Doubly Exponential Runs in Fixed Dimensional VASSes

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- unary 2-VASS in NL (Englert et al. `16)

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Brakes Conjecture II in d=4 and Conjecture III in d=3

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needs 2d dimensions to reach 2^{2^d}

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for $n_i = 2^{i-1}$ for $i \in \{1, \ldots, k\}$

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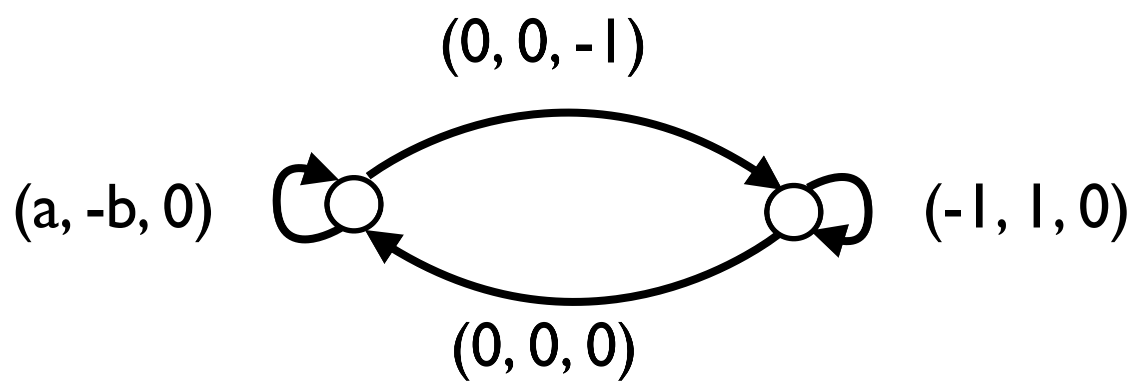

VASS construction

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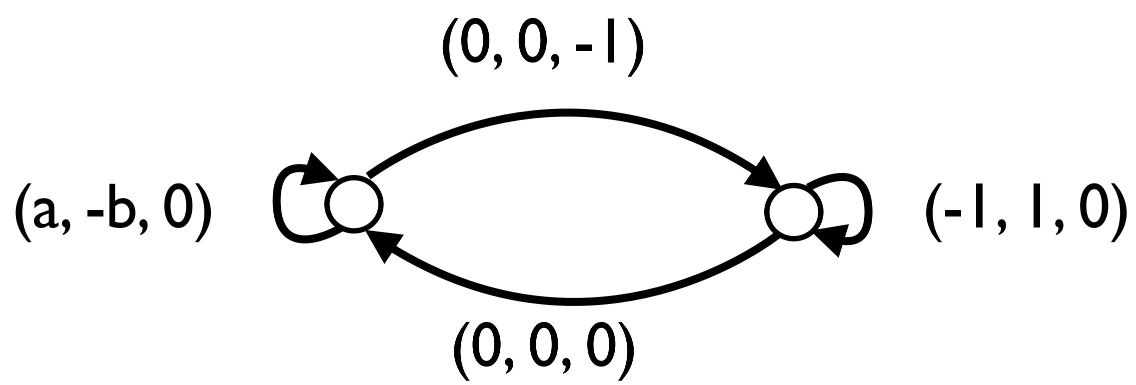

$$(Cb^{i}, 0, i) \Longrightarrow (Cab^{i-1}, 0, i-1) \Longrightarrow \dots \Longrightarrow (Ca^{i}, 0, 0)$$

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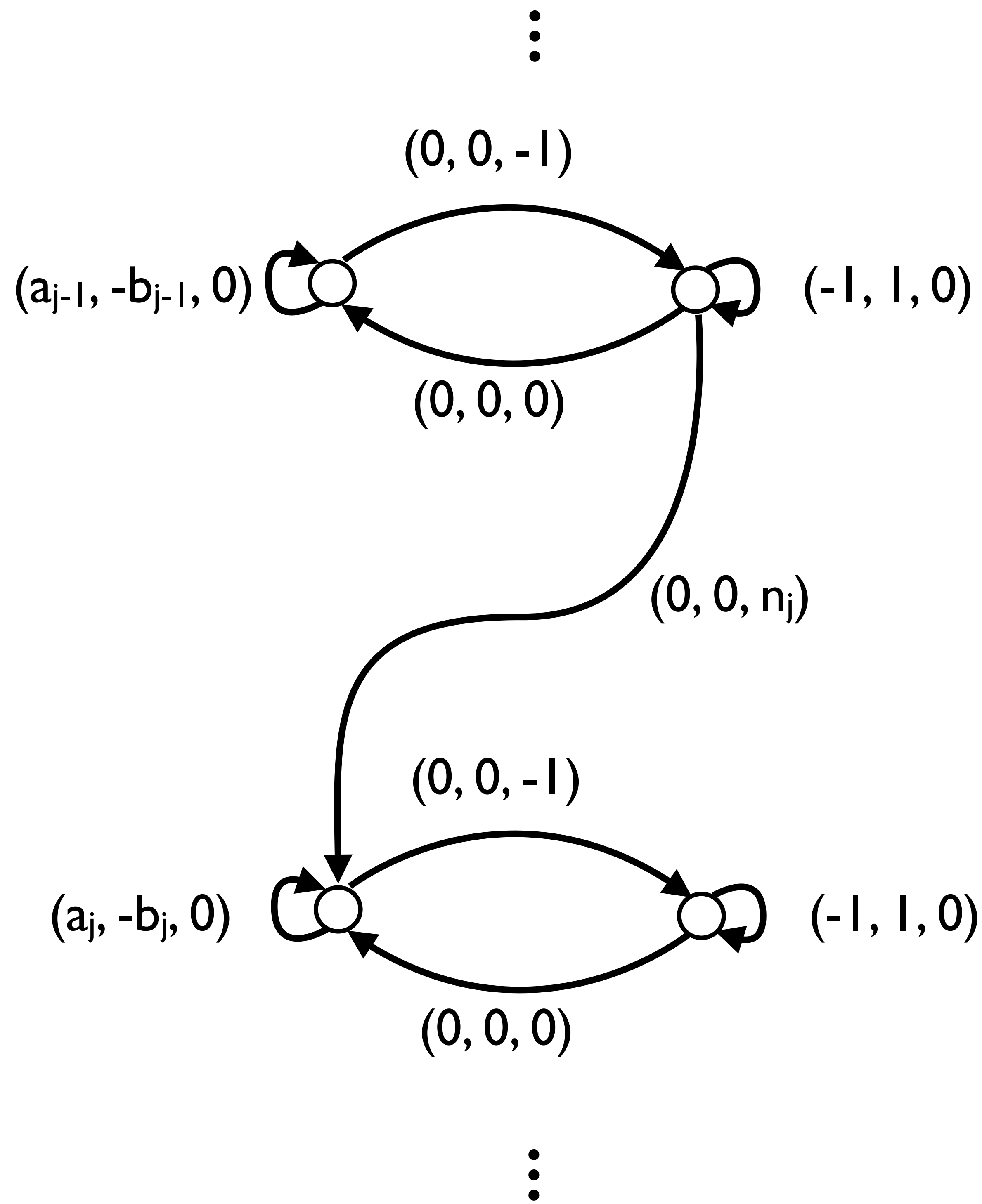

VASS construction

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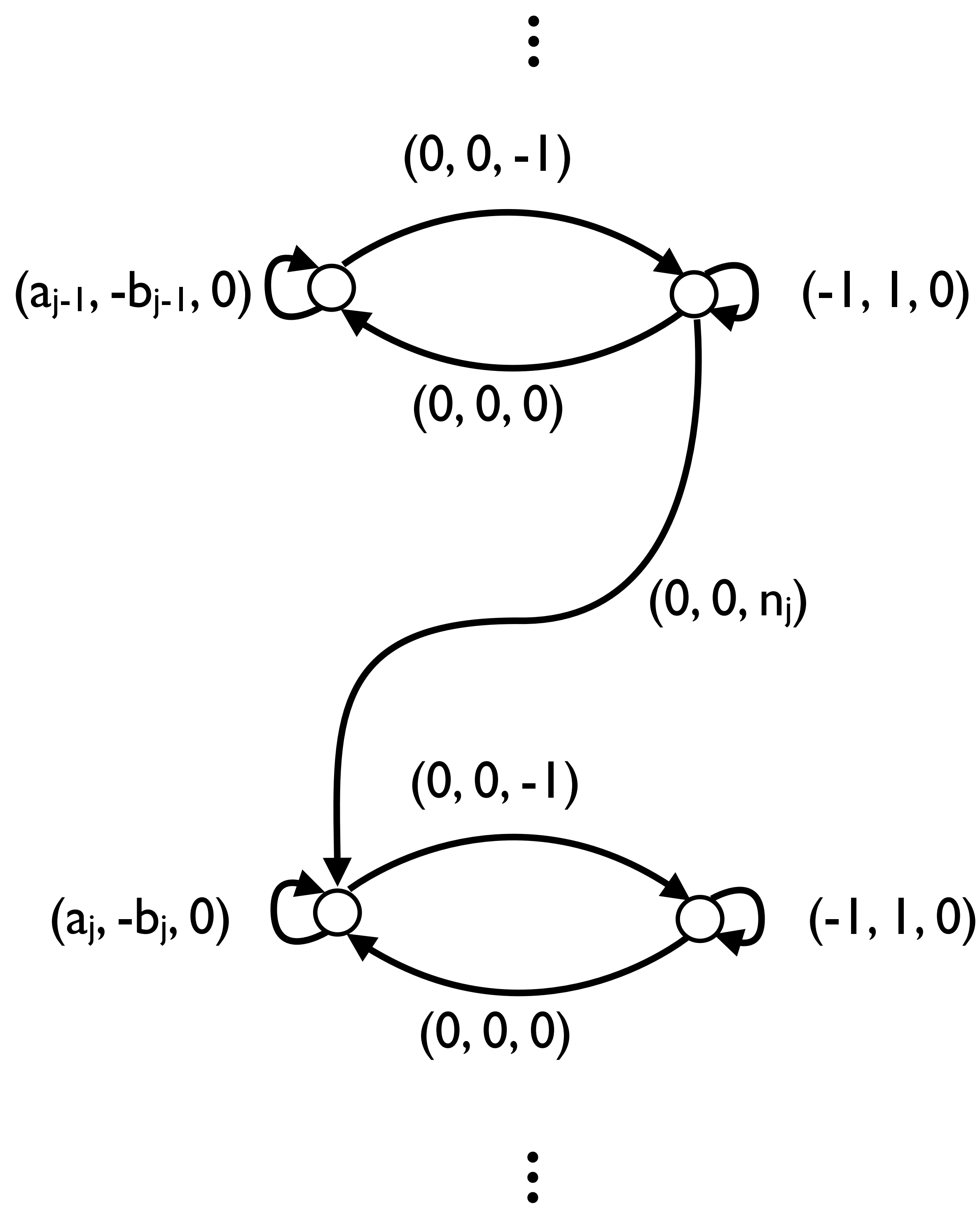

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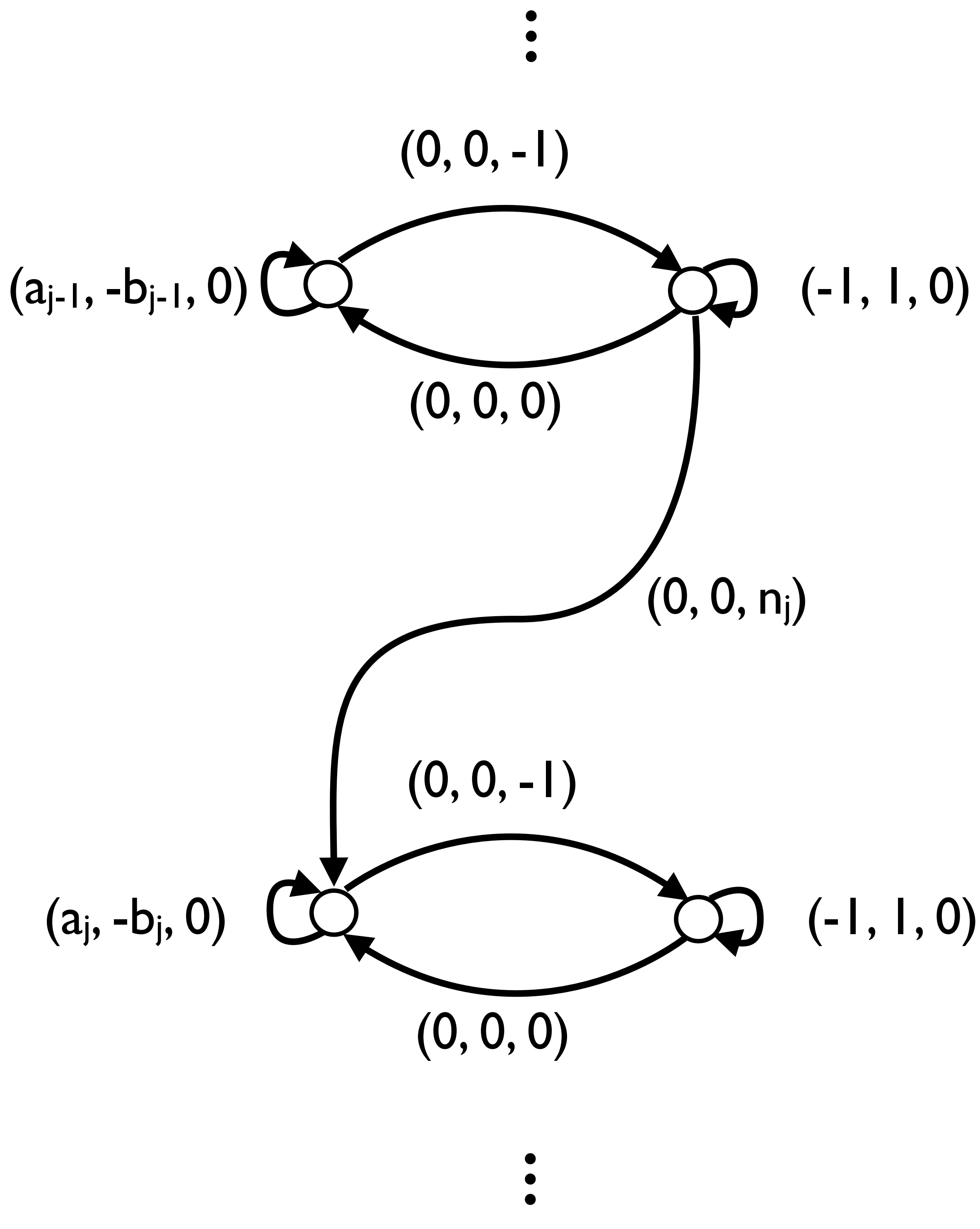

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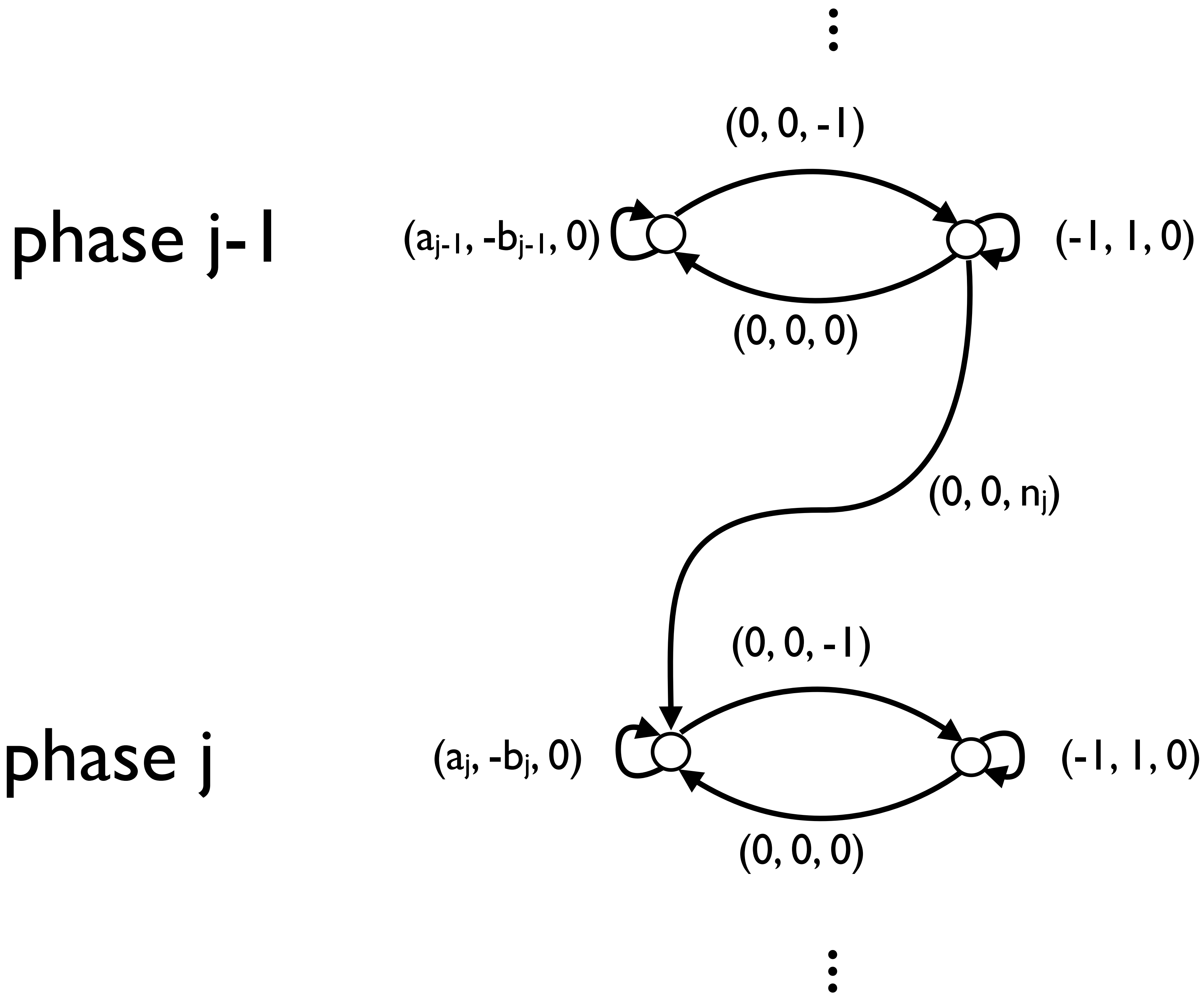

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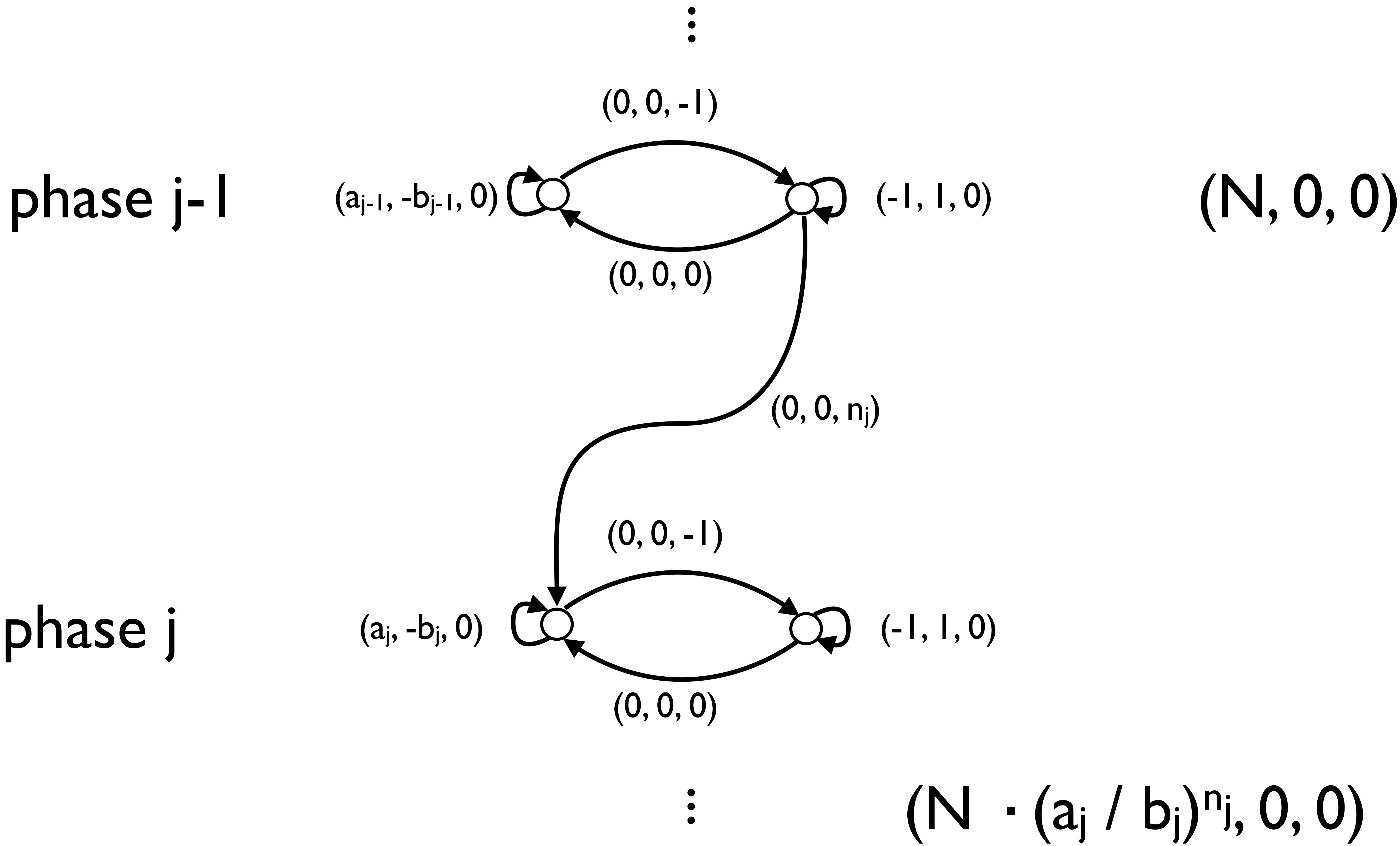

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Doubly-exponential run

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C, c_j, d_j at most exponential in k

Thank you!