QUANTUM COMPUTING 5

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5. SIMPLE QUANTUM ALGORITHMS

This chapter presents very basic techniques of designing quantum algorithms that are more efficient than their classical counterparts.

Quantum algorithms for the Deutsch, Deutsch-Jozsa and Simon problems are presented and analyzed.

Power of quantum parallelism, constructive and destructive interference and entanglement are illustrated.

QUANTUM PARALLELISM

If

$$f : \{0, 1, \ldots, 2^n - 1\} \Longrightarrow \{0, 1, \ldots, 2^n - 1\}$$

then the mapping

$$f' : (x, b) \Longrightarrow (x, b \oplus f(x)),$$

where $x, b \in \{0, 1, \ldots, 2^n - 1\}$ is one-to-one and therefore there is a unitary transformation U_f such that.

$$U_f(|x\rangle|0\rangle) \Longrightarrow |x\rangle|f(x)\rangle$$

Let

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle|0\rangle$$

With a **single application** of the mapping U_f we get

$$U_f|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle|f(i)\rangle$$

IN A SINGLE COMPUTATIONAL STEP 2^n VALUES OF f ARE COMPUTED!

MEASUREMENT — EXAMPLE

If we "measure" second register of the state

$$|\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle|f(i)\rangle$$

with respect to the standard basis $\{|z\rangle \mid z \in \{0,1\}^n\}$, then the state $|\phi\rangle$ collapses into one of the states

$$|\phi_y\rangle = \frac{1}{\sqrt{k}} \sum_{\{x \mid f(x)=y\}} |x\rangle|y\rangle,$$

where

- y is in the range of the values of the function f.
- $k = |\{x \mid f(x) = y\}|$.

The collapse into the state $|\phi_y\rangle$ happens with the probability

$$\frac{k}{2^n}$$

and into the classical world one gets information which of y in the range of f, in the second register, has been (randomly) chosen.

This fact we usually interpret that y is the (classical) result of the measurement of the second register of the state$|\phi\rangle$, with respect to the standard basis.

U_f OPERATOR versus V_f OPERATOR

Another useful operator related to functions

$$f : \{0, 1, \ldots, 2^n - 1\} \rightarrow \{0, 1\}$$

is the operator

$$V_f|x\rangle \rightarrow (-1)^{f(x)}|x\rangle,$$

where $x \in \{0, 1, 2, \ldots, 2^n - 1\}$, which can be expressed using the operator

$$U_f : |x, b\rangle \rightarrow |x, b \oplus f(x)\rangle$$

and one additional qubit, called again **ancilla**, in the state $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ as follows

$$\begin{aligned} U_f|x, \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\rangle &= \frac{1}{\sqrt{2}}(x, 0 \oplus f(x)\rangle - |x, 1 \oplus f(x)\rangle \\ &= (-1)^{f(x)}|x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

Warmup: Show how the operator V_f can be used to implement U_f.

$\boxed{\text{EXAMPLE}}$

Mapping $V_f : \{0,1\}^2 \leftrightarrow \{0,1\}$ is realized by the unitary matrix

$$V_f = \begin{pmatrix} (-1)^{f(00)} & 0 & 0 & 0 \\ 0 & (-1)^{f(01)} & 0 & 0 \\ 0 & 0 & (-1)^{f(10)} & 0 \\ 0 & 0 & 0 & (-1)^{f(11)} \end{pmatrix}.$$

DEUTSCH's PROBLEM – RANDOMIZED SOLUTION

Given a function $f : \{0,1\} \rightarrow \{0,1\}$, as a black box, the task is to determine whether f is constant or balanced.

In classical computing 2 calls of f are needed.

In quantum computing 1 call of f is sufficient.

Quantum algorithm presented below solves the problem with probability $\frac{1}{2}$ in such a way that we know whether the answer is correct. Since

$$U_f : (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0, f(0)\rangle + |1, f(1)\rangle),$$

the result can be written, in the standard and dual basis, as follows:

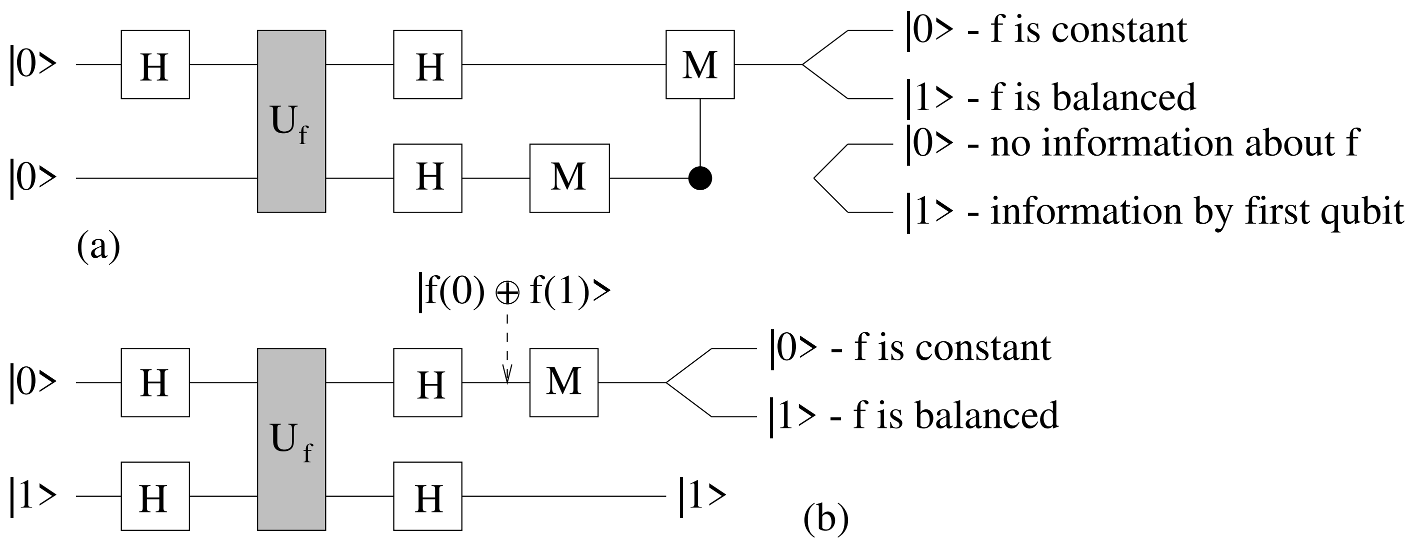
if f is constant:

$$\frac{1}{\sqrt{2}}(|0, f(0)\rangle + |1, f(1)\rangle) = \frac{1}{\sqrt{2}}(|0', 0'\rangle + (-1)^{f(0)}|0', 1'\rangle)$$

and if f is balanced:

$$\frac{1}{\sqrt{2}}(|0, f(0)\rangle + |1, f(1)\rangle) = \frac{1}{\sqrt{2}}(|0', 0'\rangle + (-1)^{f(0)}|1', 1'\rangle).$$

If the measurement of the second qubit in the dual bases provides 0 we have lost all information about f. Otherwise the measurement of the first qubit yields the correct result.
The corresponding circuit is shown in the following Figure.


|0>
|0>
H
U
f
H
H
M
M
|0> - f is constant
|1> - f is balanced
|0> - no information about f
|1> - information by first qubit
(a)
|f(0) ⊕ f(1)>
|0>
|1>
H
H
U
f
H
H
M
|0> - f is constant
|1> - f is balanced
|1>
(b)

DEUTSCH's PROBLEM – DETERMINISTIC SOLUTION

Apply first the Hadamard transform on both registers in the initial state $|0,1\rangle$ and then U_f to get

$$\begin{aligned} |0\rangle|1\rangle &\stackrel{H_2}{\longrightarrow} \frac{1}{2}(|0\rangle+|1\rangle)(|0\rangle-|1\rangle) \\ &= \frac{1}{2}(|0\rangle(|0\rangle-|1\rangle)+|1\rangle(|0\rangle-|1\rangle)) \\ &\stackrel{U_f}{\longrightarrow} \frac{1}{2}(|0\rangle(|0\oplus f(0)\rangle-|1\oplus f(0)\rangle)+|1\rangle(|0\oplus f(1)\rangle-|1\oplus f(1)\rangle)) \\ &= \frac{1}{2}(\sum_{x=0}^{1}(-1)^{f(x)}|x\rangle)(|0\rangle-|1\rangle) \\ &= \frac{1}{2}(-1)^{f(0)}(|0\rangle+(-1)^{f(0)\oplus f(1)}|1\rangle)(|0\rangle-|1\rangle). \end{aligned} \tag{1}$$

Hence

$$|0\rangle|1\rangle \stackrel{H_2}{\rightarrow} \tfrac{1}{2}(-1)^{f(0)}(|0\rangle + (-1)^{f(0)\oplus f(1)}|1\rangle)(|0\rangle - |1\rangle) \tag{2}$$

From the right side in (), the two possibilities for f to be constant lead to the left sides in (3) and (4) and two possibilities for f to be balanced lead to the left sides in (5) and (6):

$$\frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle) = |0'\rangle|1'\rangle \textbf{ if } f(0) = 0; \tag{3}$$

$$\frac{1}{2}(|0\rangle + |1\rangle)(|1\rangle - |0\rangle) = -|0'\rangle|1'\rangle \textbf{ if } f(0) = 1; \tag{4}$$

$$\frac{1}{2}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle) = |1'\rangle|1'\rangle \textbf{ if } f(0) = 0; \tag{5}$$

$$\frac{1}{2}(|0\rangle - |1\rangle)(|1\rangle - |0\rangle) = -|1'\rangle|1'\rangle \textbf{ if } f(0) = 1. \tag{6}$$

By measuring the first bit, with respect to the dual basis, we can immediately see whether f is constant or balanced.

EVEN-ODD PROBLEM

A function $f:\{0,1\}^2 \leftrightarrow \{0,1\}$ is called even (odd) if the range of f has even (odd) number of ones.

Classically, given such a function f as an oracle, one needs 4 calls of f to dtermine whether f is even or odd.

Quantumly, it holds

$$(H\otimes H)V_f(I\otimes H)V_f(H\otimes H)|00\rangle = \begin{cases} \frac{1}{\sqrt{2}}(\pm|00\rangle + |01\rangle) & \text{if f is even} \\ \frac{1}{\sqrt{2}}(\pm|10\rangle + |01\rangle) & \text{if f is odd} \end{cases}$$

and therefore using only two quantum calls of f (of V_f), the problem is transformed into the problem to distinguish two non-orthogonal quantum states.

Unfortunately, there is no projection measurement that can faithfully distinguish such non-orthogonal states. However, as discussed later, there is so called POVM measurement that either tells us whether a given function f is even or odd or the algorithm tells us "I don't know".

DEUTSCH-JOZSA PROMISE PROBLEM

Given a function $f : \{0,1\}^n \rightarrow \{0,1\}$, as a black box, that is (promised to be) balanced or constant. Decide which property f has.

Classical deterministic computers needs, in the worst case, exponential time to solve the problem. Surprisingly, there is a quantum algorithm to solve the problem by applying f only once.

Let us consider one quantum register with n qubits and apply the Hadamard transformation H_n to the first register. This yields

$$|0^{(n)}\rangle \xrightarrow{H_n} |\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle.$$

By applying the transformation V_f on the first register we get

$$V_f|\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} (-1)^{f(i)}|i\rangle = |\phi_1\rangle.$$

What has been achieved by these operations? The values of f were transferred to the amplitudes.

This can be utilized, through the power of quantum superposition and a proper observable, as follows.

Let us consider the observable $\mathcal{D} = \{E_a, E_b\}$, where E_a is the one-dimensional subspace spanned by the vector

$$|\psi_a\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle,$$

and $E_b = (E_a)^\perp$. The projection of $|\phi_1\rangle$ into E_a and E_b has the form

$$|\phi_1\rangle = \alpha|\psi_a\rangle + \beta|\psi_b\rangle \quad \textbf{with} \quad |\alpha|^2 + |\beta|^2 = 1,$$

where $|\psi_b\rangle$ is a vector in E_b such that $|\psi_a\rangle \perp |\psi_b\rangle$. A measurement by $\mathcal{D}$ provides "the value a or b" with probability $|\alpha|^2$ or $|\beta|^2$.
It is easy to determine α in

$$|\phi_1\rangle = \alpha|\psi_a\rangle + \beta|\psi_b\rangle \quad \textbf{with} \quad |\alpha|^2 + |\beta|^2 = 1,$$

using the projection of $|\phi_1\rangle$ onto E_a by the computation

$$\alpha = \langle\psi_a|\phi_1\rangle.$$

Indeed

$$\begin{aligned}\alpha &= \langle\psi_a|\phi_1\rangle = \left(\frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}\langle i|\right)\left(\frac{1}{\sqrt{2^n}}\sum_{j=0}^{2^n-1}(-1)^{f(j)}|j\rangle\right)\\ &= \frac{1}{2^n}\sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1}(-1)^{f(j)}\langle i|j\rangle = \frac{1}{2^n}\sum_{i=0}^{2^n-1}(-1)^{f(i)},\end{aligned}$$

because $\langle i|j\rangle = 1$ if and only if $i = j$ and 0 otherwise.

If f is balanced, then the sum for α contains the same number of 1s and -1s and therefore $\alpha = 0$. A measurement of $|\phi_1\rangle$, with respect to $\mathcal{D}$ therefore provides, for sure, the outcome b.

If f is constant, then either $\alpha = 1$ or $\alpha = -1$ and therefore the measurement of $|\phi_1\rangle$ with respect to $\mathcal{D}$ always gives the outcome a.

A single measurement of $|\phi_1\rangle$, with respect to $\mathcal{D}$, therefore provides the solution of the problem with probability 1.

SECOND SOLUTION

If the Hadamard transformation is applied to the state $|\phi_1\rangle$ we get the state

$$|\phi_2\rangle = \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}(-1)^{f(i)}\frac{1}{\sqrt{2^n}}\sum_{u=0}^{2^n-1}(-1)^{u\cdot i}|u\rangle = \frac{1}{2^n}\sum_{u=0}^{2^n-1}(\sum_{i=0}^{2^n-1}(-1)^{u\cdot i}(-1)^{f(i)})|u\rangle.$$

Case 1 f is constant. Then

$$\sum_{i=0}^{2^n-1}(-1)^{u\cdot i} = \begin{cases} 0 & \textbf{if } u \neq 0 \\ 2^n & \textbf{if } u = 0 \end{cases}$$

One measurement of the register therefore provides $u = 0$ with probability 1.

Case 2 f is balanced. In such a case

$$\sum_{i=0}^{2^n-1}(-1)^{u\cdot i}(-1)^{f(i)} = 0 \textbf{ if and only if } u = 0.$$

One measurement therefore shows whether f is balanced or not.

SIMON's PROBLEM

Simon has discovered a simple problem with expected polynomial time quantum algorithm, but with no polynomial time randomized algorithm.

Let $f:\{0,1\}^n \rightarrow \{0,1\}^n$ be a function such that either f is one-to-one or f is two-to-one and there exists a single $0 \neq s \in \{0,1\}^n$ such that

$$\forall x \neq x'(f(x) = f(x') \Leftrightarrow x' = x \oplus s).$$

The task is to determine which of the above conditions holds for f and, in the second case, to determine also s.

To solve the problem two registers are used, both with n qubits, and the initial states $|0^{(n)}\rangle$, and (expected) $\mathcal{O}(n)$ repetitions of the following version of the so-called Hadamard-twice scheme:

1. **Apply the Hadamard transformation on the first register, with the initial value $|0^{(n)}\rangle$, to produce the superposition $\frac{1}{\sqrt{2^n}}\Sigma_{x\in\{0,1\}^n}|x,0^{(n)}\rangle$.**
2. **Apply U_f to compute $|\psi\rangle = \frac{1}{\sqrt{2^n}}\Sigma_{x\in\{0,1\}^n}|x,f(x)\rangle$.**
3. **Apply Hadamard transformation on the first register to get**
$$\frac{1}{2^n}\sum_{x,y\in\{0,1\}^n}(-1)^{x\cdot y}|y,f(x)\rangle.$$
4. **Observe the resulting state to get a pair $(y,f(x))$.**

Case 1: f **is one-to-one. After performing the first three steps of the above procedure all possible states $|y,f(x)\rangle$ in the superposition are distinct and the absolute value of their amplitudes is the same, namely 2^{-n}.**

$n-1$ independent applications of the scheme *Hadamard-twice* therefore produce $n-1$ pairs $(y_1,f(x_1)),\ldots,(y_{n-1},f(x_{n-1}))$, distributed uniformly and independently over all pairs $(y,f(x))$.

Case 2: There is some $s \neq 0^{(n)}$ such that

$$\forall x \neq x'((f(x) = f(x') \Leftrightarrow x' = x \oplus s).$$

In such a case for each y and x the states $|y, f(x)\rangle$ and $|y, f(x \oplus s)\rangle$ are identical. Their total amplitude $\alpha(x,y)$ has the value

$$\alpha(x,y) = 2^{-n}((-1)^{x \cdot y} + (-1)^{(x \oplus s) \cdot y}).$$

If $y \cdot s \equiv 0 \bmod 2$, then $x \cdot y \equiv (x \oplus s) \cdot y \bmod 2$ and therefore $|\alpha(x,y)| = 2^{-n+1}$; otherwise $\alpha(x,y) = 0$. n independent applications of the scheme *Hadamard-twice* therefore yield $n-1$ independent pairs

$$(y_1, f(x_1)), \ldots, (y_{n-1}, f(x_{n-1})) \ \text{ such that } \ y_i \cdot s \equiv 0 \pmod 2,$$

for all $1 \leq i \leq n-1$.

In both cases, after $n-1$ repetitions of the scheme *Hadamard-twice*, $n-1$ vectors $y_i, 1 \leq i \leq n-1$, are obtained.

In both cases, after $n-1$ repetitions of the scheme *Hadamard-twice*, $n-1$ vectors $y_i, 1 \leq i \leq n-1$, are obtained.

If these vectors are linearly independent, then the system of $n-1$ linear equations in $\mathbf{Z}_2$,

$$y_i \cdot s \equiv 0 \pmod{n}$$

can be solved to obtain s.

In Case 2, if f is two-to-one, s obtained in such a way is the one to be found.

In Case 1, s obtained in such a way is a random string.

To distinguish these two cases, it is enough to compute $f(0)$ and $f(s)$.

If $f(0) \neq f(s)$, then f is one-to-one.

If the vectors obtained by the scheme *Hadamard-twice* are not linearly independent, then the whole process has to be repeated.

LOWER BOUND

We show that each classical algorithm needs to perform $\Omega(\sqrt{2^n})$ queries to solve Simon's problem.

Indeed, let us assume that f is a randomly chosen function satisfying requirements of the Simon's problem. If k f-queries are performed then the number of potential s is decreased at most by $\frac{k(k-1)}{2}$ possibilities.

In total there are 2^n potential s.

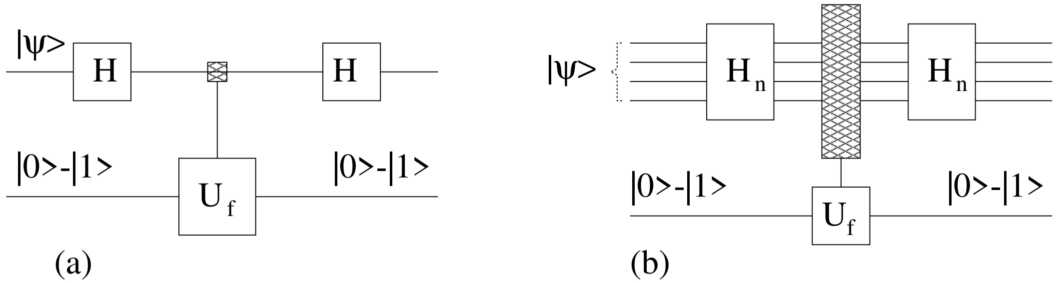
Hence at least in half of the cases any classical algorithm needs to perform $\Omega(\sqrt{2^n})$ f-queries.

HADAMARD-TWICE SCHEME

In the second algorithm for Deutsch's problem and in the algorithm for Simon's problem we have used two simple but powerful techniques which one often encounters in the design of efficient algorithms and quantum error-correcting networks:

- a change between the standard and the dual basis (for some qubits),
- the computation scheme, called **Hadamard twice** , depicted in the general form in Figure below, which uses again f-controlled NOT .

On closer examination one sees that the key point of the "Hadamard twice" scheme is again the change of the basis from standard to dual, some natural computations, and again the change of the basis back.



(Circuit for the Deutsch's problem and the "Hadamard-twice scheme". The state $|0\rangle - |1\rangle$ should be normalized.)

COMPUTATIONAL POWER of ENTANGLEMENT

As illustrated in the following examples, in some cases there is a clever way to make use of quantum entanglement to compute efficiently some global properties of a function.

Let a function $f : \{1, \ldots, n\} \to \{0, 1\}$ be given as a black box.

To determine f classically, n calls of f are needed—to get the string $w_f = f(1)f(2)\ldots f(n)$.

Quantumly, this can be done, with probability greater than 0.95, using $\frac{n}{2} + \sqrt{n}$ quantum calls of f.

Indeed, on the base of equality

$$|w_f\rangle = H_n \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot w_f} |x\rangle \tag{7}$$

in order to compute $x \cdot w_f$ one needs $hw(x)$ calls of f, where $hw(x)$ is the Hamming weight of x—the number of 1' in x.

The basic trick is to compute the sum in (7) but only for x such that $hw(x) \leq k$, for a suitable k.

If F_k is such a function that for $x \in \{0,1\}^n$,

$$F_k(x) = \begin{cases} x \cdot w_f \ \textbf{if } hw(x) \leq k \\ 0; \ \textbf{otherwise} \end{cases}$$

then

$$V_{F_k}|x\rangle = \begin{cases} (-1)^{x \cdot w_f}|x\rangle, \ \textbf{if } hw(x) \leq k \\ |x\rangle; \ \textbf{otherwise} \end{cases}$$

Therefore if V_{F_k} is applied to the (initial) state

$$|\psi_k\rangle = \frac{1}{\sqrt{M_k}} \sum_{x \in \{0,1\}^n}^{hw(x) \leq k} |x\rangle,$$

where $M_k = \Sigma_{i=0}^{k} \binom{n}{i}$, then

$$|\psi_k'\rangle = V_{F_k}|\psi_k\rangle = \frac{1}{\sqrt{M_k}} \sum_{x \in \{0,1\}^n}^{hw(x) \leq k} (-1)^{x \cdot w_f}|x\rangle.$$

In order to compute $|\psi_k'\rangle$, at most k calls of f are needed. Let us now measure all n qubits of $|\psi_k''\rangle = H_n|\psi_k'\rangle$.

The probability that this way we get w_f is

$$Pr(|\psi_k''\rangle \textbf{ yields } w_f) = |\langle w_f|\psi_k''\rangle|^2 = \frac{M_k}{2^n} = \frac{1}{2^n}\sum_{i=1}^{k}\binom{n}{i}$$

and, as one can calculate, this probability is more than 0.95 if $k = \frac{n}{2} + \sqrt{n}$.

$\boxed{\text{EXTRAS}}$

QUANTUM FOURIER TRANSFORM

The Quantum Fourier Transform is a quantum variant of the **Discrete Fourier Transform** (DFT). It maps a discrete function to another discrete one with equally distant points as its domain. For example it maps a q-dimensional complex vector

$$\{f(0), f(1), \ldots, f(q-1)\} \text{ into } \{\bar{f}(0), \bar{f}(1), \ldots, \bar{f}(q-1)\},$$

where for $c \in \{0, \ldots, q-1\}$

$$\bar{f}(c) = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} e^{2\pi iac/q} f(a), \tag{8}$$

for $c \in \{0, \ldots, q-1\}$.

The quantum version of DFT (QFT) is the unitary transformation

$$\mathsf{QFT}_q : |a\rangle \rightarrow \frac{1}{\sqrt{q}} \sum_{c=0}^{q-1} e^{2\pi iac/q} |c\rangle \tag{9}$$

for $0 \leq a < q$, with the unitary matrix

$$F_q = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{(q-1)} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(q-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{(q-1)} & \omega^{2(q-1)} & \dots & \omega^{(q-1)^2} \end{pmatrix},$$

where $\omega = e^{2\pi i/q}$ is the q**th root of unity**.
If applied to a quantum superposition, QFT_q performs as follows;

$$\mathsf{QFT}_q : \sum_{a=0}^{q-1} f(a)|a\rangle \rightarrow \sum_{c=0}^{q-1} \bar{f}(c)|c\rangle,$$

where $\bar{f}(c)$ is defined by (8).
Observe that

$$QFT_q : |\mathbf{0}\rangle \rightarrow \frac{1}{\sqrt{q}} \sum_{i=0}^{q-1} |i\rangle,$$

GENERALIZED DEUTSCH-JOZSA PROBLEM

We say that a function $f : \mathbf{Z}_N \rightarrow \mathbf{Z}_M$ is evenly distributed if, for a fixed t,

$$\{f(x) \,|\, x \in \mathbf{Z}_N\} = \{j\mu + t \,|\, j \in \mathbf{Z}_k, M = \mu k \textbf{ and } k|N\},$$

and

$$|A_0| = |A_1| = |A_2| = \ldots = |A_{k-1}| = \frac{N}{k},$$

where

$$A_j = \{x \in \mathbf{Z}_N \,|\, f(x) = j\mu + t\} \textbf{ for any } j \in \mathbf{Z}_k.$$

In other words, f is evenly distributed if f has equally spaced k values and f is a ν-to-one function where $\nu = \frac{N}{k}$.

The generalized Deutsch-Jozsa problem is to decide, given a promise that a function f, given as an oracle, is either constant or evenly distributed, which of these two properties f has.

To solve the generalized Deutsch-Jozsa problem one application of f is again sufficient. In addition, QFT will be used.

For simplicity we assume $N = 2^n, M = 2^m$. Let

$$|\psi\rangle_\chi = QFT_{2^m}(|-\chi\rangle) = \frac{1}{\sqrt{2^m}} \sum_{z=0}^{2^m-1} e^{-2\pi i\chi z/2^m}|z\rangle$$

for a $\chi \in \mathbf{Z}_{2^m}$.

In the following algorithm to the initial state

$$|0^{(n)}\rangle \otimes |\psi\rangle_\chi$$

at first the transformation $H_n \otimes I$ is applied, then the transformation U_f, transformations $z + f(x) \rightarrow z' \rightarrow z$, and, finally, again the transformation $H_n \otimes I$. This way we get

$$\begin{aligned}
|0^{(n)}\rangle \otimes |\psi\rangle_\chi &\rightarrow \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes |\psi\rangle && (10)\\
&\rightarrow \frac{1}{\sqrt{2^n 2^m}} \sum_{x=0}^{2^n-1} \sum_{z=0}^{2^m-1} e^{-2\pi i\chi z/2^m}|x\rangle|z \oplus f(x)\rangle && (11)\\
&= \frac{1}{\sqrt{2^n 2^m}} \sum_{x=0}^{2^n-1} \sum_{z=0}^{2^m-1} e^{-2\pi i\chi (z-f(x))/2^m}|x\rangle|z\rangle && (12)
\end{aligned}$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{2\pi i \chi f(x)/2^m} |x\rangle \otimes |\psi\rangle_\chi \tag{13}$$

$$\rightarrow \sum_{y=0}^{2^n-1} \left(\frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^{x \cdot y} e^{2\pi i \chi f(x)/2^m} \right) |y\rangle |\psi\rangle_\chi \tag{14}$$

Let us now denote the inner summation in the last state as

$$S_y = \frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^{x \cdot y} e^{2\pi i \chi f(x)/2^m},$$

It is now easy to see that

- **If f is constant then $S_y = 0$ if and only if $y \neq 0$.**
- **If f is evenly distributed, then $S_0 = 0$.**

Hence, when f is constant the final state of the first register is $|0^{(n)}\rangle$, whereas when f is evenly distributed the state is orthogonal. By one measurement of the first register we can therefore determine whether f is constant or evenly distributed.

DE-QUANTUMIZATION in CASE of DEUTACH PROBLEM

Surprisingly, quantum algorithms for Deutsch problem can be de-quantised as follows:

For a given $f : \{0,1\} \rightarrow \{0,1\}$ we define an oraculum mapping

$$C_f(a+bi) = (-1)^{0\oplus f(0)}a + (-1)^{1\oplus f(i)}bi$$

For the four possible functions f we get the following four functions C_f:

$$\begin{array}{ll} C_{00}(x) = x^* & \textbf{if } f(0)=0, f(1)=0 \\ C_{01}(x) = x & \textbf{if } f(0)=0, f(1)=1 \\ C_{10}(x) = -x & \textbf{if } f(0)=1, f(1)=0 \\ C_{11}(x) = -x^* & \textbf{if } f(0)=1, f(1)=1 \end{array}$$

The Deutsch problem can now be formulated as follows: A function is choosen from the set $\{C_{00}, C_{01}, C_{10}, C_{11}\}$ and the problem is to determine, with a single querry, which type of the function it is, balanced or constant.

Algorithm Given f, calculate $(i-1)f(1+i)$, if the result is real, then the function is balanced; otherwise it is constant.

Correctness:

$$\begin{aligned}
(i-1)C_{00}(1+i) &= (i-1)(1-i) = 2i \\
(i-1)C_{01}(1+i) &= (i-1)(1+i) = -2 \\
(i-1)C_{10}(1+i) &= (i-1)(-1-i) = 2 \\
(i-1)C_{11}(1+i) &= (i-1)(1-i) = -2i
\end{aligned}$$