Part I

Basic Techniques II: Concentration Bounds

Two very important inequalities

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and this is one of the main reasons why variance is used.

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At the end of the chapter we will deal with special sequences of dependent random variables, so called martingales, and also with tail bounds for martingales. That will then be applied also to the occupancy problem.

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- Of the large importance is the case X is the sum of random variables. For the case that these random variables are independent we derive so called Chernoff bound.
- For the case that random variables of the sum are dependent, but form so called martingale we get so called Azuma-Hoeffding bound.

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Very important **Chernoff bounds** on the sum of **independent Poisson trials** are obtained when the moment generating functions of X are considered.

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Taking $t = \ln(1+\delta)$ we get our Theorem (and basic Chernoff bound), that is:

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Observe three tricks that have been used in the above proof!

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By applying the last theorem ($\Pr(X > (1+\delta)\mu) > F(\mu,\delta)$), we get for $\mu = \frac{n}{3}$ and $\delta = \frac{1}{2}$,

$$Pr\left[Y_n > \frac{n}{2}\right] < F^{+}\left(\frac{n}{3}, \frac{1}{2}\right) < (0.915)^n \qquad -\text{exponentially small in } n$$

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Corollary: For $0 < \delta < 1$

$$Pr(|X-\mu| \geq \delta\mu) \leq 2e^{-\mu\delta^2/3} \tag{5}$$

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Indeed, the deviations from the mean are on the order of $\mathcal{O}(\sqrt{n\ln n})$.

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Chernoff's method therefore gives an exponentially smaller upper bound than the upper bound obtained using Chebyshev's inequality.

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In such a case $\mu = 0.75n, \delta = \frac{1}{3}$ and for $Y_n = \sum_{i=1}^{n} X_i$ we have

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and therefore the probability decreases exponentially fast in n.

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and if we want that this bound is less than an ε, then we get

$$Pr\left[|X - \mu| \geq \sqrt{3\mu \ln(2/\varepsilon)}\right] \leq \varepsilon$$

provided $\varepsilon \geq 2e^{-\mu\delta^2/3}$.

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$$
\begin{aligned}
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&= \sqrt{3\mu\cdot\mu\delta^2/3} \\
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Df.: $\Delta^{+}(\mu,\varepsilon)$ is a number such that $F^{+}(\mu,\Delta^{+}(\mu,\varepsilon)) = \varepsilon$.
$\Delta^{-}(\mu,\varepsilon)$ is a number such that $F^{-}(\mu,\Delta^{-}(\mu,\varepsilon)) = \varepsilon$.

In other words, a deviation of $\delta = \Delta^{+}(\mu,\varepsilon)$ suffices to keep $Pr[X > (1+\delta)\,\mu]$ bellow ε (irrespective of the values of n and p_i's).

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because $\Delta^{-}(\mu, \varepsilon)$ is a number such that $F^{-}(\mu, \Delta^{-}(\mu, \varepsilon)) = \varepsilon$.

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EXAMPLE 2 - MONTE CARLO METHOD - I

In this example we illustrate how Chernoff bound help us to show that a simple Monte Carlo algorithm can be used to approximate number π through sampling.

The term Monte Carlo method refers to a broad collection of tools for estimating various values through sampling and simulation.

Monte Carlo methods are used extensively in all areas of physical sciences and technologies.

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- If we perform such an experiment m times and Z_i be the value of Z at the ith run, and $W = \sum_{i=1}^{m} Z_i$, then

$$\mathbf{E}[W] = \mathbf{E}\left[\sum_{i=1}^{m} Z_i\right] = \sum_{i=1}^{m} \mathbf{E}[Z_i] = \frac{m\pi}{4}$$

and therefore $W' = (4/m)W$ is a natural estimation for π.

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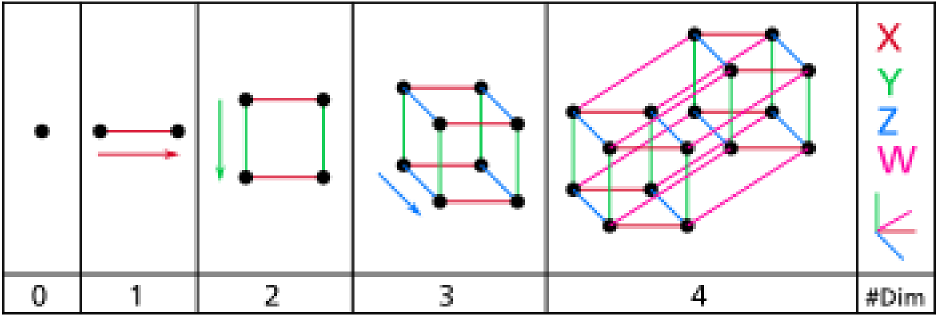
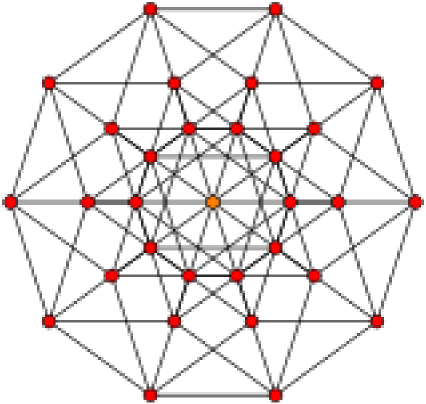
- Therefore, by taking m sufficiently large we can get an arbitrarily good approximation of π

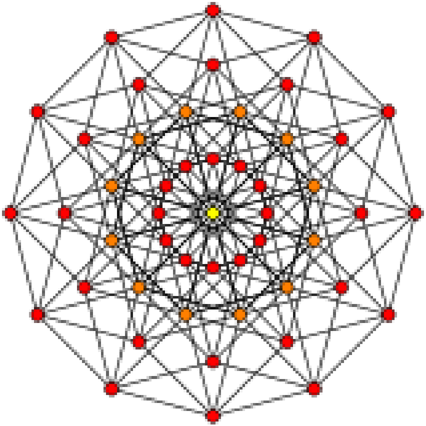
HYPERCUBES



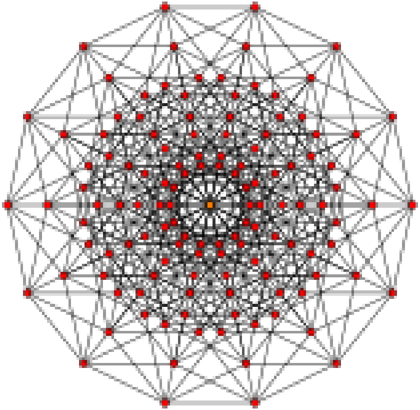
5-d hypercube



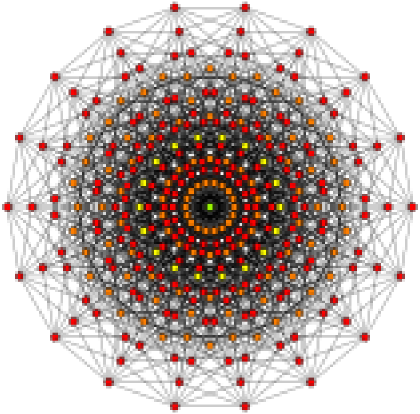
6-d hypercube



7-d hypercube



9-d hypercube



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We assume each node contains one **queue** for each edge. A routing algorithm must therefore specify also a **queueing discipline**.

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Left-to-right routing on hypercube H_d requires sometimes $\Omega\left(\sqrt{\frac{2^d}{d}}\right)$ steps.

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Queueing discipline: FIFO for each edge.
(Actually any queueing discipline is good provided at each step there is a packet ready to travel.)

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Fact: For any two packets v_i, v_j there is at most one queue q such that v_i and v_j are in the queue q at the same time.

Lemma: Let the route of a packet v_i follow the sequence of edges $\rho_i = (e_1, e_2, ..., e_k)$. Let S be the set of packets (other than v_i), whose routes pass through at least one of the edges $\{e_1, ..., e_k\}$. Then the delay the packet v_i makes is at most $|S|$.

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We show now that at each step at which the lag of v_i increases by 1, the lag can be charged to a distinct member of S.

If the lag of v_i reaches a number $l + 1$, some packet in S leaves ρ_i with lag l. (When the lag of v_i increases from l to $l + 1$, there must be at least one packet (from S) that wishes to traverse the same edge as v_i at that time step.) Thus, S contains at least one packet whose lag is l.

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Thus, each member of S whose route intersects ρ_i is charged for at most one delay, what proves the lemma.

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Fix any route $\rho_i = (e_{i,1}, e_{i,2}, ..., e_{i,k})\,, k \leq d$. Then

$$\sum_{j=1}^{N} H_{ij} \leq \sum_{j=1}^{k} T(e_{i,j}) \Rightarrow \mathbf{E}\left[\sum_{j=1}^{N} H_{ij}\right] \leq \sum_{j=1}^{k} \mathbf{E}\left[T(e_{i,j})\right]$$

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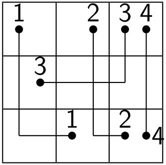

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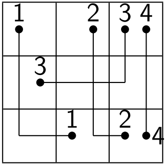

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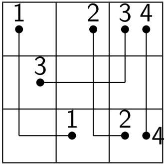

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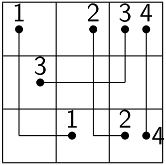

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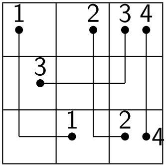

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Problem; how to decide for each net which of the following connections to use:

(that is vertical first and horizontal (right or left) next or vice verse) in order to get wiring S with minimal W_S.

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$x_{i0} = 1 \Leftrightarrow i$-th route step has the form (horiz.=vert.) ┐
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The idea of randomized rounding is to interpret the fractional solutions provided by the linear program as probabilities for the rounding process.

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$$W_S \leq \widehat{W}\left(1 + \Delta^+\left(\widehat{W}, \frac{\varepsilon}{2n}\right)\right) \leq W_0\left(1 + \Delta^+\left(W_0, \frac{\varepsilon}{2n}\right)\right)$$

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Since a $\sqrt{n} \times \sqrt{n}$ array has at most $2n$ boundaries, one has to sum the above probability of failure over all boundaries b to get an upper bound of ε on the failure probability.

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Therefore $W_S(b)$ is the sum of independent Poisson trials.

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Since $\Delta^+\left(\widehat{W}, \frac{\varepsilon}{2n}\right)$ is such that

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Theorem Let $X_1 \ldots, X_n$ be independent random variables with $\mathbf{E}[X_i] = 0$ and $|X_i| \leq c_i$ for all i and some constants c_i. Then for all t,

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In the case x_i are dependent, but form so called **martingale** Hoeffding inequality can be generalized and we get so called **Azuma-Hoeffding inequality**.

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For analysis of randomized algorithms of large importance is that, as a general rule of thumb says, most things that work for sums of independent random variables work also for martingales.

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- Because each game is fair we have $\mathbf{E}[X_i] = 0$
- $\mathbf{E}[Z_{i+1}|X_1, X_2, \ldots, X_i] = Z_i + \mathbf{E}[X_{i+1}] = Z_i$

Thus $Z_1, Z_2, \ldots, Z_n$ is martingale with respect to the sequence $X_1, X_2, \ldots, X_n$.

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Theorem For any random variables Y, Z it holds

$$\mathbf{E}[Y] = \mathbf{E}[\mathbf{E}[Y|Z]].$$

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The rule "Last time the gambler wins 4 times in a row" is not a stopping time.

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- Note that the sequence $X_0, X_1, \ldots, X_n$ relates to the counting process in a backward order - X_0 is a function of S_n,....

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- Therefore the probability of Case 1, in which candidate A leads through the account, is

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Then, for all $t \geq 0$ and any $\lambda > 0$

$$Pr(|X_t - X_0| \geq \lambda) \leq 2e^{-\lambda^2/(2\sum_{i=1}^{t} c_i^2)}$$

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- By Azuma-Hoeffding inequality/theorem,

$$Pr(|F_{P,S} - \mathbf{E}[F_{P,S}]| \geq \varepsilon) = Pr(|(Z_n - Z_0)| \geq \varepsilon) \leq 2e^{-\varepsilon^2/2nk^2}.$$

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Let each gambler start by borrowing 1$ and betting that the next coin-flip will be x_1. If she wins, she takes her 2$ and bets 2$ that next coin-flip will be x_2, continuing to play double-or-nothing until either she loses (and is out of her initial 1$) or wins her last bet on x_k (and is up $2^k - 1$ dollars).

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Suppose that $x_1x_2\dots x_n$ is the pattern we want to get.

Let us imagine we have an army of gamblers, and let one new shows up before each new coin flip.

Let each gambler start by borrowing 1$ and betting that the next coin-flip will be x_1. If she wins, she takes her 2$ and bets 2$ that next coin-flip will be x_2, continuing to play double-or-nothing until either she loses (and is out of her initial 1$) or wins her last bet on x_k (and is up $2^k - 1$ dollars).

Because each gambler's winnings form a martingale, so does their sum, and so the expected total return of all gamblers up to the **stopping time** τ at which our pattern occurs for the first time is 0.

The above facts will now be used to compute $\mathbf{E}[\tau]$.

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Let $\chi_i = 1$ if $x_1 \ldots x_i = x_{k-i+1} \ldots x_k$, and 0 otherwise. Then, using the stopping time theorem,

$$\mathbf{E}[X_\tau] = \mathbf{E}\left[-\tau + \sum_{i=1}^{k} \chi_i 2^i\right] = -\mathbf{E}[\tau] + \sum_{i=1}^{k} \chi_i 2^i = 0$$

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Examples: if pattern is HTHH (HHHH) [THHH], then $\mathbf{E}[\tau]$ equals 18 (30) [16].

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For $0 \leq t \leq m$ let Z_t be the expectation at time t of the number of bins that are empty at time m. The sequence of random variables

$$Z_0, Z_1, \ldots, Z_m$$

is a martingale, $Z_0 = \mathbf{E}[Z]$ and $Z_m = Z$.

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Kolmogorov-Doob inequality Let $X_0, X_1, \ldots$ be a martingale. Then for any $\lambda > 0$

$$\Pr[\max_{0 \leq i \leq n} X_i \geq \lambda] \leq \frac{\mathbf{E}[|X_n|]}{\lambda}.$$

Azuma inequality Let $X_0, X_1, \ldots$ be a martingale sequence such that for each k

$$|X_k - X_{k-1}| \leq c_k,$$

then for all $t \geq 0$ and any $\lambda > 0$

$$\Pr[|X_t - X_0| \geq \lambda] \leq 2\exp\left(-\frac{\lambda^2}{2\sum_{k=1}^{t} c_k^2}\right).$$

Corollary Let $X_0, X_1, \ldots$ be a martingale sequence such that for each k

$$|X_k - X_{k-1}| \leq c$$

where c is independent of k. Then, for all $t \geq 0$ and any $\lambda > 0$

$$\Pr[|X_t - X_0| \geq \lambda c\sqrt{t}] \leq 2e^{-\lambda^2/2},$$

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Azuma inequality allows to show:

$$\mu = \mathbf{E}[Z] = n(1 - \frac{1}{n})^m \approx ne^{-m/n}$$

and for $\lambda > 0$

$$\Pr[|Z - \mu| \geq \lambda] \leq 2e^{-\frac{\lambda^2(n-1/2)}{n^2-\mu^2}}.$$

APPENDIX

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$$e^{\pi/e-1} > \pi/e ==> e^{\pi/e} > \pi ==> e^{\pi} >$$