Model-Checking LTL with Regular Valuations for Pushdown Systems[1]

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Version: February 27, 2002

Recent works have proposed pushdown systems as a tool for analyzing programs with (recursive) procedures, and the model-checking problem for LTL has been devoted special attention. However, all these works impose a strong restriction on the possible valuations of atomic propositions: whether a configuration of the pushdown system satisfies an atomic proposition or not can only depend on the current control state of the pushdown automaton and on its topmost stack symbol. In this paper we consider LTL with regular valuations: the set of configurations satisfying an atomic proposition can be an arbitrary regular language. The model-checking problem is solved via two different techniques, with an eye on efficiency. The resulting algorithms are proved to be asymtotically optimal. We show that the extension to regular valuations allows to model problems in different areas, like

[1]This work was partially supported by the project "Advanced Validation Techniques for Telecommunication Protocols" of the Information Societies Technology Programme of the European Union.

[2]On leave at the Institute for Informatics, TU Munich. Supported by a Research Fellowship granted by the Alexander von Humboldt Foundation and by a grant GA ČR No. 201/00/1023.

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data-flow analysis and analysis of systems with checkpoints. We claim that our model-checking algorithms provide a general, unifying and efficient framework for solving them.

Key Words: Model-checking; linear-time logic; pushdown automata

1. INTRODUCTION

Pushdown systems can be seen as a natural abstraction of programs written in procedural, sequential languages such as C. They generate infinite-state transition systems whose states are pairs consisting of a control location (which stores global information about the program) and stack content (which keeps the track of activation records, i.e., previously called procedures and their local variables).

Previous research has established applications of pushdown systems for the analysis of Boolean Programs [1, 8] and certain data-flow analysis problems [7]. The model-checking problem has been considered for various logics, and quite efficient algorithms have emerged for linear time logics [2, 6, 9].

In this paper we revisit the model-checking problem for LTL and pushdown systems. The problem is undecidable for arbitrary valuations, i.e., the functions that map the atomic propositions of a formula to the respective sets of pushdown configurations that satisfy them. However, it remains decidable for some restricted classes of valuations. In [2, 6, 9] valuations were completely determined by the control location and/or the topmost stack symbol (we call these valuations 'simple' in the following). Here we study (and solve) the problem for valuations depending on regular predicates over the complete stack content. We argue that this solution provides a general, efficient, and unifying framework for problems from different areas (e.g., data-flow analysis, analysis of systems with checkpoints, etc.)

We proceed as follows. Section 2 contains basic definitions. Most of the technical content is in Section 3, where we formally define simple and regular valuations and propose our solutions to the model-checking problem. The solutions are based on a reduction to the case of simple valuations, which allows us to re-use most of the theory from [6]. While the reduction itself is based on a standard method, we pay special attention to ensuring its efficiency. This requires to modify the algorithm of [6] to take advantage of specific properties of our constructions. We propose two different techniques – one for regular valuations in general and another for a restricted subclass – both of which increase the complexity by only a linear factor (in the size of an automaton for the atomic regular predicates). By contrast, a blunt reduction and analysis would yield up to a quadric ('n^4') blowup. Even though one technique is more powerful than the other at the same asymptotic complexity, we present them both, because their efficiency in practice may depend on the concrete application.

In Section 4 we show that the extension to regular valuations allows to model problems in different application areas. The first one (Section 4.1) is interprocedural data-flow analysis. Here, regular valuations can be used to compute data-flow information which dynamically depends on the history of procedure calls. As an example, we show how to decide whether a given variable is dead at a given point of a recursive program with dynamic scoping. A second application area are (Section 4.2) systems with checkpoints. In these systems computation is suspended at certain points to allow for a property of the stack content to be checked; resumption of the computation depends on the result of this inspection. This part of our work is motivated by the advent of programming languages which can enforce security requirements. Newer versions of Java, for instance, enable programs to perform local security checks in which the methods on the stack are checked for correct permissions. Jensen et al [10] have proposed a formal framework for such systems. Using their techniques one can prove the validity of control flow based global security properties as well as to detect (and remove) redundant checkpoints. The formal framework of [10] can be reformulated in terms of *pushdown systems with checkpoints*, to which our model-checking algorithms can be applied. Our results are more general than those of [10]: we are not restricted to safety properties, our model can also represent data-flow, and we can handle mutually recursive methods. Moreover, we provide a detailed complexity analysis. In Section 4.3 we present a third application of our results of a more theoretical nature: the extension to regular valuations leads to an elegant model-checking algorithm for CTL*. In the context of finite-state systems it is well-known that model-checking the more powerful logic CTL* can be reduced to checking LTL [5]. We show that for pushdown systems model-checking CTL*can be reduced to model checking LTL *with regular valuations.*

Our model-checking algorithms are polynomial in the size of certain parameters of the problem which are usually small. However, in the worst case those parameters can be *exponential* in the size of a problem instance, and so it is natural to ask if polynomial algorithms in the size of the problem instance exist. In Section 5 we provide a negative answer by establishing **EXPTIME** lower bounds. Hence, all of our algorithms are asymptotically optimal. We draw our conclusions in Section 6.

2. PRELIMINARIES

2.1. The Logic LTL

Let $At = \{A, B, C, \ldots\}$ be a (countable) set of *atomic propositions*. LTL formulae are built according to the following abstract syntax equation (where A ranges over At):

$$\varphi \quad ::= \quad \mathtt{tt} \mid A \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathcal{X}\varphi \mid \varphi_1 \,\mathcal{U}\, \varphi_2$$

where $\mathcal{X}$ and $\mathcal{U}$ are the *next* and *until* operators, respectively. Let $At(\varphi)$ be the set of atomic propositions which appear in φ (note that $At(\varphi)$ is finite). Formulae are interpreted on infinite words over the alphabet $2^{At(\varphi)}$ (formally, an infinite word w is a function $w : \mathbb{N}_0 \to 2^{At(\varphi)}$). We denote that w satisfies a formula φ by $w \models \varphi$. The satisfaction relation is defined inductively on the structure of φ as follows, where w_i denotes the suffix of w starting with $w(i)$:

$$\begin{array}{lcl}
w \models \mathtt{tt} & & \\
w \models A & \iff & A \in w(0) \\
w \models \neg\varphi & \iff & w \not\models \varphi \\
w \models \varphi_1 \wedge \varphi_2 & \iff & w \models \varphi_1 \text{ and } w \models \varphi_2 \\
w \models \mathcal{X}\varphi & \iff & w_1 \models \varphi \\
w \models \varphi_1 \,\mathcal{U}\, \varphi_2 & \iff & \exists i\colon (w_i \models \varphi_2) \wedge (\forall j < i\colon w_j \models \varphi_1)
\end{array}$$

We also define $\Diamond\varphi \equiv \mathtt{tt}\,\mathcal{U}\,\varphi$ and $\Box\varphi \equiv \neg(\Diamond\neg\varphi)$.

Hence, every LTL formula φ defines a language $L(\varphi)$ consisting of all infinite words w over the alphabet $2^{At(\varphi)}$ such that $w \models \varphi$. It is well known (see, e.g., [12]) that given an LTL formula φ, one can effectively construct a Büchi automaton $\mathcal{B}_\varphi$

of size $\mathcal{O}(2^{|\varphi|})$ which recognizes the language $L(\varphi)$. This Büchi automaton is a tuple $\mathcal{B} = (Q, 2^{At(\varphi)}, \delta, q_0, F)$, where Q is the set of states, $2^{At(\varphi)}$ is the alphabet, q_0 is the initial state, F is the set of accepting states, and $\delta\colon Q \times 2^{At(\varphi)} \to 2^Q$ is the transition function. An *infinite* word w over the alphabet $2^{At(\varphi)}$ is accepted by $\mathcal{B}$ iff there is an (infinite) run of $\mathcal{B}$ over w that visits some accepting state infinitely often.

2.2. Transition Systems

A *transition system* is a triple $\mathcal{T} = (S, \to, r)$ where S is a set of *states* (not necessarily finite), $\to \subseteq S \times S$ is a *transition relation*, and $r \in S$ is a distinguished state called *root*.

As usual, we write $s \to t$ instead of $(s,t) \in \to$. The reflexive and transitive closure of $\to$ is denoted by $\to^*$. We say that a state t is *reachable from a state* s if $s \to^* t$. A state t is *reachable* if it is reachable from the root.

A *run* of $\mathcal{T}$ is an infinite sequence of states $w = s_0 s_1 s_2 \ldots$ such that $s_i \to s_{i+1}$ for each $i \geq 0$. To interpret the logic LTL over runs and states of $\mathcal{T}$, we first need to fix the meaning of atomic predicates by a *valuation*, which is a function $\nu : At \to 2^S$. Given a valuation ν and a formula φ, each run $w = s_0 s_1 s_2 \ldots$ determines a unique infinite word $w[\nu,\varphi]$ over the alphabet $2^{At(\varphi)}$ given by $w[\nu,\varphi](i) = \{A \in At(\varphi) \mid s_i \in \nu(A)\}$. The run w satisfies φ w.r.t. ν, denoted by $w \models^\nu \varphi$, iff $w[\nu,\varphi] \models \varphi$. Similarly, φ is true at a state $s \in S$ w.r.t. ν, written $s \models^\nu \varphi$, iff for each run w starting in s we have that $w \models^\nu \varphi$.

Observe that whether φ holds for runs and states of $\mathcal{T}$ is influenced only by the restriction of ν to $At(\varphi)$. In the next sections we denote this restriction by ν_φ (i.e., $\nu_\varphi : At(\varphi) \to 2^S$ is a function from a *finite* domain which agrees with ν on every argument).

2.3. Pushdown systems

A *pushdown system* is a tuple $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$ where P is a finite set of *control locations*, Γ is a finite *stack alphabet*, $\Delta \subseteq (P \times \Gamma) \times (P \times \Gamma^*)$ is a finite set of *transition rules*, $q_0 \in P$ is an *initial control location*, and $\omega \in \Gamma$ is a *bottom stack symbol*.

We use Greek letters $\alpha, \beta, \ldots$ to denote elements of Γ, and small letters $v, w, \ldots$ from the end of the alphabet to denote elements of Γ^*. We also adopt a more intuitive notation for transition rules, writing $\langle p, \alpha \rangle \hookrightarrow \langle q, w \rangle$ instead of $((p,\alpha),(q,w)) \in \Delta$.

A *configuration* of $\mathcal{P}$ is an element of $P \times \Gamma^*$. To $\mathcal{P}$ we associate a unique transition system $\mathcal{T}_\mathcal{P}$ whose states are configurations of $\mathcal{P}$, the root is $\langle q_0, \omega \rangle$, and the transition relation is the least relation $\to$ satisfying the following:

$$\langle p, \alpha \rangle \hookrightarrow \langle q, v \rangle \implies \langle p, \alpha w \rangle \to \langle q, vw \rangle \text{ for every } w \in \Gamma^*$$

Without loss of generality we require that ω is never removed from the stack, i.e., whenever $\langle p, \omega \rangle \hookrightarrow \langle q, w \rangle$ then w is of the form $v\omega$.

Pushdown systems can be conveniently used as a model of recursive sequential programs. In this setting, the (abstracted) stack of activation records increases if a new procedure is invoked, and decreases if the current procedure terminates. In particular, it means that the height of the stack can increase at most by one in a single transition. Therefore, from now on we assume that all pushdown systems we work with have this property. This assumption does not influence the expressive power of pushdown systems, but it has some impact on the complexity analysis carried out in Section 3.2.

3. LTL ON PUSHDOWN SYSTEMS

Let $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$ be a pushdown system, φ an LTL formula, and $\nu: At \to 2^{P \times \Gamma^*}$ a valuation. We deal with the following variants of the *model-checking problem*:

(I) The model-checking problem for the initial configuration: does $\langle q_0, \omega\rangle \models^\nu \varphi$?

(II) The global model-checking problem: compute (a finite description of) the set of all configurations, reachable or not, that violate φ.

(III) The global model-checking problem for reachable configurations: compute (a finite description of) the set of all reachable configurations that violate φ.

In this paper we use so-called $\mathcal{P}$-*automata* to encode infinite sets of configurations of a pushdown system $\mathcal{P}$. As we shall see, in some cases we can solve the problems (II) and (III) by computing $\mathcal{P}$-automata recognizing the above defined sets of configurations.

DEFINITION 1. Let $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$ be a pushdown system. A $\mathcal{P}$-automaton is a tuple $\mathcal{A} = (Q, \Gamma, \delta, P, F)$ where Q is a finite set of *states*, Γ (i.e., the stack alphabet of $\mathcal{P}$) is the *input alphabet*, $\delta: Q \times \Gamma \to 2^Q$ is the *transition function*, P (i.e., the set of control locations of $\mathcal{P}$) is the set of *initial states*, and $F \subseteq Q$ is a finite set of *accepting states*. We extend δ to elements of $Q \times \Gamma^*$ in the standard way. A configuration $\langle p, w\rangle$ of $\mathcal{P}$ is *recognized* by $\mathcal{A}$ iff $\delta(p, w) \cap F \neq \emptyset$. A set of configurations is *regular* if it is recognized by some $\mathcal{P}$-automaton.

In general, all of the above mentioned variants of the model-checking problem are undecidable – if there are no 'effectivity assumptions' about valuations (i.e., if a valuation is an *arbitrary* function $\nu: At \to 2^{P \times \Gamma^*}$), one can easily express undecidable properties of pushdown configurations just by atomic propositions. This motivates the search for 'reasonable' restrictions which do not limit the expressive power too much but allow to construct efficient model-checking algorithms at the same time. In the valuations considered in [2, 6]), whether a configuration satisfies an atomic proposition or not depends only on its control location and the topmost stack symbol (it can be safely assumed that the stack is always nonempty). We define these valuations formally:

DEFINITION 2. Let $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$ be a pushdown system. A set of configurations of $\mathcal{P}$ is *simple* if it has the form $\{\langle p, \alpha w\rangle \mid w \in \Gamma^*\}$ for some $p \in P$, $\alpha \in \Gamma$. A valuation $\nu: At \to 2^{P \times \Gamma^*}$ is *simple* if $\nu(A)$ is a union of simple sets for every $A \in At$.

In this paper, we propose a more general kind of valuations which are encoded by finite-state automata. We advocate this approach in the next sections by providing several examples of its applicability to practical problems; moreover, we show that this technique often results in rather efficient (or, at least, asymptotically optimal) algorithms by presenting relevant complexity results.

Definition 3. Let $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$ be a pushdown system. A valuation $\nu\colon At \to 2^{P\times\Gamma^*}$ is *regular* if $\nu(A)$ is a regular set for every $A \in At$ and it does not contain any configuration with empty stack.

Notice that, since we assume a bottom stack symbol which is never removed, the requirement that configurations with empty stack cannot satisfy any atomic proposition is not a real restriction.

Since regular sets can be infinite, we need to represent regular valuations by finite means. We fix an adequate representation for our purposes.

Notation 1. Let ν be a regular valuation. For every atomic proposition A and control location p, we denote by $\mathcal{M}_A^p$ a deterministic finite-state automaton over the alphabet Γ having a total transition function and satisfying

$$\nu(A) = \{\langle p, w\rangle \mid p \in P, w^R \in L(\mathcal{M}_A^p)\}$$

where w^R denotes the reverse of w.

Hence, A is true at $\langle p, w\rangle$ iff the automaton $\mathcal{M}_A^p$ enters a final state after reading the stack contents bottom-up. Since $\nu(A)$ does not contain any configuration with empty stack, the initial state of $\mathcal{M}_A^p$ is not accepting. This will simplify our constructions.

3.1. Model-Checking with Simple Valuations

The model-checking problems (I) to (III) of the previous section have been solved in [6] for simple valuations. In this section we briefly recall the solutions. The model-checking problem for the initial configuration (problem (I)) is solved in three steps as follows:

(1) The problem is reduced to the emptiness problem for Büchi pushdown systems. A *Büchi pushdown system* is a pushdown system with a subset of *accepting* control locations. A run of a Büchi pushdown system is *accepting* if it visits some accepting control location infinitely often. A Büchi pushdown system is *empty* if it has no accepting run.

(2) The emptiness problem for Büchi pushdown systems is reduced to computing the set of predecessors of certain regular sets of configurations. The set of predecessors of a set C of configurations, denoted by $pre^*(C)$, contains the configurations $\langle p, w\rangle$ such that $\langle p, w\rangle \to^* \langle p', w'\rangle$ for some $\langle p', w'\rangle \in C$.

(3) An algorithm is presented which, given an arbitrary $\mathcal{P}$-automaton $\mathcal{A}$ recognizing a regular set C of configurations, computes another $\mathcal{P}$-automaton $\mathcal{A}_{pre^*}$ recognizing the set $pre^*(C)$. (It can be shown that if C is regular then so is $pre^*(C)$.)

We now give some more details about these steps.

Step 1. Let $\mathcal{P} = (P, \Gamma, \Delta, p_0, \omega)$ be a pushdown system and let φ be a formula of LTL. We first construct a Büchi automaton $\mathcal{B} = (Q, 2^{At(\varphi)}, \delta, q_0, F)$ recognizing $L(\neg\varphi)$. We then construct a Büchi pushdown system $\mathcal{BP} = ((P \times Q), \Gamma, \Delta', (p_0, q_0), \omega, G)$ (where G is the set of accepting control locations) by "synchronizing" $\mathcal{P}$ and $\mathcal{B}$ as follows:

- $\langle (p,q),\alpha\rangle \hookrightarrow \langle (p',q'),v\rangle \in \Delta'$ if
  - $\langle p,\alpha\rangle \hookrightarrow \langle p',v\rangle$; and
  - $q' \in \delta(q,\sigma)$ where σ is the set of all atomic propositions of $At(\varphi)$ which are true at the configuration $\langle p,\alpha\rangle$.
- $(p,q) \in G$ if $q \in F$.

If valuations are simple, then, whenever the rule $\langle (p,q),\alpha\rangle \hookrightarrow \langle (p',q'),v\rangle \in \Delta'$ is applied to derive a transition $\langle (p,q),\alpha w\rangle \rightarrow \langle (p',q'),vw\rangle$, we always have that A is true at the configuration $\langle p,\alpha w\rangle$. Using this property, it is easy to show that $\mathcal{P}$ has a run violating φ if and only if $\mathcal{BP}$ is nonempty.

Step 2. Let $\mathcal{BP} = (P,\Gamma,\Delta,p_0,\omega,G)$ be an arbitrary Büchi pushdown system. The *head* of a transition rule $\langle p,\alpha\rangle \hookrightarrow \langle p',w\rangle$ of Δ is the configuration $\langle p,\alpha\rangle$. A head $\langle p,\alpha\rangle$ is *repeating* if there exists $v \in \Gamma^*$ such that $\langle p,\alpha v\rangle$ can be reached from $\langle p,\alpha\rangle$ by means of a sequence of transitions that visits some control location of G. Let *Rep* be the set of repeating heads, and let $Rep\Gamma^*$ denote the set $\{\langle p,\alpha w\rangle \mid \langle p,\alpha\rangle \in Rep, w \in \Gamma^*\}$. It is shown in [6] that $\mathcal{BP}$ is nonempty if and only if $\langle p_0,\omega\rangle \in pre^*(Rep\Gamma^*)$. Therefore, emptiness can be decided by computing first *Rep* and then $pre^*(Rep\Gamma^*)$.

In order to compute *Rep* we construct a *head reachability graph* having all heads of Δ as nodes. There is an edge from $\langle p,\alpha\rangle$ to $\langle p',\beta\rangle$ if there is a rule $\langle p,\alpha\rangle \hookrightarrow \langle p'',v_1\beta v_2\rangle$ in δ such that $\langle p'',v_1\rangle \in pre^*(\{\langle p',\varepsilon\rangle\})$. If $\langle p',\varepsilon\rangle$ can be reached from $\langle p'',v_1\rangle$ visiting a final control location along the way, then we say that the edge is *marked*. It is shown in [6] that a head is repeating if and only if it belongs to a strongly connected component of the head reachability graph containing at least one marked edge.

These results show that the emptiness problem reduces to computing $pre^*(\{\langle p,\varepsilon\rangle \mid p \in P\}$ for each control location p, and $pre^*(Rep\Gamma^*)$.

Step 3. Without loss of generality, we assume that $\mathcal{A}$ has no transition leading to an initial state. We compute $pre^*(C)$ by means of a saturation procedure. The procedure adds new transitions to $\mathcal{A}$, but no new states. New transitions are added according to the following *saturation rule*:

> If $\langle p,\gamma\rangle \hookrightarrow \langle p',w\rangle$ and $p' \xrightarrow{w} q$ in the current $\mathcal{P}$-automaton, add a transition (p,γ,q).

The saturation procedure eventually reaches a fixpoint because the number of possible new transitions is finite. An efficient implementation of the procedure is presented in [6].

This finishes the discussion of problem (I). Problem (II) has the same solution, because the automaton recognizing $pre^*(Rep\Gamma^*)$ is a finite representation of all the configurations violating φ. Problem (III) is solved by showing that the set of reachable configurations of $\mathcal{P}$ is regular, and by proposing an algorithm – very similar to that for $pre^*(C)$ – that constructs a $\mathcal{P}$-automaton recognizing this set. The product of this automaton and that for $pre^*(Rep\Gamma^*)$ is a finite representation of all the reachable configurations of $\mathcal{P}$ that violate φ. The following theorems are taken from [6].

THEOREM 1. *Let $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$ be a pushdown system, φ an LTL formula, and ν a simple valuation. Let $\mathcal{B}$ be a Büchi automaton for $\neg\varphi$. Then one can compute*

- *a $\mathcal{P}$-automaton $\mathcal{R}$ with $\mathcal{O}(|P| + |\Delta|)$ states and $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|))$ transitions in $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|))$ time and space such that $\mathcal{R}$ recognizes exactly the reachable configurations of $\mathcal{P}$;*
- *a $\mathcal{P}$-automaton $\mathcal{A}$ of size $\mathcal{O}(|P| \cdot |\Delta| \cdot |\mathcal{B}|^2)$ in $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |\mathcal{B}|^3)$ time using $\mathcal{O}(|P| \cdot |\Delta| \cdot |\mathcal{B}|^2)$ space such that $\mathcal{A}$ recognizes exactly those configurations $\langle p, w\rangle$ of $\mathcal{P}$ (reachable or not) such that $\langle p, w\rangle \not\models^\nu \varphi$;*
- *a $\mathcal{P}$-automaton $\mathcal{A}'$ of size $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |\mathcal{B}|^2)$ in $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |\mathcal{B}|^3)$ time using $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |\mathcal{B}|^2)$ space such that $\mathcal{A}'$ recognizes exactly those reachable configurations $\langle p, w\rangle$ of $\mathcal{P}$ such that $\langle p, w\rangle \not\models^\nu \varphi$.*

THEOREM 2. *Let $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$ be a pushdown system, φ an LTL formula, and ν a simple valuation. Let $\mathcal{B}$ be a Büchi automaton which corresponds to $\neg\varphi$.*

- *Problems (I) and (II) can be solved in $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |\mathcal{B}|^3)$ time and $\mathcal{O}(|P| \cdot |\Delta| \cdot |\mathcal{B}|^2)$ space.*
- *Problem (III) can be solved in either $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |\mathcal{B}|^3)$ time and $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |\mathcal{B}|^2)$ space, or $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |\mathcal{B}|^3)$ time and $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |\mathcal{B}|^2)$ space.*

3.2. Model-Checking with Regular Valuations

Our aim here is to design efficient model-checking algorithms for regular valuations. We show that one can actually build on top of Theorem 1.

For the rest of this section we fix a pushdown system $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$, an LTL formula φ, and a regular valuation ν. The Büchi automaton which corresponds to $\neg\varphi$ (see Section 2.1) is denoted by $\mathcal{B} = (R, 2^{At(\varphi)}, \eta, r_0, G)$. Let $At(\varphi) = \{A_1, \ldots, A_n\}$, and let $\mathcal{M}^p_{A_i} = (Q^p_i, \Gamma, \varrho^p_i, s^p_i, F^p_i)$ be the deterministic finite-state automaton associated to (A_i, p) for all $p \in P$ and $1 \le i \le n$.

In the next subsections we design two algorithms for model-checking with regular valuations. Actually, both of them reduce the problem to model-checking with simple valuations discussed in Section 3.1; and in both cases, the idea is to encode the $\mathcal{M}^p_{A_i}$ automata in the structure of $\mathcal{P}$ and simulate them 'on-the-fly' during the computation of $\mathcal{P}$. In practice, some of the $\mathcal{M}^p_{A_i}$ automata can be identical (we shall see examples in Section 4.2). Of course, it does not have much sense to simulate the execution of the *same* automaton within $\mathcal{P}$ twice and our constructions should reflect this fact. For simplicity, we assume that whenever $i \neq j$ or $p \neq q$, then the $\mathcal{M}^p_{A_i}$ and $\mathcal{M}^q_{A_j}$ automata are either identical or have disjoint sets of states. So, let $\{\mathcal{M}_1, \ldots, \mathcal{M}_m\}$ be the *set* of all $\mathcal{M}^p_{A_i}$ automata where $1 \le i \le n$ and $p \in P$ (hence, if some of the $\mathcal{M}^p_{A_i}$ automata are identical then $m < n \cdot |P|$), and let Q_j be the set of states of $\mathcal{M}_j$ for each $1 \le j \le m$. The Cartesian product $\prod_{1 \le j \le m} Q_j$ is denoted by *States*. For given $\mathbf{r} \in \mathit{States}$, $p \in P$, and $1 \le i \le n$, we denote by $\mathbf{r}^p_i$ the element of Q^p_i which appears in $\mathbf{r}$ (observe that we can have $\mathbf{r}^p_i = \mathbf{r}^q_j$ even if $i \neq j$ or $p \neq q$). The vector of initial states (i.e., the only element of *States* where each component is the initial state of some $\mathcal{M}^p_{A_i}$) is denoted by $\mathbf{s}$. Furthermore, we write $\mathbf{t} = \varrho(\mathbf{r}, \alpha)$ if $\mathbf{t}^p_i = \varrho^p_i(\mathbf{r}^p_i, \alpha)$ for all $1 \le i \le n$, $p \in P$. Now we present and evaluate two techniques for solving the model-checking problems with $\mathcal{P}$, φ, and ν.

Remark 1 (On the complexity measures). The size of an instance of the model-checking problem for pushdown systems and LTL with regular valuations is given by $|\mathcal{P}|+|\varphi|+|\nu_\varphi|$, where $|\nu_\varphi|$ is the total size of all employed automata. However, in practice we usually work with small formulae and a small number of rather simple automata (see Section 4); therefore, we measure the complexity of our algorithms in $|\mathcal{B}|$ and $|\mathit{States}|$ rather than in $|\varphi|$ and $|\nu_\varphi|$ (in general, $\mathcal{B}$ and States can be *exponentially* larger than φ and ν_φ). This allows for a detailed complexity analysis whose results better match the reality because $|\mathcal{B}|$ and $|\mathit{States}|$ stay usually 'small'. This issue is discussed in greater detail in Section 5 where we provide some lower bounds showing that all algorithms developed in this paper are also essentially optimal from the point of view of worst-case analysis.

3.2.1. Technique 1 – extending the finite control

The idea behind this technique is to evaluate the atomic propositions $A_1, \ldots, A_n$ 'on the fly' by storing the (product of) $\mathcal{M}^p_{A_i}$ automata in the finite control of $\mathcal{P}$ and updating the vector of states after each transition according to the (local) change of stack contents. As we will see, we conveniently use the assumptions that the $\mathcal{M}^p_{A_i}$ automata are deterministic, have total transition functions, and read the stack bottom-up. However, we also need one *additional* assumption to make the construction work:

> Each automaton $\mathcal{M}^p_{A_i}$ is also *backward deterministic*, i.e., for every $u \in Q^p_i$ and $\alpha \in \Gamma$ there is at most one state $t \in Q^p_i$ such that $\varrho^p_i(t, \alpha) = u$.

This assumption is truly restrictive – there are quite simple regular languages which cannot be recognized by finite-state automata which are both deterministic and backward deterministic (for example the language $\{a^i \mid i > 0\}$).

We define a pushdown system $\mathcal{P}' = (P', \Gamma, \Delta', q'_0, \omega)$ where $P' = P \times \mathit{States}$, $q'_0 = (q_0, \varrho(\mathbf{s}, \omega))$, and the transition rules Δ' are determined as follows: $\langle (p, \mathbf{r}), \alpha\rangle \hookrightarrow' \langle (q, \mathbf{u}), w\rangle$ iff the following conditions hold:

- $\langle p, \alpha\rangle \hookrightarrow \langle q, w\rangle$,
- there is $\mathbf{t} \in \mathit{States}$ such that $\varrho(\mathbf{t}, \alpha) = \mathbf{r}$ and $\varrho(\mathbf{t}, w^R) = \mathbf{u}$.

Observe that due to the backward determinism of $\mathcal{M}^p_{A_i}$ there is at most one $\mathbf{t}$ with the above stated properties; and thanks to determinism and the totality of the transition functions of $\mathcal{M}^p_{A_i}$, we further obtain that for given $\langle p, \alpha\rangle \hookrightarrow \langle q, w\rangle$ and $\mathbf{r} \in \mathit{States}$ there is exactly one $\mathbf{u} \in \mathit{States}$ such that $\langle (p, \mathbf{r}), \alpha\rangle \hookrightarrow' \langle (q, \mathbf{u}), w\rangle$. From this it follows that $|\Delta'| = |\Delta| \cdot |\mathit{States}|$.

A configuration $\langle (q, \mathbf{r}), w\rangle$ of $\mathcal{P}'$ is *consistent* iff $\mathbf{r} = \varrho(\mathbf{s}, w^R)$ (remember that $\mathbf{s}$ is the vector of initial states of $\mathcal{M}^p_{A_i}$ automata). In other words, $\langle (q, \mathbf{r}), w\rangle$ is consistent iff $\mathbf{r}$ 'reflects' the stack contents w. Let $\langle p, w\rangle$ be a configuration of $\mathcal{P}$ and $\langle (p, \mathbf{r}), w\rangle$ be the (unique) associated consistent configuration of $\mathcal{P}'$. Now we can readily confirm that

(A) if $\langle p, w\rangle \to \langle q, v\rangle$, then $\langle (p, \mathbf{r}), w\rangle \to \langle (q, \mathbf{u}), v\rangle$ where $\langle (q, \mathbf{u}), v\rangle$ is the unique consistent configuration associated to $\langle q, v\rangle$;

(B) if $\langle (p, \mathbf{r}), w\rangle \to \langle (q, \mathbf{u}), v\rangle$, then $\langle (q, \mathbf{u}), v\rangle$ is consistent and $\langle p, w\rangle \to \langle q, v\rangle$ is a transition of $\mathcal{T}_{\mathcal{P}}$.

As the initial configuration of $\mathcal{P}'$ is consistent, we see (due to (B)) that each reachable configuration of $\mathcal{P}'$ is consistent (but not each consistent configuration is necessarily reachable). Furthermore, due to (A) and (B) we also have the following:

(C) let $\langle p, w\rangle$ be a configuration of $\mathcal{P}$ (not necessarily reachable) and let $\langle (p, \mathbf{r}), w\rangle$ be its associated consistent configuration of $\mathcal{P}'$. Then the parts of $\mathcal{T}_{\mathcal{P}}$ and $\mathcal{T}_{\mathcal{P}'}$ which are reachable from $\langle p, w\rangle$ and $\langle (p, \mathbf{r}), w\rangle$, respectively, are isomorphic.

The simple valuation ν' is defined by

$$\nu'(A_i) = \{\langle (p, \mathbf{r}), w\rangle \mid (p, \mathbf{r}) \in P', \mathbf{r}_i^p \in F_i^p, w \in \Gamma^+\}$$

for all $1 \leq i \leq n$. Now it is easy to see (due to (C)) that for all $p \in P$ and $w \in \Gamma^*$ we have

$$\langle p, w\rangle \models^{\nu} \varphi \iff \langle (p, \mathbf{r}), w\rangle \models^{\nu'} \varphi \text{ where } \mathbf{r} = \varrho(\mathbf{s}, w^R) \quad (1)$$

During the construction of $\mathcal{P}'$ we observed that $|P'| = |P| \cdot |States|$ and $|\Delta'| = |\Delta| \cdot |States|$. Applying Theorem 2 naïvely, we obtain that using Technique 1, the model-checking problems (I) and (II) can be solved in cubic time and quadratic space (w.r.t. $|States|$), and that model-checking problem (III) takes even quadric time and space. However, closer analysis reveals that we can do much better.

Theorem 3. *Technique 1 (extending the finite control) gives us the following bounds on the model-checking problems with regular valuations:*

1. *Problems (I) and (II) can be solved in* $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |States| \cdot |\mathcal{B}|^3)$ *time and* $\mathcal{O}(|P| \cdot |\Delta| \cdot |States| \cdot |\mathcal{B}|^2)$ *space.*
2. *Problem (III) can be solved in either* $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |States| \cdot |\mathcal{B}|^3)$ *time and* $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |States| \cdot |\mathcal{B}|^2)$ *space, or* $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |States| \cdot |\mathcal{B}|^3)$ *time and* $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |States| \cdot |\mathcal{B}|^2)$ *space.*

In other words, all problems take only *linear* time and space in $|States|$. Before proving the theorem, we need to introduce further notions and formulate two auxiliary lemmata.

We say that a $\mathcal{P}'$-automaton is *well-formed* iff its set of states is of the form $Q \times States$ where Q is a set such that $P \subseteq Q$. A transition $((p, \mathbf{t}), \alpha, (p', \mathbf{u}))$ of a well-formed $\mathcal{P}'$-automaton is *consistent* (w.r.t. ϱ) iff $\varrho(\mathbf{u}, \alpha) = \mathbf{t}$. A $\mathcal{P}'$-automaton is consistent iff it is well-formed and contains only consistent transitions.

Now we revisit the algorithms presented in [6]. Algorithm 1 (see Table 1) shows the computation of *pre** from [6], restated for the special case of consistent $\mathcal{P}'$-automata. We first show that computation of *pre** upholds the consistency of a $\mathcal{P}'$-automaton.

Lemma 1. *When receiving a consistent* $\mathcal{P}'$*-automaton as input, Algorithm 1 will output a consistent* $\mathcal{P}'$*-automaton.*

Proof. Recall that Algorithm 1 implements the saturation procedure of [2], in which transitions are added to the original automaton $\mathcal{A}$ according to the following rule:

> If $\langle (p, \mathbf{u}), \alpha\rangle \hookrightarrow \langle (p', \mathbf{u}'), w\rangle$ in Δ' and $(p', \mathbf{u}') \xrightarrow{w} (q, \mathbf{u}'')$ in the current automaton, then add a transition $((p, \mathbf{u}), \alpha, (q, \mathbf{u}''))$.

Algorithm 1
Input: a pushdown system $\mathcal{P}' = (P \times \mathit{States}, \Gamma, \Delta', q_0, \omega)$;
a consistent $\mathcal{P}'$-automaton $\mathcal{A} = (Q \times \mathit{States}, \Gamma, \delta, P \times \mathit{States}, F)$
Output: the set of transitions of $\mathcal{A}_{pre^*}$

1 $\mathit{rel} \leftarrow \emptyset$; $\mathit{trans} \leftarrow \delta$; $\Delta'' \leftarrow \emptyset$;
2 **for all** $\langle (p, \mathbf{u}), \alpha \rangle \hookrightarrow \langle (p', \mathbf{u}'), \varepsilon \rangle \in \Delta'$ **do** $\mathit{trans} \leftarrow \mathit{trans} \cup \{((p, \mathbf{u}), \alpha, (p', \mathbf{u}'))\}$;

3 **while** $\mathit{trans} \neq \emptyset$ **do**
4   pop $t = ((q, \mathbf{u}), \alpha, (q', \mathbf{u}'))$ from *trans*;
5   **if** $t \notin \mathit{rel}$ **then**
6     $\mathit{rel} \leftarrow \mathit{rel} \cup \{t\}$;
7   **for all** $\langle (p_1, \mathbf{u}_1), \alpha_1 \rangle \hookrightarrow \langle (q, \mathbf{u}), \alpha \rangle \in (\Delta' \cup \Delta'')$ **do**
8       $\mathit{trans} \leftarrow \mathit{trans} \cup \{((p_1, \mathbf{u}_1), \alpha_1, (q', \mathbf{u}'))\}$;
9   **for all** $\langle (p_1, \mathbf{u}_1), \alpha_1 \rangle \hookrightarrow \langle (q, \mathbf{u}), \alpha\alpha_2 \rangle \in \Delta'$ **do**
10       $\Delta'' \leftarrow \Delta'' \cup \{\langle (p_1, \mathbf{u}_1), \alpha_1 \rangle \hookrightarrow \langle (q', \mathbf{u}'), \alpha_2 \rangle\}$;
11   **for all** $((q', \mathbf{u}'), \alpha_2, (q'', \mathbf{u}'')) \in \mathit{rel}$ **do**
12         $\mathit{trans} \leftarrow \mathit{trans} \cup \{((p_1, \mathbf{u}_1), \alpha_1, (q'', \mathbf{u}''))\}$;
13 **return** *rel*

TABLE 1
An algorithm for the computation of pre^*

From the existence of the transition rule in Δ' we know that there exists $\mathbf{t} \in \mathit{States}$ such that $\varrho(\mathbf{t}, \alpha) = \mathbf{u}$ and $\varrho(\mathbf{t}, w^R) = \mathbf{u}'$. Provided that the automaton is consistent, we know that $\varrho(\mathbf{u}'', w^R) = \mathbf{u}'$. Exploiting the backward determinism we get $\mathbf{t} = \mathbf{u}''$, and hence the added transition is consistent. ∎

The fact that the algorithm only has to deal with consistent transitions influences the complexity analysis:

LEMMA 2. *Given a consistent $\mathcal{P}'$-automaton $\mathcal{A} = (Q \times \mathit{States}, \Gamma, \delta, P \times \mathit{States}, F)$, we can compute $\mathit{pre}^*(L(\mathcal{A}))$ in $\mathcal{O}(|Q|^2 \cdot |\Delta| \cdot |\mathit{States}|)$ time and $\mathcal{O}(|Q| \cdot |\Delta| \cdot |\mathit{States}| + |\delta|)$ space.*

Proof. The lemma can be proved by adding to the proof of Theorem 1 in [6] the following considerations holding for the special case of consistent automata.

- Line 10 will be executed once for every combination of a rule $\langle (p_1, \mathbf{u}_1), \alpha_1 \rangle \hookrightarrow \langle (q, \mathbf{u}), \alpha\alpha_2 \rangle$ and a (consistent) transition $((q, \mathbf{u}), \alpha, (q', \mathbf{u}'))$. Since $\mathbf{u}'$ is the single state for which $\varrho(\mathbf{u}', \alpha) = \mathbf{u}$ holds, there are $\mathcal{O}(|\Delta'| \cdot |Q|) = \mathcal{O}(|Q| \cdot |\Delta| \cdot |\mathit{States}|)$ such combinations. Thus, the size of Δ'' is also $\mathcal{O}(|Q| \cdot |\Delta| \cdot |\mathit{States}|)$.

- For the loop starting at line 11, $(q', \mathbf{u}')$ and α_2 (and hence $\mathbf{u}''$) are fixed, so line 12 is executed $\mathcal{O}(|Q|^2 \cdot |\Delta| \cdot |\mathit{States}|)$ times.

- Line 8 is executed once for every combination of rules $\langle (p_1, \mathbf{u}_1), \alpha_1 \rangle \hookrightarrow \langle (q, \mathbf{u}), \alpha \rangle$ in $\Delta' \cup \Delta''$ and transitions $((q, \mathbf{u}), \alpha, (q', \mathbf{u}'))$. Since the size of Δ'' is $\mathcal{O}(|Q| \cdot |\Delta| \cdot |\mathit{States}|)$ and $\mathbf{u}'$ is unique, we have $\mathcal{O}(|Q|^2 \cdot |\Delta| \cdot |\mathit{States}|)$ such combinations. ∎

LEMMA 3. *The repeating heads of the product of $\mathcal{P}'$ and $\mathcal{B}$ can be computed in $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|^3)$ time and $\mathcal{O}(|P| \cdot |\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ space.*

Proof. (analogous to [6]) As we saw in Section 3.1, we first compute the set $pre^*(\{\langle p', \varepsilon\rangle \mid p' \in P' \times R\})$ (recall that R is the set of states of the Büchi automaton). Since this set can be represented by a consistent automaton with $|P| \cdot |R| \cdot |\mathit{States}|$ many states and no transitions, this step is bounded by the limitations on time and space of the lemma. From the results a head reachability graph of size $\mathcal{O}(|P| \cdot |\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ is constructed. To find the repeating heads, we identify its strongly connected components, which takes time linear in its size. ∎

We can now conclude the proof of Theorem 3. The steps required to solve the model-checking problems are as follows:

- Compute the set of repeating heads Rep of the product of $\mathcal{P}'$ and $\mathcal{B}$. According to Lemma 3, this takes $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|^3)$ time and $\mathcal{O}(|P| \cdot |\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ space, and we have $|\mathit{Rep}| = \mathcal{O}(|\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|)$.

- Construct an automaton $\mathcal{A}$ accepting exactly the consistent subset of $\mathit{Rep}\Gamma^*$. Take $\mathcal{A} = (((P \times R) \cup \{s\}) \times \mathit{States}, \Gamma, \delta, P \times \{r_0\} \times \mathit{States}, \{(s, \mathbf{s})\})$. For every repeating head $\langle (p, r, \mathbf{u}), \alpha\rangle$, add to δ the unique transition $((p, r, \mathbf{u}), \alpha, (s, \mathbf{u}'))$ with $\varrho(\mathbf{u}', \alpha) = \mathbf{u}$. For every triple $(\mathbf{u}, \alpha, \mathbf{u}')$ such that $\varrho(\mathbf{u}', \alpha) = \mathbf{u}$ add to δ the transition $((s, \mathbf{u}), \alpha, (s, \mathbf{u}'))$. There are at most $\mathcal{O}(|\mathit{States}| \cdot |\Gamma|) \subseteq \mathcal{O}(|\Delta| \cdot |\mathit{States}|)$ such triples. This automaton is consistent and has $\mathcal{O}(|P| \cdot |\mathit{States}| \cdot |\mathcal{B}|)$ states and $\mathcal{O}(|\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|)$ transitions.

- Compute the automaton $\mathcal{A}' = (((P \times R) \cup \{s\}) \times \mathit{States}, \Gamma, \delta', P \times \{r_0\} \times \mathit{States}, \{(s, \mathbf{s})\})$ corresponding to $pre^*(L(\mathcal{A}))$. According to Lemma 2, this takes $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|^3)$ time and $\mathcal{O}(|P| \cdot |\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ space. (Recall that the size of $\mathcal{B}$ is also a factor in the size of the product transition rules.)

- Due to Lemma 1, $\mathcal{A}'$ is consistent and accepts only consistent configurations. According to Proposition 3.1 in [6] we have $c \not\models^{\nu'} \varphi$ for every configuration in $L(\mathcal{A}')$. According to (1), we then have $\langle p, w\rangle \not\models^{\nu'} \varphi$ for every $c = \langle (p, \mathbf{u}), w\rangle$ in $L(\mathcal{A}')$. Hence, we can solve the problem (II) by modifying $\mathcal{A}'$ slightly; let $\mathcal{A}''$ be the automaton $\mathcal{A}'' = (((P \times R) \cup \{s\}) \times \mathit{States} \cup P, \Gamma, \delta'', P, \{(s, \mathbf{s})\})$ where $\delta'' = \delta' \cup \{(p, \alpha, q) \mid ((p, r, \mathbf{u}), \alpha, q) \in \delta'\}$. Problem (I) is solved by checking whether $\langle q_0, \omega\rangle \in L(\mathcal{A}'')$. Since none of the steps required to compute $\mathcal{A}''$ takes more than $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|^3)$ time and $\mathcal{O}(|P| \cdot |\Delta| \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ space, the first part of our theorem is proven.

- To prove the second part we simply need to synchronize $\mathcal{A}''$ with an automaton $\mathcal{R}$ recognizing all reachable configurations of $\mathcal{P}$. Computing $\mathcal{R}$ takes $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|))$ time and space according to Theorem 1. The synchronization is performed as follows: For all transitions (p, α, p') of $\mathcal{A}''$ and (q, α, q') of $\mathcal{R}$ we add a transition $((p, q), \alpha, (p', q'))$ to the product. A straightforward procedure however requires too much computation time. We can do better by employing the following trick from [6]: first all transitions (q, α, q') of $\mathcal{R}$ are sorted into buckets labeled by α. Then each transition of $\mathcal{A}''$ is synchronized with the transitions in the respective bucket. As $\mathcal{R}$ has $\mathcal{O}(|P| + |\Delta|)$ states, each bucket contains $\mathcal{O}((|P| + |\Delta|)^2)$ items. Hence, the product can be computed in $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ time and space. Alternatively, we can sort the transitions of $\mathcal{A}''$ into buckets of size $\mathcal{O}(|P|^2 \cdot |\mathcal{B}|^2 \cdot |\mathit{States}|)$ (exploiting the consistency of $|\mathcal{A}''|$) and construct the product in $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |States| \cdot |\mathcal{B}|^2)$ time and space. If we add the time and space which is needed to construct $\mathcal{A}''$ and $\mathcal{R}$, we get the results stated in the second part of Theorem 3.

3.2.2. Technique 2 – extending the stack

An alternative approach to model-checking with regular valuations is to store the vectors of *States* in the stack of $\mathcal{P}$. This technique works without any additional limitations, i.e., we do *not* need the assumption of backward determinism of $\mathcal{M}^p_{A_i}$ automata.

We define a pushdown system $\mathcal{P}' = (P, \Gamma', \Delta', q_0, \omega')$ where $\Gamma' = \Gamma \times States$, $\omega' = (\omega, \mathbf{s})$ where $\mathbf{s}$ is the vector of initial states of $\mathcal{M}^p_{A_i}$ automata, and the set of transition rules Δ' is determined as follows:

- $\langle p, (\alpha, \mathbf{r})\rangle \hookrightarrow' \langle q, \varepsilon\rangle \iff \langle p, \alpha\rangle \hookrightarrow \langle q, \varepsilon\rangle$
- $\langle p, (\alpha, \mathbf{r})\rangle \hookrightarrow' \langle q, (\beta, \mathbf{r})\rangle \iff \langle p, \alpha\rangle \hookrightarrow \langle q, \beta\rangle$
- $\langle p, (\alpha, \mathbf{r})\rangle \hookrightarrow' \langle q, (\beta, \mathbf{u})(\gamma, \mathbf{r})\rangle \iff \langle p, \alpha\rangle \hookrightarrow \langle q, \beta\gamma\rangle \wedge \mathbf{u} = \varrho(\mathbf{r}, \gamma)$

Intuitively, the reason why we do not need the assumption of backward determinism in our second approach is that the stack carries complete information about the computational history of the $\mathcal{M}^p_{A_i}$ automata.

A configuration $\langle q, (\alpha_k, \mathbf{r}_k) \cdots (\alpha_1, \mathbf{r}_1)\rangle$ is called *consistent* iff $\mathbf{r}_1 = \mathbf{s}$ and $\mathbf{r}_{j+1} = \varrho(\mathbf{r}_j, \alpha_j)$ for all $1 \leq j < k$.

The simple valuation ν' is defined by

$$\nu(A_i) = \{\langle p, (\alpha, \mathbf{r})w\rangle \mid p \in P, \alpha \in \Gamma, \varrho^p_i(\mathbf{r}^p_i, \alpha) \in F^p_i, w \in (\Gamma')^*\}$$

It is easy to see that $\langle q, \alpha_k \cdots \alpha_1\rangle \models^\nu \varphi \iff \langle q, (\alpha_k, \mathbf{r}_k) \cdots (\alpha_1, \mathbf{r}_1)\rangle \models^{\nu'} \varphi$ where $\langle q, (\alpha_k, \mathbf{r}_k) \cdots (\alpha_1, \mathbf{r}_1)\rangle$ is consistent.

THEOREM 4. *Technique 2 (extending the stack) gives us the same bounds on the model-checking problems with regular valuations as Technique 1, i.e.:*

1. *Problems (I) and (II) can be solved in* $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |States| \cdot |\mathcal{B}|^3)$ *time and* $\mathcal{O}(|P| \cdot |\Delta| \cdot |States| \cdot |\mathcal{B}|^2)$ *space.*
2. *Problem (III) can be solved in either* $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |States| \cdot |\mathcal{B}|^3)$ *time and* $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |States| \cdot |\mathcal{B}|^2)$ *space, or* $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |States| \cdot |\mathcal{B}|^3)$ *time and* $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |States| \cdot |\mathcal{B}|^2)$ *space.*

Proof. Since $|\Delta'| = |\Delta| \cdot |States|$ (here we use the fact that each $\mathcal{M}^p_{A_i}$ is deterministic), we can compute a $\mathcal{P}$-automaton $\mathcal{D} = (D, \Gamma', \gamma, P, G)$ of size $\mathcal{O}(|P| \cdot |\Delta| \cdot |States| \cdot |\mathcal{B}|^2)$ in $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |States| \cdot |\mathcal{B}|^3)$ time and $\mathcal{O}(|P| \cdot |\Delta| \cdot |States| \cdot |\mathcal{B}|^2)$ space such that $\mathcal{D}$ recognizes all configurations of $\mathcal{P}'$ which violate φ (see Theorem 1); then, to solve problem (I), we just look if $\mathcal{D}$ accepts $\langle q_0, \omega'\rangle$.

The problem with $\mathcal{D}$ is that it can also accept inconsistent configurations of $\mathcal{P}'$. Fortunately, it is possible to perform a kind of 'synchronization' with the reversed $\mathcal{M}^p_{A_i}$ automata. We define $\mathcal{A} = (Q, \Gamma, \delta, P, F)$ where $Q = (D \times States) \cup P$, $F = G \times \{\mathbf{s}\}$, and δ is defined as follows:

- if $g \xrightarrow{(\alpha, \mathbf{r})} h$ is a transition of γ, then δ contains a transition $(g, \mathbf{t}) \xrightarrow{\alpha} (h, \mathbf{r})$ where $\mathbf{t} = \varrho(\mathbf{r}, \alpha)$;

- if $(p, \mathbf{r}) \xrightarrow{\alpha} (g, \mathbf{t})$, $p \in P$, is a transition of δ, then $p \xrightarrow{\alpha} (g, \mathbf{t})$ is also a transition of δ.

Notice that $\mathcal{A}$ is the same size as $\mathcal{D}$ since in every transition $\mathbf{t}$ is uniquely determined by $\mathbf{r}$ and α. Now, for every configuration $\langle p, (\alpha_k, \mathbf{r}_k) \cdots (\alpha_1, \mathbf{r}_1)\rangle$, one can easily prove (by induction on k) that

$$\gamma(p, (\alpha_k, \mathbf{r}_k) \cdots (\alpha_1, \mathbf{r}_1)) = q \text{ where } \mathbf{r}_{j+1} = \varrho(\mathbf{r}_j, \alpha_j) \text{ for all } 1 \leq j < k$$

iff

$$\delta((p, \mathbf{r}), \alpha_k \cdots \alpha_1) = (q, \mathbf{r}_1) \text{ where } \mathbf{r} = \varrho(\mathbf{r}_k, \alpha_k).$$

From this we immediately obtain that $\mathcal{A}$ indeed accepts exactly those configurations of $\mathcal{P}$ which violate φ. Moreover, the size of $\mathcal{A}$ and the time and space bounds to compute it are the same as for $\mathcal{D}$ which proves the first part of the theorem.

To solve problem (III), one can try out the same strategies as in Theorem 3. Again, it turns out that the most efficient way is to synchronize $\mathcal{A}$ with the $\mathcal{P}$-automaton $\mathcal{R}$ which recognizes all reachable configurations of $\mathcal{P}$. Employing the same trick as in Theorem 3 (i.e., sorting transitions of $\mathcal{R}$ into buckets according to their labels), we obtain that the size of $\mathcal{A}'$ is $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ and it can be computed in $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |\mathit{States}| \cdot |\mathcal{B}|^3)$ time using $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ space. Using the alternative method (sorting transitions of $\mathcal{A}$ into buckets instead and exploiting determinism) we get an automaton of size $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ in $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |\mathit{States}| \cdot |\mathcal{B}|^3)$ in time and $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |\mathit{States}| \cdot |\mathcal{B}|^2)$ space. ∎

4. APPLICATIONS

4.1. Interprocedural Data-Flow Analysis

Pushdown systems provide a very natural formal model for programs with recursive procedures. Hence, it should not be surprising that efficient analysis techniques for pushdown automata can be applied to some problems of interprocedural data-flow analysis (see, e.g., [7, 11]). Here we briefly discuss the convenience of regular valuations in this application area. We do not present any detailed results about the complexity of concrete problems, because this would necessarily lead to a quite complicated and lengthy development which is beyond the scope of our work (though the associated questions are very interesting on their own). Our aim is just to provide convincing arguments demonstrating the importance of the technical results achieved in Section 3.

A standard way of abstracting recursive programs for purposes of interprocedural data-flow analysis it to represent each procedure P by its associated *flow graph*. Intuitively, the flow graph of P is a labeled binary graph whose nodes correspond to 'program points', and an arc $n \xrightarrow{c} n'$ indicates that the control flow is shifted from the point n to n' by performing the instruction c. The entry and exit points of P are modeled by distinguished nodes. To avoid undecidabilities, the `if-then-else` command (and related instructions) are modeled by nondeterminism, i.e., there can be several arcs from the same node. Moreover, there are special arcs with labels of the form *call* $Q(\mathit{args})$ which model procedure calls (where *args* is a vector of terms which are passed as parameters). Flow graphs can be easily translated to pushdown systems; as transitions of pushdown systems are not labeled, we first perform a 'technical' modification of the flow graph, replacing each arc $n \xrightarrow{c} n'$ where n

is a nondeterministic node (i.e., a node with more than one successor) by two arcs $n \stackrel{\varepsilon}{\longrightarrow} n'' \stackrel{c}{\longrightarrow} n'$ where n'' is a new node and ε is a 'dummy' instruction without any effect. This allows to associate the instruction of each arc $n \stackrel{c}{\longrightarrow} n'$ directly to n (some nodes are associated to the dummy instruction). Now suppose there is a recursive system of procedures $P_1, \ldots, P_n, S$, where S is a distinguished starting procedure which cannot be called recursively. Their associated flow graphs can be translated to a pushdown automaton in the following way:

- for each node n of each flowgraph we introduce a fresh stack symbol X_n;
- for each arc of the form $n \stackrel{c}{\longrightarrow} n'$ we add a rule $\langle \cdot, X_n \rangle \hookrightarrow \langle \cdot, X_{n'} \rangle$, where $\cdot$ is the (only) control location;
- for each arc of the form $n \xrightarrow{call\ Q(args)} n'$ we add the rule $\langle \cdot, X_n \rangle \hookrightarrow \langle \cdot, Q_{entry} X_{n'} \rangle$ where Q_{entry} is the stack symbol for the entry node of (the flow graph of) Q. Observe that one can also push special symbols corresponding to arguments if needed;
- for each procedure P different from S we add the rule $\langle \cdot, P_{exit} \rangle \hookrightarrow \langle \cdot, \varepsilon \rangle$, where P_{exit} corresponds to the exit node of P. For the starting procedure S we add the rule $\langle \cdot, S_{exit} \rangle \hookrightarrow \langle \cdot, S_{exit} \rangle$.

In other words, the top stack symbol corresponds to the current program point (and to the instruction which is to be executed), and the stack carries the information about the history of activation calls. Now, many of the well-known properties of data-flow analysis (e.g., liveness, reachability, very business, availability) can be expressed in LTL and verified by a model-checking algorithm (in some cases the above presented construction of a pushdown automaton must be modified so that all necessary information is properly reflected – but the principle is still the same). For example, if we want to check that a given variable Y is *dead* at a given program point n (i.e., whenever the program point n is executed, in each possible continuation we have that Y is either not used or it is redefined before it is used), we can model-check the formula

$$\Box(top_n \Longrightarrow ((\neg used_Y \,\mathcal{U}\, def_Y) \;\vee\; (\Box \neg used_Y)))$$

in the configuration $\langle \cdot, S_{entry} \rangle$, where $used_Y$, $used_Y$, and def_Y are atomic propositions which are valid in exactly those configurations where the topmost stack symbol corresponds to the program point n, to an instruction which uses the variable Y, or to an instruction which defines Y, respectively. Regular valuations become useful even in this simple example – if we have a language with dynamic scoping (e.g., LISP), we *cannot* resolve to which Y the instruction $Y := 3$ at a program point n refers to without examining the stack of activation records (the Y refers to a local variable Y of the topmost procedure in the stack of activation records which declares it). So, $used_Y$ and def_Y would be interpreted by regular valuations in this case.

The example above is quite simple. The 'real' power of regular valuations would become apparent in a context of more complicated problems where we need to examine complex relationships among dynamically gathered pieces of information. This is one of the subjects of intended future work.

4.2. Pushdown Systems with Checkpoints

Another area where the results of Section 3.2 find a natural application is the analysis of recursive computations with local security checks. Modern programming languages contain methods for performing run-time inspections of the stack of activation records, and processes can thus take dynamic decisions based on the gathered information. An example is the class `AccessController` implemented in Java Development Kit 1.2 offering the method `checkPermission` which checks whether all methods stored in the stack are granted a given permission. If not, the method rises an exception.

We propose a (rather general) formal model of such systems called *pushdown system with checkpoints*. Our work was inspired by the paper [10] which deals with the same problem. Our model is more general, however. The model of [10] is suitable only for checking safety properties, does not model data-flow, and forbids mutually recursive procedure calls whereas our model has none of these restrictions. Properties of pushdown systems with checkpoints can be expressed in LTL and we provide an efficient model-checking algorithm for LTL with regular valuations.

Definition 4. A *pushdown system with checkpoints* is a triple $\mathcal{C} = (\mathcal{P}, \xi, \eta)$ where

- $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$ is a pushdown system.
- $\xi \subseteq P \times \Gamma$ is a set of *checkpoints*. Each checkpoint (p, α) is *implemented* by its associated deterministic finite-state automaton $\mathcal{M}^p_\alpha = (Q^p_\alpha, \Gamma, \delta^p_\alpha, s^p_\alpha, F^p_\alpha)$. For technical convenience, we assume that δ^p_α is total, $s^p_\alpha \notin F^p_\alpha$, and $L(\mathcal{M}^p_\alpha) \subseteq \{w\alpha \mid w \in \Gamma^*\}$.
- $\eta\colon \Delta \to \{+, -, 0\}$ is a function which partitions the set of transition rules into *positive*, *negative*, and *independent* ones. We require that if (p, α) is *not* a checkpoint, then all rules of the form $\langle p, \alpha\rangle \hookrightarrow \langle q, v\rangle$ are independent.

The function η determines whether a rule can be applied when an inspection of the stack at a checkpoint yields a positive or negative result, or whether it is independent of such tests. Using positive and negative rules, we can model systems which perform `if-then-else` commands where the condition is based on dynamic checks; hence, these checks can be nested to an arbitrary level. The fact that a rule $\langle p, \alpha\rangle \hookrightarrow \langle q, v\rangle$ is positive, negative, or independent is denoted by $\langle p, \alpha\rangle \hookrightarrow^+ \langle q, v\rangle$, $\langle p, \alpha\rangle \hookrightarrow^- \langle q, v\rangle$, or $\langle p, \alpha\rangle \hookrightarrow^0 \langle q, v\rangle$, respectively.

To $\mathcal{C}$ we associate a unique transition system $\mathcal{T}_\mathcal{C}$ where the set of states is the set of all configurations of $\mathcal{P}$, $\langle q_0, \omega\rangle$ is the root, and the transition relation is the least relation $\to$ satisfying the following:

- if $\langle p, \alpha\rangle \hookrightarrow^+ \langle q, v\rangle$, then $\langle p, \alpha w\rangle \to \langle q, vw\rangle$ for all $w \in \Gamma^*$ s.t. $w^R\alpha \in L(\mathcal{M}^p_\alpha)$;
- if $\langle p, \alpha\rangle \hookrightarrow^- \langle q, v\rangle$, then $\langle p, \alpha w\rangle \to \langle q, vw\rangle$ for all $w \in \Gamma^*$ s.t. $w^R\alpha \notin L(\mathcal{M}^p_\alpha)$;
- if $\langle p, \alpha\rangle \hookrightarrow^0 \langle q, v\rangle$, then $\langle p, \alpha w\rangle \to \langle q, vw\rangle$ for all $w \in \Gamma^*$.

Some natural problems for pushdown processes with checkpoints are listed below.

- The reachability problem: given a pushdown system with checkpoints, is a given configuration reachable?

- The checkpoint-redundancy problem: given a pushdown system with checkpoints and a checkpoint (p, α), is there a reachable configuration where the checkpoint (p, α) is (or is not) satisfied?

  This problem is important because redundant checkpoints can be safely removed together with all negative (or positive) rules, declaring all remaining rules as independent. Thus, one can decrease the runtime overhead.

- The global safety problem: given a pushdown system with checkpoints and a formula φ of LTL, do all reachable configurations satisfy φ?

  An efficient solution to this problem allows to make 'experiments' with checkpoints with the aim of finding a solution with a minimal runtime overhead.

Actually, it is quite easy to see that all these problems (and many others) can be encoded by LTL formulae and regular valuations. For example, to solve the reachability problem, we take a predicate A which is satisfied only by the configuration $\langle p, w\rangle$ whose reachability is in question (the associated automaton $\mathcal{M}_A^p$ has $\mathit{length}(w)$ states) and then we check the formula $\Box(\neg A)$. Observe that this formula in fact says that $\langle p, w\rangle$ is *unreachable*; the reachability itself is not directly expressible in LTL (we can only say that $\langle p, w\rangle$ is reachable in *every run*). However, it does not matter because we can simply negate the answer of the model-checking algorithm.

4.2.1. Model-Checking LTL for Pushdown Systems with Checkpoints

Let $\mathcal{C} = (\mathcal{P}, \xi, \eta)$ be a pushdown system with checkpoints, where $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$. We define a pushdown system $\mathcal{P}' = (P \times \{+, -, 0\}, \Gamma, \Delta', (q_0, 0), \omega)$ where Δ' is the least set of rules satisfying the following (for each $x \in \{+, -, 0\}$);

- if $\langle p, \alpha\rangle \hookrightarrow^{+} \langle q, v\rangle \in \Delta$, then $\langle (p, x), \alpha\rangle \hookrightarrow \langle (q, +), v\rangle \in \Delta'$;
- if $\langle p, \alpha\rangle \hookrightarrow^{-} \langle q, v\rangle \in \Delta$, then $\langle (p, x), \alpha\rangle \hookrightarrow \langle (q, -), v\rangle \in \Delta'$;
- if $\langle p, \alpha\rangle \hookrightarrow^{0} \langle q, v\rangle \in \Delta$, then $\langle (p, x), \alpha\rangle \hookrightarrow \langle (q, 0), v\rangle \in \Delta'$.

Intuitively, $\mathcal{P}'$ behaves in the same way as the underlying pushdown system $\mathcal{P}$ of $\mathcal{C}$, but it also 'remembers' what kind of rule (positive, negative, independent) was used to enter the current configuration.

Let ν be a regular valuation for configurations of $\mathcal{C}$ (see Definition 3), and let φ be an LTL formula. Let *Check*, *Neg*, and *Pos* be fresh atomic propositions which do not appear in φ. We define a regular valuation ν' for configurations of $\mathcal{P}'$ as follows:

- If $A \in At \setminus \{\mathit{Check}, \mathit{Neg}, \mathit{Pos}\}$, then $\nu'(A) = \{\langle (p, x), w\rangle \mid \langle p, w\rangle \in \nu(A), x \in \{+, -, 0\}\}$. Hence, the automaton $\mathcal{M}_A^{(p,x)}$ is the same as the automaton $\mathcal{M}_A^p$ for each $x \in \{+, -, 0\}$.
- $\nu'(\mathit{Check}) = \bigcup_{(p,\alpha)\in\xi} \{\langle (p, x), w\rangle \mid w^R \in L(\mathcal{M}_\alpha^p), x \in \{+, -, 0\}\}$. Hence, for each $x \in \{+, -, 0\}$, $\mathcal{M}_{\mathit{Check}}^{(p,x)}$ is the product automaton (constructed out of all $\mathcal{M}_\alpha^p$) which accepts the union of all $L(\mathcal{M}_\alpha^p)$. Notice that we need to perform a product because $\mathcal{M}_{\mathit{Check}}^{(p,x)}$ has to be deterministic.
- $\nu'(\mathit{Neg}) = \{\langle (p, -), w\rangle \mid p \in P, w \in \Gamma^+\}$. So, $\mathcal{M}_{\mathit{Neg}}^{(p,-)}$ is an automaton with two states which accepts Γ^+.

- $\nu'(Pos) = \{\langle (p,+), w\rangle \mid p \in P, w \in \Gamma^+\}$.

Now we can readily confirm the following:

THEOREM 5. *Let $\langle p, w\rangle$ be a configuration of $\mathcal{C}$. We have that*

$$\langle p, w\rangle \models^{\nu} \varphi \quad \Longleftrightarrow \quad \langle (p,0), w\rangle \models^{\nu'} \psi \Longrightarrow \varphi$$

where $\psi \equiv \Box((Check \Longrightarrow \mathcal{X}(\neg Neg)) \wedge (\neg Check \Longrightarrow \mathcal{X}(\neg Pos)))$.

Proof. It suffices to observe that

$$\langle p, w\rangle \equiv \langle p_0, w_0\rangle \to \langle p_1, w_1\rangle \to \langle p_2, w_2\rangle \to \langle p_3, w_3\rangle \to \cdots$$

is an infinite path in $\mathcal{T}_{\mathcal{C}}$ iff

$$\langle (p,0), w\rangle \equiv \langle (p_0, x_0), w_0\rangle \to \langle (p_1, x_1), w_1\rangle \to \langle (p_2, x_2), w_2\rangle \to \cdots$$

is an infinite path in $\mathcal{T}_{\mathcal{P}'}$ satisfying ψ (where each x_i for $i > 0$ is either $+$, $-$, or 0; notice that all x_i are determined uniquely). Indeed, ψ ensures that all transitions in the latter path are 'consistent' with possible checkpoints in the former path. As all atomic propositions which appear in φ are evaluated identically for pairs $\langle p_i, w_i\rangle$, $\langle (p_i, x_i), w_i\rangle$ (see the definition of ν' above), we can conclude that both paths either satisfy or do not satisfy φ. ∎

The previous theorem in fact says that the model-checking problem for LTL and pushdown systems with checkpoints can be reduced to the model-checking problem for LTL and 'ordinary' pushdown systems. As the formula ψ is fixed and the atomic propositions *Check*, *Neg*, and *Pos* are regular, we can evaluate the complexity bounds for the resulting model-checking algorithm using the results of Section 3.2. Let $\{A_1, \ldots, A_n\}$ be the set of all atomic propositions which appear in φ, and let $\mathcal{N} = \{M_1, \ldots, \mathcal{M}_m\}$ be the set of all $\mathcal{M}^p_{A_i}$ automata. Let *States* be the Cartesian product of the sets of states of all $\mathcal{M}^p_{\alpha}$ automata and the automata of $\mathcal{N}$. Let $\mathcal{B}$ be a Büchi automaton which corresponds to $\neg\varphi$. Now we can state our theorem (remember that P is the set of control states and Δ the set of rules of the underlying pushdown system $\mathcal{P}$ of $\mathcal{C}$).

THEOREM 6. *We have the following bounds on the model-checking problems for LTL with regular valuations and pushdown systems with checkpoints:*

1. *Problems (I) and (II) can be solved in* $\mathcal{O}(|P|^2 \cdot |\Delta| \cdot |States| \cdot |\mathcal{B}|^3)$ *time and* $\mathcal{O}(|P| \cdot |\Delta| \cdot |States| \cdot |\mathcal{B}|^2)$ *space.*
2. *Problem (III) can be solved in either* $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |States| \cdot |\mathcal{B}|^3)$ *time and* $\mathcal{O}(|P| \cdot |\Delta| \cdot (|P| + |\Delta|)^2 \cdot |States| \cdot |\mathcal{B}|^2)$ *space, or* $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |States| \cdot |\mathcal{B}|^3)$ *time and* $\mathcal{O}(|P|^3 \cdot |\Delta| \cdot (|P| + |\Delta|) \cdot |States| \cdot |\mathcal{B}|^2)$ *space.*

Proof. We apply Theorem 5, which says that we can equivalently consider the model-checking problem for the pushdown system $\mathcal{P}'$, formula $\psi \Longrightarrow \varphi$, and valuation ν'. First, let us realize that the Büchi automaton which corresponds to $\neg(\psi \Longrightarrow \varphi)$ can be actually obtained by 'synchronizing' $\mathcal{B}$ with the Büchi automaton for ψ, because $\neg(\psi \Longrightarrow \varphi) \equiv \psi \wedge \neg\varphi$. As the formula ψ is *fixed*, the synchronization increases the size of $\mathcal{B}$ just by a constant factor. Hence, the automaton for $\neg(\psi \Longrightarrow \varphi)$ is asymptotically of the same size as $\mathcal{B}$. The same can be said about the sizes of P' and P, and about the sizes of Δ' and Δ. Moreover, if we collect all the automata which represent atomic predicates of $At(\psi \Longrightarrow \varphi)$ (see above) and consider the state space of their product, we see that it has *exactly* the size $2 \cdot \mathit{States}$ because all of the automata associated to *Pos* and *Neg* are the same and have only two states. ∎

4.3. Model-checking CTL*

In this section, we apply the model-checking algorithm to the logic CTL* which extends LTL with existential path quantification [4]. More precisely, CTL* formulae are built according to the following abstract syntax equation:

$$\varphi \quad ::= \quad \mathtt{tt} \mid A \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathcal{E}\varphi \mid \mathcal{X}\varphi \mid \varphi_1 \,\mathcal{U}\, \varphi_2$$

where A ranges over the atomic propositions (interpreted, say, by a regular valuation represented by a finite automaton of size $|\mathit{States}|$).

For finite-state systems, model-checking CTL* can be reduced to checking LTL as follows [5]: For a CTL* formula φ, call the *path depth* of φ the maximal nesting depth of existential path quantifiers within φ. Subformulae of φ can be checked in ascending order of path depth; subformulae of the form $\mathcal{E}\,\varphi'$ where φ' is $\mathcal{E}$-free are checked with an LTL algorithm which returns the set of states $S_{\varphi'}$ satisfying φ'. Then $\mathcal{E}\,\varphi'$ is replaced by a fresh atomic proposition whose valuation yields true exactly on $S_{\varphi'}$, and the procedure is repeated for subformulae of higher path depth. The method can be transferred to the case of pushdown systems; running the LTL algorithm on $\mathcal{E}\,\varphi'$ returns an automaton $\mathcal{M}_{\varphi'}$. We can then replace $\mathcal{E}\,\varphi'$ by a fresh atomic proposition whose valuation is given by $\mathcal{M}_{\varphi'}$. This method was already proposed in [9], but without any complexity analysis.

Let us review the complexity of this procedure. For the rest of this subsection fix a pushdown system $\mathcal{P} = (P, \Gamma, \Delta, q_0, \omega)$. Given an $\mathcal{E}$-free formula φ, let $\mathcal{B} = (R, 2^{At}, \eta, r_0, G)$ be a Büchi automaton corresponding to φ, and let $|\mathit{States}|$ be the size of the $\mathcal{M}^p_{A_i}$ automata encoding the regular valuations of propositions in At.

The algorithms from section 3.2 (in general we can only use Technique 2) yield an automaton $\mathcal{M}_\varphi$ which accepts exactly the configurations satisfying $\mathcal{E}\,\varphi$. Observe that $\mathcal{M}_\varphi$ is nondeterministic, reads the stack top-down, and has $\mathcal{O}(|P|\cdot|\mathcal{B}|\cdot|\mathit{States}|)$ states. We need to modify the automaton before we can use it as an encoding for the regular valuation of $\mathcal{E}\,\varphi$. More precisely, we need to reverse the automaton (i.e. make it read the stack bottom-up) and then determinize it. Reversal does not increase the size, and due to the determinism of $\mathcal{M}^p_{A_i}$ (in bottom-up direction) the determinization explodes only the '$P \times R$ part' of the states, i.e. we get an automaton $\mathcal{M}'_\varphi$ of size $O(|\mathit{States}| \cdot 2^{|P|\cdot|R|})$.

To check subformulae of higher path depth we replace $\mathcal{E}\,\varphi$ by a fresh atomic proposition A_φ. With (A_φ, p) we associate the automaton $\mathcal{M}^p_{A_\varphi}$ which is a copy of $\mathcal{M}'_\varphi$ where the set F^p_φ of accepting states is taken as $\{\, (q, s) \mid q \in 2^{P\times R}, q \ni (p, r_0), s \in \mathit{States}\,\}$. The cross product of these new automata with the 'old' $\mathcal{M}^p_{A_i}$ automata takes only $O(|\mathit{States}| \cdot 2^{|P|\cdot|R|})$ states again; we need just one copy of the new automaton, and all reachable states are of the form $((q, s), s)$ where $q \in 2^{P\times R}$ and $s \in \mathit{States}$.

As we go up in path depth, we can repeat this procedure: First we produce a deterministic valuation automaton by taking the cross product of the automata corresponding the atomic propositions and those derived from model-checking formulae of lower path depth. Then we model-check the subformula currently under consideration, and reverse and determinize the resulting automaton. By the previous arguments, each determinization only blows up the nondeterministic part of the automaton, i.e. after each stage the size of the valuation automaton increases by a factor of $2^{|P|\cdot|\mathcal{B}_i|}$ where $\mathcal{B}_i$ is a Büchi automaton for the subformula currently under consideration.

With this in mind, we can compute the complexity for formulae of arbitrary path depth. Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be the Büchi automata corresponding to the individual subformulae of a formula φ. Adding the times for checking the subformulae and using Theorem 4 we get that the model-checking procedure takes at most

$$\mathcal{O}\Big(|P|^2 \cdot |\Delta| \cdot |\mathit{States}| \cdot 2^{|P|\cdot\Sigma_{i=1}^{n}|\mathcal{B}_i|} \cdot \sum_{i=1}^{n} |\mathcal{B}_i|^3\Big)$$

time and

$$\mathcal{O}\Big(|P| \cdot |\Delta| \cdot |\mathit{States}| \cdot 2^{|P|\cdot\Sigma_{i=1}^{n}|\mathcal{B}_i|} \cdot \sum_{i=1}^{n} |\mathcal{B}_i|^2\Big)$$

space. The algorithm hence remains linear in both $|\Delta|$ and $|\mathit{States}|$. The algorithm of Burkart and Steffen [3], applied to CTL* formulae which are in the second level of the alternation hierarchy, would yield an algorithm which is cubic in $|\Delta|$. On the other hand, the performance of our algorithm in terms of the formula is less clear. In practice, it would depend strongly on the size of the Büchi automata for the subformulae, and on the result of the determinization procedures.

5. LOWER BOUNDS

In previous sections we established reasonably-looking upper bounds for the model-checking problem for pushdown systems (first without and then also with checkpoints) and LTL with regular valuations. However, the algorithms are polynomial in $|\mathcal{P}|+|\mathcal{B}|+|\mathit{States}|$, and not in the size of *problem instance* which is (as we already mentioned in Remark 1) $|\mathcal{P}|+|\varphi|+|\nu_\varphi|$. In Remark 1 we also explained *why* we use these parameters – it has been argued that typical formulae (and their associated Büchi automata) are small, hence the size of $\mathcal{B}$ is actually more relevant (a model-checking algorithm whose complexity is exponential *just* due to the blowup caused by the transformation of φ into $\mathcal{B}$ is usually efficient in practice). The same can be actually said about $|\mathit{States}|$ – in Section 4 we have seen that there are interesting practical problems where the size of $|\mathit{States}|$ does not explode. Nevertheless, from the point of view of worst-case analysis (where we measure the complexity in the size of a problem instance) our algorithms are *exponential*. A natural question is whether this exponential blowup is indeed necessary, i.e., whether we could (in principle) solve the model-checking problems more efficiently by some other technique. In this section we show it is *not* the case, because all of the considered problems are **EXPTIME**-hard (even in rather restricted forms).

We start with the natural problems for pushdown systems with checkpoints mentioned in the previous section (the reachability problem, the checkpoint redundancy problem, etc.) All of them are (polynomially) reducible to the model-checking problem for pushdown systems with checkpoints and LTL with regular valuations and therefore are solvable in **EXPTIME**. The next theorem says that this strategy is essentially optimal, because even the reachability problem provably requires exponential time.

THEOREM 7. *The reachability problem for pushdown systems with checkpoints (even for those with just three control states and no negative rules) is **EXPTIME**-complete.*

Proof. The membership to **EXPTIME** follows from Theorem 6. We show **EXPTIME**-hardness by reduction from the acceptance problem for alternating linearly bounded automata (which is known to be **EXPTIME**-complete). An *alternating linearly bounded automaton (LBA)* is a tuple $\mathcal{M} = (Q, \Sigma, \delta, q_0, \vdash, \dashv, p)$ where $Q, \Sigma, \delta, q_0, \vdash$, and $\dashv$ are defined as for ordinary nondeterministic LBA (in particular, $\vdash \in \Sigma$ and $\dashv \in \Sigma$ are the left-end and the right-end markers, resp.), and $p: Q \to \{\forall, \exists, acc, rej\}$ is a function which partitions the states of Q into *universal*, *existential*, *accepting*, and *rejecting*, respectively. We assume (w.l.o.g.) that δ is defined so that 'terminated' configurations (i.e., the ones from which there are no further computational steps) are exactly accepting and rejecting configurations. Moreover, we also assume that $\mathcal{M}$ always halts and that its branching degree is 2 (i.e., each universal and existential configuration has exactly two immediate successors). A *computational tree* for $\mathcal{M}$ on a word $w \in \Sigma^*$ is any (finite) tree T satisfying the following: the root of T is (labeled by) the initial configuration $q_0{\vdash}w{\dashv}$ of $\mathcal{M}$, and if N is a node of $\mathcal{M}$ labeled by a configuration uqv where $u, v \in \Sigma^*$ and $q \in Q$, then the following holds:

- if q is accepting or rejecting, then N is a leaf;
- if q is existential, then N has one successor labeled by a configuration reachable from uqv in one computational step (according to δ);
- if q is universal, then N has two successors labeled by the two configurations reachable from uqv in one computational step.

$\mathcal{M}$ *accepts* w iff there is a computational tree T such that all leaves of T are accepting configurations.

Now we describe a polynomial algorithm which for a given alternating LBA $\mathcal{M} = (Q, \Sigma, \delta, q_0, \vdash, \dashv, p)$ and a word $w \in \Sigma^*$ of length n constructs a pushdown system with checkpoints $\mathcal{C} = (\mathcal{P}, \xi, \eta)$ and a configuration $\langle a, \omega \rangle$ such that $\langle a, \omega \rangle$ is reachable from the initial configuration of $\mathcal{C}$ iff $\mathcal{M}$ accepts w. Intuitively, the underlying system $\mathcal{P}$ of $\mathcal{C}$ simulates the execution of $\mathcal{M}$ and the checkpoints are used to verify that there was no cheating during the simulation. We start with the definition of $\mathcal{P}$. Configurations of $\mathcal{M}$ will be represented by words over the alphabet $\Sigma \times (Q \cup \{-\})$ of length $n+2$ (notice that w is surrounded by the '$\vdash$' and '$\dashv$' markers). Each such word contains exactly one symbol $(X, t) \in \Sigma \times Q$ which encodes the current control state and the current position of the head. We put $\mathcal{P} = (\{g, a, r\}, \Gamma, \Delta, g, \omega)$ where

- $\Gamma = \Sigma \times (Q \cup \{-\}) \cup \{\beta_1, \cdots, \beta_{n+3}\} \cup \{\gamma_1, \cdots, \gamma_n\} \cup \{\#_1^e, \#_2^e, \#_1^u, \#_2^u, A, R, \omega\}$
- Δ contains the following (families of) rules:
  1. $\langle g, \omega \rangle \hookrightarrow \langle g, \beta_1 \omega \rangle$
  2. $\langle g, \beta_i \rangle \hookrightarrow \langle g, \beta_{i+1} \varrho \rangle$ for all $1 \leq i \leq n+2$ and $\varrho \in \Sigma \times (Q \cup \{-\})$
  3. $\langle g, \beta_{n+3} \rangle \hookrightarrow \langle g, \gamma_1 \varrho \rangle$ for every $\varrho \in \{\#_1^e, \#_1^u, A, R\}$
  4. $\langle g, \gamma_i \rangle \hookrightarrow \langle g, \gamma_{i+1} \rangle$ for every $1 \leq i \leq n-1$
  5. $\langle g, \gamma_n \rangle \hookrightarrow \langle g, \varepsilon \rangle$

6. $\langle g, A\rangle \hookrightarrow \langle a, \varepsilon\rangle$, $\langle g, R\rangle \hookrightarrow \langle r, \varepsilon\rangle$, $\langle g, \#_1^e\rangle \hookrightarrow \langle g, \beta_1\#_1^e\rangle$, $\langle g, \#_1^u\rangle \hookrightarrow \langle g, \beta_1\#_1^u\rangle$
7. $\langle a, \varrho\rangle \hookrightarrow \langle a, \varepsilon\rangle$ for every $\varrho \in \Sigma \times (Q \cup \{-\})$
8. $\langle a, \#_1^u\rangle \hookrightarrow \langle g, \beta_1\#_2^u\rangle$, $\langle a, \#_2^u\rangle \hookrightarrow \langle a, \varepsilon\rangle$, $\langle a, \#_1^e\rangle \hookrightarrow \langle a, \varepsilon\rangle$, $\langle a, \#_2^e\rangle \hookrightarrow \langle a, \varepsilon\rangle$
9. $\langle r, \varrho\rangle \hookrightarrow \langle r, \varepsilon\rangle$ for every $\varrho \in \Sigma \times (Q \cup \{-\})$
10. $\langle r, \#_1^e\rangle \hookrightarrow \langle g, \beta_1\#_2^e\rangle$, $\langle r, \#_2^e\rangle \hookrightarrow \langle r, \varepsilon\rangle$, $\langle r, \#_1^u\rangle \hookrightarrow \langle r, \varepsilon\rangle$, $\langle r, \#_2^u\rangle \hookrightarrow \langle r, \varepsilon\rangle$

Intuitively, the execution of $\mathcal{P}$ starts by entering the state $\langle g, \beta_1\omega\rangle$ (rule 1). Then, exactly $n+2$ symbols of $\Sigma \times (Q \cup \{-\})$ are pushed to the stack. During this phase, the family of β_i symbols is used as a 'counter' (rules 2). The last symbol β_{n+3} is then rewritten to $\gamma_1\varrho$, where ϱ is one of $\#_1^e, \#_1^u, A, R$ (rules 3). The purpose of ϱ is to keep information about the just stored configuration (whether it is existential, universal, accepting, or rejecting) and the index of a rule which is to be used to obtain the next configuration (always the first one; remember that accepting and rejecting configurations are terminal). After that, γ_1 is successively rewritten to all of the γ_i symbols and disappears (rules 4,5). The only purpose of γ_i symbols is to invoke several consistency checks – as we shall see, each pair (g, γ_i) is a checkpoint and all rules of 4,5 are positive. Depending on the previously stored ϱ (i.e., on the type of the just pushed configuration), we either continue with guessing the next one, or change the control state to a or r (if the configuration is accepting or rejecting, resp.) Hence, the guessing goes on until we end up with an accepting or rejecting configuration. This must happen eventually, because $\mathcal{M}$ always halts. If we find an accepting configuration, we successively remove all existential configurations and those universal configuration for which we have already checked both successors. If we find a universal configuration with only one successor checked – it is recognized by the '$\#_1^u$' symbol – we change '$\#_1^u$' to '$\beta_1\#_2^u$' and check the other successor (rules 7 and 8). Similar things are done when a rejecting configuration is found. The control state is switched to r and then we remove all configurations until we (possibly) find an existential configuration for which we can try out the other successor (rules 9 and 10). We see that w is accepted by $\mathcal{M}$ iff we eventually pop the initial configuration when the control state is 'a', i.e., iff the state $\langle a, \omega\rangle$ is reachable.

To make all that work we must ensure that $\mathcal{P}$ cannot gain anything by 'cheating', i.e., by pushing inconsistent sequences of symbols which do *not* model a computation of $\mathcal{M}$ in the above described way. This is achieved by declaring all pairs (g, γ_i) for $1 \le i \le n+1$ as checkpoints. The automaton $\mathcal{M}_{\gamma_i}^g$ for $1 \le i \le n$ accepts those words of the form $\omega v_1 \varrho_1 v_2 \varrho_2 \cdots v_m \varrho_m \gamma_i$, where $\mathit{length}(v_j) = n+2$, $\varrho_j \in \#_1^e, \#_2^e, \#_1^u, \#_2^u, A, R$ for every $1 \le j \le m$, such that the triples of symbols at positions $i, i+1, i+2$ in each pair of successive substrings v_k, v_{k+1} are consistent with the symbol ϱ_k w.r.t. the transition function δ of $\mathcal{M}$. Furthermore, the first configuration must be the initial one, and the last configuration v_m must be consistent with ϱ_m. Observe that $\mathcal{M}_{\gamma_i}^g$ needs just $\mathcal{O}(|\mathcal{M}|^6)$ states to store the two triples (after checking subwords v_k, ϱ_k, v_{k+1}, the triple of v_k is 'forgotten') the initial configuration, a 'counter' of capacity $n+2$, and some auxiliary information. Moreover, $\mathcal{M}_{\gamma_i}^g$ is deterministic and we can also assume that its transition function is total. As all rules associated with checkpoints are positive, any cheating move eventually results in entering a configuration where the system 'gets stuck', i.e., cheating cannot help to reach the configuration $\langle a, \omega\rangle$. ∎

From the (technical) proof of Theorem 7 we can easily deduce the following:

Theorem 8. *The model-checking problem (I) for pushdown systems with checkpoints (even for those with just three control states and no negative rules) is* ***EXPTIME****-complete even for a fixed LTL formula* $\Box(\neg \mathit{fin})$ *where fin is an atomic predicate interpreted by a* simple *valuation* ν.

Proof. Let us consider the pushdown system with checkpoints $\mathcal{C} = (\mathcal{P}, \xi, \eta)$ constructed in the proof of Theorem 7. To ensure that each finite path in $\mathcal{T}_\mathcal{C}$ is a prefix of some run, we extend the set of transition rules of Δ by a family of independent rules of the form $\langle s, \alpha\rangle \hookrightarrow \langle s, \alpha\rangle$ for each control state s and each stack symbol α. Now it suffices to realize that the initial configuration $\langle g, \omega\rangle$ cannot reach the state $\langle a, \omega\rangle$ iff it cannot reach *any* state of the form $\langle a, \omega v\rangle$ (where $v \in \Gamma^*$) iff $\langle g, \omega\rangle \models^\nu \Box(\neg \mathit{fin})$ where ν is a simple valuation such that $\nu(\mathit{fin}) = \{\langle a, \omega w\rangle \mid w \in \Gamma^*\}$. ∎

Hence, model-checking LTL for pushdown systems with checkpoints is **EXPTIME**-complete even when we have only *simple* valuations.

Now we analyze the complexity of model-checking for (ordinary) pushdown systems and LTL formulae with regular valuations. First, realize that if we take any fixed formula and a subclass of pushdown systems where the number of control states is bounded by some constant, the model-checking problem is decidable in *polynomial* time. Now we prove that if the number of control states is not bounded, the model-checking problem becomes **EXPTIME**-complete even for a fixed formula. At this point, one is tempted to apply Theorem 5 to the formula $\Box(\neg \mathit{fin})$ of Theorem 8. Indeed, it allows to reduce the model-checking problem for pushdown systems with checkpoints and $\Box(\neg \mathit{fin})$ to the model-checking problem for ordinary pushdown systems and another fixed formula $\psi \Longrightarrow \Box(\neg \mathit{fin})$. Unfortunately, this reduction is *not* polynomial because the atomic proposition *Check* occurring in ψ is interpreted with the help of several *product* automata constructed out of the original automata which implement checkpoints (see the previous section). Therefore we need one more technical proof.

Theorem 9. *The model-checking problem (I) for pushdown systems and LTL formulae with regular valuations is* ***EXPTIME****-complete even for a fixed formula* $(\Box \mathit{correct}) \Longrightarrow (\Box \neg \mathit{fin})$.

Proof. This proof is similar to the proof of Theorem 7. Again, we construct a pushdown system $\mathcal{P}$ which simulates the execution of an alternating LBA $\mathcal{M} = (Q, \Sigma, \delta, q_0, \vdash, \dashv, p)$ on an input word $w \in \Sigma^*$ of length n. The difference is that, since there are no checkpoints, we must find a new way of 'cheating-detection', i.e., we must be able to recognize situations when the next configuration of $\mathcal{M}$ has not been guessed correctly. It is achieved by adding a family of control states $c_1, \ldots, c_n$; after guessing a new configuration, $\mathcal{P}$ successively switches its control state to $c_1, \ldots, c_n$ without modifying its stack. The constructed regular valuation ν assigns to each pair $(\mathit{correct}, c_i)$ a deterministic automaton $\mathcal{M}^{c_i}_{correct}$ which checks that the triples of symbols at positions $i, i+1, i+2$ in each pair of successive configurations previously pushed to the stack are 'consistent' ($\mathcal{M}^{c_i}_{correct}$ is almost the same automaton as the $\mathcal{M}^g_{\gamma_i}$ of the proof of Theorem 7). All other pairs of the form $(\mathit{correct}, p)$ are assigned an automaton accepting Γ^+. The $\mathcal{P}$ is formally defined as follows: $\mathcal{P} = (\{g, a, r, c_1, \ldots, c_n\}, \Gamma, \Delta, g, \omega)$ where

- $\Gamma = \Sigma \times (Q \cup \{-\}) \cup \{\beta_1, \cdots, \beta_{n+3}\} \cup \{\#^e_1, \#^e_2, \#^u_1, \#^u_2, A, R, \omega\}$
- Δ contains the following (families of) rules:

1. $\langle g, \omega \rangle \hookrightarrow \langle g, \beta_1 \omega \rangle$
2. $\langle g, \beta_i \rangle \hookrightarrow \langle g, \beta_{i+1} \varrho \rangle$ for all $1 \leq i \leq n+2$ and $\varrho \in \Sigma \times (Q \cup \{-\})$
3. $\langle g, \beta_{n+3} \rangle \hookrightarrow \langle c_1, \varrho \rangle$ for every $\varrho \in \{\#_1^e, \#_1^u, A, R\}$
4. $\langle c_i, \varrho \rangle \hookrightarrow \langle c_{i+1}, \varrho \rangle$ for every $1 \leq i \leq n-1$ and $\varrho \in \{\#_1^e, \#_1^u, A, R\}$
5. $\langle c_n, \varrho \rangle \hookrightarrow \langle g, \varrho \rangle$ for every $\varrho \in \{\#_1^e, \#_1^u, A, R\}$
6. $\langle g, A \rangle \hookrightarrow \langle a, \varepsilon \rangle$, $\langle g, R \rangle \hookrightarrow \langle r, \varepsilon \rangle$, $\langle g, \#_1^e \rangle \hookrightarrow \langle g, \beta_1 \#_1^e \rangle$, $\langle g, \#_1^u \rangle \hookrightarrow \langle g, \beta_1 \#_1^u \rangle$
7. $\langle a, \varrho \rangle \hookrightarrow \langle a, \varepsilon \rangle$ for every $\varrho \in \Sigma \times (Q \cup \{-\})$
8. $\langle a, \#_1^u \rangle \hookrightarrow \langle g, \beta_1 \#_2^u \rangle$, $\langle a, \#_2^u \rangle \hookrightarrow \langle a, \varepsilon \rangle$, $\langle a, \#_1^e \rangle \hookrightarrow \langle a, \varepsilon \rangle$, $\langle a, \#_2^e \rangle \hookrightarrow \langle a, \varepsilon \rangle$
9. $\langle r, \varrho \rangle \hookrightarrow \langle r, \varepsilon \rangle$ for every $\varrho \in \Sigma \times (Q \cup \{-\})$
10. $\langle r, \#_1^e \rangle \hookrightarrow \langle g, \beta_1 \#_2^e \rangle$, $\langle r, \#_2^e \rangle \hookrightarrow \langle r, \varepsilon \rangle$, $\langle r, \#_1^u \rangle \hookrightarrow \langle r, \varepsilon \rangle$, $\langle r, \#_2^u \rangle \hookrightarrow \langle r, \varepsilon \rangle$
11. $\langle x, \varrho \rangle \hookrightarrow \langle x, \varrho \rangle$ for every control state x and every $\varrho \in \Gamma$.

Hence, the rules are almost the same as in the proof of Theorem 7, except for some changes in 3.,4.,5., and 11. Now we put $\nu(\mathit{fin}) = \{\langle a, \omega w \rangle \mid w \in \Gamma^*\}$. We see that $\mathcal{M}$ accepts w iff there is a run starting in $\langle g, \omega \rangle$ along which *correct* holds and *fin* holds in at least one of its states. In other words, $\mathcal{M}$ accepts w iff $\langle g, \omega \rangle \not\models^\nu (\Box \mathit{correct}) \Longrightarrow (\Box \neg \mathit{fin})$ where ν is the constructed regular valuation. ∎

Observe that model-checking with pushdown systems and any fixed LTL formula whose predicates are interpreted by a *simple* valuation is already *polynomial* (see Theorem 1).

6. CONCLUSION

We have presented two different techniques for checking LTL with regular valuations on pushdown systems. Both techniques rely on a reduction to (and slight modification of) the problem for simple valuations discussed in [6]. Both techniques take linear time and space in $|\mathit{States}|$ where *States* is the set of states of an automaton representing the regular predicates used in the formula. Since both take the same asymptotic time it would be interesting to compare their efficiency in practice (for cases where both techniques can be used).

The solution can be seamlessly combined with the concept of symbolic pushdown systems in [8]. These are used to achieve a succinct representation of Boolean Programs, i.e., programs with (recursive) procedures in which all variables are boolean.

The ability to represent data is a distinct advantage over the approaches hitherto made in our areas of application, namely data-flow analysis [7] and security properties [10]. For the latter, we have indicated that our model is more general. Our approach provides a unifying framework for these applications without losing efficiency. Both techniques take linear time in $|\mathit{States}|$ whereas the methods used in [7] were cubic (though erroneously reported as linear there, too). In [10] no complexity analysis was conducted.

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