Monadic Second-Order Logic

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1. Structures and logics

A *signature* Σ is a set of relation symbols and function symbols, each of which has an associated *arity*. A *Σ-structure* $\mathfrak{A} = \langle A, (\xi^{\mathfrak{A}})_{\xi \in \Sigma} \rangle$ consists of a set A together with

- one n-ary relation $R^{\mathfrak{A}} \subseteq A^n$, for every relation symbol $R \in \Sigma$ of arity n and
- one n-ary function $f^{\mathfrak{A}} : A^n \to A$, for every function symbol $f \in \Sigma$ of arity n.

Note that we allow functions of arity 0, which correspond to *constants*.

Example. (a) The field of real numbers $\langle \mathbb{R}, +, \cdot, 0, 1, \leq \rangle$ is a structure with signature $\{+, \cdot, 0, 1, \leq\}$, where $+$ and $\cdot$ are binary function symbols, 0 and 1 are 0-ary function symbols, and $\leq$ is a binary relation symbol.

(b) A graph is a structure $\langle V, E \rangle$ with a single binary relation $E \subseteq V \times V$.

Recall that *first-order logic* $\mathrm{FO}[\Sigma]$ consist of formulae that are built up from *atomic formulae* of the form $s = t$ and $Rt_0 \dots t_{n-1}$, where $R \in \Sigma$ is an n-ary relation symbol and $s, t, t_0, \dots, t_{n-1}$ are terms built up from variables and the function symbols in Σ. Such atomic formulae can be combined using boolean operations $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), and first-order quantifiers $\exists x$ and $\forall x$.

Definition 1.1. Let Σ be a signature. The formulae of *monadic second-order logic* $\mathrm{MSO}[\Sigma]$ are built up from atomic formulae of the form $s =$

t, Zt, and $Rt_0 \dots t_{n-1}$, where $R \in \Sigma$ is an n-ary relation symbol, Z is a *set variable,* and $s, t, t_0, \dots, t_{n-1}$ are terms built up from first-order variables and the function symbols in Σ. Such atomic formulae can be combined using boolean operations $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), and quantifiers $\exists x$, $\forall x$, $\exists Z$, and $\forall Z$, where x is a first-order variable and Z is a set variable.

The semantics of such a formula is defined as follows. Given a formula $\varphi(\bar{x}, \bar{Z}) \in \mathrm{MSO}[\Sigma]$ with free first-order variables $\bar{x}$ and free set variables $\bar{Z}$ and given a Σ-structure $\mathfrak{A}$, a tuple of elements $\bar{a}$ of $\mathfrak{A}$, and a tuple of subsets $\bar{P}$ of $\mathfrak{A}$, we define the *satisfaction relation*

$$\mathfrak{A} \models \varphi(\bar{a}, \bar{P})$$

by induction on φ. The definition is analogous to that for first-order logic. An atomic formula Zt holds in $\mathfrak{A}$ if the element denoted by the term t belongs to the set denoted by Z. A formula of the form $\exists Z\psi$ holds if there exists a set satisfying ψ, and $\forall Z\psi$ holds if every set satisfies ψ.

Definition 1.2. Let Σ be a signature.

(a) Let $\mathfrak{A}$ be a Σ-structure. A tuple $\bar{a} \in A^n$ is *guarded* if there exists a relation R of $\mathfrak{A}$ containing a tuple $\bar{c} \in R$ with $\bar{a} \subseteq \bar{c}$. Here, we allow R to be the equality relation $=$, even though it is not present in the signature. A relation $S \subseteq A^n$ is *guarded* if every tuple $\bar{a} \in S$ is guarded.

(b) *Guarded second-order logic* $\mathrm{GSO}[\Sigma]$ extends first-order logic by atomic formulae of the form $Zt_0 \dots t_{n-1}$, where $t_0, \dots, t_{n-1}$ are terms and Z is a relation variable, and by quantifiers $\exists Z$ and $\forall Z$ over relation variables. A formula of the form $\exists Z\psi$ holds if there exists a guarded relation satisfying ψ, and $\forall Z\psi$ holds if every guarded relation satisfies ψ.

Example. (a) For a linear order $\mathfrak{A} = \langle A, \leq, \rangle$, we can say that y is the immediate successor of x by the FO-formula

$$\varphi(x, y) := x \leq y \wedge x \neq y \wedge \forall z[x \leq z \wedge z \leq y \rightarrow (z = x \vee z = y)] .$$

(b) For a tree $\mathfrak{T} = \langle T, \leq \rangle$ where $\leq$ is the predecessor order, we can express that a set variable X contains an infinite branch by the MSO-formula

$$\exists Z\big[Z \subseteq X \wedge Z \neq \varnothing \wedge \forall x \forall y[Zx \wedge Zy \rightarrow (x \leq y \vee y \leq x)] \\ \wedge \forall x \exists y[Zx \rightarrow x < y \wedge Zy]\big].$$

Example. We consider graphs as structures over the signature $\{E\}$ consisting of one binary edge relation.

(a) The MSO-formula

$$\varphi(x, y) := \forall Z[Zx \wedge \forall u \forall v(Zu \wedge Euv \rightarrow Zv) \rightarrow Zy]$$

expresses that there exists a path from x to y.

(b) We can say that the graph is connected by the formula

$$\forall x \forall y \varphi(x, y),$$

where φ is the formula from (a).

(c) To express that a graph has a Hamiltonian cycle we can write down a GSO-formula stating that there is a guarded binary relation Z (i.e., a set of edges) such that

- for every vertex x there are unique vertices y and z with $(y, x) \in Z$ and $(x, z) \in Z$,
- every two vertices are connected by a sequence of Z-edges.

(d) A *minor* of a graph $\mathfrak{G}$ is a graph $\mathfrak{H}$ obtained from the first graph by deleting vertices and edges and by contracting edges. To say that a fixed finite graph $\mathfrak{H}$ is a minor of the given graph, we can use an MSO-formula stating that, for each vertex v of $\mathfrak{H}$, there exists a set X_v such that

- the subgraph induced by X_v is connected and
- for every edge (u, v) of $\mathfrak{H}$ there is an edge connecting some vertex of X_u with some vertex of X_v.

As defined above the logic MSO is not always convenient to use in proofs. Therefore, we introduce a simplified version that still has the same expressive power.

Definition 1.3. Let Σ be a relational signature. The logic $\mathrm{MSO}_0[\Sigma]$ has atomic formulae of the form

$$\begin{aligned}&X \subseteq Y\,,\\ &X \cap Y = \varnothing\,,\\ &\mathrm{cover}(X_0, \dots, X_{n-1})\,,\\ &RX_0 \dots X_{n-1}\,,\end{aligned}$$

where $R \in \Sigma$ is an n-ary relation symbol and $X, Y, X_0, \dots, X_{n-1}$ are set variables. The logic is closed under boolean operations and set quantifiers. An atomic formula of the form $\mathrm{cover}(X_0, \dots, X_{n-1})$ holds if the union $X_0 \cup \dots \cup X_{n-1}$ contains the whole universe, while a formula of the form $RX_0 \dots X_{n-1}$ holds if we can choose from each set X_i some element a_i such that the tuple $\langle a_0, \dots, a_{n-1}\rangle$ belongs to R.

Remark. (a) We frequently use abbreviations like:

$$\begin{aligned}&(X = Y) := (X \subseteq Y) \wedge (Y \subseteq X)\,,\\ &(X \subset Y) := (X \subseteq Y) \wedge \neg(Y \subseteq X)\,,\\ &(X = \varnothing) := (X \cap X = \varnothing)\,.\end{aligned}$$

The formula

$$\mathrm{sing}(X) := X \neq \varnothing \wedge \forall Y[Y \subset X \to Y = \varnothing]$$

states that the set X is a singleton.

(b) Note that every MSO_0-formula is equivalent to one that does not contain atomic formulae of the form $X \cap X' = \varnothing$ or $\mathrm{cover}(\bar{X})$ since we can replace the former by

$$\forall Y[Y \subseteq X \wedge Y \subseteq X' \to \forall Z(Y \subseteq Z)]\,,$$

and we can replace the latter by

$$\forall Z\Big[\mathrm{sing}(Z) \to \bigvee_i Z \subseteq X_i\Big]\,.$$

Lemma 1.4. *Let Σ be a relational signature. For every* MSO$[\Sigma]$*-formula* $\varphi(\bar{x}, \bar{Z})$*, there is an* MSO$_0[\Sigma]$*-formula* $\varphi'(\bar{X}, \bar{Z})$ *such that*

$$\mathfrak{A} \vDash \varphi'(\{a_0\}, \dots, \{a_{m-1}\}, \bar{P}) \quad \text{iff} \quad \mathfrak{A} \vDash \varphi(a_0, \dots, a_{m-1}, \bar{P}) .$$

Proof. We define φ' by induction as follows.

$$\begin{aligned}
(x = y)' &:= (X \subseteq Y) \wedge (Y \subseteq X) , \\
Rx_0 \dots x_{n-1} &:= RX_0 \dots X_{n-1} , \\
(\varphi \wedge \psi)' &:= \varphi' \wedge \psi' , & (\exists x \psi)' &:= \exists X[\operatorname{sing}(X) \wedge \psi'] , \\
(\varphi \vee \psi)' &:= \varphi' \vee \psi' , & (\forall x \psi)' &:= \forall X[\operatorname{sing}(X) \rightarrow \psi'] , \\
(\neg \varphi)' &:= \neg \varphi' , & (\exists Z \psi)' &:= \exists Z \psi' , \\
& & (\forall Z \psi)' &:= \forall Z \psi' .
\end{aligned}$$

□

Exercise 1.1. We consider coloured linear orders of the form $\langle A, \leq, P \rangle$ where $P \subseteq A$ is a unary predicate. Find MSO-formulae expressing the following statements:

(a) The set P is dense, i.e., it is nonempty and between any two elements of the order there is an element of P.

(b) The set P contains infinitely many elements.

(c) The set P is finite and it has an even number of elements.

Exercise 1.2. An $(m \times n)$-*grid* is a graph $\mathfrak{G} = \langle V, E \rangle$ where

$$\begin{aligned}
V &:= [m] \times [n] , \\
E &:= \{ \langle \langle i, k \rangle, \langle j, l \rangle \rangle \mid |i - j| + |k - l| = 1 \} .
\end{aligned}$$

(a) Construct an MSO-formula expressing that a graph is a grid.

(b) For each of the following functions $f : \omega \rightarrow \omega$, find an MSO-formula stating that the given graph is an $(n \times f(n))$-grid, for some n.

(i) $f(n) = n$, (ii) $f(n) = n^2$, (iii) $f(n) = 2^n$.

Exercise 1.3. We can encode a finite word $w = a_0 \dots a_{n-1} \in \Sigma^*$ over the alphabet Σ by a *word structure*

$$\hat{w} := \left\langle [n], \leq, (P_a)_{a \in \Sigma} \right\rangle,$$

where the universe $[n] = \{0, \dots, n-1\}$ is the set of positions in the word w and the predicates

$$P_a := \{\, i < n \mid a_i = a \,\}$$

contain all positions carrying the corresponding letter. Prove that, for every regular expression α, there exists an MSO-formula φ such that

$$\hat{w} \vDash \varphi \quad \text{iff} \quad w \in L(\alpha)\,.$$

Hint. First construct, for each regular expression α, an MSO-formula $\varphi(x, y)$ such that

$$\hat{w} \vDash \varphi(x, y) \quad \text{iff} \quad w[x, y] \in L(\alpha)\,,$$

where $w[x, y]$ denotes the factor of w between positions x and y.

2. *Simple translations between logics*

In this section we relate the various logics introduced above to each other. We start with MSO and FO.

Definition 2.1. Let Σ be a relational signature. The *power-set structure* of a Σ-structure $\mathfrak{A}$ is the structure $\wp(\mathfrak{A})$ with signature $\Sigma \cup \{\subseteq\}$ whose universe is the power set $\wp(A)$ of the universe of $\mathfrak{A}$. The relation symbol $\subseteq$ denotes the usual subset relation on $\wp(A)$. For each n-ary relation symbol $R \in \Sigma$, $\wp(\mathfrak{A})$ has the relation

$$R^{\wp(A)} := \{\, \bar{P} \in \wp(A)^n \mid \text{there are } a_i \in P_i \text{ such that } \bar{a} \in R^{\mathfrak{A}} \,\}\,.$$

It is straightforward to check that MSO over Σ-structures corresponds to FO over their power-set structures.

Lemma 2.2. *Let Σ be a relational signature.*

(a) *For every* MSO$[\Sigma]$*-formula $\varphi(\bar{X})$, there exists an* FO$[\Sigma \cup \{\subseteq\}]$*-formula $\varphi'(\bar{x})$ such that*

$$\mathfrak{A} \vDash \varphi(\bar{P}) \quad \text{iff} \quad \wp(\mathfrak{A}) \vDash \varphi'(\bar{P}),$$

for all Σ-structures $\mathfrak{A}$ and all sets $\bar{P}$ in $\mathfrak{A}$.

(b) *For every* FO$[\Sigma \cup \{\subseteq\}]$*-formula $\varphi(\bar{x})$, there exists an* MSO$[\Sigma]$*-formula $\varphi'(\bar{X})$ such that*

$$\wp(\mathfrak{A}) \vDash \varphi(\bar{P}) \quad \text{iff} \quad \mathfrak{A} \vDash \varphi'(\bar{P}),$$

for all Σ-structures $\mathfrak{A}$ and all sets $\bar{P}$ in $\mathfrak{A}$.

Proof. (a) By Lemma 1.4 and the remark after Definition 1.3, we may assume that φ is an MSO_0-formula without subformulae of the form $X \cap Y = \varnothing$ and $\mathrm{cover}(\bar{X})$. Then we obtain the desired formula φ' from φ by replacing every set variable X by a corresponding first-order variable x.

(b) It is sufficient to construct an MSO_0-formula. We obtain it from φ by replacing every first-order variable x by a corresponding set variable X. ☐

We can also relate MSO to GSO via a suitable operation.

Definition 2.3. Let Σ be a relational signature. The *incidence structure* of a Σ-structure $\mathfrak{A}$ is the structure $\mathcal{I}(\mathfrak{A})$ whose universe consists of the union

$$I(A) := A \cup \bigcup_{R \in \Sigma} R^{\mathfrak{A}}.$$

The signature is $\mathcal{I}(\Sigma) := \Sigma \cup \{\mathrm{sing}\} \cup \{\, \mathrm{in}_n \mid n < \omega \,\}$ where we now consider each relation symbol in Σ to be unary. We have a unary relation

$$\mathrm{sing} := A$$

marking the elements of the original structure, binary relations

$$\mathrm{in}_n^{\mathcal{I}(\mathfrak{A})} := \{\, \langle a, \bar{c} \rangle \in A \times I(A) \mid c_n = a \,\}$$

allowing access to the n-th component of a tuple, and, for each symbol $R \in \Sigma$, a unary relation

$$R^{\mathcal{I}(\mathfrak{A})} := \{\, \bar{c} \in I(A) \mid \bar{c} \in R^{\mathfrak{A}} \,\}$$

selecting all elements of the universe corresponding to tuples in $R^{\mathfrak{A}}$.

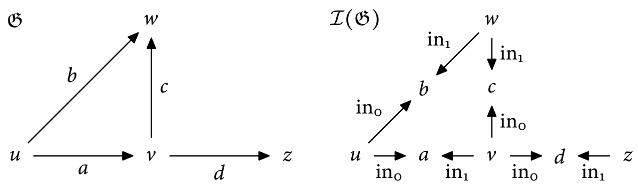
Example. Let $\mathfrak{G} = \langle V, E \rangle$ be a graph. Then

$$\mathcal{I}(\mathfrak{G}) = \langle I(V), E, \mathrm{sing}, \mathrm{in}_0, \mathrm{in}_1 \rangle$$

where

$$\begin{aligned} I(V) &:= V \cup E\,, \\ \mathrm{sing} &:= V\,, \\ \mathrm{in}_0 &:= \{\, \langle u, \langle u, v \rangle \rangle \mid \langle u, v \rangle \in E \,\}\,, \\ \mathrm{in}_1 &:= \{\, \langle v, \langle u, v \rangle \rangle \mid \langle u, v \rangle \in E \,\}\,, \end{aligned}$$

and $E \subseteq I(V)$ is now considered to be a unary relation.



Let us check that GSO over Σ-structures corresponds to MSO over their incidence structures.

Lemma 2.4. *Let Σ be a finite relational signature.*

(a) *For every* GSO$[\Sigma]$*-formula* φ*, there is an* MSO$[\mathcal{I}(\Sigma)]$*-formula* φ' *such that*

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad \mathcal{I}(\mathfrak{A}) \vDash \varphi' , \quad \textit{for all } \Sigma\textit{-structures } \mathfrak{A} .$$

(b) *For every* MSO$[\mathcal{I}(\Sigma)]$*-formula* φ*, there is a* GSO$[\Sigma]$*-formula* φ' *such that*

$$\mathcal{I}(\mathfrak{A}) \vDash \varphi \quad \text{iff} \quad \mathfrak{A} \vDash \varphi' , \quad \textit{for all } \Sigma\textit{-structures } \mathfrak{A} .$$

Proof. (a) For every n-ary relation variable Z of φ, the formula φ' will use a tuple $\bar{Z} = (Z_\sigma)_\sigma$ of variables that is indexed by functions $\sigma : [n] \to [m]$, where m is smaller or equal to the maximal arity of a relation in Σ. We define φ' by induction on φ as follows.

$$\begin{aligned}
(x = y)' &:= x = y , \\
(Rx_0 \dots x_{n-1})' &:= \exists z \Big[Rz \wedge \bigwedge_{k<n} \mathrm{in}_k(x_k, z) \Big] , \\
(Zx_0 \dots x_{n-1})' &:= \exists z \bigvee_{\sigma} \Big[Z_\sigma z \wedge \bigwedge_{k<n} \mathrm{in}_{\sigma(k)}(x_k, z) \Big] ,
\end{aligned}$$

$$\begin{aligned}
(\varphi \wedge \psi)' &:= \varphi' \wedge \psi' , & (\exists x \psi)' &:= \exists x [\mathrm{sing}(x) \wedge \psi'] , \\
(\varphi \vee \psi)' &:= \varphi' \vee \psi' , & (\forall x \psi)' &:= \forall x [\mathrm{sing}(x) \to \psi'] , \\
(\neg \varphi)' &:= \neg \varphi' , & (\exists Z \psi)' &:= \exists \bar{Z} \psi' , \\
& & (\forall Z \psi)' &:= \forall \bar{Z} \psi' .
\end{aligned}$$

(b) We may assume that φ is an MSO$_0$-formula without subformulae of the form $X \cap Y = \varnothing$ or $\mathrm{cover}(\bar{X})$. For every variable X of φ, the formula φ' will use a tuple $\bar{X} = \langle X_1, \dots, X_m \rangle$ where X_n is an n-ary relation variable and m is the maximal arity of a relation symbol in Σ. We define φ' by induction on φ as follows.

$$\begin{aligned}
(X \subseteq Y)' &:= \bigwedge_{n=1}^{m} X_n \subseteq Y_n\,, \\
(RX)' &:= \exists\bar{x}[X_n\bar{x} \wedge R\bar{x}]\,, \quad \text{where } n \text{ is the arity of } R\,, \\
(\operatorname{sing}(X))' &:= \exists x X_1 x\,, \\
(\operatorname{in}_k(X,Y))' &:= \bigvee_{n=k}^{m} \exists\bar{x}[Y_n\bar{x} \wedge X_1 x_k]\,, \\
(\varphi \wedge \psi)' &:= \varphi' \wedge \psi'\,, \qquad (\exists X\psi)' := \exists\bar{X}\psi'\,, \\
(\varphi \vee \psi)' &:= \varphi' \vee \psi'\,, \qquad (\forall X\psi)' := \forall\bar{X}\psi'\,. \\
(\neg\varphi)' &:= \neg\varphi'\,,
\end{aligned}$$

□

3. Theories and back-and-forth arguments

Definition 3.1. (a) The *quantifier rank* $\operatorname{qr}(\varphi)$ of a formula φ is the number of nested (first-order and second-order) quantifiers in φ.

(b) Let $\mathfrak{A}$ be a structure. A *parameter* in $\mathfrak{A}$ is a value α that can be assigned to a variable. There are three kinds of parameters:

- *first-order parameters* are elements $\alpha \in A$;
- *monadic parameters* are sets $\alpha \subseteq A$; and
- *guarded parameters* are guarded relations $\alpha \subseteq A^n$.

We say that α is a

- *second-order parameter,* if it is a monadic parameter or a guarded parameter;
- FO-*parameter,* if it is a first-order parameter;
- MSO_0-*parameter,* if it is a monadic parameter;
- MSO-*parameter,* if it is a first-order parameter or a monadic parameter;

- GSO-*parameter,* if it is a first-order parameter, a monadic parameter, or a guarded parameter.

Definition 3.2. Let L be one of the logics defined above.

(a) Let $\mathfrak{A}$ be a Σ-structure and $\bar{\alpha}$ L-parameters in $\mathfrak{A}$. The L-*theory* of $\mathfrak{A}, \bar{\alpha}$ is the set

$$\mathrm{Th}_L(\mathfrak{A}, \bar{\alpha}) := \{ \varphi(\bar{x}) \in L \mid \mathfrak{A} \vDash \varphi(\bar{\alpha}) \}$$

of all L-formulae satisfied by $\mathfrak{A}$. If we only consider formulae of quantifier rank at most m, we obtain the set

$$\mathrm{Th}_L^m(\mathfrak{A}, \bar{\alpha}) := \{ \varphi(\bar{x}) \in L \mid \mathrm{qr}(\varphi) \leq m,\ \mathfrak{A} \vDash \varphi(\bar{\alpha}) \}.$$

(b) Two Σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ with L-parameters $\bar{\alpha}$ and $\bar{\beta}$, respectively, are L-*equivalent* if they have the same L-theory. We denote this fact by

$$\mathfrak{A}, \bar{\alpha} \equiv_L \mathfrak{B}, \bar{\beta}.$$

Similarly, we define for $m < \omega$

$$\mathfrak{A}, \bar{\alpha} \equiv_L^m \mathfrak{B}, \bar{\beta} \quad :\text{iff} \quad \mathrm{Th}_L^m(\mathfrak{A}, \bar{\alpha}) = \mathrm{Th}_L^m(\mathfrak{B}, \bar{\beta}).$$

If the logic L is understood, we will speak of m-*equivalence* in this case. If we want to indicate the logic in question, we will use the terms *first-order m-equivalence, monadic m-equivalence,* or *guarded m-equivalence* instead.

Proposition 3.3. *Let L be one of the logics defined above, Σ a finite relational signature, $k, m < \omega$, and let $L_m^k[\Sigma]$ be the set of all $L[\Sigma]$-formulae of quantifier-rank at most m with at most k free variables..*

(a) *Up to logical equivalence, there are only finitely many $L_m^k[\Sigma]$-formulae. Furthermore, given $m, k < \omega$, we can compute a finite set Φ_m^k of $L_m^k[\Sigma]$-formulae such that every $L_m^k[\Sigma]$-formula is equivalent to one in Φ_m^k.*

(b) *There are only finitely many $L[\Sigma]$-theories of quantifier rank m with k parameters.*

(c) *Every $L[\Sigma]$-theory of quantifier rank m and k parameters is equivalent to a single $L^k_m[\Sigma]$-formula.*

Proof. (a) We prove the claim by induction on m. First, consider the case where $m = 0$. Every quantifier-free L-formula can be written in disjunctive normal form. Since the signature and the number of variables is finite, there are only finitely many atomic formulae and only finitely many negated atomic formulae. (For $L = \mathrm{MSO}_0$, we may assume w.l.o.g. that all variables in an atom $\mathrm{cover}(\bar{X})$ are distinct.) Since, up to logical equivalence, a conjunction of such formulae is uniquely determined by the set of formulae appearing in it, it follows that there are only finitely many such conjunctions. In the same way, we see that, up to logical equivalence, there are only finitely many disjunctions of such conjunctions. Hence, there are only finitely many quantifier-free formulae in disjunctive normal form.

For the inductive step, suppose that $m > 0$. As above, every L-formula of quantifier-rank at most m can be written as a boolean combination of (i) atomic formulae and (ii) formulae of the form $\exists x\psi$ or $\exists X\psi$ with $\mathrm{qr}(\psi) < m$. By induction hypothesis, there are only finitely many formulae of these two forms. Writing the boolean combination of them in disjunctive normal form, we can use the same argument as above to show that there are only finitely many such combinations.

For the desired set Φ^k_m of representatives, we can take the set of all formulae that can be written as disjunctive normal forms (without repetitions) of formulae of the form $\exists x\psi$ and $\exists X\psi$ with $\psi \in \Phi^{k+1}_{m-1}$.

(b) By (a), we can fix a finite set Φ^k_m of $L[\Sigma]$-formulae such that every $L^k_m[\Sigma]$-formula is equivalent to one in Φ^k_m. Then every $L^k_m[\Sigma]$-theory T is uniquely determined by the intersection $T \cap \Phi^k_m$. Since there are only finitely many sets of the form $T \cap \Phi^k_m$, the number of theories is finite.

(c) By (a), we can compute a finite set Φ^k_m of $L^k_m[\Sigma]$-formulae such that every $L^k_m[\Sigma]$-formula is equivalent to one in Φ^k_m. For every $L^k_m[\Sigma]$-

theory T, the formula

$$\chi_T := \bigwedge (T \cap \Phi^k_m)$$

is equivalent to T. $\square$

Proposition 3.4. *Let L be one of the logics defined above, Σ a finite signature, $\mathfrak{A}$ and $\mathfrak{B}$ Σ-structures with L-parameters $\bar{\alpha}$ and $\bar{\beta}$, and $m < \omega$. Then*

$$\mathfrak{A}, \bar{\alpha} \equiv^{m+1}_L \mathfrak{B}, \bar{\beta}$$

if, and only if, the following two properties are satisfied:

(Forth Property) For every L-parameter α' in $\mathfrak{A}$, there exists an L-parameter β' in $\mathfrak{B}$ such that

$$\mathfrak{A}, \bar{\alpha}\alpha' \equiv^m_L \mathfrak{B}, \bar{\beta}\beta' .$$

(Back Property) For every L-parameter β' in $\mathfrak{B}$, there exists an L-parameter α' in $\mathfrak{A}$ such that

$$\mathfrak{A}, \bar{\alpha}\alpha' \equiv^m_L \mathfrak{B}, \bar{\beta}\beta' .$$

Proof. ($\Leftarrow$) Suppose that both properties are satisfied. We have to show that

$$\mathfrak{A} \vDash \varphi(\bar{\alpha}) \quad \text{iff} \quad \mathfrak{B} \vDash \varphi(\bar{\beta}) ,$$

for all L-formulae $\varphi(\bar{x})$ of quantifier rank at most $m + 1$. Every such formula is a boolean combination of formulae of the form $\exists x' \psi(\bar{x}, x')$ where $\mathrm{qr}(\psi) \leq m$ and the variable x' is either first-order, monadic, or guarded. Therefore, it is sufficient to prove the claim for such formulae. By symmetry, it is therefore enough to prove that

$$\mathfrak{A} \vDash \exists x' \psi(\bar{\alpha}, x') \quad \text{implies} \quad \mathfrak{B} \vDash \exists x' \psi(\bar{\beta}, x') ,$$

for all $\psi(\bar{x}, x')$ with $\mathrm{qr}(\psi) \leq m$. Hence, suppose that $\mathfrak{A} \vDash \exists x' \psi(\bar{\alpha}, x')$. Then there exists an L-parameter α' in $\mathfrak{A}$ such that $\mathfrak{A} \vDash \psi(\bar{\alpha}, \alpha')$. By assumption, we can find an L-parameter β' in $\mathfrak{B}$ with

$$\mathfrak{A}, \bar{\alpha}\alpha' \equiv_L^m \mathfrak{B}, \bar{\beta}\beta' .$$

Consequently, $\mathfrak{B} \vDash \psi(\bar{\beta}, \beta')$, which implies that $\mathfrak{B} \vDash \exists x' \psi(\bar{\beta}, x')$.

($\Rightarrow$) By Proposition 3.3 (a), there exists a finite set Φ of L-formulae of quantifier rank at most m such that every formula of quantifier rank at most m is equivalent to some formula in Φ.

Suppose that there exists an L-parameter α' in $\mathfrak{A}$ such that

$$\mathfrak{A}, \bar{\alpha}\alpha' \not\equiv_L^m \mathfrak{B}, \bar{\beta}\beta' , \quad \text{for all } \beta' \text{ in } B .$$

We have to prove that $\mathfrak{A}, \bar{\alpha} \not\equiv_L^{m+1} \mathfrak{B}, \bar{\beta}$. Set

$$\Theta := \{ \psi(\bar{x}, x') \in \Phi \mid \mathfrak{A} \vDash \psi(\bar{\alpha}, \alpha') \} ,$$

and let $\vartheta := \bigwedge \Theta$ be the conjunction of all formulae in Θ. It is sufficient to show that

$$\mathfrak{A} \vDash \exists x' \vartheta(\bar{\alpha}, x') \quad \text{and} \quad \mathfrak{B} \nvDash \exists x' \vartheta(\bar{\beta}, x') .$$

Since $\mathfrak{A} \vDash \vartheta(\bar{\alpha}, \alpha')$, we have $\mathfrak{A} \vDash \exists x' \vartheta(\bar{\alpha}, x')$. Furthermore, for every β' in B, we have

$$\mathfrak{A}, \bar{\alpha}\alpha' \not\equiv_L^m \mathfrak{B}, \bar{\beta}\beta' .$$

Hence, there exists a formula $\eta(\bar{x}, x')$ of quantifier rank at most m such that

$$\mathfrak{A} \vDash \eta(\bar{\alpha}, \alpha') \quad \text{and} \quad \mathfrak{B} \nvDash \eta(\bar{\beta}, \beta') .$$

By choice of the set Φ, we may choose $\eta \in \Phi$. Hence, $\eta \in \Theta$ and

$$\mathfrak{B} \nvDash \eta(\bar{\beta}, \beta') \quad \text{implies} \quad \mathfrak{B} \nvDash \vartheta(\bar{\beta}, \beta') .$$

We have shown that

$$\mathfrak{B} \not\models \vartheta(\bar{\beta}, \beta'), \quad \text{for every } L\text{-parameter } \beta' \text{ in } \mathfrak{B}.$$

Hence, $\mathfrak{B} \not\models \exists x' \vartheta(\bar{\beta}, x')$. □

Exercise 3.1. Let $m < \omega$ and suppose that $\bar{a}, \bar{b} \in \mathbb{Q}^n$ are tuples such that

$$a_i \leq a_j \quad \iff \quad b_i \leq b_j, \quad \text{for all } i, j < n.$$

Prove that $\langle \mathbb{Q}, \leq \rangle, \bar{a} \equiv^m_{\mathrm{FO}} \langle \mathbb{Q}, \leq \rangle, \bar{b}$.

Exercise 3.2. Find MSO-formulae distinguishing the following structures:

$$\mathfrak{N} := \langle \mathbb{N}, \leq \rangle, \quad \mathfrak{Z} := \langle \mathbb{Z}, \leq \rangle, \quad \mathfrak{Q} := \langle \mathbb{Q}, \leq \rangle, \quad \mathfrak{R} := \langle \mathbb{R}, \leq \rangle.$$

Which of them are FO-equivalent?

4. *Operations on structures*

Disjoint unions

Definition 4.1. Let Σ be a relational signature. The *disjoint union* of two Σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is the $(\Sigma \cup \{\mathrm{Left}, \mathrm{Right}\})$-structure $\mathfrak{A} \oplus \mathfrak{B}$ with universe $A \cup B$ and relations

$$\begin{aligned} R^{\mathfrak{A} \oplus \mathfrak{B}} &:= R^{\mathfrak{A}} \cup R^{\mathfrak{B}}, \quad \text{for } R \in \Sigma, \\ \mathrm{Left}^{\mathfrak{A} \oplus \mathfrak{B}} &:= A, \\ \mathrm{Right}^{\mathfrak{A} \oplus \mathfrak{B}} &:= B. \end{aligned}$$

Proposition 4.2. *Let L be one of the logics defined above, let Σ be a finite relational signature, $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{B}$ and $\mathfrak{B}'$ Σ-structures with first-order parameters $\bar{a}, \bar{a}', \bar{b}, \bar{b}'$ and second-order parameters $\bar{P}, \bar{P}', \bar{Q}, \bar{Q}'$, respectively, and let $m < \omega$. Then*

$$\mathfrak{A}, \bar{P}, \bar{a} \equiv^m_L \mathfrak{A}', \bar{P}', \bar{a}' \quad \text{and} \quad \mathfrak{B}, \bar{Q}, \bar{b} \equiv^m_L \mathfrak{B}', \bar{Q}', \bar{b}'$$

implies

$$\mathfrak{A} \oplus \mathfrak{B}, \bar{P} \cup \bar{Q}, \bar{a}\bar{b} \equiv_L^m \mathfrak{A}' \oplus \mathfrak{B}', \bar{P}' \cup \bar{Q}', \bar{a}'\bar{b}' .$$

(We write $\bar{P} \cup \bar{Q}$ for the tuple whose i-th component is $P_i \cup Q_i$. We assume that the parameters are appropriate for the logic L, i.e., if $L = \mathrm{MSO}_0$ there are no first-order parameters and if L is FO, there are only first-order parameters.)

Proof. We prove the claim by induction on m. First, consider the case where $m = 0$. Since quantifier-free formulae are boolean combinations of atomic formulae, it is sufficient to consider such formulae. By symmetry, we therefore only need to show that

$$\mathfrak{A} \oplus \mathfrak{B} \vDash \varphi(\bar{P} \cup \bar{Q}, \bar{a}, \bar{b}) \quad \text{implies} \quad \mathfrak{A}' \oplus \mathfrak{B}' \vDash \varphi(\bar{P}' \cup \bar{Q}', \bar{a}', \bar{b}') ,$$

for every atomic formula $\varphi(\bar{X}, \bar{x}, \bar{y})$.

First, suppose that L is one of FO, MSO, or GSO. We distinguish several cases. If φ is an equality $z = z'$ and $\mathfrak{A} \oplus \mathfrak{B} \vDash \varphi(\bar{P} \cup \bar{Q}, \bar{a}, \bar{b})$, then the variables z and z' are either both among the $\bar{x}$ or among the $\bar{y}$. By symmetry, we may assume the former, i.e., $\varphi = (x_i = x_j)$. Then $a_i = a_j$ and

$$\mathfrak{A}, \bar{P}, \bar{a} \equiv_L^0 \mathfrak{A}', \bar{P}', \bar{a}'$$

implies that $a_i' = a_j'$. Hence, $\mathfrak{A}' \oplus \mathfrak{B}' \vDash \varphi(\bar{P}' \cup \bar{Q}', \bar{a}', \bar{b}')$.

If $\varphi = R\bar{z}$ for $R \in \Sigma$, then again $\bar{z}$ must be a subtuple of $\bar{x}$ or of $\bar{y}$. Say it is the former. Then $\mathfrak{A}, \bar{P}, \bar{a} \vDash R\bar{z}$ and

$$\mathfrak{A}, \bar{P}, \bar{a} \equiv_L^0 \mathfrak{A}', \bar{P}', \bar{a}'$$

implies that $\mathfrak{A}', \bar{P}', \bar{a}' \vDash R\bar{z}$. Hence, $\mathfrak{A}' \oplus \mathfrak{B}' \vDash \varphi(\bar{P}' \cup \bar{Q}', \bar{a}', \bar{b}')$.

For $\varphi = \mathrm{Left}(x)$ or $\varphi = \mathrm{Right}(x)$, the proof is similar.

It remains to consider the case where $L = \mathrm{MSO}_0$. Again, we distinguish several cases.

If $\varphi = X_i \subseteq X_j$, then

$$\begin{aligned}
& \mathfrak{A} \oplus \mathfrak{B}, \bar{P} \cup \bar{Q} \vDash X_i \subseteq Y_j \\
\Rightarrow \quad & P_i \cup Q_i \subseteq P_j \cup Q_j \\
\Rightarrow \quad & P_i \subseteq P_j \text{ and } Q_i \subseteq Q_j \\
\Rightarrow \quad & P'_i \subseteq P'_j \text{ and } Q'_i \subseteq Q'_j \\
\Rightarrow \quad & P'_i \cup Q'_i \subseteq P'_j \cup Q'_j \\
\Rightarrow \quad & \mathfrak{A}' \oplus \mathfrak{B}', \bar{P}' \cup \bar{Q}' \vDash X_i \subseteq Y_j .
\end{aligned}$$

The proofs for $\varphi = X \cap Y = \varnothing$ and $\operatorname{cover}(\bar{X})$ are analogous.

Suppose that $\varphi = RX_0 \dots X_{n-1}$, i.e., that

$$\mathfrak{A} \oplus \mathfrak{B}, \bar{P} \cup \bar{Q} \vDash R\bar{X} .$$

Then there are elements $a_i \in P_i \cup Q_i$ such that $\bar{a} \in R^{\mathfrak{A} \oplus \mathfrak{B}}$. Since $R^{\mathfrak{A} \oplus \mathfrak{B}} = R^{\mathfrak{A}} \cup R^{\mathfrak{B}}$, it follows that $\bar{a} \in R^{\mathfrak{A}}$ or $\bar{a} \in R^{\mathfrak{B}}$. By symmetry, we may assume the former. Then $\mathfrak{A}, \bar{P} \vDash R\bar{X}$, which implies that $\mathfrak{A}', \bar{P}' \vDash R\bar{X}$. It follows that

$$\mathfrak{A}' \oplus \mathfrak{B}', \bar{P}' \cup \bar{Q}' \vDash R\bar{X} .$$

For $\varphi = \operatorname{Left}(X)$ or $\varphi = \operatorname{Right}(X)$, the proof is similar.

For the inductive step, suppose that the claim holds for m. Consider structures $\mathfrak{A}, \bar{P}, \bar{a} \equiv_L^{m+1} \mathfrak{A}', \bar{P}', \bar{a}'$ and $\mathfrak{B}, \bar{Q}, \bar{b} \equiv_L^{m+1} \mathfrak{B}', \bar{Q}', \bar{b}'$. We have to show that

$$\mathfrak{A} \oplus \mathfrak{B}, \bar{P} \cup \bar{Q}, \bar{a}\bar{b} \ \equiv_L^{m+1} \ \mathfrak{A}' \oplus \mathfrak{B}', \bar{P}' \cup \bar{Q}', \bar{a}'\bar{b}' .$$

By symmetry and Proposition 3.4, it is sufficient to prove that, for every L-parameter α in $\mathfrak{A} \oplus \mathfrak{B}$, there exists an L-parameter α' in $\mathfrak{A}' \oplus \mathfrak{B}'$ such that

$$\mathfrak{A} \oplus \mathfrak{B}, \bar{P} \cup \bar{Q}, \bar{a}\bar{b}, \alpha \equiv_L^m \mathfrak{A}' \oplus \mathfrak{B}', \bar{P}' \cup \bar{Q}', \bar{a}'\bar{b}', \alpha' .$$

Hence, consider a parameter α in $\mathfrak{A} \oplus \mathfrak{B}$. We distinguish two cases.

If α is a first-order parameter, then $\alpha \in A$ or $\alpha \in B$. By symmetry, we may assume the former. According to Proposition 3.4, we can find an L-parameter α' in $\mathfrak{A}'$ such that

$$\mathfrak{A}, \bar{P}, \bar{a}\alpha \equiv_L^m \mathfrak{A}', \bar{P}', \bar{a}'\alpha' .$$

By inductive hypothesis, this implies that

$$\mathfrak{A} \oplus \mathfrak{B}, \bar{P} \cup \bar{Q}, \bar{a}\bar{b}\alpha \equiv_L^m \mathfrak{A}' \oplus \mathfrak{B}', \bar{P}' \cup \bar{Q}', \bar{a}'\bar{b}'\alpha' ,$$

as desired.

Hence, suppose that α is a second-order parameter. According to Proposition 3.4, we can find parameters α'_0 in $\mathfrak{A}'$ and α'_1 in $\mathfrak{B}'$ such that

$$\mathfrak{A}, \bar{P}, \bar{a}, \alpha \upharpoonright A \equiv_L^m \mathfrak{A}', \bar{P}', \bar{a}', \alpha'_0$$

and $$\mathfrak{B}, \bar{Q}, \bar{b}, \alpha \upharpoonright B \equiv_L^m \mathfrak{B}', \bar{Q}', \bar{b}', \alpha'_1 .$$

By inductive hypothesis, this implies that

$$\mathfrak{A} \oplus \mathfrak{B}, \bar{P} \cup \bar{Q}, \bar{a}\bar{b}, \alpha \equiv_L^m \mathfrak{A}' \oplus \mathfrak{B}', \bar{P}' \cup \bar{Q}', \bar{a}'\bar{b}', \alpha'_0 \cup \alpha'_1 .$$

Hence, we can set $\alpha' := \alpha'_0 \cup \alpha'_1$. □

Let us present some applications. We start with structures over the empty signature.

Lemma 4.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures over the empty signature. Then*

(a) $\mathfrak{A} \equiv_{\mathrm{FO}}^m \mathfrak{B}$ iff $|A| = |B|$ *or* $|A|, |B| \geq m$.

(b) $\mathfrak{A} \equiv_{\mathrm{MSO}}^{m+1} \mathfrak{B}$ iff $|A| = |B|$ *or* $|A|, |B| \geq 2^m$.

Proof. (a) ($\Rightarrow$) Suppose that $k := |A| < |B|$ with $k < m$. Then the formula

$$\exists x_0 \cdots \exists x_k \bigwedge_{0 \leq i < j \leq k} x_i \neq x_j$$

holds in $\mathfrak{B}$, but not in $\mathfrak{A}$. It quantifier-rank is $k + 1 \leq m$.

($\Leftarrow$) Clearly, if $|A| = |B|$ then $\mathfrak{A} \cong \mathfrak{B}$ and both structures have the same theory. Hence, it remains to consider the case where $|A|, |B| \geq m$. We have to show that $\mathfrak{A} \equiv^m_{\mathrm{FO}} \mathfrak{B}$. For $m = 0$ the claim is trivial, since there are no quantifier-free FO[$\varnothing$]-formulae without free variables. Hence, we may assume that $m > 0$. In this case, according to Proposition 3.4, it is enough to prove the Back-and-Forth Property. By symmetry, it is sufficient to consider only the Forth Property. Hence, let $a \in A$. We pick an arbitrary element $b \in B$. Let $\mathfrak{A}_0$ and $\mathfrak{B}_0$ be the substructures of $\mathfrak{A}$ and $\mathfrak{B}$ induced by the sets $A_0 := A \smallsetminus \{a\}$ and $B_0 := B \smallsetminus \{b\}$, and let $\mathfrak{A}_1$ and $\mathfrak{B}_1$ be the substructures induced by $\{a\}$ and $\{b\}$. By inductive hypothesis,

$$|A_0|, |B_0| \geq m - 1 \quad \text{implies} \quad \mathfrak{A}_0 \equiv^{m-1}_{\mathrm{FO}} \mathfrak{B}_0 \,.$$

Since $\mathfrak{A}_1 \cong \mathfrak{B}_1$, we also have $\mathfrak{A}_1 \equiv^{m-1}_{\mathrm{FO}} \mathfrak{B}_1$. Consequently, it follows by Proposition 4.2 that

$$\mathfrak{A}, a \cong \mathfrak{A}_0 \oplus \mathfrak{A}_1, a \equiv^{m-1}_{\mathrm{FO}} \mathfrak{B}_0 \oplus \mathfrak{B}_1, b \cong \mathfrak{B}, b \,.$$

(Strictly speaking, instead of the disjoint unions $\mathfrak{A}_0 \oplus \mathfrak{A}_1$ and $\mathfrak{B}_0 \oplus \mathfrak{B}_1$, we have to take their reducts that omit the new relations Left and Right.)

(b) ($\Rightarrow$) Suppose that $k := |A| < |B|$ with $k < 2^m$ and let n be the largest number such that $2^n \leq k$. For $i \leq 2^{n+1}$, set

$$\vartheta_i(X_0, \dots, X_n) := \exists y \bigwedge_{0 \leq j \leq n} \chi_{i,j}(\bar{X}, y)$$

where

$$\chi_{i,j}(\bar{X}, y) := \begin{cases} X_j y & \text{if the } j\text{-th bit of } i \text{ is } 1\,, \\ \neg X_j y & \text{if the } j\text{-th bit of } i \text{ is } 0\,. \end{cases}$$

The formula

$$\exists X_0 \cdots \exists X_n \bigwedge_{0 \leq i \leq k} \vartheta_i$$

holds in $\mathfrak{B}$, but not in $\mathfrak{A}$. It has quantifier rank $n + 2 \leq m + 1$.

($\Leftarrow$) Again it is sufficient to consider the case where $|A|, |B| \geq 2^m$. We prove the claim by induction on m. First, suppose that $m = 0$. Every formula φ of quantifier rank 1 contains only one bound variable. As the signature is empty, it follows that the only atomic formulae appearing in φ are of the form $x = x$. Consequently, φ either states that the structure is non-empty, or that it is empty. Since $\mathfrak{A}$ and $\mathfrak{B}$ are both non-empty, such formulae therefore hold either in both structures, or in none of them.

For the inductive step, suppose that $m > 0$. Again it is sufficient to check the Forth Property. We distinguish two cases, depending on whether we deal with a first-order parameter or with a monadic one. First, consider a monadic parameter $P \subseteq A$. If $P = \varnothing$, we choose $Q := \varnothing$. If $P = A$, we choose $Q := B$. In both cases it follows by inductive hypothesis that

$$\mathfrak{A}, P \equiv^m_{\mathrm{MSO}} \mathfrak{B}, Q\,.$$

Hence, we may assume that P is neither empty, nor all of A. If $|P| \leq 2^{m-1}$, choose a subset $Q \subseteq B$ of size $|Q| = |P|$. Otherwise, choose a subset $Q \subseteq B$ with $|B \setminus Q| = |A \setminus P|$. Let $\mathfrak{A}_0$ and $\mathfrak{B}_0$ be the substructures of $\mathfrak{A}$ and $\mathfrak{B}$ induced by P and Q, and let $\mathfrak{A}_1$ and $\mathfrak{B}_1$ be the substructures induced by $A \setminus P$ and $B \setminus Q$. It follows that

- $|P| = |Q|$ or $|P|, |Q| \geq 2^{m-1}$;
- $|A \setminus P| = |B \setminus Q|$ or $|A \setminus P|, |B \setminus Q| \geq 2^{m-1}$.

By inductive hypothesis, this implies that

$$\mathfrak{A}_0 \equiv^m_{\mathrm{MSO}} \mathfrak{B}_0 \quad \text{and} \quad \mathfrak{A}_1 \equiv^m_{\mathrm{MSO}} \mathfrak{B}_1\,.$$

By Proposition 4.2, it follows that

$$\mathfrak{A}, P \cong \mathfrak{A}_0, P \oplus \mathfrak{A}_1, \varnothing \equiv^m_{\mathrm{MSO}} \mathfrak{B}_0, Q \oplus \mathfrak{B}_1, \varnothing \cong \mathfrak{B}, Q\,.$$

(Again, we have to omit the relations Left and Right.)

For a first-order parameter $a \in A$, we choose an arbitrary element $b \in B$. We denote by $\mathfrak{A}_0$ and $\mathfrak{B}_0$ the substructures of $\mathfrak{A}$ and $\mathfrak{B}$ induced by $\{a\}$ and $\{b\}$, and we write $\mathfrak{A}_1$ and $\mathfrak{B}_1$ for the substructures induced by $A \smallsetminus \{a\}$ and $B \smallsetminus \{b\}$. Then

$$\mathfrak{A}_0, a \cong \mathfrak{B}_0, b \quad \text{implies} \quad \mathfrak{A}_0, a \equiv^m_{\mathrm{MSO}} \mathfrak{B}_0, b\,.$$

Furthermore, it follows by inductive hypothesis that

$$\mathfrak{A}_1 \equiv^m_{\mathrm{MSO}} \mathfrak{B}_1\,.$$

By Proposition 4.2, this implies that

$$\mathfrak{A}, a \cong \mathfrak{A}_0, a \oplus \mathfrak{A}_1 \equiv^m_{\mathrm{MSO}} \mathfrak{B}_0, b \oplus \mathfrak{B}_1 \cong \mathfrak{B}, b\,. \qquad \square$$

Example. There is no $\mathrm{MSO}[\Sigma]$-formula φ such that, for every finite Σ-structure $\mathfrak{A}$,

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad |A| \text{ is even.}$$

For the proof, let $m := \mathrm{qr}(\varphi)$ and let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures of size 2^m and $2^m + 1$, respectively, where every relation is empty. By Lemma 4.3, we have

$$\mathfrak{A} \equiv^{m+1}_{\mathrm{MSO}} \mathfrak{B}\,.$$

Consequently,

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad \mathfrak{B} \vDash \varphi\,.$$

A contradiction.

Exercise 4.1. We consider structures of the form $\mathfrak{A} = \langle A, E\rangle$ where E is an equivalence relation. For an equivalence relation E, we denote by $N^=_k(E)$ the number of E-classes $[a]_E$ of size $|[a]_E| = k$ and $N^>_k(E)$ denotes the number of classes of size $|[a]_E| > k$. We write $m =_k n$ iff $m = n$ or $m, n \geq k$.

Let E and F be equivalence relations on the sets A and B, respectively. Prove that $\langle A, E\rangle \cong_m \langle B, F\rangle$ if, and only if, for all $k \leq m$,

$$N^=_k(E) =_{m-k} N^=_k(F) \quad \text{and} \quad N^>_k(E) =_{m-k} N^>_k(F)\,.$$

Interpretations

Definition 4.4. Let L be either FO or MSO and let Σ and Γ be relational signatures. An *L-interpretation* from Σ to Γ is an operation τ transforming Σ-structures into Γ-structures that is defined by a list

$$\left\langle \delta(x), (\varphi_R(\bar{x}))_{R\in\Gamma} \right\rangle$$

of L-formulae as follows. τ maps a Σ-structure $\mathfrak{A}$ to the Γ-structure

$$\tau(\mathfrak{A}) := \left\langle \delta^{\mathfrak{A}}, (\varphi_R^{\mathfrak{A}})_{R\in\Gamma} \right\rangle$$

whose universe is the set

$$\delta^{\mathfrak{A}} := \{\, a \in A \mid \mathfrak{A} \vDash \delta(a) \,\}$$

defined by δ and whose relations are

$$\varphi_R^{\mathfrak{A}} := \{\, \bar{a} \mid \mathfrak{A} \vDash \varphi_R(\bar{a}) \,\}\,, \quad \text{for } R \in \Sigma\,.$$

We call the list $\langle \delta(x), (\varphi_R(\bar{x}))_{R\in\Gamma} \rangle$ the *definition scheme* of τ. The *quantifier rank* of τ is the maximal quantifier rank of a formula in its definition scheme.

Lemma 4.5. *Let L be* FO *or* MSO*, and let $\tau = \langle \delta(x), (\varphi_R(\bar{x}))_{R\in\Gamma} \rangle$ be an L-interpretation from Σ to Γ with quantifier rank m. For every $L[\Gamma]$-formula $\psi(\bar{X})$, there exists an $L[\Sigma]$-formula $\psi^\tau(\bar{X})$ with quantifier rank at most* $\mathrm{qr}(\psi) + m$ *such that*

$$\tau(\mathfrak{A}) \vDash \psi(\bar{\alpha}) \quad \text{iff} \quad \mathfrak{A} \vDash \psi^\tau(\bar{\alpha})\,,$$

for all Σ-structures $\mathfrak{A}$ such that $\tau(\mathfrak{A})$ is defined and all parameters $\bar{\alpha}$ in $\tau(\mathfrak{A})$.

Proof. We define ψ^τ by induction on ψ as follows.

$$\begin{aligned} (x = y)^\tau &:= x = y\,, & (\varphi \wedge \psi)^\tau &:= \varphi^\tau \wedge \psi^\tau\,,\\ (Xy)^\tau &:= Xy\,, & (\varphi \vee \psi)^\tau &:= \varphi^\tau \vee \psi^\tau\,,\\ (R\bar{x})^\tau &:= \varphi_R(\bar{x})\,, & (\neg\varphi)^\tau &:= \neg\varphi^\tau\,, \end{aligned}$$

$$(\exists y\psi)^\tau := \exists y[\delta(y) \wedge \psi^\tau], \qquad (\exists Y\psi)^\tau := \exists Y\psi^\tau,$$
$$(\forall y\psi)^\tau := \forall y[\delta(y) \to \psi^\tau], \qquad (\forall Y\psi)^\tau := \forall Y\psi^\tau. \qquad \square$$

Corollary 4.6. *Let L be* FO *or* MSO *and let τ be an L-interpretation from Σ to Γ with quantifier rank m. For Σ-structures $\mathfrak{A}$ and $\mathfrak{A}'$ such that $\tau(\mathfrak{A})$ and $\tau(\mathfrak{A}')$ are defined,*

$$\mathfrak{A} \equiv_L^{k+m} \mathfrak{A}' \quad \textit{implies} \quad \tau(\mathfrak{A}) \equiv_L^k \tau(\mathfrak{A}').$$

Proof. By symmetry, it is sufficient to prove that

$$\tau(\mathfrak{A}) \vDash \varphi \quad \text{implies} \quad \tau(\mathfrak{A}') \vDash \varphi, \quad \text{for all } \varphi \text{ with } \mathrm{qr}(\varphi) \leq k.$$

Suppose that $\tau(\mathfrak{A}) \vDash \varphi$ and let φ^τ be the formula from Lemma 4.5. Then

$$\mathfrak{A} \vDash \varphi^\tau \quad \text{and} \quad \mathrm{qr}(\varphi^\tau) \leq k + m.$$

Hence, $\mathfrak{A} \equiv_L^{k+m} \mathfrak{A}'$ implies that $\mathfrak{A}' \vDash \varphi^\tau$. It follows that $\tau(\mathfrak{A}') \vDash \varphi$. $\square$

Definition 4.7. Let C be a set of *colours.*

(a) A *C-coloured order* is a structure of the form $\mathfrak{A} = \langle A, \leq, (P_c)_{c\in C}\rangle$ where $\leq$ is a linear ordering on A and the P_c are unary predicates.

(b) Let $\mathfrak{I} = \langle I, \leq\rangle$ be a linear order and let $\mathfrak{A}_i := \langle A_i, \leq_i, \bar{P}_i\rangle$, $i \in I$, be a family or C-coloured linear orders indexed by I. The *ordered sum*

$$\sum_{i\in I} \mathfrak{A}_i$$

is the linear order with universe

$$L := \{ \langle i, a\rangle \mid i \in I, a \in A_i \}$$

and order

$$\langle i, a\rangle \leq \langle j, b\rangle \quad :\text{iff} \quad i < j \text{ or } (i = j \text{ and } a \leq_i b).$$

The colour predicates are

$$P_c := \bigcup_{i\in I} (P_i)_c.$$

If $I = [2]$, we simply write $\mathfrak{A}_0 + \mathfrak{A}_1$ for the ordered sum.

Lemma 4.8. *Let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{B}_0, \mathfrak{B}_1$ be C-coloured linear orders and let L be* FO *or* MSO. *Then*

$$\mathfrak{A}_0 \equiv_L^m \mathfrak{B}_0 \quad \textit{and} \quad \mathfrak{A}_1 \equiv_L^m \mathfrak{B}_1 \quad \textit{implies} \quad \mathfrak{A}_0 + \mathfrak{A}_1 \equiv_L^m \mathfrak{B}_0 + \mathfrak{B}_1 .$$

Proof. We have

$$\mathfrak{A}_0 + \mathfrak{A}_1 \cong \tau(\mathfrak{A}_0 \oplus \mathfrak{A}_1) ,$$

where τ is a quantifier-free L-interpretation that corrects the order relation. It has the definition scheme

$$\begin{aligned} \delta(x) &:= \mathit{true} , \\ \varphi_{\leq}(x, y) &:= x \leq y \vee (\mathrm{Left}(x) \wedge \mathrm{Right}(y)) , \\ \varphi_{P_c}(x) &:= P_c x . \end{aligned}$$

□

Exercise 4.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be finite linear orders. Prove that

$$\mathfrak{A} \equiv_{\mathrm{FO}}^m \mathfrak{B} \quad \text{iff} \quad |A| = |B| \text{ or } |A|, |B| \geq 2^m - 1.$$

Example. (a) There does not exist an FO-formula φ that holds in an undirected graph if, and only if, the graph is connected.

For a contradiction, suppose that such a formula φ exists. We will construct a new formula ψ that holds in a finite linear order if, and only if, this order has an even number of elements. Let m be the quantifier rank of ψ and let $\mathfrak{A}$ and $\mathfrak{B}$ be linear orders of size 2^m and $2^m + 1$, respectively. Then

$$\mathfrak{A} \models \psi \quad \text{and} \quad \mathfrak{B} \not\models \psi ,$$

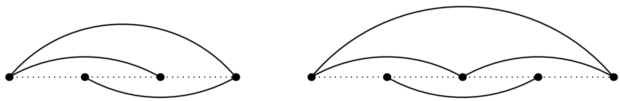
in contradiction to the statement in the above exercise.

To construct the desired formula ψ, we define an FO-interpretation $\tau = \langle \delta, \varphi_E \rangle$ mapping linear orders to undirected graphs as follows. The formula δ is *true* while $\varphi_E(x, y)$ states that

- in the order $\leq$ there is exactly one element between x and y, or

- x is the first element and y is the last one, or
- y is the first element and x is the last one.

Then τ maps finite linear orders of even size to paths and finite linear orders of odd size (at least 3) to the disjoint union of a path and a cycle. Orders of size 1 are mapped to a loop.



Hence,

$$\tau(\mathfrak{A}) \vDash \varphi \quad \text{iff} \quad \mathfrak{A} \text{ has either exactly one, or an even number of elements.}$$

Consequently, the formula $\psi := \varphi^\tau \wedge \exists xy(x \neq y)$ has the desired properties.

(b) There does not exist an FO-formula $\varphi(x, y)$ such that

$$\mathfrak{G} \vDash \varphi(u, v) \quad \text{iff} \quad \text{the graph } \mathfrak{G} \text{ contains a path from } u \text{ to } v\,.$$

Otherwise, the formula

$$\forall x \forall y \varphi(x, y)$$

would express that the graph is connected.

Example. We consider undirected graphs as structures over the signature $\{E\}$.

(a) There does not exist an MSO-formula φ that holds in a finite complete bipartite graph $K_{m,n}$ if, and only if, $m = n$. The proof is similar to that of Lemma 4.8. Suppose that such a formula φ exists and let k be its quantifier rank. Let $\mathfrak{A}$ and $\mathfrak{B}$ be graphs without any edges that have, respectively, $m := 2^k$ and $n := 2^k + 1$ vertices. Then

$$K_{m,m} := \tau(\mathfrak{A} \oplus \mathfrak{A}) \quad \text{and} \quad K_{m,n} := \tau(\mathfrak{A} \oplus \mathfrak{B})\,,$$

where τ is a quantifier-free interpretation that adds all edges between a vertex in Left and a vertex in Right. Since

$$\mathfrak{A} \equiv^k_{\mathrm{MSO}} \mathfrak{B}$$

it follows that

$$K_{m,m} = \tau(\mathfrak{A} \oplus \mathfrak{A}) \equiv^k_{\mathrm{MSO}} \tau(\mathfrak{A} \oplus \mathfrak{B}) = K_{m,n} \,.$$

A contradiction, since φ distinguishes between these two graphs.

(b) There does not exist an MSO-formula φ that holds in a finite graph if, and only if, all vertices have the same number of neighbours. For a contradiction, suppose that such a formula φ exists. For a complete bipartite graph $K_{m,n}$ it follows that

$$K_{m,n} \vDash \varphi \quad \text{iff} \quad m = n \,.$$

This contradicts (a).

(c) There does not exist an MSO-formula φ that holds in a finite undirected graph if, and only if, the graph has a Hamiltonian cycle. For a contradiction, suppose that such a formula φ exists. Since a complete bipartite graph $K_{m,n}$ contains an Hamiltonian cycle if, and only if, $m = n$, it follows that

$$K_{m,n} \vDash \varphi \quad \text{iff} \quad m = n \,.$$

This contradicts (a).

Generalised sums

Definition 4.9. Let Γ and Σ be relational signatures, $\mathfrak{I}$ a Γ-structure and, for every element $i \in I$, let $\mathfrak{A}_i$ be a Σ-structure. The *generalised sum* of the $\mathfrak{A}_i$ over $\mathfrak{I}$ is the $(\Gamma \cup \Sigma \cup \{\sim\})$-structure $\sum_{i \in \mathfrak{I}} \mathfrak{A}_i$ with universe

$$S := \{ \langle i, a \rangle \mid i \in I \,,\; a \in A_i \}$$

and the following relations. For every n-ary relation symbol $R \in \Sigma$, it has the relation

$$R := \{ \langle\langle i, a_0\rangle, \dots, \langle i, a_{n-1}\rangle\rangle \mid i \in I, \langle a_0, \dots, a_{n-1}\rangle \in R^{\mathfrak{A}_i} \},$$

for every n-ary relation symbol $R \in \Gamma$, it has the relation

$$R := \{ \langle\langle i_0, a_0\rangle, \dots, \langle i_{n-1}, a_{n-1}\rangle\rangle \mid \langle i_0, \dots, i_{n-1}\rangle \in R^{\mathfrak{I}}, a_j \in A_j \},$$

and additionally it has the equivalence relation

$$\sim := \{ \langle\langle i, a\rangle, \langle i, b\rangle\rangle \mid i \in I, \ a, b \in A_i \}.$$

Example. Given linear orders $\mathfrak{I} = \langle I, \sqsubseteq\rangle$ and $\mathfrak{A}_i := \langle A_i, \leq_i\rangle$, for $i \in I$, the generalised sum is the structure $\langle S, \sqsubseteq, \leq, \sim\rangle$ with relations

$$\begin{aligned}
\langle i, a\rangle \sqsubseteq \langle j, b\rangle \quad &\text{iff} \quad i \sqsubseteq j\,,\\
\langle i, a\rangle \leq \langle j, b\rangle \quad &\text{iff} \quad i = j \text{ and } a \leq_i b\,,\\
\langle i, a\rangle \sim \langle j, b\rangle \quad &\text{iff} \quad i = j\,.
\end{aligned}$$

We can interpret the ordered sum of the $\mathfrak{A}_i$ in this structure via the interpretation $\tau = \langle \delta, \varphi_\leq\rangle$ with

$$\delta(x) := \mathit{true} \quad \text{and} \quad \varphi_\leq(x, y) := x \sqsubseteq y \wedge [x \sim y \rightarrow x \leq y]\,.$$

For the next operation, we need to take a closer look at the quantifier structure of a formula.

Definition 4.10. Let $\bar{n} \in \omega^*$.

(a) We define the set $\mathrm{MSO}_0^{\bar{n}}[\Sigma]$ of all MSO_0-formulae with *quantifier structure $\bar{n}$* as follows. The set $\mathrm{MSO}_0^{\langle\rangle}[\Sigma]$ contains all quantifier-free $\mathrm{MSO}_0[\Sigma]$-formulae and $\mathrm{MSO}_0^{m\bar{n}}[\Sigma]$ contains all formulae that can be written as boolean combinations of formulae of the form

$$\exists X_0 \cdots \exists X_{m-1} \psi \quad \text{with} \quad \psi \in \mathrm{MSO}_0^{\bar{n}}[\Sigma]\,.$$

(b) We denote by $\mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}, \bar{P})$ the $\mathrm{MSO}_0^{\bar{n}}$-theory of $\mathfrak{A}, \bar{P}$ and we set

$$\mathfrak{A}, \bar{P} \equiv^{\bar{n}}_{\mathrm{MSO}_0} \mathfrak{B}, \bar{Q} \quad :\text{iff} \quad \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}, \bar{P}) = \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{B}, \bar{Q})\,.$$

Definition 4.11. Let $\sum_{i \in \mathfrak{I}} \mathfrak{A}_i$ be a generalised sum and let $\bar{P}$ be a tuple of monadic parameters. For an $\mathrm{MSO}_0[\Sigma]$-formula $\chi(\bar{X})$, we define

$$[\![\chi(\bar{P})]\!] := \left\{\, i \in I \;\middle|\; \mathfrak{A}_i \vDash \chi(\bar{P} \restriction A_i) \,\right\}.$$

Theorem 4.12. *Let Γ and Σ be relational signatures and let $\bar{n} \in \omega^*$. There exists a tuple $\bar{r} \in \omega^*$ of length $|\bar{r}| = |\bar{n}|$ such that, given a formula $\varphi(\bar{X}) \in \mathrm{MSO}_0^{\bar{n}}[\Gamma \mathbin{\dot\cup} \Sigma \mathbin{\dot\cup} \{\sim\}]$, we can compute formulae*

$$\begin{aligned} &\chi_0(\bar{X}), \dots, \chi_{m-1}(\bar{X}) \in \mathrm{MSO}_0^{\bar{n}}[\Sigma] \\ \textit{and} \quad &\varphi'(Z_0, \dots, Z_{m-1}) \in \mathrm{MSO}_0^{\bar{r}}[\Gamma] \end{aligned}$$

such that

$$\sum_{i \in \mathfrak{I}} \mathfrak{A}_i \vDash \varphi(\bar{P}) \quad \text{iff} \quad \mathfrak{I} \vDash \varphi'\big([\![\chi_0(\bar{P})]\!], \dots, [\![\chi_{m-1}(\bar{P})]\!]\big),$$

for all Γ-structures $\mathfrak{I}$, Σ-structures $\mathfrak{A}_i$, and monadic parameters $\bar{P}$.

Proof. We construct $\varphi'(\bar{Z})$ and $\chi_0(\bar{X}) \dots, \chi_{m-1}(\bar{X})$ by induction on φ. Furthermore, we assume by induction that the only occurrences of the variables $\bar{Z}$ in atomic subformulae of φ' are of the form

$$\mathrm{cover}(Z_{i_0}, \dots, Z_{i_k}), \quad Z_i \cap Z_j = \varnothing, \quad \text{and} \quad RZ_{i_0} \dots Z_{i_k}.$$

First, suppose that φ is atomic. We distinguish several cases. For $\varphi = (X \subseteq Y)$ we set

$$\varphi' := \mathrm{cover}(Z_0) \quad \text{and} \quad \chi_0(X, Y) := X \subseteq Y.$$

For $\varphi = (X \cap Y = \varnothing)$ we set

$$\varphi' := \mathrm{cover}(Z_0) \quad \text{and} \quad \chi_0(X, Y) := X \cap Y = \varnothing.$$

For $\varphi = \mathrm{cover}(\bar{X})$ we set

$$\varphi' := \mathrm{cover}(Z_0) \quad \text{and} \quad \chi_0(\bar{X}) := \mathrm{cover}(\bar{X}).$$

For $\varphi = R\bar{X}$ with $R \in \Sigma$, we set

$$\varphi' := Z_0 \neq \varnothing \quad \text{and} \quad \chi_0(\bar{X}) := R\bar{X}\,.$$

For $\varphi = R\bar{X}$ with $R \in \Gamma$, we set

$$\varphi' := R\bar{Z} \quad \text{and} \quad \chi_i(\bar{X}) := (X_i \neq \varnothing)\,.$$

For $\varphi = (X \sim Y)$, we set

$$\begin{aligned} \varphi' := Z_0 \cap Z_1 \neq \varnothing \quad \text{and} \quad & \chi_0(X,Y) := (X \neq \varnothing)\,, \\ & \chi_1(X,Y) := (Y \neq \varnothing)\,. \end{aligned}$$

For the inductive step, we also distinguish several cases. First, suppose that $\varphi = \psi \vee \vartheta$. By inductive hypothesis, we have already constructed formulae ψ', $\eta_0, \dots, \eta_{m-1}$ for ψ and formulae ϑ', $\zeta_0, \dots, \zeta_{n-1}$ for ϑ. Let ϑ'' be the formula obtained from ϑ' by replacing every variable Z_i by Z_{m+i}. Then we can set $\varphi' := \psi' \vee \vartheta''$ and we can use

$$\eta_0, \dots, \eta_{m-1}, \zeta_0, \dots, \zeta_{n-1}$$

for the formulae χ_i, $i < m + n$.

The construction for $\varphi = \psi \wedge \vartheta$ is analogous. For $\varphi = \neg\psi$, suppose that we have already constructed formulae ψ', $\eta_0, \dots, \eta_{m-1}$ for ψ. Then we set $\varphi' := \neg\psi'$ and we can use $\eta_0, \dots, \eta_{m-1}$ for the formulae χ_i, $i < m$.

It remains to consider the case where $\varphi(\bar{X}) = \exists\bar{Y}\psi(\bar{X},\bar{Y})$. Again we may assume that we have already constructed formulae ψ', $\eta_0, \dots, \eta_{m-1}$ for ψ. Set $k := |\bar{Y}|$ and let $\bar{n}$ be the quantifier structure of ψ. Let $\Theta_{\bar{n}}$ be the set of all $\mathrm{MSO}_0^{\bar{n}}[\Sigma]$-theories with free variables $\bar{X}$, $\bar{Y}$, and let $\Theta_{k\bar{n}}$ be the set of $\mathrm{MSO}_0^{k\bar{n}}[\Sigma]$-theories with free variables $\bar{X}$. Let ψ'' be the formula that, intuitively, is obtained from ψ' by replacing each free variable Z_i by the union

$$\bigcup \{\, Z'_\sigma \mid \sigma \in \Theta_{\bar{n}}\,,\ \eta_i \in \sigma \,\}\,.$$

Formally, we define ψ'' as follows. Recall that, by inductive hypothesis, the only atomic subformulae of ψ' containing the variables $Z_0, \dots, Z_{m-1}$ are of the form

$$\operatorname{cover}(Z_{i_0}, \dots, Z_{i_k}), \quad Z_i \cap Z_j = \varnothing, \quad \text{and} \quad RZ_{i_0} \dots Z_{i_k}.$$

We can replace

$$\begin{array}{lll} & Z_i \cap Z_j = \varnothing & \text{by} \quad \bigwedge \{ Z'_\sigma \cap Z'_\upsilon = \varnothing \mid \eta_i \in \sigma, \eta_j \in \upsilon \}, \\ & RZ_{i_0} \dots Z_{i_k} & \text{by} \quad \bigvee \{ RZ'_{\sigma_0} \dots Z'_{\sigma_k} \mid \eta_{i_j} \in \sigma_j \}, \\ \text{and} & \operatorname{cover}(Z_{i_0}, \dots, Z_{i_k}) & \text{by} \\ & & \quad \operatorname{cover}(Z'_{\sigma^0_0}, \dots, Z'_{\sigma^0_{l_0}}, \dots, Z'_{\sigma^k_0}, \dots, Z'_{\sigma^k_{l_k}}), \end{array}$$

where in the last line $\sigma^j_0, \dots, \sigma^j_{l_j}$ is an enumeration of all theories σ with $\eta_{i_j} \in \sigma$.

For φ' and χ_τ, we choose the formulae

$$\begin{aligned} \varphi' := \exists (Z'_\sigma)_{\sigma \in \Theta_{\bar{n}}} \Big[& \operatorname{part}(\bar{Z}') \wedge \psi''(\bar{Z}') \\ & \wedge \bigwedge \Big\{ Z'_\sigma \cap Z_\tau = \varnothing \;\Big| \\ & \qquad \sigma \in \Theta_{\bar{n}}, \ \tau \in \Theta_{k\bar{n}}, \ \exists \bar{Y} \bigwedge \sigma \notin \tau \Big\} \Big], \end{aligned}$$

$$\text{and} \quad \chi_\tau := \bigwedge \tau, \quad \text{for } \tau \in \Theta_{k\bar{n}},$$

where

$$\operatorname{part}(X_0, \dots, X_{n-1}) := \operatorname{cover}(X_0, \dots, X_{n-1}) \wedge \bigwedge_{i \neq k} X_i \cap X_k = \varnothing,$$

states that the sets $X_0, \dots, X_{n-1}$ form a partition of the universe (where some classes might be empty).

It follows that

$$\begin{aligned}
&\sum_{i\in\mathfrak{I}} \mathfrak{A}_i \vDash \exists \bar{Y} \psi(\bar{P}, \bar{Y}) \\
\text{iff}\quad &\sum_{i\in\mathfrak{I}} \mathfrak{A}_i \vDash \psi(\bar{P}, \bar{Q})\,, \quad \text{for some sets } \bar{Q} \\
\text{iff}\quad &\mathfrak{I} \vDash \psi'\big([\![\eta_0(\bar{P}, \bar{Q})]\!], \dots, [\![\eta_{m-1}(\bar{P}, \bar{Q})]\!]\big)\,, \quad \text{for some } \bar{Q}\,.
\end{aligned}$$

It remains to show that the latter condition holds if, and only if,

$$\mathfrak{I} \vDash \varphi'\big([\![\chi_\tau(\bar{P})]\!]\big)_{\tau\in\Theta_{k\bar{n}}}\,.$$

First suppose that there exist sets $\bar{Q}$ as above. For $i \in I$, let

$$\begin{aligned}
\sigma_i &:= \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}_i, \bar{P} \restriction A_i, \bar{Q} \restriction A_i)\,, \\
\tau_i &:= \mathrm{Th}^{k\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}_i, \bar{P} \restriction A_i)\,.
\end{aligned}$$

Then

$$[\![\chi_\tau(\bar{P})]\!] = \{\, i \in I \mid \tau_i = \tau \,\} \quad \text{and} \quad \exists \bar{Y} \bigwedge \sigma_i \in \tau_i\,.$$

Hence, setting $Z'_\sigma := \{\, i \in I \mid \sigma_i = \sigma \,\}$, we see that

$$\mathfrak{I} \vDash \varphi'\big([\![\chi_\tau(\bar{P})]\!]\big)_{\tau\in\Theta_{k\bar{n}}}\,.$$

Conversely, suppose that φ' is satisfied. Let $Z'_\sigma \subseteq I$, for $\sigma \in \Theta_{\bar{n}}$, be the corresponding witnesses. Then the sets Z'_σ form a partition of I and, for every $\sigma \in \Theta_{\bar{n}}$ and every $i \in Z'_\sigma$, there exists some $\tau_i \in \Theta_{k\bar{n}}$ such that $i \in \tau_i$ and $\exists \bar{Y} \bigwedge \sigma \in \tau_i$. Hence, we can choose subsets $\bar{Q}_i$ of A_i such that

$$\mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}, \bar{P} \restriction A_i, \bar{Q}_i) = \sigma\,.$$

Let $\bar{Q}$ be the component-wise union of the sets $\bar{Q}_i$, $i \in I$. Then

$$[\![\eta_j(\bar{P}, \bar{Q})]\!] = \bigcup \{\, Z'_\sigma \mid \eta_j \in \sigma \,\}\,,$$

and $\mathfrak{I} \vDash \psi''\big([\![\chi_\tau(\bar{P})]\!]\big)_{\tau\in\Theta_{k\bar{n}}}$ implies that

$$\mathfrak{I} \vDash \psi'\big([\![\eta_0(\bar{P}, \bar{Q})]\!], \dots, [\![\eta_{m-1}(\bar{P}, \bar{Q})]\!]\big)\,. \qquad \square$$

Lemma 4.13. *Let $\mathfrak{I}$ be a linear order and let $(\mathfrak{A}_i)_{i\in I}$ and $(\mathfrak{B}_i)_{i\in I}$ be two families of linear orders indexed by I. If*

$$\mathfrak{A}_i \equiv^{\bar{n}}_{\mathrm{MSO}_0} \mathfrak{B}_i\,, \quad \textit{for all } i \in I\,,$$

then $\sum_{i\in I} \mathfrak{A}_i \equiv^{\bar{n}}_{\mathrm{MSO}_0} \sum_{i\in I} \mathfrak{B}_i$.

Proof. Let φ be a formula with quantifier structure $\bar{n}$ and let φ' and $\chi_0, \dots, \chi_{n-1}$ be the formulae obtained via Theorem 4.12. By assumption, the sets $[\![\chi_i]\!]$ have the same value when evaluated for the sequence $(\mathfrak{A}_i)_{i\in I}$ and for $(\mathfrak{B}_i)_{i\in I}$. Consequently, we have

$$\begin{aligned} \sum_{i\in\mathfrak{I}} \mathfrak{A}_i \vDash \varphi \quad &\text{iff} \quad \mathfrak{I} \vDash \varphi'([\![\chi_0]\!], \dots, [\![\chi_{n-1}]\!]) \\ &\text{iff} \quad \sum_{i\in\mathfrak{I}} \mathfrak{B}_i \vDash \varphi\,. \end{aligned}$$

□

Exercise 4.3. Let $0 < k < \omega$ and let L be one of the logics above. The *copy operation* copy_k maps a Σ-structure $\mathfrak{A}$ to the structure

$$\mathrm{copy}_k(\mathfrak{A}) := \big\langle [k] \times A, (R')_{R\in\Sigma}, I, P_0, \dots, P_{k-1} \big\rangle$$

where

$$\begin{aligned} R' &:= \big\{ \langle\langle i, a_0\rangle, \dots, \langle i, a_{n-1}\rangle\rangle \bigm| i < k,\ \langle a_0, \dots, a_{n-1}\rangle \in R^{\mathfrak{A}} \big\}\,, \\ I &:= \big\{ \langle\langle i, a\rangle, \langle j, a\rangle\rangle \bigm| i, j < k,\ a \in A \big\}\,, \\ P_i &:= \{i\} \times A\,. \end{aligned}$$

Show that, for every MSO-formula $\varphi(x_0, \dots, x_{n-1})$ and all tuples $\bar{\imath} \in [k]^n$ of indices, there exists an MSO-formula $\varphi'_{\bar{\imath}}(\bar{x})$ with $\mathrm{qr}(\varphi'_{\bar{\imath}}) = \mathrm{qr}(\varphi)$ such that

$$\begin{aligned} &\mathrm{copy}_k(\mathfrak{A}) \vDash \varphi\big(\langle i_0, a_0\rangle, \dots, \langle i_{n-1}, a_{n-1}\rangle\big) \\ \text{iff} \qquad &\mathfrak{A} \vDash \varphi'_{i_0,\dots,i_{n-1}}(a_0, \dots, a_{n-1})\,. \end{aligned}$$

Exercise 4.4. Let L be one of the logics above, let $\mathfrak{A}$ be a Σ-structure, and let $\varepsilon(x, y) \in L[\Sigma]$ be a formula defining an equivalence relation on A. Suppose that the arity of all relations in Σ is at most k. The *quotient* of $\mathfrak{A}$ by ε is the structure

$$\mathfrak{A}/\varepsilon := \langle A/\varepsilon^{\mathfrak{A}}, (R')_{R \in \Sigma} \rangle$$

where

$$R' := \{ \langle [a_0]_\varepsilon, \dots, [a_{n-1}]_\varepsilon \rangle \mid \langle a_0, \dots, a_{n-1} \rangle \in R^{\mathfrak{A}} \}.$$

($[a]_\varepsilon$ denotes the equivalence class of a.)

Prove that, for every MSO-formula $\varphi(\bar{x})$, there is an MSO-formula $\varphi'(\bar{x})$ such that

$$\mathfrak{A}/\varepsilon \vDash \varphi\big([a_0]_\varepsilon, \dots, [a_{n-1}]_\varepsilon\big) \quad \text{iff} \quad \mathfrak{A} \vDash \varphi'(a_0, \dots, a_{n-1}).$$

Exercise 4.5. We consider a variant of an interpretation, called an MSO-*transduction,* that does not operate on single structures, but on classes of structures. Such a transduction τ has the form

$$\tau = \tau_0 \circ \mathrm{copy}_k \circ \exp_m ,$$

where τ_0 is an MSO-interpretation, copy_k the copy operation from Exercise 4.3, and

$$\exp_m(\mathcal{C}) := \{ \langle \mathfrak{A}, \bar{P} \rangle \mid \mathfrak{A} \in \mathcal{C},\ \bar{P} \in \wp(A)^m \}.$$

Hence,

$$\tau(\mathcal{C}) = \{ \tau_0(\mathrm{copy}_k(\mathfrak{A}, \bar{P})) \mid \mathfrak{A} \in \mathcal{C},\ \bar{P} \in \wp(A)^m \}.$$

(a) Find MSO-transductions with the following properties.

(i) A transduction mapping the class of all finite paths to the class of all circles.

(ii) A transduction mapping the class of all circles to the class of all finite paths.

(iii) A transduction mapping the class of all finite paths to the class of all complete bipartite graphs.

(iv) For every $n < \omega$, a transduction τ_n mapping the class of all finite paths to the class of all finite trees of height n.

(v) A transduction mapping the class of all finite grids to the class of all finite graphs.

(b) A GSO-*transduction* τ is an operation of the form

$$\tau(\mathcal{C}) = \{\, \mathfrak{A} \mid \mathcal{I}(\mathfrak{A}) \in \tau_0(\mathcal{I}(\mathfrak{B})) \text{ for some } \mathfrak{B} \in \mathcal{C} \,\},$$

where τ_0 is an MSO-transduction and $\mathcal{I}(\mathfrak{A})$ denotes the incidence structure of $\mathfrak{A}$. Which of the maps in (a) are GSO-transductions?

Exercise 4.6. For a graph $\mathfrak{G} = \langle V, E\rangle$, we call a pair $\langle A, B\rangle$ of subsets $A, B \subseteq V$ a *separation* of $\mathfrak{G}$ if $A \cup B = V$ and there is no edge between a vertex in $A \setminus B$ and one in $B \setminus A$.

Let $\mathfrak{G} = \langle V, E\rangle$ and $\mathfrak{G}' = \langle V', E'\rangle$ be two graphs with separations $\langle A, B\rangle$ and $\langle A', B'\rangle$, respectively. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}'$, and $\mathfrak{B}$ be the subgraphs of $\mathfrak{G}$ and $\mathfrak{G}'$ induced by the sets A, B, A', and B', respectively, let $\bar{c}$ be an enumeration of $A \cap B$, and let $\bar{c}'$ be one of $A' \cap B'$. Suppose that $A \cap B$ and $A' \cap B'$ are finite. Prove that

$$\mathfrak{A}, \bar{c} \equiv^m_{\mathrm{MSO}} \mathfrak{A}', \bar{c}' \quad \text{and} \quad \mathfrak{B}, \bar{c} \equiv^m_{\mathrm{MSO}} \mathfrak{B}', \bar{c}' \quad \text{implies} \quad \mathfrak{G} \equiv^m_{\mathrm{MSO}} \mathfrak{G}'.$$

Hint. Express $\mathfrak{G}$ and $\mathfrak{G}'$ as a generalised sums followed by a quantifier-free interpretation.

Exercise 4.7. For which of the following classes $\mathcal{C}$ does there exist an MSO-formula $\varphi(x, y)$ defining a linear order on every structure of $\mathcal{C}$? More generally, in which of the cases does there exist an MSO-formula $\psi(x, y; \bar{Z})$ such that, for every structure $\mathfrak{A} \in \mathcal{C}$, there are MSO-parameters $\bar{P}$ such that $\psi(x, y; \bar{P})$ defines a linear order on $\mathfrak{A}$?

(a) The class of all trees.

(b) The class of all binary trees.

(c) The class of all circles.

(d) The class of all complete bipartite graphs.

(e) The class of all connected graphs of degree at most 7.

5. * Sparse structures

The larger expressive power of GSO over MSO stems from the fact that GSO can quantify over sets of guarded tuples. Thus, the more guarded tuples a structure has the higher the expressive power. Below we will make this intuition precise by showing that, if the number of guarded tuples is linear in the size of the structure, we can replace every quantifier of a guarded relation by a set quantifier. Hence, in this case the expressive power of GSO collapses to that of MSO.

As an example, let us consider the case of trees, which for our purposes are considered as undirected graphs that are connected and acyclic. Every tree $\mathfrak{T}$ can be oriented by fixing some vertex $v \in T$ as the *root* and orienting every edge such that it points away from v. Having chosen such an orientation, we can represent every edge by the vertex it points to. In this way, we can replace every set of edges by a corresponding set of vertices. Hence, every GSO-quantifier can be replaced by an MSO-quantifier. The same idea works in a much more general setting.

Definition 5.1. Let $k < \omega$. A structure $\mathfrak{A} = \langle A, \bar{R} \rangle$ is *k-sparse* if

$$|R_i \upharpoonright X| \leq k \cdot |X| , \quad \text{for all finite } X \subseteq A \text{ and all } i .$$

Theorem 5.2 (Courcelle). *Let $k < \omega$ and let Σ be a signature. Given a* GSO$[\Sigma]$*-formula φ, we can compute an* MSO$[\Sigma]$*-formula φ' such that*

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad \mathfrak{A} \vDash \varphi' , \quad \textit{for every k-sparse } \Sigma\textit{-structure } \mathfrak{A} .$$

Notes

The logic MSO_0 was invented by Thomas to simplify composition arguments [17]. Guarded second-order logic was introduced by Grädel, Hirsch, and Otto in [7]. It generalises a logic for graphs defined by Courcelle, which is usually called MSO_2 or MS_2, cf. [6].

Composition arguments were first emphasised by Shelah [15]. This article also introduced generalised sums (in a slightly different form). Good surveys include [8, 2, 6].

The collapse of guarded second-order logic to monadic second-order logic on sparse structures was proved by Courcelle [5]. A corrected and generalised version was provided by Blumensath in [1]. The proof can also be found in [6].

Exercise 1.2 on axiomatisations of grids was inspired by an article of Matz, Schweikardt, and Thomas [9].

II. Linear Orders

1. Basic notions

Definition 1.1. (a) A binary relation $\leq \subseteq A \times A$ is a *partial ordering* if it has the following properties:

Reflexivity. $a \leq a$, for all $a \in A$.

Anti-Symmetry. $a \leq b$ and $b \leq a$ implies $a = b$, for all $a, b \in A$.

Transitivity. $a \leq b \leq c$ implies $a \leq c$, for all $a, b, c \in A$.

A *partial order* is a structure $\langle A, \leq \rangle$ where $\leq$ is a partial ordering on A.
(b) A *linear order* is a partial order $\langle A, \leq \rangle$ where

$$a \leq b \text{ or } b \leq a\,, \quad \text{for all } a, b \in A\,.$$

(c) A linear order $\langle A, \leq \rangle$ is a *well-order* if every non-empty subset $X \subseteq A$ has a minimal element, that is, if there exists no infinite, strictly descending sequence $a_0 > a_1 > a_2 > \cdots$. An *ordinal* is the isomorphism type of a well-order.
(d) We denote by ω the first infinite ordinal. It is isomorphic to the linear order of the natural numbers.

Formal language theory deals with linear orders whose positions are labelled with elements of a given set Σ.

Definition 1.2. (a) An *alphabet* is a finite set Σ whose elements are called *letters.*
(b) A (finite) *word* over an alphabet Σ is a finite sequence

$$w = \langle a_0, \dots, a_{n-1} \rangle$$

of elements $a_i \in \Sigma$. The *empty word* is the empty sequence $\langle\rangle$. We denote the *length* of a word w by $|w|$. We write Σ^* for the set of all finite words over Σ, and Σ^+ for the set of all non-empty finite words.

(c) An ω-*word* over an alphabet Σ is an infinite sequence $w = (a_i)_{i<\omega}$ of elements $a_i \in \Sigma$. The set of all ω-words is denoted by Σ^ω. We also set $\Sigma^\infty := \Sigma^* \cup \Sigma^\omega$.

(d) A *language* is a set of words or a set of ω-words.

Definition 1.3. Let Σ be an alphabet.

(a) We can associate with every word $w \in \Sigma^*$ a relational structure

$$\langle W, \leq, (P_a)_{a\in\Sigma}\rangle$$

over the signature $\{\leq\} \cup \{\, P_a \mid a \in \Sigma \,\}$ where W is the set of positions of w, $\leq$ is the ordering of the positions, and P_a is a set containing all positions labelled by the letter a. Structures of this form are called *word structures.*

(b) A language $K \subseteq \Sigma^\infty$ is *definable* in a logic L if there exists a formula $\varphi \in L$ such that

$$w \in K \quad \text{iff} \quad w \vDash \varphi\,.$$

(In the right-hand side, we have identified w with the associated word structure.)

Example. The language of all words over the alphabet $\Sigma := \{a, b, c\}$ with an even number of letters a can be defined by the MSO-formula

$$\begin{aligned}\varphi := \exists X\big[\forall x \forall y\, [x < y \wedge P_a x \wedge P_a y \wedge \neg\exists z(x < z \wedge z < y \wedge P_a z)\\ \rightarrow (Xx \leftrightarrow \neg Xy)]\\ \wedge\, \forall x[P_a x \wedge \neg\exists y(y < x \wedge P_a y) \rightarrow Xx]\\ \wedge\, \forall x[P_a x \wedge \neg\exists y(x < y \wedge P_a y) \rightarrow \neg Xy]\big]\,.\end{aligned}$$

Definition 1.4. Let Σ be an alphabet.

(a) If $w \in \Sigma^\infty$ and $i \leq k < |w|$, we write $w(i)$ for the element of w at position i and

$$w[i, k) := \langle w(i), w(i+1), \dots, w(k-1) \rangle$$

for the *factor* of w from position i to $k - 1$.

(b) The *concatenation* of two words $u \in \Sigma^*$ and $v \in \Sigma^\infty$ is the word $u^\frown v$ that consists of the elements of u, followed by the elements of v. Formally,

$$(u^\frown v)(i) = \begin{cases} u(i) & \text{if } i < |u|, \\ v(i - |u|) & \text{if } i \geq |u|. \end{cases}$$

Frequently, we omit the symbol $^\frown$ and simply write uv instead.

(c) A word u is a *prefix* of a word $w \in \Sigma^\infty$ if $w = u^\frown v$, for some $v \in \Sigma^\infty$ Similarly, u is a *suffix* of w if $w = v^\frown u$, for some v. Finally, u is a *factor* of w if $w = x^\frown u^\frown y$, for some x, y.

2. *Semigroups and Green's relations*

Definition 2.1. (a) A *semigroup* is a structure $\mathfrak{S} = \langle S, \cdot \rangle$ where the multiplication $\cdot : S \times S \to S$ is associative:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \, .$$

As usual we omit the dot and simply write ab.

(b) A *monoid* $\mathfrak{M} = \langle M, \cdot, 1 \rangle$ is a semigroup with a neutral element 1, i.e., an element satisfying

$$1 \cdot a = a = a \cdot 1 \, , \quad \text{for all } a \in M \, .$$

To each semigroup $\mathfrak{S}$ we can associate a monoid $\mathfrak{S}^1$ by adding a new neutral element 1.

(c) An element e of a semigroup is *idempotent* if $ee = e$.

Example. Let Σ be an alphabet. Then $\langle \Sigma^+, \cdot \rangle$ and $\langle \Sigma^*, \cdot \rangle$ are semigroups and $\langle \Sigma^*, \cdot, \langle\rangle \rangle$ is a monoid.

Lemma 2.2. *Let $\mathfrak{S}$ be a finite semigroup. There exists a number $0 < n < \omega$ such that a^n is idempotent, for every $a \in S$.*

Proof. Let $m := |S|$ and set $n := m!$. Given $a \in S$, consider the sequence $a, a^2, a^3, \dots, a^m, a^{m+1}$. Since S has only m elements, there are numbers $i < k \leq m + 1$ such that $a^i = a^k$. Set $p := k - i$ and let l be the minimal number such that $lp \geq i$. Note that $lp < k$ since $lp \geq k$ would imply that $(l - 1)p \geq k - p \geq i$, which would contradict our choice of l. It follows that

$$\begin{aligned} a^{lp} a^{lp} = a^{lk+(2lp-lk)} &= (a^k)^l a^{2lp-lk} \\ &= (a^i)^l a^{2lp-lk} \\ &= a^{li+(2lp-lk)} \\ &= a^{2lp-l(k-i)} = a^{2lp-lp} = a^{lp} . \end{aligned}$$

Consequently, $(a^{lp})^x = a^{lp}$, for all x, and we have

$$a^n a^n = a^{2m!} = (a^{lp})^{2m!/lp} = a^{lp} = (a^{lp})^{m!/lp} = a^{m!} = a^n. \qquad \square$$

Definition 2.3. *Greene's relations* consist of the divisibility relations

$$\begin{aligned} a \leq_{\mathsf{L}} b \quad &:\text{iff} \quad a = xb \quad \text{for some } x \in S^1 , \\ a \leq_{\mathsf{R}} b \quad &:\text{iff} \quad a = bx \quad \text{for some } x \in S^1 , \\ a \leq_{\mathsf{J}} b \quad &:\text{iff} \quad a = xby \quad \text{for some } x, y \in S^1 , \\ a \leq_{\mathsf{H}} b \quad &:\text{iff} \quad a \leq_{\mathsf{L}} b \quad \text{and} \quad a \leq_{\mathsf{R}} b , \end{aligned}$$

together with the associated equivalence relations

$$\begin{aligned} a \equiv_{\mathsf{L}} b \quad &:\text{iff} \quad a \leq_{\mathsf{L}} b \quad \text{and} \quad b \leq_{\mathsf{L}} a , \\ a \equiv_{\mathsf{R}} b \quad &:\text{iff} \quad a \leq_{\mathsf{R}} b \quad \text{and} \quad b \leq_{\mathsf{R}} a , \\ a \equiv_{\mathsf{J}} b \quad &:\text{iff} \quad a \leq_{\mathsf{J}} b \quad \text{and} \quad b \leq_{\mathsf{J}} a , \\ a \equiv_{\mathsf{H}} b \quad &:\text{iff} \quad a \leq_{\mathsf{H}} b \quad \text{and} \quad b \leq_{\mathsf{H}} a . \end{aligned}$$

We call the equivalence classes of the relations $\equiv_{\mathsf{L}}$, $\equiv_{\mathsf{R}}$, $\equiv_{\mathsf{J}}$, $\equiv_{\mathsf{H}}$ L-*classes,* R-*classes,* J-*classes,* and H-*classes,* respectively.

Example. Let $M = \{1, a, b, ab, ba, aba, 0\}$ be the monoid with unit 1, zero 0 and relations

$$aa = 1\,, \qquad bab = 0\,, \qquad bb = 0\,.$$

The Greene's relations are depicted in the following schema:

$$\boxed{1, a}$$

$$\begin{array}{|c|c|}\hline b & ba \\ \hline ab & aba \\ \hline\end{array}$$

$$\boxed{0}$$

Here each field represents a single H-class, each group of fields a J-class, each column inside a group an L-class, and each row an R-class. So we have three J-classes

$$\{0\} \leq_{\mathsf{J}} \{b, ba, ab, aba\} \leq_{\mathsf{J}} \{1, a\}\,,$$

where the middle J-class is divided into two L-classes: $\{ba, aba\}$ and $\{b, ab\}$; and into two R-classes: $\{b, ba\}$ and $\{ab, aba\}$. The only non-trivial H-class is $\{1, a\}$.

Lemma 2.4. *Let $\mathfrak{S}$ be a semigroup and $a, b, c \in S$.*

(a) *$a \leq_{\mathsf{L}} b$ implies $ac \leq_{\mathsf{L}} bc$.*

(b) *$a \leq_{\mathsf{R}} b$ implies $ca \leq_{\mathsf{R}} cb$.*

(c) *$\leq_{\mathsf{L}} \circ \leq_{\mathsf{R}} \;=\; \leq_{\mathsf{R}} \circ \leq_{\mathsf{L}}$.*

Proof. (a) $a = xb$ implies $ac = xbc$.

(b) $a = bx$ implies $ca = cbx$.

(c) Suppose that $a \leq_{\mathsf{L}} b \leq_{\mathsf{R}} c$. Then there are elements $x, y \in S^1$ such that $a = xb$ and $b = cy$. Hence, $a = xcy \leq_{\mathsf{R}} xc \leq_{\mathsf{L}} c$. Therefore, $\leq_{\mathsf{L}} \circ \leq_{\mathsf{R}} \subseteq \leq_{\mathsf{R}} \circ \leq_{\mathsf{L}}$. The other inclusion follows in the same way. $\square$

Lemma 2.5. *Let $\mathfrak{S}$ be a finite semigroup and $a \in S$, $s, t, u, v \in S^1$. Then*

$$a = stauv \quad \textit{implies} \quad ta \equiv_{\mathsf{L}} a \equiv_{\mathsf{R}} au\,.$$

Proof. Iterating the equation $a = (st)a(uv)$, we obtain

$$a = (st)^n a(uv)^n\,, \quad \text{for all } n < \omega\,.$$

By Lemma 2.2, there is some $n < \omega$ such that $(st)^n$ and $(uv)^n$ are both idempotents. It follows that

$$(st)^n a = (st)^n (st)^n a(uv)^n = (st)^n a(uv)^n = a\,.$$

Similarly, we obtain $a(uv)^n = a$. Consequently, $a \leq_{\mathsf{L}} ta$ and $a \leq_{\mathsf{R}} au$. Since the converses hold trivially, it follows that $a \equiv_{\mathsf{L}} ta$ and $a \equiv_{\mathsf{R}} au$. □

Proposition 2.6. *Let $\mathfrak{S}$ be a finite semigroup. Then*

$$\leq_{\mathsf{J}} \;=\; \leq_{\mathsf{L}} \circ \leq_{\mathsf{R}} \quad \textit{and} \quad \equiv_{\mathsf{J}} \;=\; \equiv_{\mathsf{L}} \circ \equiv_{\mathsf{R}}\,.$$

Proof. We start by proving the first equation. Suppose that $a \leq_{\mathsf{J}} b$. Then there are elements $x, y \in S^1$ with $a = xby$. Hence, $a = xby \leq_{\mathsf{L}} by \leq_{\mathsf{R}} b$.

Conversely, suppose that $a \leq_{\mathsf{L}} c \leq_{\mathsf{R}} b$. Then there are elements $x, y \in S^1$ with $a = xc$ and $c = by$. Hence, $a = xby$, which implies that $a \leq_{\mathsf{J}} b$.

For the second equation, suppose that $a \equiv_{\mathsf{L}} c \equiv_{\mathsf{R}} b$. Then $a \leq_{\mathsf{L}} c \leq_{\mathsf{R}} b$ and $a \geq_{\mathsf{L}} c \geq_{\mathsf{R}} b$. By the first equation, it follows that $a \leq_{\mathsf{J}} b$ and $a \geq_{\mathsf{J}} b$. Hence, $a \equiv_{\mathsf{J}} b$.

Conversely, suppose that $a \equiv_{\mathsf{J}} b$. Then there are elements $x, y, s, t \in S^1$ such that $a = xby$ and $b = sat$. Hence, $a = xsaty$ and it follows by Lemma 2.5 that $sa \equiv_{\mathsf{L}} a \equiv_{\mathsf{R}} at$. Consequently, $b = sat \equiv_{\mathsf{L}} at \equiv_{\mathsf{R}} a$. □

Remark. Since L-equivalence and R-equivalence both imply J-equivalence, we can partition every J-class into L-classes and into R-classes. By the above lemma, these two partitions intersect such that every L-class meets every R-class. Hence, we always obtain a picture as in the above example.

Lemma 2.7. *Let $\mathfrak{S}$ be a finite semigroup and $a, b \in S$.*

(a) *$b \leq_{\mathsf{J}} ab$ implies $b \equiv_{\mathsf{L}} ab$.*

(b) *$a \leq_{\mathsf{J}} ab$ implies $a \equiv_{\mathsf{R}} ab$.*

Proof. (a) Suppose that $b \leq_{\mathsf{J}} ab$. Then there are $x, y \in S^1$ such that $b = xaby$. By Lemma 2.5, it follows that $ab \equiv_{\mathsf{L}} b \equiv_{\mathsf{R}} by$.

(b) follows in exactly the same way. □

Corollary 2.8. *Let $\mathfrak{S}$ be a finite semigroup and $a, b, c \in S$.*

$$ab = ac = a \quad \text{and} \quad a \equiv_{\mathsf{J}} b \quad \text{implies} \quad bc = b\,.$$

Proof. By Lemma 2.7 (a),

$$b \equiv_{\mathsf{J}} ab \quad \text{implies} \quad b \equiv_{\mathsf{L}} ab = a\,.$$

Hence, $b = xa$, for some $x \in S^1$. Consequently, $bc = xac = xa = b$. □

3. *Finite words*

Before considering infinite words, we start with finite ones. For these it is quite simple to characterise which languages are MSO-definable.

Definition 3.1. (a) A *nondeterministic automaton* $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ consists of a finite set Q of *states,* a finite *input alphabet* Σ, an *initial state* $q_0 \in Q$, a set $F \subseteq Q$ of *final states,* and a *transition relation* $\Delta \subseteq Q \times \Sigma \times Q$. Instead of $\langle p, a, q \rangle \in \Delta$, we also write $p \xrightarrow{a} q$.

(b) A *run* of an automaton $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ on an input word $w = a_0 \dots a_{n-1} \in \Sigma^*$ is a sequence $p_0, \dots, p_n$ of states starting with $p_0 = q_0$ such that

$$\langle p_i, a_i, p_{i+1} \rangle \in \Delta\,, \quad \text{for all } i < n\,.$$

A run $p_0, \dots, p_n$ is *accepting* if $p_n \in F$.

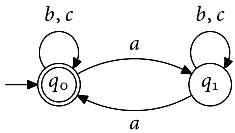
(c) An automaton $\mathcal{A}$ *accepts* an word w if there exists an accepting run of $\mathcal{A}$ on w. The *language recognised* by $\mathcal{A}$ is the set $L(\mathcal{A})$ of all ω-words it accepts.

Example. The language of all words over the alphabet $\Sigma := \{a, b, c\}$ with an even number of letters a is recognised by the automaton

$$\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$$

with states $Q = \{q_0, q_1\}$, final state $F = \{q_0\}$, and transitions

$$q_0 \xrightarrow{a} q_1, \quad q_1 \xrightarrow{a} q_0, \quad q_0 \xrightarrow{b,c} q_0, \quad q_1 \xrightarrow{b,c} q_1.$$



Definition 3.2. Let $L \subseteq \Sigma^*$.

(a) A homomorphism $\eta : \Sigma^+ \to \mathfrak{S}$ into a semigroup $\mathfrak{S}$ *recognises* L if

$$L \smallsetminus \{\langle\rangle\} = \eta^{-1}[P], \quad \text{for some } P \subseteq S.$$

(b) The *syntactic congruence* of L is the relation on Σ^* defined by

$$x \sim_L y \quad :\text{iff} \quad uxw \in L \Leftrightarrow uyw \in L \quad \text{for all } u, w \in \Sigma^*.$$

Example. Let L be the language of all words over the alphabet $\Sigma := \{a, b, c\}$ with an even number of letters a. L is recognised by the homomorphism $\eta : \Sigma^+ \to \mathbb{Z}/2\mathbb{Z}$ that maps a to 1 and b, c to 0. Its syntactic congruence is

$$\begin{aligned} x \sim_L y \quad &\text{iff} \quad \text{modulo } 2,\ x \text{ and } y \text{ have the same number of letters } a \\ &\text{iff} \quad \eta(x) = \eta(y). \end{aligned}$$

Lemma 3.3. *The syntactic congruence $\sim_L$ is a congruence relation.*

Proof. $\sim_L$ is obviously reflexive and symmetric. For transitivity, suppose that $x \sim_L y \sim_L z$. To show that $x \sim_L z$, let $u, w \in \Sigma^*$. Then

$$uxw \in L \Leftrightarrow uyw \in L \Leftrightarrow uzw \in L\,.$$

Finally, suppose that $x \sim_L x'$ and $y \sim_L y'$. To show that $xy \sim_L x'y'$, consider words $u, w \in \Sigma^*$. Then

$$uxyw \in L \Leftrightarrow ux'yw \in L \Leftrightarrow ux'y'w \in L\,. \qquad \square$$

Theorem 3.4. *Let $L \subseteq \Sigma^*$. The following statements are equivalent:*

(1) *L is definable in* MSO.

(2) *L is recognised by a homomorphism to a finite semigroup.*

(3) *L is recognised by an automaton.*

(4) *The syntactic congruence of L has finite index.*

Proof. (1) ⇒ (4) Suppose that there is an MSO-formula φ defining L. Set $m := \mathrm{qr}(\varphi)$. Since

$$v \equiv^m_{\mathrm{MSO}} v' \quad \text{and} \quad w \equiv^m_{\mathrm{MSO}} w' \quad \text{implies} \quad v^\frown w \equiv^m_{\mathrm{MSO}} v'^\frown w'\,,$$

for $v, v', w, w' \in \Sigma^+$, the relation $\equiv^m_{\mathrm{MSO}}$ is a congruence relation on Σ^+. Furthermore, if $x \equiv^m_{\mathrm{MSO}} y$ then $uxw \equiv^m_{\mathrm{MSO}} uyw$ implies that

$$uxw \in L \quad \text{iff} \quad uxw \vDash \varphi \quad \text{iff} \quad uyw \vDash \varphi \quad \text{iff} \quad uyw \in L\,.$$

Hence, $\equiv^m_{\mathrm{MSO}} \subseteq \sim_L$. We have seen in Proposition I.3.3 that there are only finitely many $\equiv^m_{\mathrm{MSO}}$-classes. As every $\equiv^m_{\mathrm{MSO}}$-class is contained in a $\sim_L$-class, it follows that $\sim_L$ also has only finitely many classes.

(4) ⇒ (2) If $\sim_L$ has only finitely many classes, the quotient $\Sigma^+/{\sim_L}$ is a finite semigroup. We claim that $L \setminus \{\langle\rangle\} = \pi^{-1}(P)$ where $\pi : \Sigma^+ \to \Sigma^+/{\sim_L}$

is the canonical projection mapping every word w to its $\sim_L$-class $[w]$ and

$$P := \{ [w] \mid w \in L \} .$$

Clearly, $w \in L$ implies $\pi(w) \in P$. Conversely, if $\pi(w) \in P$, there is some $v \sim_L w$ with $v \in L$. By definition of $\sim_L$, it follows that $w \in L$.

(2) $\Rightarrow$ (3) Suppose that $L \setminus \{\langle\rangle\} = \eta^{-1}(P)$, where $\eta : \Sigma^+ \to \mathfrak{S}$ is a homomorphism to a finite semigroup $\mathfrak{S}$ and $P \subseteq S$. We obtain an automaton $\mathcal{A}$ recognising L by setting $\mathcal{A} := \langle S^1, \Sigma, \Delta, 1, F\rangle$ where

$$F := \begin{cases} P & \text{if } \langle\rangle \notin L , \\ P \cup \{1\} & \text{if } \langle\rangle \in L , \end{cases}$$

and $\quad \Delta := \{ \langle s,\ a,\ s \cdot \eta(a)\rangle \mid s \in S ,\ a \in \Sigma \} .$

(3) $\Rightarrow$ (1) Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F\rangle$ be an automaton recognising L. We obtain a formula φ defining L as follows. φ guesses a run of $\mathcal{A}$ on the given word. It encodes this run by a tuple $(Z_q)_{q \in Q}$ of set variables, where Z_q contains all positions such that the automaton is in state q after having read that position. In the case where $\langle\rangle \notin L$, we set

$$\varphi := \exists (Z_q)_{q\in Q} [\mathrm{ADM} \wedge \mathrm{INIT} \wedge \mathrm{TRANS} \wedge \mathrm{ACC}]$$

where ADM states that every position is labelled by at most one state:

$$\mathrm{ADM} := \forall x \bigwedge_{p \neq q} \neg(Z_p x \wedge Z_q x) ,$$

INIT states that the first state is correct:

$$\mathrm{INIT} := \exists x \Big[\forall y (x \leq y) \wedge \bigvee_{\langle q_0, a, q\rangle \in \Delta} (Z_q x \wedge P_a x) \Big] ,$$

TRANS states that at every position a valid transition is used:

$$\mathrm{TRANS} := \forall x \forall y \Big[\mathrm{suc}(x, y) \to \bigvee_{\langle p, a, q\rangle \in \Delta} (Z_p x \wedge Z_q y \wedge P_a y) \Big] ,$$

and ACC states that the last state is final:

$$\mathrm{ACC} := \bigvee_{q \in F} \exists x [Z_q x \wedge \forall y (y \leq x)] .$$

If $\langle\rangle \in L$, we use the formula $\forall z \varphi$ instead, where φ is defined as above. □

Exercise 3.1. Give direct proofs for the following implications in the above theorem:

$$\begin{aligned} &(1) \Rightarrow (2), \quad (2) \Rightarrow (1), \quad (3) \Rightarrow (2), \\ &(3) \Rightarrow (4), \quad (4) \Rightarrow (1), \quad (4) \Rightarrow (3). \end{aligned}$$

Exercise 3.2. For each of the following languages over the alphabet $\Sigma := \{a, b, c\}$, find (i) an MSO-formula defining them, (ii) an automaton recognising them, and (iii) a homomorphism recognising them.

(a) The language of all words of the form $a^m b^n$ with $m, n < \omega$.

(b) The language of all words such that, after every letter a, there is some later position with a b.

(c) The language of all words such that between any two a there is at least one b.

(d) The language of all words with exactly 2 occurrences of the letter a.

(e) The language of all words of the from xay with $x, y \in \Sigma^*$ and $|y| = n$.

Exercise 3.3. Prove that the following languages over $\Sigma := \{a, b\}$ are not MSO-definable.

(a) The language of all words of the form $a^n b^n$ for $n < \omega$.

(b) The language of all words of the form ww for $w \in \Sigma^*$.

(c) The language of all words of length n^2 for $n < \omega$.

(d) The language of all words that contain more letters a than b.

(e) The language of all well-bracketed words where we consider a as an opening bracket and b as a closing one.

Exercise 3.4. Let L be an MSO-definable language over the alphabet Σ. Prove that there exists a constant $0 < n < \omega$ such that every word $w \in L$ of length $|w| \geq n$ has a factorisation $w = xyz$ such that

$$|xy| \leq n\,, \quad y \neq \langle\rangle\,, \quad \text{and} \quad xy^k z \in L \quad \text{for all } k < \omega\,.$$

4. Ramsey theory

Definition 4.1. Let A be a linear order.

(a) We denote by $[A]^2$ the set of all pairs $\langle i, k\rangle \in A^2$ with $i < k$.

(b) A *finite colouring* of A is a function $\lambda : [A]^2 \to C$ where C is a finite set of *colours*.

(c) Let $\mathfrak{S}$ be a finite semigroup. A finite colouring $\lambda : [A]^2 \to S$ is *additive* if

$$\lambda(x, y) \cdot \lambda(y, z) = \lambda(x, z)\,, \quad \text{for all } x < y < z\,.$$

Theorem 4.2 (Ramsey). *Let $\lambda : [\omega]^2 \to C$ be a finite colouring of ω. There exists an infinite subset $I \subseteq \omega$ such that*

$$\lambda(i, k) = \lambda(j, l)\,, \quad \textit{for all } i < k \textit{ and } j < l \textit{ in } I\,.$$

Proof. We construct an increasing sequence $n_0 < n_1 < \cdots$ of indices, a sequence $c_0, c_1, \ldots \in C$ of colours, and a decreasing sequence $J_0 \supseteq J_1 \supseteq \cdots$ of infinite sets such that, for every $i < \omega$,

$$n_i \in J_i \quad \text{and} \quad \lambda(n_i, k) = c_i\,, \quad \text{for all } k \in J_{i+1}\,.$$

We start with $n_0 := 0$ and $J_0 := \omega$. By induction, suppose that we have already defined n_i and J_i. For $c \in C$, set

$$L_c := \{\, k \in J_i \mid k > n_i \text{ and } \lambda(n_i, k) = c \,\}\,.$$

Then $J_i \setminus [n_i + 1] = \bigcup_{c \in C} L_c$. As J_i is infinite and C is finite, there is some element $c_i \in C$ such that L_{c_i} is infinite. We set

$$J_{i+1} := L_{c_i} \quad \text{and} \quad n_{i+1} := \min J_{i+1} \,.$$

Having defined $(n_i)_{i<\omega}$, $(c_i)_{i<\omega}$, and $(J_i)_{i<\omega}$, we consider the sets

$$M_c := \{\, i < \omega \mid c_i = c \,\} \,, \quad \text{for } c \in C \,.$$

Note that $n_j \in J_j \subseteq J_{i+1}$, for $j > i$, implies that

$$\lambda(n_i, n_j) = c \,, \quad \text{for all } i < j \text{ in } M_c \,.$$

Since $\bigcup_{c \in C} M_c = \omega$, there is some $c \in C$ such that M_c is infinite. We set $I := \{\, n_i \mid i \in M_c \,\}$. □

Exercise 4.1. (a) Let $\mathfrak{G} = \langle V, E \rangle$ be an infinite undirected graph. Prove that there exists an infinite set $X \subseteq V$ such that either all vertices in X are adjacent, or none of them are.

(b) Let $\mathfrak{G} = \langle V, E \rangle$ be an undirected graph with at least 6 vertices. Prove that there exists three vertices $x, y, z \in V$ that are either all connected by an edge, or none of them are.

Exercise 4.2. An *ω-semigroup* $\mathfrak{S}$ consists of two domains S and S_ω and three multiplication operations

$$\cdot : S \times S \to S \,, \quad \cdot : S \times S_\omega \to S_\omega \,, \quad \text{and} \quad \pi : S^\omega \to S_\omega$$

that satisfy the following associative laws:

$$\begin{aligned}
&(ab)c = a(bc) \,, \\
&(ab)u = a(bu) \,, \\
&b \cdot \pi(a_0, a_1, \dots) = \pi(b, a_0, a_1, \dots) \,, \\
&\pi(a_0, a_1, \dots) = \pi((a_0 \cdots a_{k_0 - 1}), (a_{k_0} \cdots a_{k_1 - 1}), \dots)
\end{aligned}$$

for all $a, b, c, a_0, a_1, \dots \in S$ and $u \in S_\omega$, and all increasing sequences $0 < k_0 < k_1 < \dots < \omega$.

Suppose that $\mathfrak{S}$ is a finite ω-semigroup. Prove that, for every sequence $a_0, a_1, \dots \in S$, there are two elements $s, e \in S$ such that

$$se = s\,, \quad ee = e\,, \quad \text{and} \quad \pi(a_0, a_1, a_2, \dots) = \pi(s, e, e, e, \dots)\,.$$

Exercise 4.3. Let $\mathfrak{S}$ be a finite semigroup, $\eta : \Sigma^+ \to \mathfrak{S}$ a homomorphism, and let $L \subseteq \Sigma^\omega$ be a language of the form

$$L = \bigcup_{i<n} \eta^{-1}(b_i)\big(\eta^{-1}(e_i)\big)^\omega\,, \quad \text{for } n < \omega \text{ and } b_i, e_i \in S\,,$$

where $X^\omega := \{\, x_0 x_1 x_2 \dots \mid x_i \in X \,\}$.

Prove that the complement $\Sigma^\omega \setminus L$ is also of the form

$$\Sigma^\omega \setminus L = \bigcup_{i<m} \eta^{-1}(c_i)\big(\eta^{-1}(f_i)\big)^\omega\,,$$

for suitable $m < \omega$ and $c_i, f_i \in S$.

Exercise 4.4. A *well-quasi-order* is a partial order $\langle A, \leq \rangle$ that does not contain any infinite descending sequence and any infinite antichain (i.e., a set of pairwise incomparable elements).

(a) Prove that every infinite partial order contains an infinite set that is either an ascending chain, a descending chain, or an antichain.

(b) Prove that $\langle A, \leq \rangle$ is a well-quasi-order if, and only if, for every infinite sequence $a_0, a_1, a_2, \dots$ in A there are indices $i < k$ with $a_i \leq a_k$.

(c) Let Σ be a finite alphabet. We define an ordering on Σ^* by setting $x \leq y$ if the word x can be obtained from y by deleting some letters. Prove that $\langle \Sigma^*, \leq \rangle$ is a well-quasi-order.

Hint. Assume that $\langle \Sigma^*, \leq \rangle$ is not a well-quasi-order and find words $w_0, w_1, \dots$ such that, for every $n < \omega$, the sequence $w_0, \dots, w_n$ can be continued to an infinite sequence violating the condition in (b).

For additive colourings, we can improve the Theorem of Ramsey. The starting point is the following problem: given an additive colouring $\lambda : [I]^2 \to S$ of a linear order $\mathfrak{I}$, we would like to find a colouring $\chi : I \to C$ of the elements of $\mathfrak{I}$ such that we can recover λ from χ. That is, we want to reduce a labelling of *pairs* to a labelling of *singletons.*

Definition 4.3. Let $\mathfrak{S}$ be a finite semigroup.

(a) A J-*chain* of $\mathfrak{S}$ is a tuple $\bar{a} = \langle a_0, \dots, a_m \rangle \in S^+$ such that

$$a_0 <_{\mathrm{J}} \cdots <_{\mathrm{J}} a_m \,.$$

We denote the set of all J-chains of $\mathfrak{S}$ by $\mathrm{Chain}_{\mathrm{J}}(\mathfrak{S})$.

(b) Let $\alpha \leq \omega$ and let $\lambda : [\alpha]^2 \to S$ be an additive colouring. A function $\chi : \alpha \to \mathrm{Chain}_{\mathrm{J}}(\mathfrak{S})$ is a J-*chain labelling* for λ if, for all $0 < i < \alpha$,

$$\chi(i-1) = \langle a_0, \dots, a_m \rangle$$

implies $\quad \chi(i) = \langle a_0, \dots, a_{k-1}, (a_k \cdots a_m b) \rangle \,,$

where $b := \lambda(i-1, i)$ and $0 \leq k \leq m+1$ is the maximal index such that the above tuple is a J-chain.

Example. Let $M = \{1, a, b, ab, ba, aba, 0\}$ be the monoid from the example on page 41 with three J-classes:

$$\{0\} \leq_{\mathrm{J}} \{b, ba, ab, aba\} \leq \{1, a\} \,,$$

For the colouring $\lambda : [10]^2 \to M$ with

$$0 \xrightarrow{b} 1 \xrightarrow{a} 2 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{b} 5 \xrightarrow{a} 6 \xrightarrow{a} 7 \xrightarrow{a} 8 \xrightarrow{b} 9 \,,$$

we obtain a J-chain labelling

$$\begin{vmatrix} a \end{vmatrix} \xrightarrow{b} \begin{vmatrix} ab \end{vmatrix} \xrightarrow{a} \begin{vmatrix} a \\ ab \end{vmatrix} \xrightarrow{a} \begin{vmatrix} 1 \\ ab \end{vmatrix} \xrightarrow{b} \begin{vmatrix} 0 \end{vmatrix} \xrightarrow{b} \begin{vmatrix} b \\ 0 \end{vmatrix} \xrightarrow{a}$$

$$\xrightarrow{a} \begin{vmatrix} a \\ b \\ 0 \end{vmatrix} \xrightarrow{a} \begin{vmatrix} 1 \\ b \\ 0 \end{vmatrix} \xrightarrow{a} \begin{vmatrix} a \\ b \\ 0 \end{vmatrix} \xrightarrow{b} \begin{vmatrix} 0 \end{vmatrix}$$

where we have written a J-chain $\langle a_0, \dots, a_{m-1} \rangle$ as a column

$$\begin{vmatrix} a_{m-1} \\ \vdots \\ a_0 \end{vmatrix} .$$

We will prove that we can recover λ from a J-chain labelling.

Definition 4.4. Let $\chi : \alpha \to \mathrm{Chain}_{\mathrm{J}}(\mathfrak{S})$ be a J-chain labelling for $\lambda : [\alpha]^2 \to S$.

(a) Let $\mu : S^+ \to S$ be the function mapping a tuple $\langle a_0, \dots, a_{m-1} \rangle$ to its last element a_{m-1}, and let $\pi : S^* \to S^1$ be the function mapping a tuple $\langle a_0, \dots, a_{m-1} \rangle$ to the product $a_0 \cdots a_{m-1}$ of its components.

(b) For positions $i, k < \alpha$, we define

$$i \sqsubset^{\circ}_{\chi} k \quad \text{:iff} \quad i < k \quad \text{and} \quad \mu\chi(j) \not<_{\mathrm{J}} \mu\chi(i)\,, \quad \text{for all } i \leq j \leq k\,,$$

$$i \sqsubset_{\chi} k \quad \text{:iff} \quad i \sqsubset^{\circ}_{\chi} k \quad \text{and} \quad \mu\chi(i) = \mu\chi(k)\,.$$

Lemma 4.5. *Let $\alpha \leq \omega$, let $\chi : \alpha \to \mathrm{Chain}_{\mathrm{J}}(\mathfrak{S})$ be a J-chain labelling for $\lambda : [\alpha^2] \to S$, and let $i \leq k < \alpha$ be positions with*

$$i \sqsubseteq^{\circ}_{\chi} k\,, \quad \chi(i) = sa\,, \quad \textit{and} \quad \chi(k) = t\,, \quad \textit{for } s, t \in S^*,\ a \in S\,.$$

(a) *There exist $b \in S$ and $x \in S^*$ such that*

$$t = s^\frown b^\frown x\,, \quad b \equiv_{\mathrm{J}} a\,, \quad \textit{and} \quad \pi(b^\frown x) = a \cdot \lambda(i, k)\,.$$

(b) *If $t = s^\frown b^\frown x$ with $x \neq \langle\rangle$, there is some position $i \leq j < k$ such that $\lambda(j, k) = \pi(x)$.*

(c) *If $t = s^\frown b$ and $i < k$, then $a \equiv_{\mathrm{J}} \lambda(i, k)$.*

Proof. (a) We prove the statement by induction on the number of positions between i and k. If $i = k$, then

$$t = \chi(k) = \chi(i) = s^\frown a\,,$$

and we can set $b := a$ and $x := \langle\rangle$.

For the inductive step, let k' be the immediate predecessor of k and assume that $i \leq k'$. By inductive hypothesis, it follows that

$$\chi(k') = s^\frown b^\frown x\,,$$

for some $b \in S$ and $x \in S^*$ such that

$$b \equiv_{\mathrm{J}} a \quad \text{and} \quad \pi(b^\frown x) = a \cdot \lambda(i, k')\,.$$

By definition of χ, there is a factorisation $u^\frown v$ of $s^\frown b^\frown x$ such that

$$t = u^\frown \pi(v^\frown c)\,, \quad \text{where} \quad c := \lambda(k', k)\,.$$

We claim that s is a prefix of u. For a contradiction, suppose otherwise. Then $s = u^\frown d^\frown y$ and $v = d^\frown y^\frown b^\frown x$ where $y \in S^*$ and $d \in S$ is the first element of v. Setting $d' := \pi(v^\frown c)$ we obtain

$$d' = \pi(d^\frown y^\frown b^\frown x^\frown c) \leq_{\mathrm{J}} d\,.$$

Since $s^\frown b^\frown x = u^\frown d^\frown y^\frown b^\frown x$ is a valid configuration, it follows that $d <_{\mathrm{J}} b$. Consequently,

$$\mu\chi(k') = d' \leq_{\mathrm{J}} d <_{\mathrm{J}} b \equiv_{\mathrm{J}} a\,.$$

A contradiction.

We have shown that t is of the form

$$t = s^\frown z^\frown d' \quad \text{where} \quad d' := \pi(v^\frown c) \quad \text{and} \quad z^\frown v = b^\frown x\,.$$

By definition of χ and by inductive hypothesis, it further follows that

$$\begin{aligned} a \cdot \lambda(i, k) &= a \cdot \lambda(i, k') \cdot \lambda(k', k) \\ &= \pi(b^\frown x) \cdot c = \pi(z^\frown v^\frown c) = \pi(z) \cdot \pi(v^\frown c) = \pi(z^\frown d')\,. \end{aligned}$$

It therefore remains to prove that the first element of $z^\frown d'$ is J-equivalent to a. If $z \neq \langle\rangle$, then $z = b^\frown z'$, for some $z' \in S^*$. Hence, $t = s^\frown b^\frown z'^\frown d'$ where $b \equiv_{\mathrm{J}} a$. If $z = \langle\rangle$, then $t = s^\frown d'$ where

$$d' = \pi(v^\frown c) = \pi(z^\frown v^\frown c) = \pi(b^\frown x^\frown c) \leq_{\mathrm{J}} b \equiv_{\mathrm{J}} a\,.$$

Since $d' = \mu\chi(k') \nless_{\mathrm{J}} a$, it follows that $d' \equiv_{\mathrm{J}} a$.

(b), (c) We prove both statements by induction on the number of positions between i and k If $i = k$, then $b = a$, $x = \langle\rangle$, and (b) and (c) hold trivially.

For the inductive step, let k' be the immediate predecessor of k and suppose that $i \leq k'$. By (a), it follows that $\chi(k') = s^\frown b'^\frown x'$ where

$$b' \equiv_{\mathrm{J}} a \equiv_{\mathrm{J}} b \quad \text{and} \quad \pi(b^\frown x) = a \cdot \lambda(i,k)\,.$$

Set $c := \lambda(k',k)$. The definition of χ implies that either

$$x = \langle\rangle \quad \text{and} \quad b = \pi(b'^\frown x'^\frown c)\,,$$

$$\text{or} \qquad b = b'\,, \quad x' = y^\frown z\,, \quad \text{and} \quad x = y^\frown \pi(z^\frown c)\,, \quad \text{for some } y,z \in S^*.$$

To prove (b), suppose that $x \neq \langle\rangle$. Then $\pi(x) = \pi(y^\frown z^\frown c) = \pi(x'^\frown c)$. Thus, it is sufficient to find a position $i \leq j < k$ such that $\lambda(j,k) = \pi(x'^\frown c)$. If $x' = \langle\rangle$, we can take $j := k'$. If $x' \neq \langle\rangle$, we can use the inductive hypothesis to obtain a position $i \leq j < k'$ with $\lambda(j,k') = \pi(x')$. Then $\lambda(j,k) = \lambda(j,k') \cdot c = \pi(x'^\frown c)$.

To prove (c), suppose that $x = \langle\rangle$. Set $d' := \pi(x'^\frown c)$. Then $b = b' \cdot d'$ and, by definition of χ, the sequence $s^\frown b'^\frown d'$ is not a J-chain, while $s^\frown b$ is one. Furthermore, we have $b' \equiv_{\mathrm{J}} a \equiv_{\mathrm{J}} b = b'd'$, which implies that $b' \leq_{\mathrm{J}} d'$. Hence, the only possible reason for $s^\frown b'^\frown d'$ not being a J-chain is that $b' \nless_{\mathrm{J}} d'$. Consequently, $d' \equiv_{\mathrm{J}} b' \equiv_{\mathrm{J}} a$. We choose a position $i \leq j < k$ as follows. If $x' = \langle\rangle$, we set $j := k'$. If $x' \neq \langle\rangle$, we use (b) to choose j such that $\lambda(j,k') = \pi(x')$. In both cases it follows that

$$\lambda(i,k) \leq_{\mathrm{J}} \lambda(j,k) = \pi(x'^\frown c) = d' \equiv_{\mathrm{J}} a\,.$$

Moreover, we have seen above that $b = \pi(b^\frown x) = a \cdot \lambda(i,k)$. Therefore, $\lambda(i,k) \geq_{\mathrm{J}} b \equiv_{\mathrm{J}} a$ and it follows that $a \equiv_{\mathrm{J}} \lambda(i,k)$. ☐

Corollary 4.6. *Let $\alpha \leq \omega$ and let $\chi : \alpha \to \mathrm{Chain}_{\mathrm{J}}(\mathfrak{S})$ be a* J*-chain labelling for $\lambda : [\alpha^2] \to S$. If $i \sqsubset_\chi k$, then*

$$\mu\chi(i) \cdot \lambda(i,k) = \mu\chi(i) \quad \textit{and} \quad \mu\chi(i) \equiv_{\mathrm{J}} \lambda(i,k)\,.$$

Proof. By Lemma 4.5 (a), we have

$$\chi(i) = s^\frown a \quad \text{and} \quad \chi(k) = s^\frown b^\frown x\,,$$

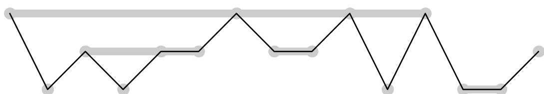
for some $s, x \in S^*$ and $a, b \in S$ such that $b \equiv_{\mathrm{J}} a$ and $\pi(b^\frown x) = a \cdot \lambda(i,k)$. Note that, if $x \neq \langle\rangle$, then $x = y^\frown a$, for some $y \in S^*$, and $b \not<_{\mathrm{J}} a$ implies that $t^\frown a = s^\frown b^\frown y^\frown a$ is not a J-chain. Hence, $x = \langle\rangle$. This implies that

$$t = s\,, \quad b = a\,, \quad \text{and} \quad a \cdot \lambda(i,k) = \pi(b^\frown x) = b = a\,.$$

Furthermore, it follows by Lemma 4.5 (c) that $a \equiv_{\mathrm{J}} \lambda(i,k)$. □

Definition 4.7. Let A and B be linear orders.
(a) Given a function $\sigma : A \to B$, we define

$$\begin{aligned} x \approx^\circ_\sigma y \quad &:\text{iff} \quad \sigma(z) \leq \sigma(x) \text{ for all } z \text{ such that} \\ &\qquad\quad x \leq z \leq y \text{ or } y \leq z \leq x\,, \\ x \approx_\sigma y \quad &:\text{iff} \quad x \approx^\circ_\sigma y \text{ and } \sigma(x) = \sigma(y)\,. \end{aligned}$$



(b) Let $\lambda : [A]^2 \to S$ be an additive colouring. A function $\sigma : A \to B$ is a *weak Ramseyan split* of λ if

$$\lambda(x,y) = \lambda(x,y) \cdot \lambda(x',y')\,, \quad \text{for all } x < y \text{ and } x' < y' \text{ in } A \text{ such that } x \approx_\sigma y \approx_\sigma x' \approx_\sigma y'\,.$$

Lemma 4.8. *Let $\mathfrak{S}$ be a finite semigroup of size $N := |S|$ and $\chi : \alpha \to \mathrm{Chain}_{\mathrm{J}}(\mathfrak{S})$ a J-chain labelling for an additive colouring $\lambda : [\alpha]^2 \to S$. If $v : S \to [N]$ is a bijection such that*

$$a <_{\mathrm{J}} b \quad \textit{implies} \quad v(a) > v(b)\,,$$

then $\sigma := v \circ \mu \circ \chi$ is a weak Ramseyan split for λ.

Proof. Consider positions $x < y$ and $x' < y'$ with $x \approx_\sigma y \approx_\sigma x' \approx_\sigma y'$. Then

$$a := \mu\chi(x) = \mu\chi(y) = \mu\chi(x') = \mu\chi(y')$$

and $\quad \mu\chi(z) \nleq_{\mathrm{J}} a\,, \quad$ for all z between any two of x, y, x', y'.

Consequently, $x \sqsubset_\chi y$ and $x' \sqsubset_\chi y'$. By Corollary 4.6 it follows that

$$\begin{aligned} a \cdot \lambda(x, y) &= a \quad \text{and} \quad a \equiv_{\mathrm{J}} \lambda(x, y)\,, \\ a \cdot \lambda(x', y') &= a \quad \text{and} \quad a \equiv_{\mathrm{J}} \lambda(x', y')\,. \end{aligned}$$

Applying Corollary 2.8 to the values $b := \lambda(x, y)$ and $c := \lambda(x', y')$, we obtain

$$\lambda(x, y) \cdot \lambda(x', y') = b \cdot c = b = \lambda(x, y)\,.$$ □

We can compute weak Ramseyan splits by an automaton.

Definition 4.9. (a) A *deterministic finite-state transducer*

$$\mathcal{T} = \langle Q, \Sigma, \Gamma, q_0, \delta, \eta \rangle$$

consists of a finite set Q of *states,* an *input alphabet* Σ, an *output alphabet* Γ, an *initial state* $q_0 \in Q$, an *output function* $\eta : Q \to \Gamma$, and a *transition function* $\delta : Q \times \Sigma \to Q$.

Let $\mathcal{T} = \langle Q, \Sigma, \Gamma, q_0, \delta, \eta \rangle$ be a transducer. The *run* of $\mathcal{T}$ on a word $w = (c_i)_{i<\alpha} \in \Sigma^\infty$ is the sequence $(q_i)_{i<\beta}$ of states where q_0 is the initial state and

$$q_{i+1} := \delta(q_i, c_i)\,, \quad \text{for all } i\,, \quad \text{and} \quad \beta := \begin{cases} \alpha + 1 & \text{if } \alpha < \omega\,, \\ \omega & \text{if } \alpha = \omega\,. \end{cases}$$

Every transducer $\mathcal{T}$ defines a function $\mathcal{T} : \Sigma^\infty \to \Gamma^\infty$ that maps an input word $w \in \Sigma^\infty$ to the word

$$\mathcal{T}(w) = (\eta(q_i))_{i<\beta}\,, \quad \text{where } (q_i)_{i<\beta} \text{ is the run of } \mathcal{T} \text{ on } w.$$

(b) Let $\mathfrak{S}$ a finite semigroup and $N < \omega$ a natural number. We say that a transducer $\mathcal{T} = \langle Q, S, [N], q_0, \delta, \eta\rangle$ *computes weak Ramseyan splits for* $\mathfrak{S}$ if, for every additive colouring $\lambda : [\alpha]^2 \to S$ with $\alpha \leq \omega$, the function $\sigma : \alpha \to [N]$ defined by

$$\sigma = \mathcal{T}(w) \quad \text{where} \quad w := (\lambda(i, i+1))_{i+1<\alpha}\,,$$

is a weak Ramseyan split of λ.

Theorem 4.10 (Colcombet)**.** *Given a finite semigroup $\mathfrak{S}$ of size $N := |S|$, we can effectively construct a deterministic finite-state transducer*

$$\mathcal{T} = \langle Q, S, [N], q_0, \delta, \eta\rangle$$

that computes weak Ramseyan splits for $\mathfrak{S}$.

Proof. We use the set $Q := \mathrm{Chain}_{\mathrm{J}}(\mathfrak{S})$ of all J-chains as states of the transducer. Note that this set is finite, since there are at most $|S|$ J-classes. The initial state q_0 is an arbitrary J-chain. We define the transition function δ such that the run of $\mathcal{T}$ on a given input λ is a J-chain labelling χ for λ. This can be done, since the value of $\chi(i+1)$ can be computed from $\chi(i)$. Fixing a bijection $v : S \to [N]$ as in Lemma 4.8, we can define the output function $\eta : Q \to [N]$ by $\eta(q) := v(\mu(q))$. ☐

Exercise 4.5. (a) Let Σ be a finite alphabet, $w \in \Sigma^\omega$, and $k < \omega$. Prove that there are sets $Q_0, \dots, Q_{n-1} \subseteq \omega$ such that, for every MSO-theory θ of quantifier rank k, there exists an FO-formula $\varphi_\theta(x, y)$ such that

$$\langle \omega, \bar{Q}\rangle \vDash \varphi_\theta(x, y) \quad \text{iff} \quad \mathrm{Th}^k_{\mathrm{MSO}}(w[x, y)) = \theta\,.$$

(b) Let τ be an MSO-interpretation τ and $P_0, \dots, P_{m-1} \subseteq \omega$ monadic parameters. Prove that there exist an FO-interpretation σ and sets $Q_0, \dots, Q_{n-1} \subseteq \omega$ such that

$$\tau(\langle \omega, \bar{P}\rangle) = \sigma(\langle \omega, \bar{Q}\rangle)\,.$$

5. The theory of ω

In this section we prove that the monadic second-order theory of $\langle \omega, \leq \rangle$ is decidable.

Definition 5.1. For $\bar{n} \in \omega^*$ and $m < \omega$, set

$$\Theta_{\bar{n}}(m) := \wp(\mathrm{MSO}_0^{\bar{n}}[\leq, P_0, \dots, P_{m-1}])\,,$$

and $\quad \Phi_{\bar{n}}(m) := \left\{ \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}) \;\middle|\; \mathfrak{A} = \langle A, \leq, P_0, \dots, P_{m-1} \rangle \text{ a finite linear order} \right\}.$

As a first step, we show how to compute the sets $\Phi_{\bar{n}}(m)$.

Lemma 5.2. *We can equip $\Theta_{\bar{n}}(m)$ with two operations $\cdot$ and $^{\omega}$ such that*

$$\mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A} + \mathfrak{B}) = \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}) \cdot \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{B})$$

and $\quad \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}\big(\sum_{i<\omega} \mathfrak{A}\big) = \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A})^{\omega}\,,$

for all linear orders $\mathfrak{A}$ and $\mathfrak{B}$.

Proof. This follows immediately from Lemma I.4.13. □

Proposition 5.3. *Given $\bar{n} \in \omega^*$ and $m < \omega$, we can compute $\Phi_{\bar{n}}(m)$.*

Proof. The set

$$\Psi := \left\{ \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}) \;\middle|\; \mathfrak{A} = \langle A, \leq, \bar{P} \rangle \text{ a linear order with } |A| \leq 1 \right\}$$

is a finite subset of $\Theta_{\bar{n}}(m)$ which we can compute from $\bar{n}$ and m. As every finite linear order can be written as a finite ordered sum of one-element linear orders, it follows that $\Phi_{\bar{n}}(m)$ is the subsemigroup of $\langle \Theta_{\bar{n}}(m), \cdot \rangle$ generated by the set Ψ. Hence, we can compute $\Phi_{\bar{n}}(m)$ from $\Theta_{\bar{n}}(m)$ and Ψ. □

Lemma 5.4. *Let $\bar{n} \in \omega^*$ and $m < \omega$. There exists a tuple $\bar{r} \in \omega^*$ of length $|\bar{r}| = |\bar{n}|$ such that, for every $\sigma \in \Phi_{\bar{n}}(m)$, we can compute $\sigma^{\omega} \in \Theta_{\bar{n}}(m)$ from σ and $\mathrm{Th}^{\bar{r}}_{\mathrm{MSO}_0}(\omega, \leq)$.*

Proof. Note that

$$\sigma^{\omega} = \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}\big(\textstyle\sum_{i<\omega} \mathfrak{A}\big)\,,$$

where $\mathfrak{A}$ is any finite linear order with $\mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}) = \sigma$. Let $\tau \in \Theta_{\bar{n}}(m)$ and let ϑ_τ be an $\mathrm{MSO}^{\bar{n}}_0$-formula equivalent to τ. It follows that

$$\sigma^{\omega} = \tau \quad \text{iff} \quad \sum_{i<\omega} \mathfrak{A} \vDash \vartheta_\tau\,.$$

According to Theorem I.4.12, we can compute formulae $\vartheta'_\tau(\bar{Z}) \in \mathrm{MSO}^{\bar{r}}_0$ and $\chi_0, \dots, \chi_{l-1} \in \mathrm{MSO}^{\bar{n}}_0$ such that

$$\sum_{i<\omega} \mathfrak{A} \vDash \vartheta_\tau \quad \text{iff} \quad \langle \omega, \leq \rangle \vDash \vartheta'_\tau\big(\llbracket \chi_0 \rrbracket, \dots, \llbracket \chi_{l-1} \rrbracket\big)\,.$$

As all terms in the sum above are equal, we have

$$\llbracket \chi_i \rrbracket = \begin{cases} \omega & \text{if } \chi_i \in \sigma\,, \\ \varnothing & \text{if } \chi_i \notin \sigma\,. \end{cases}$$

Let ϑ''_τ be the formula obtained from $\vartheta'_\tau(\bar{Z})$ by replacing every variable Z_i by

$$\begin{cases} \mathit{true} & \text{if } \chi_i \in \sigma\,, \\ \mathit{false} & \text{if } \chi_i \notin \sigma\,. \end{cases}$$

Then it follows that

$$\sigma^{\omega} = \tau \quad \text{iff} \quad \langle \omega, \leq \rangle \vDash \vartheta''_\tau \quad \text{iff} \quad \vartheta''_\tau \in \mathrm{Th}^{\bar{r}}_{\mathrm{MSO}_0}(\omega, \leq)\,. \qquad \square$$

The key argument in our decidability proof below is the following lemma, which states that every labelling of ω is equivalent to an ultimately periodic one.

Lemma 5.5. *Let $\bar{n} \in \omega^*$ and $m < \omega$. Then*

$$\big\{ \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\omega, \leq, \bar{P}) \;\big|\; P_0, \dots, P_{m-1} \subseteq \omega \big\} = \big\{ \sigma\tau^{\omega} \;\big|\; \sigma, \tau \in \Phi_{\bar{n}}(m) \big\}\,.$$

Proof. ($\supseteq$) Given $\sigma, \tau \in \Phi_{\bar{n}}(m)$, fix finite linear orders $\mathfrak{A}$ and $\mathfrak{B}$ with theories σ and τ, respectively. Then

$$\mathfrak{A} + \sum_{i<\omega} \mathfrak{B} \cong \langle \omega, \leq, \bar{P} \rangle, \quad \text{for some } P_0, \dots, P_{m-1} \subseteq \omega .$$

Consequently,

$$\sigma\tau^\omega = \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}) \cdot \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{B})^\omega = \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\omega, \leq, \bar{P}) .$$

($\subseteq$) Let $P_0, \dots, P_{m-1} \subseteq \omega$. For $i < k < \omega$, we define

$$\mathfrak{A}_{i,k} := \big\langle \{i, \dots, k-1\}, \leq, \bar{P} \restriction \{i, \dots, k-1\} \big\rangle .$$

By the Theorem of Ramsey, there exist a theory $\tau \in \Phi_{\bar{n}}(m)$ and an infinite sequence $k_0 < k_1 < \cdots$ of positions such that

$$\mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}_{k_i,k_j}) = \tau , \quad \text{for all } i < j < \omega .$$

Setting $\sigma := \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\mathfrak{A}_{0,k_0})$, it follows that

$$\mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\omega, \leq, \bar{P}) = \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}\Big(\mathfrak{A}_{0,k_0} + \sum_{i<\omega} \mathfrak{A}_{k_i,k_{i+1}}\Big) = \sigma\tau^\omega . \qquad \square$$

Theorem 5.6 (Büchi). $\mathrm{Th}_{\mathrm{MSO}}(\omega, \leq)$ *is decidable.*

Proof. We prove by induction on $|\bar{n}|$ that, given $\bar{n} \in \omega^*$, we can compute $\mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\omega, \leq)$. For $\bar{n} = \langle\rangle$, we have

$$\mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(\omega, \leq) = \varnothing .$$

Hence, suppose that $\bar{n} = m\bar{n}'$ and that we already know how to compute $\mathrm{Th}^{\bar{r}}_{\mathrm{MSO}_0}(\omega, \leq)$, for all $\bar{r} \in \omega^*$ with $|\bar{r}| = |\bar{n}'|$.

To compute $\mathrm{Th}^{m\bar{n}'}_{\mathrm{MSO}_0}(\omega, \leq)$ it is sufficient to decide whether or not

$$\langle \omega, \leq \rangle \vDash \exists X_0 \cdots \exists X_{m-1} \psi ,$$

for all $\mathrm{MSO}_0^{\bar n}$-formulae $\psi(\bar X)$. Hence, given an $\mathrm{MSO}_0^{\bar n}$-formula $\psi(\bar X)$, we have to decide whether there are sets $P_0, \dots, P_{m-1} \subseteq \omega$ such that

$$\langle \omega, \leq \rangle \vDash \psi(\bar P)\,.$$

By Lemma 5.5, this is equivalent to the question of whether there are theories $\sigma, \tau \in \Phi_{\bar n}(m)$ such that

$$\psi(\bar X) \in \sigma\tau^{\omega}.$$

Therefore, it is sufficient to compute $\sigma\tau^{\omega}$, for all of the finitely many possible choices of σ and τ. This we can do with the help of Proposition 5.3 and Lemma 5.4 since, by inductive hypothesis, we can compute $\mathrm{Th}^{\bar r}_{\mathrm{MSO}_0}(\omega, \leq)$, for all $\bar r \in \omega^*$ with $|\bar r| = |\bar n'|$. ☐

6. ω-semigroups

Definition 6.1. (a) An *ω-semigroup* is a structure $\mathfrak{S} = \langle S, S_\omega \rangle$ with three products

$$\cdot : S \times S \to S\,, \quad \cdot : S \times S_\omega \to S_\omega\,, \quad \text{and} \quad \pi : S^\omega \to S_\omega$$

that satisfy the following associative laws:

$$\begin{aligned}
&(ab)c = a(bc)\,,\\
&(ab)u = a(bu)\,,\\
&b \cdot \pi(a_0, a_1, \dots) = \pi(b, a_0, a_1, \dots)\,,\\
&\pi(a_0, a_1, \dots) = \pi((a_0 \cdots a_{k_0 - 1}), (a_{k_0} \cdots a_{k_1 - 1}), \dots)
\end{aligned}$$

for all $a, b, c, a_0, a_1, \dots \in S$ and $u \in S_\omega$, and all increasing sequences $0 < k_0 < k_1 < \dots < \omega$.

(b) The *ω-power* of an element $a \in S$ is

$$a^\omega := \pi(a, a, a, \dots)\,.$$

(c) A *homomorphism* $\eta : \mathfrak{S} \to \mathfrak{T}$ of ω-semigroups consists of two maps

$$\eta_0 : S \to T \quad \text{and} \quad \eta_\omega : S_\omega \to T_\omega$$

that commute with products, i.e., for $a, b, a_0, a_1, \dots \in S$ and $u \in S_\omega$,

$$\begin{aligned}
&\eta_0(a) \cdot \eta_0(b) = \eta_0(ab)\,, \\
&\eta_0(a) \cdot \eta_\omega(u) = \eta_\omega(au)\,, \\
&\pi(\eta_0(a_0), \eta_0(a_1), \dots) = \eta_\omega(\pi(a_0, a_1, \dots))\,.
\end{aligned}$$

Definition 6.2. Let Σ be a set.

(a) The *free ω-semigroup* over Σ is $\langle \Sigma^+, \Sigma^\omega \rangle$. By abuse of notation we also denote it simply by Σ^∞.

(b) A language $L \subseteq \Sigma^\omega$ is *recognised* by a homomorphism $\eta : \Sigma^\infty \to \mathfrak{S}$ to an ω-semigroup $\mathfrak{S}$ if there exists a set $P \subseteq S$ such that $L = \eta^{-1}[P]$.

Example. Let $\mathfrak{S} = \langle S, S_\omega \rangle$ be the ω-semigroup with $S := \{0, 1\}$ and $S_\omega := \{0, 1\}$ where

$$\begin{aligned}
a \cdot b &:= \max\{a, b\}\,, && \text{for } a, b \in S\,, \\
a \cdot u &:= u\,, && \text{for } a \in S\,,\ u \in S_\omega\,, \\
\pi(a_0, a_1, \dots) &:= \limsup_{n \to \infty} a_n\,, && \text{for } a_0, a_1, \dots \in S\,.
\end{aligned}$$

The language L of all ω-words w containing infinitely many letters a is recognised by the morphism sending the letter a to 1 and every other letter to 0.

Example. In the previous section we have already introduced the ω-semigroup $\langle \Theta_{\bar{n}}(m), \Theta_{\bar{n}}(m) \rangle$ of all $\mathrm{MSO}_0^{\bar{n}}$-theories over the signature $\{\leq, P_0, \dots, P_{m-1}\}$. Note that the function $\eta : \Sigma^\infty \to \Theta_{\bar{n}}(m)$ mapping a word $u \in \Sigma^\infty$ to its theory, is a homomorphism.

If $L \subseteq \Sigma^\omega$ is a language defined by an $\mathrm{MSO}_0^{\bar{n}}$-formula φ, then

$$L = \left\{ w \in \Sigma^\omega \;\middle|\; \varphi \in \mathrm{Th}^{\bar{n}}_{\mathrm{MSO}_0}(w) \right\} = \eta^{-1}[P]$$

where $P := \{\, \theta \in \Theta_{\bar{n}}(m) \mid \varphi \in \theta \,\}$. Thus, every definable language is recognised by η.

Exercise 6.1. Find homomorphisms into finite ω-semigroups that recognise the following languages over the alphabet $\{a, b, c\}$.

(a) The language of all ω-words containing infinitely many a, but only finitely many b.

(b) The language of all ω-words where immediately after or immediately before every letter a there is another a.

(c) The language of all ω-words containing an even (and finite) number of a.

(d) The language of all ω-words where after every letter a there is a later position with a letter b.

(e) The language of all ω-words where, for every prefix p, the numbers of the letters a, b, c differ by at most 1.

Exercise 6.2. Prove that a language $L \subseteq \Sigma^\omega$ is recognised by homomorphisms into a finite ω-semigroup if, and only if, it is of the form

$$L = \bigcup_{i<m} U_i V_i^\omega \,,$$

where $m < \omega$, $U_i, V_i \subseteq \Sigma^+$ are MSO-definable languages of finite words, and $V^\omega := \{\, v_0 v_1 v_2 \dots \mid v_i \in V \,\}$.

Exercise 6.3. Let $\mathfrak{S}$ be a finite semigroup, $\eta : \Sigma^+ \to \mathfrak{S}$ a homomorphism, and let $L \subseteq \Sigma^\omega$ be a language of the form

$$L = \bigcup_{i<n} \eta^{-1}(b_i)\big(\eta^{-1}(e_i)\big)^\omega \,, \quad \text{for } n < \omega \text{ and } b_i, e_i \in S \,,$$

where $X^\omega := \{\, x_0 x_1 x_2 \dots \mid x_i \in X \,\}$.

Prove that the complement $\Sigma^\omega \setminus L$ is also of the form

$$\Sigma^\omega \setminus L = \bigcup_{i<m} \eta^{-1}(c_i)\big(\eta^{-1}(f_i)\big)^\omega \,,$$

for suitable $m < \omega$ and $c_i, f_i \subseteq S$.

Exercise 6.4. Prove that the free ω-semigroup $\langle \Sigma^+, \Sigma^\omega \rangle$ really is free: show that, for every ω-semigroup $\mathfrak{S}$ and every map $h : \Sigma \to S$, there exists a unique homomorphism $\eta : \Sigma^\infty \to \mathfrak{S}$ such that $\eta(a) = h(a)$, for all $a \in \Sigma$.

In order to prove that a language is recognisable precisely when it is MSO-definable, we employ the Theorem of Ramsey.

Definition 6.3. (a) Let $\mathfrak{S}$ be a semigroup and $(a_n)_{n<\omega}$ a sequence of semigroup elements. For $i < k < \omega$, we write

$$a_{[i,k)} := a_i a_{i+1} \cdots a_{k-1} \,.$$

A *Ramsey factorisation* of $(a_n)_{n<\omega}$ is a sequence of indices $0 < k_0 < k_1 < \cdots < \omega$ such that

$$a_{[k_i,k_j)} = a_{[k_{i'},k_{j'})} \,, \quad \text{for all } i < j \text{ and } i' < j' \,.$$

The *type* of such a factorisation is the pair

$$\langle a_{[0,k_1)},\ a_{[k_1,k_2)} \rangle \,.$$

(b) Let $\mathfrak{S}$ be an ω-semigroup and $\eta : \Sigma^\infty \to \mathfrak{S}$ a homomorphism. A *Ramsey factorisation* of a word $w \in \Sigma^\omega$ is a Ramsey factorisation of the sequence $(\eta(w(n)))_{n<\omega}$.

Lemma 6.4. *Let $\mathfrak{S}$ be a finite semigroup and $(a_n)_{n<\omega}$ a sequence of semigroup elements.*

(a) *$(a_n)_{n<\omega}$ has a Ramsey factorisation.*

(b) *If $\langle b, e \rangle$ is the type of a Ramsey factorisation of $(a_n)_{n<\omega}$ then*

$$be = b \quad \text{and} \quad ee = e \,.$$

(c) *If $\langle b, e \rangle$ and $\langle c, f \rangle$ are the types of two Ramsey factorisations of $(a_n)_{n<\omega}$ then there are elements $u, v \in S^1$ such that*

$$c = bu \,, \quad e = uv \,, \quad \text{and} \quad f = vu \,.$$

Proof. (a) We define a colouring $\lambda : [\omega]^2 \to S$ by assigning to a pair $i < j$ of indices the colour $\lambda(i, j) := a_{[i,j)}$. By the Theorem of Ramsey, there exists an infinite set $I \subseteq \omega$ such that $\lambda(i, j) = \lambda(i', j')$, for all $i < j$ and $i' < j'$ in I. We can choose for $k_0 < k_1 < \cdots$ an increasing enumeration of I.

(b) Let $k_0 < k_1 < \cdots$ be a Ramsey factorisation with type $\langle b, e \rangle$. Set $c := a_{[0,k_0)}$ and $d_n := a_{[k_n,k_{n+1})}$, for $n < \omega$. Then $b = cd_0$ and $e = d_1$. Furthermore,

$$d_i d_{i+1} \cdots d_j = a_{[k_i,k_{j+1})} = a_{[k_{i'},k_{j'+1})} = d_{i'} d_{i'+1} \cdots d_{j'} ,$$

for all $i \leq j$ and $i' \leq j'$. In particular, $d_i = d_j$, for all i, j. Consequently, we have

$$be = cd_0 d_1 = cd_0 = b \quad \text{and} \quad ee = d_1 d_1 = d_1 d_2 = d_1 = e .$$

(c) Let $k_0 < k_1 < \cdots$ and $l_0 < l_1 < \cdots$ be Ramsey factorisations with types $\langle b, e \rangle$ and $\langle c, f \rangle$, respectively. Replacing $(k_n)_{n<\omega}$ and $(l_n)_{n<\omega}$ by suitable subsequences we may assume without loss of generality that $k_0 \leq l_0 \leq k_1 \leq l_1 \leq \cdots$. For $n < \omega$, set

$$u_n := a_{[k_n,l_n)} \quad \text{and} \quad v_n := a_{[l_n,k_{n+1})} .$$

Then $c = bu_1$ and

$$e = u_0 v_0 = u_1 v_1 = \cdots \quad \text{and} \quad f = v_0 u_1 = v_1 u_2 = \cdots .$$

Since the number of possible pairs $\langle u_n, v_n \rangle$ is finite, there exist elements $u, v \in S$ and an infinite set $I \subseteq \omega$ such that

$$\langle u_n, v_n \rangle = \langle u, v \rangle , \quad \text{for all } n \in I .$$

Fix elements $m, n \in I$ with $n > m + 1$. Then

$$\begin{aligned} c = cf^m &= (bu_0)(v_0 u_1)\cdots(v_{m-1}u_m) \\ &= b(u_0 v_0)\cdots(u_{m-1}v_{m-1})u_m = be^m u = bu\,, \\ e = e^2 &= u_m v_m e = uve\,, \\ f = f^{n-m} &= (v_m u_{m+1})\cdots(v_{n-1}u_n) \\ &= v_m(u_{m+1}v_{m+1})\cdots(u_{n-1}v_{n-1})u_n = ve^{n-m-1}u = veu\,. \end{aligned}$$

Consequently, the elements u and ve have the desired properties. □

Lemma 6.5. *Let $\eta : \Sigma^\infty \to \mathfrak{S}$ be a homomorphism into a finite ω-semigroup $\mathfrak{S}$. For $u \in S_\omega$, set*

$$F_u := \{\, \langle a, b \rangle \in S^2 \mid ab^\omega = u \,\}\,.$$

Then $\eta(w) = u$ if, and only if, w has a Ramsey factorisation with type in F_u.

Proof. ($\Leftarrow$) If w has a Ramsey factorisation $k_0 < k_1 < \cdots$ of type $\langle a, b \rangle \in F_u$ then

$$\begin{aligned} \eta(w) &= \eta\big(w[0, k_1)w[k_1, k_2)w[k_2, k_3)\cdots\big) \\ &= \eta(w[0, k_1)) \cdot \eta(w[k_1, k_2)) \cdot \eta(w[k_2, k_3))\cdots \\ &= abb\cdots = ab^\omega = u\,. \end{aligned}$$

($\Rightarrow$) Suppose that $\eta(w) = u$. By Lemma 6.4 (a), w has a Ramsey factorisation $k_0 < k_1 < \cdots$. Let $\langle b, e \rangle$ be its type. Then

$$\begin{aligned} u = \eta(w) &= \eta(w[0, k_1)) \cdot \eta(w[k_1, k_2)) \cdot \eta(w[k_2, k_3))\cdots \\ &= bee\cdots = be^\omega\,. \end{aligned}$$

Hence, $\langle b, e \rangle \in F_u$. □

Lemma 6.6. *Let $\mathfrak{S}$ be a finite ω-semigroup. For every pair $\langle b, e\rangle \in S^2$, there exists an* MSO*-formula $\varphi_{b,e}$ defining the set of all ω-words with a Ramsey factorisation of type $\langle b, e\rangle$.*

Proof. Let $c \in S$. We start by defining a formula $\psi_c(x, y)$ stating that the factor from position x to $y - 1$ is mapped to c. We use set variables $(Z_d)_{d\in S}$ containing all positions $x \leq z < y$ such that the factor from x to z is mapped to d. The formula $\psi_c(x, y)$ states that there are sets $(Z_d)_{d\in S}$ such that

- $Z_d \cap Z_{d'} = \varnothing$ for $d \neq d'$,
- if a is the letter at position x, then $x \in Z_{\eta(a)}$,
- if $x < z < y$ and the letter at position z is a, then $z-1 \in Z_d$ implies $z \in Z_{d\eta(a)}$, and
- $y - 1 \in Z_c$.

Clearly, each of these statements can be expressed in MSO.

Having defined the formulae ψ_c, the desired formula $\varphi_{b,e}$ states that there exists an infinite set Z such that

- $\psi_b(0, x)$ holds, where x is the second element of Z, and
- $\psi_e(x, y)$ holds for all elements $x < y$ of Z. □

Theorem 6.7. *Let Σ be a finite alphabet. A language $L \subseteq \Sigma^\omega$ is* MSO*-definable if, and only if, there exists a homomorphism $\eta : \Sigma^\infty \to \mathfrak{S}$ into a finite ω-semigroup $\mathfrak{S}$ recognising L.*

Proof. ($\Rightarrow$) We have seen in the example after Definition 6.2 that every $\mathrm{MSO}_0^{\bar{n}}$-definable language is recognised by a homomorphism into the ω-semigroup $\Theta_{\bar{n}}(m)$ of all $\mathrm{MSO}_0^{\bar{n}}$-theories.

($\Leftarrow$) Let $\eta : \Sigma^\infty \to \mathfrak{S}$ be a morphism such that $L = \eta^{-1}[P]$, for some $P \subseteq S_\omega$. We have seen in Lemma 6.5 that

$$\eta(w) = u \quad \text{iff} \quad w \text{ has a Ramsey factorisation with type in } F_u\,.$$

Consequently, the formula

$$\psi := \bigvee_{u \in P} \bigvee_{\langle b,e \rangle \in F_u} \varphi_{b,e}$$

defines L, where $\varphi_{b,e}$ are the formulae from Lemma 6.6. □

If we want to compute with an ω-semigroup, we face the problem that we cannot write down the multiplication table of the infinite product since it is infinite. For algorithmic applications we need to represent this table in a finite way.

Definition 6.8. A *Wilke algebra* is a structure $\langle S, S_\omega, \cdot, {}^\omega \rangle$ with two products

$$\cdot : S \times S \to S \quad \text{and} \quad \cdot : S \times S_\omega \to S_\omega$$

and one unary *ω-power* operation

$${}^\omega : S \to S_\omega .$$

These operations satisfy the following associative laws:

$$\begin{aligned} (ab)c &= a(bc)\,, & (ab)^\omega &= a(ba)^\omega\,, \\ a(bu) &= (ab)u\,, & (a^n)^\omega &= a^\omega\,, \end{aligned}$$

for $a, b, c \in S$, $u \in S_\omega$, and $0 < n < \omega$.

Theorem 6.9. *Let $\mathfrak{S}$ be a finite Wilke algebra. There exists a unique function $\pi : S^\omega \to S_\omega$ turning $\mathfrak{S}$ into an ω-semigroup with*

$$\pi(a, a, a, \dots) = a^\omega .$$

Proof. Given a sequence $(a_n)_{n<\omega}$ we define

$$\pi(a_0, a_1, \dots) := be^\omega,$$

where $\langle b, e\rangle$ is the type of a Ramsey factorisation of $(a_n)_{n<\omega}$. To see that this is well-defined, first note that, according to Lemma 6.4 (a), every sequence does have a Ramsey factorisation. Furthermore, if $\langle b, e\rangle$ and $\langle c, f\rangle$ are the types of two Ramsey factorisations of $(a_n)_{n<\omega}$ then we can use Lemma 6.4 (c) to find elements $u, v \in S^1$ such that

$$c = bu\,, \quad e = uv\,, \quad \text{and} \quad f = vu\,.$$

Hence,

$$cf^\omega = bu(vu)^\omega = b(uv)^\omega = be^\omega.$$

To prove that this operation turns $\mathfrak{S}$ into an ω-semigroup, we have to show associativity. For the first equation, let $(a_n)_{n<\omega}$ be a sequence of semigroup elements and let $c \in S$. If $k_0 < k_1 < \cdots$ is a Ramsey factorisation of $(a_n)_{n<\omega}$, then $k_0 + 1 < k_1 + 1 < \cdots$ is a Ramsey factorisation of the sequence $c, a_0, a_1, \dots$ and we have

$$\begin{aligned} c \cdot \pi(a_0, a_1, \dots) &= c(a_0 \cdots a_{k_1-1})(a_{k_1} \cdots a_{k_2-1})^\omega \\ &= (ca_0 \cdots a_{k_1-1})(a_{k_1} \cdots a_{k_2-1})^\omega \\ &= \pi(c, a_0, a_1, \dots)\,. \end{aligned}$$

For the second equation, let $(a_n)_{n<\omega}$ be a sequence of semigroup elements and let $l_0 < l_1 < \cdots < \omega$ be a sequence of indices. Suppose that $k_0 < k_1 < \cdots$ is a Ramsey factorisation of the sequence $(a_{l_n} \cdots a_{l_{n+1}-1})_{n<\omega}$. Then $l_{k_0} < l_{k_1} < \cdots$ is a Ramsey factorisation of $(a_n)_{n<\omega}$ and we have

$$\begin{aligned} \pi(a_0 \cdots a_{l_0-1},\ a_{l_0} \cdots a_{l_1-1},\ \dots) &= (a_0 \cdots a_{l_{k_1}-1})(a_{l_{k_1}} \cdots a_{l_{k_2}-1})^\omega \\ &= \pi(a_0, a_1, \dots)\,. \end{aligned}$$

It remains to show that the product π is unique. Suppose that $\pi' : S^\omega \to S_\omega$ is any associative operation such that $\pi'(a, a, a, \dots) = a^\omega$, for all $a \in S$. To prove that $\pi' = \pi$, consider a sequence $(a_n)_{n<\omega}$ in S and let $k_0 < k_1 < \cdots$ be a Ramsey factorisation of $(a_n)_{n<\omega}$ of type $\langle b, e\rangle$. Then it follows that

$$\begin{aligned}\pi'(a_0, a_1, \dots) &= \pi'(a_{[0,k_1)}, a_{[k_1,k_2)}, a_{[k_2,k_3)}, \dots) \\ &= \pi'(b, e, e, e, \dots) \\ &= b \cdot \pi'(e, e, e, \dots) = be^\omega = \pi(a_0, a_1, \dots). \qquad \square\end{aligned}$$

Exercise 6.5. (a) Let L_0 and L_1 be languages that are recognised by homomorphisms $\eta_0 : \Sigma^\infty \to \mathfrak{S}_0$ and $\eta_1 : \Sigma^\infty \to \mathfrak{S}_1$ into finite ω-semigroups. Prove that the languages $L_0 \cap L_1$, $L_0 \cup L_1$, and $\Sigma^\omega \setminus L_0$ are also recognised by a homomorphism into some finite ω-semigroup.

(b) Let $L \subseteq \Sigma^\omega$ be recognised by a homomorphism $\eta : \Sigma^\infty \to \mathfrak{S}$ into a finite ω-semigroup $\mathfrak{S}$ and let $\pi : \Sigma \to \Gamma$ be a function. Prove that

$$\pi[L] := \{\, \pi(w) \mid w \in L \,\}$$

is also recognised by a homomorphism into a finite ω-semigroup. (This exercise is a bit more involved.)

(c) Use (a) and (b) to give an alternative proof of the fact that every MSO-definable language is recognised by a homomorphism into a finite ω-semigroup. Show furthermore that, given a formula φ one can effectively construct a description of the corresponding homomorphism and Wilke algebra.

(d) Use (c) to give an alternative decidability proof for the monadic theory of $\langle \omega, \leq \rangle$.

Exercise 6.6. The *syntactic congruence* of a language $L \subseteq \Sigma^\omega$ is the relation

$$\begin{aligned} x \sim_L y \quad :\text{iff} \quad & u(xv)^\omega \in L \Leftrightarrow u(yv)^\omega \in L \\ \text{and} \quad & ux \in L \Leftrightarrow uy \in L\,, \quad \text{for all } u, v \in \Sigma^*. \end{aligned}$$

(a) Prove that the syntactic congruence is a congruence of the free ω-semigroup.

(b) Prove that a language $L \subseteq \Sigma^\omega$ is MSO-definable if, and only if, its syntactic congruence has finite index.

7. ω-automata

Definition 7.1. (a) An (*nondeterministic*) *ω-automaton* is a tuple $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega \rangle$ where where Q is a finite set of *states,* Σ is a finite *input alphabet,* $q_0 \in Q$ is the *initial state,* $\Omega : Q \to \omega$ is a *priority function,* and $\Delta \subseteq Q \times \Sigma \times Q$ is the *transition relation.* Instead of $\langle p, a, q \rangle \in \Delta$, we also write $p \xrightarrow{a} q$.

(b) A *run* of an ω-automaton $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega \rangle$ on an ω-word $w \in \Sigma^\omega$ is an ω-word $\rho \in Q^\omega$ such that

$$\big\langle \rho(n), w(n), \rho(n+1) \big\rangle \in \Delta\,, \quad \text{for all } n < \omega\,.$$

A run ρ is *accepting* if $\rho(0) = q_0$ and ρ satisfies the *parity condition*

$$\liminf_{n \to \infty} \Omega(\rho(n)) \text{ is even}\,.$$

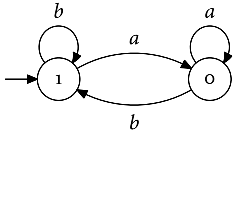
(c) An ω-automaton $\mathcal{A}$ *accepts* an ω-word w if there exists an accepting run of $\mathcal{A}$ on w. The *language recognised* by $\mathcal{A}$ is the set $L(\mathcal{A})$ of all ω-words it accepts.

(d) An ω-automaton $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega \rangle$ is *deterministic* if, for all states $q \in Q$ and all letters $a \in \Sigma$, there is a unique state $q' \in Q$ with $\langle q, a, q' \rangle \in \Delta$. In this case, we can replace the transition relation Δ by a *transition function* $\delta : Q \times \Sigma \to Q$ such that

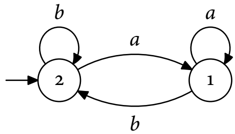
$$\Delta = \big\{\, \langle q, a, \delta(q,a) \rangle \;\big|\; q \in Q,\ a \in \Sigma \,\big\}\,.$$

(e) A *Büchi automaton* is an ω-automaton $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega \rangle$ where $\operatorname{rng} \Omega \subseteq \{0, 1\}$.

Example. The language of all words $w \in \{a, b\}^\omega$ with infinitely many a is recognised by the ω-automaton



where the numbers labelling the states indicate their priority. We obtain an ω-automaton for the language of all words $w \in \{a, b\}^\omega$ with only finitely many a by changing the priorities:



Exercise 7.1. Find ω-automata recognising the following languages over the alphabet $\{a, b, c\}$.

(a) The language of all ω-words containing infinitely many a, but only finitely many b.

(b) The language of all ω-words where immediately after or immediately before every letter a there is another a.

(c) The language of all ω-words containing an even (and finite) number of a.

(d) The language of all ω-words where after every letter a there is a later position with a letter b.

(e) The language of all ω-words where, for every prefix p, the numbers of the letters a, b, c differ by at most 1.

Exercise 7.2. (a) Prove that, for every ω-automaton $\mathcal{A}$, there exists a Büchi automaton $\mathcal{B}$ recognising the same language.

(b) Prove that a language $L \subseteq \Sigma^\omega$ is recognised by a Büchi automaton if, and only if, it is of the form

$$L = \bigcup_{i<m} U_i V_i^\omega ,$$

where $m < \omega$ and $U_i, V_i \subseteq \Sigma^+$ are MSO-definable languages of finite words.

(c) Find a Büchi automaton recognising the language of all ω-words with only finitely many letters a. Prove that this language is not recognised by a deterministic Büchi automaton, i.e., one where the transition relation Δ is the graph of a function $Q \times \Sigma \to Q$.

(d) Prove that the class of languages recognised by Büchi automata is closed under union, intersection, complement, and projection.

(e) Prove that a language is recognised by a Büchi automaton if, and only if, it is MSO-definable.

Definition 7.2. Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega \rangle$ be an ω-automaton and let $D := \operatorname{rng} \Omega$ be the set of priorities used. The ω-semigroup $\mathfrak{S}_{\mathcal{A}} := \langle S, S_\omega \rangle$ *associated* with $\mathcal{A}$ has domains

$$S := \mathscr{P}(Q \times D \times Q) \quad \text{and} \quad S_\omega := \mathscr{P}(Q)\,.$$

Binary multiplication is defined by

$$\begin{aligned} A \cdot B &:= \{ \langle p, \min\{d, d'\}, r \rangle \mid \langle p, d, q \rangle \in A\,,\ \langle q, d', r \rangle \in B \}\,, \\ A \cdot U &:= \{ p \mid \langle p, d, q \rangle \in A\,, q \in U \}\,, \end{aligned}$$

for $A, B \in S$ and $U \in S_\omega$. The infinite product is given by

$$\begin{aligned} \pi(A_0, A_1, \dots) := \{ p_0 \mid{} & \text{there are } \langle p_n, d_n, p_{n+1} \rangle \in A_n\,,\ \text{for } n < \omega\,, \\ & \text{such that } \liminf_{n \to \infty} d_n \text{ is even} \}\,. \end{aligned}$$

Theorem 7.3. *Let Σ be a finite alphabet and $L \subseteq \Sigma^\omega$ a language of ω-words. The following statements are equivalent:*

(1) *L is recognised by an ω-automaton.*

(2) *L is recognised by a Büchi automaton.*

(3) *L is recognised by a homomorphism $\eta : \Sigma^\infty \to \mathfrak{S}$ into a finite ω-semigroup.*

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) ⇒ (3) Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega \rangle$ be an ω-automaton recognising L and let $\mathfrak{S}_{\mathcal{A}}$ be the associated ω-semigroup. We define a homomorphism $\eta : \Sigma^\infty \to \mathfrak{S}_{\mathcal{A}}$ by mapping finite words $w \in \Sigma^*$ to the set of all triples $\langle p, d, q \rangle$ such that there exists a run of $\mathcal{A}$ on w starting in state p, ending in state q, and having the minimal priority d. Similarly, infinite words $w \in \Sigma^\omega$ are mapped to the set of all states p such that there exists a run of $\mathcal{A}$ on w starting in state p and satisfying the parity condition. Then

$$L = \eta^{-1}[P] \quad \text{where} \quad P := \{\, U \subseteq Q \mid q_0 \in U \,\}\,.$$

Hence, η recognises L.

(3) ⇒ (2) Suppose that $L = \eta^{-1}[P]$ for some $\eta : \Sigma^\infty \to \mathfrak{S}$. We have seen in Lemma 6.5 that

$$\eta(w) = u \quad \text{iff} \quad w \text{ has a Ramsey factorisation with type in } F_u\,.$$

Consequently, we can construct a Büchi automaton that, on input w, guesses a value $u \in P$ and a type $\langle b, e \rangle \in F_u$ and then checks that w has a Ramsey factorisation with type $\langle b, e \rangle$. This can be done as follows. After reading a prefix v the automaton remembers the image $\eta(v)$. The automaton can do this since, if the current value is $\eta(v)$ and the next letter is c, the next value will be $\eta(vc) = \eta(v)\eta(c)$. Hence, when reading a letter c it only needs to multiply the current value by $\eta(c)$. If the current value is equal to b, the automaton can nondeterministically decide that it has read the first factor of the factorisation. In this case, it resets the stored value and reads letters until it reaches the value e. After having found a factor with value e, the automaton can again nondeterministically decide that it has found the next factor of the factorisation. It resets its stored value and reads the next factor. The automaton accepts if this reset was performed infinitely many times. Formally, we have states

$$Q := \{0, 1, 2\} \times S \cup \{q_0\}$$

and the following transitions, for $a \in S$ and $c \in \Sigma$,

$$\begin{aligned}
q_0 &\xrightarrow{c} \langle 0, \eta(c)\rangle\,, \\
\langle 0, a\rangle &\xrightarrow{c} \langle 0, a\eta(c)\rangle\,, &\qquad \langle 0, b\rangle &\xrightarrow{c} \langle 1, \eta(c)\rangle\,, \\
\langle 1, a\rangle &\xrightarrow{c} \langle 2, a\eta(c)\rangle\,, &\qquad \langle 1, e\rangle &\xrightarrow{c} \langle 1, \eta(c)\rangle\,, \\
\langle 2, a\rangle &\xrightarrow{c} \langle 2, a\eta(c)\rangle\,, &\qquad \langle 2, e\rangle &\xrightarrow{c} \langle 1, \eta(c)\rangle\,.
\end{aligned}$$

The initial state is q_0 and the priority function is

$$\Omega(q_0) := 1 \quad \text{and} \quad \Omega(\langle k, a\rangle) := \begin{cases} 0 & \text{if } k = 1\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

□

Lemma 7.4. *Let $\mathfrak{S}$ be a finite ω-semigroup and let $e_0, e_1, \dots \in S$ be elements such that*

$$e_i e_k = e_i\,, \quad \textit{for all } i, k < \omega\,.$$

Then $\pi(e_0, e_1, e_2, \dots) = e_0^\omega$.

Proof.

$$\begin{aligned}
\pi(e_0,\ e_1,\ e_2, \dots) &= \pi(e_0,\ e_1 e_0,\ e_2 e_0, \dots) \\
&= \pi(e_0 e_1,\ e_0 e_2,\ e_0 e_3, \dots) \\
&= \pi(e_0,\ e_0,\ e_0, \dots) = e_0^\omega\,.
\end{aligned}$$

□

Theorem 7.5 (McNaughton). *For every homomorphism $\eta : \Sigma^\infty \to \mathfrak{S}$ into a finite ω-semigroup and every set $P \subseteq S_\omega$, we can construct a deterministic ω-automaton $\mathcal{A}$ recognising $\eta^{-1}[P]$.*

Proof. Given an ω-word $w \in \Sigma^\omega$, we consider the colouring

$$\lambda(i, k) := \eta(w[i, k))\,, \quad \text{for } i < k < \omega\,.$$

This colouring is additive. Hence, we can use Theorem 4.10 to construct a deterministic finite-state transducer $\mathcal{T} = \langle Q, S, [N], q_0, \delta, \mu \rangle$ computing weak Ramseyan splits for $\mathfrak{S}$. Suppose that, on input λ, $\mathcal{T}$ outputs the weak Ramseyan split $\sigma : \omega \to [N]$. For each $k < N$, we define two functions

$$\zeta_k : \omega \to \omega^* \quad \text{and} \quad \chi_k : \omega \to S^1 \times S^1 \times S^1$$

as follows. Given $n < \omega$, let $z_0 < \cdots < z_{l-1}$ be an increasing enumeration of all positions $0 < z \leq n$ with $z \approx^0_\sigma n$ and $\sigma(z) = k$. We set

$$\zeta_k(n) := \langle z_0, \dots, z_{l-1} \rangle$$
$$\chi_k(n) := \begin{cases} \langle 1, 1, \lambda(0, n) \rangle & \text{if } \zeta_k(n) = \langle\rangle\,, \\ \langle \lambda(0, z_0), 1, \lambda(z_0, n) \rangle & \text{if } \zeta_k(n) = \langle z_0 \rangle\,, \\ \langle \lambda(0, z_0), \lambda(z_0, z_1), \lambda(z_1, n) \rangle & \text{if } \zeta_k(n) = \langle z_0, z_1, \dots \rangle \end{cases}$$

(using the convention that $\lambda(x, x) = 1$). Note that, if

$$\chi_k(n) = \langle a_k, b_k, c_k \rangle \quad \text{and} \quad \lambda(n, n+1) = d\,,$$

then

$$\chi_k(n+1) = \langle 1, 1, a_k b_k c_k d \rangle\,, \qquad \text{for } k < \sigma(n+1)\,,$$
$$\chi_k(n+1) = \langle a_k, b_k, c_k d \rangle\,, \qquad \text{for } k > \sigma(n+1)\,,$$
$$\chi_k(n+1) = \begin{cases} \langle c_k d, 1, 1 \rangle & \text{if } a_k = 1,\ b_k = 1\,, \\ \langle a_k, c_k d, 1 \rangle & \text{if } a_k \neq 1,\ b_k = 1\,, \\ \langle a_k, b_k, c_k d \rangle & \text{if } a_k \neq 1,\ b_k \neq 1\,, \end{cases} \qquad \text{for } k = \sigma(n+1)\,.$$

Consequently, we can compute $\chi_k(n+1)$ from $\chi_k(n)$, $\sigma(n+1)$, and d.

The desired deterministic automaton $\mathcal{A}$ for $\eta^{-1}[P]$ has states

$$Q' := Q \times (S^1 \times S^1 \times S^1)^N$$

After reading the first n letters of its input, $\mathcal{A}$ is in state

$$\langle q_n, \chi_0(n), \dots, \chi_{N-1}(n) \rangle ,$$

where q_n is the state of $\mathcal{T}$. The initial state is

$$\big\langle q_0, \langle 1,1,1 \rangle, \dots, \langle 1,1,1 \rangle \big\rangle .$$

We assign to a state

$$p = \big\langle q, \langle a_0, b_0, c_0 \rangle, \dots, \langle a_{N-1}, b_{N-1}, c_{N-1} \rangle \big\rangle$$

with $\mu(q) = k$ the priority

$$\Omega(p) := \begin{cases} 2(N-k) & \text{if } a_k, b_k \neq 1 \text{ and } a_k b_k^\omega \in P , \\ 2(N-k)-1 & \text{if } a_k = 1 , b_k = 1 , \text{ or } a_k b_k^\omega \notin P . \end{cases}$$

We claim that this automaton $\mathcal{A}$ accepts an ω-word w if, and only if, $\eta(w) \in P$.

($\Leftarrow$) Suppose that $\eta(w) \in P$. Let $k := \limsup_{n\to\infty} \sigma(n)$ and let $z_0 < z_1 < \dots$ be an increasing enumeration of all positions in $\sigma^{-1}(k)$. By choice of k, there is some index $l < \omega$ such that $\sigma(x) \leq k$, for all $x \geq z_l$. Set

$$a := \lambda(0, z_l) \quad \text{and} \quad e_i := \lambda(z_{l+i}, z_{l+i+1}) , \quad \text{for } i < \omega .$$

At position z_{l+i}, the automaton is in a state of the form

$$\big\langle q, \dots, \underbrace{\langle a, e_0, c_k \rangle}_{k\text{-th component}}, \dots \big\rangle \quad \text{with} \quad \mu(q) = k .$$

Hence, the minimal priority seen infinitely often is either $2(N-k)$ or $2(N-k)-1$ depending on whether or not $ae_0^\omega \in P$. As σ is a weak Ramseyan spilt, we have $e_i e_k = e_i$, for all i, k. Therefore, it follows by Lemma 7.4 that

$$ae_0^\omega = \pi(a, e_0, e_1, e_2, \dots) = \eta(w) \in P .$$

Hence, $\mathcal{A}$ accepts w.

($\Rightarrow$) Suppose that there exists an accepting run ρ of $\mathcal{A}$ on w and let $2(N-k)$ be the minimal priority occurring infinitely often in it. Then ρ contains infinitely many states of the form

$$\langle q, \dots, \underbrace{\langle a_k, e_k, c_k \rangle}_{k\text{-th component}}, \dots \rangle \quad \text{where} \quad a_k e_k^\omega \in P \text{ and } \mu(q) = k\,.$$

Let $z_0 < z_1 < \dots$ be an enumeration of all positions with such a state. Since ρ does not contain infinitely many states with priority smaller than $2(N-k)$, it follows that there is some index $n < \omega$ such that $\sigma(x) \leq k$, for all $x \geq z_n$. Since $\sigma(z_i) = k$, for all i, we therefore have

$$z_i \approx_\sigma z_k\,, \quad \text{for } i, k \geq n\,.$$

Setting $a := \lambda(0, z_n)$ and $e_i := \lambda(z_{n+i}, z_{n+i+1})$ it follows that

$$\rho(z_{n+i}) = \langle q, \dots, \langle a, e_0, c_i \rangle, \dots \rangle\,, \quad \text{for some } c_i \in S\,,$$

and that $e_i e_k = e_i$, for all i, k. Hence, Lemma 7.4 implies that

$$\eta(w) = \pi(a, e_0, e_1, e_2, \dots) = a e_0^\omega \in P\,. \qquad \square$$

The results of the previous sections are summarised in the following theorem. We also add one further logical characterisation.

Definition 7.6. *Weak monadic second-order logic* WMSO has the same syntax as MSO, but all set quantifiers range over finite sets only.

Theorem 7.7. *Let $L \subseteq \Sigma^\omega$ be a language of ω-words. The following statements are equivalent:*

(1) *L is definable in* MSO.

(2) *L is definable in* WMSO.

(3) *L is recognised by a homomorphism into a finite ω-semigroup.*

(4) *L is recognised by a nondeterministic ω-automaton.*

(5) *L is recognised by a deterministic ω-automaton.*

(6) *L is recognised by a nondeterministic Büchi automaton.*

Furthermore, all translations between these formalisms are effective.

Proof. The equivalences (1) ⇔ (3) ⇔ (4) ⇔ (6) were already proved in Theorems 6.7 and 7.3, respectively. (2) ⇒ (1) is trivial, and (3) ⇒ (5) follows from Theorem 7.5. Hence, it remains to prove (5) ⇒ (2).

Let $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega \rangle$ be a deterministic ω-automaton that recognises L. We start by constructing formulae $\mathrm{STATE}_q(x)$, for $q \in Q$, stating that $\rho(x) = q$, where ρ is the unique run of $\mathcal{A}$ on the input word. These formulae guess finite sets Z_p, $p \in Q$, containing all positions (up to x) with state p. We set

$$\mathrm{STATE}_q(x) := \exists (Z_p)_{p \in Q} [\mathrm{ADM} \wedge \mathrm{INIT} \wedge \mathrm{TRANS}(x) \wedge Z_q x],$$

where

$$\mathrm{ADM} := \forall y \bigwedge_{p \neq p'} \neg (Z_p y \wedge Z_{p'} y)$$

states that every position is labelled by at most one state,

$$\mathrm{INIT} := \exists y [\forall z (y \leq z) \wedge Z_{q_0} y]$$

states that the first state is q_0,

$$\mathrm{TRANS}(x) := \forall y \forall z \Big[\mathrm{suc}(y, z) \wedge z \leq x \rightarrow \bigvee_{p \in Q} \bigvee_{a \in \Sigma} (Z_p y \wedge P_a y \wedge Z_{\delta(p,a)} z) \Big]$$

states that at every position a valid transition is used, and

$$\mathrm{suc}(x, y) := x < y \wedge \neg \exists z [x < z \wedge z < y]$$

states that y is the immediate successor of x.

Using these formulae $\mathrm{STATE}_q(x)$, we can construct a formula ψ defining L as follows. Let

$$H_k := \{ q \in Q \mid \Omega(q) \leq k \}$$

be the set of all states with priority at most k. We use the formula

$$\mathrm{INF}_q := \forall x \exists y [x \leq y \wedge \mathrm{STATE}_q(y)]$$

stating that the run contains infinitely many occurrences of the state q, and the formula

$$\mathrm{MIN}_k := \bigvee_{q \in H_k} \mathrm{INF}_q \wedge \bigwedge_{q \in H_{k-1}} \neg \mathrm{INF}_q$$

stating that the minimal priority seen infinitely often is k. Then we can set

$$\psi := \bigvee_{k<n} \mathrm{MIN}_{2k} \,,$$

where n is chosen such that the maximal priority of $\mathcal{A}$ is smaller than $2n$.

□

Exercise 7.3. Let Σ be a finite alphabet. The *Cantor topology* on Σ^ω is given by the following basis of open sets:

$$O_w := \{ x \in \Sigma^\omega \mid w \text{ is a prefix of } x \}\,, \quad \text{for } w \in \Sigma^*.$$

(a) Show that every basic open set O_w is also closed.

(b) Show that a set $U \subseteq \Sigma^\omega$ is open if, and only if, there exists a set $W \subseteq \Sigma^*$ such that

$$U = \{ x \in \Sigma^\omega \mid \text{some } w \in W \text{ is a prefix of } x \}\,.$$

(c) Show that a set $C \subseteq \Sigma^\omega$ is closed if, and only if, there exists a set $W \subseteq \Sigma^*$ such that

$$C = \{ x \in \Sigma^\omega \mid \text{every finite prefix of } x \text{ belongs to } W \}.$$

(d) Prove that Σ^ω is a compact Hausdorff space.

(e) A set $U \subseteq \Sigma^\omega$ is a $\mathbf{\Pi}^0_2$-set if it can be written as a countable intersection of open sets. Prove that every language $L \subseteq \Sigma^\omega$ recognised by a deterministic ω-automaton is a finite boolean combination of $\mathbf{\Pi}^0_2$-sets.

Exercise 7.4. (a) Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be ω-automata. Show that there are ω-automata recognising the languages $L(\mathcal{A}_0) \cap L(\mathcal{A}_1)$, $L(\mathcal{A}_0) \cup L(\mathcal{A}_1)$, and $\Sigma^\omega \setminus L(\mathcal{A}_0)$. (The case of the complement is a bit more involved.)

(b) Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega \rangle$ be an ω-automaton and let $\pi : \Sigma \to \Gamma$ be a function. Prove that there exists an ω-automaton recognising the language

$$\pi[L(\mathcal{A})] := \{ \pi(w) \mid w \in L(\mathcal{A}) \}.$$

(c) Use (a) and (b) to prove that, given an MSO-formula φ, we can effectively construct an ω-automaton recognising the language defined by φ.

(d) Show that we can decide whether a given ω-automaton $\mathcal{A}$ recognises the empty language.

(e) Use (c) and (d) to give an alternative decidability proof for the monadic theory of $\langle \omega, \leq \rangle$.

Notes

A comprehensive account on the theory of linear orders from a logician's perspective is given by [13]. For a good introduction to formal language theory for ω-words, ω-semigroups, and automata see [11].

Ramseyan splits were introduced by Colcombet, extending results by Simon on so-called factorisation forests. An exposition that also includes a proof of Theorem 4.10 can be found in [4].

The original proof of the decidability of the theory of ω is due to Büchi [3]. It combines automata-theoretic techniques with a Ramsey argument, see also [16] for a survey. The proof we presented is due to Shelah [15]. An exposition can be found in [17].

The Theorem of McNaughton (Theorem 7.5) is from [10]. A good exposition is [18]. Our proof is new and based on ideas by Colcombet.

III. Trees

1. Parity games

A parity game is a game where two players move a token along edges of a graph. The set of positions of the graph is divided into to classes, one for each player. The owner of the current position chooses where to move the token next. Furthermore, each position is assigned a priority. The sequence of these priorities seen during a play of the game determines who has won.

Definition 1.1. (a) A *parity game* is a tuple $\mathcal{G} = \langle V_0, V_1, E, \Omega \rangle$ where V_i is the set of *positions of Player* i, E is a binary relation on the set $V := V_0 \cup V_1$ of all positions, and $\Omega : V \to \omega$ is a *priority function.* We always assume that the range of Ω is *finite.* If $\langle v, w \rangle \in E$, we say that w is a *successor* of v. We write $\mathrm{Suc}(v)$ for the set of successors of v.

(b) A *play* of $\mathcal{G}$ is a finite or infinite sequence $(v_n)_{n<\alpha}$ of positions such that every position in the sequence is a successor of the previous one. We say that such a play *starts at* v_0.

(c) A play $(v_n)_{n<\alpha}$ is *maximal* if either $\alpha = \omega$, or if $\alpha < \omega$ and the last position $v_{\alpha-1}$ has no successors.

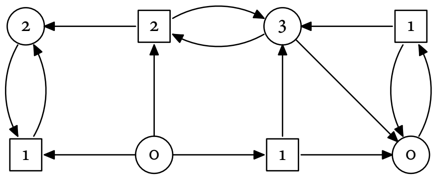
A maximal play is *winning* for Player 0 if either

- $\alpha < \omega$ and the last position belongs to Player 1, or
- $\alpha = \omega$ and the play satisfies the *parity condition:*

$$\liminf_{n\to\infty} \Omega(v_n) \text{ is even.}$$

Otherwise, the play is considered to be winning for Player 1.

Example. Consider the following game



where we have denoted positions of Player 0 by a circle and those of Player 1 by a box. The numbers indicate the priorities of the positions.

Definition 1.2. Let $\mathcal{G} = \langle V_0, V_1, E, \Omega \rangle$ be a parity game.

(a) A *strategy* for Player i is a function mapping each finite play ending in a position v of Player i to some successor w of v. A play $\pi = (v_n)_{n<\alpha}$ *follows* a strategy σ for Player i if, for every prefix $p = (v_n)_{n<m}$ that ends in a position of Player i, we have

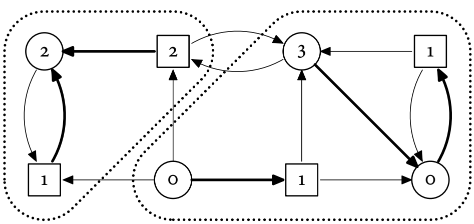
$$\sigma(p) = v_m \,.$$

(b) A stragey σ for Player i is *winning* from a position v if every play that starts at v and follows σ is winning for Player i.

(c) A strategy σ is *positional* if the value of $\sigma(p)$ only depends on the last position of p.

(d) We say that the game $\mathcal{G}$ is *determined* if, for every position v, one of the players has a winning strategy from that position. Analogously, we say that $\mathcal{G}$ is *positionally determined* if, for every position v, one of the players has a positional winning strategy.

Example. The game



from the example above is positionally determined. We have indicated the two winning strategies by bold arrows and the winning regions by dotted lines.

The aim of this section is to prove that every parity game is positionally determined. Furthermore, we would like to define the winning positions of each player by an MSO-formula. To do so, we derive a concrete description of the winning regions by a fixed-point equation.

Definition 1.3. Let $f : \wp(A) \to \wp(A)$ be a monotone function, i.e., a function such that $X \subseteq Y$ implies $f(X) \subseteq f(Y)$.

(a) The *least fixed point* of f is the least (with respect to $\subseteq$) set $X \subseteq A$ such that $f(X) = X$. We denote it by

$$[\operatorname{lfp} X : f(X)] .$$

The *greatest fixed point* of f is the greatest set $X \subseteq A$ such that $f(X) = X$. We denote it by

$$[\operatorname{gfp} X : f(X)] .$$

(b) For an ordinal α, we define the *α-th stage $f^{(\alpha)}$ of the least fixed-point induction* of f as

$$\begin{aligned} f^{(0)} &:= \varnothing , \\ f^{(\alpha+1)} &:= f(f^{(\alpha)}) , \\ f^{(\delta)} &:= \bigcup_{\alpha<\delta} f^{(\alpha)}, \quad \text{for limit ordinals } \delta . \end{aligned}$$

Similarly, the *α-th stage $f^{[\alpha]}$ of the greatest fixed-point induction* of f is

$$\begin{aligned} f^{[0]} &:= A , \\ f^{[\alpha+1]} &:= f(f^{[\alpha]}) , \\ f^{[\delta]} &:= \bigcap_{\alpha<\delta} f^{[\alpha]}, \quad \text{for limit ordinals } \delta . \end{aligned}$$

Lemma 1.4. *Let $f : \wp(A) \to \wp(A)$ be a monotone function and set $\kappa := |A|^+$.*

(a) $f^{(\alpha)} \subseteq f^{(\beta)}$, *for $\alpha \leq \beta$.*

(b) $f^{(\alpha)} \subseteq X$, *for every fixed point X of f and all ordinals α.*

(c) $f^{(\kappa)} = f^{(\alpha)}$, *for all $\alpha \geq \kappa$.*

Proof. (a) First, we consider the case where $\beta = \alpha + 1$. The proof is by induction on α. For $\alpha = 0$, we have $f^{(0)} = \varnothing \subseteq f^{(1)}$. For the successor step, note that, by monotonicity of f,

$$f^{(\alpha)} \subseteq f^{(\alpha+1)} \quad \text{implies} \quad f^{(\alpha+1)} = f(f^{(\alpha)}) \subseteq f(f^{(\alpha+1)}) = f^{(\alpha+2)}.$$

For the limit step, suppose that δ is a limit ordinal. For every $\alpha < \delta$, we have

$$f^{(\alpha+1)} = f(f^{(\alpha)}) \subseteq f\big(\textstyle\bigcup_{\alpha<\delta} f^{(\alpha)}\big) = f(f^{(\delta)}) = f^{(\delta+1)}.$$

This implies that

$$f^{(\delta)} = \bigcup_{\alpha<\delta} f^{(\alpha)} \subseteq f^{(\delta+1)}.$$

Having shown that $f^{(\alpha)} \subseteq f^{(\alpha+1)}$, we prove the general case by induction on β. For $\beta = \alpha$ the claim is trivial. For the successor step, suppose that we have already proved that $f^{(\alpha)} \subseteq f^{(\beta)}$. Together with the above claim it then follows that $f^{(\alpha)} \subseteq f^{(\beta)} \subseteq f^{(\beta+1)}$. For the limit step, let δ be a limit ordinal and suppose that we have already shown that $f^{(\alpha)} \subseteq f^{(\beta)}$, for all $\alpha \leq \beta < \delta$. Then it follows that

$$f^{(\alpha)} \subseteq \bigcup_{\beta<\delta} f^{(\beta)} = f^{(\delta)}.$$

(b) We prove the claim by induction on α. Clearly, $f^{(0)} = \varnothing \subseteq X$. Furthermore,

$$f^{(\alpha)} \subseteq X \quad \text{implies} \quad f^{(\alpha+1)} = f(f^{(\alpha)}) \subseteq f(X) = X.$$

Finally, suppose that α is a limit ordinal and that we have already shown that $f^{(\beta)} \subseteq X$, for all $\beta < \alpha$. Then it follows that

$$f^{(\alpha)} = \bigcup_{\beta<\alpha} f^{(\beta)} \subseteq X\,.$$

(c) We start by showing that

$$f^{(\alpha)} = f^{(\alpha+1)} \quad \text{implies} \quad f^{(\alpha)} = f^{(\beta)}, \quad \text{for all } \beta \geq \alpha\,.$$

Hence, suppose that $f^{(\alpha)} = f^{(\alpha+1)} = f(f^{(\alpha)})$. Then $f^{(\alpha)}$ is a fixed point of f and it follows by (b) that $f^{(\beta)} \subseteq f^{(\alpha)}$, for $\beta \geq \alpha$. Together with (a), this implies that

$$f^{(\beta)} = f^{(\alpha)}, \quad \text{for } \beta \geq \alpha\,.$$

To conclude the proof, suppose that $f^{(\kappa)} \neq f^{(\kappa+1)}$. By what we have just shown, this implies that

$$f^{(\alpha)} \neq f^{(\alpha+1)}, \quad \text{for all } \alpha \leq \kappa\,.$$

Consequently, for every $\alpha \leq \kappa$, there exists an element $a_\alpha \in f^{(\alpha+1)} \smallsetminus f^{(\alpha)}$. Since, by (a), these elements are distinct, it follows that

$$|A| \geq \kappa = |A|^+ > |A|\,.$$

A contradiction. □

Theorem 1.5 (Knaster–Tarski). *Every monotone function $f : \wp(A) \to \wp(A)$ has a least and a greatest fixed point. Furthermore,*

$$[\mathrm{lfp}\, X : f(X)] = f^{(\kappa)} \quad \textit{and} \quad [\mathrm{gfp}\, X : f(X)] = f^{[\kappa]}$$

where $\kappa := |A|^+$.

Proof. First, note that Lemma 1.4 (c) implies that $f(f^{(\kappa)}) = f^{(\kappa+1)} = f^{(\kappa)}$. Hence, $f^{(\kappa)}$ is a fixed point of f. The fact that it is the least one is a consequence of Lemma 1.4 (b).

To show that f also has a greatest fixed point, we consider the function

$$f'(X) := A \smallsetminus f(A \smallsetminus X) .$$

This function is also monotone and it follows from the first part of the proof that it has a least fixed point Z. Since a set $X \subseteq A$ is a fixed point of f' if, and only if, $A \smallsetminus X$ is a fixed point of f, it follows that $A \smallsetminus Z$ is the greatest fixed point of f. Furthermore

$$f^{[\alpha]} = A \smallsetminus (f')^{(\alpha)} .$$

Consequently, the equation $[\mathrm{gfp}\, X : f(X)] = f^{[\kappa]}$ also follows from the first part of the proof. $\square$

We can express the winning regions of a parity game in the following way as a fixed point.

Definition 1.6. Let $\mathcal{G} = \langle V_0, V_1, E, \Omega \rangle$ be a parity game and fix a number $m < \omega$ such that $\mathrm{rng}\, \Omega \subseteq [m]$.

(a) For $X \subseteq V$, we write

$$\begin{aligned} \Diamond X &:= \{\, v \in V \mid \mathrm{Suc}(v) \cap X \neq \varnothing \,\} , \\ \Box X &:= \{\, v \in V \mid \mathrm{Suc}(v) \subseteq X \,\} . \end{aligned}$$

(b) We define functions

$$\Phi_i : \wp(V)^{m-i} \to \wp(V) , \quad \text{for } 0 \leq i \leq m ,$$

as follows. For $W_0, \dots, W_{m-1} \subseteq V$, let

$$\Gamma(W_0, \dots, W_{m-1}) := \bigcup_{k<m} \big(\Omega^{-1}(k) \cap W_k\big),$$

$$\Phi_0(W_0, \dots, W_{m-1}) := [V_0 \cap \Diamond\Gamma(\bar{W})] \cup [V_1 \cap \Box\Gamma(\bar{W})],$$

$$\Phi_{i+1}(W_0, \dots, W_{m-i-2}) := \begin{cases} [\text{lfp}\, X : \Phi_i(\bar{W}, X)] & \text{if } i \text{ is even}, \\ [\text{gfp}\, X : \Phi_i(\bar{W}, X)] & \text{if } i \text{ is odd}. \end{cases}$$

Furthermore, for $0 \le i < m$, we denote by $\Phi_{i+1}^{(\alpha)}(W_0, \dots, W_{m-i-2})$ the α-th stage of the least fixed-point induction of the function

$$X \mapsto \Phi_i(W_0, \dots, W_{m-i-2}, X).$$

To understand the above definition, note that one property of the winning region W for Player 0 is the fact that, once the game reaches a position in W, she can ensure that the game will remain in W. This means that every V_0-position has a successor in W and every V_1-position only has successors in W. Consequently, W must satisfy the equation

$$W = [V_0 \cap \Diamond W] \cup [V_1 \cap \Box W].$$

This equation has several solutions (for instance, $W = V$ and $W = \varnothing$). To pick the right one, we distinguish the positions in W according to their priority, i.e., we consider the sets $W_k := W \cap \Omega^{-1}(k)$. Then we obtain a system of m equations of the form

$$\begin{aligned} W_0 &= \Big[V_0 \cap \Diamond \bigcup_{k<m} (\Omega^{-1}(k) \cap W_k)\Big] \\ &\quad \cup \Big[V_1 \cap \Box \bigcup_{k<m} (\Omega^{-1}(k) \cap W_k)\Big], \\ &\ \vdots \\ W_{m-1} &= \Big[V_0 \cap \Diamond \bigcup_{k<m} (\Omega^{-1}(k) \cap W_k)\Big] \\ &\quad \cup \Big[V_1 \cap \Box \bigcup_{k<m} (\Omega^{-1}(k) \cap W_k)\Big], \end{aligned}$$

which in our terminology can be written as

$$\begin{aligned} W_0 &= \Phi_0(W_0, \dots, W_{m-1}), \\ &\vdots \\ W_{m-1} &= \Phi_0(W_0, \dots, W_{m-1}). \end{aligned}$$

It turns out that, in order to obtain the correct solution, we have to take the least solution W_i, for odd priorities i, and the largest one, for even i. This leads to the above definition of Φ_i.

Definition 1.7. Let $\mathcal{G} = \langle V_0, V_1, E, \Omega\rangle$ be a parity game and let $m < \omega$ be a number such that $\operatorname{rng} \Omega \subseteq [m]$.

(a) We denote the lexicographic order on tuples of ordinals by $\leq_{\text{lex}}$ and its strict variant by $<_{\text{lex}}$. For two m-tuples $\bar{\alpha} = \langle \alpha_0, \dots, \alpha_{m-1}\rangle$ and $\bar{\beta} = \langle \beta_0, \dots, \beta_{m-1}\rangle$ of ordinals and $k < m$, we define

$$\begin{aligned} &\bar{\alpha} \leq_k \bar{\beta} \quad &&:\text{iff} \quad \langle \alpha_0, \dots, \alpha_k\rangle \leq_{\text{lex}} \langle \beta_0, \dots, \beta_k\rangle, \\ \text{and}\quad &\bar{\alpha} <_k \bar{\beta} \quad &&:\text{iff} \quad \langle \alpha_0, \dots, \alpha_k\rangle <_{\text{lex}} \langle \beta_0, \dots, \beta_k\rangle. \end{aligned}$$

(b) A *progress measure* is a partial function μ mapping positions of $\mathcal{G}$ to m-tuples of ordinals such that

- for every $v \in V_0 \cap \operatorname{dom}(\mu)$, there exists some $u \in \operatorname{Suc}(v) \cap \operatorname{dom}(\mu)$ such that

$$\begin{aligned} \mu(u) &\leq_{\Omega(u)} \mu(v), \quad && \text{if } \Omega(u) \text{ is even}, \\ \mu(u) &<_{\Omega(u)} \mu(v), \quad && \text{if } \Omega(u) \text{ is odd}. \end{aligned}$$

- for every $v \in V_1 \cap \operatorname{dom}(\mu)$ and all $u \in \operatorname{Suc}(v)$, we have $u \in \operatorname{dom}(\mu)$ and

$$\begin{aligned} \mu(u) &\leq_{\Omega(u)} \mu(v), \quad && \text{if } \Omega(u) \text{ is even}, \\ \mu(u) &<_{\Omega(u)} \mu(v), \quad && \text{if } \Omega(u) \text{ is odd}. \end{aligned}$$

(c) The *canonical measure* μ maps a position $v \in V$ to the lexicographically least tuple $\bar{\alpha}$ of ordinals such that

$$v \in \Phi_0(W_0^{\bar{\alpha}}, \dots, W_{m-1}^{\bar{\alpha}}),$$

where

$$W_i^{\bar{\alpha}} := \begin{cases} \Phi_{m-i}(W_0^{\bar{\alpha}}, \dots, W_{i-1}^{\bar{\alpha}}) & \text{if } i \text{ is even}, \\ \Phi_{m-i}^{(\alpha_i)}(W_0^{\bar{\alpha}}, \dots, W_{i-1}^{\bar{\alpha}}) & \text{if } i \text{ is odd}. \end{cases}$$

If there is no such tuple α, we leave $\mu(v)$ undefined. (Note that the ordinals $\alpha_0, \alpha_2, \dots, \alpha_{m-2}$ are not used in the definition of $W_i^{\bar{\alpha}}$. Since we choose them minimal, this means that they are all equal to 0.)

Lemma 1.8. *Let $\mathcal{G} = \langle V_0, V_1, E, \Omega \rangle$ be a parity game and let μ be the canonical measure.*

(a) $\operatorname{dom}(\mu) = \Phi_m$.

(b) *μ is a progress measure.*

(c) *From every position $v \in \operatorname{dom}(\mu)$, Player 0 has a positional winning strategy.*

(d) *From every position $v \notin \operatorname{dom}(\mu)$, Player 1 has a positional winning strategy.*

Proof. (a) By induction on $i < m$, we define

$$W_i^\infty := \Phi_{m-i}(W_0^\infty, \dots, W_{i-1}^\infty).$$

Then

$$\begin{aligned} W_i^\infty &= \Phi_{m-i}(W_0^\infty, \dots, W_{i-1}^\infty) \\ &= \begin{cases} [\operatorname{lfp} X : \Phi_{m-i-1}(\bar{W}^\infty, X)] & \text{if } m-i \text{ is odd}, \\ [\operatorname{gfp} X : \Phi_{m-i-1}(\bar{W}^\infty, X)] & \text{if } m-i \text{ is even}, \end{cases} \end{aligned}$$

which implies that

$$W_i^\infty = \Phi_{m-i-1}(W_0^\infty, \dots, W_{i-1}^\infty, W_i^\infty) = W_{i+1}^\infty .$$

Consequently,

$$\Phi_m = W_0^\infty = W_1^\infty = \dots = W_{m-1}^\infty = \Phi_0(W_0^\infty, \dots, W_{m-1}^\infty) .$$

($\subseteq$) Let $v \in \mathrm{dom}(\mu)$. Then there is some tuple $\bar\alpha := \mu(v)$ such that $v \in \Phi_0(W_0^{\bar\alpha}, \dots, W_{m-1}^{\bar\alpha})$. Since $\Phi_{m-i}^{(\alpha_i)}(\bar W^{\bar\alpha}) \subseteq \Phi_{m-i}(\bar W^{\bar\alpha})$, it follows by monotonicity of Φ_i that $W_i^{\bar\alpha} \subseteq W_i^\infty$. Consequently,

$$v \in \Phi_0(W_0^{\bar\alpha}, \dots, W_{m-1}^{\bar\alpha}) \subseteq \Phi_0(W_0^\infty, \dots, W_{m-1}^\infty) = \Phi_m .$$

($\supseteq$) Given $v \in \Phi_m$, consider the sets $\bar W^{\kappa\dots\kappa}$ where $\kappa := |V|^+$. According to the Theorem of Knaster and Tarski, we have

$$\Phi_i^{(\kappa)}(\bar W^{\kappa\dots\kappa}) = [\mathrm{lfp}\, X : \Phi_{i-1}(\bar W^{\kappa\dots\kappa}, X)] = \Phi_i(\bar W^{\kappa\dots\kappa}) ,$$

for odd $i < m$. Consequently, $W_i^{\kappa\dots\kappa} = \Phi_{m-i}(W_0^{\kappa\dots\kappa}, \dots, W_{i-1}^{\kappa\dots\kappa})$ and it follows by induction that $W_i^{\kappa\dots\kappa} = W_i^\infty$. Hence,

$$v \in \Phi_m = \Phi_0(W_0^\infty, \dots, W_{m-1}^\infty) = \Phi_0(W_0^{\kappa\dots\kappa}, \dots, W_{m-1}^{\kappa\dots\kappa}) .$$

This implies that $\mu(v) \leq_{\mathrm{lex}} \langle\kappa, \dots, \kappa\rangle$. In particular, $\mu(v)$ is defined.

(b) Let $v \in V_0 \cap \mathrm{dom}(\mu)$. Then there are ordinals $\bar\alpha := \mu(v)$ such that $v \in \Phi_0(\bar W^\alpha)$. By definition of Φ_0, this implies that there is some position $u \in \mathrm{Suc}(v) \cap \Gamma(\bar W^\alpha)$. We claim that u is the desired successor. By definition of Γ, it follows that $u \in W_k^{\bar\alpha}$ where $k := \Omega(u)$. Hence, $\mu(u) \leq_{\mathrm{lex}} \bar\alpha$. In particular,

$$\mu(u) \leq_k \bar\alpha = \mu(v) .$$

Finally, suppose that k is odd. Note that

$$\Phi^\alpha = \bigcup_{\beta<\alpha} \Phi^{\beta+1} = \bigcup_{\beta<\alpha} \Phi(\Phi^\beta) ,$$

for every function Φ. Hence,

$$\begin{aligned} u \in W_k^{\bar{\alpha}} &= \Phi_{m-k}^{(\alpha_k)}(W_0^{\bar{\alpha}}, \dots, W_{k-1}^{\bar{\alpha}}) \\ &= \bigcup_{\beta<\alpha_k} \Phi_{m-k-1}\big(W_0^{\bar{\alpha}}, \dots, W_{k-1}^{\bar{\alpha}}, \Phi_{m-k}^{(\beta)}(W_0^{\bar{\alpha}}, \dots, W_{k-1}^{\bar{\alpha}})\big) \\ &= \bigcup_{\beta<\alpha_k} W^{\langle \alpha_0, \dots, \alpha_{k-1}, \beta, \kappa, \dots, \kappa \rangle} \end{aligned}$$

which implies that $\mu(u) \leq_{\text{lex}} \langle \alpha_0, \dots, \alpha_{k-1}, \beta, \kappa, \dots, \kappa \rangle$, for some $\beta < \alpha_k$. Hence, $\mu(u) <_k \bar{\alpha} = \mu(v)$.

It remains to consider the case where $v \in V_1 \cap \operatorname{dom}(\mu)$. Again there are ordinals $\bar{\alpha} := \mu(v)$ such that $v \in \Phi_0(\bar{W}^\alpha)$. By definition of Φ_0, this implies that every successor $u \in \operatorname{Suc}(v)$ is contained in $\Gamma(\bar{W}^\alpha)$. As above, it follows that

$$\begin{aligned} &\mu(u) \leq_{\Omega(u)} \mu(v) \qquad \text{if } \Omega(u) \text{ is even,} \\ &\mu(u) <_{\Omega(u)} \mu(v) \qquad \text{if } \Omega(u) \text{ is odd.} \end{aligned}$$

(c) By (b), we can choose, for every vertex $v \in V_0 \cap \operatorname{dom}(\mu)$, a successor $\sigma(v)$ with

$$\begin{aligned} &\mu(\sigma(v)) \leq_{\Omega(\sigma(v))} \mu(v)\,, \quad \text{if } \Omega(\sigma(v)) \text{ is even,} \\ &\mu(\sigma(v)) <_{\Omega(\sigma(v))} \mu(v)\,, \quad \text{if } \Omega(\sigma(v)) \text{ is odd.} \end{aligned}$$

We claim that this function σ is a winning strategy. Let $(v_n)_{n<\omega}$ be a play following σ and let k be the minimal priority occurring infinitely often in $(v_n)_{n<\omega}$. We have to show that v is even. For a contradiction, suppose that it is odd. Fix some index $N < \omega$ such that no priority less than k appears in $v_N, v_{N+1}, v_{N+2}, \dots$. Note that, by definition of σ,

$$v_n \in \operatorname{dom}(\mu) \quad \text{implies} \quad v_{n+1} \in \operatorname{dom}(\mu)\,.$$

Consequently, it follows that the whole play is contained in $\operatorname{dom}(\mu)$. By choice of N, we have

$$\mu(v_N) \geq_k \mu(v_{N+1}) \geq_k \mu(v_{N+2}) \geq_k \cdots,$$

where the order is strict if the priority is odd. Let $n_0 < n_1 < \dots$ be the sequences of all indices $n \geq N$ such that $\Omega(v_{n+1}) = k$. Then we obtain an infinite strictly decreasing sequence

$$\mu(v_{n_0}) >_k \mu(v_{n_1}) >_k \mu(v_{n_2}) >_k \cdots.$$

A contradiction, since $\leq_k$ is a well-order.

(d) Let $\mathcal{G}' := \langle V_1, V_0, E, \Omega' \rangle$ be the game obtained from $\mathcal{G}$ by exchanging the positions of the players and by increasing all priorities by 1:

$$\Omega'(v) := \Omega(v) + 1\,.$$

Let μ' be the canonical measure of $\mathcal{G}'$. By (c) it is sufficient to prove that

$$\mathrm{dom}(\mu') = V \setminus \mathrm{dom}(\mu)\,.$$

Let $\Gamma', \Phi'_0, \dots, \Phi'_m$ be the functions and sets used to define μ'. Inspecting the definition, we see that

$$\begin{aligned} \Gamma'(W_0, \dots, W_m) &= \Gamma(W_1, \dots, W_m)\,, \\ V \setminus \Gamma'(W_0, \dots, W_m) &= \Gamma(V \setminus W_1, \dots, V \setminus W_m)\,. \end{aligned}$$

Consequently,

$$\begin{aligned} &\Phi'_0(W_0, \dots, W_m) \\ &\qquad = \big\{ v \in V_1 \bigm| \mathrm{Suc}(v) \cap \Gamma'(W_0, \dots, W_m) \neq \varnothing \big\} \\ &\qquad \cup \big\{ v \in V_0 \bigm| \mathrm{Suc}(v) \subseteq \Gamma'(W_0, \dots, W_m) \big\}\,, \\ &\qquad = \big\{ v \in V_1 \bigm| \mathrm{Suc}(v) \nsubseteq V \setminus \Gamma'(W_0, \dots, W_m) \big\} \\ &\qquad \cup \big\{ v \in V_0 \bigm| \mathrm{Suc}(v) \cap V \setminus \Gamma'(W_0, \dots, W_m) = \varnothing \big\}\,, \\ &\qquad = \big\{ v \in V_1 \bigm| \mathrm{Suc}(v) \nsubseteq \Gamma(V \setminus W_1, \dots, V \setminus W_m) \big\} \\ &\qquad \cup \big\{ v \in V_0 \bigm| \mathrm{Suc}(v) \cap \Gamma(V \setminus W_1, \dots, V \setminus W_m) = \varnothing \big\}\,, \\ &\qquad = V \setminus \Phi_0(V \setminus W_1, \dots, V \setminus W_m)\,. \end{aligned}$$

By induction on i, it therefore follows that

$$\Phi'_i(W_0, \dots, W_{m-i-1}) = V \setminus \Phi_i(V \setminus W_1, \dots, V \setminus W_{m-i-1}) .$$

By (a), this implies that

$$\begin{aligned} \operatorname{dom}(\mu') = \Phi'_{m+1} &= [\operatorname{lfp} X : \Phi'_m(X)] \\ &= [\operatorname{lfp} X : V \setminus \Phi_m] \\ &= V \setminus \Phi_m = V \setminus \operatorname{dom}(\mu) . \end{aligned}$$

□

By part (c) and (d) of the previous lemma, we immediately see that every parity game is positionally determined.

Theorem 1.9. *Every parity game is positionally determined.*

We can also use this lemma to prove that the winning regions can be defined by an MSO-formula. To do so, we have to encode parity games as relational structures.

Definition 1.10. Let $\mathcal{G} = \langle V_0, V_1, E, \Omega \rangle$ be a parity game. The *game structure* associated with $\mathcal{G}$ is the graph

$$\mathfrak{G} := \langle V, E, (P_k)_k, V_0, V_1 \rangle$$

with universe $V := V_0 \cup V_1$, edge relation E, unary predicates V_0 and V_1 to specify the owner of a position, and unary predicates

$$P_k := \Omega^{-1}(k) , \quad \text{for } k \in \operatorname{rng} \Omega ,$$

that specify the priorities.

Theorem 1.11. *Let $m < \omega$. There exists an* MSO*-formula $\varphi(x)$ such that, for every parity game $\mathcal{G}$ with priorities in $[m]$ and with associated game structure $\mathfrak{G}$,*

$$\mathfrak{G} \vDash \varphi(v) \quad \text{iff} \quad \textit{Player } 0 \textit{ has a winning strategy starting at } v .$$

Proof. It is sufficient to construct formulae $\psi_i(x, Z_0, \dots, Z_{m-i-1})$, for $i \leq m$, such that

$$\mathfrak{G} \models \psi_i(v, \bar{W}) \quad \text{iff} \quad v \in \Phi_i(\bar{W}) \,.$$

By Lemma 1.8, it then follows that

$$\begin{aligned} \mathfrak{G} \models \psi_m(v) \quad &\text{iff} \quad v \in \Phi_m = \mathrm{dom}(\mu) \\ &\text{iff} \quad \text{Player 0 has a winning strategy from } v. \end{aligned}$$

For $i = 0$, we define

$$\begin{aligned} \psi_0(x, Z_0, \dots, Z_{m-1}) := {} & [V_0 x \to \exists y [Exy \wedge \vartheta(y, \bar{Z})]] \\ & \wedge [V_1 x \to \forall y [Exy \to \vartheta(y, \bar{Z})]] \,, \end{aligned}$$

where

$$\vartheta(y, \bar{Z}) := \bigwedge_{i<m} (P_i y \to Z_i y) \,.$$

For odd $i > 0$, we set

$$\begin{aligned} & \psi_i(x, Z_0, \dots, Z_{m-i-1}) := \\ & \quad \forall Y \big[\forall y [Yy \leftrightarrow \psi_{i-1}(y, Z_0, \dots, Z_{m-i-1}, Y)] \to Yx \big] \,, \end{aligned}$$

and for even $i > 0$, we set

$$\begin{aligned} & \psi_i(x, Z_0, \dots, Z_{m-i-1}) := \\ & \quad \exists Y \big[\forall y [Yy \leftrightarrow \psi_{i-1}(y, Z_0, \dots, Z_{m-i-1}, Y)] \wedge Yx \big] \,. \end{aligned}$$

□

Remark. A very short proof of the preceding theorem is the following one. The MSO-formula in the above is equivalent to a formula of the modal μ-calculus:

$$\begin{aligned} \mu Z_0 . \nu Z_1 . \dots \mu Z_{m-2} . \nu Z_{m-1} \Big[\Big[V_0 \to \Diamond \bigwedge_{i<m} (P_i \to Z_i) \Big] \\ \wedge \Big[V_1 \to \Box \bigwedge_{i<m} (P_i \to Z_i) \Big] \Big] \,. \end{aligned}$$

It turns out that the model checking game for this formula is equivalent to the given parity game. Consequently, the above formula holds if, and only if, Verifier wins the corresponding model checking game if, and only if, Player 0 wins the parity game.

We conclude this section with a method to transfer winning strategies from one game to another.

Definition 1.12. Let $\mathcal{G} = \langle V_0, V_1, E, \Omega \rangle$ and $\mathcal{G}' = \langle V_0', V_1', E', \Omega' \rangle$ be parity games. A *bisimulation* between $\mathcal{G}$ and $\mathcal{G}'$ is a binary relation $Z \subseteq V \times V'$ such that, for all $\langle v, v' \rangle \in Z$,

- $v \in V_0 \quad \text{iff} \quad v' \in V_0'$;
- $\Omega(v) = \Omega(v')$;
- (*Forth Property*) for every $u \in \mathrm{Suc}(v)$, there is some $u' \in \mathrm{Suc}(v')$ with $\langle u, u' \rangle \in Z$;
- (*Back Property*) for every $u' \in \mathrm{Suc}(v')$, there is some $u \in \mathrm{Suc}(v)$ with $\langle u, u' \rangle \in Z$.

We say that $\mathcal{G}$ and $\mathcal{G}'$ are *bisimilar* if there exists a bisimulation Z between them that relates every position of $\mathcal{G}$ to some position of $\mathcal{G}'$ and every position of $\mathcal{G}'$ to some of $\mathcal{G}$.

Lemma 1.13. *Let Z be a bisimulation between two parity games $\mathcal{G} = \langle V_0, V_1, E, \Omega \rangle$ and $\mathcal{G}' = \langle V_0', V_1', E', \Omega' \rangle$ and let $\langle v, v' \rangle \in Z$. Player 0 has a positional winning strategy in $\mathcal{G}$ starting at v if, and only if, she has a positional winning strategy in $\mathcal{G}'$ starting at v'.*

Proof. Let σ be a positional strategy for Player 0 in $\mathcal{G}$ that is winning when starting at postition v. We define a strategy σ' for Player 0 in $\mathcal{G}'$ as follows. For every position $u' \in V_0'$, choose, if possible, some position $u \in V_0$ with $\langle u, u' \rangle \in Z$. By the Forth Property, there is some successor $w' \in \mathrm{Suc}(u')$ with $\langle \sigma(u), w' \rangle \in Z$. We set $\sigma'(u') := w'$. If there is no such position u, we leave $\sigma'(u')$ undefined.

We claim that σ' is a winning strategy. Let $(v'_n)_{n<\omega}$ be a play of $\mathcal{G}'$ following σ' with $v'_0 = v'$. First, note that we can show by induction on n that $\sigma'(v'_n)$ is defined for all indices $n < \omega$ with $v'_n \in V'_0$. To prove that $(v'_n)_{n<\omega}$ is winning, we construct a play $(v_n)_{n<\omega}$ in $\mathcal{G}$ such that $\langle v_n, v'_n \rangle \in Z$, for all $n < \omega$. We start with $v_0 := v$. Suppose that we have already defined $v_0, \dots, v_n$. If $v_n \in V_0$, we set $v_{n+1} := \sigma(v_n)$. By definition of σ', it then follows that

$$\langle v_{n+1}, v'_{n+1} \rangle = \langle \sigma(v_n), \sigma'(v'_n) \rangle \in Z\,.$$

If $v_n \in V_1$, we use the Back Property to select an arbitrary position $v_{n+1} \in V$ with $\langle v_{n+1}, v'_{n+1} \rangle \in Z$. The play $(v_n)_{n<\omega}$ constructed in this way follows the strategy σ. As σ is winning for Player 0, the play satisfies the parity condition. Note that $\langle v_n, v'_n \rangle \in Z$ implies $\Omega'(v'_n) = \Omega(v_n)$. Hence, $(v'_n)_{n<\omega}$ also satisfies the parity condition. $\square$

Alternating parity games

In games where the player take turns alternatingly, we can simplify the formula defining the winning region.

Definition 1.14. (a) A parity game $\mathcal{G} = \langle V_0, V_1, E, \Omega \rangle$ is *alternating* if

$$\begin{aligned} & v \in V_0 \quad \text{implies} \quad \mathrm{Suc}(v) \subseteq V_1\,, \\ \text{and} \quad & v \in V_1 \quad \text{implies} \quad \mathrm{Suc}(v) \subseteq V_0\,. \end{aligned}$$

(b) An alternating parity game is *normalised* if all positions of Player 1 have maximal priority, while all positions of Player 0 have some priority that is not maximal.

Lemma 1.15. *For every alternating parity game $\mathcal{G} = \langle V_0, V_1, E, \Omega \rangle$, there exists a normalised alternative parity game $\mathcal{G}' = \langle V'_0, V'_1, E', \Omega' \rangle$ and a surjective function $\pi : V' \to V$ such that Player 0 has a winning strategy starting at a position v' in $\mathcal{G}'$ if, and only if, he has a winning strategy starting at $\pi(v')$ in $\mathcal{G}$.*

Proof. To be able to remove the priorities of V_1-positions, we will store them in the V_0-positions. Choose some number m with $\operatorname{rng}\Omega \subseteq [m]$. To define $\mathcal{G}'$, we set

$$\begin{aligned}
V_0' &:= V_0 \times [m]\,, \\
V_1' &:= V_1\,, \\
E' &:= \{\, \langle\langle u,k\rangle, v\rangle \mid \langle u,v\rangle \in E \cap (V_0 \times V_1) \,\} \\
&\quad \cup \{\, \langle u, \langle v, \Omega(u)\rangle\rangle \mid \langle u,v\rangle \in E \cap (V_1 \times V_0) \,\}\,, \\
\Omega'(\langle v,k\rangle) &:= \min\{\Omega(v), k\}\,, \qquad \text{for } \langle v,k\rangle \in V_0'\,, \\
\Omega'(v) &:= m\,, \qquad \text{for } v \in V_1'\,.
\end{aligned}$$

Let $\pi : V' \to V$ be the function that maps positions $v \in V_1' = V_1$ to itself and positions $\langle v,k\rangle \in V_0'$ to the corresponding V_0-position v. We claim that π is the desired surjection.

Let σ' be a positional winning strategy for Player 0 in $\mathcal{G}'$. We define a (non-positional) strategy σ in $\mathcal{G}$ by setting

$$\sigma(v_0,\dots,v_n) := \begin{cases} \sigma'(\langle v_n, \Omega(v_{n-1})\rangle) & \text{if } n > 0\,, \\ \sigma'(\langle v_n, m-1\rangle) & \text{if } n = 0\,. \end{cases}$$

If $(v_n)_{n<\omega}$ is a play in $\mathcal{G}$ following σ, there is a unique play $(v_n')_{n<\omega}$ in $\mathcal{G}'$ such that $\pi(v_n') = v_n$, for all n. This play follows σ' and is therefore winning for Player 0. By definition of π, it follows that the play $(v_n')_{n<\omega}$ has the form

$$\langle v_0, k\rangle, v_1, \langle v_2, \Omega(v_1)\rangle, v_3, \langle v_4, \Omega(v_3)\rangle, \dots, \qquad \text{for some } k < m\,.$$

(For simplicity, we have assumed that the play starts in V_0'. Otherwise, we have to remove the first entry.) The corresponding sequence of priorities is

$$\begin{aligned}
&\min\{\Omega(v_0), k\},\ m,\ \min\{\Omega(v_1), \Omega(v_2)\}, \\
&\qquad\qquad m,\ \min\{\Omega(v_3), \Omega(v_4)\}, \dots\,.
\end{aligned}$$

As $(v_n)_{n<\omega}$ satisfies the parity condition, it follows that so does $(v'_n)_{n<\omega}$.

Conversely, consider a positional winning strategy σ for Player 0 in $\mathcal{G}$. We define a positional strategy σ' in $\mathcal{G}'$ by setting

$$\begin{aligned} \sigma'(\langle v, k\rangle) &:= \sigma(v)\,, && \text{for } \langle v, k\rangle \in V'_0\,,\\ \sigma'(v) &:= \langle \sigma(v), \Omega(v)\rangle\,, && \text{for } v \in V'_1\,. \end{aligned}$$

Let $(v'_n)_{n<\omega}$ be a play in $\mathcal{G}'$ following σ'. Then $(\pi(v'_n))_{n<\omega}$ is a play in $\mathcal{G}$ following σ. As above it follows that the fact that $(\pi(v'_n))_{n<\omega}$ satisfies the parity condition implies that so does $(v'_n)_{n<\omega}$. ☐

For normalised alternating games we can compute the winning regions in a different way taking two steps at a time.

Definition 1.16. Let $\mathcal{G}$ be a normalised alternating parity game and let m be its maximal priority. We define functions

$$\tilde{\Phi}_i : \wp(V)^{m-i+1} \to \wp(V)\,, \quad \text{for } 0 \le i \le m+1\,,$$

by

$$\begin{aligned} \tilde{\Phi}_0(W_0, \dots, W_m) &:= [V_0 \cap \Diamond\Box\Gamma(\bar{W})] \cup [V_1 \cap \Box\Gamma(\bar{W})]\,,\\ \tilde{\Phi}_{i+1}(W_0, \dots, W_{m-i-1}) &:= \begin{cases} [\operatorname{lfp} X : \tilde{\Phi}_i(\bar{W}, X)] & \text{if } i \text{ is even}\,,\\ [\operatorname{gfp} X : \tilde{\Phi}_i(\bar{W}, X)] & \text{if } i \text{ is odd}\,. \end{cases} \end{aligned}$$

Lemma 1.17. *Let $f : \wp(A) \to \wp(A)$ be monotone. Then*

$$[\operatorname{lfp} X : f(X)] = [\operatorname{lfp} X : f(f(X))]\,.$$

Proof. Set $F(X) := f(f(X))$ and let $f^{(\alpha)}$ and $F^{(\alpha)}$ be the stages of the corresponding least fixed-point inductions. We claim that

$$F^{(\omega\alpha+n)} = f^{(\omega\alpha+2n)}\,, \quad \text{for all } n < \omega \text{ and all ordinals } \alpha\,.$$

Clearly,

$$F^{(0)} = \varnothing = f^{(0)},$$

and $F^{(\omega\alpha+n)} = f^{(\omega\alpha+2n)}$ implies that

$$\begin{aligned} F^{(\omega\alpha+n+1)} &= F(F^{(\omega\alpha+n)}) \\ &= F(f^{(\omega\alpha+2n)}) = f(f(f^{(\omega\alpha+2n)})) = f(f^{(\omega\alpha+2n+2)}) \,. \end{aligned}$$

For limit ordinals $\delta = \omega\alpha$, we have

$$\begin{aligned} F^{(\delta)} = \sup_{\beta<\delta} F^{(\beta)} = \sup_{\beta<\omega\alpha} F^{(\beta)} &= \sup_{\beta<\alpha} \sup_{n<\omega} F^{(\omega\beta+n)} \\ &= \sup_{\beta<\alpha} \sup_{n<\omega} f^{(\omega\beta+2n)} \\ &= \sup_{\beta<\omega\alpha} f^{(\beta)} = f^{(\delta)} \,. \end{aligned}$$

In particular, for $\kappa := |A|^+$, it follows that $F^{(\kappa)} = f^{(\kappa)}$. Consequently, the claim follows by the Theorem of Knaster and Tarski. □

Lemma 1.18. *Let $\mathcal{G}$ be a normalised alternating parity game and let m be its maximal priority.*

(a) $\tilde{\Phi}_0(W_0, \dots, W_m) = \Phi_0(W_0, \dots, W_{m-1}, \Phi_0(W_0, \dots, W_m))$.

(b) $\tilde{\Phi}_i(W_0, \dots, W_{m-i}) = \Phi_i(W_0, \dots, W_{m-i})$, *for $i > 0$.*

Proof. (a) Since $V_0 = \Omega^{-1}(0) \cup \dots \cup \Omega^{-1}(m-1)$ and $V_1 = \Omega^{-1}(m)$ and since $\mathcal{G}$ is alternating, we have

$$\begin{aligned}
&\Phi_0(W_0, \dots, W_{m-1}, \Phi_0(W_0, \dots, W_m)) \\
&= \Big[V_0 \cap \Diamond\Big[\bigcup_{k<m} (\Omega^{-1}(k) \cap W_k) \cup (\Omega^{-1}(m) \cap \Phi_0(\bar{W}))\Big]\Big] \\
&\cup \Big[V_1 \cap \Box\Big[\bigcup_{k<m} (\Omega^{-1}(k) \cap W_k) \cup (\Omega^{-1}(m) \cap \Phi_0(\bar{W}))\Big]\Big] \\
&= [V_0 \cap \Diamond(V_1 \cap \Phi_0(\bar{W}))] \cup \Big[V_1 \cap \Box \bigcup_{k<m} (\Omega^{-1}(k) \cap W_k)\Big] \\
&= [V_0 \cap \Diamond(V_1 \cap \Box\Gamma(\bar{W})))] \\
&\cup \Big[V_1 \cap \Box\Big[\bigcup_{k<m} (\Omega^{-1}(k) \cap W_k) \cup (\Omega^{-1}(m) \cap W_m)\Big]\Big] \\
&= [V_0 \cap \Diamond \Box \Gamma(\bar{W}))] \cup [V_1 \cap \Box\Gamma(\bar{W})] \\
&= \tilde{\Phi}_0(\bar{W}) .
\end{aligned}$$

(b) It is sufficient to prove the claim for $i = 1$. Then the other cases will follow by induction. By (a) and Lemma 1.17, we have

$$\begin{aligned}
\tilde{\Phi}_1(\bar{W}) &= [\mathrm{lfp}\, X : \tilde{\Phi}_0(\bar{W}, X)] \\
&= [\mathrm{lfp}\, X : \Phi_0(\bar{W}, \Phi_0(\bar{W}, X))] \\
&= [\mathrm{lfp}\, X : \Phi_0(\bar{W}, X)] = \Phi_1(\bar{W}) .
\end{aligned}$$
□

Lemma 1.19. *Let $f : \wp(A) \to \wp(A)$ be a monotone function and let $C \subseteq A$ be a set such that*

$$C \cap f(X) = C \cap f(C \cap X), \quad \textit{for all } X \subseteq A .$$

Then

$$\begin{aligned}
& C \cap [\mathrm{lfp}\, X : f(X)] = [\mathrm{lfp}\, X : C \cap f(X)] , \\
\textit{and} \quad & C \cap [\mathrm{gfp}\, X : f(X)] = [\mathrm{gfp}\, X : C \cap f(X)] .
\end{aligned}$$

Proof. Set $f_C(X) := C \cap f(X)$ and let $f^{(\alpha)}$ and $f_C^{(\alpha)}$ be the stages of the corresponding least fixed-point inductions. To show that

$$C \cap [\mathrm{lfp}\, X : f(X)] = [\mathrm{lfp}\, X : C \cap f(X)]\,,$$

we prove by induction on α that

$$C \cap f^{(\alpha)} = f_C^{(\alpha)}\,, \quad \text{for all ordinals } \alpha\,.$$

For $\alpha = 0$, we have

$$C \cap f^{(0)} = C \cap \varnothing = \varnothing = f_C^{(0)}\,.$$

For the successor step, we have

$$\begin{aligned} C \cap f^{(\alpha+1)} &= C \cap f(f^{(\alpha)}) \\ &= C \cap f(C \cap f^{(\alpha)}) \\ &= C \cap f(f_C^{(\alpha)}) = f_C(f_C^{(\alpha)}) = f_C^{(\alpha+1)}\,. \end{aligned}$$

For the limit step, suppose that δ is a limit ordinal. Then

$$C \cap f^{(\delta)} = C \cap \bigcup_{\alpha<\delta} f^{(\alpha)} = \bigcup_{\alpha<\delta} (C \cap f^{(\alpha)}) = \bigcup_{\alpha<\delta} f_C^{(\alpha)} = f_C^{(\delta)}\,.$$

The proof of the second equation is analogous using the greatest fixed-point induction. □

Definition 1.20. Let $\mathcal{G}$ be a normalised alternating parity game and let m be its maximal priority. We define functions

$$\hat{\Phi}_i : \wp(V)^{m-i+1} \to \wp(V)\,, \quad \text{for } 0 \le i \le m+1\,,$$

by

$$\hat{\Phi}_0(W_0, \dots, W_m) := V_0 \cap \Diamond \Box \Gamma(\bar{W})\,,$$

$$\hat{\Phi}_{i+1}(W_0, \dots, W_{m-i-1}) := \begin{cases} [\mathrm{lfp}\, X : \hat{\Phi}_i(\bar{W}, X)] & \text{if } i \text{ is even}\,, \\ [\mathrm{gfp}\, X : \hat{\Phi}_i(\bar{W}, X)] & \text{if } i \text{ is odd}\,. \end{cases}$$

Lemma 1.21. *Let $\mathcal{G}$ be a normalised alternating parity game and let m be its maximal priority. Then*

$$\hat{\Phi}_i(W_0, \dots, W_{m-i}) = V_0 \cap \tilde{\Phi}_i(W_0, \dots, W_{m-i}) .$$

Proof. For $i = 0$, the claim follows by definition of $\hat{\Phi}_0$ and $\tilde{\Phi}_0$ and the fact that $\mathcal{G}$ is alternating. For $i > 0$, we can use Lemma 1.19 and the inductive hypothesis to show that

$$\begin{aligned}
\hat{\Phi}_i(\bar{W}) &= \begin{cases} [\mathrm{lfp}\, X : \hat{\Phi}_{i-1}(\bar{W}, X)] & \text{if } i \text{ is odd}, \\ [\mathrm{gfp}\, X : \hat{\Phi}_{i-1}(\bar{W}, X)] & \text{if } i \text{ is even}, \end{cases} \\
&= \begin{cases} [\mathrm{lfp}\, X : V_0 \cap \tilde{\Phi}_{i-1}(\bar{W}, X)] & \text{if } i \text{ is odd}, \\ [\mathrm{gfp}\, X : V_0 \cap \tilde{\Phi}_{i-1}(\bar{W}, X)] & \text{if } i \text{ is even}, \end{cases} \\
&= V_0 \cap \tilde{\Phi}_i(\bar{W}) .
\end{aligned}$$

□

Theorem 1.22. *Let $m < \omega$. There exists an* MSO*-formula $\varphi(x)$ where all set quantifiers range over subsets of V_0 such that, for every normalised alternating parity game $\mathcal{G}$ with maximal priority m and with associated game structure $\mathfrak{G}$,*

$$\mathfrak{G} \vDash \varphi(v) \quad \text{iff} \quad v \in V_0 \text{ and Player } 0 \text{ has a winning strategy starting at } v .$$

Proof. As in Theorem 1.11 we construct formulae $\hat{\psi}_i(x, Z_0, \dots, Z_{m-i})$ such that

$$\mathfrak{G} \vDash \hat{\psi}_i(v, W_0, \dots, W_{m-i}) \quad \text{iff} \quad v \in \hat{\Phi}_i(W_0, \dots, W_{m-i}) .$$

It then follows by Lemmas 1.18 and 1.21 that

$$\mathfrak{G} \vDash \hat{\psi}_m(v) \quad \text{iff} \quad v \in \hat{\Phi}_m = V_0 \cap \tilde{\Phi}_m = V_0 \cap \Phi_m ,$$

as desired.

For $i = 0$, we define

$$\hat{\psi}_0(x, Z_0, \dots, Z_m) := V_0 x \wedge \exists y \forall z [Exy \wedge Eyz \to \vartheta(z, \bar{Z})]] ,$$

where

$$\vartheta(y, \bar{Z}) := \bigwedge_{i \leq m} (P_i y \to Z_i y) .$$

For odd $i > 0$, we set

$$\begin{aligned} &\hat{\psi}_i(x, Z_0, \dots, Z_{m-i}) := \\ &\quad (\forall Y \subseteq V_0)\big[\forall y[Yy \leftrightarrow \hat{\psi}_{i-1}(y, Z_0, \dots, Z_{m-i}, Y)] \to Yx\big] , \end{aligned}$$

and for even $i > 0$, we set

$$\begin{aligned} &\hat{\psi}_i(x, Z_0, \dots, Z_{m-i}) := \\ &\quad (\exists Y \subseteq V_0)\big[\forall y[Yy \leftrightarrow \hat{\psi}_{i-1}(y, Z_0, \dots, Z_{m-i}, Y)] \wedge Yx\big] . \end{aligned}$$

For correctness of these definitions, note that the fixed point is a subset of V_0 since $\hat{\Phi}_i(\bar{W}) = V_0 \cap \tilde{\Phi}_i(\bar{W}) \subseteq V_0$, by Lemma 1.21. □

2. *Tree automata*

Definition 2.1. Let D be a set.

(a) The *prefix ordering* on D^* is defined by

$$x \leq y \quad :\text{iff} \quad y = xz , \quad \text{for some } z \in D^* .$$

(b) A *tree domain* is a subset $T \subseteq D^*$ that is *prefix-closed*, i.e., such that $x \leq y \in T$ implies $x \in T$. If T is a tree domain and $x, y \in T$, we call x an (immediate) *successor* of y if $y = xd$, for some $d \in D$. In this case, we also say that y is an (immediate) *predecessor* of x. We write $\mathrm{Suc}(x)$ for the set of all successors of x and $\mathrm{Suc}_*(x)$ for $\{x\} \cup \mathrm{Suc}(x)$.

(c) A *tree* is a structure isomorphic to one of the form $\langle T, \leq \rangle$ where T is a tree domain. The elements of a tree are called *vertices*.

(d) An *enriched tree* is a structure $\langle T, \leq, \bar{R} \rangle$ where $\langle T, \leq \rangle$ is a tree and each relation R_i only contains tuples $\bar{a}$ where all components a_i have the same predecessor, i.e., there are $w \in T$ and $c_0, c_1, \dots \in D$ such that $a_i = wc_i$, for all i.

(e) A *branch* β of an (enriched) tree $\mathfrak{T}$ is a maximal linearly ordered set of vertices. For a branch β of $\mathfrak{T}$, we write $\beta(n)$ for the nth vertex of β and we write $w < \beta$ to indicate that w is some vertex of β. That is, sometimes we identify a branch $\beta \subseteq T$ of length α with a function $\alpha \to T$ or with a word in D^α.

In this section we prove that, over enriched trees, monadic second-order logic is equivalent to tree automata.

Definition 2.2. (a) Let Σ be a signature of enriched trees and let Q be a set. We write $\Sigma_Q := \Sigma \cup \{\mathrm{rt}\} \cup \{ P_q \mid q \in Q \}$ for the expanded signature where rt is a constant symbol and the P_q are unary predicates. The *transition logic* $\mathrm{TL}[\Sigma, Q]$ consists of those $\mathrm{MSO}[\Sigma_Q]$-formulae where all subformulae of the form $P_q x$, for $q \in Q$, only appear positively, i.e., under an even number of negation signs.

(b) Let $\mathfrak{T}$ be an enriched tree over the signature Σ and let $\rho : T \to Q$ be a function. The *successor structure* $\mathfrak{Suc}_*(v; \rho)$ of a vertex $v \in T$ is the Σ_Q-structure

$$\mathfrak{Suc}_*(v; \rho) := \langle \mathfrak{S}, (P_q)_{q \in Q}, v \rangle ,$$

where $\mathfrak{S}$ is the substructure of $\mathfrak{T}$ induced by the set $\mathrm{Suc}_*(v)$ and

$$P_q := \{ x \in \mathrm{Suc}(v) \mid \rho(x) = q \} , \quad \text{for } q \in Q .$$

(c) A *nondeterministic tree automaton* is a tuple $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega \rangle$ where Q is a finite set of *states,* Σ is a finite *input signature* of enriched trees, $q_0 \in Q$ is the *initial state,* $\Omega : Q \to \omega$ is a *priority function,* and $\delta : Q \to \mathrm{TL}[\Sigma, Q]$ is the *transition function.*

(d) A *run* of a tree automaton $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega \rangle$ on an enriched tree $\mathfrak{T}$ over the signature Σ is a function $\rho : T \to Q$ such that

$$\mathfrak{Suc}_*(v; \rho) \models \delta(\rho(v)) , \quad \text{for all } v \in T .$$

A run ρ is *accepting* if $\rho(\langle\rangle) = q_0$ and, for every branch β of $\mathfrak{T}$,

$$\liminf_{n\to\infty} \Omega(\rho(\beta(n))) \quad \text{is even.}$$

(e) A tree automaton $\mathcal{A}$ *accepts* an enriched tree $\mathfrak{T}$ over the signature Σ if there exists an accepting run of $\mathcal{A}$ on $\mathfrak{T}$. The *language recognised* by $\mathcal{A}$ is the set $L_{\mathrm{nd}}(\mathcal{A})$ of all trees it accepts.

Example. The following nondeterministic tree automaton recognises the language of all trees over the alphabet $\{a, b\}$ the contain at least one letter a.

$$\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega\rangle$$

where $Q := \{q_0\}$, $\Omega(q_0) := 1$, and

$$\delta(q_0, a) := \mathit{true} \quad \text{and} \quad \delta(q_0, b) := \exists x[x \neq \mathrm{rt} \wedge P_{q_0} x]\,.$$

Definition 2.3. An *alternating tree automaton* $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega\rangle$ is defined in the same way as a nondeterministic one, except that the notions of a run and a successor structure are defined differently.

A *run* of an alternating automaton $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega\rangle$ on an enriched tree $\mathfrak{T}$ over the signature Σ is a function $\rho : T \to \wp(Q \times Q)$ such that

$$\mathfrak{Suc}_*(v; \rho/q) \vDash \delta(q)\,, \quad \text{for all } \langle p, q\rangle \in \rho(v) \text{ and all } v \in T\,,$$

where the *successor structure* $\mathfrak{Suc}_*(v; \rho/q)$ is obtained from the substructure of $\mathfrak{T}$ induced by the set $\mathrm{Suc}_*(v)$ by adding the predicates

$$P_p := \{\, x \in \mathrm{Suc}(v) \mid \langle q, p\rangle \in \rho(x)\,\}\,, \quad \text{for } p \in Q\,.$$

A *trace* of a run ρ is a sequence $(p_n)_{n<\omega}$ of states such that, for some branch β of ρ,

$$\langle p_n, p_{n+1}\rangle \in \rho(\beta(n))\,, \quad \text{for all } n < \omega\,.$$

A run ρ is *accepting* if $\langle q_0, q_0 \rangle \in \rho(\langle\rangle)$ and, for every trace $(p_n)_{n<\omega}$ of ρ,

$$\liminf_{n\to\infty} \Omega(p_n) \text{ is even.}$$

The *language recognised* by $\mathcal{A}$ is denoted $L_{\mathrm{alt}}(\mathcal{A})$

Example. The following alternating tree automaton recognises the language of all trees over the alphabet $\{a, b, c\}$ that contain at least one letter a and at least one letter b.

$$\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega \rangle$$

where $Q := \{q_0, q_a, q_b\}$, $\Omega(q_0) = \Omega(q_a) = \Omega(q_b) = 1$, and

$$\delta(q_0, d) := \begin{cases} \exists x[x \neq \mathrm{rt} \wedge P_{q_b} x] & \text{if } d = a\,, \\ \exists x[x \neq \mathrm{rt} \wedge P_{q_a} x] & \text{if } d = b\,, \\ \exists x[x \neq \mathrm{rt} \wedge P_{q_a} x] \wedge \exists x[x \neq \mathrm{rt} \wedge P_{q_b} x] & \text{if } d = c\,, \end{cases}$$

$$\delta(q_a, d) := \begin{cases} \mathit{true} & \text{if } d = a\,, \\ \exists x[x \neq \mathrm{rt} \wedge P_{q_a} x] & \text{otherwise}\,, \end{cases}$$

$$\delta(q_b, d) := \begin{cases} \mathit{true} & \text{if } d = b\,, \\ \exists x[x \neq \mathrm{rt} \wedge P_{q_b} x] & \text{otherwise}\,. \end{cases}$$

Exercise 2.1. Find tree automata recognising the following languages over the alphabet $\{a, b\}$.

(a) The language of all trees containing infinitely many letters a.

(b) The language of all trees such that below every vertex there is some vertex with the letter a.

(c) The language of all trees such that there is some vertex below which there are only letters a.

(d) The language of all trees such that some branch contains only letters a.

(e) The language of all trees such that every branch contains at least one a.

(f) The language of all trees such that every branch contains only finitely many a.

Exercise 2.2.
(a) Let $L \subseteq \Sigma^\omega$ be an MSO-definable language. Construct a tree automaton $\mathcal{A}$ accepting the language of all trees where each infinite branch belongs to L.

(b) Let $L \subseteq \{0,1\}^*$ be an MSO-definable language. Construct a tree automaton $\mathcal{A}$ accepting the language of all trees over the alphabet $\{a,b\}$ such that a vertex v is labelled by a if, and only if, $v \in L$.

Exercise 2.3. Let $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega\rangle$ be a nondeterministic tree automaton. Find an MSO-formula defining $L(\mathcal{A})$.

In the following we will prove several closure properties for languages recognised by alternating automata. We start with the closure under union.

Proposition 2.4. *Given two alternating tree automata $\mathcal{A}$ and $\mathcal{A}'$, we can compute an alternating tree automaton that recognises the language $L_{\text{alt}}(\mathcal{A}) \cup L_{\text{alt}}(\mathcal{A}')$.*

Proof. Let $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega\rangle$ and $\mathcal{A}' = \langle Q', \Sigma, \delta', q_0', \Omega'\rangle$. We set

$$\mathcal{B} := \left\langle Q \cup Q' \uplus \{q_+\}, \Sigma, \delta_+, q_+, \Omega_+ \right\rangle$$

where

$$\delta_+(q) := \begin{cases} \delta(q) & \text{if } q \in Q\,, \\ \delta'(q) & \text{if } q \in Q'\,, \\ \delta(q_0) \vee \delta'(q_0') & \text{if } q = q_+\,, \end{cases}$$

$$\Omega(q) := \begin{cases} \Omega(q) & \text{if } q \in Q\,, \\ \Omega'(q) & \text{if } q \in Q'\,, \\ 0 & \text{if } q = q_+\,. \end{cases}$$

We claim that $L_{\mathrm{alt}}(\mathcal{B}) = L_{\mathrm{alt}}(\mathcal{A}) \cup L_{\mathrm{alt}}(\mathcal{A}')$.

($\supseteq$) Let ρ be an accepting run of $\mathcal{A}$ on some tree $\mathfrak{T}$. We obtain an run of $\mathcal{B}$ on $\mathfrak{T}$ by replacing every occurrence of the state q_0 in $\rho(\langle\rangle)$ by q_+. This run is again accepting.

In the same way, we can turn an accepting run of $\mathcal{A}'$ on $\mathfrak{T}$ into one of $\mathcal{B}$.

($\subseteq$) Let ρ be an accepting run of $\mathcal{B}$ on some tree $\mathfrak{T}$. By definition of $\delta_+(q_+)$, we have

$$\mathfrak{Suc}_*(\langle\rangle; \rho/q_+) \vDash \delta(q_0) \vee \delta'(q'_0)\,.$$

By symmetry, we may suppose that

$$\mathfrak{Suc}_*(\langle\rangle; \rho/q_+) \vDash \delta(q_0)\,.$$

Then we can replace every occurrence of the state q_+ in $\rho(\langle\rangle)$ by q_0, and we can remove all occurrences of states of $\mathcal{A}'$ from ρ. This results in a run of $\mathcal{A}$ on $\mathfrak{T}$ which is again accepting. $\square$

Closure under projection

For the closure under projection, we make a detour via nondeterministic automata. Therefore, we need to prove that they are equivalent to alternating ones.

Proposition 2.5. *For every nondeterministic tree automaton $\mathcal{A}$, we can compute an alternating tree automaton $\mathcal{B}$ recognising the same language.*

Proof. Let $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega\rangle$ be a nondeterministic automaton. We construct the automaton $\widetilde{\mathcal{A}} := \langle Q, \Sigma, \tilde{\delta}, q_0, \Omega\rangle$ with transition function

$$\tilde{\delta}(q) := \exists \bar{Z}\big[\mathrm{REFINE}(\bar{Z}, \bar{P}) \wedge \delta(q)[\bar{Z}/\bar{P}]\big]$$

where the formula

$$\mathrm{REFINE}(\bar{Z}, \bar{P}) := \bigwedge_{p \in Q} Z_p \subseteq P_p \wedge \bigwedge_{p \neq q} Z_p \cap Z_q = \varnothing \\ \wedge \forall x\Big[x \neq \mathrm{rt} \rightarrow \bigvee_{p \in Q} Z_p\Big]$$

states that $\bar{Z}$ is a partition of all non-root vertices that is contained in $\bar{P}$. ($\varphi[\bar{Z}/\bar{P}]$ denotes the formula obtained from φ by replacing each atomic subformula of the form $P_i x$ by the corresponding formula $Z_i x$.) We claim that

$$L_{\mathrm{nd}}(\mathcal{A}) = L_{\mathrm{nd}}(\widetilde{\mathcal{A}}) = L_{\mathrm{alt}}(\widetilde{\mathcal{A}})\,.$$

For the first equation, let ρ be a run of a nondeterministic automaton. Then the predicates $\bar{P}$ of the structure $\mathfrak{Suc}_*(v;\rho)$ form a partition of the non-root vertices. Hence, $\bar{Z} = \bar{P}$ is the unique family of sets satisfying the formula $\mathrm{REFINE}(\bar{Z},\bar{P})$. Consequently,

$$\mathfrak{Suc}_*(v;\rho) \vDash \tilde{\delta}(q) \quad \text{iff} \quad \mathfrak{Suc}_*(v;\rho) \vDash \delta(q)\,.$$

It follows that ρ is an accepting run of $\widetilde{\mathcal{A}}$ if, and only if, it is an accepting run of $\mathcal{A}$.

For the second equation, let ρ be an accepting run of $\widetilde{\mathcal{A}}$, considered as a nondeterministic automaton. Then we obtain an accepting run ρ' of the corresponding alternating automaton by setting

$$\rho'(\langle\rangle) := \{\langle q_0, q_0\rangle\} \quad \text{and} \quad \rho'(v) := \{\langle \rho(u), \rho(v)\rangle\}\,,$$

for every vertex v with immediate predecessor u.

Conversely, let ρ be an accepting run of $\widetilde{\mathcal{A}}$, considered as an alternating automaton. We construct an accepting run ρ' of the corresponding nondeterministic automaton by induction. We start with $\rho'(\langle\rangle) := q_0$. For the inductive step, suppose that $\rho'(v)$ has already been defined. Since

$$\mathfrak{Suc}_*(v;\rho/\rho'(v)) \vDash \tilde{\delta}(\rho'(v))$$

there exists a family $\bar{P}'$ of sets such that

$$\mathfrak{Suc}_*(v;\rho/\rho'(v)) \vDash \mathrm{REFINE}(\bar{P}',\bar{P}) \wedge \delta(\rho'(v))[\bar{P}'/\bar{P}]\,.$$

For each $u \in \mathrm{Suc}(v)$, we define

$$\rho'(u) := q \quad \text{where} \quad u \in P'_p\,.$$

(As $\bar{P}'$ is a partition of $\mathrm{Suc}(v)$, this is well-defined.) Then ρ' is a run of $\widetilde{\mathcal{A}}$. It is accepting since, in the above construction, the sets $\bar{P}'$ are chosen as subsets of $\bar{P}$. $\square$

Definition 2.6. Let Q be a set and $\Omega : Q \to \omega$ a priority function. The *trace semigroup* is the semigroup $\mathfrak{S}_\Omega(Q) := \langle S, S_\omega \rangle$ where

$$S := \wp(Q \times Q) \quad \text{and} \quad S_\omega := \wp(Q)\,.$$

For $A, B \in S$ and $P \in S_\omega$, the binary products are defined by

$$A \cdot B := \{\, \langle p, r\rangle \mid \langle p, q\rangle \in A \text{ and } \langle q, r\rangle \in B \,\}\,,$$
$$A \cdot P := \{\, p \in Q \mid q \in P \text{ for all } q \text{ with } \langle p, q\rangle \in A \,\}\,.$$

Given an infinite sequence $A_0, A_1, \dots \in S$, we define the infinite product $\pi(A_0, A_1, \dots)$ as follows. We call a sequence $(p_n)_{n<\omega}$ of states a *trace* of $(A_n)_{n<\omega}$ if

$$\langle p_n, p_{n+1}\rangle \in A_n\,, \quad \text{for all } n < \omega\,.$$

The set $\pi(A_0, A_1, \dots)$ consists of all states $p \in Q$ such that every trace $(p_n)_{n<\omega}$ of $(A_n)_{n<\omega}$ with $p_0 = p$ satisfies the parity condition Ω.

Proposition 2.7. *For every alternating tree automaton $\mathcal{A}$, we can compute a nondeterministic tree automaton $\mathcal{B}$ recognising the same language.*

Proof. Let $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega\rangle$ be an alternating tree automaton. We start by constructing a nondeterministic tree automaton $\mathcal{C}$ with set of states $Q' := \wp(Q \times Q)$ such that every run $\rho : T \to \wp(Q \times Q)$ of $\mathcal{A}$ is also a run $T \to Q'$ of $\mathcal{C}$, and vice verse. This can be done by defining the transition function $\delta' : Q' \to \mathrm{TL}[\Sigma, Q']$ as

$$\delta'(A) := \bigwedge_{\langle p,q\rangle \in A} \delta(q)[\vartheta^q_r / P_r]_{r \in Q}$$

where

$$\vartheta^q_r(x) := \bigvee \big\{\, P_A x \;\big|\; A \in Q' \text{ with } \langle q, r\rangle \in A \,\big\}\,,$$

and where, for a formula $\varphi \in \mathrm{TL}[\Sigma, Q]$, we denote by $\varphi[\vartheta_r/P_r]_{r\in Q}$ the formula obtained from φ by replacing every subformula of the form $P_r x$, for $r \in Q$, by the corresponding formula $\vartheta_r(x)$.

Clearly, every run of $\mathcal{A}$ is a run of $\mathcal{C}$ and every run of $\mathcal{C}$ is one of $\mathcal{A}$. Unfortunately, the same is not true for accepting runs. Therefore, we modify $\mathcal{C}$ as follows.

Let $\mathfrak{S}_\Omega(Q) = \langle Q', \wp(Q)\rangle$ be the trace semigroup of Q. According to Theorem II.7.5 we can effectively construct a deterministic automaton $\mathcal{D} = \langle \tilde{Q}, Q', \tilde{\delta}, \tilde{q}_0, \tilde{\Omega}\rangle$ that recognises the language of all ω-words $w \in (Q')^\omega$ whose product $\pi(w) \in \wp(Q)$ contains the state q_0.

The automaton $\mathcal{B} = \langle Q'', \Sigma, \delta'', q''_0, \Omega''\rangle$ is the product of $\mathcal{C}$ and $\mathcal{D}$. The set of states is $Q'' := Q' \times \tilde{Q}$, the initial state is $q''_0 := \big\langle \{\langle q_0, q_0\rangle\},\ \tilde{q}_0\big\rangle$, the priority function is

$$\Omega''(\langle A, p\rangle) := \tilde{\Omega}(p)\,,$$

and the transition function is defined by

$$\delta''(\langle A, p\rangle) := \delta'(A)[\vartheta_B/P_B]_{B\in Q'} \wedge \forall x\big[x \neq \mathrm{rt} \rightarrow \eta_{\tilde{\delta}(p,A)}(x)\big]\,.$$

where

$$\vartheta_A(x) := \bigvee_{p\in\tilde{Q}} P_{\langle A,p\rangle}x \quad \text{and} \quad \eta_p(x) := \bigvee_{A\in Q'} P_{\langle A,p\rangle}x\,.$$

We claim that $\mathcal{B}$ accepts the same trees as $\mathcal{A}$. Suppose that $\mathfrak{T} \in L(\mathcal{A})$ and let $\rho : T \rightarrow \wp(Q \times Q)$ be an accepting run of $\mathcal{A}$ on $\mathfrak{T}$. Then ρ is also a run of $\mathcal{C}$. We define a function $\tau : T \rightarrow \tilde{Q}$ by setting

$$\tau(v) := \tilde{\delta}^*(\tilde{q}_0, v)\,,$$

where $\tilde{\delta}^*$ is the extension of $\tilde{\delta} : \tilde{Q}\times Q' \rightarrow \tilde{Q}$ to a function $\tilde{Q}\times(Q')^* \rightarrow \tilde{Q}$. We obtain a run $\rho' : T \rightarrow Q''$ by setting

$$\rho'(v) := \langle \rho(v), \tau(v)\rangle\,.$$

To show that ρ' is accepting, consider a branch β of $\mathfrak{T}$. Since every trace of ρ satisfies the parity condition, the product

$$\pi(\rho(v))_{v<\beta}$$

evaluates to a set containing the state q_0. Consequently, $\mathcal{D}$ accepts the word $(\rho(v))_{v<\beta}$ and the run $(\tau(v))_{v<\beta}$ is accepting, i.e., it satisfies the parity condition. By definition of Ω'', it follows that $(\rho'(v))_{v<\beta}$ also satisfies the parity condition.

Conversely, suppose that $\mathfrak{T} \in L(\mathcal{B})$ and let $\rho : T \to Q''$ be a corresponding accepting run. Let $\rho' : T \to Q'$ and $\tau : T \to \tilde{Q}$ be the functions such that

$$\rho(v) = \langle \rho'(v), \tau(v) \rangle .$$

Then ρ' is a run of $\mathcal{C}$ and, hence, one of $\mathcal{A}$. To show that it is accepting, let β be a branch of $\mathfrak{T}$. Since the sequence $(\rho(v))_{v<\beta}$ satisfies the parity condition, it follows by definition of Ω'' that so does the projection $(\tau(v))_{v<\beta}$. By definition of the trace semigroup, this implies that every trace of $(\rho'(v))_{v<\beta}$ satisfies the parity condition. $\square$

Proposition 2.8. *Let Σ be a signature of enriched trees and let $P \notin \Sigma$ be an additional unary predicate. For every nondeterministic tree automaton $\mathcal{A}$ over the signature $\Sigma \cup \{P\}$, we can compute a nondeterministic tree automaton $\mathcal{B}$ over the signature Σ such that*

$$L(\mathcal{B}) = \{ \mathfrak{T} \mid \langle \mathfrak{T}, U \rangle \in L(\mathcal{A}) \text{ for some } U \subseteq T \} .$$

Proof. Given $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega \rangle$, we set

$$\mathcal{B} := \langle Q \times \{0,1\} \cup \{q_+\}, \Sigma, \delta', q_+, \Omega' \rangle$$

where the priority function is

$$\Omega'(\langle q, b \rangle) := \Omega(q)$$

and the transition function is

$$\begin{aligned}\delta'(q_+) &:= \exists U \delta(q_0)[\vartheta_p/P_p]_{p \in Q}\,,\\ \delta'(\langle q, b\rangle) &:= \exists U \delta(q)[\eta^b/U][\vartheta_p/P_p]_{p\in Q}\end{aligned}$$

with

$$\begin{aligned}\eta^0(x) &:= (x \neq \mathrm{rt}) \wedge Ux\\ \eta^1(x) &:= (x = \mathrm{rt}) \vee Ux\\ \vartheta_p(x) &:= [\neg Ux \wedge P_{\langle p,0\rangle}x] \vee [Ux \wedge P_{\langle p,1\rangle}x]\,.\end{aligned}$$

We claim that $\mathcal{B}$ accepts a tree $\mathfrak{T}$ if, and only if, $\mathcal{A}$ accepts $\langle T, U\rangle$, for some $U \subseteq U$.

($\Leftarrow$) Suppose that $\langle \mathfrak{T}, U\rangle \in L(\mathcal{A})$ and let ρ be a corresponding accepting run. We obtain an accepting run ρ' of $\mathcal{B}$ on $\mathfrak{T}$ by setting

$$\rho'(v) := \begin{cases}\langle \rho(v), 0\rangle & \text{if } v \notin U\,,\\ \langle \rho(v), 1\rangle & \text{if } v \in U\,.\end{cases}$$

($\Rightarrow$) Suppose that $\mathfrak{T} \in L(\mathcal{B})$ and let ρ be a corresponding accepting run. For every vertex $v \in T$, there exists a set $U_v \subseteq \mathrm{Suc}_*(v)$ such that

$$\langle \mathfrak{Suc}_*(v;\rho), U_v\rangle \vDash \delta(q)[\eta^b/U][\vartheta_p/P_p]_{p\in Q}\,,$$

where $\langle q, b\rangle = \rho(v)$. By choice of the formulae η^b, we may choose U_v such that $v \notin U_v$, for $v \neq \langle\rangle$. Set $U := \bigcup_{v\in T} U_v$ and let $\rho' : T \to Q$ be the function such that $\rho'(\langle\rangle) := q_0$ and, for $v \neq \langle\rangle$,

$$\rho(v) = \langle \rho'(v), b_v\rangle\,, \quad \text{for some } b_v \in \{0,1\}\,.$$

Then ρ' is an accepting run of $\mathcal{A}$ on $\langle \mathfrak{T}, U\rangle$. ☐

Closure under complement

It remains to prove the closure under complement. To do so, we first introduce a game-based characterisation of acceptance.

Definition 2.9. Let $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega \rangle$ be an alternating automaton and let $\mathfrak{T}$ be an enriched tree over Σ. The *automaton game* $\mathcal{G}(\mathcal{A}, \mathfrak{T})$ for $\mathcal{A}$ on $\mathfrak{T}$ is the parity game where the positions of Player 0 (called *Automaton*) are

$$V_0 := T \times Q$$

and the positions of Player 1 (called *Trace-Finder*) are

$$V_1 := \bigcup_{v \in T} \wp(\mathrm{Suc}(v) \times Q) .$$

The initial position is $\langle \langle \rangle, q_0 \rangle$.

The edge relation is defined as follows. From a V_1-position H there are edges to every V_0-position $\langle u, p \rangle \in H$. From a V_0-position $\langle v, q \rangle$ there are edges to every V_1-position $H \subseteq \mathrm{Suc}(v) \times Q$ such that

$$\langle \mathfrak{S}, \bar{P}, v \rangle \vDash \delta(q) ,$$

where, similarly to the definition of $\mathfrak{Suc}_*(v; \rho)$, $\mathfrak{S}$ is the substructure of $\mathfrak{T}$ induced by the set $\mathrm{Suc}_*(v)$ and

$$P_p := \{ u \in \mathrm{Suc}(v) \mid \langle u, p \rangle \in H \} , \quad \text{for } p \in Q .$$

Finally, we assign to positions $\langle v, q \rangle \in V_0$ the priority $\Omega(q)$ and to positions $H \in V_1$ an arbitrary priority greater than all priorities in $\mathrm{rng}\, \Omega$.

Lemma 2.10. *Automaton has a winning strategy in $\mathcal{G}(\mathcal{A}, \mathfrak{T})$ if, and only if, $\mathfrak{T} \in L_{\mathrm{alt}}(\mathcal{A})$.*

Proof. ($\Leftarrow$) Given an accepting run ρ of $\mathcal{A}$ on $\mathfrak{T}$, we construct a winning strategy σ for Automaton in $\mathcal{G}(\mathcal{A}, \mathfrak{T})$ as follows. In a position $\langle v, q \rangle \in V_0$, Automaton chooses the new position

$$H := \{ \langle u, p \rangle \in \mathrm{Suc}(v) \times Q \mid \langle q, p \rangle \in \rho(u) \} .$$

To see that this strategy is winning, consider a play

$$\langle v_0, q_0 \rangle, H_0, \langle v_1, q_1 \rangle, H_1, \dots$$

where Automaton follows the strategy σ. By definition of σ, it follows that the sequence $q_0, q_1, \dots$ of states appearing in this play is a trace of ρ. As ρ is accepting, this trace satisfies the parity condition. Consequently, the above play also satisfies the parity condition and Automaton wins the game.

($\Rightarrow$) Let σ be a winning strategy for Automaton in $\mathcal{G}(\mathcal{A}, \mathfrak{T})$. We construct a run ρ of $\mathcal{A}$ on $\mathfrak{T}$ inductively as follows. We start with $\rho(\langle\rangle) := \{\langle q_0, q_0\rangle\}$. For the inductive step, suppose that $\rho(v)$ has already been defined. Let

$$U := \{\, q \in Q \mid \langle p, q\rangle \in \rho(v) \text{ for some } p \,\}\,.$$

For $u \in \mathrm{Suc}(v)$, set

$$\rho(u) := \{\, \langle q, p\rangle \mid q \in U\,, \langle u, p\rangle \in \sigma(\langle v, q\rangle) \,\}\,.$$

Then ρ is a run of $\mathcal{A}$ on $\mathfrak{T}$.

To see that it is accepting, consider a trace $(q_n)_{n<\omega}$ of ρ belonging to a branch β. Let

$$\langle v_0, q_0\rangle, H_0, \langle v_1, q_1\rangle, H_1, \dots$$

be the play of $\mathcal{G}(\mathcal{A}, \mathfrak{T})$ where Automaton follows the strategy σ and Trace-Finder chooses in step n the pair $\langle v_n, q_n\rangle \in H_{n-1}$ where v_n is the successor of v_{n-1} that lies on the branch β. Since σ was assumed to be winning, this play satisfies the parity condition. Hence, so does the trace $(q_n)_{n<\omega}$. □

Definition 2.11. Let $\varphi \in \mathrm{TL}[\Sigma, Q]$. The *dual* of φ is the formula $\overline{\varphi}$ obtained from $\neg\varphi$ by negating all atomic formulae of the form $P_q x$ with $q \in Q$.

Lemma 2.12. *Let $\mathfrak{S}$ be a Σ-structure, $a \in S$, and $\varphi \in \mathrm{TL}[\Sigma, Q]$. For a family $\bar{P}$ of subsets $P_q \subseteq S$, we have*

$$\langle \mathfrak{S}, \bar{P}, a\rangle \vDash \overline{\varphi} \quad \text{iff} \quad \textit{for all } \bar{P}' \textit{ in } \mathfrak{S} \textit{ with } \langle \mathfrak{S}, \bar{P}', a\rangle \vDash \varphi \textit{ there is some } q \in Q \textit{ with } P_q \cap P'_q = \varnothing\,.$$

Proof. Set $P_q^{\mathrm{c}} := S \setminus P_q$. Then

$$\begin{aligned}\langle \mathfrak{S}, \bar{P}, a \rangle \vDash \overline{\varphi} \quad &\text{iff} \quad \langle \mathfrak{S}, \bar{P}^{\mathrm{c}}, a \rangle \vDash \neg\varphi \\ &\text{iff} \quad \text{there is no } \bar{P}' \text{ in } \mathfrak{S} \text{ with } \langle \mathfrak{S}, \bar{P}', a \rangle \vDash \varphi \\ &\qquad\ \text{and } P'_q \subseteq P_q^{\mathrm{c}} \text{ for all } q \in Q \\ &\text{iff} \quad \text{for all } \bar{P}' \text{ in } \mathfrak{S} \text{ with } \langle \mathfrak{S}, \bar{P}', a \rangle \vDash \varphi\,, \text{ there} \\ &\qquad\ \text{is some } q \in Q \text{ with } P_q \cap P'_q \neq \varnothing\,,\end{aligned}$$

where the second step follows from the fact that the formula φ is monotone in $\bar{P}$. ☐

Proposition 2.13. *Given an alternating tree automaton $\mathcal{A}$, we can compute an alternating tree automaton that recognises the complement of $L_{\mathrm{alt}}(\mathcal{A})$.*

Proof. Suppose that $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \Omega \rangle$. We construct the new automaton $\mathcal{B} := \langle Q, \Sigma, \tilde{\delta}, q_0, \tilde{\Omega} \rangle$ where, for $q \in Q$,

$$\tilde{\delta}(q) := \overline{\delta(q)} \quad \text{and} \quad \tilde{\Omega}(q) := \Omega(q) + 1\,.$$

We claim that $\mathcal{B}$ recognises the complement of $L_{\mathrm{alt}}(\mathcal{A})$.

First, let $\mathfrak{T} \in L_{\mathrm{alt}}(\mathcal{A})$ and let ρ be an accepting run of $\mathcal{A}$ on $\mathfrak{T}$. Consider a run ρ' of $\mathcal{B}$ on $\mathfrak{T}$. We have to show that ρ' is not accepting. We define a branch $(v_n)_{n<\omega}$ of $\mathfrak{T}$ and a sequence $(p_n)_{n<\omega}$ of states such that

$$\langle p_n, p_{n+1} \rangle \in \rho(v_{n+1}) \cap \rho'(v_{n+1})\,, \quad \text{for all } n < \omega\,.$$

We start with $v_0 := \langle\rangle$ and $p_0 := q_0$. Suppose that v_n and p_n have already been defined. Then

$$\mathfrak{Suc}_*(v_n; \rho/p_n) \vDash \delta(p_n) \quad \text{and} \quad \mathfrak{Suc}_*(v_n; \rho'/p_n) \vDash \overline{\delta(p_n)}\,.$$

Hence, we can use Lemma 2.12 to find $v_{n+1} \in \mathrm{Suc}(v_n)$ and $p_{n+1} \in Q$ with

$$\langle p_n, p_{n+1} \rangle \in \rho(v_{n+1}) \cap \rho'(v_{n+1})\,.$$

Note that the sequence $(p_n)_{n<\omega}$ we constructed is a trace of both runs. Since ρ is accepting, it follows that this trace satisfies the parity condition, that is,

$$\liminf_{n\to\infty} \Omega(p_n) \text{ is even.}$$

Consequently,

$$\liminf_{n\to\infty} \Omega'(p_n) = \liminf_{n\to\infty} \Omega(p_n) + 1 \text{ is odd.}$$

We found a trace of ρ' that does not satisfy the parity condition. Consequently, ρ' is not accepting.

To conclude the proof we have to show that every tree $\mathfrak{T} \notin L_{\mathrm{alt}}(\mathcal{A})$ has an accepting run for the automaton $\mathcal{B}$. By Lemma 2.10, Trace-Finder has a winning strategy σ in $\mathcal{G}(\mathcal{A},\mathfrak{T})$ and it is sufficient to construct a winning strategy σ' for Automaton in $\mathcal{G}(\mathcal{B},\mathfrak{T})$. For $\langle v, q\rangle \in V_0$, we set

$$\begin{aligned}\sigma'(\langle v,q\rangle) := \big\{ \langle u,p\rangle \in \mathrm{Suc}(v) \times Q \;\big|\; &\text{in } \mathcal{G}(\mathcal{A},\mathfrak{T}) \text{ there is some } K \in \mathrm{Suc}(\langle v,q\rangle) \\ &\text{such that } \sigma(K) = \langle u,p\rangle \big\}\,.\end{aligned}$$

First, let us prove that $H := \sigma'(\langle v,q\rangle)$ is actually a successor of $\langle v,q\rangle$ in $\mathcal{G}(\mathcal{B},\mathfrak{T})$. Let $\mathfrak{S}$ be the substructure of $\mathfrak{T}$ induced by $\mathrm{Suc}_*(v)$ and set

$$P_p := \{\, u \in \mathrm{Suc}(v) \mid \langle u,p\rangle \in H \,\}\,.$$

We have to show that

$$\langle \mathfrak{S}, \bar P, v\rangle \vDash \overline{\delta(q)}\,.$$

By Lemma 2.12, it is sufficient to prove that, for all $\bar P'$ in $\mathfrak{S}$,

$$\langle \mathfrak{S}, \bar P', v\rangle \vDash \delta(q) \quad \text{implies} \quad P_p \cap P'_p \neq \varnothing\,,\ \text{for some } p \in Q\,.$$

Hence, suppose that $\langle \mathfrak{S}, \bar P', v\rangle \vDash \delta(q)$. Set

$$K := \{\, \langle u,p\rangle \in \mathrm{Suc}(v) \times Q \mid u \in P'_p \,\}\,.$$

By choice of $\bar{P}'$, the position K is a successor of $\langle v, q \rangle$ in $\mathcal{G}(\mathcal{A}, \mathfrak{T})$. Suppose that

$$\sigma(K) = \langle u, p \rangle \,.$$

By definition of H, it follows that $\langle u, p \rangle \in H$, i.e., $u \in P_p$. As $\langle u, p \rangle$ is a successor of K in $\mathcal{G}(\mathcal{A}, \mathfrak{T})$, we further have $\langle u, p \rangle \in K$, i.e., $u \in P'_p$. Consequently, $u \in P_p \cap P'_p \neq \varnothing$, as desired.

It remains to prove that σ' is a winning strategy for Automaton. Let

$$\langle v_0, q_0 \rangle, H_0, \langle v_1, q_1 \rangle, H_1, \dots$$

be a play in $\mathcal{G}(\mathcal{B}, \mathfrak{T})$ following σ'. Then $\langle v_n, q_n \rangle \in H_{n-1}$, for $n < \omega$. By definition of σ', it follows that in $\mathcal{G}(\mathcal{A}, \mathfrak{T})$ there are positions

$$K_{n-1} \in \mathrm{Suc}(\langle v_{n-1}, q_{n-1} \rangle)\,, \quad \text{for } n < \omega\,,$$

such that $\sigma(K_{n-1}) = \langle v_n, q_n \rangle$. Consequently,

$$\langle v_0, q_0 \rangle, K_0, \langle v_1, q_1 \rangle, K_1, \dots$$

is a play in $\mathcal{G}(\mathcal{A}, \mathfrak{T})$ following the strategy σ. Since σ is winning for Trace-Finder, it follows that the sequence $(q_n)_{n<\omega}$ does not satisfy the parity condition in $\mathcal{G}(\mathcal{A}, \mathfrak{T})$. As the priorities in $\mathcal{G}(\mathcal{B}, \mathfrak{T})$ are shifted by 1, this implies that $(q_n)_{n<\omega}$ does satisfy the parity condition in $\mathcal{G}(\mathcal{B}, \mathfrak{T})$. Hence, the play is winning for Automaton. $\square$

Equivalence to MSO

Theorem 2.14. *For every* $\mathrm{MSO}_0[\Sigma]$*-formula* $\varphi(\bar{X})$*, we can effectively construct an alternating tree automaton* $\mathcal{A}_\varphi$ *such that*

$$L_{\mathrm{alt}}(\mathcal{A}_\varphi) = \{\, \langle \mathfrak{T}, \bar{P} \rangle \mid \mathfrak{T} \vDash \varphi(\bar{P}) \,\}\,.$$

Proof. We construct $\mathcal{A}_\varphi$ by induction on φ. We may assume that φ does not contain subformulae of the form $X \cap Y = \varnothing$ or $\mathrm{cover}(\bar{X})$. Thus, there are the following cases.

If $\varphi = (X \subseteq Y)$, we set $\mathcal{A}_\varphi := \langle \{q_0\}, \Sigma, \delta, q_0, \Omega \rangle$ where

$$\delta(q_0) := \forall x[Xx \to Yx] \wedge \forall x P_{q_0} x \quad \text{and} \quad \Omega(q_0) := 0 .$$

If $\varphi = RX_0 \dots X_{n-1}$, we set $\mathcal{A}_\varphi := \langle \{q_0\}, \Sigma, \delta, q_0, \Omega \rangle$ where

$$\delta(q_0) := \exists \bar{x}\Big[R\bar{x} \wedge \bigwedge_{i<n} X_i x_i\Big] \vee \exists x[x \neq \mathrm{rt} \wedge P_{q_0} x]$$

and $\quad \Omega(q_0) := 1 .$

Suppose that $\varphi = \psi \vee \vartheta$. By inductive hypothesis, we can construct automata $\mathcal{A}_\psi$ and $\mathcal{A}_\vartheta$ for ψ and ϑ. Hence, we can use Proposition 2.4 to construct the desired automaton for φ.

Suppose that $\varphi = \neg\psi$. By inductive hypothesis, we can construct an automaton for $\mathcal{A}_\psi$. Hence, we can use Proposition 2.13 to construct the desired automaton for φ.

Suppose that $\varphi = \exists X\psi$. By inductive hypothesis, we can construct an automaton for $\mathcal{A}_\psi$. Since we can translate between alternating automata and nondeterministic automata, we can use Proposition 2.8 to construct the desired automaton for φ. □

Theorem 2.15. *For every alternating tree automaton $\mathcal{A}$ over the signature Σ, we can effectively construct an* MSO$[\Sigma]$*-formula $\varphi_{\mathcal{A}}$ such that*

$$\mathfrak{T} \vDash \varphi \quad \text{iff} \quad \mathfrak{T} \in L_{\mathrm{alt}}(\mathcal{A}) .$$

Proof. Let $\mathcal{A} := \langle Q, \Sigma, \delta, q_0, \Omega \rangle$. The formula $\varphi_{\mathcal{A}}$ guesses sets encoding a run of the automaton and then checks that the guessed run is accepting. We set

$$\varphi_{\mathcal{A}} := \exists (Z_{p,q})_{p,q \in Q}\big[\mathrm{INIT} \wedge \mathrm{TRANS} \wedge \mathrm{ACC}\big]$$

where we use the following formulae.

$$\mathrm{INIT} := \exists x[Z_{q_0,q_0} x \wedge \forall y(x \leq y)]$$

states that the root contains the pair $\langle q_0, q_0 \rangle$.

$$\mathrm{TRANS} := \forall x \bigwedge_{p,q \in Q} \big[Z_{p,q} x \to \hat{\delta}(q)(x) \big]$$

states that at every vertex a correct transition is used. Here, $\hat{\delta}(q)(x)$ denotes the restriction of the formula $\delta(q)$ to the set $\mathrm{Suc}_*(x)$.

$$\mathrm{ACC} := \forall Y \big[\mathrm{BRANCH}(Y) \to \mathrm{PARITY}(\bar{Z}, Y) \big]$$

checks the parity condition. The formula

$$\begin{aligned} \mathrm{BRANCH}(Y) := Y \neq \varnothing \wedge \forall x \forall y [Yx \wedge Yy \to (x \leq y \vee y \leq x)] \\ \wedge \forall x \exists y [Yx \to x < y \wedge Yy] \end{aligned}$$

states that the elements in Y form an infinite branch and

$$\begin{aligned} &\mathrm{PARITY}(\bar{Z}, Y) := \\ &\quad \forall (X_{p,q})_{p,q \in Q} \Big[\mathrm{TRACE}(\bar{Z}, \bar{X}, Y) \to \bigvee_{k<n} \mathrm{MIN}_{2k}(\bar{X}, Y) \Big] \end{aligned}$$

states that every trace for the branch Y satisfies the parity condition. Here, n is any number such that the maximal priority of $\mathcal{A}$ is smaller than $2n$, the formula

$$\begin{aligned} \mathrm{TRACE}(\bar{Z}, \bar{X}, Y) := \bigwedge_{p,q} \big[X_{p,q} \subseteq Z_{p,q} \wedge X_{p,q} \subseteq Y \big] \\ \wedge \forall x \Big[Yx \to \bigvee_{p,q \in Q} X_{p,q} x \Big] \end{aligned}$$

states that the sets $\bar{X}$ encode a trace of the branch Y, and the formula

$$\mathrm{MIN}_k(\bar{X}, Y) := \bigvee_{q \in H_k} \mathrm{INF}_q(\bar{X}, Y) \wedge \bigwedge_{q \in H_{k-1}} \neg \mathrm{INF}_q(\bar{X}, Y)$$

states that the minimal priority seen infinitely often in the trace encoded by X is equal to k. The set

$$H_k := \{ q \in Q \mid \Omega(q) \leq k \}$$

contains all states with priority at most k and the formula

$$\mathrm{INF}_q(\bar{X}, Y) := \forall x \exists y \Big[x \leq y \wedge Yy \wedge \bigvee_{p \in Q} X_{p,q} y \Big]$$

states that the trace contains infinitely many occurrences of the state q. □

3. Decidability

We consider the theory of enriched trees of the following kind.

Definition 3.1. Let $\mathfrak{A} = \langle A, \bar{R} \rangle$ be a structure. The *Muchnik iteration* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* := \langle A^*, \bar{R}^*, \preceq, \mathrm{cl} \rangle$$

where

$$\begin{aligned} R_i^* &:= \{ \langle wa_0, \dots, wa_{n-1} \rangle \mid w \in A^*,\ \bar{a} \in R \}, \\ \mathrm{cl} &:= \{ waa \mid w \in A^*,\ a \in A \}, \end{aligned}$$

and $\preceq$ is the prefix ordering on A^*. cl is called the *clone relation.*

Example. The unravelling of a graph $\mathfrak{G} = \langle V, E \rangle$ is definable in $\mathfrak{G}^*$. To do so, we only need to find a formula $\varphi(x)$ stating that a sequence $x \in V^*$ of vertices corresponds to a path of $\mathfrak{G}$. Such a formula is

$$\varphi(x) := \forall y [y \preceq x \rightarrow \exists z (\mathrm{cl}(z) \wedge Ezy)] .$$

For an enriched tree of the form $\mathfrak{A}^*$, the automaton game $\mathcal{G}(\mathcal{A}, \mathfrak{A}^*)$ can be simplified since we do not need to remember the precise vertex $v \in A^*$ we are in, but only its last letter.

Definition 3.2. Let $\mathcal{A} := \langle Q, \Sigma \cup \{\mathrm{cl}\}, \delta, q_0, \Omega \rangle$ be an alternating tree automaton and let $\mathfrak{A}$ be a Σ-structure. The *reduced game*

$$\mathcal{G}_0(\mathcal{A}, \mathfrak{A}) = \langle V_0, V_1, E, \Omega' \rangle$$

has positions

$$V_0 := A \times Q \cup \{\langle\langle\rangle, q_0\rangle\} \quad \text{and} \quad V_1 := \wp(A \times Q) .$$

The initial position is $\langle\langle\rangle, q_0\rangle$. The edge relation E is defined as follows. From a V_1-position H there are edges to all V_0-positions $\langle b, q\rangle \in H$. From a V_0-position $\langle a, q\rangle$ with $a \in A \cup \{\langle\rangle\}$, there are edges to every V_1-position H such that

$$\langle \mathfrak{A} + a, \bar{P}, a\rangle \vDash \delta(q) ,$$

where

$$P_p := \{ b \in A \mid \langle b, p\rangle \in H \} , \quad \text{for } p \in Q ,$$

and $\mathfrak{A} + a$ is isomorphic to the substructure $\mathfrak{Suc}_*(a)$ of $\mathfrak{A}^*$, that is, $\mathfrak{A} + a$ is the disjoint union of $\mathfrak{A}$ with the single element a where we have added the order $\leq$ and the clone relation cl.

The priority function is defined in the same way as for $\mathcal{G}(\mathcal{A}, \mathfrak{A}^*)$, i.e., we set

$$\Omega'(\langle v, q\rangle) := \Omega(q) , \quad \text{for } \langle v, q\rangle \in V_0 ,$$

while $\Omega'(H)$, for $H \in V_1$, is an arbitrary number larger than all priorities used by Ω.

Lemma 3.3. *Let $\mathcal{A}$ be an alternating tree automaton and let $\mathfrak{A}$ be a Σ-structure. Automaton has a winning strategy for $\mathcal{G}_0(\mathcal{A}, \mathfrak{A})$ if, and only if, he has one for $\mathcal{G}(\mathcal{A}, \mathfrak{A}^*)$.*

Proof. Suppose that $\mathcal{A} = \langle Q, \Sigma \cup \{\text{cl}\}, \delta, q_0, \Omega\rangle$,

$$\mathcal{G}(\mathcal{A}, \mathfrak{A}^*) = \langle V_0, V_1, E, \Omega\rangle \quad \text{and} \quad \mathcal{G}_0(\mathcal{A}, \mathfrak{A}) = \langle V_0^0, V_1^0, E^0, \Omega^0\rangle .$$

Let $r : A^+ \to A$ be the function mapping a word to its last letter:

$$r(va) := a .$$

We define a projection function $h : V \to V^{\circ}$ by

$$\begin{aligned} h(\langle v, q\rangle) &:= \begin{cases} \langle\langle\rangle, q_0\rangle & \text{if } v = \langle\rangle\,, \\ \langle r(v), q\rangle & \text{otherwise}\,, \end{cases} && \text{for } \langle v, q\rangle \in V_0\,, \\ h(H) &:= \{\, \langle r(v), q\rangle \mid \langle v, q\rangle \in H \,\}\,, && \text{for } H \in V_1\,. \end{aligned}$$

(Note that, in a play of $\mathcal{G}(\mathcal{A}, \mathfrak{A}^*)$, we will never see a position of the form $\langle\langle\rangle, q\rangle$ with $q \neq q_0$. Hence, in the above definition the replacement of q by q_0 is harmless.)

It is straightforward to check that the relation

$$Z := \{\, \langle v, h(v)\rangle \mid v \in V \,\}$$

is a bisimulation between $\mathcal{G}(\mathcal{A}, \mathfrak{A}^*)$ and $\mathcal{G}_0(\mathcal{A}, \mathfrak{A})$. Consequently, the claim follows by Lemma 1.13. □

Lemma 3.4. *Let $\mathcal{A}$ be an alternating tree automaton, let $\mathfrak{A}$ be a Σ-structure, and let $\mathfrak{G}$ be the game structure associated with $\mathcal{G}_0(\mathcal{A}, \mathfrak{A})$. Given an* MSO$[E, V_0, V_1, \bar{P}]$*-formula φ where each set quantifier only ranges over subsets of V_0, we can compute an* MSO$[\Sigma]$*-formula φ' such that*

$$\mathfrak{G} \models \varphi \quad \text{iff} \quad \mathfrak{A} \models \varphi'\,.$$

Proof. Suppose that $\mathcal{A} = \langle Q, \Sigma \cup \{\preceq, \mathrm{cl}\}, \delta, q_0, \Omega\rangle$. We will employ a kind of generalised interpretation that encodes

- positions in $V_0 = A \times Q \cup \{\langle\langle\rangle, q_0\rangle\}$ by an element of $\mathfrak{A}$ together with an index in Q, and
- positions in $V_1 = \wp(A \times Q)$ by a $|Q|$-tuple of subsets of A.

Formally this means that,

- for each first-order variable x in φ, the formula φ' will use a first-order variable x and a tuple of set variables $\bar{X} = (X^q)_{q \in Q}$, and

- for each set variable X in φ (which, by assumption, ranges over subsets of V_0), it will use a tuple of set variables $\bar{X} = (X^q)_{q \in Q}$.

For every subformula $\psi(w, \bar{x}, \bar{y}, \bar{Z})$ of φ whose free first-order variables w, $\bar{x} = \langle x_0, \dots, x_{m-1} \rangle$, and $\bar{y} = \langle y_0, \dots, y_{n-1} \rangle$ are divided into three parts (one for the initial position, one for the rest of V_0, and one for V_1) and for every m-tuple $\bar{q} \in Q^m$ of states, we will construct a formula

$$\psi'_{\bar{q}}(\bar{x}, \bar{Y}_0, \dots, \bar{Y}_{n-1}, \bar{Z}_0, \dots, \bar{Z}_{l-1})$$

such that, for all positions $\bar{v} \in V_0^m$ and $\bar{H} \in V_1^n$ and all subsets $\bar{K} \in \wp(V_0)^l$,

$$\begin{aligned} & \mathfrak{G} \vDash \psi(\langle\langle\rangle, q_0\rangle, \bar{v}, \bar{H}, \bar{K}) \\ \text{iff} \quad & \mathfrak{A} \vDash \psi'_{\bar{q}}(\bar{a}, \bar{B}_0, \dots, \bar{B}_{n-1}, \bar{C}_0, \dots, \bar{C}_{l-1}), \end{aligned}$$

where $\bar{a} \in A^m$, $\bar{q} \in Q^m$, and $\bar{B}_0, \dots, \bar{B}_{n-1}, \bar{C}_0, \dots, \bar{C}_{l-1} \in \wp(A)^Q$ are chosen such that

$$\begin{aligned} v_i &= \langle a_i, q_i \rangle, \\ B_i^p &= \{ b \in A \mid \langle b, p \rangle \in H_i \}, \\ \text{and} \quad C_i^p &= \{ b \in A \mid \langle b, p \rangle \in K_i \}. \end{aligned}$$

The construction of ψ' is by induction on ψ. We start with the special case where $\psi(w, x, y) = Exy$ or $\psi(w, y) = Ewy$, that is, we have to find a formulae $\psi'_q(x, \bar{Y})$ and $\psi''_q(\bar{Y})$ such that

$$\begin{aligned} \mathfrak{A} \vDash \psi'_q(a, \bar{B}) \quad & \text{iff} \quad \text{the } V_1\text{-position encoded in } \bar{B} \text{ is a} \\ & \qquad \text{successor of the } V_0\text{-position } \langle a, q \rangle, \\ \mathfrak{A} \vDash \psi''_q(\bar{B}) \quad & \text{iff} \quad \text{the } V_1\text{-position encoded in } \bar{B} \text{ is a} \\ & \qquad \text{successor of the initial position } \langle\langle\rangle, q_0\rangle. \end{aligned}$$

In order to check whether a position $H \in V_1$ is a successor of some position $\langle a, q \rangle \in V_0$ (with $a \in A \cup \{\langle\rangle\}$), we have to evaluate the transition formula $\delta(q)$ in the structure $\langle \mathfrak{A} + a, \bar{P}, a\rangle$, where $\bar{P}$ is derived from H. Note that the tuple $\bar{P} \in \wp(A)^Q$ is in fact the encoding of the position H used by our interpretation. Hence, we are looking for formulae $\psi'_q(x, \bar{Y})$ and $\psi''_q(\bar{Y})$ such that

$$\begin{aligned} \mathfrak{A} &\vDash \psi'_q(a, \bar{B}) &&\text{iff}\quad \langle \mathfrak{A} + a, \bar{B}, a\rangle \vDash \delta(q)\,, \\ \mathfrak{A} &\vDash \psi''_q(\bar{B}) &&\text{iff}\quad \langle \mathfrak{A} + \langle\rangle, \bar{B}, \langle\rangle\rangle \vDash \delta(q)\,. \end{aligned}$$

We can write the successor structure on the right-hand side as

$$\langle \mathfrak{A} + a, \bar{B}, a\rangle \cong \tau(\langle \mathfrak{A}, \bar{B}, a\rangle \oplus *)\,,$$

where $*$ denotes a one-element structure and τ is an interpretation that adds the order relation $\leq$ and the clone relation cl. Its definition scheme consist of the formulae

$$\begin{aligned} &\delta(x) := \mathit{true}\,, && \varphi_R(\bar{x}) := R\bar{x}\,, \qquad \text{for } R \in \Sigma\,, \\ &\varphi_\leq(x, y) := \mathrm{Right}(x) \wedge \mathrm{Left}(y)\,, && \varphi_{\mathrm{cl}}(x) := \begin{cases} x = a & \text{if } a \neq \langle\rangle\,, \\ \mathit{false} & \text{if } a = \langle\rangle\,. \end{cases} \end{aligned}$$

By Proposition I.4.2 and Lemma I.4.5 we can find formulae $\delta'_q(x, \bar{Y})$ and $\delta''_q(\bar{Y})$ such that

$$\begin{aligned} \mathfrak{A} &\vDash \delta'_q(a, \bar{B}) &&\text{iff}\quad \tau(\langle \mathfrak{A}, \bar{B}, a\rangle \oplus *) \vDash \delta(q)\,, \\ \mathfrak{A} &\vDash \delta''_q(\bar{B}) &&\text{iff}\quad \tau(\langle \mathfrak{A}, \bar{B}\rangle \oplus \langle *, \langle\rangle\rangle) \vDash \delta(q)\,. \end{aligned}$$

These formulae have the desired properties.

Having found the formulae $\delta'_q(x, \bar{Y})$ and $\delta''_q(\bar{Y})$, we can proceed with the induction. First, we translate the atomic formulae. The interesting cases are the following ones.

$$\begin{aligned} (x_i = x_k)'_{\bar{q}} &:= \begin{cases} x_i = x_k & \text{if } q_i = q_k\,, \\ \mathit{false} & \text{otherwise}\,, \end{cases} \\ (y_i = y_k)'_{\bar{q}} &:= \bigwedge_{p \in Q} Y_i^p = Y_k^p\,, \end{aligned}$$

$$(Z_i x_k)'_{\bar q} := Z_i^{q_k} x_k\,,$$

$$(P_k x'_i)_{\bar q} := \begin{cases} \mathit{true} & \text{if } \Omega(q_i) = k\,, \\ \mathit{false} & \text{otherwise}\,, \end{cases}$$

$$(P_k y'_i)_{\bar q} := \begin{cases} \mathit{true} & \text{if } k > \max \operatorname{rng} \Omega\,, \\ \mathit{false} & \text{otherwise}\,, \end{cases}$$

$$(Ewy_k)'_{\bar q} := \delta''_q(\bar Y_k)$$
$$(Ex_i y_k)'_{\bar q} := \delta'_q(x_i, \bar Y_k)$$
$$(Ey_i x_k)'_{\bar q} := x_k \in Y_i^{q_k}\,.$$

The remaining cases are trivial.

$$(w = w)'_{\bar q} := \mathit{true}\,, \quad (x_i = w)'_{\bar q} := \mathit{false}\,, \quad (y_i = w)'_{\bar q} := \mathit{false}\,,$$
$$(w = x_i)'_{\bar q} := \mathit{false}\,, \quad (x_i = y_k)'_{\bar q} := \mathit{false}\,, \quad (y_i = x_k)'_{\bar q} := \mathit{false}\,,$$
$$(w = y_i)'_{\bar q} := \mathit{false}\,,$$

$$(V_0 w)'_{\bar q} := \mathit{true}\,, \quad (V_1 w)'_{\bar q} := \mathit{false}\,,$$
$$(V_0 x_i)'_{\bar q} := \mathit{true}\,, \quad (V_1 x_i)'_{\bar q} := \mathit{false}\,,$$
$$(V_0 y_i)'_{\bar q} := \mathit{false}\,, \quad (V_1 y_i)'_{\bar q} := \mathit{true}\,,$$

$$(Eww)'_{\bar q} := \mathit{false}\,, \quad (Ex_i w)'_{\bar q} := \mathit{false}\,, \quad (Ey_i w)'_{\bar q} := \mathit{false}\,,$$
$$(Ewx_i)'_{\bar q} := \mathit{false}\,, \quad (Ex_i x_k)'_{\bar q} := \mathit{false}\,, \quad (Ey_i y_k)'_{\bar q} := \mathit{false}\,.$$

Boolean operations are left unchanged by the translation $\psi \mapsto \psi'_{\bar q}$. For the existential first-order quantifier, we have to distinguish three cases depending on which part the newly quantified element belongs to.

$$\begin{aligned}
\big(\exists x' \psi(w,\bar{x},\bar{y},\bar{Z})\big)'_{\bar{q}} :=& \\
&(\psi[w/x'])'_{\bar{q}}(\bar{x},\bar{Y}_0,\dots,\bar{Y}_{n-1},\bar{Z}_0,\dots,\bar{Z}_{l-1}) \\
&\vee \bigvee_{p\in Q} \exists x' \psi'_{\bar{q}p}(\bar{x}x',\bar{Y}_0,\dots,\bar{Y}_{n-1},\bar{Z}_0,\dots,\bar{Z}_{l-1}) \\
&\vee \exists \bar{X}' \psi'_{\bar{q}}(\bar{x},\bar{Y}_0,\dots,\bar{Y}_{n-1},\bar{X}',\bar{Z}_0,\dots,\bar{Z}_{l-1})\,,
\end{aligned}$$

where $\psi[w/x']$ is the formula obtained from ψ by replacing every occurrence of the variables x' by the variable w. For the existential set quantifier, we distinguish two cases depending on whether or not the initial position w belongs to Z'.

$$\big(\exists Z' \psi(w,\bar{x},\bar{y},\bar{Z})\big)'_{\bar{q}} := \exists \bar{Z}' \tilde{\psi}'_{\bar{q}}(\bar{x},\bar{Y}_0,\dots,\bar{Y}_{n-1},\bar{Z}_0,\dots,\bar{Z}_{l-1},\bar{Z}')\,,$$

where the formula

$$\tilde{\psi} := \psi[Z'z \vee z = w / Z'z] \vee \psi[Z'z \wedge z \neq w / Z'z]$$

is obtained from two copies of ψ by replacing every atom of the form $Z'z$ by one of the formulae $Z'z \vee z = w$ and $Z'z \wedge z \neq w$. The definitions for the universal quantifiers follow by duality. □

Theorem 3.5 (Muchnik). *Let Σ be a finite relational signature. For every formula $\varphi \in \mathrm{MSO}[\Sigma \cup \{\leq, \mathrm{cl}\}]$, we can effectively construct a formula $\varphi^* \in \mathrm{MSO}[\Sigma]$ such that*

$$\mathfrak{A}^* \vDash \varphi \quad \text{iff} \quad \mathfrak{A} \vDash \varphi^*, \quad \textit{for all } \Sigma\textit{-structures } \mathfrak{A}\,.$$

Proof. Given the formula φ, we can use Theorem 2.14 to construct a tree automaton $\mathcal{A}$ such that

$$\mathfrak{A}^* \vDash \varphi \quad \text{iff} \quad \mathfrak{A}^* \in L(\mathcal{A})\,.$$

By Lemmas 2.10 and 3.3, the latter is equivalent to Automaton having a winning strategy for the game $\mathcal{G}_0(\mathcal{A},\mathfrak{A})$.

Let $\mathfrak{G}$ be the game structure associated with $\mathcal{G}_0(\mathcal{A}, \mathfrak{A})$. Note that the game $\mathcal{G}_0(\mathcal{A}, \mathfrak{A})$ is alternating and normalised. Hence, we can apply Theorem 1.22 to construct an MSO-formula χ where all set quantifiers range over subsets of V_0 such that

$$\text{Automaton wins } \mathcal{G}_0(\mathcal{A}, \mathfrak{A}) \quad \text{iff} \quad \mathfrak{G} \vDash \chi(\langle\langle\rangle, q_0\rangle)\,.$$

By Lemma 3.4 we can construct an MSO-formula φ^* such that

$$\mathfrak{G} \vDash \chi(\langle\langle\rangle, q_0\rangle) \quad \text{iff} \quad \mathfrak{A} \vDash \varphi^*.$$

It follows that

$$\begin{aligned} \mathfrak{A}^* \vDash \varphi \quad & \text{iff} \quad \text{Automaton wins } \mathcal{G}_0(\mathcal{A}, \mathfrak{A}) \\ & \text{iff} \quad \mathfrak{G} \vDash \chi(\langle\langle\rangle, q_0\rangle) \\ & \text{iff} \quad \mathfrak{A} \vDash \varphi^*. \end{aligned}$$

□

Definition 3.6. The *infinite binary tree* is the structure

$$\mathfrak{T} := \langle [2]^*, \preceq, P_0, P_1 \rangle\,,$$

where $\preceq$ is the prefix ordering and $P_c := \{\, wc \mid w \in [2]^* \,\}$, for $c < 2$.

Corollary 3.7 (Rabin). $\mathrm{Th}_{\mathrm{MSO}}(\mathfrak{T})$ *is decidable where* $\mathfrak{T}$ *is the infinite binary tree.*

Proof. Consider the two-element structure $\mathfrak{A} := \langle \{0,1\}, U_0, U_1 \rangle$ where $U_c := \{c\}$. The binary tree $\mathfrak{T}$ can be obtained from $\mathfrak{A}^*$ by removing the clone relation. By Theorem 3.5, it follows that, for every formula φ, we can compute a formula φ^* such that

$$\mathfrak{T} \vDash \varphi \quad \text{iff} \quad \mathfrak{A} \vDash \varphi^*.$$

As $\mathfrak{A}$ is finite, the latter property is decidable. □

Corollary 3.8. $\mathrm{Th}_{\mathrm{MSO}}(\mathbb{Q}, \leq)$ *is decidable.*

Proof. It is sufficient to construct an interpretation

$$\tau = \big\langle \delta(x), \varphi_{\leq}(x,y) \big\rangle$$

mapping the binary tree $\mathfrak{T} := \langle 2^*, \preceq, P_0, P_1 \rangle$ to the order of the rationals $\mathfrak{Q} := \langle \mathbb{Q}, \leq \rangle$. We set

$$\begin{aligned} \delta(x) &:= P_1 x\,, \\ \varphi(x,y) &:= x \preceq y \vee \exists z \exists u \exists v [\mathrm{suc}_0(z,u) \wedge \mathrm{suc}_1(z,v) \\ &\qquad\qquad\qquad\qquad \wedge u \preceq x \wedge v \preceq y]\,, \end{aligned}$$

where

$$\mathrm{suc}_i(x,y) := x \prec y \wedge P_i y \wedge \neg\exists z [x \prec z \prec y]$$

states that y is the i-successor of x. □

Notes

Good introductions to the theory of automata on ω-words and infinite trees are [16, 18]. The latter also contains a proof that parity games are positionally determined. Corollary 3.7 is from Rabin [12].

The Theorem of Muchnik was announced in [14], but a proof was never published. The proof we have presented above was provided by Walukiewicz [19]. This paper is also the source of the determinacy proof of Section 1 and the automata constructions presented in Section 2.

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Symbol Index

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The Roman and Fraktur alphabets							
A	a	$\mathfrak{A}$	$\mathfrak{a}$	N	n	$\mathfrak{N}$	$\mathfrak{n}$
B	b	$\mathfrak{B}$	$\mathfrak{b}$	O	o	$\mathfrak{O}$	$\mathfrak{o}$
C	c	$\mathfrak{C}$	$\mathfrak{c}$	P	p	$\mathfrak{P}$	$\mathfrak{p}$
D	d	$\mathfrak{D}$	$\mathfrak{d}$	Q	q	$\mathfrak{Q}$	$\mathfrak{q}$
E	e	$\mathfrak{E}$	$\mathfrak{e}$	R	r	$\mathfrak{R}$	$\mathfrak{r}$
F	f	$\mathfrak{F}$	$\mathfrak{f}$	S	s	$\mathfrak{S}$	ſ $\mathfrak{s}$
G	g	$\mathfrak{G}$	$\mathfrak{g}$	T	t	$\mathfrak{T}$	$\mathfrak{t}$
H	h	$\mathfrak{H}$	$\mathfrak{h}$	U	u	$\mathfrak{U}$	$\mathfrak{u}$
I	i	$\mathfrak{I}$	$\mathfrak{i}$	V	v	$\mathfrak{V}$	$\mathfrak{v}$
J	j	$\mathfrak{J}$	$\mathfrak{j}$	W	w	$\mathfrak{W}$	$\mathfrak{w}$
K	k	$\mathfrak{K}$	$\mathfrak{k}$	X	x	$\mathfrak{X}$	$\mathfrak{x}$
L	l	$\mathfrak{L}$	$\mathfrak{l}$	Y	y	$\mathfrak{Y}$	$\mathfrak{y}$
M	m	$\mathfrak{M}$	$\mathfrak{m}$	Z	z	$\mathfrak{Z}$	$\mathfrak{z}$

The Greek alphabet					
A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ε	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	ϑ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega