Asymptotic Monadic Second-Order Logic

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Abstract. In this paper we introduce so-called *asymptotic logics,* logics that are meant to reason about weights of elements in a model in a way inspired by topology. Our main subject of study is Asymptotic Monadic Second-Order Logic over infinite words. This is a logic talking about ω-words labelled by integers. It contains full monadic second-order logic and can express asymptotic properties of integers labellings.

We also introduce several variants of this logic and investigate their relationship to the logic MSO+$\mathbb{U}$. In particular, we compare their expressive powers by studying the topological complexity of the different models. Finally, we introduce a certain kind of tiling problems that is equivalent to the satisfiability problem of the weak fragment of asymptotic monadic second-order logic, i.e., the restriction with quantification over finite sets only.

1 Introduction

In this paper we consider logics that are able to express asymptotic properties about structures whose elements are labelled by weights. We call such logics 'asymptotic logics'. In general, these logics refer to a structure $\mathfrak{A}$ together with a labelling function d, called the 'weight map', that maps elements or tuples of elements to non-negative reals. A typical example of such an object is a metric structure, i.e., a structure $\mathfrak{A}$ equipped with a distance map $d : \mathcal{U}_{\mathfrak{A}} \times \mathcal{U}_{\mathfrak{A}} \to [0, \infty)$. In general, we refer to such structures as 'weighted structures'.

We are interested in the formalisation of properties of asymptotic nature over weighted structures. Typical examples, in the case of a metric structure, are:

– Continuity of a function f:

$$(\forall x)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y)\big[d(x, y) < \delta \to d(f(x), f(y)) < \varepsilon\big] .$$

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– Uniform continuity of f:

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x)(\forall y)\big[d(x, y) < \delta \to d(f(x), f(y)) < \varepsilon\big] \ .$$

– Cauchy convergence of a sequence $(a_i)_{i\in\mathbb{N}}$:

$$(\forall\varepsilon > 0)(\exists k)(\forall i, j > k)\big[d(a_i, a_j) < \varepsilon\big] \ .$$

– Density of a set Y: $\quad (\forall x)(\forall\varepsilon > 0)(\exists y \in Y)\big[d(x, y) < \varepsilon\big] \ .$

Inspecting the syntax of these formulae, we note the following properties. First, there are two sorts: the objects that live in the universe of the structure, such as elements, series, functions, etc.. . . , and the objects living in $\mathbb{R}$ that are used to refer to distances. The map d is the only way to relate these two sorts, and all tests in which elements of $\mathbb{R}$ are involved are comparisons with variables ε, δ. More interesting is the remark that if a variable, say ε, ranging over $\mathbb{R}^+$ is quantified universally, it is always used as an upper bound, i.e, positively in a test of the form $d(-) < \varepsilon$ (positively in the sense that an even number of negations separate the quantifier from its use). Dually, if it is quantified existentially, it is always used as a lower bound, i.e., positively in a test of the form $d(-) \geq \varepsilon$. In particular, this is the case for the test $d(x, y) < \delta$ in the sentences expressing continuity and uniform continuity, since it occurs in the left hand-side of an implication.

This syntactic property witnessed in the above examples can be turned into a definition. An *asymptotic formula* is a formula in which it is possible to quantify over quantities $\exists\varepsilon$, $\forall\delta$, and the only way to use the map d is in tests of the form $d(-) \geq \varepsilon$ positively below $\exists\varepsilon$ and $d(-) < \delta$ positively below $\forall\delta$.

This restriction captures the intuition that variables ranging over $\mathbb{R}^+$ are always thought as 'tending to 0' or 'to be very small'. In other words, they are only used to state properties of a topological nature. Our objective is to understand the expressive power and the decidability status of logics to which we have added this asymptotic capability.

Link with topological logics. Of course, logics as described above are related to topological notions, and as such these logics are not very far from the topological logics as studied in the seventies and eighties. These were logics (variants of first-order logic) in which it is possible to quantify over open sets. There are several variants. Flum and Ziegler introduced a logic in which it is possible to quantify over open sets, but it is only allowed to test the membership in these sets under a positivity assumption with respect to the quantifier [11] (in a way very similar to our case). Rabin proved, as a consequence of the decidability of the theory of the infinite binary tree that the theory of the real line $(\mathbb{R}, <)$ with quantification over open sets is decidable [15]. On the other hand, Shelah and Gurevich showed that monadic formulas over Cantor space equipped with an 'is open' predicate is undecidable [12].

Our approach is slightly different. Our base object is not, as above, a topology of open sets, but a weight map d. Of course, if d is required to be a distance, it induces a topology. However, there is no such assumption in general (and d may even be non-binary). Nevertheless, we can consider the topology over the non-negative reals in which the open sets are the neighbourhoods of 0 (as well as $\emptyset$ of course). Then the quantifiers $\forall\varepsilon$, $\exists\delta$,... can be replaced by quantifiers ranging over open sets, and tests of the form $d(-) < \varepsilon$ by membership tests of $d(-)$ in an open set. Furthermore, these tests respect the positivity assumption as defined by Flum and Ziegler. However, this relationship of our logic with those from the literature does not seem to help with solving the questions raised in the present paper.

Monadic second-order logic and asymptotic monadic second-order logic. In this paper, we consider the asymptotic variant of monadic second-order logic, though certainly this notion of asymptoticity can be combined with other formalisms. Let us recall that monadic second-order logic is the extension of first-order logic by set quantifiers. There is a long history of works dealing with the decidability of monadic second-order logic over some classes of structures, the prominent examples being the results over ω by Büchi [7] and over the infinite binary tree by Rabin [15]. These results can be regarded as foundations for a theory of 'regular languages' of infinite words and trees. We are interested in knowing whether this logic can be 'made asymptotic' while keeping these strong decidability properties. We have good hopes that – at least some of – these results can be generalised to more general ones, in which monadic logic is extended with asymptotic capabilities.

Before continuing, let us formalise what is 'asymptotic monadic second-order logic' (AMSO for short). The first aspect is that weight maps range over the elements of the structure, and not tuples. This is a design choice, our goal being to concentrate our attention on the simplest situation. The second aspect is cosmetic: instead of considering quantities ranging over $\mathbb{R}^+$, we consider quantities ranging over $\mathbb{N}$. Essentially, this amounts for weights to exchange $d(-)$ with $\lceil 1/d(-)\rceil$ and, for quantifiers $\exists\varepsilon$, $\forall\delta$ ranging over $\mathbb{R}^+$, to exchange them for $\exists r, \forall s$ ranging over $\mathbb{N}$. As a consequence, existentially quantified numbers are used as upper bounds, while universally quantified ones are used as lower bounds. Hence, the syntax of 'asymptotic monadic second-order logic' is the one of MSO, extended by number quantifiers $\exists r$, $\forall s$ ranging over $\mathbb{N}$ and by predicates of the form $d(x) < r$ and $d(x) \geq s$, where x is a first-order variable, under the assumption that there is an even number of negations between the quantifier and the use.

Let us give some examples. The structure here is ω and $f : \omega \to \mathbb{N}$ is a weight map. The convention is that variables x, y, z range over elements of ω, upper case variables X, Y, Z over subsets of ω, and r, s over $\mathbb{N}$.

- f is bounded: $\exists s\forall x[f(x) \leq s]$.
- f tends to ∞: $\forall r\exists x(\forall y > x)[f(y) > r]$.
- f takes infinitely many values infinitely many times:
  $\forall r\exists s\forall x(\exists y > x)[r \leq f(y) < s]$.

The subject of this paper is to analyse the expressive power of AMSO, as well as its variants, and study its decidability status.

Link with MSO+$\mathbb{U}$. A logic closely related to AMSO is MSO+$\mathbb{U}$ [1, 3]. MSO+$\mathbb{U}$ is monadic second-order logic extended by a quantifier $\mathbb{U}X\varphi(X)$ stating that 'there are arbitrarily large finite sets X satisfying $\varphi(X)$', i.e., $\mathbb{U}X\varphi(X)$ is equivalent to $\forall s\exists X[|X| \geq s \wedge \varphi(X)]$. Thus, MSO+$\mathbb{U}$ can be regarded as a fragment of an asymptotic logic, if the weight map is chosen to be 'the cardinality map' that associates to each set its size (and, say, 0 for infinite sets).

So far, the precise decidability status of MSO+$\mathbb{U}$ is unknown. The most expressive decidable fragment over infinite words corresponds (essentially) to Boolean combination of formulas in which the $\mathbb{U}$-quantifier occurs positively [3] (in fact a bit more). On the negative side, it is known that over infinite trees MSO+$\mathbb{U}$ is undecidable [4] under the set-theoretic assumption $V = L$. This proof is inspired from the undecidability proof of MSO over the real line by Shelah [16], and it is absolutely not adaptable to infinite words as such. Hence, there is a very large gap in our knowledge of the decidability of MSO+$\mathbb{U}$. The case of the weak fragment of MSO+$\mathbb{U}$, i.e., where set quantifiers do only range over finite sets, has been positively settled in [5, 6] over infinite words and trees. In terms of the expressive power, this weak fragment still falls in the classes that are understood from [3].

In some sense, this paper can be seen as an attempt to better understand the logic MSO+$\mathbb{U}$. This is also the subject of another branch of research: the theory of regular cost functions [8, 10, 9]. However, that approach concentrates on how to measure the cardinality of sets (the quantifier $\mathbb{U}$ involves such a computation), and does not give any asymptotic analysis of quantities.

Contributions of the paper. In this paper, we study AMSO and some of its variants over infinite words. These variants are: BMSO in which number quantifiers are replaced by a boundedness predicate; EAMSO which extends AMSO with quantification over weight functions; and EBMSO that combines these two modifications. We also study the weak fragment WAMSO of AMSO, and its 'number prenex' fragment $\mathrm{AMSO}^{\mathrm{np}}$. The contributions are in several directions: expressive power, topological complexity, and decidability.

Concerning the expressive power we show that EAMSO is equivalent to EBMSO, AMSO is equivalent to BMSO, and WAMSO is equivalent to $\mathrm{AMSO}^{\mathrm{np}}$. All other pairs of logics can be separated. However, more interestingly, we can show that as far as the decidability of satisfiability is concerned, AMSO, BMSO, EAMSO, EBMSO and MSO+$\mathbb{U}$ are all equivalent, and WAMSO is equivalent to $\mathrm{AMSO}^{\mathrm{np}}$. We are hence confronted with only two levels of difficulty.

Concerning topological complexity, we perform an analysis in terms of descriptive set theory. We prove that AMSO reaches all levels of the projective hierarchy, while WAMSO reaches all finite levels of the Borel hierarchy. This separates the two classes. In particular, this shows that – as far as topological complexity is concerned – WAMSO is far simpler than AMSO, and at the same time far more complex than any variant of MSO known to be decidable (for instance the weak fragment of MSO+$\mathbb{U}$ remains at the third level of the Borel hierarchy).

On the decidability front, the case of MSO+$\mathbb{U}$ is notoriously open and difficult, and as explained above (in particular, it is known to be undecidable over infinite trees, though this gives no clue about the infinite word case). AMSO is not easier. In this paper, we advocate the importance of the weak fragment WAMSO as a logic of intermediate difficulty. Though we have to leave its decidability status open as well, we are able to disclose new forms of tiling problems that are equivalent to the decidability of the satisfiability of WAMSO. This provides a promising line of attack for understanding the decidability status of AMSO and MSO+$\mathbb{U}$.

We believe that these numerous results perfectly describe how the asymptotic notions relate to other notions from the literature, the prominent one being MSO+$\mathbb{U}$. In particular we address and answer the most important questions: expressive power, topological complexity, and – in some very preliminary form – decidability. We are finally convinced that the tiling problems that we introduce deserve to be studied on their own.

Structure of the paper. In Section 2, asymptotic monadic second-order logic is introduced as well as several fragments. Some first results are proved: the weak fragment is introduced and it is shown to be equivalent to the number prenex-form of AMSO. The extended version of asymptotic monadic second-order logic (EAMSO) is introduced and its relation to MSO+$\mathbb{U}$ is established. Section 2.3 characterises our logics in terms of Borel complexity. In Section 3, we introduce certain tiling problems and we show their equivalence with the satisfiability problem for WAMSO.

2 Asymptotic monadic second-order logic and variants

In this section, we quickly recall the definition of monadic second-order logic and we introduce the new asymptotic variant AMSO (which happens to be equivalent to another formalism, called BMSO, see below). We then introduce the weak fragment WAMSO, mention some of its basic properties. We conclude with a comparison of the expressive power of these logics.

We assume that the reader is familiar with the basic notions of logic. We consider relational structures $\mathfrak{A} = \langle \mathcal{U}, R_1, \dots, R_k \rangle$ with universe $\mathcal{U}$ and relations $R_1, \dots, R_k$. A *word* (finite or infinite) over the alphabet Σ is regarded as a structure whose universe is the set of positions and where the relations consist of the ordering $\leq$ of positions and unary relations a, for each $a \in \Sigma$, containing those positions carrying the letter a.

Monadic second-order logic (MSO) is the extension of first-order logic (FO) by *set variables* $X, Y, \dots$ ranging over sets of positions, quantifiers $\exists X$, $\forall X$ over such variables, and membership tests $x \in Y$.

2.1 Weighted structures and asymptotic monadic second-order logic

The subject of this paper is *asymptotic monadic second-order logic.* This logic expresses properties of structures whose elements have a weight which is a natural number. Formally, a *weighted structure* is a pair $\langle \mathfrak{A}, \bar{f} \rangle$ consisting of a relational structure $\mathfrak{A}$ with universe $\mathcal{U}$ and a tuple of functions $f_i : \mathcal{U} \to \mathbb{N}$ called *weight functions*. A *weighted finite word* (*resp.* a *weighted ω-word*) corresponds to the case where $\mathfrak{A}$ is a finite word (*resp.*, an ω-word).

Asymptotic monadic second-order logic (AMSO) extends MSO with the following constructions:

– quantifiers over variables of a new type, *number variables* (written $r, s, t, \dots$) that range over natural numbers, and
– atomic formulae $f(x) \leq r$ where x is a first-order variable and r a number variable. These formulae must appear positively inside the existential quantifier binding r, *i.e.*, the predicate and the quantifier are separated by an even number of negations. As a commodity of notation, the dual predicate $f(x) > r$ can be used positively below the universal quantifier $\forall r$.

Example 1. It is possible to express in AMSO that:

– the weights in a structure are bounded: $\exists r \forall x [f(x) \leq r]$,
– an ω-word has weights tending to infinity: $\forall s \exists x (\forall y > x)[f(y) > s]$,
– infinitely many weights occur infinitely often in a weighted ω-word:
$\forall s \exists r \forall x (\exists y > x)[f(y) > s \wedge f(y) \leq r]$.

On the other hand, the formula $\forall r \exists x [f(x) \leq r]$ is ill-formed since it does not respect the positivity constraint separating the introduction of r and its use.

There is an alternative way to define this logic, in a spirit closer to MSO+$\mathbb{U}$: the logic BMSO extends MSO with *boundedness predicates* of the form $f[X] < \infty$ where X is a set variable. Such a predicate holds if the function f restricted to the set X is bounded by some natural number. Hence $f[X] < \infty$ can be seen as a shorthand for the AMSO formula $\exists r (\forall x \in X)[f(x) \leq r]$. It follows that BMSO is a fragment of AMSO. In fact, both logics are equivalent, as shown by the following theorem.

Theorem 2. AMSO *and* BMSO *are effectively equivalent over all weighted structures.*

Finally, let us mention an important invariance of the logic AMSO. Two functions $f, g : \mathcal{U} \to \mathbb{N}$ are *equivalent*, noted $f \approx g$, if they are bounded over the same subsets of their domain (this is expressible in BMSO as $\forall X (f[X] < \infty \leftrightarrow g[X] < \infty)$). We extend this equivalence to weighted structures by $\langle \mathfrak{A}, \bar{f} \rangle \approx \langle \mathfrak{B}, \bar{g} \rangle$ if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, and $f_i \approx g_i$ for all i.

Proposition 3. *$\approx$-equivalent weighted structures have same* AMSO*-theory.*

This is obviously true for the logic BMSO, and hence also for AMSO according to Theorem 2. In fact, Proposition 3 also holds for the logic EAMSO introduced in Section 4 below and, more generally, for every logic that would respect syntactic constraints similar to AMSO in the use of weights. An immediate consequence is that there is no formula defining $f(x) = f(y)$ in AMSO or its variants. This rules out all classical arguments yielding undecidability in similar contexts of 'weighted logics'.

2.2 The weak and the number prenex fragments of AMSO

The use of quantifiers over infinite sets combined with number quantifiers induces intricate phenomena (the complexity analysis performed in Section 2.3 will make this obvious: AMSO reaches all levels of the projective hierarchy). There are two ways to avoid it. Either we allow only quantifiers over finite sets, thus obtaining WAMSO (which defines only Borel languages), or we prevent the nesting of monadic and number quantifiers by requiring all number quantifiers to be at the head of the formula, thereby obtaining the number-prenex fragment of AMSO, named AMSO$^{\mathrm{np}}$. We will see that these two logics have the same expressive power. To avoid confusion, let us immediately point out that WAMSO and WMSO+$\mathbb{U}$, the weak fragment of MSO+$\mathbb{U}$, are very different logics. This is due to the fact that the syntax of AMSO is not the one of MSO+$\mathbb{U}$, and as a consequence assuming the sets to be finite has dramatically different effects. In particular, we will see that WAMSO inhabits all finite levels of the Borel hierarchy, while it is known that WMSO+$\mathbb{U}$ is confined in the third level [13].

Weak asymptotic monadic second-order logic (WAMSO) is obtained by restricting set quantification to finite sets (the syntax remains the same). We will write $\exists^w X$ and $\forall^w X$ when we want to emphasize that the quantifiers are weak, *i.e.*, range over finite sets. Let us remark that, as usual, the weak logic is not strictly speaking a fragment of the full logic since, in general, AMSO is not able to express that a set is infinite. However, on models such as words, ω-words, or even infinite trees, the property of 'being finite' is expressible, even in MSO.

It turns out that, in a certain sense, weak quantifiers commute with number quantifiers.

Lemma 4. *There exists a* WAMSO*-formula* $\psi(X, r)$ *such that, for every sequence* $\bar{Q}\bar{t}$ *of number quantifiers, every* WAMSO*-formula* $\bar{Q}\bar{t}\,\varphi(X, \bar{t})$*, and all weighted* ω*-words* w*,*

$$w \models \exists^w X \bar{Q}\bar{t}\varphi(X, \bar{t}) \quad \textit{iff} \quad w \models \exists r \bar{Q}\bar{t}\exists^w X[\varphi(X, \bar{t}) \wedge \psi(X, r)] \ .$$

By this lemma, it follows that we can transform every WAMSO-formula into *number-prenex form, i.e.*, into the form $\bar{Q}\bar{t}\varphi$, where $\bar{Q}\bar{t}$ is a sequence of number quantifiers while φ does not contain such quantifiers. However, this translation adds new number variables in the formula. The fragment of AMSO-formulae in number prenex form is denoted AMSO$^{\mathrm{np}}$. For weak quantifiers, we obtain the logic WAMSO$^{\mathrm{np}}$ in the same way.

Theorem 5. *The logics* WAMSO*,* AMSO$^{\mathrm{np}}$ *and* WAMSO$^{\mathrm{np}}$ *effectively have the same expressive power over weighted* ω*-words.*

2.3 Separation results

To separate the expressive power of the logics introduced so far, we employ topological arguments. One way to show that a logic is strictly more expressive than one of its fragments is to prove that it can define languages of a topological complexity the fragment cannot define. In our case we use the Borel hierarchy and the projective hierarchy to measure topological complexity.

Theorem 6. *Languages definable in* AMSO *strictly inhabit all levels of the projective hierarchy, and not more. Languages definable in* WAMSO *strictly inhabit all finite levels of the Borel hierarchy, and not more.*

This is proved using standard reduction techniques. We obtain the following picture:

$$\overbrace{\mathrm{MSO} \;=\; \mathrm{WMSO}}^{\mathrm{Bool}(\boldsymbol{\Sigma}^0_2)} \subsetneq \overbrace{\mathrm{WAMSO} \;=\; \mathrm{AMSO}^{\mathrm{np}}}^{\text{all Borel levels of finite rank}} \subsetneq \overbrace{\mathrm{AMSO}}^{\text{all projective levels}} .$$

As an immediate consequence, we obtain the corollary that AMSO is strictly more expressive than WAMSO and AMSO$^{\mathrm{np}}$, that AMSO$^{\mathrm{np}}$ is strictly more expressive than MSO, and that WAMSO is strictly more expressive than WMSO.

3 Weak asymptotic monadic second-order logic and tiling problems

We have introduced in the previous sections several logics with quantitative capabilities. The analysis performed shows that WAMSO (or equivalently AMSO$^{\mathrm{np}}$) offers a good compromise in difficulty in the quest for solving advanced logics like MSO+$\mathbb{U}$. Indeed, in terms of Borel complexity, it is significantly simpler than other logics like AMSO and EAMSO, and hence MSO+$\mathbb{U}$. Despite its relative simplicity, this logic is, still in terms of Borel complexity, significantly more complex than any other extensions of WMSO known to be decidable over infinite words, e.g., WMSO+$\mathbb{U}$[4] and WMSO+$\mathbb{R}$[5] [2, 5]. Both of these logics can define Boolean combinations of languages at the third level of the Borel hierarchy.

In this section, we develop techniques for attacking the satisfiability problem of WAMSO over weighted ω-words, though we are not able to solve this problem itself. Our contribution in this direction is to reduce the satisfiability problem of WAMSO to a natural kind of tiling problem, new to our knowledge, the decidability of which is unknown, even in the simplest cases. As a teaser, let us show the simplest form of such tiling problems:

Open problem 7 *Given two regular languages K and L over an alphabet Σ where K is closed under letter removal, can we decide whether, for every n, there exists a Σ-labelled picture of height n such that all rows belong to L and all columns to K?*

Note that this problem would clearly be undecidable if K was not required to be sub-word closed. In the remainder of the section, we first introduce these problems in a more general setting (a multidimensional version of it), and give the essential ideas explaining why the decidability of satisfiability for WAMSO reduces to such tiling problems.

[4] The weak fragment of MSO+$\mathbb{U}$, where set quantifiers range over finite sets.

[5] An extension of WMSO with an unusual recurrence operator. Adding this operator to MSO yields a logic equivalent to MSO+$\mathbb{U}$.

3.1 Lossy tiling problems

A *picture* $p : [h] \times [w] \to \Sigma$ is a rectangle labelled by a (fixed) finite alphabet Σ, where $h \in \mathbb{N}$ is the *height* and $w \in \mathbb{N}$ the *width* of the picture. For $0 \leq i < w$, the ith *column* of the picture is the word $p(0,i)p(1,i)\dots p(h-1,i)$. A *band of height* m in a picture is obtained by erasing all but m-many rows from a picture. We regard bands of height m as words over the alphabet Σ^m. Formally, for $0 \leq j_1 < j_2 < \cdots < j_m < h$, the *band for rows* $j_1, \dots, j_m$ is the word $(p(j_1,0),\dots,p(j_m,0))\dots(p(j_1,w-1),\dots,p(j_m,w-1))$. Our *tiling problems* have the following form. Fix an alphabet Σ and a *dimension* $m \in \mathbb{N}$.

Input: A *column language* $K \subseteq A^*$ and a *row language* $L \subseteq (A^m)^*$, both regular.
Question: Does there exist, for all $h \in \mathbb{N}$, a picture p of height h such that

- all columns in p belong to K,
- all bands of height m in p belong to L?

Such a picture is called a *solution* of the *tiling system* (K, L).

Of course, in general such problems are undecidable, even in dimension $m = 1$. Consequently, we consider two special cases of tiling systems: *monotone* and *lossy* ones. A tiling system (K, L) is *lossy* if K is closed under sub-words: for all words u, v and all letters a, $uav \in K$ implies $uv \in K$. A tiling system (K, L) is *monotone* if there exists a partial order $\leq$ on the alphabet Σ (which we extend component-wise, i.e., letter-by-letter, to Σ^* and to $(\Sigma^m)^*$) such that $u \leq v$ and $u \in L$ implies $v \in L$, and $uabv \in K$ implies $ucv \in K$, for some c with $c \geq a$ and $c \geq b$. Consequently, if we have a solution p of a lossy tiling system, we can obtain new solutions (of smaller height) by *removing* arbitrarily many rows of p. For a monotone tiling system, we obtain a new solution by *merging* two rows.

Example 8. (a) Consider the one-dimensional lossy tiling problem defined by $L = a^*ba^*$ and $K = a^*b^?a^*$. There are solutions of every height n: take a picture that has label a everywhere but for one b in each row, and at most one b per column (see Figure 1 (a)). The width of such a solution is at least n.

$$\begin{array}{ccccccc}
a & \boldsymbol{b} & a & a & a & a & a \\
\boldsymbol{b} & a & a & a & a & a & a \\
a & a & \boldsymbol{b} & a & a & a & a \\
a & a & a & a & a & a & \boldsymbol{b} \\
a & a & a & a & \boldsymbol{b} & a & a
\end{array}$$

(a)

$$\begin{array}{ccccccc}
\boldsymbol{b} & c & c & c & c & c & c \\
a & \boldsymbol{b} & c & c & c & c & c \\
a & a & \boldsymbol{b} & c & c & c & c \\
a & a & a & a & \boldsymbol{b} & c & c \\
a & a & a & a & a & a & \boldsymbol{b}
\end{array}$$

(b)

$$\begin{array}{c|ccc|c|ccc|c}
\boldsymbol{d} & \boldsymbol{b} & a & a & c & \boldsymbol{b} & a & a & c \\
c & a & \boldsymbol{b} & a & \boldsymbol{d} & a & \boldsymbol{b} & a & c \\
c & a & a & \boldsymbol{b} & c & a & a & \boldsymbol{b} & \boldsymbol{d}
\end{array}$$

(c)

$$\begin{array}{cccccccc}
\boldsymbol{a} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
b & 1 & 1 & 1 & \boldsymbol{a} & 1 & 1 & 1 \\
b & 1 & \boldsymbol{a} & 1 & b & 1 & \boldsymbol{a} & 1 \\
b & \boldsymbol{a} & b & \boldsymbol{a} & b & \boldsymbol{a} & b & \boldsymbol{a}
\end{array}$$

(d)

Fig. 1. Some solutions to tiling problems

(b) A similar example uses the languages $L = a^*bc^*$ and $K = a^*b^?c^*$. Again, there exist solutions for all heights n, and the corresponding width is at least n

too. However, the solution is more constrained since it involves occurrences of b letters to describe some sort of diagonal in the solution (see Figure 1 (b)).

(c) More complex is the system with $L = (ca^*ba^*)^*d(a^*ba^*c)^*$ and $K = a^*b^?a^*+c^*d^?c^*$. There are also solutions of all heights, but this time, the minimal width for a solution is quadratic in its height (see Figure 1 (c)).

(d) Our final example is due to Paweł Parys. It consists of $L = a1^*+(b1^*a1^*)^+$ and $K = b^*a^?1^*$. All solutions of this system have exponential length (see Figure 1 (d)).

Theorem 9. *The satisfiability problem for* WAMSO *and the monotone tiling problem are equivalent. Both reduce to the lossy tiling problem.*

Conjecture 10. Monotone tiling problems and lossy ones are decidable.

This is the main open problem raised in this paper, even in dimension one. In the remainder, we will sketch some ideas on how to reduce the satisfiability of WAMSO to lossy tiling problems.

3.2 From ω-words to finite words

Using Ramsey arguments in the spirit of Büchi's seminal proof [7], we can reduce WAMSO over ω-words to the following question concerning sequences of finite words. Consider a formula $\bar{Q}\bar{t}\varphi(\bar{t})$ in AMSO^{np} and a sequence $\bar{u} = u_1, u_2, \dots$ of weighted finite words. We say that $\bar{u}$ $(\bar{Q}\bar{t})$*-ultimately satisfies* $\varphi(\bar{t})$ if

$$\bar{Q}\bar{t}[u_i \models \varphi(\bar{t}) \text{ for all but finitely many } i] .$$

The *limit satisfiability problem* for AMSO^{np} is to decide, given a formula $\bar{Q}\bar{t}\varphi(\bar{t})$, whether $\varphi(\bar{t})$ is $(\bar{Q}\bar{t})$-ultimately satisfied by some sequence $\bar{u}$.

Lemma 11. *The satisfiability problem for* AMSO^{np} *and the limit satisfiability problem for* AMSO^{np} *can be reduced one to the other. Furthermore, the prefix of number quantifiers is preserved by these reductions.*

Of course, the interesting reduction is from satisfiability of AMSO^{np} on infinite words to limit satisfiability. We follow here an approach similar to Büchi's technique or, more precisely, its compositional variant developed by Shelah [16]. It amounts to use Ramsey's Theorem for chopping ω-words into infinitely many pieces that have the same theory. However, in this weighted situation, this kind of argument requires significantly more care.

A typical example would be to solve the satisfiability of the AMSO^{np}-formula $\varphi := \forall s \exists r \forall x (\exists y > x)[s < f(y) \leq r]$ stating that there are infinitely many values that occur infinitely often (Example 1). It reduces to solving the limit satisfiability of the formula $\psi := \forall s \exists r \exists y[s < f(y) \leq r]$. A limit model for this formula would be the sequence (in which we omit the letters and only mention the weights) $\bar{u} = 0, 01, 012, 0123, \dots$. Indeed, for all s, fixing $r = s + 1$, the formula $\exists y[s < f(y) \leq r]$ holds for almost all u_i. If we concatenate this sequence of words, we obtain the weighted ω-word $0010120123\dots$ which satisfies φ. Conversely, every ω-word satisfying φ can be chopped into an infinite sequence of finite weighted words that satisfy ψ in the limit. In fact, this last reduction is more complex since it requires us to take care of the values contained in the finite prefixes. This is just an example, since in general the reduction is 'one-to-many' and involves regular properties of the finite prefixes.

4 Extended asymptotic monadic logic

In this section we prove that the decidability problem for AMSO over ω-words is equivalent to the corresponding problem for MSO+$\mathbb{U}$. To do so we introduce an extension of AMSO called *extended asymptotic monadic second-order logic* (EAMSO). This logic extends AMSO by quantifiers over weight functions. Inside a quantifier $\exists f$ we can use the function f in the usual constructions of AMSO. Note that variables for weight functions are not subject to any positivity constraint. Only number variables do have to satisfy such constraints.

Example 12. Let L_S be the language of ω-words over the alphabet $\{a, b, c\}$ such that, either there are finitely many occurrences of the letter b, or the number of a appearing between consecutive b tends to infinity. Consider the EAMSO-formula

$$\psi := \exists f \forall r \exists s \exists w \forall x \forall z \big[(w < x < z \wedge b(x) \wedge b(z)) \rightarrow \\ \exists y (x < y < z \wedge a(y) \wedge r < f(y) \leq s)\big] \ .$$

This formula defines L_S as follows. It guesses a weight function f and expresses that, for every number r, there exists a number s such that, ultimately, every two b-labeled positions $x < z$ are separated by an a-labeled position y with weight in $(r, s]$. It is easy to see that, if the number of a in an ω-word separating consecutive b tends to infinity, the weight function f defined by

$$f(x) = \begin{cases} 0 & \text{if the letter at } x \text{ is not } a \\ r & \text{if } x \text{ is the } r\text{-th occurrence of the letter } a \text{ after the last} \\ & \quad \text{occurrence of the letter } b \text{ or the beginning of the word} \end{cases}$$

witnesses that the ω-word is a model of ψ. One can show that the converse also holds, *i.e.*, an ω-word satisfies ψ if and only if the number of a occurring between b tends to infinity (or there are finitely many occurrences of b).

The interesting point concerning EAMSO is that we can prove that, as far as satisfiability over infinite words is concerned, this logic is essentially equivalent to both AMSO and MSO+$\mathbb{U}$. Let us recall that MSO+$\mathbb{U}$ is the extension of MSO with a new quantifier $\mathbb{U}X\varphi$ which signifies that 'there exists sets of arbitrarily large finite size such that φ holds'. For instance, it is straightforward to define the above language L_S in MSO+$\mathbb{U}$.

Theorem 13. (a) *For every* MSO+$\mathbb{U}$*-sentence, we can compute an* EAMSO*-sentence equivalent to it over* ω*-words. Conversely, for every* EAMSO*-sentence,* *there effectively exists an* MSO+$\mathbb{U}$*-sentence such that the former is satisfiable over* ω*-words if, and only if, the latter is.*

(b) *For every* EAMSO*-sentence, we can compute an* AMSO*-sentence such that the former is satisfiable over* ω*-words if, and only if, the latter is.*

To compare the expressive power of EAMSO and AMSO, we again employ topological arguments. It is easy to show that, over ω-words without weights, AMSO collapses to MSO and, therefore, defines only Borel sets. However, according to Theorem 13, EAMSO is at least as expressive as MSO+$\mathbb{U}$ which reaches all levels of the projective hierarchy [13], even over non-weighted ω-words. Consequently, EAMSO is strictly more expressive than AMSO.

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Appendices

In the remainder of this document we present the proofs that had to be omitted from the main part due to space considerations. These appendices follow the structure of the main part of the paper. Appendix A presents the proofs missing from Section 2, Appendix B the ones missing from Section 2.3, and Appendix C the ones missing from Section 3.

A Logics

We successively provide complementary proofs for AMSO in Section A.1, for WAMSO and $\mathrm{AMSO}^{\mathrm{np}}$ in Section A.2, and for EAMSO in Section A.3.

A.1 Some more on AMSO

The positivity assumption on the use of number variables results in the fact that whenever some number r makes a formula $\exists r\varphi$ true, then all larger values would also make the formula true. A consequence of this is that some syntactic transformations are valid in formulae, that are not allowed for general logics.

Proposition 14. *In* AMSO *the following equivalences hold:*

$$\begin{aligned} \exists r\exists s\varphi(r,s) &\equiv \exists r\varphi(r,r)\,, \qquad & (\exists r\varphi)\wedge(\exists r\,\psi) &\equiv \exists r(\varphi\wedge\psi)\,,\\ \forall r\forall s\varphi(r,s) &\equiv \forall r\varphi(r,r)\,, \qquad & (\forall r\varphi)\vee(\forall r\psi) &\equiv \forall r(\varphi\vee\psi)\,. \end{aligned}$$

Proof. In the first equivalence, the right to left implication always holds. For the converse, just note that, if values for r and s are chosen such that φ holds, the formula would still be true if we would choose $\max(r,s)$ as value of both r and s, by the above remark. Hence, there exists a value of r (namely $\max(r,s)$) such that $\varphi(r,r)$ holds.

The second equivalence follows from the first one since $\exists r\varphi(r)\wedge\exists s\psi(s)$ is equivalent to $\exists r\exists s(\varphi(r)\wedge\psi(s))$ (this is true in any logic). The two remaining equivalences follow by duality. ☐

We next prove Theorem 2 stating that AMSO and BMSO are equivalent.

Lemma 15. (a) *For every* BMSO*-formula* $\varphi(\bar{X})$*, there is an* AMSO*-formula* $\varphi^*(\bar{X})$ *such that*

$$\mathfrak{A}\models\varphi(\bar{P}) \quad\text{iff}\quad \mathfrak{A}\models\varphi^*(\bar{P}),$$

for all weighted structures $\mathfrak{A}$ *and sets* $\bar{P}$*.*

(b) *For every* AMSO*-formula* $\varphi(\bar{X})$ *without free number variables, there is a* BMSO*-formula* $\varphi^*(\bar{X})$ *such that*

$$\mathfrak{A}\models\varphi(\bar{P}) \quad\text{iff}\quad \mathfrak{A}\models\varphi^*(\bar{P}),$$

for all weighted structures $\mathfrak{A}$ *and sets* $\bar{P}$*.*

Proof. (a) We obtain φ^* from φ by replacing every atom of the form $f[X] < \infty$ by the AMSO-formula

$$\exists r(f[X] \leq r)\,.$$

(b) Intuitively, instead of quantifying over a number r, we quantify over sets of the form

$$Z := \{\, a \mid f(a) \leq r \,\}\,.$$

Formally, the translation is as follows. For every number variable r and every weight function f, we fix a set variable $Z_{f,r}$. Let $f_0, \dots, f_l$ be an enumeration of all weight functions.

$$\begin{aligned}
(X \subseteq Y)^* &:= X \subseteq Y\,, & (\psi \wedge \vartheta)^* &:= \psi^* \wedge \vartheta^*\,,\\
(X \cap Y = \emptyset)^* &:= X \cap Y = \emptyset\,, & (\psi \vee \vartheta)^* &:= \psi^* \vee \vartheta^*\,,\\
(R\bar{X})^* &:= R\bar{X}\,, & (\neg\psi)^* &:= \neg\psi^*\,,\\
(f[X] \leq r)^* &:= X \subseteq Z_{f,r}\,, & (\exists X\psi)^* &:= \exists X\psi^*\,,\\
(f[X] > s)^* &:= X \cap Z_{f,r} = \emptyset\,, & (\forall X\psi)^* &:= \forall X\psi^*\,,
\end{aligned}$$

$$\begin{aligned}
(\exists r\psi)^* &:= \exists Z_{f_0,r} \cdots \exists Z_{f_l,r}\Big[\textstyle\bigwedge_{i \leq l} f_i[Z_{f_i,r}] < \infty \wedge \psi^*\Big]\,,\\
(\forall s\psi)^* &:= \forall Z_{f_0,r} \cdots \forall Z_{f_l,r}\Big[\textstyle\bigwedge_{i \leq l} f_i[Z_{f_i,r}] < \infty \rightarrow \psi^*\Big]\,.
\end{aligned}$$

A.2 The logic WAMSO

We need to prove Lemma 4, and then complete the proof of Theorem 5. The following operation on formulae provides a translation from AMSO to MSO.

Definition 16. *Let $\bar{r}, \bar{s}, \bar{t}$ be tuples of number variables and let $\varphi(\bar{r}, \bar{s})$ be an* AMSO*-formula. We denote by $\varphi{\downarrow_{\bar{t}}}$ the formula obtained from φ by substituting*

- *the formula* `true` *for every atom of the form $f[X] \leq r_i$ with $r_i \in \bar{t}$, and*
- *the formula* `false` *for every atom of the form $f[X] > s_i$ with $s_i \in \bar{t}$.*

In the case where $\bar{t} = \bar{r}\bar{s}$, we simply write $\varphi{\downarrow}$ for $\varphi{\downarrow_{\bar{t}}}$.

The following observation follows immediately from the definitions.

Lemma 17. *Let $\bar{r}, \bar{r}', \bar{s}, \bar{s}'$ be number variables and let $\bar{m}, \bar{m}', \bar{n}, \bar{n}'$ be tuples of numbers such that every component of $\bar{m}$ and $\bar{n}$ is greater than or equal to the maximal weight of $\mathfrak{A}$. Then*

$$\mathfrak{A} \models \varphi(\bar{m}\bar{m}', \bar{n}\bar{n}') \quad \text{iff} \quad \mathfrak{A} \models \varphi{\downarrow_{\bar{r}',\bar{s}'}}(\bar{m}', \bar{n}')\,.$$

The next lemma summarises the key property of the operation $\varphi{\downarrow_{\bar{t}}}$. It is a direct consequence of the preceding observation.

Lemma 18. *Let $\varphi(\bar{t})$ be an* AMSO*-formula and $\bar{Q}\bar{t}$ a prefix a number quantifiers. Then*

$$\mathfrak{A} \models \bar{Q}\bar{t}\varphi \quad \text{iff} \quad \mathfrak{A} \models \varphi{\downarrow_{\bar{t}}}\,,$$

for every structure $\mathfrak{A}$ whose weights are bounded.

Proof. It is sufficient to prove the following two equivalences

$$\begin{aligned} \mathfrak{A} \models \exists r\psi(r) \quad &\text{iff} \quad \mathfrak{A} \models \psi{\downarrow_r}\,, \\ \mathfrak{A} \models \forall s\psi(s) \quad &\text{iff} \quad \mathfrak{A} \models \psi{\downarrow_s}\,, \end{aligned}$$

for every formula ψ.

Let k be some number larger than all weights of $\mathfrak{A}$. First, suppose that $\mathfrak{A} \models \exists r\psi(r)$. Then there is some number m such that $\mathfrak{A} \models \psi(m)$. By monotonicity, it follows that $\mathfrak{A} \models \psi(m + k)$. Consequently, Lemma 17 implies that $\mathfrak{A} \models \psi{\downarrow_r}$.

Conversely, suppose that $\mathfrak{A} \models \psi{\downarrow_r}$. By Lemma 17 it follows that $\mathfrak{A} \models \psi(k)$. Hence, $\mathfrak{A} \models \exists r\psi(r)$.

For the second claim, suppose that $\mathfrak{A} \models \forall s\psi(s)$. Then $\mathfrak{A} \models \psi(k)$ which implies, by Lemma 17, that $\mathfrak{A} \models \psi{\downarrow_s}$.

Conversely, suppose that $\mathfrak{A} \models \psi{\downarrow_s}$. By Lemma 17 it follows that $\mathfrak{A} \models \psi(n)$, for all $n \geq k$. By monotonicity, it follows that $\mathfrak{A} \models \psi(n)$, for all n. Hence, $\mathfrak{A} \models \forall s\psi(s)$.

The following proposition is a refinement of Lemma 4 from the main part of the paper.

Proposition 19. *There exists a* WAMSO*-formula $\vartheta(X, r)$ such that, for every* AMSO*-formula $\varphi(X, \bar{t})$ and all sequences $\bar{Q}\bar{t}$ of number quantifiers,*

$$w \models \exists^w X\bar{Q}\bar{t}\varphi \quad \text{iff} \quad w \models \exists r\bar{Q}\bar{t}\exists^w X[\vartheta(X, r) \wedge \varphi]\,,$$

for all weighted ω-words w.

Proof. We claim that the formula

$$\vartheta(X, r) := (\forall x \in X)(\forall y < x) \bigwedge_f f(y) \leq r$$

has the desired properties where the conjunction ranges over all weight functions f.

For the proof, we distinguish two cases. First, suppose that the all weights are bounded in w. Then Lemma 18 implies that

$$\begin{aligned} w \models \exists r\bar{Q}\bar{t}\exists^w X[\vartheta(X, r) \wedge \varphi] \quad &\text{iff} \quad w \models \exists^w X[\vartheta(X, r) \wedge \varphi]{\downarrow} \\ &\text{iff} \quad w \models \exists^w X[\vartheta(X, r){\downarrow} \wedge \varphi{\downarrow}] \\ &\text{iff} \quad w \models \exists^w X\varphi{\downarrow} \\ &\text{iff} \quad w \models \exists^w X\bar{Q}\bar{t}\varphi\,. \end{aligned}$$

It remains to consider the case where the weights of w are unbounded. We claim that, in this case,

$$w \models (\exists^w X\, \vartheta(X,m))\bar{Q}\bar{t}\varphi \quad \text{iff} \quad w \models \bar{Q}\bar{t}(\exists^w X\, \vartheta(X,m))\varphi\,,$$

for all $m < \omega$. For the proof, consider the weight function

$$g(x) := \max\{\, f(y) \mid y \leq x \text{ and } f \text{ a weight function}\,\}\,.$$

This function is unbounded and (not necessarily strictly) increasing. For $m < \omega$ and a finite nonempty set $P \subseteq \omega$, it follows that

$$w \models \vartheta(P,m) \quad \text{iff} \quad g(\max P) \leq m\,,$$

Consequently, for every $m < \omega$, there are only finitely many finite sets $P \subseteq \omega$ such that $w \models \vartheta(P,m)$. For $m < \omega$, it follows by Proposition 14 that

$$\begin{aligned}
&w \models (\exists X\, \vartheta(X,m))\bar{Q}\bar{t}\varphi(X,\bar{t}) \\
\text{iff}\quad &w \models \bigvee\{\, \bar{Q}\bar{t}\varphi(P,\bar{t}) \mid P \subseteq \omega,\ w \models \vartheta(P,m)\,\} \\
\text{iff}\quad &w \models \bar{Q}\bar{t}\bigvee\{\, \varphi(P,\bar{t}) \mid P \subseteq \omega,\ w \models \vartheta(P,m)\,\} \\
\text{iff}\quad &w \models \bar{Q}\bar{t}(\exists X\, \vartheta(X,m))\varphi\,.
\end{aligned}$$

Note that, for every finite set P, there is some number m with $w \models \vartheta(P,m)$. Hence, having proved the claim, it follows that

$$\begin{aligned}
w \models \exists X\, \bar{Q}\bar{t}\varphi \quad &\text{iff} \quad w \models \exists r(\exists X\, \vartheta(X,r))\bar{Q}\bar{t}\varphi \\
&\text{iff} \quad w \models \exists r\bar{Q}\bar{t}(\exists X\, \vartheta(X,r))\varphi \\
&\text{iff} \quad w \models \exists r\bar{Q}\bar{t}\exists X\, [\vartheta(X,r) \wedge \varphi]\,.
\end{aligned}$$

Corollary 20. *On ω-words, every* WAMSO*-formula is equivalent to a* WAMSO*-formula in number prenex form.*

Let us now complete the proof of Theorem 5, stating that, over infinite words, $\mathrm{WAMSO}^{\mathrm{np}}$, WAMSO and $\mathrm{AMSO}^{\mathrm{np}}$ have the same expressive power. The only inclusion that is missing is to prove that, over infinite words, every $\mathrm{AMSO}^{\mathrm{np}}$-formula can be turned into a $\mathrm{WAMSO}^{\mathrm{np}}$-formula. This last direction relies on McNaughton's result stating that, over infinite words, every MSO-formula over infinite words can be turned into a WMSO-formula.

Lemma 21. *For every* $\mathrm{AMSO}^{\mathrm{np}}$*-formula* $\bar{Q}\bar{t}\varphi$*, there exists a* $\mathrm{WAMSO}^{\mathrm{np}}$*-formula* $\bar{Q}\bar{t}\varphi'$ *that is equivalent to* $\bar{Q}\bar{t}\varphi$ *over weighted ω-words.*

Proof. For the proof, we use the machinery of AMSO^0_h-types introduced in Section C.1 below. We will need the fact that AMSO^0_h-types are partially ordered by $\subseteq$, that, for each such type p, there is a formula χ_p stating that a word has some type $q \supseteq p$, and that we have two operations $\oplus$ and ${}^\omega$ on types corresponding to the concatenation of words and to their ω-power.

Let h be the quantifier rank of φ. By the Theorem of Ramsey and Lemmas 37 and 38, we obtain the following equivalence over ω-words:

$$\varphi \equiv \bigvee_{(p,e):\varphi\in p\oplus e^\omega} \chi_{p,e} \,,$$

where the disjunction is over all AMSO^0_h-types p and e such that $\varphi \in p \oplus e^\omega$ and the formula $\chi_{p,e}$ states that the ω-word w has a factorisation $w_0w_1w_2\dots$ such that w_0 has some type containing p and all other factors have a type (not necessarily the same) containing e.

Hence, it is sufficient to translate the formulae $\chi_{p,e}$ into WAMSO. Given an infinite word w, let λ be the function mapping a pair $x < y$ of positions of w to the AMSO^0_h-type of the factor from position x to position $y-1$. Using predicates of the form $\lambda(x,y) \supseteq q$, for types q, we can write an MSO-formula $\vartheta_{p,e}$ stating there are positions $x_0 < x_1 < x_2 < \dots$ such that

$$\lambda(0,k_0) \supseteq p \quad \text{and} \quad \lambda(x_i,x_{i+1}) \supseteq e\,, \quad \text{for every } i\,.$$

Since, over ω-words, MSO and WMSO have the same expressive power, $\vartheta_{p,e}$ is equivalent to some WMSO-formula $\vartheta^0_{p,e}$.

Let $\eta_q(x,y)$ be an WAMSO-formula stating that $\lambda(x,y) \supseteq q$. The desired WAMSO-formula $\chi_{p,e}$ is obtained from the formula $\vartheta^0_{p,e}$ by replacing

- every positive occurrence of an atomic formula $\lambda(x,y) \supseteq q$ by η_q and
- every negative occurrence of an atomic formula $\lambda(x,y) \supseteq q$ by $\bigwedge_r \neg\eta_r$, where the conjunction is over all types r such that $r \cup q$ is inconsistent.

A.3 The logic EAMSO and MSO+$\mathbb{U}$

We begin showing that the decidability of model-checking/satisfiability of EAMSO can be reduced to the one of AMSO, thus showing that EAMSO and AMSO have the same decidability status. We begin with a preparatory lemma. In this statement EBMSO represents the logic obtained by extending BMSO by quantifiers over weight functions: there are no number quantifiers, but there is a predicate $f[X] < \infty$ where f is a function variable and X a set variable.

Lemma 22. EAMSO *and* EBMSO *are effectively equivalent over ω-words.*

Proof. The translation from EBMSO to EAMSO is straightforward: it suffices to substitute each occurrence of $f[X] < \infty$ by the EAMSO-formula defining it. For the converse direction some ideas from the translation from AMSO to BMSO are used. However, these are not sufficient by themselves. The essential problem is that this construction does not consider the case of a quantifier over weight functions appearing in the scope of a number quantifier.

Our goal is to transform an EAMSO-formula φ into an equivalent EBMSO formula φ^*. The transformation is syntactic, and is done inductively on the structure. We begin our description by explaining how the free variables F^* of φ^* are obtained from the set of free variables F of φ:

– For every set variables X in F, the same variable is in F^*.
– For every first-order variable x in F, the same variable is in F^*.
– For every weight function variable f in F, the same variable occurs in F^*.
– For every weight function variable f and each number variable r, a new set variable $L_{f,r}$ is in F^* (to be thought of as the set of positions x for which $f(x) \leq r$).

We assume that the bound variables are numbered, or equivalently totally ordered. This order is required to be consistent with the sequence of quantifiers binding the corresponding variables. This means that whenever we consider some formula which quantifies two bound variables r, s such that s is quantified inside the scope of r, then $r < s$ with respect to this order/numbering. The intuition behind that is that we can safely assume that the valuations of these variables respect the same order, $r_1 < r_2 < \cdots < r_\ell$. We refer to this order as the *quantification order.*

When translating formulae, every construct is preserved unmodified but the ones involving weight functions and number variables. These are translated according to the following rules: Atomic MSO-formulae, boolean operations, and quantifiers over elements and sets are unchanged.

$$(f(x) \leq r)^* := x \in L_{f,r}\,, \quad \text{for existentially quantified variables } r\,,$$
$$(f(x) > s)^* := x \notin L_{f,s}\,, \quad \text{for universally quantified variables } s\,,$$

$$(\exists s\varphi)^* := \exists L_{f_1,s} \cdots \exists L_{f_k,s}\Big[\varphi^* \wedge \overbrace{\bigwedge_{i=1}^{k} f_i[L_{f_i,s}] < \infty}^{A} \wedge \underbrace{\bigwedge_{i=1}^{k}\bigwedge_{j=1}^{\ell} L_{f_i,r_j} \subseteq L_{f_i,s}}_{B}\Big]$$

where $f_1, \ldots, f_k$ are the function variables in F and $r_1, \ldots, r_\ell$ are the number variables in F,

$$(\forall s\varphi)^* := \forall L_{f_1,s} \cdots \forall L_{f_k,s}[A \wedge B \rightarrow \varphi^*]\,,$$

$$(\exists f\varphi)^* := \exists f \exists L_{f,r_1} \cdots \exists L_{f,r_\ell}\Big[\varphi^* \wedge \overbrace{\bigwedge_{j=1}^{\ell} f[L_{f,r_1}] < \infty}^{C} \wedge \underbrace{\bigwedge_{j=1}^{\ell-1} L_{f,r_i} \subseteq L_{f,r_{i+1}}}_{D}\Big]\,,$$

where $r_1, r_2, \ldots, r_\ell$ are the number variables in F in their order of quantification,

$$(\forall f\varphi)^* := \forall f \forall L_{f,r_1} \cdots \forall L_{f,r_\ell}[C \wedge D \rightarrow \varphi^*]\,.$$

Let us first note that membership tests to the variable $L_{f,r}$ always appear under the same number of negations as the statement $f(x) \leq r$ it originates from.

Of course, in the correctness proof of this construction, we will have to relate valuations v over the free variables F of φ to valuations v^* over the free variables F^* of φ^* in order to model the transfer of semantics. We also have to impose some structural constraints on these valuations. We assume that the bound variables in F are $r_1, \ldots, r_\ell$ in this quantification order.

- We call the valuation v *valid* of $v(r_1) < v(r_2) < \cdots < v(r_\ell)$.
- We call the valuation v^* *valid* if, for all weight variables f, $v^*(L_{f,r_1}) \subseteq v^*(L_{f,r_2}) \subseteq \cdots \subseteq v^*(L_{f,r_\ell})$, and furthermore f is bounded over L_{f,r_ℓ} (note that this is EBMSO-definable, and that in particular the above formulae A, B, C and D are expressing these constraints).
- Given a valuation valid valuation v of F, define v^* to be the valuation of F^* such that
  - v and v^* coincide over all set variables, all first-order variables, and all weight variables,
  - for all weight variables f and all number variables r from F^*,

$$v^*(L_{f,r}) = \{i \in \omega \mid v(f)(i) \leq v(r)\} \ .$$

  Let us note that if v is valid, then v^* is valid.

Let us comment first on the role of the subformulae A, B, C and D. Consider the case in which φ is $\exists s\psi$, and the inductive definition of φ^*. Assume given a valid valuation v^* of φ^*. The formula φ^* is introducing new variables, and immediately requiring them to satisfy A and B, yielding a new valuation w^*. It is easy to check that A and B imply that w^* is also valid. The same goes for the rule defining $(\exists f\psi)^*$, as well as for the two dual constructs.

We shall now prove by structural induction on a formula φ of EAMSO with free variables F that, for all valid valuations v of F,

$$u, v \models \varphi \qquad \text{implies} \qquad u, v^* \models \varphi^* \ .$$

We assume that the negations are pushed to the leaves (this means that we have to treat all constructions as well as their dual version).

Let us consider some formula φ with free variables F and a valid valuation v of F such that $u, v \models \varphi$. Our goal is to show that $u, v^* \models \varphi^*$. This is done by case distinction. In all the cases, we assume that the free number variables are $r_1, \ldots, r_\ell$ in quantification order.

Bases cases. The only interesting base cases are when number variables are used. Assume first that φ is $f(x) \leq r$ for some upper-bound variable r and that $u, v \models f(x) \leq r$. This means that $v(f)(v(x)) \leq v(r)$ and, hence, by definition of v^*, $v^*(x) = v(x) \in v^*(L_{f,r})$. Therefore, $u, v^* \models \varphi^*$.

Assume now that φ is $f(x) > r$ for some lower-bound variable r and that $u, v \not\models f(x) > r$. Then $v(f)(v(x)) > v(r)$ and, by definition of v^*, it follows that $v^*(x) = v(x) \notin v^*(L_{f,r})$. Hence, $u, v^* \models \varphi^*$.

The other base cases as well as disjunction and conjunction are all straightforward. The case of negation does not exist (these were pushed to the leaves). The case of quantifiers over set variables and first-order variables are immediate.

Case of number quantifiers. The next interesting case is the one of an existential quantifier, i.e., that φ is $\exists s\psi$. The free variables of ψ are $F \cup \{s\}$. The free variables of ψ^* are $F^* \cup \{L_{f,s} \mid f \in F\}$. Since we assume that $u, v \models \exists s\psi$, there is a value n such that $u, w \models \psi$ where w is v extended with $s = n$. Without loss of generality, using the monotonicity in the use of s in ψ, we can assume n to be larger than $v(r)$ for all other number variables in F. This makes v a valid valuation. By induction hypothesis, we obtain $u, w^* \models \psi^*$. This proves that $u, w \models \varphi^*$.

Let us consider now the case in which φ is $\forall s\psi$. Assume that $u, v \models \forall s\psi$. Let w' be any extension of v^* by valuations of $L_{f,s}$ for a weight function variable $f \in F$. We have to prove that $u, w' \models (A \wedge B) \rightarrow \psi^*$. Firstly, if u, w^* does not satisfy A or B, then $u, w' \models (A \wedge B) \rightarrow \psi^*$ by definition. Hence, we assume from now $u, w' \models A \wedge B$. Since w' extends the valid valuation v^*, this means that w' is a valid valuation. Let n be some number larger than $v(r)$ for all other number variables in F and larger than $v(f)(i)$ for all weight function variables $f \in F$ and $i \in w^*(L_{f,r})$ (this is possible since $u, w' \models A$, and hence $v(f)(i)$ is bounded when f ranges in F and i in $w^*(L_{f,r})$). Let now w be a valuation extending v with $w(s) = n$. By construction w is valid. Furthermore, since $u, v \models \forall s\psi$, $u, w \models \psi$. Hence by inductive hypothesis, $u, w^* \models \psi^*$. Note that w^* coincides with w' on all variables in F^*. For the variables $L_{f,s}$, because n has been chosen sufficiently large, $w'(L_{f,s}) \subseteq w^*(L_{f,s})$. Using now the fact that ψ^* uses the variable $L_{f,s}$ only negatively, we deduce that $u, w^* \models \psi^*$ implies $u, w' \models \psi^*$. Hence, once more $u, w' \models (A \wedge B) \rightarrow \psi^*$. It follows that $u, v^* \models \varphi^*$.

Cases of weight function quantifiers. The next case is when φ is $\exists f\psi$. Let g be the weight function such that the valuation w obtained by extending v with $w(f) = g$ is such that $u, w \models \psi$. By induction hypothesis, $u, w^* \models \psi^*$. Since furthermore w^* is valid, $u, w^* \models C \wedge D$. Hence w^* is a witness that $u, v^* \models \varphi^*$.

The last case is when φ is $\forall f\psi$. Let w' be any valuation of F^* that coincides with v^* over F. If either $u, w' \not\models C$ or $u, w' \not\models D$, then $u, w' \models (C \wedge D) \rightarrow \psi^*$. Hence we assume that $u, w' \models C \wedge D$. This implies that w' is valid. We have to prove that $u, w' \models \psi'$ using the inductive hypothesis. Let us construct a map g as follows. For all $i \in \omega$,

- if $i \notin w'(L_{f,r_\ell})$ (recall that r_ℓ is the largest number variable in scope), then $g(i) = w'(f)(i) + w(r_\ell) + 1$,
- otherwise, $g(i) = w(r_j)$ for j in $1 \ldots \ell$ minimal such that $i \in w'(L_{f,r_j})$.

Let us consider the valuation w that extends v with $w(f) = g$. Since v is valid, the same holds for w. Note now that, by choice of g, w' and w^* do only differ (possibly) on the variable f. However, $w'(f)$ and $w^*(f) = g$ are $\approx$-equivalent (i.e, bounded over the same subsets of ω). This comes from the fact that the first line in the definition of g amounts to just taking $w'(f)$ and shifting it with the constant $w(r_\ell)+1$ (an operation invisible up to $\approx$), and that the second line replaces only values in the set $w'(L_{f,r_\ell})$, over which $w'(f)$ is bounded, by values that are no greater than $v(e_\ell)$. Since the semantics of EBMSO is invariant under replacing weight variables by $\approx$-equivalent ones, it follows that $u, w' \models \psi^*$ if and only if $u, w^* \models \psi^*$. Hence, $u, w^* \models \psi^*$ implies $u, w' \models \psi^*$. As a consequence we obtain once more $u, w' \models (C \wedge D) \to \psi^*$. Since this holds for all w' extending v^*, we obtain $u, v^* \models \varphi^*$.

Hence, we have established the inductive hypothesis for all formulae φ.

Consider now a sentence φ. Then φ^* has no free variables. Let us show that $u \models \varphi$ if and only if $u \models \varphi^*$. Clearly, by the above arguments, if $u \models \varphi$ then $u \models \varphi^*$. Otherwise assume that $u \not\models \varphi$. Then this means that $u \models \psi$ where ψ is obtained from $\neg\varphi$ by pushing the negation to the leaves. We can apply the above result to ψ and we obtain $u \models \psi^*$. However, the * construction commutes with negation (in fact in the definition, we just gave half of the rules, letting the others be obtained by duality: this directly implies that * commutes with negation) which means that ψ^* is equal to $\neg(\varphi^*)$ after the negation have been pushed to the leaves. Hence, $u \not\models \varphi^*$. It follows, using excluded third, that $u \models \varphi$ if and only if $u \models \varphi^*$. $\square$

Proposition 23. *There exists a formula ξ of* AFO *with the following properties:*

- *There exists a weighted ω-word that satisfies ξ.*
- *Given an* EBMSO*-formula ψ, we can effectively construct an* AMSO*-formula ψ^* such that, for all models u of ξ,*

$$\omega \models \psi \qquad \textit{if and only if} \qquad u \models \psi^* \ .$$

Proof. Let us start with the description of the formula ξ. It states that:

- There are infinitely many letters $\sharp$ in u, expressed as $\forall x(\exists y > x)\sharp(y)$, and the first symbol is $\sharp$. This means that u can be uniquely decomposed as $\sharp u_1 \sharp u_2 \ldots$ such that no symbol $\sharp$ does occur in the factors u_i. Furthermore, each of the u_i have length at least one. This is obtained for instance with $\forall x \forall y[(\sharp(x) \wedge \sharp(y) \wedge x < y) \to \exists z(z > x \wedge z < y)]$.
- For all r, there exists s such that almost all the u_i contain a position with weight in $(r, s]$. This is formalized as

$$\forall r \exists s \exists w (\forall x > w)[\sharp(x) \to (\exists y > x)\ ((\forall z \in (x, y])\ \neg(\sharp x)) \wedge r < f(y) \leq s]\ .$$

It is easy to provide a model for ξ. It looks like (with some abuse of notation):

$$\sharp\, 0 \,\sharp\, 01 \,\sharp\, 012 \,\sharp\, 0123 \,\sharp\, 01234 \,\sharp\, \ldots\ .$$

Let us consider some model $u = u_0 \sharp u_1 \sharp u_2 \sharp \ldots$ of ξ. Let f be its weight function. We call a subset $X = \{x_i \mid i \in \omega\}$ of ω an *encoding* if x_i is a position in u_i (with respect to the above decomposition). Note that this decomposition of X into x_i is unique. Note also that the fact that X is an encoding is expressible in MSO. An encoding X induces a function g_X defined by $g_X(i) = f(x_i)$ for all $i \in \omega$.

We claim that, for all weight functions g, there exists an encoding X such that $g_X \approx g$. First, using skolemization, there exists a map $\alpha : \omega \to \omega$ such that, for all r, almost all the u_i contain a position with weight in $(r, \alpha(r)]$. (Note in particular that this implies $r < \alpha(r)$.) Without loss of generality, this map α can be chosen non-decreasing and such that all the u_i contain a position with weight at most $\alpha(0)$. (This is possible because we have guaranteed that all the u_i are non-empty.) Consider some $i \in \omega$ and chose x_i to be a position in u_i such that $f(x_i) \leq \alpha(g(i))$ that maximizes $f(x_i)$ (such an element exists because we have chosen α with suitable properties). We shall prove that $X = \{x_i \mid i \in \omega\}$ satisfies the claim. First it is clear that $g_X \leq g$. This means that whenver g is bounded over some set, the same goes for g_X. For the opposite direction, assume that that g is not bounded over some set Y. Let $n \in \omega$ be fixed. We shall prove that g_X is not bounded by n. For this, we know that almost all the u_i contain a position with weight in $(n, \alpha(n)]$. Furthermore, since g is not bounded, there is a position in Y as large as we want, say i, such that $g(i) > \alpha(n)$. For such an i, maximality of $f(x_i)$ in the definition of x_i implies that $g_X(i) > n$. Since this holds for all n, g_X is not bounded either over Y. The claim is established.

The remainder of the proof is straightforward. The principle is to use encodings to quantify over weight functions. We do it as 'an interpretation'. Let us call a position of u an *element* if it carries the letter $\sharp$. Being an element is FO-definable. There is a bijection π between the elements of u and the positions in ω: x_i is mapped to i. The elements are ordered as positions in u. This order is of course definable. π is monotonic w.r.t. this order. Sets are interpreted as sets of elements. This is of course definable. Sets of elements are in bijection with sets of positions of ω by extension of the mapping π. Finally weighted variables are interpreted as encodings. The only thing to do is provide a formula which, given a set of elements X and an encoding Y, tests whether g_Y is bounded over the set of elements represented as $\pi(Y)$. For this, we use the intermediate formula $\mathrm{int}(x, y)$ which expresses that x carries a $\sharp$, and y lies in the u_i that follows this $\sharp$: $\sharp(x) \wedge \forall z[(x < z \wedge z \leq y) \to \neg\sharp(z)]$. The predicate B is now replaced by $\exists r(\forall x \in X)(\forall y \in Y)[\mathrm{int}(x, y) \to f(y) \leq r]$. The only argument worth mentioning in this proof is that indeed, quantifying over encodings is as good as quantifying over weight functions. This comes on one side from the claim that every weight function is equivalent to some encoding, and from the other side that we are starting from the logic EBMSO, and in this logic (as opposed to AMSO) replacing the valuation of any variable with an $\approx$-equivalent one does not change the semantics. $\square$

Next, we present the translations from MSO+$\mathbb{U}$ to EAMSO and back again.

Proposition 24. *For every* (MSO+$\mathbb{U}$)*-formula* φ*, there is an* EAMSO*-formula* φ^* *such that*

$$\langle \omega, \bar{P}, \bar{f} \rangle \models \varphi \quad \text{iff} \quad \langle \omega, \bar{P}, \bar{f} \rangle \models \varphi^*,$$

for all sets $\bar{P}$ *and weight functions* $\bar{f}$*.*

Proof. For every subformula of the form $\mathbb{U}X\psi$, we have to find an equivalent EAMSO-formula. The proof is by induction on ψ. First, note that

$$\mathbb{U}X\psi(X) \equiv \mathbb{U}X\exists Y[\psi(Y) \wedge X \subseteq Y]\,.$$

Hence, we may assume that the formula $\psi(X)$ is closed under subsets, i.e., if a set P satisfies ψ then every subset of P also satisfies it.

Let $\psi^*(X)$ be the translation of ψ we obtain from the induction hypothesis. We set

$$(\mathbb{U}X\psi)^* := \exists X\exists Y\exists f[\vartheta_0 \wedge \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3]\,,$$

where the formulae $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3$ are as follows.

$$\begin{aligned}\vartheta_0 := \forall Z\forall x\forall y[\,Yx \wedge Yy \wedge \forall z(x < z < y \to \neg Yz)& \\ \wedge \forall z(Zz \leftrightarrow Xz \wedge x \leq z < y)& \\ \to \psi^*(Z)]&\end{aligned}$$

states that, on every factor of the word between two consecutive elements of Y, the restriction of the set X to this factor satisfies ψ.

$$\vartheta_1 := \exists r\forall x[Yx \to f(x) \leq r]$$

states that the f-weight of all elements of Y is bounded.

$$\vartheta_2 := \forall s\exists x[Y(x+1) \wedge f(x) > s]$$

states that the f-weight of all elements whose successor is in Y is unbounded.

$$\begin{aligned}\vartheta_3 := \forall s\exists r\forall x\forall y[x < y& \\ \wedge \neg\exists u\exists v(x \leq u < v \leq y \wedge Xu \wedge Xv)& \\ \wedge \forall z(x < z \leq y \to \neg Yz)& \\ \to (f(x) \leq s \to f(y) \leq r)]&\end{aligned}$$

states that there is a function $\alpha : \omega \to \omega$ such that $f(y) \leq \alpha(f(x))$, for all positions $x < y$ such that the interval $[x, y]$ contains at most one element of X and no element of Y (except possibly for x).

To show that the above formula is equivalent to the original one, let us first assume that there are arbitrarily large finite sets satisfying ψ. Since we assume that ψ is closed under subsets, we can find a sequence $(P_i)_{i<\omega}$ of finite sets of unbounded size, each satisfying ψ and such that every element of P_i is less than all elements of P_{i+1}. Setting

$$\begin{aligned} X &:= \bigcup_{i<\omega} P_i\,, \\ Y &:= \{\, \min P_i \mid i < \omega \,\}\,, \\ f(x) &:= \big|P_i \cap [\min P_i, x]\big|\,, \quad \text{where } i \text{ is the index such that} \\ &\qquad\qquad\qquad\qquad\quad\ \min P_i \leq x < \min P_{i+1}\,, \end{aligned}$$

we can satisfy $\vartheta_0 \wedge \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3$.

Conversely, suppose that X, Y, f satisfy $\vartheta_0 \wedge \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3$. Let $y_0 < y_1 < \dots$ be an enumeration of Y and set $P_i := X \cap [y_i, y_{i+1})$. Then each P_i satisfies φ. Hence, it remains to show that the size of the sets P_i is unbounded.

For a contradiction, suppose otherwise. Let

$$m := \max \{ |P_i| \mid i < \omega \} .$$

Since f satisfies the formula ϑ_3, there exists a function $\alpha : \omega \to \omega$ such that $f(y) \leq \alpha(f(x))$, for all pairs $x < y$ as in ϑ_3. For every $i < \omega$, we choose a sequence $y_i = z_0^i \leq \dots \leq z_m^i = y_{i+1} - 1$ of positions such that each interval $[z_k^i, z_{k+1}^i)$ contains at most one element of P_i. Then $f(z_{k+1}^i) \leq \alpha(f(z_k^i))$ implies $f(z_m^i) \leq \alpha^m(f(z_0^i))$. By ϑ_1, there is some number n such that $f(z_0^i) \leq n$, for all i. Hence, $f(z_m^i) \leq \alpha^m(n)$, for all i. But this number is unbounded since f satisfies ϑ_2. A contradiction.

Let us now consider the opposite direction. Since we have already shown that EAMSO and EBMSO are equivalent, it is sufficient to translate EBMSO-formulae.

Proposition 25. *For every* EBMSO-*sentence* φ*, there exists an* (MSO+$\mathbb{U}$)-*sentence* ψ *such that*

$$\langle \omega, \leq \rangle \models \varphi \quad \text{iff} \quad \langle \omega, \leq \rangle \models \psi .$$

Proof. To construct the desired formula ψ we choose an infinite set W of positions, each followed by an interval of positions not in W. The positions in W will represent the elements of ω while those not in W will be needed to code the values of the weight functions. For every weight functions f, we introduce a set variable Z_f (disjoint from W) that codes f as follows. If $f(x) = n$, then Z_f contains an interval of length n starting immediately after the position x. We use the following auxiliary formula

$$\begin{aligned}
&\vartheta(X, Y) := \\
&\quad X \subseteq W \wedge Y \cap W = \emptyset \\
&\qquad\qquad \wedge \mathbb{U}Z\big[\, Z \subseteq Y \wedge \\
&\qquad\qquad\qquad \exists x \exists y[Xx \wedge \forall z(Zz \leftrightarrow x < z \leq y)]\big] .
\end{aligned}$$

which states, for sets $X \subseteq W$ and $Y \subseteq \omega \setminus W$, that there are arbitrarily large intervals in Y that begin directly after some position in X.

We set

$$\begin{aligned}
\psi := \exists W \big[&\forall x \exists y(x < y \wedge Wy) \wedge \forall x(Wx \to \neg W(x+1)) \\
&\wedge \vartheta(W, \omega \setminus W) \wedge \varphi^*(W) \big]
\end{aligned}$$

where $\varphi^*(W)$ states that the formula φ holds on the positions in W. We define φ^* by induction on φ. The nontrivial cases are:

$$\begin{aligned}
(f[X] < \infty)^* &:= \neg\vartheta(X, Z_f)\,, \\
(f \sqsubseteq g)^* &:= \forall X[\vartheta(X, Z_f) \to \vartheta(X, Z_g)]\,, \\
(\exists f \chi(f))^* &:= \exists Z_f[Z_f \cap W = \emptyset \wedge \chi^*(Z_f)]\,, \\
(\exists X \chi)^* &:= \exists X[X \subseteq W \wedge \chi^*]\,.
\end{aligned}$$

To see that this construction is correct note that, if Z_f is a set disjoint from W that satisfies χ^*, we can define a corresponding weight function f by taking as value of $f(x)$ the length of the interval of Z_f starting at the position $x + 1$. Conversely, given a weight function f, we can define the set Z_f as the union of all intervals $[x+1, x+f(x)]$, but only if the elements $x+1, \dots, x+f(x)$ are not members of W. Hence, consider the function $w : W \to \omega$ defined by

$$w(x) := \max \{\, n \mid x+1, \dots, x+n \notin W \,\}\,.$$

Note that, by choice of the set W, this function is unbounded. We have seen that any weight function $f \leq w$ can be encoded by a suitable set Z_f. To conclude the proof, let f be an arbitrary weight function. We consider the function $f_0(x) := \min\{f(x), w(x)\}$. Then $f_0 \leq f$ implies $f_0 \sqsubseteq f$. For the converse, note that w is unbounded. Therefore, if $f_0[X]$ is bounded then so is $f[X]$. Consequently, $f \sqsubseteq f_0$. Hence, $f \approx f_0$ and it follows by Proposition 3 that, if f satisfies a formula χ, then so does f_0. Since $f_0 \leq w$ we can find a set Z_f as above.

B Borel complexity

In this section, we give the missing proof of Theorem 6. This theorem has two independent parts, one concerning the complexity of the logic WAMSO, and the other concerning AMSO. The first one is presented in Section B.1 and the second one in Section B.2.

B.1 The Borel complexity of the logic WAMSO

In this section, we establish the following proposition.

Proposition 26. *The logic* WAMSO *inhabits strictly all levels of finite rank of the Borel hierarchy.*

We start with the obvious upper bound on complexity.

Proposition 27. *Every* WAMSO*-definable language of weighted ω-words is of finite Borel rank.*

Proof. By induction on the complexity of a formula φ, we show that the language defined by φ is Borel. The claim clearly holds for atomic formulae. For boolean operations, the inductive step follows from the fact that the Borel sets form a boolean algebra. For quantifiers, it is sufficient to note that every WMSO-quantifier ranges over a countable domain (finite sets or natural numbers). Consequently, each quantifier corresponds to a countable union or a countable intersection.

Let us now turn to the lower bound showing that WAMSO reaches arbitrarily high finite ranks of the Borel hierarchy. For the proof we use the following languages, which are complete for $\boldsymbol{\Pi}^0_{2k}$.

Definition 28. *$C^0_{2k} \subseteq (\omega^{2k})^\omega$ is the language consisting of all words u over the alphabet ω^{2k} such that*

$$\forall n_0 \exists m_0 \cdots \forall n_{k-1} \exists m_{k-1} \\ [(m_0, \ldots, m_{k-1}, n_0, \ldots, n_{k-1}) \text{ appears in } u] \,.$$

We recall the following classical fact.

Theorem 29. *C^0_{2k} is complete for $\boldsymbol{\Pi}^0_{2k}$.*

Proof. A set of level $\boldsymbol{\Pi}^0_{2k}$ is a countable intersection of countable unions, ..., of basic open sets, which are sets of words that have a fixed prefix. Hence, a Borel set $A \in C^0_{2k}$ can be written as the set of words u such that:

$$\forall n_0 \exists m_0 \cdots \forall n_{k-1} \exists m_{k-1} \\ [v(m_0, \ldots, m_{k-1}, n_0, \ldots, n_{k-1}) \text{ is a prefix of } u]$$

where v is some function from $\mathbb{N}^{2k}$ to finite words. To prove completeness, let us provide a continuous reduction from A to C^0_{2k}.

Assume a finite word given u, then let $[u]_i$ be the list of the i first tuples $\bar{m}$ (for some enumeration of them by $\mathbb{N}$) such that $v(\bar{m})$ is a prefix of u. One can assume that $[u]_i$ is non-empty. Now, given an infinite sequence $u = a_1 \ldots a_n \ldots$, let us set:

$$f(u) = [a_1]_1 [a_1 a_2]_2 \ldots [a_1 a_2 \ldots a_n]_n \ldots$$

Clearly this function is continuous, and clearly too, u has $v(\bar{m})$ as prefix if and only if $\bar{m}$ appears in $f(u)$.

Proposition 30. *There exists a* WAMSO*-definable set of weighted ω-words that does not belong to $\boldsymbol{\Sigma}^0_{2k}$.*

Proof. We will construct a WAMSO-formula ψ and a continuous function h mapping a sequence $u \in (\omega^{2k})^*$ to a weighted ω-word $h(u)$ such that

$$h(u) \models \psi \quad \text{iff} \quad u \in C^0_{2k} \,.$$

Since C^0_{2k} is complete for $\boldsymbol{\Pi}^0_{2k}$, it follows that the language defined by ψ is not in $\boldsymbol{\Sigma}^0_{2k}$.

Note that a sequence u is in C^0_{2k} if, and only if, there exists Skolem functions $f_i : \omega^i \to \omega$, for $i < k$, such that, for all $n_0, \ldots, n_{k-1}$,

$$(f_0(n_0), f_1(n_0, n_1), \ldots, f_{k-1}(n_0, \ldots, n_{k-1}), n_0, \ldots, n_{k-1})$$

appears in u. For such a tuple $\bar{f}$, we write

$$\bar{f}(\bar{n}) := (f_0(n_0), f_1(n_0, n_1), \dots, f_{k-1}(n_0, \dots, n_{k-1}))\,.$$

The *s-th approximation* of $\bar{f}$ are the functions $\bar{f}^{(s)}$ where $f_i^{(s)} : [s]^i \to \omega$ is the restriction of f_i to $[s]^i$. Such a tuple $\bar{f}^{(s)}$ will be called a *partial Skolem function* or an *s-Skolem function.* We say that an s-Skolem function $\bar{f}$ *occurs before position* j in a word u if, for all $\bar{n} \in [s]^k$, the tuple $(\bar{f}(\bar{n}), \bar{n})$ appears in u before position j.

An s-Skolem function $\bar{f}$ can be encoded into a finite weighted word $[\bar{f}]$, simply by enumerating the tuples in its graph and by, say, adding the number s. Fix an enumeration $\bar{S}^0, \bar{S}^1, \dots$ of all partial Skolem functions such that every function occurs infinitely often in this sequence (this is possible since there are only countably many partial Skolem functions). Given a word u, we define

$$h(u) := \#h_0(u)\#h_1(u)\#\dots$$

where

$$h_i(u) := \begin{cases} [\bar{S}^i] & \text{if } \bar{S}^i \text{ occurs before } i \text{ in } u\,, \\ \varepsilon & \text{otherwise}\,. \end{cases}$$

Note that h is continuous.

We can write down a formula ψ stating that

$$\begin{aligned} &\forall n_0 \exists m_0 \cdots \forall n_{k-1} \exists m_{k-1} \forall n' \\ &\quad \big[\text{the word has a factor of the form } \#[\bar{f}]\# \text{ where} \\ &\quad \bar{f} \text{ is an } s\text{-Skolem function with } s > n' \text{ such that} \\ &\quad f_l(i_0, \dots, i_l) < m_l \text{ for all } l < k \text{ and } i_0 < n_0, \dots, i_l < n_l\big]\,. \end{aligned}$$

We claim that

$$h(u) \models \psi \quad \text{iff} \quad u \in C^0_{2k}\,.$$

($\Leftarrow$). Suppose that $u \in C^0_{2k}$. Then there exists a tuple of Skolem functions $\bar{f}$ such that

$$\forall \bar{n}[(\bar{f}(\bar{n}), \bar{n}) \text{ occurs in } u]\,.$$

Let $\bar{f}^{(s)}$ be the s-th approximation of $\bar{f}$. Fix a tuple $\bar{n}$. For every large enough number s, it follows that $(\bar{f}^{(s)}(\bar{n}), \bar{n})$ appears in u. Suppose that $\bar{f}^{(s)} = \bar{S}^i$. Since every function appears infinitely often in the enumeration $\bar{S}^0, \bar{S}^1, \dots$, we can choose the index i large enough such that $\bar{f}^{(s)}$ occurs before i in u. Then $h_i(u) = [\bar{f}_s]$ and $\bar{f}_s$ satisfies the condition

$$(f_s)_l(i_0, \dots, i_l) < m_l \text{ for all } l < k \text{ and } i_0 < n_0, \dots, i_l < n_l\,.$$

Consequently, $h(u) \models \psi$.

($\Rightarrow$) Suppose that $h(u) \models \psi$. Then there exist functions $\bar{g}$ such that, for all tuples $\bar{n}$ and every number n', the word $h(u)$ has a factor of the form $\#[\bar{f}_{\bar{n},n'}]\#$ such that $\bar{f}_{\bar{n},n'}$ is an s-Skolem function with $s > n'$ and

$$(f_{\bar{n},n'})_l(i_0, \dots, i_l) < g_l(n_0, \dots, n_l)\,,$$

for all $l < k$ and $i_0 < n_0, \dots, i_l < n_l$. Let us abbreviate this last condition by $\bar{f}_{\bar{n},n'} \leq \bar{g}(\bar{n})$.

To show that $u \in C^0_{2k}$, fix $\bar{n}$. Note that, for each s, there are only finitely many s-Skolem functions $\bar{f}$ with $\bar{f} \leq \bar{g}(\bar{n})$. Ordering the set of all functions $\bar{f}_{\bar{n},n'}$, for $n' < \omega$, by the extension relation, we obtain an infinite tree that is finitely branching. By Kőnig's Lemma, this tree contains an infinite branch. Consequently, there exists a sequence $n'_0 < n'_1 < \dots$ such that $\bar{f}_{\bar{n},n'_0} \subseteq \bar{f}_{\bar{n},n'_1} \subseteq \dots$. Let $\bar{f}$ be the limit of this sequence. It follows that, for every $s < \omega$, the s-th approximation $\bar{f}^{(s)}$ of $\bar{f}$ appears as factor $\#[\bar{f}^{(s)}]\#$ in $h(u)$. By definition of h, this implies that $\bar{f}^{(s)}$ occurs in u. In particular, the tuple $(\bar{f}^{(s)}(\bar{n}), \bar{n})$ appears in u. Since $\bar{f}^{(s)}(\bar{n}) = \bar{f}(\bar{n})$, it follows that

$$(\bar{f}(\bar{n}), \bar{n}) \text{ appears in } u\,, \quad \text{for all } \bar{n}\,.$$

Hence, $u \in C^0_{2k}$.

B.2 The Borel complexity of the logic AMSO

In this section, we establish the following proposition.

Proposition 31. *The logic* WAMSO *inhabits strictly all levels of the projective hierarchy.*

The arguments are inspired by the corresponding results for MSO+$\mathbb{U}$ [13]. Let us consider the topological space $[2]^{\omega^*}$ (where we consider $[2]$ as a discrete space). Note that this space is homeomorphic to Cantor space $[2]^\omega$, via a suitable bijection $\omega^* \to \omega$. For notational simplicity, we will identify functions $f : \omega^* \to [2]$ with subsets $f^{-1}(1) \subseteq \omega^*$. Such a set $T \subseteq \omega^*$ is a *tree* if

- $u \in T$ implies that $v \in T$, for every prefix v of u and
- $uk \in T$ implies that $ui \in T$, for all $u \in \omega^*$ and $i < k < \omega$.

We use the following languages of trees, which are complete for $\boldsymbol{\Sigma}^1_{2k}$.

Definition 32. *We denote by C^1_{2k} the language consisting of all trees $T \subseteq (\omega^{2k})^*$ such that*

$$\begin{aligned}&\exists u_0 \forall v_0 \cdots \forall u_{k-1} \exists v_{k-1}\\ &\quad [u_0 \times \cdots \times u_{k-1} \times v_0 \times \cdots \times v_{k-1} \textit{ is a branch of } T]\,,\end{aligned}$$

where $u_0, v_0, \dots, u_{k-1}, v_{k-1}$ are ω-words over the alphabet ω and $u_0 \times \cdots \times u_{k-1} \times v_0 \times \cdots \times v_{k-1}$ denotes the ω-word over ω^{2k} obtained by pairing them componentwise.

Theorem 33. *C^1_{2k} is complete for $\boldsymbol{\Sigma}^1_{2k}$.*

Proof. This is the language described in the proof of Theorem (37.7) of [14].

Proposition 34. *There exists an* AMSO*-definable set of weighted ω-words that does not belong to $\boldsymbol{\Pi}^1_{2k}$.*

Proof. The proof is completely inspired by the work of Szczepan Hummel and Michał Skrzypczak [13]. Given a finite sequence $u \in \omega^*$, let us denote by $[u]$ the weighted word over the unary alphabet with two weight functions f and d defined by $f(x) := u(x)$ and $d(x) := x$. Hence, the weight function f over $[u]$ merely codes u, and d is an extra weight function that encodes the position in the word. Note that a sequence $u_0, u_1, \dots$ in ω^* converges if, and only if, the corresponding sequence $[u_0], [u_1], \dots$ converges.

Let $T \subseteq (\omega^{2k})^*$ be a tree. The *n-th approximation* of T is the finite tree of height $2k + 1$ such that

- all branches have length exactly $2k + 1$,
- there exists a number n such that every node but the root is labelled by $[u]$ for some word u of length n,
- there exists a branch with label $[u_0], [v_0], \dots, [u_{k-1}], [v_{k-1}]$ if, and only if, $u_0 \times \dots \times u_{k-1} \times v_0 \times \dots \times v_{k-1} \in T$.

We denote the label of an node x by $T(x)$.

Given a tree T, let $[T]_n$ be some encoding of the n-th approximation of T as a finite weighted word. (We omit the details. One can use, e.g., brackets to encode the tree structure. Since the height of the tree is bounded, we can assume that all relevant concepts, like being a node, being a branch, the successor relation, etc., are expressible in MSO.)

Given a tree T, define the weighted ω-word

$$h(T) := [T]_1 \# [T]_2 \# \dots$$

which enumerates the approximations of T. Clearly, h is a continuous mapping. Our goal is to construct an AMSO-formula ψ which is satisfied by $h(T)$ if, and only if, $T \in C^1_{2k}$. This would immediately imply that the language defined by this formula is at level $\boldsymbol{\Sigma}^1_{2k}$ or higher in the analytic hierarchy. In particular, it is not in $\boldsymbol{\Pi}^1_{2k}$.

The idea of the construction is to replace in the description of C^1_{2k} each quantification over an infinite word, say, in the ith quantifier, by a quantification over an infinite set Z over $h(T)$ which selects nodes at level i in infinitely many approximations occurring in $h(T)$ (what exactly 'selecting' means depends on the chosen coding, but it should be clear).

We will make use of the following auxiliary formulae. For each $i < 2k$, there exists a formula

$$\mathrm{lvl}_i(Z)$$

expressing that the set Z selects infinitely many nodes of the successive approximations, that each selected node is at level i in the approximation (the root having level 0), and that no two nodes belong to the same approximation. Given some Z satisfying lvl_i we will often refer to the sequence $x_0, x_1, \dots$ of nodes *selected* by Z. We will also say that Z' is a *subsequence* of Z if it corresponds to extracting a subsequence of $x_0, x_1, \dots$.

We define a successor relation on sets Z by

$$\begin{aligned}\mathrm{suc}_i(Z, Z') := {} & \mathrm{lvl}_i(Z) \wedge \mathrm{lvl}_{i+1}(Z') \wedge \\ & \text{`All nodes selected by } Z' \text{ are children of} \\ & \text{nodes selected by } Z.\text{'}\end{aligned}$$

Recall that we would like to replace a quantification over an infinite word by a set Z. Hence, what we would like to have is that the nodes $x_0, x_1, \dots$ selected by Z be such that $T(x_0), T(x_1), \dots$ converges to some infinite word. However, it is not possible to express this property in AMSO. All we can express is a sufficient condition guaranteeing that there some is some subsequence $T(y_0), T(y_1), \dots$ of $T(x_0), T(x_1), \dots$ that converges to some infinite word in ω^ω. For this we introduce a formula $\mathrm{conv}(Z)$ expressing that the nodes selected by Z contain, at the limit, an infinite word. We define

$$\mathrm{conv}(Z) := \forall s \exists r (\forall x \in Z)[d(x) \leq s \to f(x) \leq r] \,.$$

(Here, $x \in Z$ means that x is a position in the weighted word representing a node selected by Z. Recall that d is simply a depth stamp numbering the positions in the word.)

We claim that

(i) If $h(T) \models \mathrm{conv}(Z)$, there is a subset of Z selecting nodes $x_0, x_1, \dots$ such that $T(x_0), T(x_1), \dots$ has a limit in ω^ω.
(ii) If Z selects $x_0, x_1, \dots$ and the sequence $T(x_0), T(x_1), \dots$ converges, then $h(T) \models \mathrm{conv}(Z)$.

(i) Let $x_0, x_1, \dots$ be the nodes selected by Z. For $s = 0$, we obtain that the weight of the first position is bounded. Hence, it is possible to extract a subsequence such that all $T(x_i)$ have the same weight over the first position. Then, one can continue with $s = 1$ and extract a subsequence such that the weight of the second position is constant. By iterating this extraction process we obtain, for each number $s < \omega$, a subsequence $x_0^s, x_1^s, \dots$ of $x_0, x_1, \dots$ such that the weights of the first s positions is constant. The 'diagonal' sequence $x_0^0, x_1^1, x_2^2, \dots$ is the desired converging subsequence.

(ii) Suppose that $T(x_0), T(x_1), \dots$ converges. To show that $\mathrm{conv}(Z)$ holds, fix a number $s < \omega$. There exists an index n such that, in the subsequence $T(x_n), T(x_{n+1}), T(x_{n+2}), \dots$, the first s positions are constants. Choose a number r that is greater than the weights of the first s positions in the words $T(x_0), \dots, T(x_n)$. For $x \in Z$, it follows that $d(x) \leq s$ implies $f(x) \leq r$.

To simplify terminology, we will say that Z *converges toward* $u \in \omega^\omega$ if Z selects nodes $x_0, x_1, \dots$ such that $T(x_0), T(x_1), \dots$ converges to u. In this terminology the above statements read:

(i) If $\mathrm{conv}(Z)$ holds, then there is some subset of Z which converges.
(ii) If Z converges, then $\mathrm{conv}(Z)$ holds.

Consider the AMSO-formula

$$\begin{aligned}
&\psi := \\
&\quad (\exists X_0.\mathrm{lvl}_1(X_0) \wedge \mathrm{conv}(X_0)) \\
&\qquad (\forall Y_0.\mathrm{suc}_1(X_0, Y_0) \wedge \mathrm{conv}(Y_0)) \\
&\qquad\quad (\exists X_1.\mathrm{suc}_2(Y_0, X_1) \wedge \mathrm{conv}(X_1)) \\
&\qquad\qquad (\forall Y_1.\mathrm{suc}_3(X_1, Y_1) \wedge \mathrm{conv}(Y_1)) \\
&\qquad\qquad\quad \dots \\
&\qquad\qquad\qquad (\exists X_{k-1}.\mathrm{suc}_{2k-2}(Y_{k-2}, X_{k-1}) \wedge \mathrm{conv}(X_{k-1})) \\
&\qquad\qquad\qquad\quad (\forall Y_{k-1}.\mathrm{suc}_{2k-1}(X_{k-1}, Y_{k-1}) \wedge \mathrm{conv}(Y_{k-1})) \\
&\qquad\qquad\qquad\qquad \texttt{true}\,.
\end{aligned}$$

We claim that

$$h(T) \models \psi \quad \text{iff} \quad T \in C^1_{2k}\,.$$

Note that, by Skolemising the definition, we see that a tree T belongs to C^1_{2k} if, and only if, there are functions $S_i : (\omega^\omega)^i \to \omega^\omega$, for $i < k$, such that, for all $v_0, \dots, v_{k-1} \in \omega^\omega$, the sequence

$$S_0 \times S_1(v_0) \times \cdots \times S(v_0, v_1, \dots, v_{k-1}) \times v_0 \times \cdots \times v_{k-1}$$

is a branch of T.

First, suppose that $T \in C^1_{2k}$. We have to show that $h(T) \models \psi$. By the above remark there are Skolem functions $S_0, \dots, S_{k-1}$ witnessing that T belongs to C^1_{2k}. Fix a set X_0 that satisfies $\mathrm{lvl}_1(X_0)$ and that converges to S_0. Let Y_0 be an arbitrary set satisfying $\mathrm{suc}_1(X_0, Y_0)$ and $\mathrm{conv}(X_0)$. By (i), there exists a subsequence Y_0' which converges toward a word v_0.

Choose a set X_1 that converges to $S_1(v_0)$ and satisfies $\mathrm{suc}_2(Y_0', X_1)$. (This is possible by construction of the approximations.) Then $\mathrm{conv}(X_1)$ holds by (ii). To proceed, consider some set Y_1 satisfying $\mathrm{suc}_3(X_1, Y_1)$ and $\mathrm{conv}(Y_1)$. Again, there exists a subsequence X_1' of X_1 that converges to some word v_1. Repeating this argument for $2k$ steps we see that $h(T)$ satisfies ψ.

It remains to prove the converse. Hence, suppose that $h(T)$ satisfies ψ. We have to show that $T \in C^1_{2k}$, i.e., that

$$\begin{aligned}
&\exists u_0 \forall v_0 \cdots \exists u_{k-1} \forall v_{k-1} \\
&\quad \big[u_0 \times \cdots \times u_{k-1} \times v_0 \times \cdots \times v_{k-1} \text{ is a branch of } T\big]\,.
\end{aligned}$$

Since ψ holds, there exists a set X_0 satisfying $\mathrm{lvl}_1(X_0)$ and $\mathrm{conv}(X_0)$. By (i), there exists a subset X_0' converging to some word u_0. Given an arbitrary word $v_0 \in \omega^\omega$, we choose a set Y_0 satisfying $\mathrm{suc}_1(X_0', Y_0)$ that converges to v_0. By (ii), Y_0 satisfies $\mathrm{conv}(Y_0)$.

In the next step, we find a set X_1 satisfying $\mathrm{suc}_2(Y_0, X_1)$ and $\mathrm{conv}(X_1)$. Again, there exists a subset X_1' converging to some word u_1. Given a word $v_1 \in \omega^\omega$, we choose Y_1 satisfying $\mathrm{suc}_1(X_1', Y_1)$ and converging to v_1.

Repeating this argument, we obtain a definition of $u_0, \dots, u_{k-1}$. Since X_i' converges to u_i and Y_i converges to v_i, there are arbitrarily long prefixes u_i' of u_i and v_i' of v_i such that

$$u_0' \times \cdots \times u_{k-1}' \times v_0' \times \cdots \times v_{k-1}' \text{ is a branch of } T\,.$$

This implies that

$$u_0 \times \cdots \times u_{k-1} \times v_0 \times \cdots \times v_{k-1} \text{ is a branch of } T\,.$$

Corollary 35. *Over weighted ω-words* AMSO *is strictly more expressive than* WAMSO*.*

C Tiling systems

C.1 Restriction to limit satisfiability

The first proof we present is the one of Lemma 11, which states that the satisfiability problem for $\mathrm{AMSO}^{\mathrm{np}}$ over infinite words is equivalent to the limit satisfiability problem.

Composition theorems We start by developing composition theorems for $\mathrm{AMSO}^{\mathrm{np}}$. This part is similar to the techniques introduced by Feferman-Vaught and Shelah, but paying furthermore attention to positivity consideration in the use of weights. Our analysis is based on the notion of a type and two operations $\oplus$ and $^\omega$ on them.

Definition 36. *Let $h < \omega$.*

(a) We denote by AMSO^0_h the set of all AMSO*-formulae of quantifier-rank at most h that do not contain number quantifiers.*

(b) Let w be a word and $\bar m, \bar n$ numbers. The AMSO^0_h-type of $\langle w, \bar m, \bar n\rangle$ is the set

$$\mathrm{tp}_h(w, \bar m, \bar n) := \{\, \varphi(\bar r, \bar s) \in \mathrm{AMSO}^0_h \mid w \models \varphi(\bar m, \bar n)\,\}\,.$$

Lemma 37. *There exists a monotone binary operation $\oplus$ on AMSO^0_h-types such that*

$$\mathrm{tp}_h(u, \bar m, \bar n) \oplus \mathrm{tp}_h(v, \bar m, \bar n) = \mathrm{tp}_h(uv, \bar m, \bar n)\,,$$

for all words u and v and all numbers $\bar m, \bar n$.

Proof. It is sufficient to prove the following claim. For every AMSO_h^0-formula $\varphi(\bar{r},\bar{s})$ in negation normal form, there exist two finite lists $\psi_0(\bar{r},\bar{s}),\dots,\psi_{l-1}(\bar{r},\bar{s})$ and $\vartheta_0(\bar{r},\bar{s}),\dots,\vartheta_{l-1}(\bar{r},\bar{s})$ of AMSO_h^0-formulae such that, for all words u and v and all numbers $\bar{m},\bar{n}$,

$$\begin{aligned} uv \models \varphi(\bar{m},\bar{n}) \quad \text{iff} \quad & u \models \psi_i(\bar{m},\bar{n}) \text{ and } v \models \vartheta_i(\bar{m},\bar{n})\,, \\ & \text{for some } i < l\,. \end{aligned}$$

We prove the claim by induction on φ. If φ is of the form

$$X \subseteq Y\,,\ X \cap Y = \emptyset\,,\ \neg PX\,,\ f[X] \leq r\,,\ \text{or } f[X] > s\,,$$

then we can set $l = 1$ and

$$\psi_0 := \varphi\,, \quad \vartheta_0 := \varphi\,.$$

Similarly, if φ is of the form

$$\begin{aligned} &\neg(X \subseteq Y)\,,\ \neg(X \cap Y = \emptyset)\,,\ PX\,,\ \neg(f[X] \leq r)\,, \\ &\quad \text{or } \neg(f[X] > s)\,, \end{aligned}$$

then we can set $l = 2$ and

$$\begin{aligned} \psi_0 &:= \varphi\,, & \psi_1 &:= \mathtt{true}\,, \\ \vartheta_0 &:= \mathtt{true}\,, & \vartheta_1 &:= \varphi\,. \end{aligned}$$

If $\varphi = X \leq Y$, we use the formulae

$$\begin{aligned} \psi_0 &:= X \leq Y\,, & \psi_1 &:= \mathtt{true}\,, & \psi_2 &:= \neg(X \cap X = \emptyset)\,, \\ \vartheta_0 &:= \mathtt{true}\,, & \vartheta_1 &:= X \leq Y\,, & \vartheta_2 &:= \neg(Y \cap Y = \emptyset)\,. \end{aligned}$$

For $\varphi = \neg(X \leq Y)$, we use

$$\begin{aligned} \psi_0 &:= \neg(X \leq Y)\,, & \psi_1 &:= X \cap X = \emptyset \\ \vartheta_0 &:= Y \cap Y = \emptyset\,, & \vartheta_1 &:= \neg(X \leq Y)\,. \end{aligned}$$

For the inductive step, suppose that we have already proved the claim for the formulae φ' and φ'', and let

$$\psi'_0,\dots,\psi'_{l'-1}\,, \quad \vartheta'_0,\dots,\vartheta'_{l'-1}$$

and

$$\psi''_0,\dots,\psi''_{l''-1}\,, \quad \vartheta''_0,\dots,\vartheta''_{l''-1}$$

be the corresponding lists of formulae.

If $\varphi = \varphi' \vee \varphi''$, we use $l := l' + l''$ and the lists

$$\begin{aligned}&\psi'_0, \dots, \psi'_{l'-1}, \psi''_0, \dots, \psi''_{l''-1}\\&\vartheta'_0, \dots, \vartheta'_{l'-1}, \vartheta''_0, \dots, \vartheta''_{l''-1}\,.\end{aligned}$$

Similarly, if $\varphi = \varphi' \wedge \varphi''$, we use $l := l'l''$ and the lists

$$\begin{aligned}&\psi'_i \wedge \psi''_k\,, \quad \text{for } i < l' \text{ and } k < l''\,,\\&\vartheta'_i \wedge \vartheta''_k\,, \quad \text{for } i < l' \text{ and } k < l''\,.\end{aligned}$$

If $\varphi = \exists X \varphi'$, we use $l = l'$ and

$$\exists X \psi'_0, \dots, \exists X \psi'_{l'-1} \quad \text{and} \quad \exists X \vartheta'_0, \dots, \exists X \vartheta'_{l'-1}\,.$$

Similarly, if $\varphi = \forall X \varphi'$, we use $l = l'$ and

$$\forall X \psi'_0, \dots, \forall X \psi'_{l'-1} \quad \text{and} \quad \forall X \vartheta'_0, \dots, \forall X \vartheta'_{l'-1}\,.$$

Lemma 38. *There exists a monotone operation* $^{\omega}$ *on* AMSO^0_h*-types such that*

$$p^{\omega} = \mathrm{tp}_h(w_0 w_1 w_2 \dots, \bar{m}, \bar{n})\,,$$

for all words $w_0, w_1, w_2, \dots$ *with*

$$\mathrm{tp}_h(w_i, \bar{m}, \bar{n}) = p\,.$$

Proof. It is sufficient to prove the following claim. For every AMSO^0_h-formula $\varphi(\bar{r}, \bar{s})$ in negation normal form, there exists a finite list $\psi_0(\bar{r}, \bar{s}), \dots, \psi_l(\bar{r}, \bar{s})$ of AMSO^0_h-formulae such that, for all words $w_0, w_1, \dots$ with

$$\mathrm{tp}_h(w_i) = \mathrm{tp}_h(w_k)\,, \quad \text{for all } i, k < \omega\,,$$

and for all numbers $\bar{m}, \bar{n}$,

$$w_0 w_1 \dots \models \varphi(\bar{m}, \bar{n}) \quad \text{iff} \quad (\exists i < l)(\forall k < \omega)[w_k \models \psi_i(\bar{m}, \bar{n})]\,.$$

We prove the claim by induction on φ. If φ is an atomic formula, or a negated atomic formula, but not of the form $X \leq Y$ or $\neg(X \leq Y)$, we can take $l := 1$ and

$$\psi_0 := \varphi\,.$$

If $\varphi = X \leq Y$, we take $l := 2$ and

$$\psi_0 := X \leq Y\,, \quad \psi_1 := \neg(X \cap X = \emptyset) \wedge \neg(Y \cap Y = \emptyset)\,.$$

If $\varphi = \neg(X \leq Y)$, we take $l := 1$ and

$$\psi_0 := X \cap X = \emptyset \vee Y \cap Y = \emptyset\,.$$

For the inductive step, suppose that we have already proved the claim for the formulae φ' and φ'', and let

$$\psi'_0, \dots, \psi'_{l'-1} \quad \text{and} \quad \psi''_0, \dots, \psi''_{l''-1}$$

be the corresponding lists of formulae.

If $\varphi = \varphi' \vee \varphi''$, we use $l := l' + l''$ and the list

$$\psi'_0, \dots, \psi'_{l'-1}, \psi''_0, \dots, \psi''_{l''-1}\,.$$

Similarly, if $\varphi = \varphi' \wedge \varphi''$, we take $l := l'l''$ and

$$\psi'_i \wedge \psi'_k\,, \quad \text{for all } i < l' \text{ and } k < l''.$$

If $\varphi = \exists X \varphi'$, we use $l = l'$ and

$$\exists X \psi'_0, \dots, \exists X \psi'_{l'-1}\,.$$

Similarly, if $\varphi = \forall X \varphi'$, we use $l = l'$ and

$$\forall X \psi'_0, \dots, \forall X \psi'_{l'-1}\,.$$

Factorisations In the second part of this reduction, we use the theorem of Ramsey for chopping an infinite word into infinitely many pieces that have essentially the same behaviour with respect to the logic. The real difficulty here is to deal with the prefix of the word since it also contains weights.

Recall the notation $\varphi{\downarrow_{\bar t}}$ introduced in Definition 16.

Definition 39. *Let $\bar r, \bar s, \bar t$ be tuples of number variables and let $p(\bar r, \bar s)$ be an AMSO^0_h-type.*

(a) We denote by $p{\uparrow_{\bar t}}$ the set of all AMSO^0_h-formulae $\varphi(\bar r, \bar s, \bar t)$ such that $\varphi{\downarrow_{\bar t}} \in p$.

(b) We denote by $\pi(p)$ the set of all formulae $\varphi \in p$ without free number variables.

The following result is an immediate consequence of Lemma 17.

Lemma 40. *If $m_i, n_i \geq \max f[w]$, for all i, then*

$$w \models p \quad \text{iff} \quad w \models (p{\uparrow_{\bar r, \bar s}})(\bar m, \bar n)\,.$$

Proof. ($\Leftarrow$) follows form the fact that $p \subseteq p{\uparrow_{\bar r, \bar s}}$. For ($\Rightarrow$), suppose that $w \models p$ and let $\varphi \in p{\uparrow_{\bar r, \bar s}}$. Then $\varphi{\downarrow_{\langle\rangle}} \in p$. Hence, $w \models \varphi{\downarrow_{\langle\rangle}}$ implies, by Lemma 17, that $w \models \varphi(\bar m, \bar n)$.

Definition 41. *Let I and J be sets and $k < \omega$ a number. We write $I \subseteq_\infty J$ if I is an infinite subset of J and we write $I \subseteq^k_\infty J$ if $I \subseteq_\infty J$ and I contains the first k elements of J.*

Definition 42. *Let w be an ω-word and $I \subseteq_\infty \omega$.*

(a) Let $k_0 < k_1 < \dots$ be an enumeration of I and set $k_{-1} := 0$. The factorisation *of w* induced *by I is the sequence $(w_i)_{i<\omega}$ where*

$$w_i := w[k_{i-1}, k_i)$$

is the factor of w from position k_{i-1} to position $k_i - 1$. Hence, $w = w_0 w_1 w_2 \dots$ and $|w_i| = k_i - k_{i-1}$.

(b) Let $h < \omega$, let p and e be AMSO^0_h-types, let $(w_i)_{i<\omega}$ be the factorisation of w induced by I, and let $\bar{n}$ be numbers. We say that the triple $\langle w, I, \bar{n}\rangle$ is h-Ramsey of type (p, e) *if*

$$\mathrm{tp}_h(w_0, \bar{n}) = p \quad \textit{and} \quad \mathrm{tp}_h(w_i \dots w_k, \bar{n}) = e\,, \quad \textit{for all } 0 < i \le k < \omega\,.$$

Definition 43. *For a AMSO^0_h-type p, we define the formulae*

$$\begin{aligned}\mathrm{pref}(p, I) &:= \exists x(\forall y \in I)\big[y \ge x \to p^{[0,y)}\big]\,,\\ \mathrm{ult}(p, I) &:= \exists x(\forall y, z \in I)\big[x \le y < z \wedge I \cap (y, z) = \emptyset \to p^{[y,z)}\big]\,,\end{aligned}$$

where $p^{[x,y)}$ is the relativisation of p to the interval $[x, y)$.

Lemma 44. *Let $Q \in \{\exists, \forall\}$ and let $\varphi(t, \bar{t}', \bar{I})$ be an AMSO^0_h-formula such that*

$$w \models \varphi(m, \bar{m}, \bar{I}) \quad \textit{implies} \quad w \models \varphi(m, \bar{m}, \bar{J})\,,$$

for all $J_i \subseteq_\infty I_i$. Then

$$\begin{aligned}\langle w, \bar{m}\rangle \models Qt(\forall I \subseteq_\infty \omega)(\forall k < \omega)&\\ (\exists J_0 \subseteq^k_\infty I)(\exists J_1 \subseteq^k_\infty J_0) \cdots (\exists J_l \subseteq^k_\infty J_{l-1})\varphi(t, \bar{t}', \bar{J})&\end{aligned}$$

implies

$$\begin{aligned}\langle w, \bar{m}\rangle \models (\forall I \subseteq_\infty \omega)(\forall k < \omega)&\\ (\exists J_0 \subseteq^k_\infty I)(\exists J_1 \subseteq^k_\infty J_0) \cdots (\exists J_l \subseteq^k_\infty J_{l-1})Qt\varphi(t, \bar{t}', \bar{J})\,.&\end{aligned}$$

Proof. If $Q = \exists$ then

$$\begin{aligned}&\langle w, \bar{m}\rangle \models \exists t(\forall I \subseteq_\infty \omega)(\forall k < \omega)\\ &\qquad (\exists J_0 \subseteq^k_\infty I) \dots (\exists J_l \subseteq^k_\infty J_{l-1})\varphi(t, \bar{t}', \bar{J})\\ \Rightarrow\ &\langle w, \bar{m}\rangle \models (\forall I \subseteq_\infty \omega)(\forall k < \omega)\exists t\\ &\qquad (\exists J_0 \subseteq^k_\infty I) \dots (\exists J_l \subseteq^k_\infty J_{l-1})\varphi(t, \bar{t}', \bar{J})\\ \Rightarrow\ &\langle w, \bar{m}\rangle \models (\forall I \subseteq_\infty \omega)(\forall k < \omega)\\ &\qquad (\exists J_0 \subseteq^k_\infty I) \dots (\exists J_l \subseteq^k_\infty J_{l-1})\exists t\varphi(t, \bar{t}', \bar{J})\,.\end{aligned}$$

Hence, suppose that $Q = \forall$ and that

$$\langle w, \bar{m}\rangle \models \forall t(\forall I \subseteq_\infty \omega)(\forall k < \omega) \\ (\exists J_0 \subseteq^k_\infty I)\dots(\exists J_l \subseteq^k_\infty J_{l-1})\varphi(t, \bar{t}', \bar{J})\,.$$

To show that

$$\langle w, \bar{m}\rangle \models (\forall I \subseteq_\infty \omega)(\forall k < \omega) \\ (\exists J_0 \subseteq^k_\infty I)\dots(\exists J_l \subseteq^k_\infty J_{l-1})\forall t\varphi(t, \bar{t}', \bar{J})\,,$$

fix $I \subseteq_\infty \omega$ and $k < \omega$. By induction on n, we construct sets J^n_i, for $i \leq l$ and $n < \omega$, such that

$$J^0_l \subseteq^k_\infty \dots \subseteq^k_\infty J^0_0 \subseteq^k_\infty I\,, \\ J^{n+1}_l \subseteq^k_\infty \dots \subseteq^k_\infty J^{n+1}_0 \subseteq^k_\infty J^n_l\,,$$

and

$$w \models \varphi(n, \bar{m}, J^n_0 \dots J^n_l)\,, \quad \text{for all } n < \omega\,.$$

Set $J^{-1}_l := I$. For the inductive step, suppose that we have already defined J^{n-1}_l. Choosing n, J^{n-1}_l, $k+n$ for t, I, k,

$$\langle w, \bar{m}\rangle \models \forall t(\forall I \subseteq_\infty \omega)(\forall k < \omega) \\ (\exists J_0 \subseteq^k_\infty I)\dots(\exists J_l \subseteq^k_\infty J_{l-1})\varphi(t, \bar{t}', \bar{J})$$

implies that there are sets $J^n_l \subseteq^{k+n}_\infty \dots \subseteq^{k+n}_\infty J^{n-1}_l$ such that $w \models \varphi(n, \bar{m}, J^n_0 \dots J^n_l)$.

Having constructed $(J^n_i)_{n<\omega, i\leq l}$, let $K^n_i \subseteq J^n_i$ consist of the first $k+n$ elements of J^n_i and set

$$J_i := \bigcup_{n<\omega} K^n_i\,.$$

Then $J_i \subseteq^{k+n}_\infty J^n_i$, for all n and i. Hence,

$$w \models \varphi(n, \bar{m}, J^n_0 \dots J^n_l) \quad \text{implies} \quad w \models \varphi(n, \bar{m}, J_0 \dots J_l)\,.$$

Consequently, $w \models \forall t\varphi(t, \bar{m}, J)$ and it remains to prove that

$$J_l \subseteq^k_\infty \dots \subseteq^k_\infty J_0 \subseteq^k_\infty I\,.$$

Let $j \in J_{i+1}$. Then $j \in K^n_{i+1}$, for some n. Since $K^n_{i+1} \subseteq K^n_i \subseteq J_i$ it follows that $j \in J_i$. Consequently, $J_{i+1} \subseteq_\infty J_i$. Since $K^0_{i+1} \subseteq J_{i+1}$, it follows that $J_{i+1} \subseteq^k_\infty J_i$. Similarly, one can show that $J_0 \subseteq^k_\infty I$.

The induction step in the proof of the theorem below is based on the following two lemmas. The first one deals with the case of an universal number quantifier, the next one treats existential quantifiers.

Lemma 45. *Let w be an ω-word, $\bar{n}$ natural numbers, and $\varphi(t, \bar{t}')$ an AMSO^0_h-formula. We define*

$$\Phi := \{ \langle p, e\rangle \mid p \oplus \pi(e) = p \text{ and } p{\uparrow_t} \oplus e^\omega \models \varphi \}$$

where p is an AMSO_h-type with free variables $\bar{t}', J$ and e is an AMSO_h-type with free variables $t, \bar{t}', J$. The following statements are equivalent:

(1) $\langle w, \bar{n}\rangle \models \forall t \varphi(t, \bar{t}')$.

(2) $\langle w, \bar{n}\rangle \models (\forall I \subseteq_\infty \omega)(\forall k < \omega)(\exists J \subseteq^k_\infty I)$
$$\bigvee_{\langle p,e\rangle \in \Phi} \big[\mathrm{pref}(p, J) \wedge \forall t\, \mathrm{ult}(e, J)\big] .$$

(3) $\langle w, \bar{n}\rangle \models (\exists J \subseteq_\infty \omega) \bigvee_{\langle p,e\rangle \in \Phi} \big[\mathrm{pref}(p, J) \wedge \forall t\, \mathrm{ult}(e, J)\big]$.

Proof. (2) $\Rightarrow$ (3) follows by taking $I := \omega$ and $k := 0$.

(3) $\Rightarrow$ (1) Suppose that

$$\langle w, \bar{n}\rangle \models (\exists J \subseteq_\infty \omega) \bigvee_{\langle p,e\rangle \in \Phi} \big[\mathrm{pref}(p, J) \wedge \forall t\, \mathrm{ult}(e, J)\big] .$$

Fix a set $J \subseteq_\infty \omega$ and types $\langle p, e\rangle \in \Phi$ such that

$$\langle w, \bar{n}\rangle \models \mathrm{pref}(p, J) \wedge \forall t\, \mathrm{ult}(e, J) .$$

To show that

$$\langle w, \bar{n}\rangle \models \forall t \varphi(t, \bar{t}') ,$$

fix $m < \omega$. Let $(w_i)_{i<\omega}$ be the factorisation of w induced by J. There is some index $k < \omega$ such that

$$\begin{aligned} \langle w_0 \dots w_i, \bar{n}\rangle &\models p , \quad \text{for } i \geq k , \\ \langle w_i, m, \bar{n}\rangle &\models e , \quad \text{for } i > k . \end{aligned}$$

Let $M \geq m$ be a number such that $M \geq f[w_0 \dots w_k]$. Then

$$\langle w_0 \dots w_i, M, \bar{n}\rangle \models p{\uparrow_t} ,$$

which, by monotonicity, implies that

$$\langle w_0 \dots w_i, m, \bar{n}\rangle \models p{\uparrow_t} .$$

Hence,

$$\langle w, m, \bar{n}\rangle \models (p{\uparrow_t}) \oplus e \oplus e \dots = (p{\uparrow_t}) \oplus e^\omega .$$

Since $\langle p, e\rangle \in \Phi$, it follows that

$$\langle w, m, \bar{n}\rangle \models \varphi\,.$$

(1) $\Rightarrow$ (2) Suppose that

$$\langle w, \bar{n}\rangle \models \forall t\varphi(t, \bar{t}')\,,$$

To show that

$$\begin{aligned}\langle w, \bar{n}\rangle \models &(\forall I \subseteq_\infty \omega)(\forall k < \omega)(\exists J \subseteq^k_\infty I)\\ &\bigvee_{\langle p,e\rangle\in\Phi} \big[\mathrm{pref}(p, J) \wedge \forall t\,\mathrm{ult}(e, J)\big]\,,\end{aligned}$$

fix $I \subseteq_\infty \omega$ and $k < \omega$. By induction on i, we choose sets J_i, $i < \omega$ such that $J_0 \subseteq_\infty I$, $J_{i+1} \subseteq_\infty J_i$, and $\langle w, J_i, i\rangle$ is h-Ramsey of some type (p^+_i, e_i). Note that this implies that $p^+_i \oplus e_i = p^+_i$ and $e_i \oplus e_i = e_i$. Set $j_0 := -1$ and let j_{i+1} be the least element of J_i such that $j_{i+1} > j_i$. We set

$$J_\omega := \{\, j_{i+1} \mid i < \omega \,\}\,.$$

Let $(w_i)_{i<\omega}$ be the factorisation of w induced by J_ω. There exists an infinite set $K \subseteq_\infty \omega$ and types p^+ and e such that

$$p^+_i = p^+ \quad \text{and} \quad e_i = e\,, \quad \text{for all } i \in K\,.$$

Hence,

$$\begin{aligned}\langle w_0 \dots w_i, j, \bar{n}\rangle &\models p^+\,, &&\text{for all } j \leq i < \omega \text{ and } j < \omega\,,\\ \langle w_i, j, \bar{n}\rangle &\models e\,, &&\text{for all } j < i < \omega \text{ and } j < \omega\,.\end{aligned}$$

Setting $p := \pi(p^+)$ it follows that

$$\langle w, \bar{n}\rangle \models \mathrm{pref}(p, J_\omega) \wedge \forall t\,\mathrm{ult}(e, J_\omega)\,.$$

Let I_0 be the set consisting of the first k elements of I. Then

$$\langle w, \bar{n}\rangle \models \mathrm{pref}(p, I_0 \cup J_\omega) \wedge \forall t\,\mathrm{ult}(e, I_0 \cup J_\omega)\,.$$

Hence,

$$\langle w, \bar{n}\rangle \models (\exists J \subseteq^k_\infty I)\big[\mathrm{pref}(p, J) \wedge \forall t\,\mathrm{ult}(e, J)\big]\,.$$

To show that

$$\begin{aligned}\langle w, \bar{n}\rangle \models &(\forall I \subseteq_\infty \omega)(\forall k < \omega)\\ &(\exists J \subseteq^k_\infty I) \bigvee_{\langle p,e\rangle\in\Phi} \big[\mathrm{pref}(p, J) \wedge \forall t\,\mathrm{ult}(e, J)\big]\end{aligned}$$

it therefore remains to prove that $\langle p, e\rangle \in \Phi$.

Note that $p^+ \oplus e = p^+$ implies $p \oplus \pi(e) = p$. Hence, it remains to show that $p{\uparrow}_t \oplus e^\omega \models \varphi$. Fix $m \in K$. Then

$$\mathrm{tp}_h(w, m, \bar{n}) = p^+ \oplus e^\omega \quad \text{and} \quad \langle w, m, \bar{n}\rangle \models \varphi\,.$$

This implies that $p^+ \oplus e^\omega \models \varphi$. Let $m' \in K$ be some number such that $m' \geq m$ and

$$\mathrm{tp}_h(w, m', \bar{n}) = p{\uparrow}_t \oplus e^\omega\,.$$

Set $r_i := \mathrm{tp}_h(w_i, m', \bar{n})$. Then there exists some $l < \omega$ such that $r_i = e$, for all $i > l$. Since $m' \geq m$,

$$\mathrm{tp}_h(w_i, m, \bar{n}) = e \quad \text{implies} \quad r_i \subseteq e\,.$$

Hence,

$$\begin{aligned}\varphi \in \mathrm{tp}_h(w, m', \bar{n}) &= p{\uparrow}_t \oplus r_0 \oplus \cdots \oplus r_l \oplus e^\omega \\ &\subseteq p{\uparrow}_t \oplus e \oplus \cdots \oplus e \oplus e^\omega\end{aligned}$$

implies $p{\uparrow}_t \oplus e^\omega \models \varphi$.

Lemma 46. *Let w be an ω-word, $\bar{n}$ natural numbers, and $\varphi(t, \bar{t}')$ an AMSO^0_h-formula. We define*

$$\Phi := \{\, \langle p, e\rangle \mid p \oplus \pi(e) = p \text{ and } p{\uparrow}_t \oplus e^\omega \models \varphi \,\}$$

where p is an AMSO_h-type with free variables $\bar{t}', J$ and e is an AMSO_h-type with free variables $t, \bar{t}', J$. The following statements are equivalent:

(1) $\langle w, \bar{n}\rangle \models \exists t \varphi(t, \bar{t}')\,.$

(2) $\langle w, \bar{n}\rangle \models (\forall I \subseteq_\infty \omega)(\forall k < \omega)$
$$(\exists J \subseteq^k_\infty I) \bigvee_{\langle p,e\rangle \in \Phi} \big[\mathrm{pref}(p, J) \wedge \exists t\, \mathrm{ult}(e, J)\big]\,.$$

(3) $\langle w, \bar{n}\rangle \models (\exists J \subseteq_\infty \omega) \displaystyle\bigvee_{\langle p,e\rangle \in \Phi} \big[\mathrm{pref}(p, J) \wedge \exists t\, \mathrm{ult}(e, J)\big]\,.$

Proof. (2) $\Rightarrow$ (3) follows by taking $I := \omega$ and $k := 0$.

(3) $\Rightarrow$ (1) Suppose that

$$\langle w, \bar{n}\rangle \models (\exists J \subseteq_\infty \omega) \bigvee_{\langle p,e\rangle \in \Phi} \big[\mathrm{pref}(p, J) \wedge \exists t\, \mathrm{ult}(e, J)\big]\,.$$

We claim that

$$\langle w, \bar{n}\rangle \models \exists t \varphi(t, \bar{t}')\,.$$

Fix a set $J \subseteq_\infty \omega$, types $\langle p, e\rangle \in \Phi$, and a number $m < \omega$ such that

$$\langle w, \bar{n}\rangle \models \mathrm{pref}(p, J) \wedge \mathrm{ult}(e, J)(m)\,.$$

Let $(w_i)_{i<\omega}$ be the factorisation of w induced by J. There is some index $k < \omega$ such that

$$\begin{aligned} \langle w_0 \dots w_i, \bar{n}\rangle &\models p\,, \quad \text{for } i \geq k\,,\\ \langle w_i, m, \bar{n}\rangle &\models e\,, \quad \text{for } i > k\,. \end{aligned}$$

Let $M \geq m$ be a number such that $M \geq f[w_0 \dots w_k]$. By monotonicity, it follows that

$$\begin{aligned} \langle w_0 \dots w_k, M, \bar{n}\rangle &\models p{\uparrow_t}\,,\\ \langle w_i, M, \bar{n}\rangle &\models e\,, \quad \text{for } i > k\,. \end{aligned}$$

Set $s_i := \mathrm{tp}_h(w_i, M)$. Then

$$\langle w, M, \bar{n}\rangle \models (p{\uparrow_t}) \oplus s_{k+1} \oplus s_{k+2} \oplus \dots\,.$$

Since $e \subseteq s_i$ implies $s_i \models e$, it follows that

$$\langle w, M, \bar{n}\rangle \models (p{\uparrow_t}) \oplus e \oplus e \oplus \dots\,.$$

Hence, $p{\uparrow_t} \oplus e^\omega \models \varphi$ implies that

$$\langle w, M, \bar{n}\rangle \models \varphi\,.$$

Consequently, $\langle w, \bar{n}\rangle \models \exists t\varphi$.

(1) $\Rightarrow$ (2) Suppose that

$$\langle w, \bar{n}\rangle \models \exists t\varphi(t, \bar{t}')\,,$$

and fix some number $m < \omega$ such that

$$\langle w, m, \bar{n}\rangle \models \varphi\,.$$

To show that

$$\begin{aligned} \langle w, \bar{n}\rangle \models &(\forall I \subseteq_\infty \omega)(\forall k < \omega)\\ &\quad (\exists J \subseteq^k_\infty I) \bigvee_{\langle p,e\rangle \in \Phi} \big[\mathrm{pref}(p, J) \wedge \exists t\, \mathrm{ult}(e, J)\big]\,, \end{aligned}$$

fix $I \subseteq_\infty \omega$ and $k < \omega$. Let $J \subseteq_\infty I$ be a set such that $\langle w, J, m, \bar{n}\rangle$ is h-Ramsey of type (p^+, e), for some types p^+ and e. This implies that $p^+ \oplus e = p^+$. Let $(w_i)_{i<\omega}$ be the factorisation of w induced by J_ω, and set $p := \pi(p^+)$. Note that $p \oplus \pi(e) = p$ and

$$\begin{aligned} &\varphi \in \mathrm{tp}_h(w, m, \bar{n}) = p^+ \oplus e^\omega &&\text{implies} \quad p^+ \oplus e^\omega \models \varphi\,,\\ &p^+ \subseteq p{\uparrow_t} &&\text{implies} \quad p{\uparrow_t} \models p^+\,. \end{aligned}$$

Hence,

$$p{\uparrow_t} \oplus e^\omega \models p^+ \oplus e^\omega \models \varphi\,,$$

and $\langle p, e\rangle \in \varPhi$. Furthermore,

$$\begin{aligned} \langle w_0 \dots w_i, m, \bar n\rangle &\models p^+ \oplus e \oplus \dots \oplus e = p^+ \models p\,, \\ &\qquad\qquad \text{for all } i < \omega\,, \\ \langle w_i, m, \bar n\rangle &\models e\,, \qquad \text{for all } 0 < i < \omega\,. \end{aligned}$$

Consequently,

$$\langle w, m, \bar n\rangle \models \mathrm{pref}(p, J) \wedge \mathrm{ult}(e, J)\,.$$

Let I_0 be the set consisting of the first k elements of I. Then it follows that

$$\langle w, \bar n\rangle \models \mathrm{pref}(p, I_0 \cup J) \wedge \exists t\, \mathrm{ult}(e, I_0 \cup J)\,.$$

Hence,

$$\begin{aligned} \langle w, \bar n\rangle \models &(\forall I \subseteq_\infty \omega)(\forall k < \omega) \\ &\quad (\exists J \subseteq^k_\infty I) \bigvee_{\langle p, e\rangle \in \varPhi} \big[\mathrm{pref}(p, J) \wedge \exists t\, \mathrm{ult}(e, J)\big]\,. \end{aligned}$$

Combining the preceding lemmas we obtain the following theorem, which is the main result of this section.

Theorem 47. *Let w be an ω-word, $\bar n$ natural numbers, $\varphi(\bar t, \bar t')$ an AMSO^0_h-formula, and let $Q_0, \dots, Q_{l-1} \in \{\exists, \forall\}$. We define*

$$\begin{aligned} \varPhi := \big\{ \langle \bar p, \bar e\rangle \bigm| & p_i \oplus \pi(e_i) = p_i \text{ for all } i\,, \\ & p_{l-1}{\uparrow_{t_{l-1}}} \oplus e^\omega_{l-1} \models \varphi\,, \text{ and} \\ & p_i{\uparrow_{t_i}} \oplus e^\omega_i \models \mathrm{pref}(p_{i+1}, J_{i+1})\,, \text{ for } i < l-1 \big\}\,, \end{aligned}$$

where, for $i < l$, p_i is an AMSO^0_h-type with free variables $t_0, \dots, t_{i-1}, \bar t', J_{i+1}, \dots, J_{l-1}$ and e_i an AMSO^0_h-type with free variables $t_0, \dots, t_{i-1}, t_i, \bar t', J_{i+1}, \dots, J_{l-1}$.

The following statements are equivalent:

(1) $\langle w, \bar n\rangle \models Q_0 t_0 \cdots Q_{l-1} t_{l-1} \varphi(\bar t, \bar t')\,.$

(2) $$\begin{aligned} \langle w, \bar n\rangle \models\ &(\forall I \subseteq_\infty \omega)(\forall k < \omega) \\ &(\exists J_{l-1} \subseteq^k_\infty I)(\exists J_{l-2} \subseteq^k_\infty J_{l-1}) \cdots \\ &(\exists J_0 \subseteq^k_\infty J_1) \\ &\bigvee_{\langle \bar p, \bar e\rangle \in \varPhi} \Big[\mathrm{pref}(p_0, J_0) \wedge \\ &\qquad \bigwedge_{i<l} Q_0 t_0 \cdots Q_i t_i\, \mathrm{ult}(e_i, J_i)\Big]\,. \end{aligned}$$

(3) $\langle w, \bar{n}\rangle \models (\exists J_{l-1} \subseteq_\infty \omega)(\exists J_{l-2} \subseteq_\infty J_{l-1}) \cdots$
$$(\exists J_0 \subseteq_\infty J_1) \bigvee_{\langle \bar{p}, \bar{e}\rangle \in \Phi} \Big[\operatorname{pref}(p_0, J_0) \wedge \bigwedge_{i<l} Q_0 t_0 \cdots Q_i t_i \operatorname{ult}(e_i, J_i)\Big] .$$

Proof. (2) $\Rightarrow$ (3) follows by taking $I := \omega$ and $k := 0$.

(3) $\Rightarrow$ (1) We prove the implication by induction on l. The case where $l = 1$ follows directly from Lemmas 45 and 46. Hence, let $l > 1$. Suppose that

$$\langle w, \bar{n}\rangle \models (\exists J_{l-1} \subseteq_\infty \omega) \cdots (\exists J_0 \subseteq_\infty J_1) \bigvee_{\langle \bar{p}, \bar{e}\rangle \in \Phi} \Big[\operatorname{pref}(p_0, J_0) \wedge \bigwedge_{i<l} Q_0 t_0 \cdots Q_i t_i \operatorname{ult}(e_i, J_i)\Big]$$

and that we have already proved the implication (3) $\Rightarrow$ (1) for $l - 1$ quantifiers. Set

$$\begin{aligned} \Phi' := \big\{ \langle p_1, \dots, p_{l-1}, e_1, \dots, e_{l-1}\rangle \;\big|\; & p_i \oplus \pi(e_i) = p_i \text{ for all } i\,, \\ & p_{l-1}{\uparrow_{t_{l-1}}} \oplus e_{l-1}^\omega \models \varphi\,, \text{ and} \\ & p_i{\uparrow_{t_i}} \oplus e_i^\omega \models \operatorname{pref}(p_{i+1}, J_{i+1})\,, \text{ for } i < l-1 \big\}\,. \end{aligned}$$

Then

$$\begin{aligned} \langle w, \bar{n}\rangle \models & (\exists J_{l-1} \subseteq_\infty \omega) \cdots (\exists J_1 \subseteq_\infty J_2) \\ & \bigvee_{\langle \bar{p}, \bar{e}\rangle \in \Phi'} \Big[(\exists J_0 \subseteq_\infty J_1) \bigvee_{\substack{p_0, e_0: \\ p_0 \oplus \pi(e_0)^\omega = p_0 \\ p_0 \uparrow_{t_0} \oplus e_0^\omega \models \operatorname{pref}(p_1, J_1)}} \big[\operatorname{pref}(p_0, J_0) \wedge Q_0 t_0 \operatorname{ult}(e_0, J_0)\big] \\ & \qquad \wedge \bigwedge_{0<i<l} Q_0 t_0 \cdots Q_i t_i \operatorname{ult}(e_i, J_i)\Big] . \end{aligned}$$

By Lemmas 45 and 46, it follows that

$$\begin{aligned} \langle w, \bar{n}\rangle \models & (\exists J_{l-1} \subseteq_\infty \omega) \cdots (\exists J_1 \subseteq_\infty J_2) \\ & \bigvee_{\langle \bar{p}, \bar{e}\rangle \in \Phi'} \Big[Q_0 t_0 \operatorname{pref}(p_1, J_1) \wedge \bigwedge_{0<i<l} Q_0 t_0 \cdots Q_i t_i \operatorname{ult}(e_i, J_i)\Big] , \end{aligned}$$

which, by Proposition 14, is equivalent to

$$\begin{aligned}
\langle w, \bar{n} \rangle \models (\exists J_{l-1} \subseteq_\infty \omega) \cdots (\exists J_1 \subseteq_\infty J_2) Q_0 t_0 \\
\bigvee_{\langle \bar{p}, \bar{e} \rangle \in \Phi'} \Big[\operatorname{pref}(p_1, J_1) \wedge \\
\bigwedge_{0<i<l} Q_1 t_1 \cdots Q_i t_i \operatorname{ult}(e_i, J_i) \Big] .
\end{aligned}$$

This formula implies

$$\begin{aligned}
\langle w, \bar{n} \rangle \models Q_0 t_0 (\exists J_{l-1} \subseteq_\infty \omega) \cdots (\exists J_1 \subseteq_\infty J_2) \\
\bigvee_{\langle \bar{p}, \bar{e} \rangle \in \Phi'} \Big[\operatorname{pref}(p_1, J_1) \wedge \\
\bigwedge_{0<i<l} Q_1 t_1 \cdots Q_i t_i \operatorname{ult}(e_i, J_i) \Big] .
\end{aligned}$$

By induction hypothesis, it follows that

$$\langle w, \bar{n} \rangle \models Q_0 t_0 \dots Q_{l-1} t_{l-1} \varphi .$$

(1) $\Rightarrow$ (2) We prove the implication by induction on l. The case where $l = 1$ follows directly from Lemmas 45 and 46. Hence, let $l > 1$. Suppose that

$$\langle w, \bar{n} \rangle \models Q_0 t_0 \cdots Q_{l-1} t_{l-1} \varphi(\bar{t}, \bar{t}')$$

and that we have already proved the implication (1) $\Rightarrow$ (2) for $l - 1$ quantifiers. Set

$$\begin{aligned}
\Phi' := \big\{ \langle p_1, \dots, p_{l-1}, e_1, \dots, e_{l-1} \rangle \bigm| \\
p_i \oplus \pi(e_i) = p_i \text{ for all } i , \\
p_{l-1}{\upharpoonleft}_{t_{l-1}} \oplus e_{l-1}^\omega \models \varphi , \text{ and} \\
p_i{\upharpoonleft}_{t_i} \oplus e_i^\omega \models \operatorname{pref}(p_{i+1}, J_{i+1}) , \text{ for } i < l - 1 \big\} .
\end{aligned}$$

By induction hypothesis, it follows that

$$\begin{aligned}
\langle w, \bar{n} \rangle \models Q_0 t_0 (\forall I \subseteq_\infty \omega)(\forall k < \omega) \\
(\exists J_{l-1} \subseteq_\infty^k I) \cdots (\exists J_1 \subseteq_\infty^k J_2) \\
\bigvee_{\langle \bar{p}, \bar{e} \rangle \in \Phi'} \Big[\operatorname{pref}(p_1, J_1) \wedge \\
\bigwedge_{0<i<l} Q_1 t_1 \cdots Q_i t_i \operatorname{ult}(e_i, J_i) \Big] .
\end{aligned}$$

Using Lemma 44 we obtain

$$\begin{aligned}\langle w,\bar{n}\rangle \models\ &(\forall I \subseteq_\infty \omega)(\forall k<\omega)\\ &(\exists J_{l-1} \subseteq^k_\infty I)\cdots(\exists J_1 \subseteq^k_\infty J_2)Q_0t_0\\ &\bigvee_{\langle\bar{p},\bar{e}\rangle\in\Phi'}\Big[\mathrm{pref}(p_1,J_1)\wedge\\ &\qquad\bigwedge_{0<i<l} Q_1t_1\cdots Q_it_i\,\mathrm{ult}(e_i,J_i)\Big],\end{aligned}$$

which, by Proposition 14, is equivalent to

$$\begin{aligned}\langle w,\bar{n}\rangle \models\ &(\forall I \subseteq_\infty \omega)(\forall k<\omega)\\ &(\exists J_{l-1} \subseteq^k_\infty I)\cdots(\exists J_1 \subseteq^k_\infty J_2)\\ &\bigvee_{\langle\bar{p},\bar{e}\rangle\in\Phi'}\Big[Q_0t_0\mathrm{pref}(p_1,J_1)\wedge\\ &\qquad\bigwedge_{0<i<l} Q_0t_0Q_1t_1\cdots Q_it_i\,\mathrm{ult}(e_i,J_i)\Big].\end{aligned}$$

Applying Lemmas 45 and 46 to $Q_0t_0\mathrm{pref}(p_1,J_1)$, we obtain

$$\begin{aligned}\langle w,\bar{n}\rangle \models&(\forall I \subseteq_\infty \omega)(\forall k<\omega)(\exists J_{l-1} \subseteq^k_\infty I)\cdots(\exists J_1 \subseteq^k_\infty J_2)\\ &\bigvee_{\langle\bar{p},\bar{e}\rangle\in\Phi'}\Big[\quad(\forall I' \subseteq_\infty \omega)(\forall k'<\omega)(\exists J_0 \subseteq^{k'}_\infty I')\\ &\qquad\bigvee_{\substack{p_0,e_0:\\ p_0\oplus\pi(e_0)^\omega=p_0\\ p_0\uparrow_{t_0}\oplus e_0^\omega\models\mathrm{pref}(p_1,J_1)}}\big[\mathrm{pref}(p_0,J_0)\wedge Q_0t_0\mathrm{ult}(e_0,J_0)\big]\\ &\qquad\wedge\bigwedge_{0<i<l} Q_0t_0Q_1t_1\cdots Q_it_i\,\mathrm{ult}(e_i,J_i)\Big].\end{aligned}$$

Choosing $I'=J_1$ and $k'=k$, this reduces to

$$\begin{aligned}\langle w,\bar{n}\rangle \models&(\forall I \subseteq_\infty \omega)(\forall k<\omega)(\exists J_{l-1} \subseteq^k_\infty I)\cdots(\exists J_1 \subseteq^k_\infty J_2)\\ &\bigvee_{\langle\bar{p},\bar{e}\rangle\in\Phi'}\Big[\quad(\exists J_0 \subseteq^{k}_\infty J_1)\\ &\qquad\bigvee_{\substack{p_0,e_0:\\ p_0\oplus\pi(e_0)^\omega=p_0\\ p_0\uparrow_{t_0}\oplus e_0^\omega\models\mathrm{pref}(p_1,J_1)}}\big[\mathrm{pref}(p_0,J_0)\wedge Q_0t_0\mathrm{ult}(e_0,J_0)\big]\\ &\qquad\wedge\bigwedge_{0<i<l} Q_0t_0Q_1t_1\cdots Q_it_i\,\mathrm{ult}(e_i,J_i)\Big].\end{aligned}$$

It follows that

$$
\begin{aligned}
\langle w, \bar{n}\rangle \models &(\forall I \subseteq_\infty \omega)(\forall k < \omega)(\exists J_{l-1} \subseteq_\infty^k I) \cdots \\
&\quad (\exists J_1 \subseteq_\infty^k J_2)(\exists J_0 \subseteq_\infty^k J_1) \\
&\quad \bigvee_{\langle \bar{p}, \bar{e}\rangle \in \Phi'} \Big[\bigvee_{\substack{p_0, e_0 : \\ p_0 \oplus \pi(e_0)^\omega = p_0 \\ p_0 \upharpoonright t_0 \oplus e_0^\omega \models \mathrm{pref}(p_1, J_1)}} \big[\mathrm{pref}(p_0, J_0) \wedge Q_0 t_0 \mathrm{ult}(e_0, J_0)\big] \\
&\qquad \wedge \bigwedge_{0<i<l} Q_0 t_0 Q_1 t_1 \cdots Q_i t_i \, \mathrm{ult}(e_i, J_i)\Big] .
\end{aligned}
$$

This formula reduces to

$$
\begin{aligned}
\langle w, \bar{n}\rangle \models &(\forall I \subseteq_\infty \omega)(\forall k < \omega)(\exists J_{l-1} \subseteq_\infty^k I) \cdots (\exists J_0 \subseteq_\infty^k J_1) \\
&\bigvee_{\langle p_0, \bar{p}, e_0, \bar{e}\rangle \in \Phi} \Big[\mathrm{pref}(p_0, J_0) \wedge \\
&\qquad \bigwedge_{i<l} Q_0 t_0 \cdots Q_i t_i \, \mathrm{ult}(e_i, J_i)\Big] .
\end{aligned}
$$

Using the preceding theorem, we can reduce the satisfiability problem for WAMSO to the so-called limit satisfiability problem.

Definition 48. *The* limit satisfiability problem *consists in, given an* AMSO_h^0-*formula* $\varphi(\bar{t})$ *and a quantifier-prefix* $\bar{Q}\bar{t}$, *to decide whether there exists a sequence* $(w_i)_{i<\omega}$ *of finite words such that*

$$\bar{Q}\bar{t}(\exists k < \omega)(\forall i \geq k)[w_i \models \varphi(\bar{t})] .$$

Proposition 49. *The satisfiability problem for* WAMSO *over the class of* ω-*words reduces to the limit satisfiability problem.*

Proof. Let φ be a WAMSO-formula. We can use Corollary 20 to compute a formula $\bar{Q}\bar{t}\psi$ in number prenex form that is equivalent to φ on the class of all ω-words. For an ω-word w, it follows by Theorem 47 that

$$
\begin{aligned}
& w \models \bar{Q}\bar{t}\psi \\
\text{iff}\quad & w \models (\exists J_{l-1} \subseteq_\infty \omega)(\exists J_{l-2} \subseteq_\infty J_{l-1}) \cdots (\exists J_0 \subseteq_\infty J_1) \\
& \qquad \bigvee_{\langle \bar{p}, \bar{e}\rangle \in \Phi} \Big[\mathrm{pref}(p_0, J_0) \wedge \bigwedge_{i<l} Q_0 t_0 \cdots Q_i t_i \, \mathrm{ult}(e_i, J_i)\Big] \\
\text{iff}\quad & w \models (\exists J_{l-1} \subseteq_\infty \omega)(\exists J_{l-2} \subseteq_\infty J_{l-1}) \cdots (\exists J_0 \subseteq_\infty J_1) \\
& \qquad \bigvee_{\langle \bar{p}, \bar{e}\rangle \in \Phi} \Big[\mathrm{pref}(p_0, J_0) \wedge \bar{Q}\bar{t} \bigwedge_{i<l} \mathrm{ult}(e_i, J_i)\Big] .
\end{aligned}
$$

It follows that φ is satisfiable if, and only if, there exist types $\langle \bar{p}, \bar{e}\rangle \in \Phi$ and a sequence $(w_i, \bar{J}^i)_{i<\omega}$ where the w_i are finite words and $J_0^i \subseteq \cdots \subseteq J_{l-1}^i \subseteq w_i$ are unary predicates such that

– $0 \in J^i_0$ for $i > 0$
– $(\exists k < \omega)(\forall i \geq k)[(w_0, \bar{J}^0) \dots (w_i, \bar{J}^i) \models p_0]$
– $\bar{Q}\bar{t}(\exists k < \omega)(\forall i \geq k)\Big[w_i \models \bigwedge_{i<l} \mathrm{all}(e_i, J_i)\Big]$

where

$$\begin{aligned}\mathrm{all}(e, J) := (\forall x, y \in J)\big[x < y \wedge J \cap (x, y) = \emptyset \to e^{[x,y)}\big] \\ \wedge (\forall x \in J)\forall y\big[\neg\exists z(y < z) \wedge J \cap (x, y] = \emptyset \to e^{[x,y]}\big]\end{aligned}$$

states that every interval with end-points specified by J satisfies e.

Fix $\langle \bar{p}, \bar{e}\rangle \in \Phi$ and set

$$\vartheta(\bar{t}, \bar{J}) := 0 \in J_0 \subseteq \dots \subseteq J_{l-1} \wedge \bigwedge_{i<l} \mathrm{all}(e_i, J_i)\,.$$

We claim that the following two statements are equivalent:

(a) There exists a sequence $(w_i, \bar{J}^i)_{i<\omega}$ such that
- $0 \in J^i_0$ for $i > 0$
- $(\exists k < \omega)(\forall i \geq k)[(w_0, \bar{J}^0) \dots (w_i, \bar{J}^i) \models p_0]$
- $\bar{Q}\bar{t}(\exists k < \omega)(\forall i \geq k)\Big[w_i \models \bigwedge_{j<l} \mathrm{all}(e_j, J^i_j)\Big]$

(b) There exist a finite word satisfying p_0 and a sequence $(v_i, \bar{I}^i)_{i<\omega}$ such that

$$\bar{Q}\bar{t}(\exists k < \omega)(\forall i \geq k)[v_i \models \vartheta(\bar{t}, \bar{I}^i)]\,.$$

Note that satisfiability of p_0 by a finite word is decidable since p_0 is a set of WMSO-formulae. Consequently, (b) reduces to the limit satisfiability problem for the formula ϑ. To prove the proposition, it is therefore sufficient to show that (a) and (b) are equivalent.

Clearly, (a) implies (b). Hence, suppose that (b) holds. Let $(v_*, \bar{I}^*)$ be a finite word satisfying p_0 and let $(v_i, \bar{I}^i)_{i<\omega}$ be a sequence as in (b). By assumption, there is some index $k < \omega$ such that,

$$v_i \models \vartheta(0, \dots, 0, \bar{I}^i)\,, \quad \text{for all } i \geq k\,.$$

Consider the sequence $(w_i, \bar{J}^i)$ where

$$(w_i, \bar{J}^i) := \begin{cases} (v_*, \bar{I}^*) & \text{if } i = 0\,, \\ (v_{k+i}, \bar{I}^{k+i}) & \text{if } i > 1\,. \end{cases}$$

We claim that $(w_i, \bar{J}^i)$ satisfies (a). The only thing we have to check is that every long enough prefix satisfies p_0. In fact, we claim that all prefixes satisfy p_0. Hence, let $i < \omega$. Then

$$\mathrm{tp}_h((w_0, \bar{J}^0) \dots (w_i, \bar{J}^i)) = p_0 \oplus \pi(e_0) \oplus \dots \oplus \pi(e_0) = p_0\,.$$

Consequently,

$$(w_0, \bar{J}^0) \dots (w_i, \bar{J}^i) \models p_0\,.$$

The last result of this section provides a preparation step to the reduction of the limit satisfiability problem to the tiling problems. To simplify notation, for a tuple $\bar{n} = \langle n_0, \dots, n_{m-1} \rangle$ of numbers, we write $\bar{n} + 1$ for the tuple $\langle n_0 + 1, \dots, n_{m-1} + 1 \rangle$.

Lemma 50. *An* AMSO^0_h*-formula* $\varphi(\bar{r}, \bar{s})$ *and the quantifier prefix* $\forall s_0 \exists r_0 \cdots \forall s_{m-1} \exists r_{m-1}$ *are a solution to the limit satisfiability problem if, and only if, there exists a sequence* $(w_i)_{i<\omega}$ *of finite words such that, for all* $n_0 < \cdots < n_{m-1} < \omega$,

$$(\exists k < \omega)(\forall i \geq k)[w_i \models \varphi(\bar{n} + 1, \bar{n})] .$$

Proof. ($\Leftarrow$) Suppose that, for all $n_0 < \cdots < n_{m-1} < \omega$,

$$(\exists k < \omega)(\forall i \geq k)[w_i \models \varphi(\bar{n} + 1, \bar{n})] .$$

By induction on l, we will show that

$$\begin{aligned} &\forall s_{m-l} \exists r_{m-l} \cdots \forall s_{m-1} \exists r_{m-1} (\exists k < \omega)(\forall i \geq k) \\ &\quad \big[w_i \models \varphi(n_0 + 1, \dots, n_{m-l-1} + 1, r_{m-l}, \dots, r_{m-1}, \\ &\qquad\qquad n_0, \dots, n_{m-l-1}, s_{m-l}, \dots, s_{m-1})\big] . \end{aligned}$$

for all $n_0 < \cdots < n_{m-i-1}$.

By induction hypothesis, assume that

$$\begin{aligned} &(\forall n_0 < \cdots < n_{m-l-2} < n_{m-l-1}) \\ &\quad \forall s_{m-l} \exists r_{m-l} \cdots \forall s_{m-1} \exists r_{m-1} (\exists k < \omega)(\forall i \geq k) \\ &\qquad \big[w_i \models \varphi(n_0 + 1, \dots, n_{m-l-2} + 1, n_{m-l-1} + 1, \\ &\qquad\qquad r_{m-l}, \dots, r_{m-1}, \\ &\qquad\qquad n_0, \dots, n_{m-l-2}, n_{m-l-1}, \\ &\qquad\qquad s_{m-l}, \dots, s_{m-1})\big] . \end{aligned}$$

Then

$$\begin{aligned} &(\forall n_0 < \cdots < n_{m-l-2})(\forall s_{m-l-1} > n_{m-l-2}) \exists r_{m-l-1} \\ &\quad \forall s_{m-l} \exists r_{m-l} \cdots \forall s_{m-1} \exists r_{m-1} (\exists k < \omega)(\forall i \geq k) \\ &\qquad \big[w_i \models \varphi(n_0 + 1, \dots, n_{m-l-2} + 1, \\ &\qquad\qquad r_{m-l-1}, r_{m-l}, \dots, r_{m-1}, \\ &\qquad\qquad n_0, \dots, n_{m-l-2}, \\ &\qquad\qquad s_{m-l-1}, s_{m-l}, \dots, s_{m-1})\big] . \end{aligned}$$

By monotonicity, it follows that

$$\begin{aligned}
&(\forall n_0 < \cdots < n_{m-l-2}) \\
&\quad \forall s_{m-l-1} \exists r_{m-l-1} \forall s_{m-l} \exists r_{m-l} \cdots \forall s_{m-1} \exists r_{m-1} \\
&\quad (\exists k < \omega)(\forall i \geq k) \\
&\qquad \big[w_i \models \varphi(n_0 + 1, \dots, n_{m-l-2} + 1, \\
&\qquad\qquad\qquad r_{m-l-1}, r_{m-l}, \dots, r_{m-1}, \\
&\qquad\qquad\qquad n_0, \dots, n_{m-l-2}, \\
&\qquad\qquad\qquad s_{m-l-1}, s_{m-l}, \dots, s_{m-1})\big] .
\end{aligned}$$

($\Rightarrow$) Let $(w_i)_{i<\omega}$ be a sequence of finite words such that

$$\forall s_0 \exists r_0 \cdots \forall s_{m-1} \exists r_{m-1} (\exists k < \omega)(\forall i \geq k)\big[w_i \models \varphi(\bar{r}, \bar{s})\big] .$$

Then there are Skolem functions $\beta_0, \dots, \beta_{m-1}$ such that, for all $\bar{n}$,

$$\begin{aligned}
&(\exists k < \omega)(\forall i \geq k) \\
&\quad \big[w_i \models \varphi(\beta_0(n_0), \beta_1(n_0, n_1), \dots, \beta_{m-1}(n_0, \dots, n_{m-1}), \bar{n})\big] .
\end{aligned}$$

By monotonicity of φ, we may assume that each β_i is increasing in every argument. Furthermore, setting $\beta'_i(x) := \beta_i(x, \dots, x)$ and $n'_i := \max\{n_0, \dots, n_i\}$, it follows that

$$\begin{aligned}
&(\exists k < \omega)(\forall i \geq k) \\
&\quad \big[w_i \models \varphi(\beta'_0(n'_0), \beta'_1(n'_1), \dots, \beta'_{m-1}(n'_{m-1}), \bar{n})\big] .
\end{aligned}$$

Consequently, it follows for

$$\beta(x) := \max\{\beta'_0(x), \dots, \beta'_{m-1}(x)\}$$

that

$$(\exists k < \omega)(\forall i \geq k)\big[w_i \models \varphi(\beta(n_0), \dots, \beta(n_{m-1}), \bar{n})\big] ,$$

for all $n_0 < \cdots < n_{k-1}$. Let w'_i be the weighted word obtained from w_i by replacing each weight function f by the function

$$f'(x) := k \quad \text{where } \beta^k(0) \leq f(x) < \beta^{k+1}(0) .$$

For all $n_0 < \cdots < n_{m-1}$ and every i, it follows that

$$w_i \models \varphi(\beta(n_0), \dots, \beta(n_{m-1}), \bar{n})$$

implies

$$w'_i \models \varphi(n_0 + 1, \dots, n_{m-1} + 1, \bar{n}) .$$

Consequently, we have

$$(\exists k < \omega)(\forall i \geq k)\big[w'_i \models \varphi(n_0 + 1, \dots, n_{m-1} + 1, \bar{n})\big] ,$$

for all $n_0 < \cdots < n_{k-1}$.

C.2 Reductions between tiling problems

Before reducing the limit satisfiability problem to certain tiling problems, we present reductions between various versions of these tiling problems. Let us start with some terminology. For a picture $p : [h] \times [w] \to \Sigma$, we denote the ith column by

$$p(-, i) := p(0, i) \cdots p(k-1, i)\,,$$

the jth row by

$$p(j, -) := p(j, 0) \cdots p(j, n-1)\,,$$

and, the band for rows $j_1 < \cdots < j_k$ by

$$p(j_1, -) \times \cdots \times p(j_k, -)\,.$$

We will consider the following variants of tiling systems.

Definition 51. *(a) An m-dimensional tiling system (L, K) is* lossy *if the column language K is closed under subwords.*

(b) An m-dimensional tiling system (L, K) is monotone *if there is a partial order $\leq$ on the alphabet Σ such that*

- *if $uabv \in K$, for $a, b \in \Sigma$ and $u, v \in \Sigma^*$, then there exists a letter $c \in \Sigma$ such that $c \geq a, b$ and $ucv \in K$;*
- *$a_{00} \dots a_{0(k-1)} \times \cdots \times a_{(m-1)0} \dots a_{(m-1)(k-1)} \in L$ and $a_{ij} \leq b_{ij}$ implies $b_{00} \dots b_{0(k-1)} \times \cdots \times b_{(m-1)0} \dots b_{(m-1)(k-1)} \in L$.*

(c) An m-dimensional tiling system (L, K) is restricted *if the language K is a finite union of languages of the form a^*bc^*, for $a, b, c \in \Sigma$.*

The *m-dimensional tiling problem* is the problem to decide whether a given m-dimensional tiling system (L, K) has solutions of arbitrarily large height. The *lossy/monotone/restricted* m-dimensional tiling problem is the similar problems for tiling systems of the corresponding kind.

Lemma 52. *The [restricted] monotone m-dimensional tiling problem reduces to the [restricted] lossy m-dimensional tiling problem.*

Proof. Let (L, K) be a monotone m-dimensional tiling system and let K' be the closure of K under subwords. We claim that (L, K') has solutions of arbitrarily large heights if, and only if, (L, K) has such solutions.

Clearly, any solution of (L, K) is also one of (L, K'). Conversely, let $p : [n] \times [k] \to \Sigma$ be a solution of (L, K'). Then

$$p(-, i) \in K'\,, \quad \text{for all } i < k\,.$$

Hence, there exists a word $a^i_0 \dots a^i_{l_i - 1} \in K$ and an injective function $h_i : [n] \to [l_i]$ such that $p(j, i) = a^i_{h_i(j)}$. Since (K, L) is monotone, there exists a word

$b_0^i \dots b_{n-1}^i \in K$ such that $b_j^i \geq a_{h_i(j)}^i$. We define $q : [n] \times [k] \to \Sigma$ by $q(i,j) := b_j^i$. By choice of b_j^i, we have

$$q(-,i) \in K\,, \quad \text{for all } i < k\,.$$

Furthermore, as (K,L) is monotone, and $q(i,j) \geq p(i,j)$,

$$q(i_0,-) \times \dots \times q(i_{m-1},-) \in L\,,$$

for all $i_0 < \dots < i_{m-1} < n$. Hence, q is a solution of (K,L) of the same height as p.

Lemma 53. *The restricted lossy* 1*-dimensional tiling problem and the restricted monotone* 1*-dimensional tiling problem are equivalent (with respect to many-one reductions).*

Proof. We have already presented a reduction from the monotone case to the lossy case. For the other direction, let (L,K) be a restricted lossy 1-dimensional tiling system over the alphabet Σ. We set $\Sigma' := \mathcal{P}(\Sigma)$. The language K' is the closure of the set

$$\{\,\{a_0\} \dots \{a_{n-1}\} \mid \{a_0\} \dots \{a_{l-1}\} \in K\,\}$$

under the operation

$$A_0 \dots A_{n-1} \mapsto A_0 \dots A_{i-1}(A_i \cup A_{i+1})A_{i+2} \dots A_{n-1}\,.$$

The language L' consists of all words $A_0 \dots A_{k-1}$ such that there exist elements $a_i \in A_i$ with $a_0 \dots a_{k-1} \in L$.

We claim that (K,L) has solutions of arbitrary height if, and only if, (K',L') has such solutions. Clearly, if p is a solution of (K,L) then we obtain a solution p' of (K',L') by setting $p'(i,j) := \{p(i,j)\}$. Conversely, let $p' : [n] \times [k] \to \Sigma'$ be a solution of (K',L'). For each $i < n$, there are elements $a_j^i \in p'(i,j)$ such that $a_0^i \dots a_{k-1}^i \in L$. We set $p(i,j) := a_j^i$. By choice of a_j^i, we have $a_0^i \dots a_{k-1}^i \in L$, for every $i < n$. By definition of K', for each $j < k$, there exists a word $w_j \in K$ such that $a_j^0 \dots a_j^{n-1}$ is a subword of w_j. As (K,L) is lossy, it follows that $a_j^0 \dots a_j^{n-1} \in K$. Consequently, p is a solution of (K,L).

Lemma 54. *The lossy m-dimensional tiling problem and the restricted lossy m-dimensional tiling problem are equivalent (with respect to many-one reductions).*

Proof. Clearly, every restricted tiling problem is also an unrestricted one. Hence, we only have to prove one direction. Let (L,K) be an arbitrary lossy m-dimensional tiling system over the alphabet Σ. We define an equivalent restricted problem as follows. Suppose that $K = \bigcup_{i<l} T_i$ where each language T_i is of the form

$$T_i = (A_0^i)^* b_1^i (A_1^i)^* \dots (A_{n-1}^i)^* b_n^i (A_n^i)^*,$$

for $A_0^i, \dots, A_n^i \subseteq \Sigma$ and $b_1^i, \dots, b_n^i \in \Sigma$. W.l.o.g. we may assume that the number n is the same for all languages T_i. Let $>, < \notin \Sigma$ be new symbols and define the alphabet

$$\Sigma' := (\Sigma \cup \{<, >\}) \times [l] \times [2n+1] .$$

For every $a \in \Sigma$, we define the language $B_a \subseteq (\Sigma')^*$ consisting of all words

$$(c_0, i, j_0) \dots (c_s, i, j_s)$$

such that

- $c_0 \dots c_s \in >^* a <^*$;
- $c_t = a$ and $j_t = 2r$ implies $a \in A_r^i$;
- $c_t = a$ and $j_t = 2r+1$ implies $a = b_{r+1}^i$;
- $0 = j_0 \leq \dots \leq j_s = 2n$;
- $j_t = j_{t+1}$ implies that j_t is even;
- $j_t \neq j_{t+1}$ implies that $j_{t+1} = j_t + 1$.

Let L' be the language obtained from L by replacing each letter a by a word of the form B_a. The language K' is

$$K' := \bigcup_{a \in \Sigma} \bigcup_{i<l} \bigcup_{j<2n+1} \begin{array}{l} \big[(<, i, j)^*(a, i, j)(>, i, j)^* \\ \quad \cup (<, i, j)^*(>, i, j)^*\big] . \end{array}$$

We claim that (L', K') has a solution of height n if, and only if, (L, K) has such a solution.

($\Leftarrow$) Let p be a solution of (L, K) of height h. For each column $p(-, x)$, fix a language T_i containing it and fix numbers $k_0, \dots, k_n$ such that

$$p(-, x) \in (A_0^i)^{k_0} b_1^i (A_1^i)^{k_1} \dots (A_{n-1}^i)^{k_{n-1}} b_n^i (A_n^i)^{k_n} .$$

We obtain p' by replacing the column $p(-, x)$ by h columns

$$\begin{bmatrix} (>, i, 0) \\ \\ \vdots \\ \\ (>, i, 0) \\ (p(0, x), i, 0) \end{bmatrix} \cdots \begin{bmatrix} (>, i, j_s) \\ \vdots \\ (>, i, j_s) \\ (p(s, x), i, j_s) \\ (<, i, j_s) \\ \vdots \\ (<, i, j_s) \end{bmatrix} \cdots \begin{bmatrix} (p(h-1, x), i, 2n) \\ (<, i, 2n) \\ \\ \vdots \\ \\ (<, i, 2n) \end{bmatrix} ,$$

where

$$j_s := \begin{cases} 0 & \text{if } s < k_0 , \\ 2j+1 & \text{if } s = k_0 + \dots + k_j + j , \\ 2j & \text{if } k_0 + \dots + k_{j-1} + j - 1 \\ & \qquad < s < k_0 + \dots + k_j + j . \end{cases}$$

Then p' is a solution of (L', K') of height h.

($\Rightarrow$) Let p' be a solution of (L', K') of height h and length w. Let $0 = x_0 < \dots < x_t = w$ be all indices such that, for every $s < t$,

$$p'(y, x_s)p'(y, x_s + 1) \dots p'(y, x_{s+1} - 1) \in B_a \,,$$

for some $a \in \Sigma$. Given x, y with $x_s \leq x < x_{s+1}$, let $p(y, x) := a$, where $a \in \Sigma$ is the letter such that

$$p'(y, x_s)p'(y, x_s + 1) \dots p'(y, x_{s+1} - 1) \in B_a \,.$$

We claim that p is a solution of (L, K).

From the definition of L' it follows easily that every tuple of m lines of p is in L. Hence, we only have to check that every column is in K. For $s < t$, set $a_y := p(y, x)$. Then

$$p'(y, x_s)p'(y, x_s + 1) \dots p'(y, x_{s+1} - 1) \in B_{a_y}$$

and there is some index $x_s \leq x < x_{s+1}$ such that $p'(y, x) = (a_y, i_y, j_y)$, for suitable i_y, j_y. By definition of L' and K', we have $i_0 = \dots = i_{h-1}$ and

$$\begin{aligned} a_y &\in A_s^{i_y} \,, \quad \text{if } j_y = 2s \,, \\ a_y &= b_{s+1}^{i_y} \,, \quad \text{if } j_y = 2s + 1 \,. \end{aligned}$$

Consequently, $a_0 \dots a_{h-1} \in T_i \subseteq K$.

C.3 Reducing limit satisfiability to tiling problems

Theorem 55. *The limit satisfiability problem for formulae with number quantifier prefix $(\forall^*\exists^*)^m$ reduces to the restricted monotone m-dimensional tiling problem.*

Proof. First, note that, by Proposition 14, the limit satisfiability problem for formulae with number quantifier prefix $(\forall^*\exists^*)^m$ reduces to the one with number quantifier prefix $(\forall\exists)^m$. Hence, given a quantifier prefix $\forall s_0 \exists r_0 \cdots \forall s_{m-1} \exists r_{m-1}$ and a AMSO_h^0-formula $\varphi(\bar{r}, \bar{s})$, we will construct a restricted monotone m-dimensional tiling system (L, K) that has solutions of arbitrary large height if, and only if, there exists a sequence $(w_i)_{i<\omega}$ of finite words satisfying

$$\forall s_0 \exists r_0 \cdots \forall s_{m-1} \exists r_{m-1} (\exists k < \omega)(\forall i \geq k)[w_i \models \varphi(\bar{t})] \,.$$

Let Σ be the alphabet used in φ. We define a monotone tiling system (L, K) over the alphabet $\Sigma' := \Sigma \times \{<, =, >\}$ with ordering

$$(a, \sigma) < (b, \tau) \quad \text{:iff} \quad a = b, \ \tau = >, \ \text{and } \sigma \in \{<, =\} \,.$$

The column language is

$$K := \bigcup_{a \in \Sigma} (a, <)^*(a, =)(a, >)^* \cup \bigcup_{a \in \Sigma} (a, <)^*(a, >)^* \,.$$

To define the row language L let Θ be the set of all AMSO_h^0-types with free variables $\bar{r}\bar{s}$ and set

$$L_0 := \{\, p_0 \dots p_k \mid p_0, \dots, p_k \in \Theta \text{ with } p_0 \oplus \dots \oplus p_k \models \varphi \,\} \,.$$

Let μ be the function that maps an m-tuple $((a, \sigma_0), \dots, (a, \sigma_{m-1})) \in (\Sigma')^m$ to the AMSO_h-type

$$\mathrm{Th}_h(a, n_0 + 1, \dots, n_{m-1} + 1, \bar{n}) \,,$$

where a is the word consisting of the single letter a with weight $k := 2m$, and $\bar{n}$ are arbitrary numbers such that, for all i,

$$n_i + 1 < n_{i+1} \quad \text{and} \quad \begin{cases} n_i + 1 < k \text{ if } \sigma_i = < \,, \\ \quad n_i = k \text{ if } \sigma_i = = \,, \\ \quad n_i > k \text{ if } \sigma_i = > \,. \end{cases}$$

If there are no such numbers, μ remains undefined. Note that μ is well-defined since, for a word consisting of a single letter with weight k, the theory

$$\mathrm{Th}_h(a, n_0 + 1, \dots, n_{m-1} + 1, \bar{n})$$

only depends on the order type of the numbers $k, n_0, \dots, n_{m-1}$. We set

$$\begin{aligned} L := \mu^{-1}(L_0) = \{\, c_0 \dots c_{k-1} \in ((\Sigma')^m)^* \mid \\ \mu(c_0) \dots \mu(c_{k-1}) \in L_0 \,\} \,. \end{aligned}$$

To show that (L, K) has the desired properties, it is sufficient, by Lemma 50, to prove that (L, K) has solutions of arbitrary large height if, and only if, there exists a sequence $(u_i)_{i<\omega}$ of words such that, for all $n_0 < \dots < n_{m-1} < \omega$,

$$(\exists k < \omega)(\forall i \geq k)[u_i \models \varphi(\bar{n} + 1, \bar{n})] \,.$$

($\Rightarrow$) For each $i < \omega$, fix a solution p_i of height $h_i \geq i$ and length l_i. We define a word u_i as follows. The k-th column of p_i is of the form

$$(a_{i,k}, <)^{w_{i,k}} (a_{i,k}, =) (a_{i,k}, >)^{h_i - w_{i,k} - 1}$$

or

$$(a_{i,k}, <)^{w_{i,k}} (a_{i,k}, >)^{h_i - w_{i,k}} \,,$$

for some $a_{i,k} \in \Sigma$ and $w < \omega$. We define the weighted word $u_i := a_{i,0} \dots a_{i,l_i - 1}$, where the letter $a_{i,k}$ has weight $w_{i,k}$. For $n_0 < \dots < n_{m-1} < h_i$ with $n_j + 1 < n_{j+1}$, it follows that

$$\begin{aligned} &\mathrm{Th}_h(a_{i,k}, n_0 + 1, \dots, n_{m-1} + 1, \bar{n}) \\ &\quad = \mu\big(p(n_0, i) \dots p(n_{m-1}, i)\big) \,. \end{aligned}$$

Hence,

$$\begin{aligned}&\mu\big(p(n_0,0)\dots p(n_{m-1},0)\big)\dots\\&\quad\mu\big(p(n_0,l_i-1)\dots p(n_{m-1},l_i-1)\big)\in L_0\end{aligned}$$

implies

$$\begin{aligned}&\mathrm{Th}_h(u_i,\bar{n}+1,\bar{n})=\\&\quad\mathrm{Th}_h(a_{i,0},\bar{n}+1,\bar{n})\oplus\dots\oplus\mathrm{Th}_h(a_{i,l_i-1},\bar{n}+1,\bar{n})\models\varphi\,.\end{aligned}$$

Consequently,

$$u_i\models\varphi(\bar{n}+1,\bar{n})\,,\quad\text{for all } n_0<\dots<n_{m-1}\text{ with } n_j+1<n_{j+1}\text{ and } i\text{ with } n_{m-1}<h_i\,.$$

($\Leftarrow$) Let $(u_i)_{i<\omega}$ be a sequence of words such that, for all $n_0<\dots<n_{m-1}<\omega$,

$$(\exists k<\omega)(\forall i\geq k)[u_i\models\varphi(\bar{n}+1,\bar{n})]\,.$$

Then there exists an increasing sequence $(h_i)_{i<\omega}$ of numbers such that

$$u_i\models\varphi(\bar{n}+1,\bar{n})\,,\quad\text{for all } n_0<\dots<n_{m-1}<h_i\,.$$

Let l_n be the length of u_n. For each n, we define a function $p_n:[h_n]\times[l_n]\to\Sigma$ as follows. Suppose that a_j is the j-th letter of u_n and that its weight is k_j. We set

$$p_n(i,j):=\begin{cases}(a_j,<)&\text{if } i<k_j\,,\\(a_j,=)&\text{if } i=k_j\,,\\(a_j,>)&\text{if } i>k_j\,.\end{cases}$$

We claim that every p_n is a solution of (L,K). By definition, every column of p_n belongs to K. Let $n_0<\dots<n_{m-1}<h_n$ and set $\bar{n}':=(n_0+1,\dots,n_{m-1}+1)$. For $j<l_n$, let $a_{n,j}$ be the letter such that $u_n=a_{n,0}\oplus\dots\oplus a_{n,l_n-1}$. It follows that

$$\mu(p(n_0,j)\dots p(n_{m-1},j))=\mathrm{Th}_h(a_{n,j},\bar{n}',\bar{n})\,.$$

Consequently,

$$\begin{aligned}&\mathrm{Th}_h(a_{n,0},\bar{n}',\bar{n})\oplus\dots\oplus\mathrm{Th}_h(a_{n,l_n-1},\bar{n}',\bar{n})\\&\quad=\mathrm{Th}_h(u_n,\bar{n}',\bar{n})\models\varphi\end{aligned}$$

implies that

$$\begin{aligned}&\mu(p(n_0,0)\dots p(n_{m-1},0))\dots\\&\quad\mu(p(n_0,l_n-1)\dots p(n_{m-1},l_n-1))\in L_0\,.\end{aligned}$$

Consequently, the rows $n_0,\dots,n_{m-1}$ are in L.

Theorem 56. *The restricted monotone m-dimensional tiling problem reduces to the satisfiability problem for* WAMSO*-sentences with number quantifier prefix* $(\forall^*\exists^*)^m$.

Proof. Let (L, K) be a restricted monotone m-dimensional tiling problem over the alphabet Σ. We construct a formula stating that the model is an ω-word of the form $\#w_0\#w_1\#w_2\#\dots$ where $\# \notin \Sigma$ is a new letter and the words $w_i \in (\Sigma^3)^*$ encode solutions of (L, K) of unbounded height. The following conditions are easily expressed in WAMSO:

- there are infinitely many $\#$,
- the weight function f is unbounded,
- each letter other than $\#$ is of the form $(a, b, c) \in \Sigma^3$ where $a^*bc^* \subseteq K$.

The main task is finding a formula expressing that every m-tuple of lines belongs to L. We use the formula

$$\begin{aligned}\varphi :=& \forall s_0 \exists r_0 \cdots \forall s_{m-1} \exists r_{m-1} \forall x \forall y \\ &\quad [\text{`}x \text{ and } y \text{ are consecutive occurrences of } \#\text{'} \\ &\quad\quad \to \psi(x, y, \bar{r}, \bar{s})]\end{aligned}$$

where ψ is defined as follows. Let χ be an MSO-formula defining the language L where we use letter predicates of the form $P_{\geq a}$ meaning that the letter at the given position is greater or equal to a. As L is upward closed, we can choose χ such that every occurrence of such a predicate $P_{\geq a}$ is positive. For $a \in \Sigma$, set

$$\begin{aligned}\vartheta_a^i(x) := \bigvee_{b,c \in \Sigma} \Big[&(P_{(a,b,c)}(x) \wedge f(x) < r_i) \\ &\vee (P_{(b,a,c)}(x) \wedge s_i \leq f(x) < r_i) \\ &\vee (P_{(b,c,a)}(x) \wedge s_i \leq f(x)) \Big] .\end{aligned}$$

Let χ' be the formula obtained from χ by replacing every predicate $P_{\geq(a_0,\dots,a_{m-1})}$ by

$$\bigwedge_{i<m} \vartheta_{a_i}^i(x) .$$

The formula ψ states that the formula χ' holds between positions x and y.

We claim that φ is satisfiable if, and only if, (K, L) has solutions of arbitrarily large height.

($\Leftarrow$) For each $n < \omega$, fix a solution $p_n : [n] \times [k_n] \to \Sigma$ of (K, L) of height n. Set

$$w_n := a_0^n \dots a_{k_n-1}^n ,$$

where $a_i^n = (a, b, c)$ is choosen such that $p_n(0, i) \dots p_n(n-1, i) \in a^*bc^*$. We define the weight function f of w_n such that $p_n(0, i) \dots p_n(n-1, i) \in a^{f(i)}bc^*$. We claim that the ω-word

$$\#w_0\#w_1\#w_2\#\dots$$

is a model of φ.

Given the values s_i of the universally quantified number variables, we choose the value $r_i := s_i + 1$ for the existentially quantified ones. It is sufficient to prove that a subformula ϑ^i_a holds for a position j in the word w_n if, and only if, $p_n(r_i, j) \geq a$. Hence, suppose that ϑ^i_a holds at j in w_n and let (c_0, c_1, c_2) be the letter at that position. Then one of the following cases holds:

- $a = c_0$ and $p_n(s_i, j) \in \{a, c_1\}$, or
- $a = c_1$ and $p_n(s_i, j) = a$, or
- $a = c_2$ and $p_n(s_i, j) \in \{a, c_1\}$.

Since $c_1 \geq a$ it follows that $p_n(s_i, j) \geq a$.

Conversely, suppose that $p_n(s_i, j) \geq a$. Let (c_0, c_1, c_2) be the letter of w_n at position j. Then

- $c_0 \geq a$ and $f(j) \leq s_i$, or
- $c_1 \geq a$ and $f(j) = s_i$, or
- $c_2 \geq a$ and $f(j) > s_i$.

Since $r_i = s_i + 1$ it follows that ϑ^i_a holds at j.

$(\Rightarrow)$ Suppose that φ is satisfiable. Then the model has the form

$$\# w_0 \# w_1 \# w_2 \# \dots ,$$

for words $w_0, w_1, \dots$. We have to construct solutions p_n of (K, L) of arbitrarily large height.

Fix a skolem function $\beta_i(s_0, \dots, s_i)$ for the variable r_i and set

$$\beta(s) := \max \{\, \beta_i(s_0, \dots, s_i) \mid i < m \text{ and} s_0, \dots, s_i \leq s \,\} .$$

By monotonicity of $\chi'(\bar{r}, \bar{s})$, it follows that

$$w_n \models \chi'\big(\beta(s_0), \dots, \beta(s_{m-1}), s_0, \dots, s_{m-1}\big) ,$$

for all $s_0 < \dots < s_{m-1} < \omega$. Let (a^n_j, b^n_j, c^n_j) be the i-th letter of w_n. We set

$$p_n(i, j) := \begin{cases} a^n_j & \text{if } f(j) < \beta^i(0) , \\ b^n_j & \text{if } \beta^i(0) \leq f(j) < \beta^{i+1} , \\ c^n_j & \text{if } f(j) \geq \beta^{i+1}(0) . \end{cases}$$

As height of p_n we take the maximal value of $f(j)$ where j ranges over the positions in w_n. Since the function f is unbounded, it follows that the p_n have unbounded weight. Hence, it remains to prove that each p_n is a solution of (K, L). Clearly, we have

$$p_n(0, j) \dots p_n(h-1, j) \in K ,$$

for all j, where h is the height of p_n.

Fix numbers $k_0 < \dots < k_{m-1}$ and set $s_i := \beta^{k_i}(0)$. If a formula $\vartheta^i_a(x)$ holds at position j in w_n with letter (c_0, c_1, c_2) then

- $a = c_0$ and $f(j) < \beta(s_i)$, or
- $a = c_1$ and $s_i \leq f(j) < \beta(s_i)$, or
- $a = c_2$ and $s_i \leq f(j)$.

Consequently, $\vartheta^i_a(j)$ implies that $p_n(k_i, j) \geq a$. It follows that

$$(p_n(k_0, 0) \dots p_n(k_{m-1}, 0)) \dots (p_n(k_0, l) \dots p_n(k_{m-1}, l)) \in L\,,$$

as desired.