Part I

Random Walks - Markov Chains/Models

Chapter 9. RANDOM WALKS - MARKOV CHAINS/MODELS

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The concept of a random walk is closely related with that of **Markov chain**

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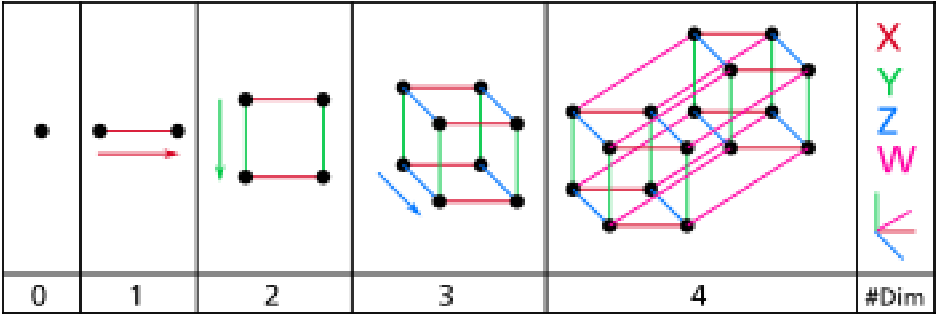
Typical problems to explore concerning walking for a given graph G are:

- What is the expected number of steps to get from a given node u to a given node v?
- What is the expected number of steps needed to visit all nodes of G at least once when starting in a given node u?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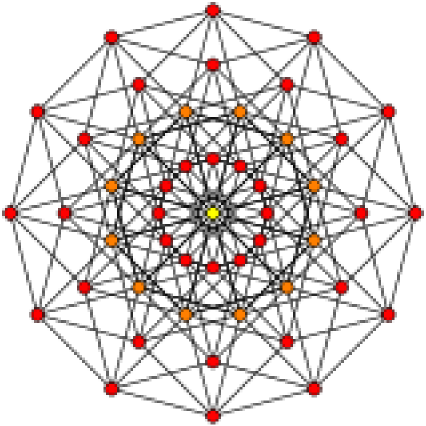
6-d hypercube



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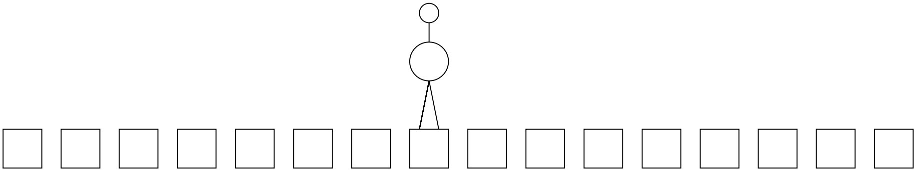

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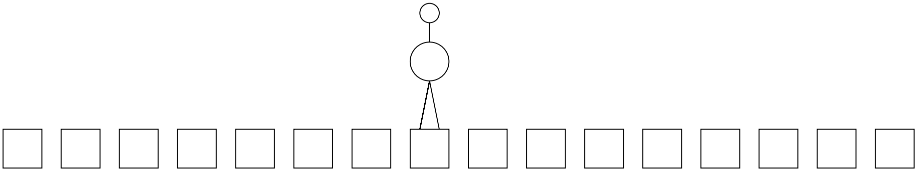

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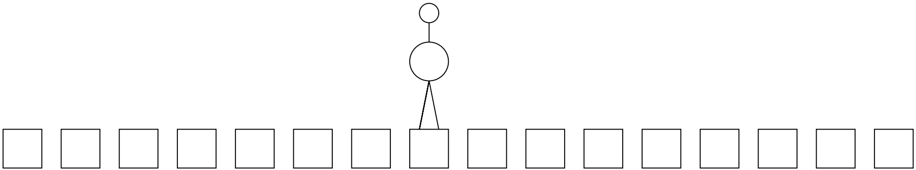

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What are probabilities for such a drunken man to be in a particular position after some steps in case he starts in some fixed initial position?

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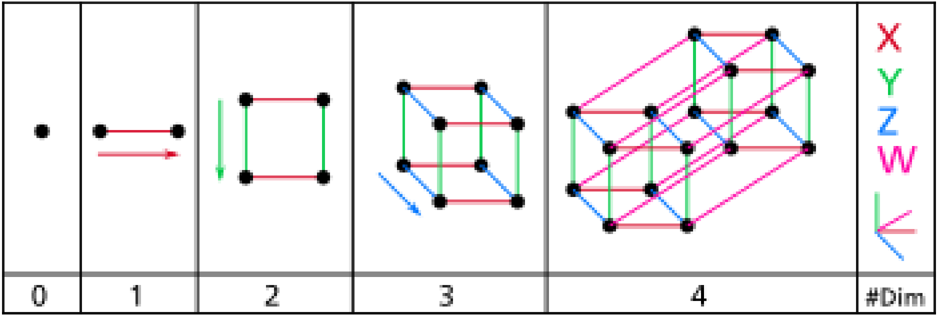
Simple hypercubes



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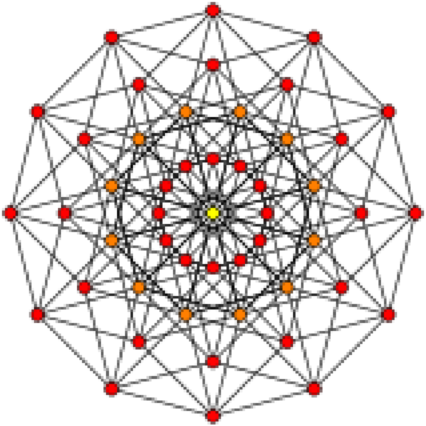

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Hence:

$$p = n - 1$$

- The expected number of steps to visit all nodes in G starting from any node u is

$$(n-1)H_n,$$

where H_n is so called **Harmonic number**

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Therefore

$$\lim_{i\to\infty} P_i = \frac{1}{3}$$

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Theorem The expected number of steps of the above algorithm at finding a satisfying assignment is $\mathcal{O}(n^2)$ (where n is the number of variables).

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STOCHASTIC PROCESSES in RANDOMIZED ALGORITHMS

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The most useful algorithmic property of Markov chains, to be explored in the next, is their convergence to a fixed (probability) distributions on states.

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$$Pr[X_{t+1} = j | X_0 = i_0, X_1 = i_1, \ldots, X_t = i_t = i] = Pr[X_{t+1} = j | X_t = i] = p_{ij}.$$

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Such a distribution is called the **initial (probability) distribution**.

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In informatics, random walks provide a general paradigm for random exploration of an exponentially large combinatorial structures (for example graphs), by a sequence of simple and local transitions.

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- For example, almost all fast speech and patterns recognition systems use HMM.

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Before that only such sequences of random experiments were considered where the result of each random experiment was fully independent from all previous experiments.

UNIVERSALITY of QUANTUM RANDOM WALKS

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It can be also shown that any

quantum evolution

can be seen as so-called

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MARKOV CHAINS –2nd DEFINITION

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The probability distribution over states of a Markov chain C with n nodes $N_1, ..., N_n$ and with a transition $n \times n$ matrix P at any given time t is given by a row vector $Q_t = (P_(1, t), \ldots, P(n, t)$, where $P(i, t)$ is the probability that chain is in the state N_i after t steps. Q_t is therefore distribution vector for time step t and therefore probability distribution of states after t steps

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Distribution vector of such a Markov chain at time t is then given by the vector $P^t Q_0$.

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- We say that **states i and j are called mutually reachable** if i is reachable from j and vice verse.
- A **Markov chain is called irreducible**, if any two of its states are mutually reachable.

GENERAL OBSERVATION

Systems represented by Markov chains change randomly and therefore it is generally impossible to determine the exact state of the system in the future.

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However, we often can determine various useful statistical properties of Markov chains.

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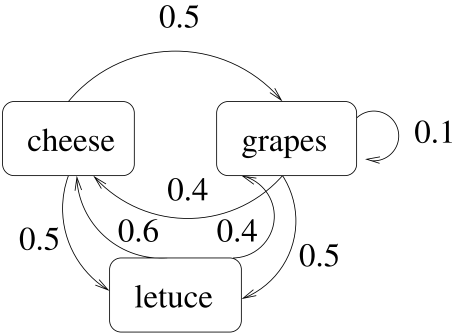
These eating habits can be modelled by Markov model in the next figure.

MARKOV CHAIN for EATING HABITS

A Markov chain is often described by a directed graph with edges labelled by probabilities for going from one state/node to another one.



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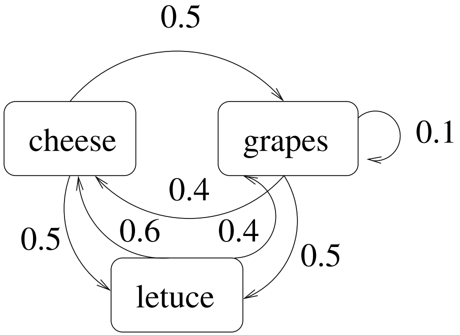

Eating habits of a creature.

One statistical property that can be computed is the percentage of days the creature eats grapes (or cheese).

EHRENFEST MODEL

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If we choose as states number of balls in the first urn, then the transition matrix, where rows and columns are labelled (from the top to bottom and from the left to right) **0, 1, 2, 3, 4** looks as follows

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$P(i, j)$ is probability that if first urn has i balls than in next step will have j bals.

ABSORBING DRUNKARD's WALK

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The transition matrix has therefore the form

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where rows and columns are labeled by $0, 1, 2, 3, 4$ and $P(i, j)$ is probability that the drunked man goes from the node i to the node j.

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If $f_{ii} < 1$, then each time the chain is in the state i, with probability $1 - f_{ii}$ will never return again to i. It therefore holds:

$$Pr[\text{The number of visits to } i \text{ from } i \text{ equals } k] = f_{ii}^k(1 - f_{ii}).$$

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If $h_{ii} < \infty$, then the state i is called **positive (non-null) recurrent/persistent**; otherwise it is called **null-recurrent/persistent**.

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A finite Markov chain all states of which are aperiodic and recurrent is called **ergodic.**

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If a state i is not transient, then it is called recurrent.

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Consider a Markov chain whose states are all positive integers.

From each state i the next states are the state $i+1$ (with probability $\frac{i}{i+1}$) and the state 1 (with probability $\frac{1}{i+1}$)

Starting at state 1, the probability of not having returned to state 1 within the first t steps is

$$\prod_{j=1}^{t} \frac{j}{j+1} = \frac{1}{t+1}.$$

Hence the probability of never returning to state 1 from 1 is 0, and state 1 is recurrent. It holds also

$$r_{1,1}^{t} = \frac{1}{t} \cdot \frac{1}{t+1} = \frac{1}{t(t+1)}.$$

However, the expected number of steps until the first return to state 1 from state 1 is

$$h_{1,1} = \sum_{t=1}^{\infty} t \cdot r_{1,1}^{t} = \sum_{t=1}^{\infty} \frac{1}{t+1}.$$

ERGODIC MARKOV MODEL REVISITED

An equivalent definition of ergodic Markov chains.

Definition A Markov chain with a transition matrix P is called ergodic if it is possible to go from every state to every state and there is an integer n such that all entries of the matrix P^n are positive.

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Let us have an ergodic Markov chain C with the set of states $S = \{1, \ldots, n\}$. Then, it holds:

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- The vector π is the only non-negative solution of the system of linear equalities

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Implications: Ergodic Markov chains always "forget", after a number of steps, their initial probability distribution.

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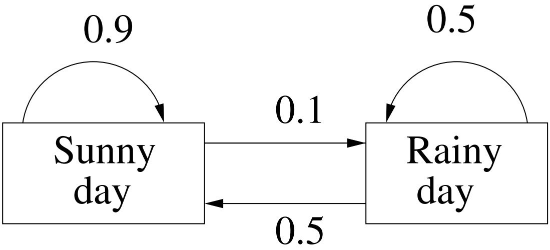
The following Markov chain *WF* depicts probabilities for going from sunny days to rainy and vice verse.



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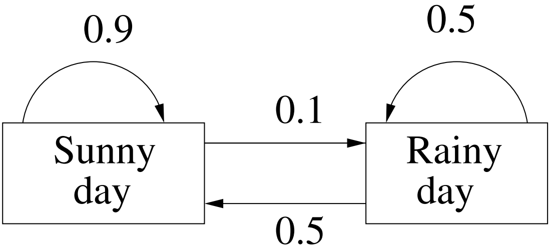

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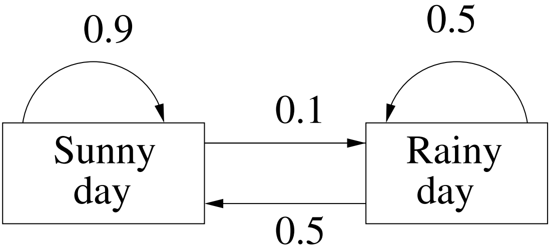

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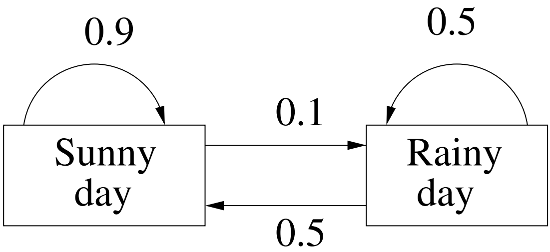

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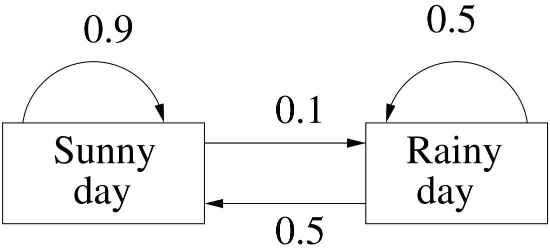

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This Markov chain is ergodic and therefore it has a unique stationary distribution π. It holds

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\begin{aligned}
\pi_0 &= \pi_0(1-\lambda) + \pi_1\mu \\
\pi_i &= \pi_{i-1}\lambda + \pi_i(1-\lambda\mu) + \pi_{i+1}\mu, 1 \leq i \leq n-1 \\
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- M_G is irreducible.
- Periodicity of M_G is the greatest common divisor of the length of all closed walks in G.
- Since G is undirected, there are closed walks of length 2;
- Since G is non-bipartite, it has odd cycles and therefore the *greatest common divisor* of all closed walks is 1. Hence, M_G **is aperiodic**.

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Lemma For all $v \in V$, $\pi_v = \frac{d(v)}{2m}$. **Proof** Let $[\pi P]_v$ be the v-th component of πP. Then

$$\pi_v = [\pi P]_v = \sum_u \pi_u P(u,v) = \sum_{(u,v)\in E} \frac{d(u)}{2m} \times \frac{1}{d(u)} = \sum_{(u,v)\in E} \frac{1}{2m} = \frac{d(v)}{2m}.$$

Corollary: For all $v \in V$,
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Comment: Google's page ranking algorithm is essentially a Markov chain over the graph of the web.

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Informally, in the time reversible Markov chains, for each pair of states i, j, the long-run rate at which the chain makes a transition from state i to state j equals the long-run rate at which the chain makes a transition from state j to state i.

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and therefore it holds $\pi P = \pi$. Hence π is the stationary distribution and can be computed from the above system of linear equations.

The reason why a reversible chain is called reversible is that if we start in the stationary distribution at time 0, then the sequence of random variables $(X_0, \ldots, X_t)$ has exactly the same distributions as the reversed sequence $(X_t, \ldots, X_0)$.

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3. P^*'s paths starting from the stationary distribution are reverse of P's paths starting from the same distribution.

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is also a Markov chain with transition probabilities

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REVERSIBLE and TIME REVERSIBLE MARKOV CHAINS

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Theorem: Suppose an ergodic irreducible Markov chain C has transition probabilities P_{ij}. If there are non-negative numbers x_i summing up to 1 and satisfying all equalities $x_i P_{ij} = x_j P_{ji}$ for all i, j, then C is time reversible and $x_i = \pi_i$ for all i.

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Let us have the graph G with nodes $\{1, 2, 3, 4, 5\}$ and with non-zero-weight edges:

$$w_{12} = 3, w_{14} = 1, w_{15} = 2$$

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and C_G is time reversible.

APPLICATIONS - SAMPLING

SAMPLING

Sampling in a set S according a given probability distribution π, on elements of S, is picking up an element $x \in S$ with probability $\pi(x)$.

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- If we perform such an experiment m times and Z_i be the value of Z at the ith run, and $W = \sum_{i=1}^{m} Z_i$, then

$$\mathbf{E}[W] = \mathbf{E}\left[\sum_{i=1}^{m} Z_i\right] = \sum_{i=1}^{m} \mathbf{E}[Z_i] = \frac{m\pi}{4}$$

and therefore $W' = (4/m)W$ **is a natural estimation for** π.

A natural question now is how good is the estimation of π that we get from

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Therefore, **taking m large enough we get an arbitrarily good approximation of π.**

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Let $X_0, X_1, \ldots$ be a run of such a Markov chain. The chain converges to the stationary distribution from any state X_0 and so after a sufficiently large number of steps r, the distribution of the state X_r will be close to the stationary distribution and so it can be used for required sampling.

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The next lemma show that this can be done in case we can introduce also self-loops.

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Notation: For any $x \in \Omega$ let $N(x)$ be the set of neighbours in the created graph and let $M > N = \max_{x \in \Omega} |N(x)|$.

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Lemma: For a finite state space Ω and a given neighbourhood structure and any $M > N$ design a Markov chain such that for any $x, y \in \Omega$

$$P_{x,y} = \begin{cases} 1/M & \text{if } x \neq y, y \in N(x); \\ 0 & \text{if } x \neq y, y \notin N(x); \\ 1 - N(x)/M & \text{if } x = y; \end{cases}$$

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It is easy to see that the chain is time reversible and so we can apply corresponding theorem from slide 211. Therefore for any x, $\pi_x = \frac{1}{|\Omega|}$.

EXAMPLE - INDEPENDENT SETS of a GRAPH - I.

A set S of nodes of a graph G is called independent if no two nodes in S are connected by an edge in G.

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Using the construction from previous lemma one can show that if x, y are neighbouring independent sets then $P_{x,y} = 1/|V|$ and the stationary distribution is the uniform distribution.

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   - a ball from the third box with probability $\frac{3}{36}$;
   - a ball from the fourth box with probability $\frac{4}{36}$;
   - a ball from the fifth b0x with probability $\frac{5}{36}$;
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Let us assume that we have already designed an irreducible state space (graph) and we want to construct a Markov chain on this state space with a stationary distribution $\pi_x = b(x)/B$, where $b(x) > 0$ for all $x \in \Omega$ and $B = \sum_{x \in \Omega} b(x)$ is finite.

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Theorem For a finite state space Ω, its neighbourhood structure $\{N(x) \mid x \in \Omega\}$, $N = \max_{x \in \Omega} |N(x)|$ let $M \geq N$ be any such number.

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Then, if this Markov chain is irreducible and aperiodic, then its stationary distribution is given by probabilities π_x.

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One way to do that is to design such a Markov chain M on the set S that has π as the stationary distribution and then to start a random walk on M and to stop when one can expect that stationary distribution is (almost) reached.

DOING SAMPLING - once more

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To find time for halting we need an estimation of the convergence rate of any initial distribution to the stationary one.

CONVERGENCE

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Lemma: Let P and Q be probability distributions over a set I of states. Then

$$||P - Q|| = \frac{1}{2} \sum_{i \in I} |P(i) - Q(i)|.$$

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It is sometimes required that $Pr(T < \infty) = 1$, or that T is almost surely finite.

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The intuition behind such a definition of the stopping rule is that at any particular time it is enough to look at the sequence of variables/states consideed so far in order to be able to tell if it is time to stop.

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- Playing until she doubles her money is not a stopping rule (here it is assured that betting systems has some limitations on doubling or tripling the money).
- Playing until she is the maximum amount ahead she will ever be is not a stopping rule (as it requires information about future, present and past).

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For a finite ergodic Markov chain, a **strong uniform stopping time** T is a stopping rule which satisfies the condition

$$\Pr[Z_k = i \mid T = k] = \pi_i,$$

where Z_k is the state at the kth step in the Markov chain, and π_i is the probability, under the stationary distribution, of being at the state i.

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Next theorem relates strong uniform stopping rule (time) and the rate of convergence of the random walk.

Theorem Let π be the stationary distribution of a random walk, and $Q^{(t)}$ be the probability distribution of that walk after t steps. In addition, let T be a strong uniform stopping time.

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Proof. Let X_t be the random variable producing states from I visited by a random walk at step t.

$$\begin{aligned}
\forall I' \subseteq I, Q^{(t)}(I') &= \Pr[X_t \in I'] \\
&= (\sum_{j \leq t} \Pr[(X_t \in I') \cap (T = j)]) + \Pr[(X_t \in I') \cap (T > t)] \\
&= \sum_{j \leq t} \Pr[X_t \in I' | T = j] \Pr[T = j] \\
&+ \Pr[X_t \in I' | T > t] \Pr[T > t] \\
&= \sum_{j \leq t} \pi(I') \Pr[T = j] + \Pr[X_t \in I' | T > t] \Pr[T > t] \\
&= \pi(I')(1 - \Pr[T > t]) + \Pr[X_t \in I' | T > t] \Pr[T > t] \\
&\leq \pi(I') + \Pr[T > t] \Pr[X_t \in I' | T > t] \leq \pi(I') + \Pr[T > t]
\end{aligned}$$

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Finally, the last equality implies, since both $\Pr[X_t \in I' | T > t]$ and $\pi(I')$ are at most 1,

$$\forall I' \subseteq I, |Q^{(t)}(I') - \pi(I')| \leq \Pr[T>t].$$

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Using Markov's inequality we get $\Pr[T > t] \leq \frac{\mathbf{E}[T]}{t} = \mathcal{O}(\frac{n \lg n}{t})$.

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Indeed, define T_i to be the number of steps until there are i cards bellow BOTTOM (including BOTTOM). Since $T = T_n$ we have

$$T = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1}).$$

Moreover, $T_{i+1} - T_i$ has a geometric distribution with parameter $\frac{n-i}{n}$.

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This is similar to the situation in coupon selection. Indeed, let V_i denote the number of steps until we see i coupons and let $V = V_n$. Similarity between V and T follows from the fact that

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Therefore, to determine how fast X approaches stationary distribution, it is sufficient to determine when such two random walks meet.

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3. If $X_i \neq Y_i$, then with probability $\frac{1}{2}$ player X moves and Y stay still, and with the same probability it is vice verse. (Move is again to the node that differs in the ith coordinate.)

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2. If $X_i = Y_i$, then with probability $\frac{1}{2}$ both X and Y move, and with probability $\frac{1}{2}$ both of them stay still. The move, if any, is to the neighbor that differs in the ith bit from the node currently visited.
3. If $X_i \neq Y_i$, then with probability $\frac{1}{2}$ player X moves and Y stay still, and with the same probability it is vice verse. (Move is again to the node that differs in the ith coordinate.)

From the point of view of both, X and Y, they perform a random walk.

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If, for some i, $X_i = Y_i$, then it always stays as such. If a coordinate i is chosen, (and $X_i \neq Y_i$), then $X_i = Y_i$ at the end of the step.

Hence, the random variable that counts the number of steps needed for X and Y to meet behaves as a coupon collector.

This way we get the same result as using the previous method.

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> *Uniformly and randomly choose a card and put it, alternatively, either to the top or to the bottom of the pack.*

To demonstrate *the coupling argument* we will consider two initial packs of cards. One that is fully ordered (first is the card number 1, last the card number n), second that is randomly ordered.

The *coordinator* chooses, randomly, a card i and in both packs the card is moved alternatively to top or to bottom.

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Observe now that the problem of picking all cards is again similar to the coupon selector problem. The expected number of steps needed to shuffle the pack is therefore $\mathcal{O}(n \lg n)$.

FIXING A SPANNING TREE 1/3

For a graph $G = (V, E)$ we want to generate randomly a spanning tree of G. (To make the exposition simple, we assume that edges of the spanning tree will always be oriented towards a specific node r, called the *root* of the tree.)

Consider a Markov chain the states of which are all possible pairs (T, r), where T is a spanning tree of G, and r is the root of T. (The number of states in this Markov chain is $s|V|$, where s is the number of spanning trees of G.) A random walk on this Markov chain will now be defined as follows:

1. Randomly choose an edge $e \in E$ that connects the root r to a node y;
2. If the edge e is in the tree (and therefore directed from y to r), then change its direction - making by that y to be the root of the (new) tree;
3. If e is not in the tree, add e to the tree and direct it from r to y and delete from the tree the (unique) outgoing edge leaving y.

Observe:

- The indegree of each state of the Markov chain is equal to the degree of the root of the corresponding tree (in the graph) - because a state (T, r) can be reached from all neighbours of r in G;
- The outdegree of each state is equal to the degree of the root of the corresponding tree (in the graph) - argument is the same as in the previous case;

FIXING A SPANNING TREE 2/3

We use now coupling argument to find out how fast the random walk approaches the the stationary distribution.

Consider two random walks: first walk starts at the stationary distribution; second starts at the arbitrary state.

Stage 1 Two walks will progress independently until they have the same node as the root (observe that trees might still be different).
Stage 2 Once both walks agree on a root, from now on, they will make the same moves.

Observe that if we imagine the root of the tree as a particle, then it (implicitly) performs a random walk in the graph G - this observation is crucial for the following analysis.

Let us now calculate the expected number of steps until two random walks meet - that is that they generate the same spanning tree.

Calculating the expected number of steps in Stage 1 is equivalent to the following problem: Two particles are moving randomly in a graph. What is the expected number of steps they meet?

FIXING A SPANNING TREE 3/3

Lemma The expected number of steps until two particles meet is at most twice the cover time of the graph.

Concerning the expected number of steps in Stage 2, observe first that at the beginning of this stage both spanning trees (in the corresponding states of the two random walks), have the same root.

Observe that when a new root is chosen, there is an edge connecting the old root to the new root, and it exists in both trees, implying that both spanning trees remain in the same root.

Suppose the root node is switched from r to r'. Notice that, from now on, the (unique) outgoing edge of node r, will remain the same in both trees. This happens, since the outgoing edge of a node r changes when either r becomes the root, or when it ceases to be the root.

It follows, that a sufficient condition for the two trees to be identical, is that each node in the graph is a root of the tree at some point in time.

This implies that an upper bound on the time for the two trees to converge at Stage II is the cover time of G.

Conclusion: the expected number of steps, in both stages, until the spanning trees become identical, depends linearly on the cover time of the graph.

RANDOM WALKS on GRAPHS - APPLICATIONS

COMMUTE and COVER TIME

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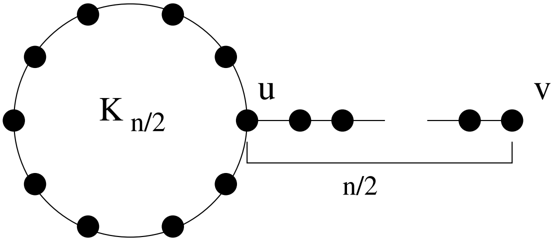
The **cover time** of G, notation $C(G)$, is defined by

$$C(G) = \max_u C_u(G).$$

EXAMPLES

For the **lollipop graph** (cukrátko) L_n



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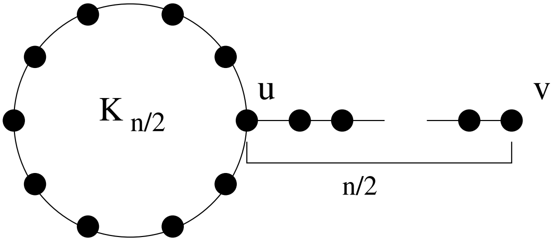

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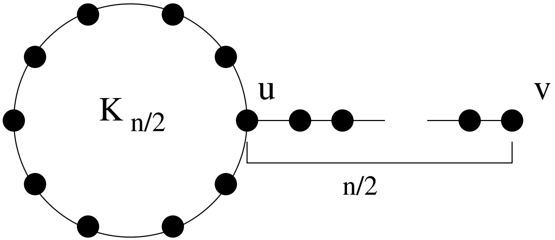

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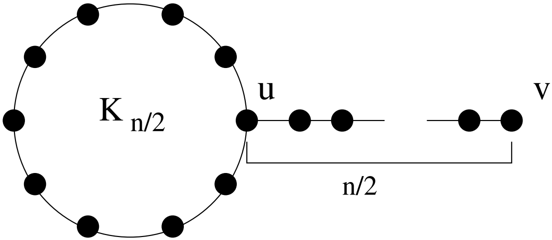

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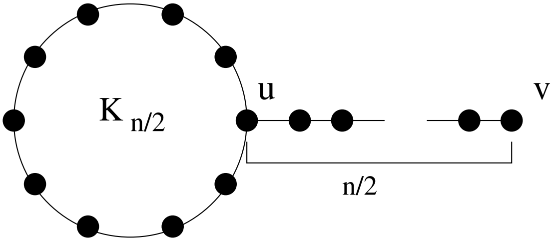

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- L_n has cover time $\theta(n^3)$.
- Complete graph K_n has the cover time $\theta(n \lg n)$.

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Proof: Let us assign to G a new Markov chain, $\bar{M}_G$, states of which are oriented versions of the edges of E ($2m$ of them), and let the transition matrix Q of $\bar{M}_G$ have only the following non-zero values

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Matrix Q is **doubly stochastic** (all rows and also columns sum-up to 1). Indeed, for each $v, w \in V$:

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It can be shown, that for any Markov chain with a doubly stochastic matrix the uniform distribution is stationary.

Therefore, stationary probability of each state (an edge of $\bar{M}_G$ is $\frac{1}{2m}$. Consequently, (by Ergodic theorem), the expected time between successive traversals of the directed edge (v, u) is $2m = 1/(1/2m)$.

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The memoryless property of Markov chains allows now to remove conditioning and to get the Lemma.

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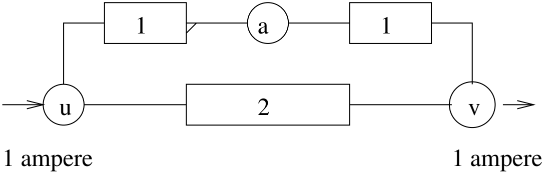

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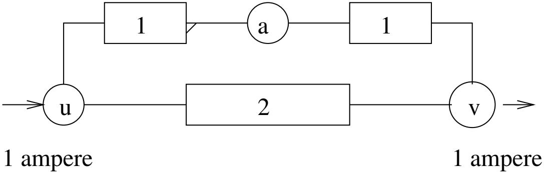

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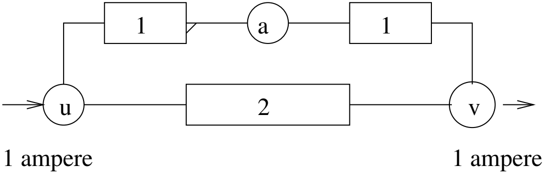

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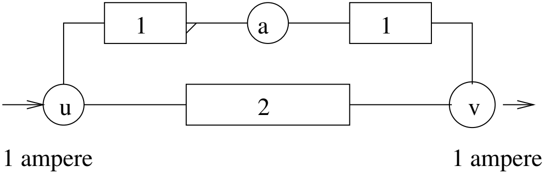

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Elektrický potenciál je práca potrebná k preneseniu jednotkového náboja z nekonečna na dané miesto silového poľa po ľubovolnej dráhe.

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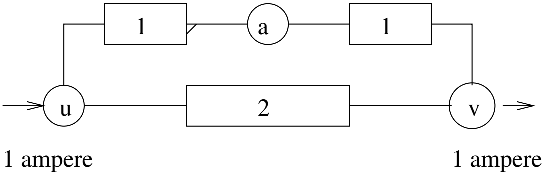

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We would like to compute the potential $\phi(v)$ for each node v and the current i_{uv} for each edge $e = (u, v)$. By Ohm's Law (V=IR),

$$i_{uv} = \frac{\phi(u) - \phi(v)}{R_e}.$$

Effective resistance R_{uv} between two nodes u and v is the voltage difference between u and v when one ampere is injected into u and removed from v. Hence $R_{uv} = \phi(u) - \phi(v)$.

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To each graph G we associate an electrical network $\mathcal{N}(G)$ at which the resistance of each edge is 1.

Theorem Commute time for any two vertices u and v in G is

$$C_{uv} = 2mR_{uv} = h_{uv} + h_{vu}.$$

Proof: Notation:

- $d(x)$ is the degree of any node x.
- Φ_{zv} is the potential of any node z, relative to v, when $d(x)$ units of current enter each node $x \in V$, and $2m$ units of the current exit v.

For every edge $(u, w) \in E$ it holds, by Ohm's law:

$$\Phi_{uv} - \Phi_{wv} = i_{uw}R_{uw} = i_{uw}.$$

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It can be shown, that both above systems of equations are in fact the same system of linear equations and therefore they have the same solution for each $u, v \in V$: $h_{uv} = \Phi_{uv}$.

DERIVATION

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Hence,

$$d(u) = \Phi_{uv} d(u) - \sum_{w \in \Gamma(u)} \Phi_{wv}$$

and therefore

$$1 = \Phi_{uv} - \frac{1}{d(u)} \sum_{w \in \Gamma(u)} \Phi_{wv}$$

and

$$\Phi_{uv} = 1 + \frac{1}{d(u)} \sum_{w \in \Gamma(u)} \Phi_{wv}.$$

Hence

$$\Phi_{uv} = \frac{1}{d(u)} \sum_{w \in \Gamma(u)} 1 + \frac{1}{d(u)} \sum_{w \in \Gamma(u)} \Phi_{wv} = \sum_{w \in \Gamma(u)} \frac{1}{d(u)} (1 + \Phi_{wv}).$$

and so we got the same equation as $h_{uv} = \sum_{w \in \Gamma(u)} \frac{1}{d(u)} (1 + h_{wv})$.

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Let us now perform a superposition of such a network with the network considered on previous slide (at which $d(x)$ units of current enter each node $x \in V$, and $2m$ units of the current exit v).

In the resulting network, all external currents cancel, except for those in vertices u (where the current of magnitude $2m$ enters) and v (where the current of magnitude $2m$ exits).

PROOF – CONTINUATION I.2

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Hence, by Ohm law ($I = VR$),

$$C_{uv} = 2mR_{uv}$$

Claim: For every pair of vertices u and v, the effective resistance R_{uv} is not more than the distance between u and v in G.

Corollary: Let $G = (V, E)$ be a graph, $n = |V|$, $m = |E|$ and $u, v \in V$.

- If $(u, v) \in \mathcal{N}(G)$, then $C_{uv} \leq 2m$;
- If $u, v \in V$, then $C_{uv} \leq 2m(n-1)$.
- If $u, v \in V$, then $C_{uv} < n^3$.

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The above bound is independent of the choice of v_0. Hence $C(G) \leq 2m(n-1)$.

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- **Lollipop graph** L_n**:** $C(L_n) = \theta(n^3)$.

EFFECTIVE RESISTANCE of GRAPHS - II.

The **effective resistance** $R(G)$ of a graph G is defined by

$$R(G) = \max_{\{u,v\} \subset V(G)} R_{uv}.$$

Theorem $mR(G) \leq C(G) \leq 2e^3 mR(G) \ln n + n.$

Proof. Lower bound: Let $R(G) = R_{uv}$ for some vertices $u, v \in V$. Then

$$C(G) \geq \max(h_{uv}, h_{vu}) \geq \frac{C_{uv}}{2} = \frac{2mR_{uv}}{2} = mR(G).$$

Upper bound. Create a random walk of the length $2e^3mR(G)\ln n$ and divide it into $\ln n$ **phases** of the same length $[= 2e^3mR(G)]$.

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For any vertices u and v, the hitting time h_{uv} is at most $2mR(G)$. (This is the average time to get through any of $\ln n$ phases.)

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The expected total time is therefore

$$2e^3mR(G)\ln n + (\frac{1}{n^2})n^3 = 2e^3mR(G)\ln n + n.$$

APPLICATION of RAYLEIGHT's MONOTONICITY LAW

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Rayleight's monotonicity law states that the effective resistance of a graph is non-increased (non-decreased), whenever the resistance of any edge of the graph is decreased (increased).

Corollary: effective resistance of graphs can not increase by adding edges.

Lemma Effective resistance of graphs is not more than its diameter diam(G).

Proof The whole graph can be generated by adding edges to the subgraph that corresponds to the diameter.

Fact: If G is a k-regular graph with n edges, then $\text{diam}(G) \leq \frac{3n}{k}$.

Theorem If G is a k-regular graph with n edges, then $C(G) = \mathcal{O}(n^2 \ln n)$.

Proof. Since

$$n \geq \frac{k \cdot \text{diam}(G)}{3},$$

and, by the last theorem, $C(G) = \mathcal{O}(mR(G) \ln n)$, we have $R(G) \leq \text{diam}(G) \leq \frac{3n}{k}$ and

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Let **RLP** be the family of languages L for which there exists a probabilistic off-line log-space TM M such that for any input x

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Theorem USTCON $\in$ **RLP**.

Proof Let a log-space bounded probabilistic TM $\mathcal{M}$ simulate a random walk of length $2n^3$ through the given graph starting from s.

If $\mathcal{M}$ encounters t during such a walk, it outputs YES, otherwise it outputs NO. The probability of the output YES instead of NO is 0.

What is the probability that $\mathcal{M}$ outputs NO instead of YES?

We know that $h_{st} \leq n^3$. By Markov inequality, if t is reachable from s, then the probability that t is not visited during $2n^3$ steps is at most $\frac{1}{2}$.

$\mathcal{M}$ needs a space to count till $2n^3$ and to keep track of its position in the graph during the walk. Therefore it needs space

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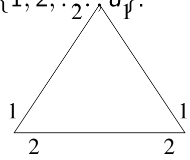

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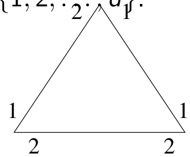

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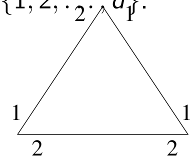

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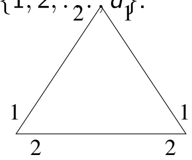

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A universal traversal sequence whose length is polynomial in n can be used by a deterministic log-space off-line TM to decide instances of USTCON.

(However, in order to be a uniform log-space algorithm, the universal traversal sequence should be constructable by a log-space TM, rather than be encoded in

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$U(\mathcal{G})$ — length of the shortest universal traversal sequence for $\mathcal{G}$.
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Theorem $U(\mathcal{G}) \leq 5mR(\mathcal{G}) \lg(n|\mathcal{G}|)$.

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Proof Given $G \in \mathcal{G}$, $v \in G$, let us consider a random walk of the length

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$R(\mathcal{G})$ — maximum resistance between any two nodes of any graph in $\mathcal{G}$,

Theorem $U(\mathcal{G}) \leq 5mR(\mathcal{G})\lg(n|\mathcal{G}|)$.

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divided into $\lg(n|\mathcal{G}|)$ sections of length $5mR(\mathcal{G})$.

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As a consequence, there is a sequence of such a length that is universal for the class $\mathcal{G}$. (We have just used the probabilistic method.)

HMM
HIDDEN MARKOV MODELS

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In general, HMM can be applied when the goal is to recover a data sequence that is not immediately observable (but other data that depend on the sequence are).

HMM - Figure

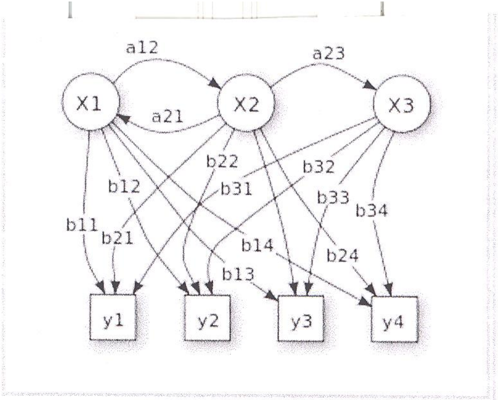

Figure 1. Probabilistic parameters of a hidden Markov model (example)

X — states

y — possible observations

a — state transition probabilities

b — output probabilities

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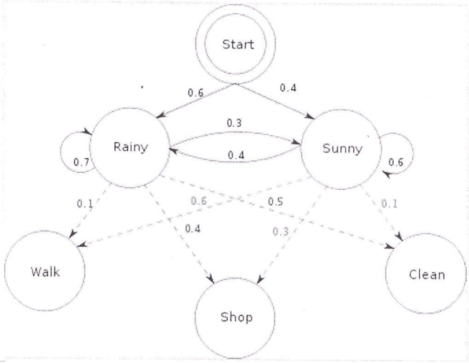
- This process can continue many times. Observes see each time only a sequence of labels $y_{i1}, y_{i2}, \ldots .y_{ik}$.
- The task for observers is to determine parameters: transition probabilities for states (of an ordinary Markov chain behind) and the number of different balls in different urns (and emission probabilities - actually number of different balls in urns).

EXAMPLE - WEATHER

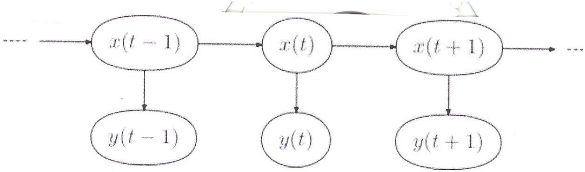
Alice and Bob live far apart from each other and talk daily about what Bob did. His actions (waking, shopping, cleaning) depend on the weather in the following way.



From their phone calls Alice tries to deduce how was and is weather in the place Bob lives.

INFERENCE PROBLEMS

In the following picture $x(t)$ is the state at time t and $y(t)$ is the output at time t.



- **Probability of observed sequence**: The probability of observing an output sequence

$$Y = y(0), y(1), \dots, y(l-1)$$

of length l is given by

$$Pr(Y) = \sum_X Pr(Y|X)Pr(X)$$

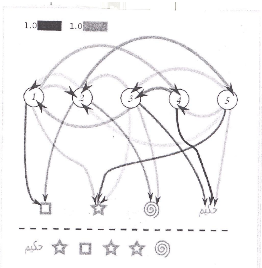
where the sum runs over all possible hidden-node sequences $X = x(0), x(1), \dots, x(l-1)$. This problem can be handled effective using so called forward algorithm.

Filtering: The task is to compute, given the chain's parameters and a sequence of observations, the last states at the end of observations:, i.e. to compute

$$Pr(x(t) \,|\, y(1), \ldots, y(t))$$

EXAMPLE

In the following HMM and its output sequence



the following state sequences are possible:

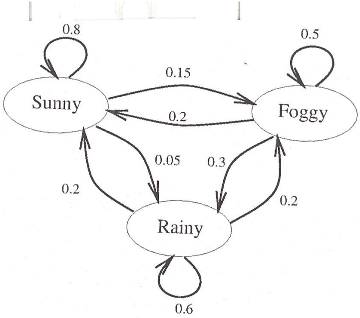
$$5, 3, 2, 5, 3, 2$$

$$4, 3, 2, 5, 3, 2$$

$$3, 1, 2, 5, 3, 2$$

EXAMPLE 2. Markov model

For the Markov model



show that:

- Provided that today is sunny, show that 0.04 is probability that tomorrow is sunny and the day after is rainy.
- Show that 0.34 is probability that it will be rainy two days from now provided it is foggy today.

EXAMPLE 2. Hidden Markov Model

Let us add to the previous model two outputs "umbrella" and "no umbrella" and let probability of having umbrella be 0.1 (0.8) [0.3] for the sunny (rainy) [foggy] day.
Supposed you were locked in a room for several days and you were asked about weather outside. The only piece of evidence you have is whether a man bringing you food carries umbrella or not.

- Suppose the day you were locked in was sunny. The next day man carrying food came with the umbrella. Assume that the prior probability of the man carrying an umbrella on any day is 0.5. Show that 0.08 is the probability that the second day was rainy.

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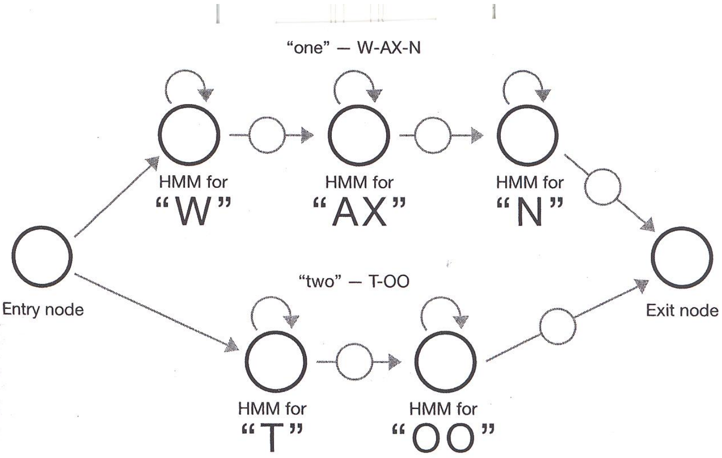
- Suppose the day you were locked in was sunny. The next day man carrying food came with the umbrella. Assume that the prior probability of the man carrying an umbrella on any day is 0.5. Show that 0.08 is the probability that the second day was rainy.
- Suppose the day you were locked in the room was sunny and that man brought an umbrella on day 2 but not on day 3. Show that 0.19 is the probability that it was foggy on day 3.

HMM - speech recognition - example



A simple hidden Markov model topology to recognize two spoken words.

HIERARCHICAL HIDDEN MARKOV MODEL

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In **Hierarchical Hidden Markov Model (HHMM)** each state can itself be a HHMM.

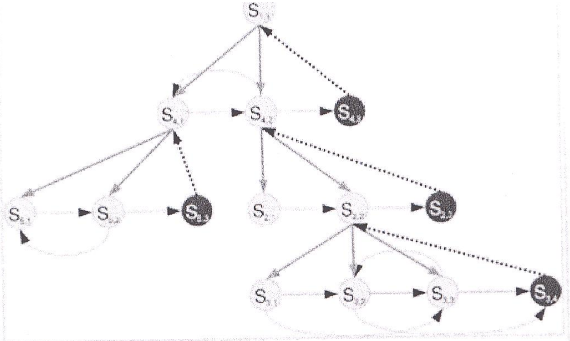

Illustration of the structure of a HHMM. Gray lines show vertical transitions. The horizontal transitions are shown as black lines. The light gray circles are the internal states and the dark gray circles are the terminal states that returns control to the activating state. The production states are not shown in this figure.

HHMM and SPEECH RECOGNITION

A huge amount of samples of speech, from many different individuals, are applied to a HHMM to infer the hierarchy of states and all transition and transmission probabilities (essentially a simulation of neocortex for producing speech), and then the resulting HHMM is used to recognize new utterances.

Appendix

APPENDIX

SECRETARY PROBLEM

The problem:

- There is a single secretariat position to fill.
- There are n applicants for the position, and the value of n is known.
- Each applicant has a unique "quality value" - the interview making committee has no knowledge of quality values of those applicants that have not been interviewed yet and no knowledge how large is the best quality value of applicants.
- The applicants are interviewed in a random order.
- After each interview, the applicant is immediately accepted or rejected.
- The decision to accept or reject an applicant can be based only on the relative "quality value" of the applicants interviewed so far.
- Rejected applicants cannot be recalled.
- The goal is to select an applicant with the best 'quality value". The payoff is 1 for the best applicant and 0 otherwise.
- How should selection committee proceeds at the best?

SOLUTION

Terminology: A candidate is an applicant who, when interviewed, is better than all the applicants interviewed previously.

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Terminology: A candidate is an applicant who, when interviewed, is better than all the applicants interviewed previously. Since the goal in the secretary problem is to select the single best applicant, only candidates will be considered for acceptance.

Optimal policy for this problem (the stopping rule): For large n the optimal policy is to interview and reject the first $\frac{n}{e}$ applicants and then accept the next one who is better than candidates interviewed till then.

As n gets larger, the probability of selecting the best applicant goes to $\frac{1}{e}$, which is around 37%.

ANDREY ANDREEVITCH MARKOV

- Russian mathematician (1856-1922)
- Introduced Markov Models in 1906
- The original motivation was to extend the law of large numbers to dependent events.
- In 1913 he applied his findings to the first 20 000 letters of Pushkin's Eugene Onegin