Uppsala Computing Science
Research Report No. 165
August 30, 1999 (Revised November 22, 1999)
ISSN 1100–0686

Simulation and Bisimulation over One-Counter Processes*

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Uppsala Computing Science Research Report Series

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http://www.csd.uu.se/papers/reports.html
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Abstract

A ***one-counter automaton*** consists of a finite-state control operating on a single counter ranging over the nonnegative integers; transitions of the automaton are labelled from some finite alphabet, and may increment, decrement, or ignore the counter value, possibly depending on whether or not the counter value is zero, but may not decrement the counter when it is zero. The class of one-counter automata is equivalent to the class of pushdown automata with a single stack symbol (apart from a special bottom-of-stack marker). A ***one-counter net*** is a one-counter automaton which cannot test for zero: any transition which can be performed when the counter is zero can equally be performed when the counter is non-zero. The class of one-counter nets is equivalent to the class of labelled Petri nets with a single unbounded place.

We show an effective construction of (a periodicity description of) the maximal simulation relation for a given one-counter net. Then we demonstrate how to reduce *simulation* problems over one-counter nets to analogous *bisimulation* problems over one-counter automata. This requires a close analysis of a recent proof of the decidability of the simulation relation over one-counter nets resulting in an effective construction of (a semilinearity description of) the (maximal) simulation relation. We use this to demonstrate the decidability of various problems, specifically testing regularity and strong regularity of one-counter nets with respect to simulation equivalence, and testing simulation equivalence between a one-counter net and a deterministic pushdown automaton. Various obvious generalisations of these problems are known to be undecidable.

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[*] Abridged version to appear in *Proceedings of STACS2000*

[†] Partially supported by the Grant Agency of the Czech Republic, grants No. 201/97/0456 and No. 201/00/0400.

[‡] Supported by a Research Fellowship granted by the Alexander von Humboldt Foundation and by a Post-Doc grant GA ČR No. 201/98/P046.

[§] Partially supported by TFR grant No. 221-98-103 and Np. 201/97/275.

1 Introduction

In concurrency theory, a ***process*** is typically defined to be a state in a ***transition system***, which is a triple $\mathsf{T} = \langle \mathsf{S}, \Sigma, \rightarrow \rangle$ where S is a set of ***states***, Σ is a set of ***actions*** (assumed to be *finite* in this paper) and $\rightarrow \subseteq \mathsf{S} \times \Sigma \times \mathsf{S}$ is a ***transition relation***. We write $s \xrightarrow{a} t$ instead of $\langle s, a, t \rangle \in \rightarrow$, and we extend this notation in the natural way to elements of Σ^*. A state t is ***reachable*** from a state s iff $s \xrightarrow{w} t$ for some $w \in \Sigma^*$. T is ***image-finite*** iff for all $s \in \mathsf{S}$ and $a \in \Sigma$ the set $\{t : s \xrightarrow{a} t\}$ is finite; T is ***deterministic*** if each such set is of size at most 1.

In this paper, we consider such processes generated by ***one-counter automata***, nondeterministic finite-state automata operating on a single counter variable ranging over the set $\mathbb{N}$ of nonnegative integers. Formally this is a tuple $M = \langle Q, \Sigma, \delta^{=}, \delta^{>} \rangle$ where Q is a finite set of ***control states***, Σ is a finite set of ***actions***, and $\delta^{=} : Q \times \Sigma \rightarrow \mathcal{P}(Q \times \{0, 1\})$, $\delta^{>} : Q \times \Sigma \rightarrow \mathcal{P}(Q \times \{-1, 0, 1\})$ are ***transition functions*** (where $\mathcal{P}(A)$ denotes the set of subsets of A). $\delta^{=}$ represents the transitions which are enabled when the counter value is zero, and $\delta^{>}$ represents the transitions which are enabled when the counter value is positive. M is a ***one-counter net*** iff $\forall q \in Q, \forall a \in \Sigma : \delta^{=}(q, a) \subseteq \delta^{>}(q, a)$. To M we associate the (image-finite) transition system $\mathsf{T}_M = \langle \mathsf{S}, \Sigma, \rightarrow \rangle$, where $\mathsf{S} = \{\, p(n) : p \in Q, n \in \mathbb{N} \}$ and $\rightarrow$ is defined as follows:

$$p(n) \xrightarrow{a} p'(n+i) \quad \text{iff} \quad \begin{cases} n{=}0, \text{ and } (p', i) \in \delta^{=}(p, a); \; \textit{or} \\ n{>}0, \text{ and } (p', i) \in \delta^{>}(p, a). \end{cases}$$

Note that any transition increments, decrements, or leaves unchanged the counter value; and a decrementing transition is only possible if the counter value is positive. Also observe that when $n{>}0$ the transitions of $p(n)$ do not depend on the actual value of n. Finally, note that a one-counter *net* can in a sense test if its counter is nonzero (that is, it can perform some transitions only on the proviso that its counter is nonzero), but it cannot test in any sense if its counter is zero.

As an example, we might take $Q = \{p\}$, $\Sigma = \{a, z\}$, and take the only non-empty transition function values to be $\delta^{>}(p, a) = \{(p, +1), (p, -1)\}$, $\delta^{=}(p, a) = \{(p, +1)\}$, and $\delta^{=}(p, z) = \{(p, 0)\}$. This one-counter automaton gives rise to the infinite-state transition system depicted in Fig. 1; if we eliminate the z-action, then this would be a one-counter net. The class of transition systems which are generated by one-counter nets is the same (up to isomorphism) as that generated by the class of labelled Petri nets with (at most) one unbounded place. (This is immediately clear if we consider Petri nets with arc weights 1; however, the correspondence is true even with general arcweights, which we formally prove in the Appendix). The class of transition systems which are generated by one-counter automata is the same (up to isomorphism) as that generated by the class of realtime pushdown automata with a single stack symbol (apart from a special bottom-of-stack marker).

Given a transition system $\mathsf{T} = \langle \mathsf{S}, \Sigma, \rightarrow \rangle$, a ***simulation*** is a binary relation $\mathcal{R} \subseteq \mathsf{S} \times \mathsf{S}$ satisfying: whenever $\langle s, t \rangle \in \mathcal{R}$, if $s \xrightarrow{a} s'$ then $t \xrightarrow{a} t'$ for some t' with $\langle s', t' \rangle \in \mathcal{R}$. s is ***simulated*** by t, written $s \preccurlyeq t$, iff $\langle s, t \rangle \in \mathcal{R}$ for some simulation $\mathcal{R}$; and s and t are ***simulation equivalent***, written $s \succcurlyeq\!\!\preccurlyeq t$, iff $s \preccurlyeq t$ and $t \preccurlyeq s$. (The relation $\preccurlyeq$, being the union of all simulation relations, is in fact the maximal simulation relation.) A ***bisimulation*** is a symmetric simulation relation, and s and t are ***bisimulation equivalent***, or ***bisimilar***, written $s \sim t$, if they are related by a bisimulation. Simulations and bisimulations can also be used to relate states of

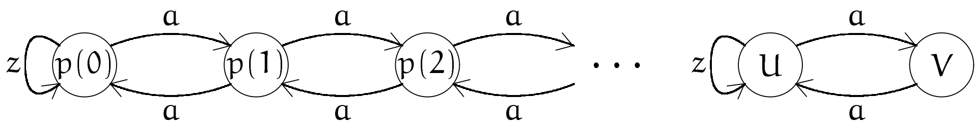

Figure 1: A one-counter automata process and a simulation-equivalent finite-state process.

different transition systems; formally, we can consider two transition systems to be a single one by taking the disjoint union of their state sets.

There are various other equivalences over processes which have been studied within the framework of concurrency theory; an overview and comparison of these is presented in [15]. Each has its specific advantages and disadvantages, and consequently none is universally accepted as the "best" one, although it seems that simulation and bisimulation equivalences are of particular importance as their accompanying theory has been intensively developed. Bisimilarity is especially mathematically tractable, having the best polynomial-time algorithms over finite-state transition systems (while all language-based equivalences by comparison are PSPACE-complete), and the only one which is decidable for various classes of infinite-state systems such as context-free processes and commutative context-free processes (see [13] for a survey of such results).

Let s be a state of a transition system T and $\approx$ be an equivalence over the class of all processes (that is, all states of all transition systems). s is ***$\approx$-regular***, or ***regular w.r.t.*** $\approx$, iff $s \approx f$ for some state f of a finite-state transition system; and s is ***strongly*** ***$\approx$-regular***, or ***strongly regular w.r.t.*** $\approx$, iff only finitely many states, up to $\approx$, are reachable from s. For bisimilarity, these two concepts coincide, but this is not true in general for other equivalences. For example, the state $p(0)$ of the infinite-state transition system depicted in Fig. 1 is $\precsim\!\succsim$-regular, being simulation equivalent to the state U of the depicted finite-state system. However, it is not strongly $\precsim\!\succsim$-regular (nor $\sim$-regular) as $p(i) \not\precsim p(j)$ whenever $i < j$. The conditions of regularity and strong regularity say that a process can in some sense be finitely represented (up to the equivalence): in the first case there is an equivalent finite-state process; and in the second case the quotient of its state-space under the equivalence is finite. As all "reasonable" process equivalences are preserved under their respective quotients [9] (that is, each state is equivalent to its equivalence class in the automaton produced by collapsing equivalent states [2]), strong regularity in fact guarantees the existence of a finite-state process whose state-space is the same (up to the equivalence); this process provides a more robust description of the original process as it preserves strictly more logical properties than a process which is just equivalent [10].

Finite descriptions of infinite-state processes are important from the point of view of automatic formal verification. Verification tools typically work only for finite-state systems, and the types of systems which they analyze, such as protocols, are typically *semantically* finite-state. However, these systems are often expressed *syntactically* as infinite-state systems, for example maintaining a count of how many unacknowledged messages have been sent, so it is advantageous to develop algorithms which replace infinite-state processes with equivalent finite-state systems (when they exist). Examples of such algorithms appear in [2, 4, 5, 9, 12]

In Section 2 we show an effective construction of (a periodicity description of) the maximal simulation relation for a given one-counter net. Then, in Section 3, we study the connection between simulation and bisimulation relations, and demonstrate the decidability of the $\precsim\!\succsim$-regularity and strong $\precsim\!\succsim$-regularity problems for *one-counter nets*, a restricted form of Petri nets; the $\precsim\!\succsim$-regularity problem is reduced to the $\sim$-regularity problem for the more general class of *one-counter automata*, which is known to be decidable [3]. Note that the $\precsim\!\succsim$-regularity problem is known to be undecidable for general Petri nets [5] and an incomparable class of PA processes [11]. Finally, we demonstrate how to decide simulation equivalence between (a process related to) a one-counter net and (a process related to) a deterministic pushdown automaton. Here note that simulation equivalence between a (nondeterministic) one-counter automaton and a deterministic one-counter automaton (i.e., a special deterministic pushdown automaton) can be demonstrated to be undecidable [7].

2 Simulation on One-Counter Nets

In this section we fix a one-counter net with control state set Q, and present an algorithm which constructs a (simple) description of the set

$$\mathcal{S} \;=\; \{\,\langle p(m), q(n)\rangle \;:\; p, q \in Q,\; m, n \in \mathbb{N},\; p(m) \preccurlyeq q(n)\,\}$$

i.e., the maximal simulation relation on the transition system associated to the net. $\mathcal{S}$ can be viewed as a collection of $|Q|^2$ subsets of $\mathbb{N}\times\mathbb{N}$: to each $p, q \in Q$ we associate $\mathcal{S}_{\langle p,q\rangle} = \{\,\langle m, n\rangle \;:\; p(m) \preccurlyeq q(n)\,\}$. Observe that if $p(m) \preccurlyeq q(n)$ then $p(m') \preccurlyeq q(n')$ for all $m'{\leq}m$ and $n'{\geq}n$ since the set $\{\,\langle p(m'), q(n')\rangle : p(m) \preccurlyeq q(n)$ for some $m{\geq}m', n{\leq}n'\,\}$ is a simulation relation.

By a ***colouring*** we mean a function $\mathbb{C} : (Q{\times}Q) \rightarrow (\mathbb{N}{\times}\mathbb{N}){\rightarrow}\{\text{black}, \text{white}\}$, where we write the function applications as $\mathbb{C}_{\langle p,q\rangle}(m, n)$. We further stipulate that a colouring must satisfy the following monotonicity condition: if $\mathbb{C}_{\langle p,q\rangle}(m, n)$=black then $\mathbb{C}_{\langle p,q\rangle}(m', n')$=black for all $m'{\leq}m$ and $n'{\geq}n$. With this proviso, each $\mathbb{C}_{\langle p,q\rangle}$ is determined by the ***frontier function*** $f^{\mathbb{C}}_{\langle p,q\rangle} : \mathbb{N} \rightarrow \mathbb{N} \cup \{\omega\}$ defined by: $f^{\mathbb{C}}_{\langle p,q\rangle}(n) = \min\{\, m : \mathbb{C}_{\langle p,q\rangle}(m, n)\text{=white}\,\}$; we put $f^{\mathbb{C}}_{\langle p,q\rangle}(n){=}\omega$ if $\mathbb{C}_{\langle p,q\rangle}(m, n)$=black for all m. Note that this function is nondecreasing, i.e., each ***step*** $f^{\mathbb{C}}_{\langle p,q\rangle}(n{+}1) - f^{\mathbb{C}}_{\langle p,q\rangle}(n)$ is nonnegative. When $f_{\langle p,q\rangle}(n) \in \mathbb{N}$, we call the pair $\langle f_{\langle p,q\rangle}(n), n\rangle$ a ***frontier point*** and the set of all frontier points constitutes the ***frontier*** (in $\mathbb{C}_{\langle p,q\rangle}$).

We use $\mathbb{G}$ to denote the following distinguished colouring:

$$\mathbb{G}_{\langle p,q\rangle}(m, n) = \begin{cases} \text{black}, & \text{if} \;\; p(m) \preccurlyeq q(n); \\ \text{white}, & \text{if} \;\; p(m) \not\preccurlyeq q(n). \end{cases}$$

The observation about $\mathcal{S}$ from above confirms that this is a valid colouring, i.e., that the required monotonicity condition holds. We use $f_{\langle p,q\rangle}$ to denote the frontier function of $\mathbb{G}_{\langle p,q\rangle}$, and we understand the terms *frontier function* and *frontier* to be related to $\mathbb{G}$ when not specified otherwise.

The following "Belt Theorem" gives a crucial fact about frontiers; by a ***belt*** we mean the set of points of the (first quadrant of the) plane lying between two parallel lines.

Belt Theorem. Every frontier lies within a belt with nonnegative rational or infinite slope.

This theorem is central for the decidability of simulation over one-counter nets. It was proven in [6] by a combination of short and intuitive arguments; the theorem is also present (though not so explicitly) in [1] but the proof outlined there is formidable.

Note that if, for a frontier function f, $f(n){=}\omega$ for some n then the respective frontier is finite and lies within a horizontal belt (i.e., with slope 0). Otherwise f (as a function $\mathbb{N} \rightarrow \mathbb{N}$) is almost linear, though its steps $\big(f(n{+}1){-}f(n)\big)$ need not be constant. Nevertheless, we shall show that f is ***periodic***, i.e., from some n_0 a finite sequence of steps is repeated forever; and moreover, its ***periodicity description***—i.e., n_0, the sequence of steps to be repeated, and the values of $f(n)$ for all $n \leq n_0$—can be effectively computed, yielding the simple description of the set $\mathcal{S}$. (Note that the decision algorithms in both [1] and [6] only approximate the set $\mathcal{S}$, or equivalently the colouring $\mathbb{G}$, to a sufficient level to answer the relevant question; effective constructability of the functions $f_{\langle p,q\rangle}$ does not follow from there.)

We now show how the frontier functions $f_{\langle p,q\rangle}$ can be stepwise approximated. First we say that a point $\langle m, n\rangle$ (in $\mathbb{N}\times\mathbb{N}$) is ***locally correct in a colouring*** $\mathbb{C}$ iff the following holds for all $p, q \in Q$: if $\mathbb{C}_{\langle p,q\rangle}(m,n)$=black and $p(m) \xrightarrow{a} p'(m')$ then there is $q(n) \xrightarrow{a} q'(n')$ with $\mathbb{C}_{\langle p',q'\rangle}(m',n')$=black. Note that the local correctness of a point $\langle m, n\rangle$ depends only on the restriction of $\mathbb{C}$ to the ***neighbourhood*** of $\langle m, n\rangle$, i.e., to the set $\{\, \langle m', n'\rangle : |m'-m| \leq 1, |n'-n| \leq 1 \,\}$; this follows from the fact that a transition in a one-counter net can change the counter value by at most 1. We say that $\mathbb{C}$ is k-***admissible***, where $k \in \mathbb{N}\cup\{\omega\}$, iff each point $\langle m, n\rangle$ with $m, n < k$ is locally correct in $\mathbb{C}$. In particular, note that $\mathbb{G}$ is ω-admissible.

The function $\mathbb{G}^k : (Q\times Q) \to (\mathbb{N}\times\mathbb{N}) \to \{$black, white$\}$ defined by

$$\mathbb{G}^k_{\langle p,q\rangle}(m,n) = \text{black iff } \mathbb{C}_{\langle p,q\rangle}(m,n) = \text{black for some } k\text{-admissible colouring } \mathbb{C}$$

is easily seen to be a k-admissible colouring, and is in fact the ***maximal*** (i.e., *maximally-black*) k-***admissible*** colouring; furthermore, the maximal ω-admissible colouring $\mathbb{G}^\omega$ is clearly $\mathbb{G}$. For $k \in \mathbb{N}$, we denote the frontier function of $\mathbb{G}^k_{\langle p,q\rangle}$ by $f^k_{\langle p,q\rangle}$, and note that the range of $f^k_{\langle p,q\rangle}$ is $\{\,0, 1, \ldots, k-1\,\} \cup \{\omega\}$ and that $f^k_{\langle p,q\rangle}(n) = \omega$ for all $n \geq k$. The description of each function $f^k_{\langle p,q\rangle}$, i.e., (a table of) its values for $0, 1, \ldots, k-1$, is effectively computable, for example, by an exhaustive search. As $\mathbb{G}^k$ is i-admissible for any $i \leq k$, we have, for each p, q, $f^0_{\langle p,q\rangle} \geq f^1_{\langle p,q\rangle} \geq f^2_{\langle p,q\rangle} \geq \ldots \geq f_{\langle p,q\rangle}$ (where $f' \geq f''$ means $\forall n \in \mathbb{N} : f'(n) \geq f''(n)$). Therefore the function $g_{\langle p,q\rangle} = \lim_{n\to\infty} f^n_{\langle p,q\rangle}$ is well-defined, and $g_{\langle p,q\rangle} \geq f_{\langle p,q\rangle}$. But since the colouring defined by these limit functions $g_{\langle p,q\rangle}$ (as the frontier functions) is ω-admissible (recall the "locality" of the local correctness condition), and $\mathbb{G}$ is the *maximal* ω-admissible colouring, we have $g_{\langle p,q\rangle} \leq f_{\langle p,q\rangle}$. Thus $g_{\langle p,q\rangle} = f_{\langle p,q\rangle}$, and therefore we get the following.

Lemma 1 *For each* $n \in \mathbb{N}$ *there is* $k \geq n$ *such that each* $f^k_{\langle p,q\rangle}$ *coincides with* $f_{\langle p,q\rangle}$ *on the set* $\{0, 1, 2, \ldots, n\}$.

Our algorithm will construct $\mathbb{G}^k$ for $k = 0, 1, 2, \ldots$; Lemma 1 guarantees that larger and larger initial portions of (the graphs of) $\mathbb{G}_{\langle p,q\rangle}$ are appearing during the run of the algorithm (though we do not know the extent of the portion of $\mathbb{G}$ in $\mathbb{G}^k$). To show when our algorithm can terminate, recognizing an initial portion of $\mathbb{G}$ and providing a description of the whole $\mathbb{G}$, we now explore a certain "repeatable pattern" which is guaranteed to appear in $\mathbb{G}$.

By the Belt Theorem, we can fix a set of belts with nonnegative rational or infinite slopes such that each frontier is contained in one of them. We assume that the belts are "sufficiently" thick; thus we can, for instance, suppose that the belt slopes are pairwise distinct (merging parallel belts into one thicker).

Now we can choose $h_1, h_2, i \in \mathbb{N}$, where $0 < h_1 < h_2 < i$, such that (see Fig. 2):

1. for each frontier function f with $f(h_2) < \omega$, all frontier points $\langle f(n), n\rangle$ between levels h_1 and h_2, (i.e., with $h_1 \leq n \leq h_2$) lie in one of the belts (this follows trivially from our assumption; note that Fig. 2 depicts just one frontier in each belt, though in general there can be several frontiers in a single belt);

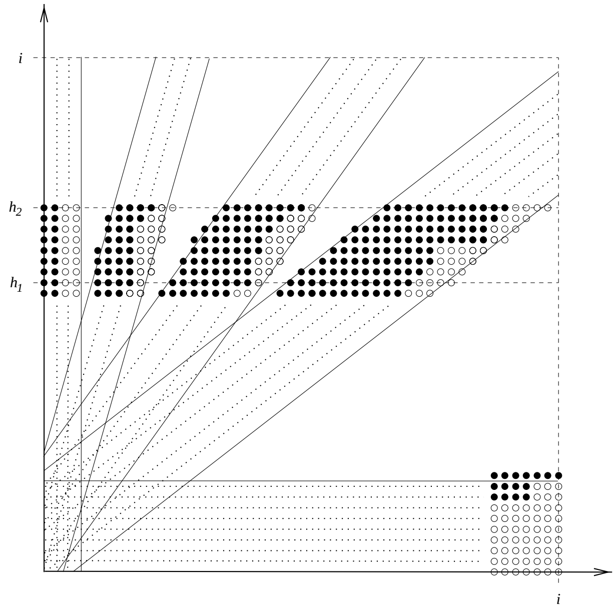

Figure 2: Graphs of $\mathbb{G}_{\langle p,q\rangle}$ displaying a repeatable pattern, superimposed onto each other

2. the belts are pairwise disjoint at and above level h_1-1 (i.e., we choose h_1 large enough so that at level h_1-1 each belt is to the right of any other belt with greater slope);
3. for each frontier function f: if $f(h_1-1)\leq 1$ then $f(h_2)=f(h_1-1)$; and if $f(h_2)=\omega$ then $f(h_1-1)=\omega$ (this is satisfied when h_1 and h_2 are chosen large enough);
4. for each frontier function f and each $n\leq h_2$: if $f(n)<\omega$ then $f(n)<i$ (this is satisfied by choosing i large enough after the choice of h_1 and h_2).

Each frontier point $\langle f(n), n\rangle$ has a certain (horizontal) *distance* to the left border line of the belt in which it lies. Since the slope of each belt is rational, it is clear that such distances range over finitely many possible values. So, by a straightforward use of the pigeonhole principle, we can additionally suppose (i.e., we could choose h_1, h_2, i so) that the frontier points of all frontiers inside a single belt have the same relative positions at levels h_2 and h_2-1 as at levels h_1 and h_1-1, respectively. More precisely:

5. for each frontier function f with $f(h_2)<\omega$, the slope of the belt in which the respective frontier appears between levels h_1 and h_2 is $(h_2-h_1)/(f(h_2)-f(h_1))$; moreover, $f(h_2)-f(h_2-1) = f(h_1)-f(h_1-1)$

The number of possible distances would allow us to calculate a bound b such that we can even suppose (i.e., choose so) that $h_2-h_1\leq b$. Note that b does not depend on how thick the belts are chosen. In particular, we can assume each belt to be so thick that for each frontier point $\langle f(n), n\rangle$ in the belt, with $n\geq h_1$, the point $\langle f(n), n+b\rangle$ is still an ***interior point*** of the belt, i.e., its whole neighbourhood lies in the belt. Informally we say that the belt has a *sufficiently thick monochromatic left subbelt* (above h_1); monochromatic means that each $\mathbb{G}_{\langle p,q\rangle}$ is constant (either black or white) on the subbelt. Therefore we could choose belts and h_1, h_2 and i so that the following additional condition is satisfied:

6. for each frontier point $\langle f(n), n\rangle$ with $h_1 \le n \le h_2$, the point $\langle f(n), n+(h_2-h_1)\rangle$ is an interior point of the belt in which the respective frontier lies between levels h_1 and h_2.

We now say that a colouring $\mathbb{C}$ has a ***repeatable pattern***, based on h_1, h_2 and i, iff there are belts such that the above conditions 1.–6. are satisfied (where the terms *frontier* and *frontier function* are understood as those related to $\mathbb{C}$). We have thus demonstrated that $\mathbb{G}$ has a repeatable pattern. Our algorithm which constructs $\mathbb{G}^0, \mathbb{G}^1, \mathbb{G}^2, \ldots$ terminates when it finds some $\mathbb{G}^j$ which has a repeatable pattern based on some h_1, h_2 and i with $i<j$; such a condition is clearly decidable; and Lemma 1, together with the fact that $\mathbb{G}$ has a repeatable pattern, guarantees termination of the algorithm. Having discovered a repeatable pattern for $\mathbb{G}^j$ based on h_1, h_2 and i with $i<j$, we define the colouring $\mathbb{G}^*$ by defining its frontier functions $f^*_{\langle p,q\rangle}$ inductively as follows:

$$f^*_{\langle p,q\rangle}(n) \;=\; \begin{cases} f^j_{\langle p,q\rangle}(n), & \text{if } n \le h_2 \\[2ex] f^*_{\langle p,q\rangle}(n-c) + d, & \text{if } n > h_2 \end{cases}$$

where $c = h_2 - h_1$ and $d = f^j_{\langle p,q\rangle}(h_2) - f^j_{\langle p,q\rangle}(h_1)$. Hence each $f^*_{\langle p,q\rangle}$ is periodic, arising from $f^j_{\langle p,q\rangle}$ by repeating the sequence of steps between h_1 and h_2 forever. Also note that if $f^j_{\langle p,q\rangle}(n)=\omega$ for some $n \le h_2$ then $f^*_{\langle p,q\rangle}=f^j_{\langle p,q\rangle}$. We shall show (Lemma 3) that $\mathbb{G}^*$ is in fact $\mathbb{G}$. To this end, we make some considerations and introduce some auxiliary notions.

First recall that the local correctness of a point $\langle m, n\rangle$ in a colouring $\mathbb{C}$ depends only on the restriction of $\mathbb{C}$ to the neighbourhood of $\langle m, n\rangle$. Also recall that the possible transitions from a state $p(m)$ do not depend on m when $m>0$. Therefore $\mathbb{G}^*$ is surely ω-admissible: each point $\langle m, n\rangle$ in the *verified area*, i.e., with $m<j$ and $n<h_2$, is locally correct since it is (by definition) locally correct in $\mathbb{G}^j$, and $\mathbb{G}^j$ and $\mathbb{G}^*$ coincide on the neighbourhood of $\langle m, n\rangle$. Furthermore, each point outside the verified area obviously has a corresponding point in the verified area whose neighbourhood is coloured identically. By the fact that $\mathbb{G}$ is the maximal ω-admissible colouring, we have $f^*_{\langle p,q\rangle} \le f_{\langle p,q\rangle}$. Since $f_{\langle p,q\rangle} \le f^j_{\langle p,q\rangle}$, we have $f^*_{\langle p,q\rangle}(n) = f_{\langle p,q\rangle}(n)$ for all $n \le h_2$ (where $f^*_{\langle p,q\rangle}$ coincides with $f^j_{\langle p,q\rangle}$). The only possibility that $\mathbb{G}^*$ and $\mathbb{G}$ are not equal is if $f^*_{\langle p,q\rangle}(n) < f_{\langle p,q\rangle}(n)$ for some $n > h_2$. Due to the next result (Lemma 2), this will be lead to a contradiction in the proof of Lemma 3.

Let $\vec{v} = \langle v_1, v_2\rangle \in \mathbb{Z}\times\mathbb{Z}$ be a vector with integer entries. A point $\langle m, n\rangle \in \mathbb{N}\times\mathbb{N}$ with $m+v_1, n+v_2 \ge 0$ is ***lit by $\vec{v}$ in*** $\mathbb{G}_{\langle p,q\rangle}$ iff $\mathbb{G}_{\langle p,q\rangle}(m,n) = \text{black}$ and $\mathbb{G}_{\langle p,q\rangle}(m+v_1, n+v_2) = \text{white}$; if $\langle m, n\rangle$ is lit by $\vec{v}$ in some $\mathbb{G}_{\langle p,q\rangle}$, then we say that $\langle m, n\rangle$ is ***lit by*** $\vec{v}$. For points $\langle m, n\rangle, \langle m', n'\rangle \in \mathbb{N}\times\mathbb{N}$ we write $\langle m, n\rangle \leftrightarrow_{\vec{v}} \langle m', n'\rangle$ iff both are lit by $\vec{v}$, and $|m-m'| \le 1$ and $|n-n'| \le 1$. The transitive closure of $\leftrightarrow_{\vec{v}}$ is denoted by $\leftrightarrow^*_{\vec{v}}$. Note that $\langle m, n\rangle \leftrightarrow^*_{\vec{v}} \langle m', n'\rangle$ can be demonstrated by giving a ***trajectory***, a sequence of points $\langle m_0, n_0\rangle, \langle m_1, n_1\rangle, \ldots, \langle m_k, n_k\rangle$ such that

$$\langle m, n\rangle = \langle m_0, n_0\rangle \leftrightarrow_{\vec{v}} \langle m_1, n_1\rangle \leftrightarrow_{\vec{v}} \cdots \leftrightarrow_{\vec{v}} \langle m_k, n_k\rangle = \langle m', n'\rangle.$$

Lemma 2 *Let $h>0$ and $\vec{v} = \langle v_1, v_2\rangle$ with $v_1 \le 0$ and $v_2 < 0$. If a point $\langle m_0, n_0\rangle$ with $n_0+v_2 > h$ is lit by $\vec{v}$ then there is a point $\langle m'_0, n'_0\rangle$ with $n'_0+v_2 = h$ such that $\langle m_0, n_0\rangle \leftrightarrow^*_{\vec{v}} \langle m'_0, n'_0\rangle$.*

Proof: Suppose $\langle m_0, n_0\rangle$ satisfies the assumption but there is no required $\langle m'_0, n'_0\rangle$; then $n'+v_2 > h$ for each $\langle m', n'\rangle$ such that $\langle m_0, n_0\rangle \leftrightarrow^*_{\vec{v}} \langle m', n'\rangle$. Define the colouring $\overline{\mathbb{G}}$ by

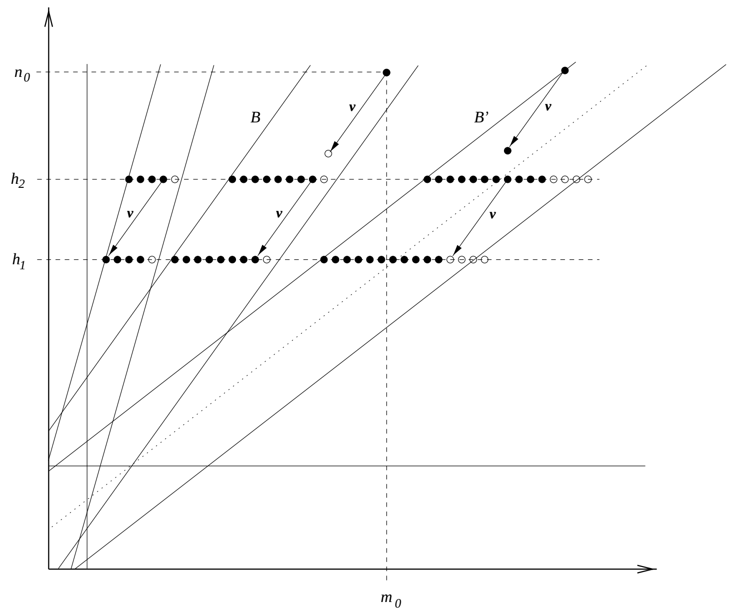

Figure 3: The assumption $\mathbb{G} \neq \mathbb{G}^*$ leads to a contradiction.

$$\overline{\mathbb{G}}_{\langle p,q\rangle}(m,n) = \text{black} \quad \text{iff} \quad \begin{array}{l} \mathbb{G}_{\langle p,q\rangle}(m,n) = \text{black, or} \\ \langle m-\nu_1, n-\nu_2\rangle \text{ is lit by } \vec{\nu} \text{ in } \mathbb{G}_{\langle p,q\rangle} \quad \text{and} \\ \langle m_0, n_0\rangle \leftrightarrow^*_{\vec{\nu}} \langle m-\nu_1, n-\nu_2\rangle. \end{array}$$

$\overline{\mathbb{G}}$ obviously satisfies the monotonicity property of colourings, and we can easily check that each point is locally correct in $\overline{\mathbb{G}}$. Hence $\overline{\mathbb{G}}$ is ω-admissible, which contradicts the fact that $\mathbb{G}$ is the *maximal* ω-admissible colouring. □

Lemma 3 $\mathbb{G}^*$ *is equal to* $\mathbb{G}$*.*

Proof: We have already shown that each $f^*_{\langle p,q\rangle}$ coincides with $f_{\langle p,q\rangle}$ on the set $\{0, 1, 2, \ldots, h_2\}$, so we only have to exclude the possibility that $f^*_{\langle p,q\rangle}(n) < f_{\langle p,q\rangle}(n)$ for some $n > h_2$.

Recall that our algorithm stops by finding a repeatable pattern, for h_1, h_2, i, in $\mathbb{G}^j$ $(i<j)$. Let us fix a corresponding set of belts required by the definition of a repeatable pattern (note that each frontier of $\mathbb{G}^*$ lies in one of the belts above h_1).

We say that a belt B is *valid* iff $\mathbb{G}^*$ coincides with $\mathbb{G}$ when restricted to B. (In particular, the horizontal belt, if it was chosen, is surely valid.) If all belts are valid, then surely $\mathbb{G}^*$ is equal to $\mathbb{G}$. Otherwise, let B be the *rightmost* belt (i.e., the belt with the least slope) which is not valid. Consider an *invalid point* $\langle m_0, n_0\rangle$ in B, i.e., $\mathbb{G}^*_{\langle p,q\rangle}(m_0, n_0)$=white and $\mathbb{G}_{\langle p,q\rangle}(m_0, n_0)$=black, for some p, q; moreover we suppose n_0 to be minimal (i.e., B is valid below n_0). Note that $n_0 > h_2$.

Let α be the slope of B, and let $\vec{\nu} = \langle \nu_1, \nu_2\rangle$, where $\nu_1 = (h_1 - h_2)/\alpha$ and $\nu_2 = h_1 - h_2$ ($\vec{\nu}$ corresponds to the "period of B" in $\mathbb{G}^*$; see Fig. 3). Due to the choice of $\vec{\nu}$ (as the period of B) we have $\mathbb{G}^*_{\langle p,q\rangle}(m_0+\nu_1, n_0+\nu_2) = \text{white}$, and since B is valid below n_0, we have $\mathbb{G}_{\langle p,q\rangle}(m_0+\nu_1, n_0+\nu_2) = \text{white}$. This means that the point $\langle m_0, n_0\rangle$ is lit by $\vec{\nu}$ in $\mathbb{G}_{\langle p,q\rangle}$. Due to Lemma 2 (for h_1 in the place of h) there is a point $\langle m_0', n_0'\rangle$ (lit by $\vec{\nu}$) such that

$\langle m_0, n_0\rangle \leftrightarrow^*_{\vec{v}} \langle m_0', n_0'\rangle$ and $n_0'+v_2=h_1$, i.e., $n_0'=h_2$. Recall that the restrictions of $\mathbb{G}^*$ and $\mathbb{G}$ to $\mathbb{N} \times \{0, 1, 2, \ldots, h_2\}$ coincide. Hence if there is no belt to the right of B then there is clearly no point $\langle m', h_2\rangle$ which would be lit by $\vec{v}$. Otherwise let B' be the first belt to the right of B. Any point $\langle m', h_2\rangle$ which is lit by $\vec{v}$ can lie only in, or to the right of, B'. Nevertheless any trajectory demonstrating $\langle m_0, n_0\rangle \leftrightarrow^*_{\vec{v}} \langle m', h_2\rangle$ would have to cross the (sufficiently thick) monochromatic left subbelt of (the valid) B', which is impossible. (The first point on such a trajectory which is in B', and is thus not an interior point of B', cannot be lit by $\vec{v}$.) $\square$

We can summarize the preceding argument in the following.

Theorem 1 *There is an algorithm which, given a one counter net, constructs a description of the respective maximal simulation relation; more concretely, it gives periodicity descriptions for the corresponding frontier functions.*

3 Applications

In this section we show how Theorem 1 can be applied to obtain new decidability results for one-counter nets. The following one comes almost for free.

Theorem 2 *The problem of strong $\preccurlyeq\succcurlyeq$-regularity of one-counter nets is decidable.*

Proof: Let $p(i)$ be a process of the one-counter net $N = \langle Q, \Sigma, \delta^=, \delta^>\rangle$. Define the set $\mathcal{M} = \{q \in Q \mid p(i) \to^* q(j)$ for infinitely many $j \in \mathbb{N}\}$. Observe that $\mathcal{M}$ is effectively constructible using standard techniques for pushdown automata. As Q is finite, we see that $p(i)$ can reach infinitely many pairwise non-equivalent states iff there is $q \in \mathcal{M}$ such that for every $i \in \mathbb{N}$ there is some $j > i$ such that $q(j) \not\preccurlyeq q(i)$. In other words, $p(i)$ is not strongly regular w.r.t. simulation equivalence iff there is $q \in \mathcal{M}$ such that the frontier function $f_{\langle q,q\rangle}$ has no ω-values ($\forall n \in \mathbb{N} : f_{\langle p,q\rangle}(n) < \omega$). $\square$

Next we show that a number of *simulation* problems for processes of one-counter nets can be reduced to the corresponding *bisimulation* problems for processes of one-counter automata. In this way we obtain further (original) decidability results. The basic tool which enables the mentioned reductions is taken from [11] and is described next.

For every image-finite transition system $T = \langle S, Act, \to\rangle$ we define the transition system $\mathcal{B}(T) = \langle S, Act, \mapsto\rangle$ where $\mapsto$ is given by

$$s \overset{a}{\mapsto} t \quad \text{iff} \quad s \overset{a}{\to} t \text{ and } \forall u \in S : (s \overset{a}{\to} u \wedge t \preccurlyeq u) \Longrightarrow u \preccurlyeq t$$

Note that $\mathcal{B}(T)$ is obtained from T by deleting certain transitions (preserving only the "maximal" ones). Also note that T and $\mathcal{B}(T)$ have the same set of states; as we often need to distinguish between processes "s of T" and "s of $\mathcal{B}(T)$", we denote the latter one by $s_{\mathcal{B}}$. A proof of the next (crucial) theorem, relating simulation equivalence and bisimulation equivalence, can be found in [11].

Theorem 3 *Let s and t be processes of image-finite transition systems T and T', respectively. It holds that $s \preccurlyeq\succcurlyeq s_{\mathcal{B}}$ and $t \preccurlyeq\succcurlyeq t_{\mathcal{B}}$; moreover, $s \preccurlyeq\succcurlyeq t$ iff $s_{\mathcal{B}} \sim t_{\mathcal{B}}$.*

The next theorem provides the technical basis for the aforementioned reductions.

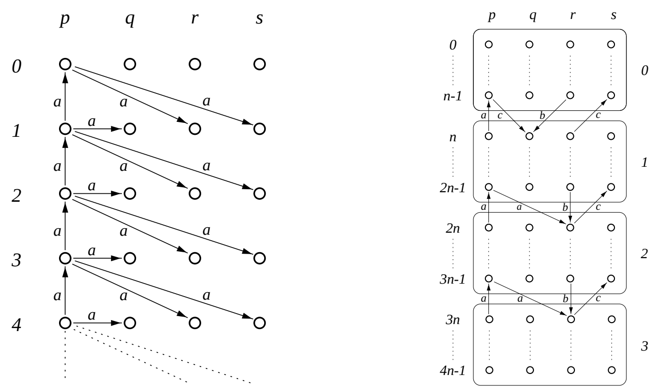

Figure 4: The structure of T_N (left) and $\mathcal{B}(\mathsf{T}_\mathsf{N})$ (right)

Theorem 4 *Let N be a one-counter net. Then the transition system $\mathcal{B}(\mathsf{T}_\mathsf{N})$ is effectively definable within the syntax of one-counter automata, i.e., one can effectively construct a one-counter automaton M such that T_M is isomorphic to $\mathcal{B}(\mathsf{T}_\mathsf{N})$. Moreover, for every state $s = p(i)$ of T_N we can effectively construct a state $p'(i')$ of T_M which is isomorphic to the state $s_\mathcal{B}$ of $\mathcal{B}(\mathsf{T}_\mathsf{N})$.*

Proof: Let $\mathsf{N} = \langle Q, \Sigma, \delta^=, \delta^> \rangle$ be a one-counter net, and let $\mapsto$ be the transition relation of $\mathcal{B}(\mathsf{T}_\mathsf{N})$. Let us define the function *MaxTran* : $Q \times \Sigma \times \mathbb{N} \to \mathcal{P}(Q \times \{-1, 0, 1\})$ as follows:

$$\langle q, j \rangle \in \textit{MaxTran}(p, a, i) \quad \text{iff} \quad p(i) \overset{a}{\mapsto} q(i + j)$$

where $\mapsto$ is the transition relation of $\mathcal{B}(\mathsf{T}_\mathsf{N})$. In fact, *MaxTran*$(p, a, i)$ represents all "maximal" a-transitions of $p(i)$. Our aim is to show that the function *MaxTran* is, in some sense, periodic—we prove that there (effectively) exists $n > 0$ such that for all $p \in Q$, $a \in \Sigma$, and $i \geq n$ we have that *MaxTran*$(p, a, i) =$ *MaxTran*$(p, a, i+n)$. It clearly suffices for our purposes because then we can construct a one-counter automaton $\mathsf{M} = \langle Q \times \{0, \ldots, n-1\}, \Sigma, \gamma^=, \gamma^> \rangle$ where $\gamma^=$ and $\gamma^>$ are the least sets satisfying the following conditions:

- if $p(i) \overset{a}{\mapsto} q(j)$ where $0 \leq i, j < n$, then $(\langle q, j \rangle, 0) \in \gamma^=(\langle p, i \rangle, a)$
- if $p(n-1) \overset{a}{\mapsto} q(n)$, then $(\langle q, 0 \rangle, +1) \in \gamma^=(\langle p, n-1 \rangle, a)$
- if $p(n+i) \overset{a}{\mapsto} q(n+j)$ where $0 \leq i, j < n$, then $(\langle q, j \rangle, 0) \in \gamma^>(\langle p, i \rangle, a)$
- if $p(n) \overset{a}{\mapsto} q(n-1)$, then $(\langle q, n-1 \rangle, -1) \in \gamma^>(\langle p, 0 \rangle, a)$
- if $p(2n-1) \overset{a}{\mapsto} q(2n)$, then $(\langle q, 0 \rangle, +1) \in \gamma^>(\langle p, n-1 \rangle, a)$

Note that the definition of M is effective, because the constant n can be effectively found and for every transition $p(i) \overset{a}{\to} p(j)$ of T_N we can effectively decide whether $p(i) \overset{a}{\mapsto} p(j)$ (here we need the decidability of simulation for one-counter nets). The fact that T_M is isomorphic to $\mathcal{B}(\mathsf{T}_\mathsf{N})$ is easy to see as soon as we realize that $\mathcal{B}(\mathsf{T}_\mathsf{N})$ can be viewed as a sequence of "blocks" of height n, where all "blocks" except for the initial one are the same. The structure of the two (types of) blocks is encoded in the finite control of M, and the number of "current" blocks is stored in its counter (see Fig. 4). Note that M indeed needs the test for zero in order to recognize that the initial block has been entered.

Now we show how to construct the constant n. First, we prove that for all $p \in Q$, $a \in \Sigma$ one can effectively find two constants $k(p,a)$ and $l(p,a)$ such that for every $i > k(p,a)$ we have $\mathit{MaxTran}(p,a,i) = \mathit{MaxTran}(p,a,i+l(p,a))$. We start by reminding ourselves that the out-going transitions of $p(i)$ and $p(j)$, where $i,j \geq 1$, are the "same" in the following sense (see Fig. 4):

$$p(i) \xrightarrow{a} q(i+m) \quad \text{iff} \quad p(j) \xrightarrow{a} q(j+m) \quad \text{iff} \quad (q,m) \in \delta^{>}(p,a).$$

Hence, the set $\mathit{MaxTran}(p,a,i)$, where $i \geq 1$, is obtained by selecting certain elements from $\delta^{>}(p,a)$. In order to find these elements, we must (by the definition of $\mathcal{B}(T)$) take all pairs $\langle\langle q,m\rangle,\langle r,n\rangle\rangle \in \delta^{>}(p,a) \times \delta^{>}(p,a)$, determine whether $q(i+m) \preccurlyeq r(i+n)$, and select only the "maximals". For each such pair $\langle\langle q,m\rangle,\langle r,n\rangle\rangle$ we define an infinite binary sequence $\mathcal{S}$ as follows: $\mathcal{S}(i) = 1$ if $\mathbb{G}_{\langle q,r\rangle}(i+m, i+n) = \text{black}$, and $\mathcal{S}(i) = 0$ otherwise. As (a description of) $\mathbb{G}_{\langle q,r\rangle}$ can be effectively constructed, and the frontier function $f_{\langle q,r\rangle}$ is periodic (see Theorem 1), we can conclude that $\mathcal{S} = \alpha\beta^{\omega}$ where α, β are finite binary strings. Note that α and β can be "read" from the constructed description of $\mathbb{G}_{\langle q,r\rangle}$ and thus they are effectively constructible. As $\delta^{>}(p,a)$ is finite, there are only finitely many pairs to consider and hence we obtain only finitely many α's and β's. Now we let $k(p,a)$ be the length of the longest α, and let $l(p,a)$ be the product of lengths of all β's. In this way we achieve that the whole information which determines the selection of "maximal" elements of $\delta^{>}(p,a)$ during the construction of $\mathit{MaxTran}(p,a,i)$ is periodic (w.r.t. i) with period $l(p,a)$ after a finite "initial segment" of length $k(p,a)$. Let $K = \max\{k(p,a) \mid p \in Q, a \in \Sigma\}$, and $L = \prod_{p\in Q, a\in\Sigma} l(p,a)$. Finally, let $n = K \cdot L$.

To finish the proof, we need to show that for every state $s = p(i)$ of T_N one can construct a state $p'(i')$ of T_M which is isomorphic to the state $s_{\mathcal{B}}$ of $\mathcal{B}(T_N)$. This is straightforward; we simply take $p' = \langle p, i \bmod n\rangle$ and $i' = i \operatorname{div} n$. □

Two concrete examples of how Theorems 3 and 4 can be applied to obtain (new and nontrivial) positive decidability results on one-counter nets are given next.

Corollary 1 *The problem of* $\lessapprox\!\gtrapprox$*-regularity of one-counter nets is decidable.*

Proof: It suffices to realize that a process s of a transition system T is $\lessapprox\!\gtrapprox$-regular iff the process $s_{\mathcal{B}}$ of $\mathcal{B}(T)$ is $\sim$-regular. As $\sim$-regularity is decidable for processes of one-counter automata [3], we are done. □

Corollary 2 *Let* $p\alpha$ *be a process of a deterministic pushdown automaton* Γ *and* $q(i)$ *be a process of a one-counter net* N*. The problem whether* $p\alpha \lessapprox\!\gtrapprox q(i)$ *is decidable.*

Proof: First, realize that if T is a deterministic transition system then $\mathcal{B}(T) = T$. Hence, $p\alpha \lessapprox\!\gtrapprox q(i)$ iff $p\alpha \sim q'(i')$ where $q'(i')$ is the process of Theorem 4. As one-counter automata are (special) pushdown automata, we can apply the result of [14] which says that bisimilarity is decidable for pushdown processes. □

The previous corollary touches, in a sense, the decidability/undecidability border for simulation equivalence, because the problem whether $p\alpha \lessapprox\!\gtrapprox q(i)$ where $p\alpha$ is a process of a deterministic PDA Γ and $q(i)$ is a process of a one-counter automaton M is undecidable [7] (in fact, it is undecidable even if we require Γ to be a deterministic one-counter automaton).

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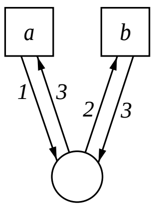

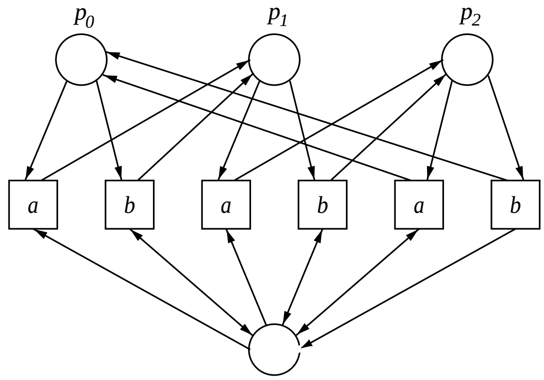

Figure 5: An example of a net $\mathcal{N}$ (left) and $\mathcal{N}'$ (right)

Appendix

In this appendix we show that the model of one-counter nets which has been introduced in Section 1 exactly corresponds to Petri nets with at most one unbounded place.

A ***Petri net*** is a tuple $\mathcal{N} = (P, T, F, Act, \ell)$ where

- P and T are finite disjoint sets of *places* and *transitions*, respectively.
- $F : (P \times T) \cup (T \times P) \to \mathbb{N}$ is the *flow function*. A place p is an *input place* for a transition t iff $F(p,t) \geq 1$. Similarly, p is an *output place* for t iff $F(t,p) \geq 1$.
- *Act* is a finite set of *actions*.
- $\ell : T \to Act$ is the *labelling* which associates an action with every transition.

A *marking* $\mathcal{M}$ is a function $\mathcal{M} : P \to \mathbb{N}$ which associates a number of *tokens* with every place. A transition t is *enabled* at a marking $\mathcal{M}$ iff $\mathcal{M}(p) \geq F(p,t)$ for every place p. A net $\mathcal{N}$ determines a unique transition system $T_{\mathcal{N}}$ where the set of states is the set of all markings, *Act* is the set of actions, and transitions are determined as follows: $\mathcal{M} \stackrel{a}{\to} \mathcal{M}'$ iff there is $t \in T$ such that t is enabled at $\mathcal{M}$, $\ell(t) = a$, and $\mathcal{M}'(p) = \mathcal{M}(p) - F(p,t) + F(t,p)$ for all $p \in P$ (we say that t *fires* at $\mathcal{M}$ reaching $\mathcal{M}'$). Let $\mathcal{M}$ be a marking. A place p is *bounded* for $\mathcal{M}$ iff there is $k \in \mathbb{N}$ such that $\mathcal{M}'(p) \leq k$ for every marking $\mathcal{M}'$ which is reachable from $\mathcal{M}$. The set of all bounded places for $\mathcal{M}$ can be effectively constructed [8].

Theorem 5 *Let $\mathcal{N} = (P, T, F, Act, \ell)$ be a Petri net, $\mathcal{M}$ a marking such that $\mathcal{N}$ has at most one unbounded place for $\mathcal{M}$. Then we can effectively construct a one-counter net $N = \langle Q, \Sigma, \delta^{=}, \delta^{>} \rangle$ and its process $p(i)$ such that the parts of T_N and $T_{\mathcal{N}}$ which are reachable from $p(i)$ and $\mathcal{M}$, respectively, are isomorphic.*

Here we give only a brief sketch of the crucial argument which allows to prove the previous theorem. Let $n = \max\{F(s,t), F(t,s) \mid s \in P, t \in T\}$. First we construct from $\mathcal{N}$ another net $\mathcal{N}'$ which is isomorphic to $\mathcal{N}$ and where each transition changes the value stored in the 'counter' (i.e., the number of tokens in the only unbounded place) at most by one, taking at most one token from it. To do that, we first add to $\mathcal{N}$ new places $p_0, \ldots, p_{n-1}$. Intuitively, the idea is to 'encode' the value i of the counter of $\mathcal{N}$ by storing $i \operatorname{div} n$ tokens in the counter of $\mathcal{N}'$ and putting a token to the place $p_{i \bmod n}$. Each transition t of $\mathcal{N}$ is then replaced with a set of transitions t_i, where $0 \leq i \leq n-1$, such that each t_i has the same label and the same 'connections' to the bounded places of $\mathcal{N}$ as t, but the operation on the counter is 're-implemented' using the newly added places $p_0, \ldots, p_{n-1}$ — each t_i takes one token from p_i, puts one token to (some) p_j, and possibly increments/decrements the counter by one. A concrete example is given is Fig. 5. Then, the net $\mathcal{N}'$ is transformed into a net $\mathcal{N}''$ where each transition has exactly one bounded place among its input and output places (observe that the net $\mathcal{N}'$ of Fig. 5 already has this property and hence it need not be further transformed). It is achieved by introducing a special place for each of the finitely many reachable 'states' of the bounded part of $\mathcal{N}'$ and replacing each transition t of $\mathcal{N}'$ with a family of transitions which have the same label and the same 'connectivity' to the counter as t, and which implement the state-change in the bounded part of $\mathcal{N}'$ caused by t 'explicitly' by shifting only one token. The net $\mathcal{N}''$ can then be 'translated' to a one-counter net N in a straightforward way.