B components. We can find parameters $c' \subseteq A'$ and $d' \subseteq B'$ such that

$$\mathfrak{A}, \bar{a}c|_A \equiv_{m-1} \mathfrak{A}', \bar{a}'c',$$
$$\mathfrak{B}, \bar{b}c|_B \equiv_{m-1} \mathfrak{B}', \bar{b}'d'.$$

With the induction hypothesis we get

$$\mathfrak{A} \oplus \mathfrak{B}, \bar{a}\bar{b}c \equiv_{m-1} \mathfrak{A}' \oplus \mathfrak{B}', \bar{a}'\bar{b}'(c' \cup d').$$

Therefore $\mathfrak{A} \oplus \mathfrak{B}, \bar{a}\bar{b} \equiv_m \mathfrak{A}' \oplus \mathfrak{B}', \bar{a}'\bar{b}'$ holds. □

We will use m-theories to define a colouring $(\chi_C)_C$ for a graph $\mathfrak{G} = \tau(\mathfrak{I})$.

Lemma 2.13. *Let $\tau = \langle \delta(x), \varphi_E(x, y) \rangle$ be an MSO interpretation where φ_E has quantifier rank m. The colouring $(\chi_C)_C$ for $C := (X, X^c)$, $x \in X$ and $y \in X^c$ where $\chi_C(x) := \mathrm{Th}_m(\mathfrak{I}|_X, x)$ and $\chi_C(y) := \mathrm{Th}_m(\mathfrak{I}|_{X^c}, y)$ is a compatible colouring for the graph $\mathfrak{G} = \tau(\mathfrak{I})$.*

Proof. Let $C = (X, X^c), x, x' \in X$ and $y, y' \in X^c$. Then the following holds:

$$\begin{aligned}
&\chi_C(x) = \chi_C(x'), \chi_C(y) = \chi_C(y') \\
&\implies \mathfrak{I}|_X, x \equiv_m \mathfrak{I}|_X, x' \text{ and } \mathfrak{I}|_{X^c}, y \equiv_m \mathfrak{I}|_{X^c}, y' \\
&\implies \mathfrak{I}|_X \oplus \mathfrak{I}|_{X^c}, xy \equiv_m \mathfrak{I}|_X \oplus \mathfrak{I}|_{X^c}, x'y' \\
&\implies \mathfrak{I} \models \varphi_E(x, y) \Leftrightarrow \mathfrak{I} \models \varphi_E(x', y') \\
&\implies \langle x, y \rangle \in E \Leftrightarrow \langle x', y' \rangle \in E
\end{aligned}$$

Let $D = (Y, Y^c)$ be a cut with $X \subset Y$, then

$$\begin{aligned}
&\chi_C(x) = \chi_C(x') \\
&\implies \mathfrak{I}|_X, x \equiv_m \mathfrak{I}|_X, x' \\
&\implies \mathfrak{I}|_X \oplus \mathfrak{I}|_{Y \setminus X}, x \equiv_m \mathfrak{I}|_X \oplus \mathfrak{I}|_{Y \setminus X}, x' \\
&\implies \mathfrak{I}|_Y, x \equiv_m \mathfrak{I}|_Y, x' \\
&\implies \chi_D(x) = \chi_D(x').
\end{aligned}$$

In the same way we can show $\chi_C(y) = \chi_C(y') \Rightarrow \chi_D(y) = \chi_D(y')$ for $Y \subset X$. Hence $(\chi_C)_C$ is a compatible colouring. $\square$

We can use these lemmas to prove the necessity of finite linear rank-width in Theorem 2.1.

Proof. Let $\mathcal{I}$ be a class of coloured linear orders and $\tau = \langle \delta(x), \varphi_E(x, y) \rangle$ an MSO interpretation such that $\mathcal{G} = \tau(\mathcal{I})$. For each graph $\mathfrak{G} \in \mathcal{G}$ we define a compatible colouring $(\chi_C)_C$ as shown in Lemma 2.13. So every colour of $(\chi_C)_C$ is an MSO theory of quantifier rank m with one parameter. Since $\mathfrak{G}$ has a finite signature, by Lemma 2.10, there are only finitely many such theories. It follows that there is a $k \in \mathbb{N}$ such that the compatible family of colourings $(\chi_C)_C$ of every graph $\mathfrak{G} \in \mathcal{G}$ has at most k colours and by Lemma 2.3 every graph $\mathfrak{G} \in \mathcal{G}$ has $\mathrm{cwd}(\langle V, \leq \rangle) \leq k$. Therefore $\mathcal{G}$ has bounded linear rank-width. $\square$

Conclusion

We proved the central theorem of this thesis which states that a class $\mathcal{G}$ of finite graphs has bounded linear rank-width if and only if there is an MSO interpretation τ and a class $\mathcal{I}$ of coloured linear orders such that $\mathcal{G} = \tau(\mathcal{I})$. For the actual proof, we used linear clique-width instead of linear rank-width, since they are equivalent concerning boundedness.

We defined compatible colourings of graphs and their encodings and linked their existence with finite linear clique-width. We used the encoding η to define an interpretation τ of a given class of graphs and showed the sufficiency of finite linear rank-width.

We introduced m-theories and some of their properties including the boundedness of their quantity if $m < \infty$. We then showed the necessity of finite linear rank-width, using m-theories to define a compatible colouring.

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Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have acknowledged all the sources of information which have been used in the thesis.

Darmstadt,

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