Effective Decomposability of Sequential Behaviours

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Abstract

A process is prime if it cannot be equivalently expressed as a parallel composition of nonempty processes. We characterize all non-prime normed BPA processes together with their prime decompositions by means of normal forms which are designed in this paper. Using this result we demonstrate decidability of the problem whether a given normed BPA process is prime; moreover, we show that non-prime normed BPA processes can be decomposed into primes effectively. Finally, we prove that bisimilarity is decidable in a natural subclass of normed PA processes.

Key words: concurrency, bisimilarity, infinite-state systems, parallelization

1 Introduction

A general problem considered by many researchers is how to improve performance of sequential programs by parallelization. In this paper we study this problem within the framework of process algebras. They provide us with a pleasant formalism which allows to specify sequential as well as parallel programs.

Here we adopt normed BPA processes as a simple model of sequential behaviours (they are equipped with a binary sequential operator). We examine the problem of effective decomposability of normed BPA processes into a parallel product of primes (a process is prime if it cannot be decomposed into nontrivial components). We design special normal forms for normed BPA processes which allow us to characterize all non-prime normed BPA processes

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together with their decompositions up to bisimilarity. As a consequence we also obtain a refinement of the result achieved in [4].

Next we show that any normed BPA process can be decomposed into a parallel product of primes effectively. We also prove several related decidability results. Finally, we prove that bisimilarity is decidable in a natural subclass of normed PA processes (see [2]), which consists of processes of the form $\Delta_1 \| \cdots \| \Delta_n$, where each Δ_i is a normed BPA or a normed BPP process.

In many parts of our paper we rely on results established by other researchers. The question of possible decomposability of processes into a parallel product of primes was first addressed by Milner and Moller in [14]. A more general result was later proved by Christensen, Hirshfeld and Moller (see [8])—it says that *each* normed process has a unique decomposition into primes up to bisimilarity. However, the proof is non-constructive.

Bisimilarity was proved to be decidable for normed BPA processes (see [1,11,9]) and normed BPP processes (see [7,10]). Blanco proved in [3] that bisimilarity is decidable even in the union of normed BPA and normed BPP processes. The same problem was independently examined by Černá, Křetínský, and Kučera in [5]. They demonstrated decidability of the problem whether for a given normed BPA (or BPP) process Δ there is some unspecified normed BPP (or BPA) process Δ' such that $\Delta \sim \Delta'$. If the answer is positive, then it is also possible to *construct* an example of such Δ'. Decidability of bisimilarity in the union of normed BPA and normed BPP processes is an immediate consequence.

Another property of normed BPA and BPP processes which is important for us is *regularity*. A process is regular if it is bisimilar to a process with finitely many states. Kučera proved in [12] that regularity is decidable for normed BPA and normed BPP processes in polynomial time.

This paper is organized as follows. In Section 3 we characterize all decomposable normed BPA processes together with their decompositions by means of special normal forms. As a consequence we also obtain a refinement of the result achieved in [4].

In Section 4 we show that any normed BPA process can be decomposed into a parallel product of primes effectively. We also prove several related decidability results. Finally, we prove that bisimilarity is decidable in a large subclass of normed PA processes (see [2]), which consists of processes of the form $\Delta_1 \| \cdots \| \Delta_n$, where each Δ_i is a normed BPA or a normed BPP process.

2 Preliminaries

2.1 BPA and BPP processes

Let $Act = \{a, b, c, \ldots\}$ be a countably infinite set of *atomic actions*. Let $Var = \{X, Y, Z, \ldots\}$ be a countably infinite set of *variables* such that $Var \cap Act = \emptyset$. The classes of BPA and BPP expressions are defined by the following abstract syntax equations:

$$
\begin{aligned}
E_{BPA} &::= \epsilon \mid X \mid aE_{BPA} \mid E_{BPA}.E_{BPA} \mid E_{BPA} + E_{BPA} \\
E_{BPP} &::= \epsilon \mid X \mid aE_{BPP} \mid E_{BPP} \| E_{BPP} \mid E_{BPA} + E_{BPA}
\end{aligned}
$$

Here a ranges over Act and X ranges over Var. In the rest of this paper we do not distinguish between expressions related by *structural congruence* which is the smallest congruence relation over process expressions such that the following laws hold:

- associativity and 'ϵ' as a unit for '.', '$\|$' and '$+$'
- commutativity for '$\|$' and '$+$'

Moreover, we often write a instead of $a\epsilon$.

As usual, we restrict our attention to *guarded* expressions. A process expression E is guarded if there is a process expression E' such that E and E' are structurally congruent and every variable occurrence in E' is within the scope of an atomic action.

A *guarded BPA (or BPP) process* is defined by a finite family Δ of recursive process equations

$$\Delta = \{X_i \stackrel{def}{=} E_i \quad | \quad 1 \leq i \leq n\}$$

where X_i are distinct elements of Var and E_i are guarded BPA (or BPP) expressions, containing variables from $\{X_1, \ldots, X_n\}$. The set of variables which appear in Δ is denoted by $Var(\Delta)$.

The variable X_1 plays a special role (X_1 is sometimes called the *leading* variable)—it is a root of a labelled transition system, defined by the process Δ and the rules of Figure 1 (note that '$\|$' and '$+$' are commutative).

Nodes of the transition system generated by Δ are BPA (or BPP) expressions, which are often called *states of* Δ, or just "states" when Δ is understood from

$$\frac{}{aE \xrightarrow{a} E} \qquad \frac{E \xrightarrow{a} E'}{E.F \xrightarrow{a} E'.F} \qquad \frac{E \xrightarrow{a} E'}{E + F \xrightarrow{a} E'}$$

$$\frac{E \xrightarrow{a} E'}{E\|F \xrightarrow{a} E'\|F} \qquad \frac{E \xrightarrow{a} E'}{X \xrightarrow{a} E'} \; (X \stackrel{def}{=} E \in \Delta)$$

Fig. 1. SOS rules

the context. We also extend the notation $E \xrightarrow{a} F$ to elements of Act^* in an obvious way (we often write $E \rightarrow^* F$ instead of $E \xrightarrow{w} F$ if $w \in Act^*$ is irrelevant). Given two states E, F, we say that F is *reachable* from E, if $E \rightarrow^* F$. States of Δ which are reachable from X_1 are said to be *reachable*.

Remark 1 *Processes are often identified with their leading variables. Furthermore, if we assume fixed processes Δ_1, Δ_2 such that $Var(\Delta_1) \cap Var(\Delta_2) = \emptyset$, then we can view any process expression E (not necessarily guarded) whose variables are defined in Δ_1, Δ_2 as a process—if we denote it by Δ, then the leading equation of Δ is $X \stackrel{def}{=} E'$, where $X \notin Var(\Delta_1) \cup Var(\Delta_2)$ and E' is a process expression obtained from E by substituting each variable in E with the right-hand side of its corresponding defining equation in Δ_1 or Δ_2 (E' must be guarded now). Moreover, defining equations of Δ_1, Δ_2 are added to Δ. All notions originally defined for processes can be used for process expressions in this sense too.*

2.1.1 Bisimulation

The equivalence between process expressions (states) we are interested in here is *bisimilarity* [15], defined as follows:

Definition 2 *A binary relation R over process expressions is a* bisimulation *if whenever $(E, F) \in R$ then for each $a \in Act$*

- *if $E \xrightarrow{a} E'$, then $F \xrightarrow{a} F'$ for some F' such that $(E', F') \in R$*
- *if $F \xrightarrow{a} F'$, then $E \xrightarrow{a} E'$ for some E' such that $(E', F') \in R$*

Processes Δ and Δ' are bisimilar*, written $\Delta \sim \Delta'$, if their leading variables are related by some bisimulation.*

2.1.2 Normed processes

An important subclass of BPA and BPP processes can be obtained by an extra restriction of *normedness*. A variable $X \in Var(\Delta)$ is *normed* if there is $w \in Act^*$ such that $X \xrightarrow{w} \epsilon$. In that case we define the *norm* of X, written $|X|$, to be the length of the shortest such w. A process Δ is *normed* if all variables of $Var(\Delta)$ are normed. The norm of Δ is then defined to be the norm of X_1.

Remark 3 *As normed processes are intensively studied in this paper, we emphasize some properties of the norm:*

- *Note the norm of a normed process is easy to compute by the following rules:* $|\epsilon| = 0$, $|aE| = |E| + 1$, $|E + F| = \min\{|E|, |F|\}$, $|E.F| = |E| + |F|$, $|E \| F| = |E| + |F|$ *and if* $X_i \stackrel{def}{=} E_i$ *and* $|E_i| = n$, *then* $|X_i| = n$.
- *Bisimilar processes must have the same norm.*

In the rest of this paper we denote the normed subclasses of BPA and BPP processes by nBPA and nBPP, respectively.

2.1.3 Greibach normal form

Any BPA or BPP process Δ can be effectively presented in a special normal form which is called 3-Greibach normal form by analogy with CF grammars (see [1] and [6]). Before the definition we need to introduce the set $Var(\Delta)^*$ of all finite sequences of variables from $Var(\Delta)$, the set $Var(\Delta)^+$ of all nonempty finite sequences over $Var(\Delta)$, and the set $Var(\Delta)^{\otimes}$ of all finite multisets over $Var(\Delta)$. Each multiset of $Var(\Delta)^{\otimes}$ denotes a BPP expression which can be obtained by combining its elements in parallel using the '$\|$' operator.

Definition 4 *A BPA (or BPP) process* Δ *is said to be in* Greibach normal form (GNF) *if all its equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j \alpha_j$$

where $n \in \mathbb{N}$, $a_j \in Act$ *and* $\alpha_j \in Var(\Delta)^*$ *(or* $\alpha_j \in Var(\Delta)^{\otimes}$*). We also require that each* $Y \in Var(\Delta)$ *appears in some reachable state of* Δ. *If* $length(\alpha_j) \leq 2$ *(or* $card(\alpha_j) \leq 2$*) for each* j, $1 \leq j \leq n$, *then* Δ *is said to be in 3-GNF.*

From now on we assume that all BPA and BPP processes we are working with are presented in GNF. This justifies also the assumption that all reachable states of a BPA process Δ are elements of $Var(\Delta)^*$ and all reachable states of a BPP process Δ' are elements of $Var(\Delta')^{\otimes}$.

2.2 Regular processes

Many proofs in this paper take advantage of the fact that regularity of nBPA and nBPP processes is decidable (even in polynomial time—see [12]). The next definition explains what is meant by the notion of regularity and introduces standard normal form for regular processes.

Definition 5 *A process* Δ *is* regular *if there is a process* Δ' *with finitely many states such that* $\Delta \sim \Delta'$*. A regular process* Δ *is said to be in* normal form *if all its equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j[X_j]$$

where $n \in \mathbb{N}$*,* $a_j \in Act$ *and* $X_j \in Var(\Delta)$*. The square brackets indicate optional occurrence—see Remark 7.*

It is easy to see that a process is regular iff it can reach only finitely many states up to bisimilarity. In [13] it is shown, that regular processes can be represented in the normal form just defined. Thus a process Δ is regular iff there is a regular process Δ' in normal form such that $\Delta \sim \Delta'$. A proof of the following proposition can be found in [12].

Proposition 6 *Let* Δ *be a nBPA or nBPP process. The problem whether* Δ *is regular is decidable in polynomial time. Moreover, if* Δ *is regular then a regular process* Δ' *in normal form such that* $\Delta \sim \Delta'$ *can be effectively constructed.*

Remark 7 (special notation) *In the rest of this paper we also use some special notation (due to the lack of general standard). To improve readability, we put all specialties to one place:*

- *if* α *is a regular state of a nBPA or nBPP process (see Remark 1), then* $\Delta^{\mathcal{R}}(\alpha)$ *denotes a bisimilar regular process in normal form, which can be effectively constructed due to Proposition 6. Furthermore, we always assume that* $\Delta^{\mathcal{R}}(\alpha)$ *contains completely fresh variables which are not contained in any other process we deal with.*
- *the class of all processes for which there is a bisimilar nBPA (or nBPP) process is denoted* $\mathcal{S}$*(nBPA) (or* $\mathcal{S}$*(nBPP)).*
- *if* $\Delta_1, \ldots, \Delta_n$ *are processes from nBPA*$\cup$*nBPP and* X_i *is the leading variable of* Δ_i *for* $1 \leq i \leq n$*, then* $\Delta_1 \| \cdots \| \Delta_n$ *denotes the process* $X_1 \| \cdots \| X_n$ *in the sense of Remark 1.*
- *square brackets '*[*' and '*]*' indicate optional occurrence—if we say that some expression is of the form* $a[A][B]$*, we mean that this expression is either* a*,* aA*,* aB *or* aAB*.*
- *upper indexes are used heavily; they appear in two forms:*

$$\alpha^i = \underbrace{\alpha \| \cdots \| \alpha}_{i} \qquad\qquad \alpha^{\bullet i} = \underbrace{\alpha . \cdots . \alpha}_{i}$$

2.3 Decidability of bisimilarity in nBPA $\cup$ nBPP

Bisimilarity is known to be decidable for nBPA [1,11,9] and nBPP [7,10] processes. The following result due to Černá, Křetínský, and Kučera [5] says that bisimilarity is decidable even in the union of nBPA and nBPP processes.

Proposition 8 *Let Δ be a nBPA (or nBPP) process. It is decidable whether $\Delta \in \mathcal{S}(nBPP)$ (or whether $\Delta \in \mathcal{S}(nBPA)$) and if the answer is positive, then a bisimilar nBPP (or nBPA) process can be effectively constructed.*

3 The Characterization of Decomposable nBPA Processes

In this section we design special normal forms for nBPA processes which allow us to characterize all decomposable nBPA processes together with their decompositions.

Definition 9 (prime processes) *Let nil be a special name for the process which cannot emit any action (i.e., nil $\sim \epsilon$). A nBPA or nBPP process Δ is* prime *if $\Delta \not\sim$ nil and whenever $\Delta \sim \Delta_1 \| \Delta_2$ we have that either $\Delta_1 \sim$ nil or $\Delta_2 \sim$ nil.*

Natural questions are, what processes have a decomposition into a finite parallel product of primes and whether this decomposition is unique. This problem was first examined by Milner and Moller in [14]. They proved that each normed finite-state process has a unique decomposition up to bisimilarity. A more general result is due to Christensen, Hirshfeld, and Moller [8]—they proved the following proposition:

Proposition 10 *Let Δ be a nBPP process. Then Δ has a unique decomposition (up to bisimilarity) into a parallel product of primes.*

Remark 11 *Proposition 10 in fact holds for* any *normed process (in particular for nBPA). The proof in [8] is independent of a concrete syntax—it could be easily formulated in terms of normed transition systems.*

Proposition 10 in fact says that each normed process Δ can be parallelized in the "best" way and that this way is in some sense unique. However, this nice theoretical result is non-constructive. It is not clear how to *construct* the decomposition and how to test whether some process is prime. This is the subject of next sections.

An immediate consequence of Proposition 10 is the following "cancelation" lemma (see [6]):

Lemma 12 *Let $\Delta, \Gamma, \Psi, \Phi$ be normed processes such that $\Delta \| \Psi \sim \Gamma \| \Phi$ and $\Psi \sim \Phi$. Then $\Delta \sim \Gamma$.*

3.1 Decomposability of nBPP Processes

Each nBPP processes Δ can be easily decomposed into a parallel product of primes—all that has to be done is a construction of a bisimilar *canonical* process (see [6]).

Theorem 13 *Let Δ be a nBPP process. It is decidable whether Δ is prime and if not, its decomposition into primes can be effectively constructed.*

PROOF. By induction on $n = |\Delta|$:

- **n=1:** each nBPP process whose norm is 1 is prime.
- **Induction step:** Suppose $\Delta \sim \Delta_1 \| \Delta_2$. As Δ_1, Δ_2 are reachable states of $\Delta_1 \| \Delta_2$, there are $\alpha_1, \alpha_2 \in \mathit{Var}(\Delta)^{\otimes}$ such that $\Delta_1 \sim \alpha_1$ and $\Delta_2 \sim \alpha_2$, thus $\Delta \sim \alpha_1 \| \alpha_2$. Furthermore, $|\Delta| = |\alpha_1| + |\alpha_2|$. We show that there are only finitely many candidates for α_1, α_2. First, there are only finitely many pairs $(k_1, k_2) \in \mathbb{N} \times \mathbb{N}$ such that $k_1 + k_2 = |\Delta|$. For each such pair (k_1, k_2) there are only finitely many pairs (β_1, β_2) such that $\beta_1, \beta_2 \in \mathit{Var}(\Delta)^{\otimes}$, $|\beta_1| = k_1$ and $|\beta_2| = k_2$. It is obvious that the set $\mathcal{M}$ of all such pairs can be effectively constructed. For each element (β_1, β_2) of $\mathcal{M}$ we check whether $\Delta \sim \beta_1 \| \beta_2$ (it can be done because bisimilarity is decidable for nBPP processes). If there is no such pair then Δ is prime. Otherwise, we check whether β_1, β_2 are prime (it is possible by induction hypothesis) and construct their decompositions. If we combine the obtained decompositions in parallel, we get a decomposition of Δ. $\square$

As each normed regular process in normal form can be seen as a nBPP process in GNF (see Definition 4 and 5), the previous theorem (and especially its constructive proof) can also be used in case of *regular* nBPA processes—remember that regularity of nBPA processes is decidable and regular nBPA processes can be transformed into normal form specified in Definition 5 effectively (see Proposition 6). However, it is not clear how to decompose *non-regular* nBPA processes; this is the problem we concentrate on in the rest of this paper.

It this section we give an exact characterization of non-prime nBPA processes. As we already know from the previous section, the problem is actually interesting only for non-regular nBPA processes, hence the main characterization theorem (Theorem 30) does not concern regular nBPA processes. Our results bring also interesting consequences; for example, we obtain a refinement of the result achieved in [4] (see Remark 25).

The layout of this subsection is as follows: first we prove two technical lemmas (Lemma 14 and 15). Then we consider the following problem: if Δ is a non-regular nBPA process such that $\Delta \sim \Delta_1 \| \Delta_2$, where Δ_1, Δ_2 are some (unspecified) processes, how do the processes $\Delta, \Delta_1, \Delta_2$ look? It is clear that $\Delta_1, \Delta_2 \in \mathcal{S}(nBPA)$, hence the assumption that Δ_1, Δ_2 are nBPA processes can be used w.l.o.g. This problem is solved by Proposition 18 and 23, with a help of several definitions. Having this, the proof of Theorem 30 is easy to complete.

Lemma 14 *Let Δ be a nBPA process. Let $\alpha, \gamma \in Var(\Delta)^+$, $Q, C \in Var(\Delta)$ such that $|Q| = |C| = 1$ and $\alpha \| Q \sim C.\gamma$. Then $\alpha \sim Q^{|\alpha|}$.*

PROOF. We prove that for each $1 \leq i \leq |\alpha| + 1$ there is $\beta \in Var(\Delta)^*$ such that $\beta \| Q^i \sim C.\gamma$. This is clearly sufficient, because then $\alpha \| Q \sim C.\gamma \sim Q^{|\alpha|+1}$ and thus $\alpha \sim Q^{|\alpha|}$ due to Lemma 12. We proceed by induction on i.

- **i=1:** choose $\beta = \alpha$.
- **Induction step:** Let $\beta \| Q^i \sim C.\gamma$. As $|C| = 1$, all states which are reachable from $\beta \| Q^i$ in one norm-decreasing step are bisimilar. As Δ is normed, there is $\beta' \in Var(\Delta)^*$ such that $\beta \xrightarrow{a} \beta'$ where $|\beta| = |\beta'| + 1$. Hence $\beta \| Q^{i-1} \sim \beta' \| Q^i$ and by substitution we obtain $\beta \| Q^i \sim \beta' \| Q^{i+1}$. $\square$

The proof of the following lemma is probably the most technical part of this paper. Diagrams of Figure 2 could ease the reading.

Lemma 15 *Let Δ be a nBPA process, $\alpha, \beta, \gamma \in Var(\Delta)^*$ such that α is non-regular and $\alpha \| \beta \sim \gamma$. Let $\beta \to^* Q$ where $|Q| = 1$. Then $\beta \sim Q^{|\beta|}$.*

PROOF. As α is non-regular, it can reach a state of an arbitrary length, i.e., for each $i \in \mathbb{N}$ there is α' such that $\alpha \to^* \alpha'$ and $length(\alpha') = i$. Let $m = \max\{|X|, X \in Var(\Delta)\}$ and let $\alpha \to^* \alpha_1$ where $length(\alpha_1) \geq m.(|\beta|+1)$. Then $\alpha_1 \| \beta \sim \gamma_1$ for some $\gamma_1 \in Var(\Delta)^*$. As $\beta \to^* Q$, we have $\alpha_1 \| Q \sim \gamma_2$ where $\gamma_2 \in Var(\Delta)^*$ and $length(\gamma_2) > 1$ — hence γ_2 is of the form $P.\omega$ where

$$
\begin{tikzcd}
\alpha\|\beta \arrow[d] \arrow[r, phantom, "\sim"] & \gamma \arrow[d] \\
\alpha_1\|\beta \arrow[d] \arrow[r, phantom, "\sim"] & \gamma_1 \arrow[d] \\
\alpha_1\|Q \arrow[d, "s"] \arrow[r, phantom, "\sim"] & \gamma_2 = P.\omega \arrow[d, "s"] \\
\alpha_2\|Q \arrow[r, phantom, "\sim"] & C.\omega
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Q^{|\eta|}\|\beta \arrow[ddl, "v"'] \arrow[d, "t"] \arrow[r, phantom, "\sim"] & A_1.\cdots.A_{|\beta|+1}.\rho \arrow[d, "t"] \arrow[dd, bend left, "v"] \\
 & Q^{|\eta|} \arrow[d, "u"] \arrow[r, phantom, "\sim"] & \varphi.A_{|\beta|+1}.\rho \arrow[d, "u"] \\
Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}\|\beta \arrow[r, phantom, "\sim"] & Q^{|\eta|-|\varphi|} \arrow[r, phantom, "\sim"] & A_{|\beta|+1}.\rho
\end{tikzcd}
$$

Fig. 2. Diagrams for the proof of Lemma 15

$\omega \in Var(\Delta)^+$. Let $\alpha_1 \xrightarrow{s} \alpha_2$ where s is a norm-decreasing sequence of actions such that $length(s) = |P| - 1$. As $\alpha_1\|Q \xrightarrow{s} \alpha_2\|Q$ and $\alpha_1\|Q \sim P.\omega$, $P.\omega \xrightarrow{s} C.\omega$ where $|C| = 1$ and $\alpha_2\|Q \sim C.\omega$. Now we can apply Lemma 14 and conclude $\alpha_2 \sim Q^{|\alpha_2|}$. As $\alpha_1 \xrightarrow{s} \alpha_2$ where $length(s) = |P| - 1 < m$, only the first $m - 1$ variables of α_1 could contribute to the sequence s — hence α_1, α_2 must have a common suffix whose length is at least $m.|\beta|$, i.e., $\alpha_1 = \nu.\eta$, $\alpha_2 = \delta.\eta$ where $length(\eta) \geq m.|\beta|$. As $\alpha_1\|\beta \sim \gamma_1$ and $\alpha_1 = \nu.\eta$, we can conclude $\eta\|\beta \sim \gamma_3$ for some $\gamma_3 \in Var(\Delta)^*$. Clearly $length(\gamma_3) > |\beta|$, because $|\,\eta\|\beta\,| > m.|\beta|$ (remember $length(\eta) \geq m.|\beta|$). Thus γ_3 is of the form $A_1.\cdots.A_{|\beta|+1}.\rho$ where $\rho \in Var(\Delta)^*$. Furthermore, $\eta \sim Q^{|\eta|}$ because $\alpha_2 \sim Q^{|\alpha_2|}$ and $\alpha_2 = \delta.\eta$. To sum up, we have $Q^{|\eta|}\|\beta \sim A_1.\cdots.A_{|\beta|+1}.\rho$. Now we prove that $\beta \sim Q^{|\beta|}$. Let $\beta \xrightarrow{t} \epsilon$ where $length(t) = |\beta|$. Then $Q^{|\eta|}\|\beta \xrightarrow{t} Q^{|\eta|}$ and the state $A_1.\cdots.A_{|\beta|+1}.\rho$ must be able to match the sequence t and enter a state bisimilar to $Q^{|\eta|}$. As $length(t) = |\beta|$, only the first $|\beta|$ variables of $A_1.\cdots.A_{|\beta|+1}.\rho$ can contribute to the sequence t, i.e., $A_1.\cdots.A_{|\beta|+1}.\rho \xrightarrow{t} \varphi.A_{|\beta|+1}.\rho$ where $\varphi \in Var(\Delta)^*$. Now let $\varphi.A_{|\beta|+1}.\rho \xrightarrow{u} A_{|\beta|+1}.\rho$ where $length(u) = |\varphi|$. The state $Q^{|\eta|}$ can match the sequence u only by removing $|\varphi|$ copies of Q — hence $Q^{|\eta|-|\varphi|} \sim A_{|\beta|+1}.\rho$. As $|\eta| \geq m.|\beta|$, it is clear that $|\eta| \geq |A_1.\cdots.A_{|\beta|}|$. Therefore there is $v \in Act^*$, $length(v) = |A_1.\cdots.A_{|\beta|}|$ such that $Q^{|\eta|} \xrightarrow{v} Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}$ and thus $Q^{|\eta|}\|\beta \xrightarrow{v} Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}\|\beta$. The state $A_1.\cdots.A_{|\beta|+1}.\rho$ can match the sequence v only by removing $A_1.\cdots.A_{|\beta|}$ — hence $Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}\|\beta \sim A_{|\beta|+1}.\rho$ and by transitivity of bisimilarity we have $Q^{|\eta|-|\varphi|} \sim Q^{|\eta|-|A_1.\cdots.A_{|\beta|}|}\|\beta$. From this we obtain $\beta \sim Q^{|\beta|}$. $\square$

Definition 16 (simple processes) *A nBPA process Δ is* simple *if $Var(\Delta)$ contains just one variable, i.e., $card(Var(\Delta)) = 1$.*

We will often identify simple processes with their leading (and only) variables in the rest of this paper. Moreover, it is easy to see that a simple process Q is non-regular iff the def. equation for Q contains a summand of the form

$aQ^{\bullet k}$ where $a \in Act$ and $k \geq 2$. The norm of Q is one, because Q could not be normed otherwise. Another important property of simple processes is presented in the remark below:

Remark 17 *Each simple nBPA process Q belongs to $\mathcal{S}$(nBPP)—a bisimilar nBPP process can be obtained just by replacing the '.' operator with the '$\|$' operator in the def. equation for Q. Consequently, any process expressions built over the same number of copies of Q using the '.' and '$\|$' operators are bisimilar (e.g., $(Q.(Q\|Q))\|Q \sim (Q\|Q).(Q\|Q)$).*

Proposition 18 *Let Δ_1, Δ_2 be non-regular nBPA processes. Then $\Delta_1 \| \Delta_2 \in \mathcal{S}$(nBPA) iff $\Delta_1 \sim Q^{|\Delta_1|}$ and $\Delta_2 \sim Q^{|\Delta_2|}$ for some non-regular simple process Q.*

PROOF.
"$\Leftarrow$" Easy—see Remark 17.
"$\Rightarrow$" Assume there is some nBPA process Δ such that $\Delta_1 \| \Delta_2 \sim \Delta$. Then there are $\alpha_1, \alpha_2 \in Var(\Delta)^*$ such that $\Delta_1 \sim \alpha_1$ and $\Delta_2 \sim \alpha_2$. Thus $\alpha_1 \| \alpha_2 \sim \Delta$ and as α_1, α_2 are non-regular, we can use Lemma 15 and conclude that there are $Q_1, Q_2 \in Var(\Delta)$ such that $|Q_1| = |Q_2| = 1$, $\alpha_1 \to^* Q_1$, $\alpha_2 \to^* Q_2$ and $\alpha_1 \sim Q_1^{|\alpha_1|}$, $\alpha_2 \sim Q_2^{|\alpha_2|}$. First we prove that $Q_1 \sim Q$ for some simple process Q. To do this, it suffices to prove that if $a\gamma$ is a summand in the def. equation for Q_1, then $\gamma \sim Q_1^{\bullet |\gamma|}$. As $\alpha_1 \| \alpha_2 \to^* Q_1 \| \alpha_2 \xrightarrow{a} \gamma \| \alpha_2$, the process $\gamma \| \alpha_2$ belongs to $\mathcal{S}$(nBPA). Let $\gamma \to^* R$ where $|R| = 1$. Then $\gamma \sim R^{|\gamma|}$ (due to Lemma 15) and as $\alpha_1 \to^* \gamma \to^* R$, we also have $\alpha_1 \sim R^{|\alpha_1|}$. Hence $R \sim Q_1$ and $\gamma \sim Q_1^{|\gamma|} \sim Q_1^{\bullet |\gamma|}$.

Similarly, we could prove that Q_2 is also bisimilar to some simple process. To finish the proof, we need to show that $Q_1 \sim Q_2$. Let $m = \max\{|X|, X \in Var(\Delta)\}$. As α_1 is non-regular, it can reach a state of an arbitrary norm—let $\alpha_1 \to^* \alpha_1'$ where $|\alpha_1'| = m$. Then $\alpha_1' \| Q_2 \sim \delta$ for some $\delta \in Var(\Delta)^*$ whose length is at least two—$\delta = A.B.\delta'$. Clearly $\alpha_1' \sim Q_1^{|\alpha_1'|}$ (we can use the same argument as in the first part of this proof—Q_2 is non-regular and α_1' plays the role of γ), hence $Q_1^{|\alpha_1'|} \| Q_2 \sim A.B.\delta'$. As $Q_1^{|\alpha_1'|-|A|} \| Q_2 \sim B.\delta'$ and $Q_1^{|\alpha_1'|-|A|+1} \sim B.\delta'$, we have $Q_1^{|\alpha_1'|-|A|} \| Q_2 \sim Q_1^{|\alpha_1'|-|A|+1}$ by transitivity of $\sim$ and thus $Q_1 \sim Q_2$. □

Proposition 18 in fact says that if Δ is a non-regular nBPA process such that $\Delta \sim \Delta_1 \| \Delta_2$, where Δ_1, Δ_2 are non-regular processes, then each of those three processes can be equivalently represented as a power of some non-regular simple process. This representation is very special and can be seen as normal form.

If Δ is a non-regular nBPA process such that $\Delta \sim \Delta_1 \| \Delta_2$, it is also possible that Δ_1 is non-regular and Δ_2 regular. Before we start to examine this possibility, we introduce a special normal form for nBPA processes (as we shall see, Δ and Δ_1 can be represented in this normal form):

Definition 19 (DNF(Q)) *Let Δ be a non-regular nBPA process in GNF, $Q \in \mathit{Var}(\Delta)$. We say that Δ is in DNF(Q) if all summands in all defining equations from Δ are of the form $a([Y].[Q^{\bullet i}])$, where $Y \in \mathit{Var}(\Delta)$, $i \in \mathbb{N}$ and $a \in \mathit{Act}$. Furthermore, all summands in the def. equation for Q must be of the form $a[Q]$, where $a \in \mathit{Act}$.*

Example 20 *The following process is in DNF(Q):*

$$
\begin{aligned}
X &\stackrel{\mathit{def}}{=} a(Y.Q.Q) + bX + a(Q.Q.Q) + c \\
Y &\stackrel{\mathit{def}}{=} bQ + cX + c(Y.Q) + b \\
Q &\stackrel{\mathit{def}}{=} aQ + bQ + a + c
\end{aligned}
$$

Remark 21 *Reachable states of a nBPA process Δ in DNF(Q) are of the form $[Y].[Q^{\bullet i}]$ where $Y \in \mathit{Var}(\Delta)$ and $i \in \mathbb{N}_0$. As Δ is non-regular, the state $Q^{\bullet k}$ is reachable for each $k \in \mathbb{N}$.*

Note that the variable Q itself is a regular simple process. The next lemma says that if Δ is a process in *DNF*(Q), then the variable Q is in some sense unique:

Lemma 22 *Let Δ and Δ' be processes in DNF(Q) and DNF(R), respectively. If $\Delta \sim \Delta'$, then $Q \sim R$.*

PROOF. Let $m = \max\{|X|,\ X \in \mathit{Var}(\Delta')\}$. As the state $Q^{\bullet m+1}$ is a reachable state of Δ, $Q^{\bullet m+1} \sim [Y].R^{\bullet i}$ for some $Y \in \mathit{Var}(\Delta')$, $i \in \mathbb{N}$ (see Remark 21). Hence $Q \sim R$. $\square$

Proposition 23 *Let Δ_1, Δ_2 be nBPA processes such that Δ_1 is non-regular and Δ_2 is regular. Then $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$ iff there is a process Δ_1' in DNF(Q) such that $\Delta_1 \sim \Delta_1'$ and $\Delta_2 \sim Q^{|\Delta_2|}$.*

PROOF.
"$\Rightarrow$" Let $\Delta_2 \to^* Q'$ where $Q' \in \mathit{Var}(\Delta_2)$, $|Q'| = 1$. Using the same kind of argument as in the proof of Proposition 18 we obtain that $Q' \sim Q$ for some regular simple process Q such that $\Delta_2 \sim Q^{|\Delta_2|}$. It remains to prove that there is a process Δ_1' in *DNF*(Q) such that $\Delta_1 \sim \Delta_1'$. We show that each summand of each defining equation from Δ_1 can be transformed into a form which is

admitted by $DNF(Q)$. First, let us realize two facts about summands—if $a\alpha$ is a summand in a def. equation from Δ_1, then

(1) If $\alpha = \beta.Y.\gamma$ where Y is a non-regular variable, then each variable P of γ is bisimilar to $Q^{|P|}$.
(2) α contains at most one non-regular variable.

The first fact is a consequence of Lemma 14—let Δ be a nBPA process such that $\Delta_1 \| \Delta_2 \sim \Delta$. As Δ_1 is normed, $\Delta_1 \rightarrow^* Y.\gamma.\delta$ for some $\delta \in Var(\Delta_1)^*$. As Y is non-regular, it can reach a state of an arbitrary length—let $m = \max\{|X|,\ X \in Var(\Delta_1)\}$ and let $Y \rightarrow^* \omega$ where $length(\omega) = m$. As $\Delta_1 \| \Delta_2 \rightarrow^* \omega.\gamma.\delta \| Q'$, there is $\varphi \in Var(\Delta)^*$ such that $\omega.\gamma.\delta \| Q' \sim \varphi$. Let $\varphi = C.\varphi'$ and let s be a norm-decreasing sequence of actions such that $length(s) = |C| - 1$ and $\omega \xrightarrow{s} \omega'$. Then $\omega'.\gamma.\delta \| Q' \sim C'.\varphi'$ where $|C'| = 1$ and due to Lemma 14 (and the fact that $Q' \sim Q$) we have $\omega'.\gamma.\delta \sim Q^{|\omega'.\gamma.\delta|}$, hence $\gamma \sim Q^{|\gamma|}$ and $P \sim Q^{|P|}$ for each variable P which appears in γ.

The second fact is a consequence of the first one—assume that $\alpha = \beta.Y.\gamma.Z.\delta$ where Y, Z are non-regular. Then $Z \sim Q^{|Z|}$ and as Q is regular, $Q^{|Z|}$ is regular too. Hence Z is regular and we have a contradiction.

Now we can describe the promised transformation of Δ_1 into Δ_1': if $X \stackrel{def}{=} \sum_{i=1}^{n} a_i\alpha_i$ is a def. equation in Δ_1, then $X \stackrel{def}{=} \sum_{i=1}^{n} a_i\mathcal{T}(\alpha_i)$ is a def. equation in Δ_1', where $\mathcal{T}$ is defined as follows:

- If α_i does not contain any non-regular variable, then $\mathcal{T}(\alpha_i) = A$, where A is the leading variable of $\Delta^{\mathcal{R}}(\alpha_i)$. Moreover, defining equations of $\Delta^{\mathcal{R}}(\alpha_i)$ are added to Δ_1'.
- If $\alpha_i = \beta.Y.\gamma$ where Y is a non-regular variable, then $\mathcal{T}(\alpha_i) = A$, where A is the leading variable of the process Δ' which is obtained by the following modification of the process $\Delta^{\mathcal{R}}(\beta)$: each summand in each def. equation of $\Delta^{\mathcal{R}}(\beta)$ which is of the form b, where $b \in Act$, is replaced with $b(Y.Q^{\bullet|\gamma|})$ — remember $\gamma \sim Q^{|\gamma|} \sim Q^{\bullet|\gamma|}$. Moreover, def. equations of Δ' are added to Δ_1'.

The defining equation for Q is also added to Δ_1'. The resulting process is in $DNF(Q)$ and as $\mathcal{T}$ preserves bisimilarity, $\Delta_1 \sim \Delta_1'$.

"$\Leftarrow$" We show how to construct a nBPA process Δ which is bisimilar to $\Delta_1' \| Q^{|\Delta_2|}$. Let $k = |\Delta_2|$. The set of variables of Δ looks as follows:

$$Var(\Delta) = \{Q\} \cup \{Y_i \mid Y \in Var(\Delta_1'), Y \neq Q \text{ and } i \in \{0, \ldots, k\}\}$$

Defining equations of Δ are constructed using the following rules:

- the def. equation for Q is the same as in Δ_1'

- if $a(Y.Q^j)$, where $j \in \mathbb{N}_0$, $Y \neq Q$, is a summand in the def. equation for $Z \in \mathit{Var}(\Delta_1')$, then $a(Y_i.Q^j)$ is a summand in the def. equation for Z_i for each $i \in \{0, \ldots, k\}$
- if $a(Q^j)$ where $j \in \mathbb{N}_0$ is a summand in the def. equation for $Z \in \mathit{Var}(\Delta_1')$, then $a(Q^{j+i})$ is a summand in the def. equation for Z_i for each $i \in \{0, \ldots, k\}$
- if aQ is a summand in the def. equation for Q and $Z \in \mathit{Var}(\Delta_1')$, $Z \neq Q$, then aZ_i is a summand in the def. equation for Z_i for each $i \in \{1, \ldots, k\}$
- if a is a summand in the def. equation for Q and $Z \in \mathit{Var}(\Delta_1')$, $Z \neq Q$, then aZ_{i-1} is a summand in the def. equation for Z_i for each $i \in \{1, \ldots, k\}$

The intuition which stands behind this construction is that lower indexes of variables indicate how many copies of Q in $Q^{|\Delta_2|}$ have not disappeared yet. The fact $\Delta_1' \| Q^{|\Delta_2|} \sim \Delta$ is easy to check. □

Example 24 *If we apply the algorithm presented in the "⇐" part of the proof of Proposition 23 to the process $X \| Q^2$, where X, Q are variables of the process presented in Example 20, we obtain the following output:*

$$\begin{aligned}
X_2 &\stackrel{\mathit{def}}{=} a(Y_2.Q.Q) + bX_2 + a(Q.Q.Q.Q.Q) + c(Q.Q) + aX_2 + bX_2 + aX_1 + cX_1 \\
X_1 &\stackrel{\mathit{def}}{=} a(Y_1.Q.Q) + bX_1 + a(Q.Q.Q.Q) + cQ + aX_1 + bX_1 + aX_0 + cX_0 \\
X_0 &\stackrel{\mathit{def}}{=} a(Y_0.Q.Q) + bX_0 + a(Q.Q.Q) + c \\
Y_2 &\stackrel{\mathit{def}}{=} b(Q.Q.Q) + cX_2 + c(Y_2.Q) + b(Q.Q) + aY_2 + bY_2 + aY_1 + cY_1 \\
Y_1 &\stackrel{\mathit{def}}{=} b(Q.Q) + cX_1 + c(Y_1.Q) + bQ + aY_1 + bY_1 + aY_0 + cY_0 \\
Y_0 &\stackrel{\mathit{def}}{=} bQ + cX_0 + c(Y_0.Q) + b \\
Q &\stackrel{\mathit{def}}{=} aQ + bQ + a + c
\end{aligned}$$

Remark 25 *Proposition 23 can also be seen as a refinement of the result achieved in [4]—Burkart and Steffen proved that PDA processes are closed under parallel composition with finite-state processes, while BPA processes lack this property. Proposition 23 says precisely what nBPA processes can remain nBPA if they are combined in parallel with a regular process. Moreover, it also characterizes all such regular processes.*

It is easy to see that the algorithm from the proof of Proposition 23 always outputs a process in $\mathit{DNF}(Q)$ (see Example 24). Moreover, the structure of this process is very specific; we can observe that each variable belongs to a special "level". This intuition is formally expressed by the following definition (it is a little complicated—but it pays because we will be able to characterize all non-prime nBPA processes):

Definition 26 *Let Δ be a nBPA process in $\mathit{DNF}(Q)$. The* level *of Δ, denoted $\mathit{Level}(\Delta)$, is the maximal $l \in \mathbb{N}$ such that the set $\mathit{Var}(\Delta) - \{Q\}$ can be divided into l disjoint linearly ordered subsets $L_1, \ldots, L_l$ of the same cardinality*

k. Moreover, the following conditions must be true (the j^{th} element of L_i is denoted $A_{i,j}$):

- $A_{l,1}$ is the leading variable of Δ.
- Defining equations for variables of L_1 contain only variables from $L_1 \cup \{Q\}$
- The defining equation for $A_{i,j}$, where $i \geq 2$, $1 \leq j \leq k$, contains exactly those summands which can be derived by one of the following rules:

(1) If aQ is a summand in the defining equation for Q, then $aA_{i,j}$ is a summand in the defining equation for $A_{i,j}$ for each $2 \leq i \leq l$, $1 \leq j \leq k$.
(2) If a is a summand in the defining equation for Q, then $aA_{i-1,j}$ is a summand in the defining equation for $A_{i,j}$ for each $2 \leq i \leq l$, $1 \leq j \leq k$.
(3) If $a(A_{1,m}.Q^{\bullet n})$ is a summand in the defining equation for $A_{1,j}$, then $a(A_{i,m}.Q^{\bullet n})$ is a summand in the defining equation for $A_{i,j}$ for each $2 \leq i \leq l$.
(4) If $aQ^{\bullet n}$ is a summand in the defining equation for $A_{1,j}$, then $aQ^{\bullet (n+i-1)}$ is a summand in the defining equation for $A_{i,j}$, where $2 \leq i \leq l$.

Example 27 *The process of Example 24 has the level 3; $L_1 = \{X_0, Y_0\}$, $L_2 = \{X_1, Y_1\}$ and $L_3 = \{X_2, Y_2\}$.*

Remark 28 *It is easy to see that any process Δ in DNF(Q) whose level is greater than one is decomposable; it holds that $\Delta \sim \Delta' \| Q^k$ where $k = Level(\Delta) - 1$ and Δ' is obtained from Δ by deleting all equations for variables of L_i where $i \geq 2$. The leading variable of Δ' is $A_{1,1}$.*

Lemma 29 *Let Q be a non-regular simple process and let Δ be a nBPA process such that $\Delta \| Q \in \mathcal{S}$(nBPA). Then $\Delta \sim Q^{|\Delta|}$.*

PROOF. Let $\Delta \to^* R$ where $|R| = 1$. As Q is non-regular, we can use Lemma 15 and conclude that $\Delta \sim R^{|\Delta|}$. Now it suffices to prove that $R \sim Q$. Let Δ' be a nBPA process such that $\Delta \| Q \sim \Delta'$ and let $m = \max\{|X|,\ X \in Var(\Delta')\}$. As Q is simple and non-regular, $Q \to^* Q^{\bullet m}$ (see Remark 21). Hence $R \| Q^{\bullet m} \sim \alpha$ for some $\alpha \in Var(\Delta')^*$ whose length is at least 2 — thus $\alpha = A.\beta$ for some $\beta \in Var(\Delta')^+$. Let $k = |A|$. Then each two states which are reachable from $R \| Q^{\bullet m}$ in k norm-decreasing steps are bisimilar—hence $R \| Q^{\bullet m-k} \sim Q^{\bullet m-k+1}$ and from this we have $R \sim Q$. □

Now we can prove the first main theorem of this paper:

Theorem 30 *Let Δ be a non-regular nBPA process and let $\Delta \sim \Delta_1 \| \cdots \| \Delta_n$, where $n \geq 2$, Δ_i is a prime process for each $1 \leq i \leq n$ and Δ_1 is non-regular. Then one of the following possibilities holds:*

- *There is a non-regular simple process Q such that $\Delta \sim Q^{\bullet |\Delta|}$ and $\Delta_i \sim Q$*

for each $1 \leq i \leq n$.

- *There are nBPA processes Δ', Δ'_1 in $DNF(Q)$ such that $\Delta \sim \Delta'$, $\Delta_1 \sim \Delta'_1$, $Level(\Delta') = n$, $Level(\Delta'_1) = 1$ and $\Delta_i \sim Q$ for each $2 \leq i \leq n$.*

PROOF. We proceed by induction on n:

- **n=2:** if Δ_2 is non-regular, we can use Proposition 18. Similarly, if Δ_2 is regular, we use Proposition 23; note that $Level(\Delta_1) = 1$ because Δ_1 would not be prime otherwise (see Remark 28).
- **Induction step:** let $\Delta \sim \Delta_1 \| \cdots \| \Delta_n$. As $\Delta_1 \| \cdots \| \Delta_n \rightarrow^* \Delta_1 \| \cdots \| \Delta_{n-1}$, there is a reachable state α of Δ such that $\alpha \sim \Delta_1 \| \cdots \| \Delta_{n-1}$ — hence we can use ind. hypothesis (note that α must be non-regular) and conclude that there are two possibilities:

(1) There is a non-regular simple process Q such that $\Delta_i \sim Q$ for each $1 \leq i \leq n-1$. We prove that $\Delta_n \sim Q$. As $\Delta \sim Q^{n-1} \| \Delta_n$ and $Q^{n-1} \| \Delta_n \rightarrow^* Q \| \Delta_n$, we can use Lemma 29 and conclude $\Delta_n \sim Q^{|\Delta_n|}$. Hence $\Delta_n \sim Q$ because Δ_n would not be prime otherwise.

(2) There is a nBPA process Δ'_1 in $DNF(Q)$ such that $\Delta_1 \sim \Delta'_1$, $Level(\Delta'_1) = 1$ and $\Delta_i \sim Q$ for each $1 \leq i \leq n-1$. First we prove that $\Delta_n \sim Q$. As $\Delta_1 \| \Delta_n$ is a reachable state of $\Delta_1 \| \cdots \| \Delta_n$, it belongs to $\mathcal{S}$*(nBPA)*. Let us realize that Δ_n is regular. Assume the converse—then we can use Proposition 18 and conclude that $\Delta_1 \sim R^{|\Delta_1|}$ for some non-regular simple process R. From this and Remark 21 we can easily prove that $R \sim Q$ and it contradicts regularity of Q.

As Δ_n is regular and $\Delta_1 \| \Delta_n \in \mathcal{S}$*(nBPA)*, we can apply Proposition 23; from this (and also from Lemma 22) we get that $\Delta_n \sim Q^{|\Delta_n|}$ and thus $\Delta_n \sim Q$ because Δ_n is prime.

It remains to prove that there is a process Δ' in $DNF(Q)$ such that $Level(\Delta') = n$ and $\Delta \sim \Delta'$. But the process Δ' can be easily constructed by the algorithm from the proof of Proposition 23 with $\Delta'_1 \| Q^{n-1}$ on input.

□

4 Decidability Results

In this section we present several positive decidability results. We show that it is decidable whether a given nBPA process is prime and if the answer is negative, then its decomposition into primes can be effectively constructed. There are also other decidable properties which are summarized in Theorem 35.

4.1 Effective Decomposability of nBPA Processes

Lemma 31 *Let Δ be a nBPA process. It is decidable whether there is a nBPA process Δ' in DNF(Q) such that $\Delta \sim \Delta'$. Moreover, if the answer to the previous question is positive, then the process Δ' can be effectively constructed.*

PROOF. We can assume (w.l.o.g.) that Δ is in 3-GNF. If there is a process Δ' in $DNF(Q)$ such that $\Delta \sim \Delta'$, then there is $R \in Var(\Delta)$ such that $R \sim Q$, because Q is a reachable state of Δ'. As Q is a regular simple process, each summand in the def. equation for R must be of the form $a[P]$, where $R \sim P$. As bisimilarity is decidable for nBPA processes, we can construct the set $\mathcal{M}$ of all variables of $Var(\Delta)$ with this property. Each variable from this set is a potential candidate for the variable which is bisimilar to Q (if the set $\mathcal{M}$ is empty, then Δ cannot be bisimilar to any process in $DNF(Q)$).

For each variable $V \in \mathcal{M}$ we now modify the process Δ slightly—we replace each summand of the form aP in the def. equation for V with aV. The resulting process is denoted Δ_V (clearly $\Delta \sim \Delta_V$). For each such Δ_V we check whether Δ_V can be transformed into a process in $DNF(V)$. To do this, we first need to realize the following fact: if there is Δ'_V in $DNF(V)$ such that $\Delta_V \sim \Delta'_V$ and $a(A.B)$ is a summand in a def. equation from Δ_V such that A is non-regular, then $B \sim V^{\bullet|B|}$. It is easy to prove by the technique we already used many times in this paper—as A is non-regular, it can reach a state of an arbitrary norm. Furthermore, there is a reachable state of Δ_V which is of the form $A.B.\gamma$ where $\gamma \in Var(\Delta_V)^*$. We choose sufficiently large α such that $A \rightarrow^* \alpha$ and $\alpha.B.\gamma$ must be bisimilar to a state of Δ'_V which is of the form $[Y].V^{\bullet i}$ where $i \geq |B.\gamma|$. From this we get $B \sim V^{\bullet|B|}$.

Now we can describe the promised transformation $\mathcal{T}$ of Δ_V into a process Δ'_V in $DNF(V)$. If this transformation fails, then there is *no* process in $DNF(V)$ bisimilar to Δ_V. $\mathcal{T}$ is invoked on each summand of each def. equation from Δ_V and works as follows:

- $\mathcal{T}(a) = a$
- $\mathcal{T}(aA) = aA$
- $\mathcal{T}(a(A.B)) = aN$ if A is regular. The variable N is the leading variable of $\Delta^{\mathcal{R}}(A)$, whose def. equations are also added to Δ'_V after the following modification: each summand in each def. equation of $\Delta^{\mathcal{R}}(A)$ which is of the form b where $b \in Act$ is replaced with bB.
- $\mathcal{T}(a(A.B)) = a(A.V^{\bullet|B|})$ if A is non-regular and $B \sim V^{\bullet|B|}$. If A is non-regular and $B \not\sim V^{\bullet|B|}$, then $\mathcal{T}$ *fails*.

If there is $V \in \mathcal{M}$ such that $\mathcal{T}$ succeeds for Δ_V, then the process $\Delta'_V \sim \Delta$ is the process we are looking for. Otherwise, there is no process in $DNF(Q)$ bisimilar to Δ. $\square$

Proposition 32 *Let $\Delta_1, \ldots, \Delta_n$, $n \geq 2$ be nBPA processes. It is decidable whether $\Delta_1 \| \cdots \| \Delta_n \in \mathcal{S}(nBPA)$. Moreover, if the answer to the previous question is positive, then a nBPA process Δ such that $\Delta_1 \| \cdots \| \Delta_n \sim \Delta$ can be effectively constructed.*

PROOF. By induction on n:

- **n=2:** we distinguish three possibilities (it is decidable which one actually holds—see Proposition 6):

(1) Δ_1 and Δ_2 are regular. Then $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$ and a bisimilar regular process Δ in normal form can be easily constructed.

(2) Δ_1 and Δ_2 are non-regular. Suppose $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$. Proposition 18 says that there is a non-regular simple process Q such that $\Delta_1 \sim Q^{|\Delta_1|} \sim Q^{\bullet|\Delta_1|}$ and $\Delta_2 \sim Q^{|\Delta_2|} \sim Q^{\bullet|\Delta_2|}$. As Q is a reachable state of $Q^{\bullet|\Delta_2|}$, there is $R \in \mathit{Var}(\Delta_1)$ such that $Q \sim R$. As reachable states of Q are of the form $Q^{\bullet i}$ where $i \in \mathbb{N}_0$, each summand $a\alpha$ in the def. equation for R has the property $\alpha \sim R^{\bullet|\alpha|}$. As bisimilarity is decidable for nBPA processes, we can find all variables of $\mathit{Var}(\Delta)$ which have this property—we obtain a set of possible candidates for R (if this set is empty, then $\Delta_1 \| \Delta_2 \notin \mathcal{S}(nBPA)$). Now we check whether the constructed set of candidates contains a variable R such that $\Delta_1 \sim R^{\bullet|\Delta_1|}$. If not, then $\Delta_1 \| \Delta_2 \notin \mathcal{S}(nBPA)$. Otherwise we have R which is bisimilar to Q.

The same procedure is now applied to Δ_2. If it succeeds, it outputs some $S \in \mathit{Var}(\Delta)$. Now we check whether $R \sim S$. If not, then $\Delta_1 \| \Delta_2 \notin \mathcal{S}(nBPA)$. Otherwise $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$ and $\Delta_1 \| \Delta_2 \sim R^{\bullet|\Delta_1|+|\Delta_2|}$.

(3) Δ_1 is non-regular and Δ_2 is regular (or Δ_1 is regular and Δ_2 is non-regular—this is symmetric). Suppose $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$. Due to Proposition 23 we know that there is a regular simple process Q and a nBPA process Δ_1' in $\mathit{DNF}(Q)$ such that $\Delta_1 \sim \Delta_1'$ and $\Delta_2 \sim Q^{|\Delta_2|} \sim Q^{\bullet|\Delta_2|}$. An existence of Δ_1' can be checked effectively (see Lemma 31). If it does not exist, then $\Delta_1 \| \Delta_2 \notin \mathcal{S}(nBPA)$. If it exists, it can be also constructed and thus the only thing which remains is to test whether $\Delta_2 \sim Q^{\bullet|\Delta_2|}$. If this test succeeds, then $\Delta_1 \| \Delta_2 \in \mathcal{S}(nBPA)$ and we invoke the algorithm from the proof of Proposition 23 with $\Delta_1' \| Q^{|\Delta_2|}$ on input—it outputs a nBPA process which is bisimilar to $\Delta_1 \| \Delta_2$.

- **Induction step:** if $\Delta_1 \| \cdots \| \Delta_n \in \mathcal{S}(nBPA)$, then also $\Delta_1 \| \cdots \| \Delta_{n-1} \in \mathcal{S}(nBPA)$ and this is decidable by ind. hypothesis—if the answer is negative, then $\Delta_1 \| \cdots \| \Delta_n \notin \mathcal{S}(nBPA)$ and if it is positive, then we can construct a nBPA process Δ' such that $\Delta_1 \| \cdots \| \Delta_{n-1} \sim \Delta'$. Now we check whether $\Delta' \| \Delta_n \in \mathcal{S}(nBPA)$ and construct a bisimilar nBPA process Δ if needed. □

As an immediate consequence of Proposition 32 we get:

Proposition 33 *Let $\Delta, \Delta_1, \ldots, \Delta_n$ be nBPA processes. It is decidable whether $\Delta \sim \Delta_1 \| \cdots \| \Delta_n$.*

Now it is easy to prove the following theorem:

Theorem 34 *Let Δ be a nBPA process. It is decidable whether Δ is prime and if not, its decomposition into primes can be effectively constructed.*

PROOF. The technique is the same as in the proof of Theorem 13. We can almost copy the whole proof—the crucial result which allows us to do so is Proposition 33. □

Decidability results which were proved in this section (and some of their immediate consequences) are summarized in the following theorem:

Theorem 35 *Let $\Delta, \Delta_1, \ldots, \Delta_n$ be nBPA processes. The following problems are decidable:*

- *Is Δ prime? (If not, its decomposition can be effectively constructed)*
- *Is Δ bisimilar to $\Delta_1 \| \cdots \| \Delta_n$?*
- *Does the process $\Delta_1 \| \cdots \| \Delta_n$ belong to $\mathcal{S}$(nBPA)?*
- *Is there any process $\Delta' \not\sim nil$ such that $\Delta \| \Delta' \in \mathcal{S}$(nBPA)? (if so, an example of such a process can be effectively constructed).*
- *Is there any process Δ' such that $\Delta \sim \Delta_1 \| \cdots \| \Delta_n \| \Delta'$? (if so, Δ' can be effectively constructed).*

4.2 Decidability of Bisimilarity for sPA Processes

A "structural" way how to construct new processes from older ones is to combine them in parallel. If we do this with nBPA and nBPP processes, we obtain a natural subclass of normed PA processes denoted sPA (simple PA processes):

Definition 36 (sPA processes) *The class of sPA processes is defined as follows:*

$$sPA = \{\Delta_1 \| \cdots \| \Delta_n \mid n \in \mathbb{N},\ \Delta_i \in nBPA \cup nBPP \text{ for each } 1 \le i \le n\}$$

The class sPA is strictly greater than the union of nBPA and nBPP processes. This is demonstrated by the following example:

Example 37 *Let Δ_1, Δ_2 be nBPA processes defined as follows:*

$$\Delta_1: \quad \begin{aligned} X &\stackrel{def}{=} zX + i(Y.X) + q \\ Y &\stackrel{def}{=} i(Y.Y) + d \end{aligned} \qquad\qquad \Delta_2: \quad \begin{aligned} A &\stackrel{def}{=} aA + b(B.A) + r \\ B &\stackrel{def}{=} b(B.B) + c \end{aligned}$$

Then there is no *nBPA or nBPP process bisimilar to the sPA process $\Delta_1 \| \Delta_2$. This can be easily proved with the help of pumping lemmas for context-free languages and for languages generated by nBPP processes—see [6].*

Theorem 38 *Let $\Phi = \varphi_1 \| \cdots \| \varphi_n$, $\Psi = \psi_1 \| \cdots \| \psi_m$ be sPA processes. It is decidable whether $\Phi \sim \Psi$.*

PROOF. As each φ_i, $1 \leq i \leq n$ and ψ_j, $1 \leq j \leq m$ can be effectively decomposed, we can also construct decompositions of Φ and Ψ. If $\Phi \sim \Psi$, then those decompositions must be the same up to bisimilarity (see Remark 11). In other words, there must be a one-to-one correspondence between primes forming the two decompositions which preserves bisimilarity. An existence of such a correspondence can be checked effectively, because bisimilarity is decidable in the union of nBPA and nBPP processes (see Proposition 8). □

5 Conclusions, Future Research

The main characterization theorem (Theorem 30) says that non-regular nBPA processes which are not prime can be divided into two groups:

(1) Processes which can be equivalently represented as a power of some non-regular simple process. It is obvious that each such nBPA process belongs to $\mathcal{S}$*(nBPP)*—see Remark 17.
(2) Processes which can be equivalently represented in *DNF*(Q) and their level is at least 2. It can be proved (with the help of results achieved in [5]) that each such process does *not* belong to $\mathcal{S}$*(nBPP)*.

From this we can observe that our division based on normal forms corresponds to the membership to $\mathcal{S}$*(nBPP)*.

The first possible generalization of our results could be the replacement of the '$\|$' operator with the parallel operator of CCS which allows synchronizations on complementary actions. This should not be hard, but we can expect more complicated normal forms. Decidability results should be the same.

A natural question is whether our results can be extended to the class of all (not necessarily normed) BPA processes. A major problem is that there are quite primitive BPA processes which do not have any (finite) decomposition at all. For example, the process $X \stackrel{def}{=} aX$ is not prime as $X \sim a\|X$. However, X cannot have any finite decomposition into primes because at least one of those primes would have to be unnormed and able to emit just an infinite sequence of a's; hence this prime is bisimilar to X and as X is decomposable, the prime is decomposable as well and we have a contradiction. Thus, we cannot expect that our results immediately generalize to the class of all BPA processes.

Another related open problem is decidability of bisimilarity for normed PA processes. It seems that it should be possible to design at least rich subclasses of normed PA processes where bisimilarity remains decidable.

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