QUANTUM COMPUTING 8

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8.QUANTUM CRYPTOGRAPHY

Security of many classical encryption systems is such how secure is their secret key distribution.

Unconditional security of quantum key generation protocols is based on the fact that, on the basis of physical laws, undectable eavesdropping is not possible.

Unconditional security of quantum protocols for classical key generation is based on quantum laws, on the fact that quantum information cannot be copied and cannot be measured without causing detectable disturbances.

Experimentally, secure quantum key distribution has been tested, using polarization or phase of photons, for distance of 150 (200) km using standard optical fibres and for the distance of 22.8 km in the open air of Alps nd for distance 144 km in the open air among Canary Islands.

Earth-to-satelite quantum bit transmissions are considered as feasible. Quantum cryptography is therefore in the advanced experimental and development stage.

FUNDAMENTAL DIFFERENCES

between classical and quantum cryptography

- Security of (public key) classical cryptography is based on unproven assumptions of computational complexity (and it can be jeopardized by progress in algorithms and/or technology).

  Security of quantum cryptography is based on laws of quantum physics that allow to build systems where undetectable eavesdropping is impossible.

- Since classical cryptography is vulnerable to technological improvements it has to be designed in such a way that a secret is secure with respect to **future technology**, during the whole period in which the secrecy is required.

  Quantum key generation, on the other hand, needs to be designed only to be secure against technology available at the moment of key generation.

CLASSICAL CRYPTOGRAPHY — ELEMENTS

Classical cryptography has four main important components:

- Secret-key cryptosystems:

  CESAR, HILL, VIGENER, AFFINE, TRANSPOSITION¡...

  DES, AES.

- Public key cryptosystems:

  RSA, ElGamal, knapsack, and elliptic curves,

- Digital signatures

- Cryptographic protocols.

  Coin-tossing, bit commitment, oblivious transfer,...

  Authentication, anonymity, voting, privacy preservation

CLASSICAL CRYPTOGRAPHY — EXAMPLE

$$e(k, w) = c \qquad\qquad d(k, c) = w$$

encoding key plaintext cryptotext algorithm

decoding algorithm

EXAMPLE

ONE-TIME PAD cryptosystem

$$e(k, w) = k \oplus w = c \qquad\qquad d(k, c) = k \oplus c = w$$

A secret-key cryptosystem is so much secure how much secure is key distribution, provided each time different random key is used.

KEY DISTRIBUTION PROBLEM

The problem of key distribution has plagued cryptographers throughout the history.

During Second World War the German High Command had to distribute each month a new book of day keys to all enigma operators (including submarines), what was an enormous logistical problem.

Around 1970 US Government keys were distributed, several tons every day, by COMSEC

Big banks had special employes that used to travel all the time around the world and to deliver keys, in special padlock briefcases, to everyone who had to get a message next week.

Informatization of society was quaestioned because if governments had problems with key distribution how smaller companies could handle key distribution problem without bankrupting.

Key distribution problem was by almost all **considered to be unsolvable.**

SOLVING UNSOLVABLE PROBLEMS

Key distribution problem was conceptually solved by Whitfield Diffie, Martin Hellman and Ralph Merkle.

Their outcome some consider as the greatest cryptographic achievement since the invention of monoalphabetic cryptosystems, over two thousand years ago.

Diffie realized that whoever could solve key-distribution problem would go down in history as one of the all-time great cryptographers.

Key distribution was main entry in his special notebook entitled "Problems for an Ambitious Theory of Cryptography".

His colleagues had told him he was crazy to do research in cryptography, because he would be competing with the NSA and their multi-bilion-dollar budget.

GREAT IDEA

Let us assume that Alice wants to send her answer to Bob's question whether she will marry him. She can proceed as follows.

- Alice puts her message in an iron box, padlocks it and sends it to Bob.
- Bob adds his own padlock and sends box back to Alice.
- Alice removes her own padlock and sends box to Bob
- Bob removes his padlock and gets the message.

Great, however how to digitilize this great idea?

DIFFICULTIES

Alice's encryption permutation.

```
a b c d e f g h i j k l m n o p q r s t u v w x y z
H F S U G T A K V D E O Y J B P N X W C Q R I M Z L
```

Bob's encryption permutation.

```
a b c d e f g h i j k l m n o p q r s t u v w x y z
C P M G A T N O J E F W I Q B U R Y H X S D Z K L V
```

Message	m	e	e	t	m	e	a	t	n	o	o	n
Alice's encrypt.	Y	G	G	C	Y	G	H	C	J	B	B	J
Bob's encrypt.	L	N	N	M	L	N	O	M	E	P	P	E
Alice's decrypt.	Z	Q	Q	X	Z	Q	L	X	K	P	P	K
Bob's decrypt.	w	n	n	t	w	n	y	t	x	b	b	x

Public Establishment of Secret Keys

Main problem of the classical secret-key cryptography: a need to make a secure distribution (establishment) of secret keys ahead of transmissions.

Diffie+Hellman solved this problem in 1976 by designing a protocol for secure key establishment (distribution) over public channels.

Protocol: If two parties, Alice and Bob, want to create a common secret key, then they first agree, somehow, on large prime p and a primitive root $q \ (\mathrm{mod} p)$ and they perform through public channel the following activities.

- Alice chooses, randomly, a large $1 \leq x < p-1$ and computes $X = q^x \bmod p$.
- Bob also chooses, randomly, a large $1 \leq y < p-1$ and computes $Y = q^y \bmod p$.
- Alice and Bob exchange X and Y, through a public channel, but keep x, y secret.
- Alice computes $Y^x \bmod p$ and Bob computes $X^y \bmod p$ and then each of them knows the key $K = q^{xy} \bmod p$

An eavesdropper seems to need, in order to determine x from X, q, p and y from Y, q, p, to have a capability to compute discrete logarithms, or to compute q^{xy} from q^x and q^y, what is believed to be infeasible.

QUANTUM KEY GENERATION

Quantum protocols for using quantum systems to achieve unconditionally secure generation of secret (classical) keys by two parties are one of the main theoretical achievments of quantum information processing and communication research.

Moreover, experimental systems for implementing such protocols are one of the main successes of the experimental quantum information processing research.

It is believed and hoped that it will be

quantum key generation (QKG)

where one can expect the first

transfer from experimental to developmental stage.

POLARIZATION of PHOTONS

Polarized photons are currently mainly used for experimental quantum key generation.

Photon, or light quantum, is a particle composing light and other forms of electromagnetic radiation.

Photons are electromagnetic waves and their electric and magnetic fields are perpendicular to the direction of propagation and also to each other.

An important property of photons is polarization—it refers to the bias of the electric field in the electromagnetic field of the photon.

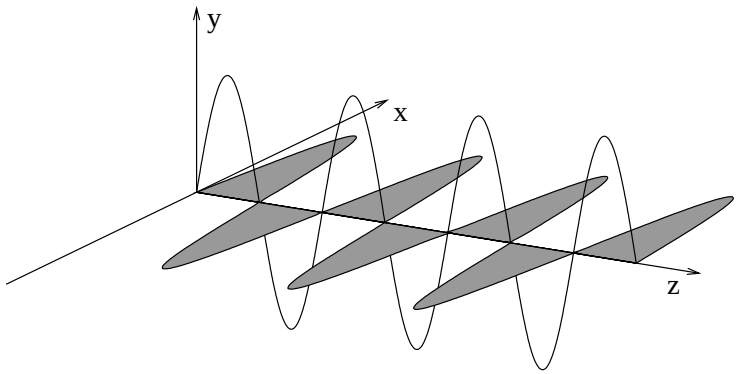


Figure 1: Electric and magnetic fields of a linearly polarized photon

If the electric field vector is always parallel to a fixed line we have **linear polarization** (see Figure).

If the electric field vector rotates about the direction of propagation forming a (right-) left-handed screw, we have (right) left **eliptic polarization** .

PHOTON POLARIZERS

There is no way to determine exactly the polarization of a single photon.

However, for any angle θ there are θ-**polarizers**—"filters"—that produce θ-polarized photons from an incoming stream of photons and they let θ_1-polarized photons to get through with probability $\cos^2(\theta - \theta_1)$.

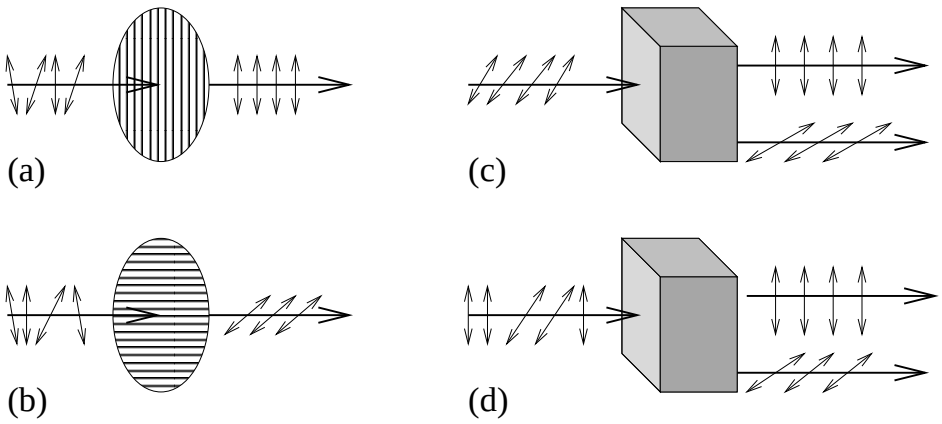


Figure 2: Photon polarizers and measuring devices-80%

Photons whose electronic fields oscillate in a plane at either $0°$ or $90°$ to some reference line are called usually **rectilinearly polarized** and those whose electric field oscillates in a plane at $45°$ or $135°$ as **diagonally polarized**. Polarizers that produce only vertically or horizontally polarized photons are depicted in Figure 3a,b.

Generation of orthogonally polarized photons.

For any two orthogonal polarizations there are generators that produce photons of two given orthogonal polarizations. For example, a calcite crystal, properly oriented, can do the job.

Fig. c — a calcite crystal that makes θ-polarized photons to be horizontally (vertically) polarized with probability $\cos^2\theta$ ($\sin^2\theta$).

Fig. d — a calcite crystal can be used to separate horizontally and vertically polarized photons.

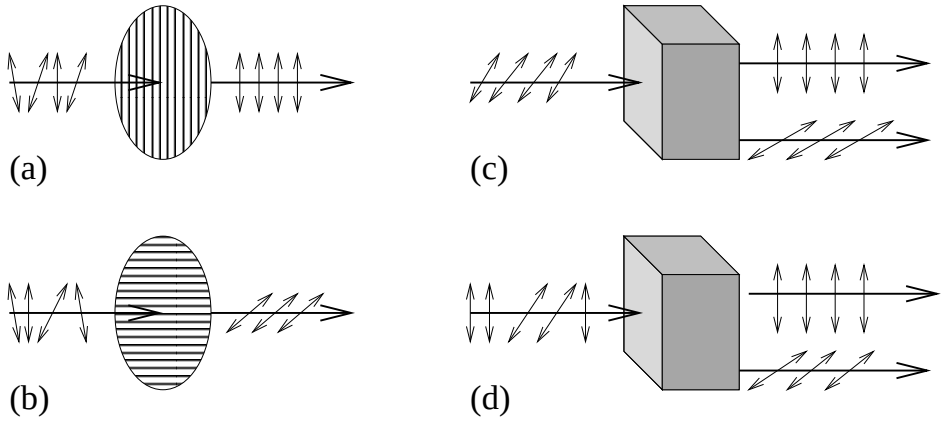


Figure 3: Photon polarizers and measuring devices-80%

QUANTUM KEY GENERATION – PROLOGUE

Quantum cryptography is, similarly as classical cryptography, a continuous fight between good and bad.

Very basic setting. Alice tries to send a quantum system to Bob and an eavesdropper tries to learn, or to change, as much as possible, without being detected.

Eavesdroppers have this time especially hard time, because quantum states cannot be copied and cannot be measured without causing, in general, a disturbance.

Key problem: Alice prepares a quantum system in a specific way, unknown to the eavesdropper, Eve, and sends it to Bob.

The question is how much **information** can Eve extract of that quantum system and how much does it cost in terms of the **disturbance** of the system.

Three special cases

1. Eve has no information about the state $|\psi\rangle$ Alice sends.
2. Eve knows that $|\psi\rangle$ is one of the states of an orthonormal basis $\{|\phi_i\rangle\}_{i=1}^n$.
3. Eve knows that $|\psi\rangle$ is one of the states $|\phi_1\rangle, \ldots, |\phi_n\rangle$ that **are not** mutually orthonormal and that p_i is the probability that $|\psi\rangle = |\phi_i\rangle$.

TRANSMISSION ERRORS INCREASE due to EAVESDROPER

If Alice sends randomly chosen bit

$$0 \text{ encoded randomly as } |0\rangle \text{ or } |0'\rangle$$

or

$$1 \text{ encoded randomly as } |1\rangle \text{ or } |1'\rangle$$

and Bob measures the encoded bit by choosing randomly the standard or the dual basis, then the probability of error is $\frac{1}{4}$

If Eve measures the encoded bit, sent by Alice, according to the randomly chosen basis, standard or dual, then she can learn the bit sent with the probability 75%.

If she then sends the state obtained after the measurement to Bob and he measures it with respect to the standard or dual basis, randomly chosen, then the probability of error for his measurement is $\frac{3}{8}$ – a 50% increase with respect to the case there was no eavesdropping.

Indeed the error is

$$\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \left(\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} \right) = \frac{3}{8}$$

BB84 QUANTUM KEY GENERATION PROTOCOL

Quantum key generation protocol BB84 (due to Bennett and Brassard), for generation of a key of length n, has several phases:

Preparation phase

Alice generates two private random binary sequences of bits of length $m \gg n$ bits and Bob generates one such private random sequence.

Quantum transmission

Alice is assumed to have four transmitters of photons in one of the following four polarizations $0, 45, 90$ and 135 degrees

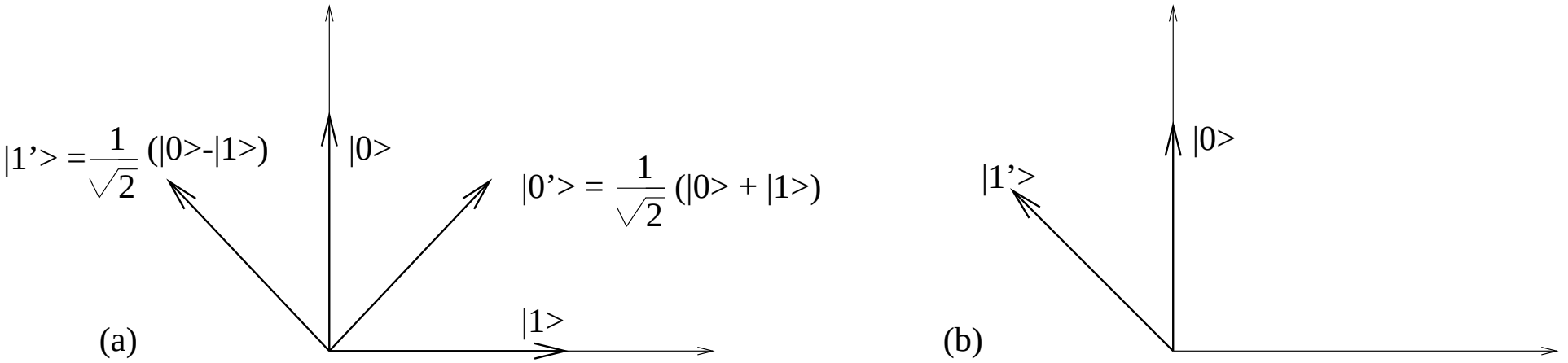


Figure 4: Polarizations of photons for BB84 and B92 protocols

Expressed in a more general form, Alice uses for encoding states from the set $\{|0\rangle, |1\rangle, |0'\rangle, |1'\rangle\}$.)

Bob has a detector that can be set up to distinguish between rectilinear polarizations (0 and 90 degrees) or can be quickly reset to distinguish between diagonal polarizations (45 and 135 degrees).

However, in accordance with the laws of quantum physics, there is no detector that could distinguish between unorthogonal polarizations.

(In a more formal setting, Bob can use either the standard observable $\mathcal{B} = \{|0\rangle, |1\rangle\}$ or the dual observable $\mathcal{D} = \{|0'\rangle, |1'\rangle\}$, to measure the incoming photon.

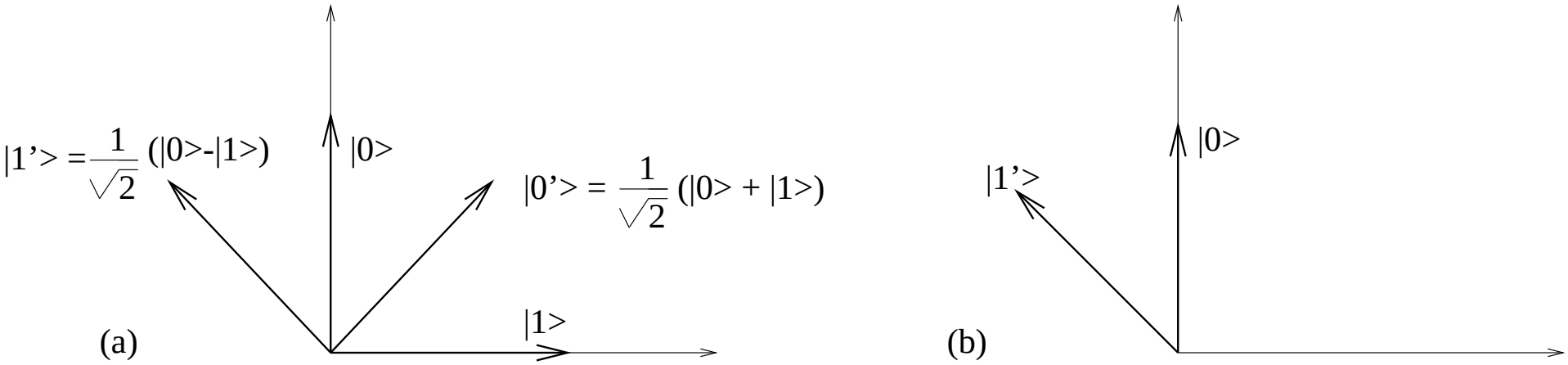


Figure 5: Polarizations of photons for BB84 and B92 protocols

Transmissions

To send a bit 0 (1) of her first random sequence through a quantum channel Alice chooses, on the basis of her second random sequence, one of the encodings $|0\rangle$ or $|0'\rangle$ ($|1\rangle$ or $|1'\rangle$), i.e., in the standard or dual basis,

Bob chooses, each time on the base of his private random sequence, one of the observables $\mathcal{B}$ or $\mathcal{D}$ to measure the photon he is to receive and he records the results of his measurements and keeps them secret.

Alice's encodings	Bob's observables	Alice's state relative to Bob	the result and its probability	correctness
$0 \rightarrow \|0\rangle$	$0 \rightarrow \mathcal{B}$	$\|0\rangle$	0 (prob. 1)	correct
	$1 \rightarrow \mathcal{D}$	$\frac{1}{\sqrt{2}}(\|0'\rangle + \|1'\rangle)$	0/1 (prob. $\frac{1}{2}$)	random
$0 \rightarrow \|0'\rangle$	$0 \rightarrow \mathcal{B}$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	0/1 (prob. $\frac{1}{2}$)	random
	$1 \rightarrow D$	$\|0'\rangle$	0 (prob. 1)	correct
$1 \rightarrow \|1\rangle$	$0 \rightarrow \mathcal{B}$	$\|1\rangle$	1 (prob. 1)	correct
	$1 \rightarrow \mathcal{D}$	$\frac{1}{\sqrt{2}}(\|0'\rangle - \|1'\rangle)$	0/1 (prob. $\frac{1}{2}$)	random
$1 \rightarrow \|1'\rangle$	$0 \rightarrow \mathcal{B}$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	0/1 (prob. $\frac{1}{2}$)	random
	$1 \rightarrow D$	$\|1'\rangle$	1 (prob. 1)	correct

Figure 6: Quantum cryptography with BB84 protocol

Figure 6 shows the possible results of the measurements and their probabilities.

An example of an encoding–decoding process is in the Figure 8.

1	0	0	0	1	1	0	0	0	1	1	Alice's random sequence
$\|1\rangle$	$\|0'\rangle$	$\|0\rangle$	$\|0'\rangle$	$\|1\rangle$	$\|1'\rangle$	$\|0'\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1'\rangle$	Alice's polarizations
0	1	1	1	0	0	1	0	0	1	0	Bob's random sequence
$\mathcal{B}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{B}$	Bob's observable
1	0	R	0	1	R	0	0	0	R	R	outcomes

Figure 7: Quantum transmissions in the BB84 protocol—R stands for the case that the result of the measurement is random

Raw key extraction and tests

Bob makes public the sequence of observables he used to measure the photons he received—but not the results of the measurements —and Alice tells Bob, through a classical channel, in which cases he has chosen the same basis for observable as she did for encoding. The corresponding bits then form the basic **raw key** both parties agree on.

1	0	0	0	1	1	0	0	0	1	1	Alice's random sequence
$\|1\rangle$	$\|0'\rangle$	$\|0\rangle$	$\|0'\rangle$	$\|1\rangle$	$\|1'\rangle$	$\|0'\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1'\rangle$	Alice's polarizations
0	1	1	1	0	0	1	0	0	1	0	Bob's random sequence
$\mathcal{B}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{B}$	Bob's observable
1	0	R	0	1	R	0	0	0	R	R	outcomes

Figure 8: Quantum transmissions in the BB84 protocol—R stands for the case that the result of the measurement is random

Test for evesdropping

Alice and Bob agree on a sequence of indices of the raw key and make the corresponding bits of their raw keys public.

Case 1. Noiseless channel. If the subsequences chosen by Alice and Bob are not completely identical eavesdropping is detected. Otherwise, the remaining bits are taken as creating the final key.

Case 2. Noisy channel. If the subsequences chosen by Alice and Bob contains more errors than the admitable error (that has to be determined from channel characteristics), then eavesdropping is assumed. Otherwise, the remaining bits are taken as the next result of the raw key generation process.

Error correction phase

In the case of a noisy channel for transmission it may happen that Alice and Bob have different keys after the key generation phase.

A way out is that before sending choosen sequence of bits Alice encodes them using some classical error correcting code.

During error correcting phase Alice sends Bob information about encoding and so Bob can use corresponding decoding procedures.

At the end of this stage both Alice and Bob share identical keys.

PRIVACY AMPLIFICATION PHASE

One problem remains. Eve can still have quite a bit of information about the key both Alice and Bob share. Privacy amplification is a tool to deal with such a case.

Privacy amplification is a method how to select a short and very secret binary string s from a longer but less secret string s'.
The main idea is simple. If $|s| = n$, then one picks up n random subsets $S_1, \ldots, S_n$ of bits of s' and let s_i, the ith bit of s, be the parity of S_i. One way to do it is to take a random binary matrix of size $|s| \times |s'|$ and to perform multiplication Ms'^T, where s'^T is the binary column vector corresponding to s'.

The point is that even in the case where an eavesdropper knows quite a few bits of s', she will have almost no information about s.

B92 PROTOCOL

A simpler protocol for quantum key generation, called B92, has been developed by Ch. Bennett in 1992.

Protocol: Alice uses for encoding of randomly chosen bit-sequence some two non-orthogonal states and encoding is deterministic. Bob uses for decoding/measurement two noncommutative measurements.

Example of an encoding/decoding procedure

Alice's bit	Alice's encoding	Bob's bit	Bob's test for	Test's result and probability	Correctness
0	$\|0\rangle$	0	$\|0'\rangle$	Yes/No (prob. $\frac{1}{2}$)	random
		1	$\|1\rangle$	No (prob. 1)	correct
1	$\|1'\rangle$	0	$\|0'\rangle$	No (prob. 1)	correct
		1	$\|1\rangle$	Yes/No (prob. $\frac{1}{2}$)	random

Figure 9: Encodings/decodings with B92 protocol

Example

1	0	0	0	1	1	0	0	0	1	1	Alice's random seq.
$\|1'\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1'\rangle$	$\|1'\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1'\rangle$	$\|1'\rangle$	Alice's polarizations
1	0	0	0	1	0	0	1	1	0	1	Bob's random sequence
$!0'\rangle$	$!0'\rangle$	$\|1\rangle$	$!1\rangle$	$\|0'\rangle$	$!1\rangle$	$\|1\rangle$	$\|0'\rangle$	$!0'\rangle$	$\|1\rangle$	$\|0'\rangle$	Bob's test for
No	R	No	No	No	R	No	R	R	R	No	outcomes of test
1	R	0	0	1	R	0	R	R	R	1	resulting Bob's bit

Figure 10: Quantum transmissions within B92 protocol

B92 protocol is also called *minimal QKG protocol.*

SECURITY of B92 PROTOCOL

We show that Eve with an undetectable probe cannot obtain any information from B92 protocol.

Let $|\phi\rangle$ and $|\psi\rangle$ be two non-orthogonal states used in B92 protocol. Thus

$$\langle\phi|\psi\rangle \neq 0$$

and let U be the unitary performed by Eve's detection probe, to be initially in state $|\Psi\rangle$.

Since Eve's probe is undetectable, we have

$$|\Psi\rangle|\phi\rangle \rightarrow U|\Psi\rangle|\phi\rangle = |\Psi'\rangle|\phi\rangle$$

and

$$|\Psi\rangle|\psi\rangle \rightarrow U|\Psi\rangle|\psi\rangle = |\Psi''\rangle|\psi\rangle$$

where $|\Psi'\rangle$ and $|\Psi''\rangle$ denote the states of Eve's probe after the detection of $|\phi\rangle$ and $|\psi\rangle$, respectively.

Note that since Eve is undetectable her probe has no effect on the states $|\phi\rangle$ and $|\psi\rangle$.

Therefore $|\phi\rangle$ appears on both sides of the first equation above and $|\psi\rangle$ appears on both sides of the second equation.

Thus

$$\langle\langle\phi|\langle\Psi|U^*|U|\Psi\rangle|\psi\rangle\rangle = \langle\phi|\langle\Psi|\Psi\rangle|\psi\rangle = \langle\phi|\psi\rangle$$

because of the unitarity of U and because $\langle\Psi|\Psi\rangle = 1$

In addition to

$$\langle\langle\phi|\langle\Psi|U^*|U|\Psi\rangle|\psi\rangle\rangle = \langle\phi|\langle\Psi|\Psi\rangle|\psi\rangle = \langle\phi|\psi\rangle$$

we have

$$\langle\phi|\langle\Psi'|\Psi''\rangle|\psi\rangle = \langle\Psi'|\Psi''\rangle\langle\phi|\psi\rangle.$$

As the result we have

$$\langle\phi|\psi\rangle = \langle\Psi'|\Psi''\rangle\langle\phi|\psi\rangle.$$

However,

$$\langle\phi|\psi\rangle \neq 0$$

implies that

$$\langle\Psi'|\Psi''\rangle = 1.$$

since $|\Psi'\rangle$ and $|\Psi''\rangle$ are normalized, this implies

$$|\Psi'\rangle = |\Psi''\rangle.$$

This implies that Eve's probe is in the same state no matter which of the states $|\phi\rangle$ and $|\psi\rangle$ is received.

Thus Eve obtains no information whatsoever.

ANOTHER VERSION of B92 PROTOCOL

- Alice keeps sending n times a quantum system in one of two randomly chosen given non-orthogonal states $|\phi\rangle$ or $|\psi\rangle$.
- Bob measures the received system randomly either in $\{|\phi\rangle, |\phi^{\perp}\rangle\}$ basis or in $\{|\psi\rangle, |\psi^{\perp}\rangle\}$ basis.
- Bob let Alice to know in which cases he got 100% clear outcome (if measurement outcomes is $|\phi^{\perp}\rangle$ or $|\psi^{\perp}\rangle$) and they discard all other outcomes.
- The rest of the protocol is as before.

DRAWBACKS of B92 PROTOCOL

B92 protocol has the following serious drawbacks.

The eavesdropper may use so-called POVM measurement to get one of the results

$$|\phi\rangle, |\psi\rangle, \textit{ don't know}$$

This means that an eavesdropper can be sure that when the result of measurement is $|\phi\rangle, |\psi\rangle$ then it corresponds to the state that was sent and therefore eavesdropping introduces no noise.

If the result of measurement is *don't know* then eavesdropping would introduce a noise. However, in this case the eavesdropper can discard the quantum system sent by Alice so that Bob receives nothing.

It is therefore of large importance to follow the number of missing quantum systems.

QUANTUM ATTACKS

Individual or intercept-resent attacks. Each quantum signal is first measured by Eve and then resent.

Coherent or **joint attacks**. Instead of measuring the particles while they are in transit from Alice to Bob, one–by–one, Eve regards all the transmitted particles as a single entity. She then couples this entity with a simple auxiliary system (ancilla), prepared in a special state, and creates the compound system. Afterwards, she sends the particles to Bob and keeps the ancilla.

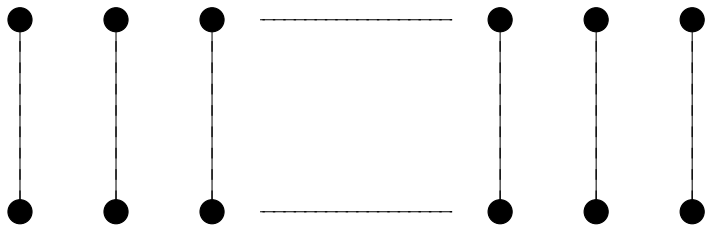
After the end of the public interactions between Alice and Bob (for error detection, error correction and privacy amplification), Eve extracts from her ancilla some information about the key. Such attack are directed against the final key. They represent the most general type of attacks that is possible. (However, no particular attack of this type has been suggested so far.)

Trojan horse attack. This refers to the possibility that an entanglement with environment can "open the door" and let a "Trojan horse to get in to gather information about quantum communications in the protocol"..

EPR METHOD for QUANTUM KEY GENERATION

Let Alice and Bob share n pairs of particles in the entangled state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$



n pairs of particles in EPR state

QUANTUM ONE-TIME PAD CRYPTOSYSTEM

CLASSICAL ONE-TIME PAD cryptosystem

plaintext: an n-bit string p

shared key: an n-bit string k

cryptotext: an n-bit string c

encoding: $c = p \oplus k$

decoding: $p = c \oplus k$

QUANTUM ONE-TIME PAD cryptosystem:

plaintext: an n-qubit string $|p\rangle = |p_1\rangle \ldots |p_n\rangle$

shared key: two n-bit strings k, k'

cryptotext: an n-qubit string $|c\rangle = |c_1\rangle \ldots |c_n\rangle$

encoding: $|c_i\rangle = \sigma_x^{k_i}\sigma_z^{k'_i}|p_i\rangle$

decoding: $|p_i\rangle = \sigma_z^{k'_i}\sigma_x^{k_i}|c_i\rangle$ where $|p_i\rangle = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$ and $|c_i\rangle = \begin{pmatrix} d_i \\ e_i \end{pmatrix}$ are qubits and

$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $\sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices.

SHANNON THEOREMS

Shannon theorem says that n bits are necessary and sufficient to encrypt securely n bits.

Quantum version of Shannon theorem says that $2n$ classical bits are necessary and sufficient to encrypt securely n qubits.

EKERT's QKG PROTOCOL

Ekert developed a 3-state protocol that uses the so-called Bell's inequalities to detect the presence, or absence, of an eavesdropper (as a so-called hidden variable).

Example — in terms of photon polarizations.

A randomly choosen sequence of pairs of photons in one of the following states

$$|\phi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\frac{3\pi}{6}\rangle - |\frac{3\pi}{6}\rangle|0\rangle)$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|\frac{\pi}{6}\rangle|\frac{4\pi}{6}\rangle - |\frac{4\pi}{6}\rangle|\frac{\pi}{6}\rangle)$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|\frac{2\pi}{6}\rangle|\frac{5\pi}{6}\rangle - |\frac{5\pi}{6}\rangle|\frac{2\pi}{6}\rangle)$$

is sent to Alice and Bob, with one photon of each pair to Alice and second to Bob.

Both Alice and Bob measure their particles by randomly choosing one of the observables

$$\mathcal{O}_1 = \{|0\rangle, |\frac{3\pi}{6}\rangle\} \quad \mathcal{O}_2 = \{|\frac{\pi}{6}\rangle, |\frac{4\pi}{6}\rangle\} \quad \mathcal{O}_3 = \{|\frac{2\pi}{6}\rangle, |\frac{5\pi}{6}\rangle\}$$

and record bit-outcomes of the measurement.

Using a communication via a public channel they determine the cases they used the same mesurement. The corresponding bits form the raw key.

The two remaining subsequences of bits , one at Alice another at Bob, form the so-called rejected key which is then used to detect eavesdropping, using the method discussed next.

BELL INEQUALITIES I.

A special type of Bell inequalities is a tool to detect the presence of an eavesdropper for the Ekert-type QKG protocols.

Einstein believed that quantum mechanics is incomplete and that using some *hidden variables* one could develop a complete quantum theory without nonlocal influences, where one could believe in the existence of an objective reality for quantum phenomena.

A Gedanken experiment by Bell (1964) and a physical experiment by Aspen (1982) demonstrated that the above Einstein's idea does not work.

BELL INEQUALITIES II.

Let a pair of electrons whose spins add up to zero is created and sent off in two directions. Let us have on each path a switch and two Stern-Gerlach magnets, set at different angles.

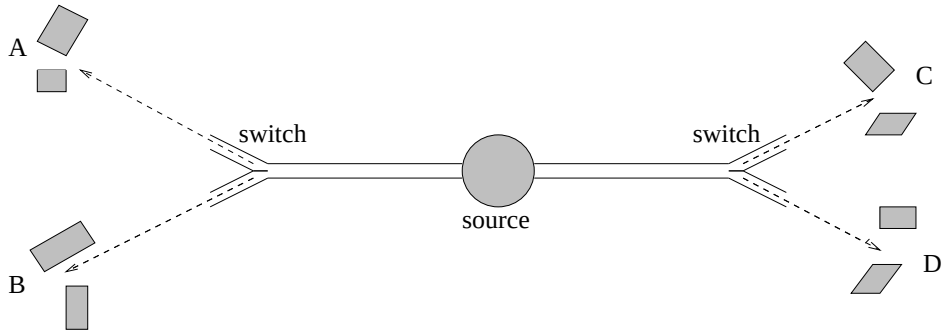


Figure 11: Aspect's experiment

For $Y \in \{A, B, C, D\}$ let $Y = 1$ or $Y = -1$ denote in which of two posible ways an electron gets out. At each particular experiment one of two variables A and B and one of C and D gets one of the values $1, -1$ and

$$X = C(A + B) + D(A - B)$$

can take on one of the values $2, -2$. Therefore, in case of many experiments

$$-2 \leq EX \leq 2.$$

Bell showed that quantum mechanics implies that EX can be up to $2\sqrt{2}$ and Aspect's experiment confirmed Bell's expectations.

WHAT CAN a BAD EVE DO?

Eve has no chance to get some information about the key from the particles while they are in transit because there is no information there.

Eve has two possibilities for distruction:

- To measure one or both particles on their way to Alice/Bob and this way to prevent Alice/Bob to share a common key.
- To substitute her own carefully prepared particles for those generated by the source.

Bell's inequalities for the Ekert protocol

Denote

$$Pr(\neq, i, j)$$

the probability that corresponding bits of Alice's and Bob's rejected sequences are different if Alice measures with observable $\mathcal{O}_i$ and Bob with observable $\mathcal{O}_j$. Moreover,

$$Pr(=, i, j) = 1 - Pr(\neq, i, j)$$
$$\Delta(i, j) = Pr(\neq, i, j) - Pr(=, i, j)$$

and

$$\beta = 1 + \Delta(2, 3) - |\Delta(1, 2) - \Delta(1, 3)|.$$

The Bell inequality for this protocol is $\beta \geq 0$ but quantum mechanics (without hidden variables) implies

$$\beta = -\frac{1}{2}$$

By measuring β one can therefore determine the presence or absence of an eavesdropping.

QUANTUM CRYPTOGRAPHIC PROTOCOLS

A variety of quantum cryptographic protocols have already been developed for basic cryptographic tasks. Some of them will now be presented.

The key issue is whether for basic cryptographic tasks there exist unconditionally secure quantum cryptographic protocols.

BASIC CLASSICAL CRYPTOGRAPIC PROTOCOLS

Cryptographic protocols are specifications how two parties, Alice and Bob, should prepare themselves for a communication and how they should behave during a communication in order to achieve their goal and be protected against an adversary.

In **coin-flipping protocols** Alice and Bob can flip a coin over a distance in such a way that neither of them can determine the outcome of the flip, but both can agree on the outcome in spite of the fact that they do not trust each other.

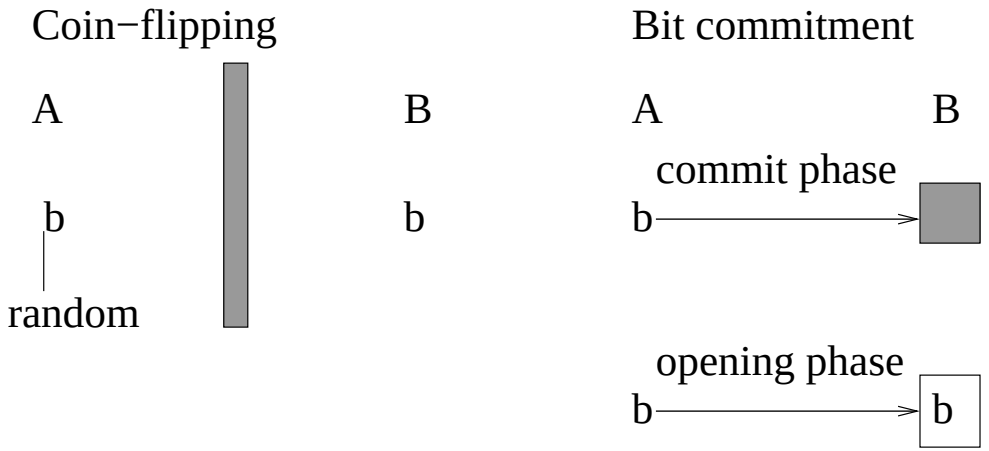
In **bit commitment protocols** Alice can choose a bit and get committed to it in the following sense: Bob has no way of learning Alice's commitment and Alice has no way of changing her commitment.

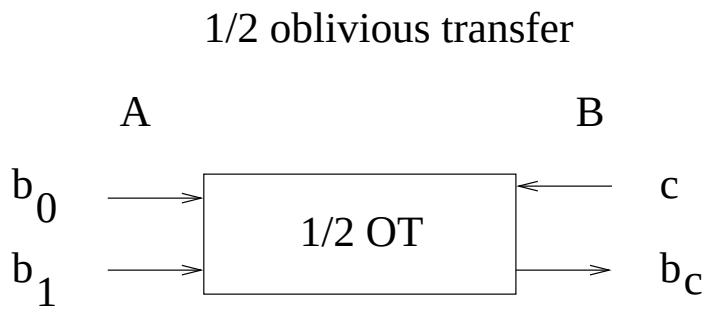
Alice commits herself to a bit x using a *commit*(x) procedure, and reveals her commitment, if needed, using *open*(x) procedure.

In **1-out-2 oblivius transfer protocols** Alice transmits two messages m_1 and m_2 to Bob who can chose whether to receive m_1 or m_2, but cannot learn both, and Alice has no idea which of them Bob has received.

In **standard oblivious transfer protocols** Alice can send a message to Bob in such a way that Bob receives the message with probability $\frac{1}{2}$ and a garbage with probability $\frac{1}{2}$. Moreover, at the end Bob knows whether he got a message or a garbage, but Alice has no idea which of them Bob has received.

PRIMITIVES of CRYPTOGRAPHIC PROTOCOLS





CLASSICAL COIN-FLIPPING (BY PHONE) PROTOCOL

The history of cryptographic protocols started with the following Blum's coin-flipping protocol (1981):

Protocol 0.1 (Coin-flipping by telephone)

1. *Alice chooses two large primes p, q, sends Bob $n = pq$, and keeps p, q secret.*
2. *Bob chooses a random number $y \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and sends Alice $x = y^2 \bmod n$.*
3. *Alice computes four square roots $(x_1, n - x_1)$ and $(x_2, n - x_2)$ of x. (Alice can compute them because she knows p and q.)*

   Let $x'_1 = \min\{x_1, n - x_1\}$, $x'_2 = \min\{x_2, n - x_2\}$. Since $y \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor)\}$, either $y = x'_1$ or $y = x'_2$.

   Alice then guesses whether $y = x'_1$ or $y = x'_2$ and tells Bob her choice (for example, by reporting the position and the value of the leftmost bit in which x'_1 and x'_2 differ).
4. *Bob tells Alice whether her guess was correct (head) or not correct (tail).*

Later, if necessary, Alice can reveal p and q, and Bob can reveal y.

QUANTUM COIN-FLIPPING PROTOCOL

Alice's random bits	1	0	1	1	0	1	1	0	0	1	1
Alice polarization choice	rectilinear										
photons sent	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
Bob's observable	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{D}$
Bob's table for $\mathcal{B}$	1			1			1		0	1	
Bob's table for $\mathcal{D}$		1	1					0			0
Bob's guess of Alice's pol.	rectilinear										
Alice's message	you WON										
Alice's original bits	1	0	1	1	0	1	1	0	0	1	1
comparison with $\mathcal{B}$	Y			Y			Y		Y	Y	
comparison with $\mathcal{D}$		N	Y					Y			N

Figure 12: Illustration of a quantum coin-flipping protocol

1. Alice randomly chooses a sequence of bits (for example 1000) and a polarization (rectilinear or diagonal—standard or dual). Finally, Alice sends the resulting sequence of the polarized photons to Bob.

2. Bob chooses, for each received photon, randomly, an observable, $\mathcal{B}$ or $\mathcal{D}$, and measures the incoming photon. He records the result into two tables—one for observable $\mathcal{B}$ and the second for observable $\mathcal{D}$. Since some photons can get lost during the transmissions, there can be holes in both tables. At the end of all transmissions, Bob makes a guess whether Alice chosed rectilinear or diagonal polarization and announces his guess to Alice. He is to win if the guess is correct; to lose otherwise.

3. Alice tells Bob whether he won or lost by telling him the polarization she choosed. She can certify her claim by sending Bob the random sequence of bits she choosed at Step 1.

4. Bob verifies Alice's claim by comparing his records in the table for the basis she claims to choose. There should be a perfect agreement with the entries in that table and no correlation with the other table.

Alice's random bits	1	0	1	1	0	1	1	0	0	1	1
Alice polarization choice						rectilinear					
photons sent	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
Bob's observable	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{D}$
Bob's table for $\mathcal{B}$	1			1			1		0	1	
Bob's table for $\mathcal{D}$		1	1					0			0
Bob's guess of Alice's pol.						rectilinear					
Alice's message						you WON					
Alice's original bits	1	0	1	1	0	1	1	0	0	1	1
comparison with $\mathcal{B}$	Y			Y			Y		Y	Y	
comparison with $\mathcal{D}$		N	Y					Y			N

Figure 13: Illustration of a quantum coin-flipping protocol

CAN BOB OR ALICE CHEAT?

1. Bob is not able to cheat.

   Bob would be able to "cheat" only would he be able to guess with probability $> \frac{1}{2}$, on the base of photons he received, which basis (polarization) Alice has chosen (what contradicts physical laws).

2. Alice could potentially cheat only in Step 1 or in Step 3.

   - **Alice cannot cheat in Step 3**

     In order to cheat she would need to send sequence of bits matching the entries of Bob's table for two possible bases (polarizarions). The probability she keeps making correct guess about Bob's measurement goes fast to 0.

- **Alice CAN CHEAT at Step 1 by making a clever use of entanglement!**

  In Step 1, instead of sending a sequence of isolated photons, polarized in one of two ways, she produces pairs of photons, each in the state

  $$\frac{1}{2}(|01\rangle + |10\rangle),$$

  she sends to Bob one photon of each pair and stores the other one.

  After Bob's guess, in Step 2, she measures her photon in the opposite basis as was Bob's guess. By that she receives a sequence of bits perfectly correlated with Bob's table corresponding to the basis he did not choose as his guess in Step 2. Alice then announces her sequence in Step 3.

CLASSICAL BIT COMMITMENT with ONE-WAY FUNCTION

Commitment phase: • Alice and Bob choose a one-way function f;

• Bob sends a randomly chosen r_1 to Alice;

• Alice chooses random r_2 and her committed bit b and sends to Bob $f(r_1, r_2, b)$.

Opening phase: • Alice sends to Bob r_2 and b;

• Bob computes $f(r_1, r_2, b)$ and compares with the value he has already received.

CLASSICAL BIT COMMITMENT with SYMMETRIC CIPHER

Commitment phase: • Alice and Bob choose a symmetric cryptosystem with an encryption algorithm e_k;

• Bob sends Alice a randomly chosen string r;

• ALice chooses a random key k, a commitment b and sends to Bob $e_k(rb)$

Opening phase: • Alice sends to Bob the key k;

• Bob performs decryption and verifies b and r.

Comment: The role of r chosen by Bob is to make infeasible for Alice to find two keys k_1 and k_2 such that $e_{k_1}(r0) = e_{k_2}(r1)$.

BENNETT-BRASSARD BIT COMMITMENT

Commit phase: **Input** b

- Alice chooses a binary random vector $\mathbf{r} = (r_1, \ldots, r_n)$.
- Bob chooses a binary random vector $\mathbf{s} = (s_1, \ldots, s_n)$
- **for** $i = 1$ **to** n Alice sends to Bob a quantum system in the state $H^b|r_i\rangle$.
- Bob measures the system he obtains in the basis $\{H^{s_i}|0\rangle, H^{s_i}|1\rangle\}$ and sets z_i to 0 (to 1) if the result of the measurement is $|0\rangle$ or $|0'\rangle$ (is $|1\rangle$ or $|1'\rangle$).

Opening phase

- Alice sends the commitment b and the vector $\mathbf{r}$ to Bob
- If there is an i such that $b = s_i$ and $r_i \neq z_i$, then Bob rejects; otherwise it accepts b as correct commitment.

QUANTUM GAMBLING

The question whether there is (unconditionally) secure quantum coin tossing protocol remains open.

The existence of unconditionally secure protocol has been shown for the so-called **quantum gambling** problem.

Quantum gambling — the rules of gambling

Alice and Bob are gamblers. Alice has a particle and two boxes, A and B, to store the particle. Let $|a\rangle$ ($|b\rangle$) denote the quantum state for the particle to be in the box A (B).

Step 1 Alice prepares the particle in some state $|\psi\rangle$ and sends the box B to Bob.

Step 2 Bob has two choices:

1. To open the box B. If he finds there the particle, Alice should pay him a coin; otherwise Bob pays Alice a coin.
2. To ask Alice to send him also the box A. Bob then checks whether

$$|\psi\rangle = |\psi_0\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle).$$

   If yes, Bob pays Alice a coin; otherwise Alice pays him R coins. (R is here a "security parameter". For the case $R = 1$ we have a coin tossing protocol.)

Quantum gambling — a protocol

Alice's strategy: To prepare $|\psi\rangle$ as $|\psi_0\rangle$ — as defined above.

Bob's strategy: After receiving the box B, he performs the unitary mapping

$$|b\rangle \rightarrow \sqrt{1-\theta(R)}|b\rangle + \sqrt{\theta(R)}|b'\rangle = |\psi'\rangle,$$

where $\langle b \,|\, b' \rangle = 0$ and $\theta(R)$ is a "splitting parameter", to be specified later.

Afterwards, Bob performs the projection measurement on $|\psi'\rangle$, with respect to the basis $\{|b\rangle, |b'\rangle\}$. If he finds, by doing that, that the particle is in the box B, he then asks Alice to pay him a coin; otherwise he asks her for the box A and verifies whether she prepared the state $|\psi_0\rangle$. If yes, he pays a coin; otherwise Alice pays him R coins.

Quantum gambling — security

Since this is a zero-sum game, and an expectation value of Alice's gain is clearly nonnegative (independently of Bob's strategy), the expectation value of Bob's gain, G_B, has to be nonpositive (independently of Alice's strategy).

It has been shown that at a proper choice of $\theta(R)$ (roughly $\frac{1}{\sqrt{2R}}$), $G_B \geq \delta(R)$, (with $\delta(R)$ approximately $-\sqrt{\frac{2}{R}}$), where $\lim_{R\rightarrow\infty} \delta(R) = 0$.

QUANTUM BIT COMMITMENT PROTOCOLS

In 1993, Brassard, Crépeau, Jozsa and Langlois develop a quantum bit commitment protocol, the so-called BCJL-protocol and provided a proof that it is unconditionally secure.

In 1995 a flaw in the proof of unconditional security of the BCJL-protocol was discovered.

In 1996 two proofs were given that there is no unconditionally secure quantum bit commitment protocol.

The point is that a variant of cheating strategy demonstrated for our quantum coin tossing protocol, a clever misuse of entanglement by Alice, can be always used, in a modified form.

The fact that there is no unconditionally secure quantum bit protocol can be seen as a bad news because bit commitment is a primitive on which many cryptographic protocols can be built.

BCJL — PROTOCOL

Let $\varepsilon > 0$ (be an upper bound on the error rate of the quantum channel being used).

Protocol 0.2 (commit (x))

1. Bob chooses a generator matrix G of a binary linear (n, k, d)-code C such that $\frac{d}{n} > 10\varepsilon$ and $\frac{k}{n} = 0.52$ and announces G to Alice.
2. Alice chooses:
   2.1. a random string r of length n and announces it to Bob;
   2.2. a random k-bit vector s, such that $r \cdot c = x$, where $c = sG$;
   2.3. a random sequence b of length n of the polarizations, $\mathcal{B}$ or $\mathcal{D}$, and sends to Bob the sequence c of bits through a sequence of n photons with the polarization of the ith photon $P_{b_i}(c_i)$, where $P_0(0) = 0°$, $P_0(1) = 90°$ and $P_1(0) = 45°$, $P_1(1) = 135°$.
3. Bob chooses a random string b' of n bits and measures the ith photon, containg encoding of c_i, according to the basis $M(b'_i)$, where $M(0) = \mathcal{B}$ and $M(1) = \mathcal{D}$. Let c' be the n-bit vector where c'_i is the result of the measurement of the ith photon.

Alice keeps the bit x and vectors c and b secret, until the opening takes place, and Bob keeps vectors b' and c' secret.

Protocol 0.3 (open (c, b, x, c', b'))

1. Alice sends vectors c, b and bit x to Bob.
2. Bob verifies that c is a codeword of C and computes $B = \Sigma_{\{i \,|\, b'_i = b_i\}} \frac{c_i \oplus c'_i}{n/2}$, in order to verify that the error rate is under the limit of those pairs of outgoing and measured bits that were polarized/measured by the same basis.
3. if $B < 1.4\varepsilon$ and $x = r \cdot c$, then Bob accepts, otherwise Bob rejects.

QUANTUM OBLIVIOUS TRANSFER PROTOCOLS I

It is easy to design a simple and perfectly secure QOTP provided transmissions and detectors are perfect and no party cheats.

Protocol 0.4 (Ideal one-photon standard QOTP)

1. Alice chooses a bit b and sends it to Bob through one photon encoded using a randomly chosen basis—standard or dual.
2. Bob measures the photon with respect to a randomly chosen basis—standard or dual.
3. Alice lets Bob know the basis she chosed.

At the end Bob has a 50% chance to know b for sure and he knows whether he knows b for sure. Alice has no information whether Bob knows the bit for sure.

Problems with this protocol.

1. Imperfect devices (Alice's source, a noisy channel, or Bob's detector) could much affect the probability of success of Bob's measurement.

2. Bob could "cheat" by making his measurement in the so called Breidbart basis

$$\mathcal{B}_0 = \{\theta_0, \theta_1\},$$

where

$$\theta_0 = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle; \quad \theta_1 = -\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle$$

This way he could learn b with large probability $\cos^2\frac{\pi}{8} \approx 0.85$.

1-out-of-2 QOTP

Parameter agreeing phase

Alice and Bob find out, or agree during their communication, on the following parameters:

ε — the expected error rate of the communication channel;

α — the fraction of photons Bob is able to detect successfully;

n — the security parameter; number of photons to be transmitted;

C — a binary linear error-correcting code capable of correcting, well, n-bit words transmitted with the expected error rate ε.

Transmission phase

Alice chooses a random binary string of length $\frac{2n}{\alpha}$ and sends each of the bits through a polarized photon using randomly either the standard or dual polarization basis.

Measurement phase

Bob measures each incoming photon by a randomly chosen basis ($\mathcal{B}$ or $\mathcal{D}$) and records the basis chosen and the results of the measurements into tables.

Bob expects to receive $2n$ photons. If he gets more, he ignores additional bits. if less, he adds randomly chosen bits.

At the end of transmittions, Bob reports to Alice arrival times of all $2n$ photons, but neither the bases nor the results of his measurements.

Bases-revealing phase.

Alice tells Bob, through a public channel, the bases she used to encode her random sequence of bits during the transmition phase.

Design of good and bad sequences phase.

Bob partitions his $2n$ bits into two sequences, each of length n.

Into the "good" sequence he puts as much as possible of bits he obtained when he used the correct basis for measurements.

The "bad" sequence contains as much as possible of other bits.

Bob then tells Alice indices of bits of both sequences, but not which one is "good" and not which is " bad".

At this point Bob shares with Alice a binary word of his good sequence (with respect to an expected error rate not greater than ε). Concerning the bad sequence, Bob shares almost nothing with Alice.

Of course, the above process is not ideal. There can be errors introduced because that Bob did not use exactly n times correct basis for measurement . However, the number of errors introduced this way should be negligible.

Error-correction phase.

Using the code C, Alice computes syndromes of her words corresponding to the good and bad sequence of Bob, and sends syndromes to Bob, who uses them to perform error correction on his good and bad sequences.

Privacy amplification phase.

Alice chooses randomly two subsets of bits, one from her "good" and one from her "bad" sequence and computes their parities. She let Bob know the "addresses" of bits she chose, but not their values.

This way Bob can compute the parity of the corresponding subset of his good sequence, but he will have no idea about the parity Alice obtained for the subset of bits corresponding to his bad sequence. Alice knows she has no idea which one Bob knows.

Let x_0, x_1 be parity bits Alice knows and $\bar{c}$ be the parity bit Bob knows.

Oblivious transfer phase for sending bits b_0 and b_1

1. Bob tells Alice whether or not $c = \bar{c}$ for c he chooses.
2. If $c = \bar{c}$, then Alice sends Bob bits $x_0 \oplus b_0$ and $x_1 \oplus b_1$, in this order. If $c \neq \bar{c}$, Alice sends Bob bits $x_0 \oplus b_1$ and $x_1 \oplus b_0$, again in the given order.
3. Bob computes b_c out of two bits he got from Alice.

QUANTUM NON-LOCALITY

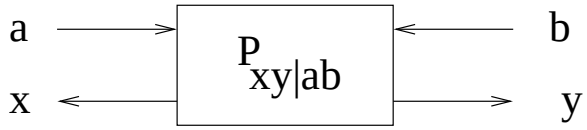
- Physics was non-local since Newton's time, with exception of the period 1915-1925.
- Newton has fully realized counterintuitive consequences of the non-locality his theory implied.
- Einstein has realized the non-locality quantum mechanics imply, but it does not seem that he realized that entanglement based non-locality does not violate no-signaling assumption.
- Recently, attempts started to study stronger non-signaling non-locality than the one quantum mechanics allows.

BEGINNINGS of MODERN STORY of NONLOCALITY

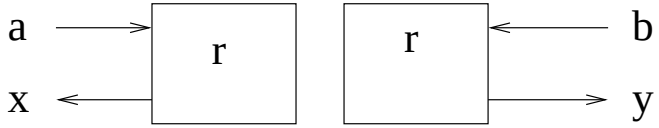
- In 1935 Einstein, Podolsky a Rosen (EPR) used entanglement to attack the validity of quantum physics as a complete theory of Nature.
- They defined an entangled state of two particles such that if a position (momentum) measurement was made on one of the particles, then position (momentum) of second particle was known.
- EPR concluded that position and momentum have to be *elements of reality*, i.e. they have to have predetermined values before measurements.
- If translated into mathematical formalism this means that Local Hidden Variable (LHV) model of Nature has to hold.
- In 1964 Bell showed that if the LHV model holds, then not all predictions of QM can be correct and he also showed a way how to test which model - LHV or QM -holds

NON-LOCAI and LOCAL SYSTEMS

The behaviour of a bipartite quantum state under measurement can be described by a conditional pro bability distribution $P_{xy|ab}$ - so called two-party information-theoretic primitive - where a and b denote the chosen bases and x with y are corresponding outputs.



John Bell was the first to recognize that there are measurement bases such that resulting behaviou r is not local, i.e. cannot be explained by shared classical information r. He showed existence of *Bell inequalities* that cannot be violated by any



local system, but ar e violated when the EPR-state is measured.

POWER OF PR-BOXES

- PR-boxes are non-local, yet they do not contradict relativity theory.
- Using one PR-box one can simulate measurement of the EPR-state.
- Using PR-boxes one can make bit commitment and 1/2-oblivious transfer unconditionally secure.
- Having PR-boxes one can simulate any secret multiparty computation and solve any multipartite communication problem by communicating a single bit (van Dam and Cleve) - what no one can believe.

APPROXIMATION of PR-BOXES

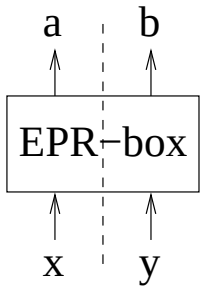
Basic questions:

- Why are the correlations achievable by quantum mechanics not maximal among those that preser ve causality?
- How well the correlations of PR-boxes can be realized or approximated by devices that follow the laws of physics?

- The availability of prior shared entanglement allows to approximate PR-boxes with success pr obability $\cos^2 \frac{\pi}{8} = 0.854$.
- In any physical world in which it is possible, without communication, to approximate PR-boxe s with probability greater than $\frac{3+\sqrt{6}}{6} \approx 90.8\%$.

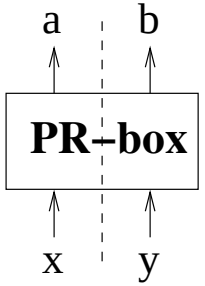
EPR-box versus RP-box

Non-locality exhibited by the measurement of the EPR state can be seen as the implementation of the following *EPR-box*



x = y implies a = b

Also non-locality exhibited by the following *PR-box* does not allow superluminal communication and therefore does not contradict special relativity.



x . y = (a + b) mod 2

However, it is unlikely that there is physical implementation of PR-boxes, as argued later.

MOTIVATION for PR-BOXES

The idea of PR-boxes arises in the following setting:

Let us have two parties, A and B, and let the party X performs two measurements on a quantum state with two outcomes m_0^x and m_1^x with 0 and 1 as potential values of each.

Let us denote a bound on correlations between two such measurements as

$$B = \sum_{x,y \in \{0,1\}} Prob(m_x^A \oplus m_y^B = x \cdot y).$$

So called Bell/CHCS inequality says that $B \leq 3$ in any classical hidden variable theory.

So-called Cirel'son's bound (Cirel'son, 1980), says that the maximum for B in quantum mechanics is $2 + \sqrt{2}$.

Popescu and Rohrlich developed a model in which the maximal possible bound of B, that is 4, is achievable.

1/2-OT versus PR-boxes

The following important relations hold between 1/2-oblivious transfer and PR-boxes:

- From the security point of view PR-boxes and 1/2-oblivious transfer are equivalent.
- From the communication point of view, the following relations hold

$$\text{PR-box}+1 \text{ classical bit} \longrightarrow 1/2\text{-OT} \longrightarrow \text{PR-box}$$

- PR-box can be seen as a classical analogue of the EPR sate (and its measurement)
- 1/2-oblivious transfer can be seen as classical analogue of quantum channel (and projective measurement of the transmitted state)

LI/BARNUM AUTHENTICATION

Let Alice and Bob share n EPR states

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(00|\rangle + |11\rangle)_{AB}$$

and let Alice has another n EPR pairs

$$|\Phi^+\rangle_{12} = \frac{1}{\sqrt{2}}(00|\rangle + |11\rangle)_{12}$$

ith step of the protocol on ith copies of the states $|\Phi^+\rangle_{AB}$ and $|\Phi^+\rangle_{12}$.

- Alice performs CNOT with qubit 1 as the control qubit and qubit A as the target qubit.
- Alice sends the state $|\Phi^+\rangle_{12}$ to Bob.
- Bob performs CNOT with qubit 2 as control qubit and qubit B as target qubit.
- Bob measures particles 1 and 2 in the Bell basis. If the outcome is $|\Phi^+\rangle$, then the current authentication round succeeds.

Afer steps 1, 3 and 4 the overall states are:

$$|\Phi^+\rangle_{AB}|\Phi^+\rangle_{12} = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{AB12}$$

$$= \frac{1}{2}(|0000\rangle + |1011\rangle + |1100\rangle + |0111\rangle)_{AB12};$$

$$= \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{AB12} = +|\Phi^+\rangle_{AB}|\Phi^+\rangle_{12}.$$

QUANTUM SECRET SHARING

There is a simple method how Alice can teleport a (secret) qubit $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob and Charles in such a way that they have to cooperate in order to have $|\phi\rangle$.

The basic idea is that Alice couples a given particle P in the state $|\phi\rangle$ with the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ of three particles P_a, P_b and P_c she shares with Bob and Charles and then performs a measurement on the state of particles P and P_a, with respect to the Bell basis $\{\Phi^{\pm}, \Psi^{\pm}\}$. Since

$$\begin{aligned} |\phi\rangle|\psi\rangle \;=\; & \frac{1}{2}(|\Phi^+\rangle(\alpha|00\rangle + \beta|11\rangle) + |\Phi^-\rangle(\alpha|00\rangle - \beta|11\rangle) \\ & +|\Psi^+\rangle(\beta|00\rangle + \alpha|11\rangle) + |\Psi^-\rangle(-\beta|00\rangle + \alpha|11\rangle)), \end{aligned}$$

the outcome of the measurement is that particles P_b and P_c get into one of the states

$$\frac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|11\rangle), \frac{1}{\sqrt{2}}(\alpha|00\rangle - \beta|11\rangle), \frac{1}{\sqrt{2}}(\beta|00\rangle + \alpha|11\rangle), \frac{1}{\sqrt{2}}(-\beta|00\rangle + \alpha|11\rangle)$$

and Alice gets two bits to tell her about which of these four cases happened. However, neither Bob nor Charles has information about which of these four states their particles are in.

Bob now performs a measurement of his particle with respect to the dual basis. He gets out of it one bit of information and Charles's particle P_c gets into one of 8 possible states , which is uniquely determined by bits both Alice and Bob got as the results of their measurements , and which can be transformed into the state $|\phi\rangle$ using one or two applications of Pauli matrices.