QUANTUM COMPUTING 6.

Jozef Gruska

Faculty of Informatics
Brno
Czech Republik

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6. QUANTUM FOURIER TRANSFORM and SHOR'S ALGORITHMS

Perhaps the most significant success of quantum computing so far has been Shor's polynomial time algorithm for factorization to be presented in this section. This is a highly nontrivial algorithm that uses a new technique, that of Quantum Fourier Transform, that will also be illustrated in this chapter.

The fastest classical algorithm to factor m bit numbers requires time $\mathcal{O}(e^{cm^{1/3}(\lg m)^{2/3}})$.

Shor's factorization algorithm requires $\mathcal{O}(m^2 \lg^2 m \lg\lg m)$ time on a quantum computer and polynomial time on a classical computer.

Of interest and importance is also another Shor's polynomial time algorithm for discrete logarithm to be also presented in this chapter.

Correctness and efficiency of Shor's algorithm is based on several simple results from number theory to be presented first.

FIRST REDUCTION

Lemma 0.1 *If there is a polynomial time deterministic (randomized) [quantum] algorithm to find a nontrivial solution of the modular quadratic equations*

$$a^2 \equiv 1 \pmod{n},$$

then there is a polynomial time deterministic (randomized) [quantum] algorithm to factorize integers.

Proof. Let $a \neq \pm 1$ be such that $a^2 \equiv 1 \pmod{n}$. Since

$$a^2 - 1 = (a+1)(a-1),$$

if n is not prime, then a prime factor of n has to be a prime factor of either $a+1$ or $a-1$.

By using Euclid's algorithm to compute

$$gcd(a+1, n) \text{ and } gcd(a-1, n)$$

we can find, in $\mathcal{O}(\lg n)$ steps, a prime factor of n.

SECOND REDUCTION

The second key concept is that of **period** of the functions

$$f_{n,x}(k) = x^k \bmod n.$$

It is the smallest integer r such that

$$f_{n,x}(k+r) = f_{n,x}(k)$$

for any k, i.e. the smallest r such that

$$x^r \equiv 1 \pmod{n}.$$

AN ALGORITHM TO SOLVE EQUATION $x^2 \equiv 1 \pmod{n}$.

1. Choose randomly $1 < a < n$.
2. Compute $gcd(a, n)$. If $gcd(a, n) \neq 1$ we have a factor.
3. Find period r of function $a^k \bmod n$.
4. If r is odd or $a^{r/2} \equiv \pm 1 \pmod{n}$,then go to step 1; otherwise stop.

If this algorithm stops, then $a^{r/2}$ is a non-trivial solution of the equation

$$x^2 \equiv 1 \pmod{n}.$$

EXAMPLE

Let $n = 15$. Select $a < 15$ such that $gcd(a, 15) = 1$.
{The set of such a is $\{2, 4, 7, 8, 11, 13, 14\}$}

Choose $a = 11$. Values of $11^x \bmod 15$ are then

$$11, 1, 11, 1, 11, 1$$

what gives $r = 2$.

Hence $a^{r/2} = 11 \pmod{15}$. Therefore

$$gcd(15, 12) = 3, \qquad gcd(15, 10) = 5$$

For $a = 14$ we get again $r = 2$, but in this case

$$14^{2/2} \equiv -1 \pmod{15}$$

and the following algorithm fails.

1. Choose randomly $1 < a < n$.
2. Compute $gcd(a, n)$. If $gcd(a, n) \neq 1$ we have a factor.
3. Find period r of function $a^k \bmod n$.
4. If r is odd or $a^{r/2} \equiv \pm 1 \pmod{n}$,then go to step 1; otherwise stop.

EFFICIENCY of REDUCTION

Lemma 0.2 *If $1 < a < n$ satisfying $gcd(n, a) = 1$ is selected in the above algorithm randomly and n is not a power of prime, then*

$$Pr\{r \text{ is even and } a^{r/2} \not\equiv \pm 1\} \geq \frac{9}{16}.$$

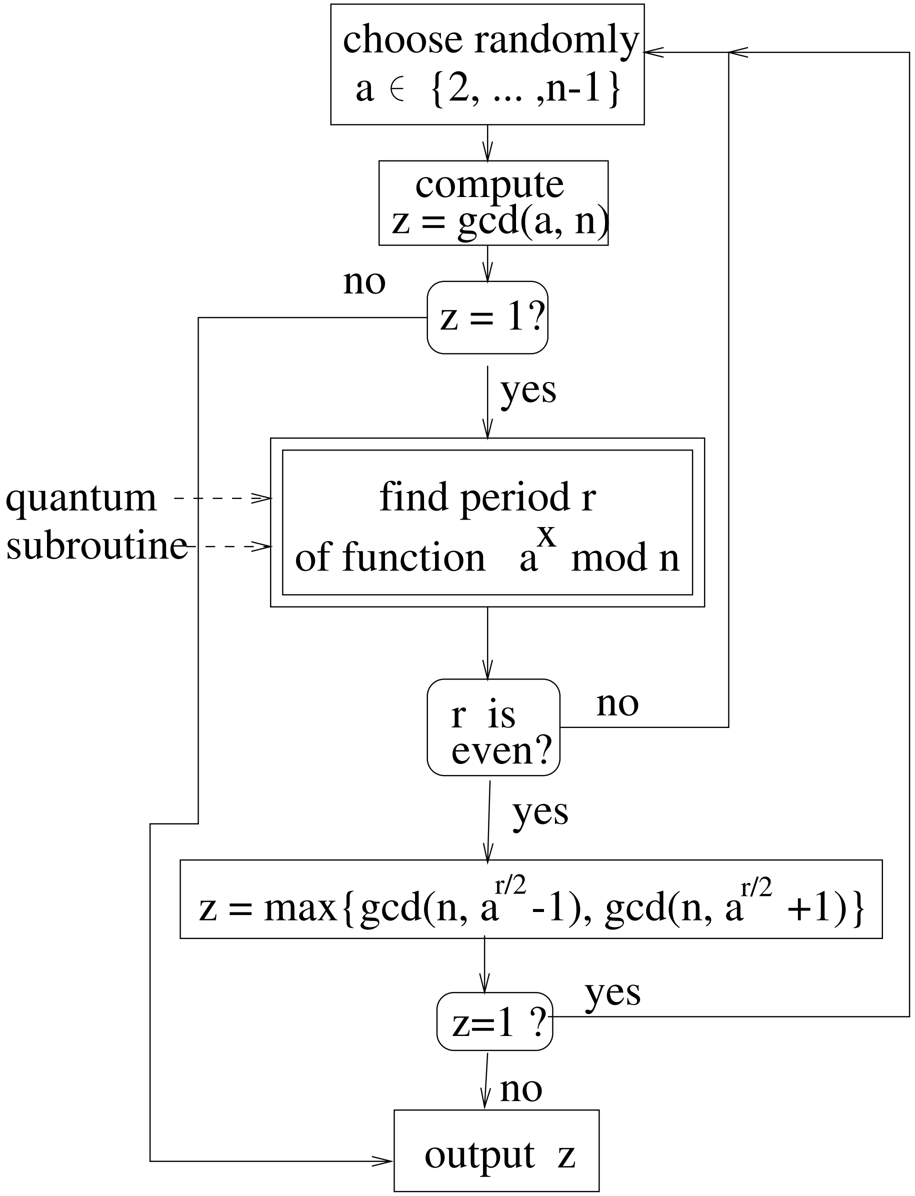
1. Choose randomly $1 < a < n$.
2. Compute $gcd(a, n)$. If $gcd(a, n) \neq 1$ we have a factor.
3. Find period r of function $a^k \bmod n$.
4. If r is odd or $a^{r/2} \equiv \pm 1 \pmod n$,then go to step 1; otherwise stop.

Corollary 0.3 *If there is a polynomial time randomized [quantum] algorithm to compute the period of the function*

$$f_{n,a}(k) = a^k \bmod n,$$

then there is a polynomial time randomized [quantum] algorithm to find non-trivial solution of the equation $a^2 \equiv 1 \pmod n$ (and therefore also to factorize integers).

A GENERAL SCHEME FOR SHOR'S ALGORITHM



choose randomly
a ∈ {2, ... ,n-1}
compute
z = gcd(a, n)
no
z = 1?
yes
quantum
subroutine
find period r
of function a^x mod n
r is
even?
no
yes
z = max{gcd(n, a^{r/2} -1), gcd(n, a^{r/2} +1)}
z=1 ?
yes
no
output z

SHOR's ALGORITHM

1. For given $n, q = 2^d, a$ create states

$$\frac{1}{\sqrt{q}}\sum_{x=0}^{q-1}|n,a,q,x,\mathbf{0}\rangle \text{ and } \frac{1}{\sqrt{q}}\sum_{x=0}^{q-1}|n,a,q,x,a^x \bmod n\rangle$$

2. By measuring the last register the state collapses into the state

$$\frac{1}{\sqrt{A+1}}\sum_{j=0}^{A}|n,a,q,jr+l,y\rangle \text{ or, shortly } \frac{1}{\sqrt{A+1}}\sum_{j=0}^{A}|jr+l\rangle,$$

   where A is the largest integer such that $l + Ar \leq q$, r is the period of $a^x \bmod n$ and l is the offset.

3. In case $A = \frac{q}{r} - 1$, the resulting state has the form.

$$\sqrt{\frac{r}{q}}\sum_{j=0}^{\frac{q}{r}-1}|jr+l\rangle$$

4. By applying quantum Fourier transformation we get then the state

$$\frac{1}{\sqrt{r}}\sum_{j=0}^{r-1}e^{2\pi ilj/r}|j\frac{q}{r}\rangle.$$

5. By measuring the resulting state we get $c = \frac{jq}{r}$ and if $gcd(j,r) = 1$, what happens with sufficient large probability, then from c and q we can determine the period r.

PERIOD COMPUTATION for $f_{n,a}(x) = a^x \bmod n, q = 2^d$

Hadamard transform applied to the state $|0^{(d)}, 0^{(d)}\rangle$ yields

$$|\phi\rangle = \frac{1}{\sqrt{2^d}} \sum_{x=0}^{q-1} |x, 0^{(d))}\rangle$$

and an application of the unitary transformation

$$U_{f_{n,a}} : |x, 0^{(d)}\rangle \rightarrow |x, a^x \bmod n\rangle$$

provides the state

$$|\phi_1\rangle = U_{f_{n,a}}|\phi\rangle = \frac{1}{\sqrt{2^d}} \sum_{x=0}^{q-1} |x, f_{n,a}(x)\rangle$$

Note 1: All possible values of the function $f_{n,a}$ are encoded in the second register in the state $|\phi_1\rangle$.

Note 2: We are interested in the period of the function $f_{n,a}$ and not in particular values of $f_{n,a}$.

Could we get period by measuring, several times, at first the second register and then the first one?

EXAMPLE

For $n = 15, a = 7$, $f_{n,a}(x) = 7^x \bmod 15$, $q = 16$, the state

$$U_{f_{n,a}}|\phi\rangle = \frac{1}{\sqrt{q}}\sum_{x=0}^{q-1}|x, f_{n,a}(x)\rangle$$

has the form

$$\frac{1}{4}(|0\rangle|1\rangle + |1\rangle|7\rangle + |2\rangle|4\rangle + |3\rangle|13\rangle + |4\rangle|1\rangle + |5\rangle|7\rangle + \ldots + |14\rangle|4\rangle + |15\rangle|13\rangle).$$

If we measure at this point the second register, then we get as the outcome one of the numbers $1, 4, 7$ or 13, and the following table shows the corresponding post-measurement states in the second column.

result	post-measurement state	offset
1	$\frac{1}{2}(\|0\rangle + \|4\rangle + \|8\rangle + \|12\rangle)\|1\rangle$	0
4	$\frac{1}{2}(\|2\rangle + \|6\rangle + \|10\rangle + \|14\rangle)\|4\rangle$	2
7	$\frac{1}{2}(\|1\rangle + \|5\rangle + \|9\rangle + \|13\rangle)\|7\rangle$	1
13	$\frac{1}{2}(\|3\rangle + \|7\rangle + \|11\rangle + \|15\rangle)\|13\rangle$	3

The corresponding sequences of values of the first register are periodic with period 4 but they have different offsets (pre-periods) listed in column 3 of the table.

GRAPHICAL REPRESENTATION of STEPS

FOR SHOR's ALGORITHM

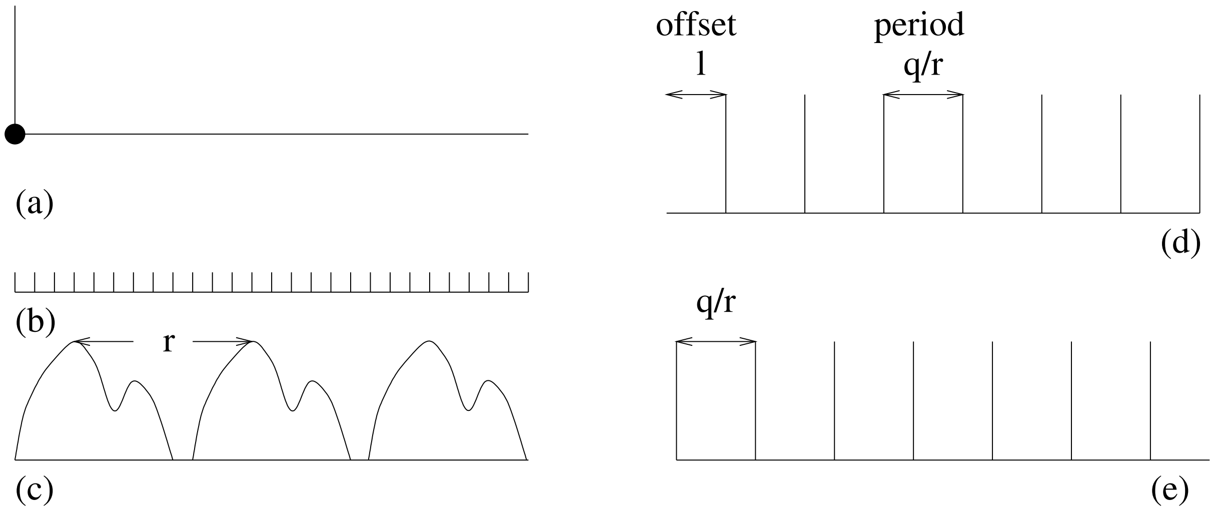


Figure 1: Graphical representation of steps of Shor's algorithm

DISCRETE FOURIER TRANSFORM

Discrete Fourier Transform maps a vector $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})^T$ into the vector $DFT(\mathbf{a}) = A_n\mathbf{a}$, where A_n is an $n \times n$ matrix such that $A_n[i,j] = \frac{1}{\sqrt{n}}\omega^{ij}$ for $0 \leq i, j < n$ and $\omega = e^{2\pi i/q}$ is the q**th root of unity**.
The matrix A_n has therefore the form

$$A_n = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{(n-1)} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)^2} \end{pmatrix}.$$

The Inverse Discrete Fourier Transform is the mapping

$$DFT^{-1}(\mathbf{a}) = A_n^{-1}\mathbf{a},$$

where

$$A_n[i,j] = \frac{1}{\sqrt{n}}\omega^{-ij}.$$

INSIDES into DFT

There is a close relation between Discrete Fourier Transform and polynomial evaluation and interpolation. Let us consider a polynomial

$$p(x) = \sum_{i=0}^{n-1} a_i x^i.$$

Such a polynomial can be uniquely represented in two ways: either by a list of its coefficients $a_0, a_1, \ldots, a_{n-1}$, or by a list of its values at n distinct points $x_0, x_1, \ldots, x_{n-1}$.

The process of finding the coefficient representation of the polynomial given its values at points $x_0, x_1, \ldots, x_{n-1}$ is called interpolation.

Computing the Discrete Fourier Transform of a vector $(a_0, a_1, \ldots, a_{n-1})$ is equivalent to converting the coefficient representation of the polynomial $\Sigma_{i=0}^{n-1} a_i x^i$ to its value representation at the points $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Likewise, the Inverse Discrete Fourier Transform is equivalent to interpolating a polynomial given its values at the n-th roots of unity.

QUANTUM FOURIER TRANSFORM

The Quantum Fourier Transform is a quantum variant of the Discrete Fourier Transform (DFT). DFT maps a discrete function to another discrete one with equally distant points as its domain. For example it maps a q-dimensional complex vector

$$\{f(0), f(1), \ldots, f(q-1)\} \ \textbf{into} \ \{\bar{f}(0), \bar{f}(1), \ldots, \bar{f}(q-1)\},$$

where for $c \in \{0, \ldots, q-1\}$

$$\bar{f}(c) = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} e^{2\pi iac/q} f(a), \qquad (1)$$

for $c \in \{0, \ldots, q-1\}$.

The quantum version of DFT (QFT) is the unitary transformation

$$\mathbf{QFT}_q : |a\rangle \rightarrow \frac{1}{\sqrt{q}} \sum_{c=0}^{q-1} e^{2\pi iac/q} |c\rangle \qquad (2)$$

The quantum version of DFT (QFT) is the unitary transformation

$$\mathbf{QFT}_q : |a\rangle \to \frac{1}{\sqrt{q}} \sum_{c=0}^{q-1} e^{2\pi iac/q} |c\rangle \tag{3}$$

for $0 \leq a < q$, with the unitary matrix

$$F_q = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{(q-1)} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(q-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{(q-1)} & \omega^{2(q-1)} & \dots & \omega^{(q-1)^2} \end{pmatrix},$$

where $\omega = e^{2\pi i/q}$ is the qth root of unity.
If applied to a quantum superposition, $\mathbf{QFT}_q$ performs as follows;

$$\mathbf{QFT}_q : \sum_{a=0}^{q-1} f(a)|a\rangle \to \sum_{c=0}^{q-1} \bar{f}(c)|c\rangle,$$

where $\bar{f}(c)$ is defined by (1).
Observe that

$$QFT_q : |\mathbf{0}\rangle \to \frac{1}{\sqrt{q}} \sum_{i=0}^{q-1} |i\rangle,$$

SHOR's ALGORITHM — PHASE 1

Design of states whose amplitudes have the same period as $f_{n,a}$

Given an m bit integer n we choose a $n^2 \leq q \leq 2n^2$ and start with five registers in states $|n, a, q, \mathbf{0}, \mathbf{0}\rangle$, where the last two registers have $m = \lceil \lg n \rceil$ qubits.
An application of the Hadamard transformation to the fourth register yields the state

$$\frac{1}{\sqrt{q}} \sum_{x=0}^{q-1} |n, a, q, x, \mathbf{0}\rangle.$$

and using quantum parallelism we compute $a^x \bmod n$ for all x in one step, to get

$$\frac{1}{\sqrt{q}} \sum_{x=0}^{q-1} |n, a, q, x, a^x \bmod n\rangle.$$

As the next step we perform a measurement on the last register. Let y be the value obtained, i.e. $y = a^l \bmod n$ for the smallest l_y with this property. If r is the period of $f_{n,a}$, then $a^{l_y} \equiv a^{jr+l_y} \pmod{n}$ for all j. Therefore, the measurement actually selects the sequence of x's values (in the fourth register), $l_y, l_y + r, l_y + 2r, \ldots, l_y + Ar$, where A is the largest integer such that $l_y + Ar \leq q - 1$. Clearly, $A \approx \frac{q}{r}$. The post-measurement state is then

$$|\phi_l\rangle = \frac{1}{\sqrt{A+1}} \sum_{j=0}^{A} |n, a, q, jr + l_y, y\rangle = \frac{1}{\sqrt{A+1}} \sum_{j=0}^{A} |jr + l_y\rangle. \tag{4}$$

after omitting some registers being fixed from now on.

SHOR's ALGORITHM — PHASE 2.

Amplitude amplification by QFT

From now on we consider in detail only a special case. Namely that r divides q. In such a case $A = \frac{q}{r} - 1$. In such a case the last state can be written in the form

$$|\phi_l\rangle = \sqrt{\frac{r}{q}} \sum_{j=0}^{\frac{q}{r}-1} |jr + l_y\rangle$$

and after QFT_q is applied on $|\phi_l\rangle$ we get:

$$\mathsf{QFT}_q|\phi_l\rangle = \frac{1}{\sqrt{q}} \sum_{c=0}^{q-1} \sqrt{\frac{r}{q}} \sum_{j=0}^{\frac{q}{r}-1} e^{2\pi ic(jr+l_y)/q}|c\rangle = \frac{\sqrt{r}}{q} \sum_{c=0}^{q-1} e^{2\pi ilc/q} \left(\sum_{j=0}^{\frac{q}{r}-1} e^{2\pi ijcr/q} \right) |c\rangle = \sum_{c=0}^{q-1} \alpha_c|c\rangle$$

If c is a multiple of $\frac{q}{r}$, then $e^{2\pi ijcr/q} = 1$ and if c is not a multiple of $\frac{q}{r}$, then

$$\sum_{j=0}^{\frac{q}{r}-1} e^{2\pi ijcr/q} = 0,$$

because the above sum is over a set of $\frac{q}{r}$ roots of unity equally spaced around the unit circle. Thus

$$\alpha_c = \begin{cases} \frac{1}{\sqrt{r}} e^{2\pi i l c/q}, & \text{if } c \text{ is a multiple of } \frac{q}{r}; \\ 0, & \text{otherwise;} \end{cases}$$

and therefore

$$|\phi_{out}\rangle = \mathsf{QFT}_q|\phi_l\rangle = \frac{1}{\sqrt{r}}\sum_{j=0}^{r-1} e^{2\pi i l_y j/r}|j\frac{q}{r}\rangle.$$

The key point is that the trouble-making offset l_y appears now in the phase factor $e^{2\pi i l_y j/r}$ and has no influence either on the probabilities or on the values in the register.

SHOR's ALGORITHM — PHASE 3

Period extraction

Each measurement of the state $|\phi_{out}\rangle$ therefore yields one of the multiples $c = \lambda\frac{q}{r}$, $\lambda \in \{0, 1, \ldots r-1\}$, where each λ is chosen with the same probability $\frac{1}{r}$.

Observe also that in this case the QFT transforms a function with the period r (and an offset l) to a function with the period $\frac{q}{r}$ and offset 0. After each measurement we therefore know c and q and

$$\frac{c}{q} = \frac{\lambda}{r},$$

where λ is randomly chosen.

If $gcd(\lambda, r) = 1$, then from q we can determine r by dividing q with $gcd(c, q)$. Since λ is chosen randomly, the probability that $gcd(\lambda, r) = 1$ is greater than $\Omega(\frac{1}{\lg \lg r})$. If the above computation is repeated $\mathcal{O}(\lg \lg r)$ times, then the success probability can be as close to 1 as desired and therefore r can be determined efficiently.[1]

In the general case, i.e., if $A \neq \frac{q}{r} - 1$, there is only a more sophisticated computation of the resulting probabilities and a more sophisticated way to determine r (using a continuous fraction method to extract the period from its approximation).

[1] As observed by Shor (1994) and shown by Cleve et al. (1998), the expected number of trials can be put down to a constant.

GENERAL CASE

Let us now sketch Shor's algorithm to compute the period of a function $f(x) = a^x \bmod n$ for the general case.

QFT_q is applied to the first register of the state $\frac{1}{\sqrt{q}}\Sigma_{x=0}^{q-1}|x\rangle|f(x)\rangle$ and afterwords the measurement is performed on the first register to provide an $y_0 \in [0, \ldots, q-1]$.

To get the period r the following algorithm is realized where $\xi = \frac{y_0}{q}$, $a_0 = \lfloor\xi\rfloor$, $\xi_0 = \xi - a_0$, $p_0 = a_0, q_0 = 1$, $p_1 = a_1 a_0 + 1, q_1 = a_1$

for $j = 1$ **until** $\xi_j = 0$ **do**

- compute p_j and q_j using the recursion (for the case $\xi_j \neq 0$);

$$a_j = \left\lfloor \frac{1}{\xi_{j-1}} \right\rfloor, \xi_j = \frac{1}{\xi_{j-1}} - a_j,$$

$$p_j = a_j p_{j-1} + p_{j-2}, \qquad q_j = a_j q_{j-1} + q_{j-2}$$

- Test whether $q_j = r$ by computing first $m^{q_j} = \Pi_i (m^{2^i})^{q_{j,i}} \bmod n$, where $q_j = \Sigma_i q_{j,i} 2^i$ is the binary expansion of q_n.

  If $a^{q_j} = 1 \bmod n$, then exit with $r = q_j$; if not continue the loop.

The non-easy task is to show, what has been done, that the above algorithm provides the period r with sufficient probability ($> \frac{0.232}{\lg\lg n}(1-\frac{1}{n})^2$).

COMMENTS on SHOR's FACTORIZATION ALGORITHM

- **Efficient implementations of QFT_q, concerning the number of gates, are known for the the case $q = 2^d$ or q is smooth (that is if factors of q are smaller than $\mathcal{O}(\lg q)$).**
- **Efficient implementation of modular operations (including exponentiation) are known.**
- **An estimation says that $300 \lg n$ gates are needed to factor n.**
- **An estimation says that to factor 130 digit integers would require two weeks on an ideal quantum computer with switching frequency 1 MHz. However to factor 260-digit number only 16 times larger time would be needed.**
- **It has been shown that there is polynomial time factorization even in the case only one pure qubit is available and the rest of quantumness available is in mixed states.**

SHOR's DISCRETE LOGARITHM ALGORITHM

Shor's quantum algorithm for discrete logarithm will be again presented only for a special case.

The task is to determine an r such that $g^r \equiv x \pmod{p}$ given a prime p, a generator g of the group $\mathbf{Z}_p^*$ and a $0 < x < p$. The special case we consider is that $p-1$ is smooth.

Using QFT$_{p-1}$ twice, on the third and fourth sub-register of the register $|x, g, \mathbf{0}, \mathbf{0}, \mathbf{0}\rangle$, we get

$$|\phi\rangle = \frac{1}{p-1} \sum_{a=0}^{p-2} \sum_{b=0}^{p-2} |x, g, a, b, \mathbf{0}\rangle,$$

a uniform distribution of all pairs (a, b), $0 \leq a, b \leq p-2$. By applying to $|\phi\rangle$ a unitary mapping

$$(x, g, a, b, \mathbf{0}) \rightarrow (x, g, a, b, g^a x^{-b} \bmod p)$$

we get

$$|\phi'\rangle = \frac{1}{p-1} \sum_{a=0}^{p-2} \sum_{b=0}^{p-2} |x, g, a, b, g^a x^{-b} \bmod p\rangle.$$

Since x, g will not be changed in the following computations we will not write them explicitly any longer.

As the next step we apply QFT$_{p-1}$ on $|\phi'\rangle$ twice, once to map $a \rightarrow c$ with amplitude $\frac{1}{\sqrt{p-1}} e^{2\pi iac/(p-1)}$ and once to map $b \rightarrow d$ with amplitude $\frac{1}{\sqrt{p-1}} e^{2\pi ibd/(p-1)}$. The resulting state is

$$|\phi_1\rangle = \frac{1}{(p-1)^2} \sum_{a,b,c,d=0}^{p-2} e^{\frac{2\pi i}{p-1}(ac+bd)} |c, d, g^a x^{-b} \bmod p\rangle.$$

$$|\phi_1\rangle = \frac{1}{(p-1)^2}\sum_{a,b,c,d=0}^{p-2} e^{\frac{2\pi i}{p-1}(ac+bd)}|c,d,g^a x^{-b} \bmod p\rangle.$$

Let us now measure last register and denote by y the value we get.

The state $|\phi_1\rangle$ then collapses into the state (before normalization)

$$|\phi_2\rangle = \sum_{c,d=0}^{p-1} \alpha(c,d)|c,d,y\rangle,$$

where

$$\alpha(c,d) = \frac{1}{(p-1)^2}\sum_{\{(a,b)\,|\,y=g^a x^{-b} \bmod p\}} e^{\frac{2\pi i}{p-1}(ac+bd)}.$$

We now claim that if $y = g^a x^{-b} \bmod p$, then $y = g^k$ for some k such that

$$a - rb \equiv k \pmod{p-1}.$$

Indeed,

$$y = g^a x^{-b} \equiv g^a (g^r)^{-b} = g^{a-rb}.$$

If $a - rb \equiv k \pmod{p-1}$, then

$$g^{a-br} \equiv g^k \pmod{p}$$

Therefore

$$\alpha(c,d) = \frac{1}{(p-1)^2} \sum_{\{(a,b)\,|\,a-rb\equiv k \pmod{p-1}\}} e^{\frac{2\pi i}{p-1}(ac+bd)}$$

For the probability *Pr* that, for fixed c and d we get by measurement of $|\phi_2\rangle$ a value y is therefore

$$\left|\frac{1}{(p-1)^2}\sum_{a,b=0}^{p-2}\{e^{\frac{2\pi i}{p-1}(ac+bd)}\,|a-rb\equiv k \pmod{p-1}\}\right|^2.$$

By substituting $a = k + rb + j_b(p-1)$ we get the probability

$$Pr = \left|\frac{1}{(p-1)^2}\sum_{b=0}^{p-2} e^{\frac{2\pi i}{p-1}(kc+cj_b(p-1)+b(d+rc))}\right|^2.$$

$$\left|\frac{1}{(p-1)^2} e^{\frac{2\pi iac}{p-1}} \sum_{b=0}^{p-2} e^{\frac{2\pi i}{p-1}(b(d+rc))}\right|^2$$

what equals

$$\left|\frac{1}{(p-1)^2}\sum_{b=0}^{p-2} e^{\frac{2\pi i}{p-1}(b(d+rc))}\right|^2$$

The above probability Pr is therefore 0 if

$$d + rc \not\equiv 0 \bmod (p-1)$$

because, as in the previous algorithm, in such a case the sum in the above expression is over a set of $(p-1)$st roots of unity equally spaced around the unit circle.

On the other hand, if

$$d \equiv -rc \pmod{p-1},$$

then the above sum does not depend on b and it is equal to

$$(p-1)^{-1}e^{(2\pi ikc/(p-1))}.$$

The square of its absolute value, the probability, is therefore $\frac{1}{(p-1)^2}$.

Consequence: the measurements on the first and second register provide a (random) $c < p-1$ and a d such that

$$d \equiv -rc \pmod{p-1}.$$

If $gcd(c, p-1) = 1$, r can now be obtained as a unique solution of the above congruence equation.

Therefore, the number of computations needed to perform in order to get the probability close to 1 for finding r is polynomial in $\lg\lg p$.

COMMENTS on SHOR's ALGORITHMS

- **To factor an integer n shor's algorithm uses $\mathcal{O}(\lg^3 n)$ steps and success probability is guarantd to be at least $\Omega(\frac{1}{\lg\lg n})$.**
- **An analysis of Shor's algorithm therefore shows that by running the alsgoritthm $\mathcal{O}(\lg\lg n)$ times, therefore in total in $\mathcal{O}(\lg^3 n \lg\lg n)$ timnes we have very high success probability.**
- **Shor's algorithms make some of the important current cryptosystems, as RSA, ElGamal and so on vulnerable to attacks using quantum computers.**
- **Shor's result have been generalized to show that a large range of cryptosystems, including elliptic curve cryptosystems, would be vulnerable to attacks using quantum computers.**

$\boxed{\text{EXTRAS}}$

HIDDEN SUBGROUP PROBLEM

Given: An (efficiently computable) function $f : G \rightarrow R$, where G group and R a finite set.
Promise: There exists a subgroup $G_0 \leq G$ such that f is constant and distinct on the cossets of G_0.
Task: Find a generating set for G_0 (in polynomial time (in $\lg|G|$) number of calls to the oracle for f and in the overall polynomial time).[2]

Deutsch's problem, $G = \mathbf{Z}_2$, $f : \{0,1\} \rightarrow \{0,1\}$, $x - y \in G_0 \Leftrightarrow f(x) = f(y)$. Decide whether $G_0 = \{0\}$ (and f is balanced) or $G_0 = \{0,1\}$ (and f is constant).

Simon's problem, $G = \mathbf{Z}_2^n$, $f : G \rightarrow R$. $x - y \in G_0 \Leftrightarrow f(x) = f(y)$, $G_0 = \{0^{(n)}, s\}$, $s \in \mathbf{Z}_2^n$. Decide whether $G_0 = \{0^{(n)}\}$ or $G_0 = \{0^{(n)}, s\}$, with an $s \neq 0^{(n)}$ (and in the second case find s).

Order-finding problem, $G = \mathbf{Z}$, $a \in \mathbf{N}$, $f(x) = a^x$, $x - y \in G_0 \Leftrightarrow f(x) = f(y)$, $G_0 = \{rk \,|\, k \in \mathbf{Z}$ for the smallest r such that $a^r = 1.\}$ Find r.

Discrete logarithm problem, $G = \mathbf{Z}_r \times \mathbf{Z}_r$, $a^r = 1$, $b = a^m$, $a, b \in \mathbf{N}$, $f(x,y) = a^x b^y$, $f(x_1,y_1) = f(x_2,y_2) \Leftrightarrow (x_1,y_1) - (x_2,y_2) \in G_0$. $G_0 = \{(-km, m) \,|\, k \in \mathbf{Z}_r\}$. Find G_0 (or m).

[2] A way to solve the problem is to show that in polynomial number of oracle calls (or time) the states corresponding to different candidate subgroups have exponentially small inner product and are therefore distinguishable.

IMPLEMENTATION Of INVERSE FOURIER TRANSFORM in $\mathbf{Z}_{2^m}$

The clue to design of a quantum circuit to implement the inverse QFT

$$|x\rangle \rightarrow \frac{1}{\sqrt{2^m}} \sum_{y=0}^{2^m-1} e^{\frac{2\pi ixy}{2^m}} |y\rangle$$

for $|x\rangle = |x_{m-1}\rangle|x_{m-2}\rangle \ldots |x_0\rangle$ is the decomposition

$$\sum_{y=0}^{2^m-1} e^{\frac{2\pi ixy}{2^m}} |y\rangle = (|0\rangle + e^{\frac{\pi ix}{2^0}}|1\rangle)(|0\rangle + e^{\frac{\pi ix}{2^1}}|1\rangle) \ldots (|0\rangle + e^{\frac{\pi ix}{2^{m-1}}}|1\rangle)$$

The l-th phase in the above decomposition can be written as follows

$$\exp(\frac{\pi i(2^{m-1}x_{m-1} + 2^{m-2}x_{m-2} + \ldots + 2x_1 + x_0)}{2^{l-1}})$$

$$= \exp(\frac{\pi i(2^{l-1}x_{l-1} + 2^{l-2}x_{l-2} + \ldots + 2x_1 + x_0)}{2^{l-1}})$$

$$= (-1)^{x_{l-1}} exp(\frac{\pi i x_{l-2}}{2^l}) \ldots exp(\frac{\pi i x_1}{2^{l-2}}) exp(\frac{\pi i x_0}{2^{l-1}})$$

DESIGN of CIRCUIT

Starting, for convenience, with the reverse representation of x **as** $x_0x_1\ldots x_{m-1}$ **we show how to implement**

$$(|0\rangle + e^{\frac{\pi i x}{2^0}}|1\rangle)(|0\rangle + e^{\frac{\pi i x}{2^1}}|1\rangle)\ldots(|0\rangle + e^{\frac{\pi i x}{2^{m-1}}}|1\rangle)$$

for qubits $m-1, m-2, \ldots, 0$, **step by step, starting with** $(m-1)$**-th qubit.**

Using Hadamard transform on the $m-1$**-th qubit we get**

$$\frac{1}{\sqrt{2}}|x_0\rangle|x_1\rangle\ldots|x_{m-2}\rangle(|0\rangle + (-1)^{x_{m-1}}|1\rangle)$$

and then we can complete the phase $(-1)^{x_{m-1}}$ **to**

$$(-1)^{x_{m-1}}exp(\frac{\pi i x_{m-2}}{2^1})\ldots exp(\frac{\pi i x_1}{2^{m-2}})exp(\frac{\pi i x_0}{2^{m-1}})$$

by using conditionally phase rotations

$$exp(\frac{\pi i}{2^1}),\ldots, exp(\frac{\pi i}{2^{m-2}})exp(\frac{\pi i}{2^{m-1}})$$

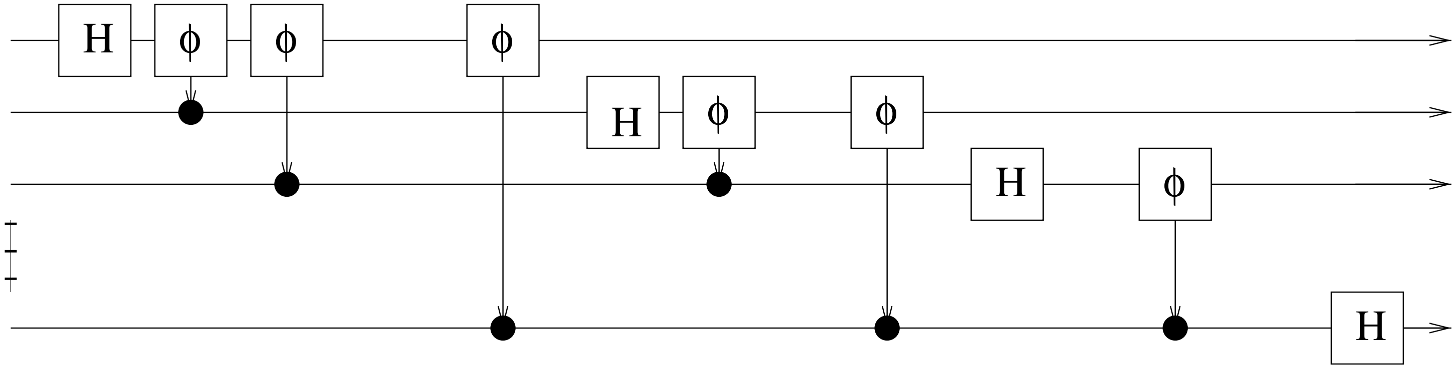
This means that for each $l \in \{1, 2, \ldots, m-1\}$ a phase factor $exp(\frac{\pi i}{2^{m-l}})$ is introduced to the m-th bit if and only if mth and lth qubits are both 1. This will provide the state

$$\frac{1}{\sqrt{2}}|x_0\rangle|x_1\rangle \ldots |x_{m-2}\rangle(|0\rangle + e^{\frac{2\pi i x}{2^{m-1}}}|1\rangle)$$

This process can be repeated with other qubits. Each time we use once the Hadamard transform and then the unitary

$$\phi_{kl} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\frac{\pi i}{2^{l-k}}} \end{pmatrix}$$

which acts on the lth and kth qubit. The resulting circuit has then the following form: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COMPLEXITY of FOURIER TRANSFORM

- **The naive algorithm to compute all elements of classical Fourier transform, element by element using basic definition, requires $\mathcal{O}(2^{2m})$ steps.**
- **A special recursive method, called Fast Fourier Transform, that recursively reduces computation of DFT in $\mathbf{Z}_{2^m}$ to computation of two DFT in $\mathbf{Z}_{2^{m-1}}$, requires $\mathcal{O}(m2^m)$ steps - a significant improvement.**
- **Quantum Fourier Transform in $\mathbf{Z}_{2^m}$ can be done in $\mathcal{O}(m^2)$ quantum steps.**

Moreover, in the classical case, physical representation of

$$(f(0), f(1), \ldots, f(2^m - 1)$$

requires $\Omega(2^m)$ bits, but in the quantum case representation of

$$c_0|0\rangle + c_1|1\rangle + \ldots + c_{2^m-1}|2^m - 1\rangle$$

requires only m qubits.

FOURIER TRANSFORM on FINITE ABELIAN GROUPS

We show now basics how the concept of Fourier Transform is defined on any finite Abelian group.

CHARACTERS of ABELIAN GROUPS

Let G be an Abelian group written additively, and $|G| = n$. A character χ of G is any morphism $\chi : G \rightarrow \mathbf{C}/0$. That is, it holds, for any $g_1, g_2 \in G$:

$$\chi(g_1 + g_2) = \chi(g_1)\chi(g)2).$$

This implies that $\chi(0) = 1$ and $1 = \chi(ng) = \chi(g)^n$ for any $g \in G$. Therefore, all values of χ are nth roots of unity.

If we define multiplication of characters χ_1 and χ_2 by $\chi_1\chi_2(g) = \chi_1(g)\chi_2(g)$, then characters form so-called dual group Ĝ. Groups G and Ĝ are isomorphic for all Abelian groups G.

Example 1 Any cyclic group of n elements is isomorphic to the group $\mathbf{Z}_n$ and all its characters have the form, for $y \in \mathbf{Z}_n$:

$$\chi_y(x) = e^{\frac{2\pi ixy}{n}}.$$

Example 2 In the additive group $\mathbf{F_2}^m$, of all binary strings of length m, all characters have the form, for binary m-bit strings x and y:

$$\chi_y(x) = (-1)^{x \cdot y},$$

where $x \cdot y = \Sigma_{i=1}^m x_i y_1 \bmod 2$

ORTHOGONALITY of CHARACTERS

Any function $f : G \rightarrow \mathbf{C}$ on an Abelian group $G = \{g_1, \ldots, g_n\}$ can be specified by the vector $(f(g_1), \ldots, f(g_n))$, and if the scalar product of two functions is defined in the standard way as

$$\langle f|g\rangle = \sum_{i=1}^{n} f^*(g_i)h(g_i),$$

then for any characters χ_1 and χ_2 on G it holds

$$\langle \chi_i|\chi_j\rangle = \begin{cases} 0, & \text{if } i \neq j \\ n, & \text{if } i = j \end{cases}$$

Therefore, the functions $\{B_i = \frac{1}{\sqrt{n}}\chi_i\}$ form an orthonormal basis on the set of all functions $f : G \rightarrow \mathbf{C}$.

FOURIER TRANSFORM

Since any $F : G \rightarrow \mathbf{C}$ has a unique representation with respect to the basis $\{B_i = \frac{1}{\sqrt{n}}\chi_i\}_{i=1}^{n}$,

$$f = \hat{\mathsf{f}}_1 B_1 + \ldots + \hat{\mathsf{f}}_n B_n$$

the function $\hat{\mathsf{f}}$: $G \rightarrow \mathbf{C}$ defined by

$$\hat{\mathsf{f}}(g_i) = \hat{\mathsf{f}}_i$$

is called the Fourier transform of f.

Since $\hat{\mathsf{f}}_i = \langle B_i | f \rangle$, we get

$$\hat{\mathsf{f}}(g_i) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \chi_i^*(g_k) f(g_k),$$

and therefore in $\mathbf{Z}_n$ the Fourier transform has the form

$$\hat{\mathsf{f}}(x) = \frac{1}{\sqrt{n}} \sum_{y \in \mathbf{Z}_n} e^{-\frac{2\pi i x y}{n}} f(y)$$

and in $\mathbf{F}_2^m$ the Fourier transform has the form

$$\hat{\mathsf{f}}(x) = \frac{1}{\sqrt{2^m}} \sum_{y \in \mathbf{F}_2^m} (-1)^{x \cdot y} f(y).$$

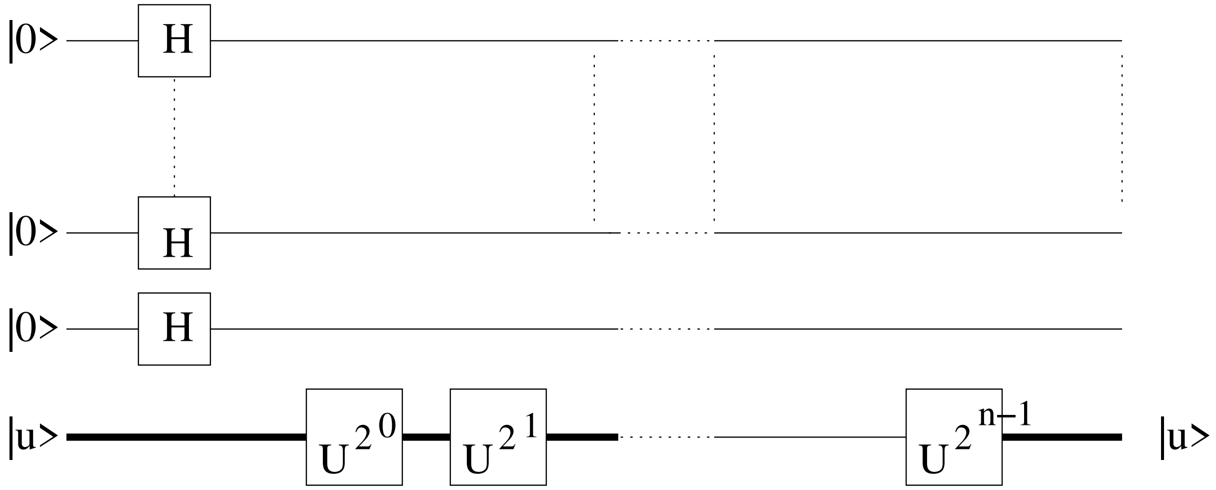
PHASE ESTIMATION

Closely related to implementation of Fourier transform is a method for phase estimation. Given is a unitary operator U with an eigenvector $|u\rangle$ and eigenvalue $e^{2\pi i\phi}$, w here $|\phi\rangle$ is unknown. The task is to determine ϕ.

For a related control-U^j-gate it holds

$$U^j(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)]|u\rangle = \frac{1}{\sqrt{2}}(|0\rangle|u\rangle + e^{2\pi ij\phi}|1\rangle|u\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi j\phi}|1\rangle)|u\rangle.$$

This menas that the first n-qubit of the circuit produces the state



$$\frac{1}{\sqrt{2^n}}\bigotimes_{t=1}^{n}(|0\rangle + e^{2\pi i 2^{t-1}\phi}|1\rangle) = \frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi ik\phi}|k\rangle$$

The last equality follows from the lemma on next slide.

LEMMA

Let $x \in \{1, \ldots, 2^n - 1\}$ and let its binary representation be $x_1 x_2 \ldots x_n$. For quantum Fourier transform

$$F|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi ijk/2^n} |k\rangle$$

it holds **Lemma**

$$F|x\rangle = \frac{1}{\sqrt{2^n}}[(|0\rangle + e^{2\pi i 0.x_n}|1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle)\ldots(|0\rangle + e^{2\pi i 0.x_1\ldots x_n}|1\rangle)].$$

Proof This follows form calculations

$$\begin{aligned}
F|x\rangle &= \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi ixk/2^n}|k\rangle = \frac{1}{\sqrt{2^n}} \sum_{k_1=0}^{1} \ldots \sum_{k_n=0}^{1} \exp(2\pi ix \sum_{l=1}^{n} k_l 2^{-l})|x_1 \ldots x_n\rangle & (6)\\
&= \frac{1}{\sqrt{2^n}} \sum_{k_1=0}^{1} \ldots \sum_{k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi ixk_l/2^l}|k_l\rangle = \frac{1}{\sqrt{2^n}} \bigotimes_{l=1}^{n} \sum_{k_l=0}^{1} e^{2\pi ixk_l/2^l}|k_l\rangle & (7)\\
&= \frac{1}{\sqrt{2^n}} \bigotimes_{l=1}^{n} (|0\rangle + e^{2\pi ix/2^l}|1\rangle) & (8)
\end{aligned}$$

PHASE INVESTIGATION

In case that phase $\phi = 0.\phi_1 \ldots \phi_n$ the last circuit produces

$$F|\phi_1\phi_2 \ldots \phi_n\rangle$$

and therefore if we apply the inverse quantum Fourier transfom on the outome of the circuit we get

$$|\phi_1\phi_2 \ldots \phi_n\rangle$$

and therefore the outcome of the measurement is ϕ.