Part I

Basic Techniques I: Moments and Deviations

Chapter 5. BASIC TECHNIQUES: MOMENTS and DEVIATIONS

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Surprisingly, **these simple probability problems are at the core of the analyses of many randomized algorithms**.

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Therefore, this inequality allows to analyse phenomena with very complicated interactions (without revealing these interactions).

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$$\binom{n}{k} \leq \frac{n^k}{k!}, \quad \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k, \quad \left(\frac{n}{k}\right)^k \leq \binom{n}{k}$$

For large n

$$\binom{n}{k} \sim \frac{n^k}{k!}.$$

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nth Harmonic number H_n is defined as follows

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \ln n + \theta(1).$$

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The probability that bin 1 receives exactly i balls is

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$$Pr[\xi_1(k^*)] \leq \left(\frac{e}{k^*}\right)^{k^*}\frac{1}{1-\frac{e}{k^*}} \leq n^{-2}.$$

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Corollary With the probability at least $1 - \frac{1}{n}$ no bin has more than

$$k^* = \frac{e \ln n}{\ln \ln n}$$

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- Using Bucket sort we can sort such numbers in the expected time $\mathcal{O}(n)$.

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Because of the assumption that the elements to be sorted are chosen uniformly, the number of elements that land uniformly in a bucket follows the binomial distribution $B(n, \frac{1}{n})$ introduced in Chapter 3.

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The expected time to do this sorting is therefore

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$$\mathbf{E}[X_1^2] = \frac{n(n-1)}{n^2} + 1 = 2 - \frac{1}{n} < 2$$

and therefore the expected time of the bucket sort is at most $2cn$.

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$$\bar{p}(n) = 1(1 - \frac{1}{365})(1 - \frac{2}{365})(1 - \frac{3}{365}) \dots (1 - \frac{k-1}{365}) = \prod_{j=1}^{k-1}(1 - \frac{j}{365})$$

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Hence the probability that some k people in the rom all have different birthdays from a set of n possible birthdays is $\frac{1}{2}$ and it is approximately given by the equation

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what gives, for the case $n = 365$, $k = 22.49$

$$\prod_{j=1}^{k-1}(1-\frac{j}{365}) = \frac{\prod_{j=1}^{k-1}(365-j)}{365^{k-1}} \cdot \frac{(365-k)!}{(365-k)!}$$

$$= \frac{(365-1)!}{365^{k-1}(365-k)!} \cdot \frac{(365)}{(365)} = \frac{365!}{365^{k}(365-k)!} \cdot \frac{k!}{k!} = k! \binom{365}{k} 365^{-k}.$$

VARIATIONS on BIRTHDAY PARADOX

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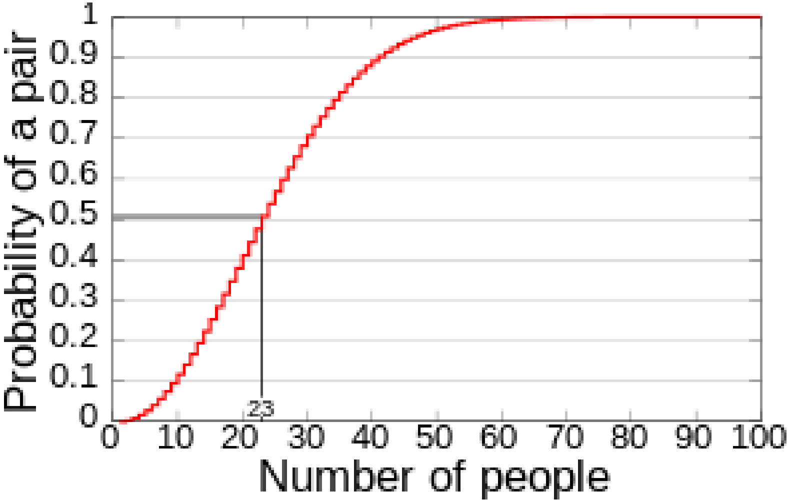
More generally, if we have n objects and r people each choosing one object (and several of them can choose the same object), then if $r \approx 1.177\sqrt{n}$ ($r \approx \sqrt{2\lambda}$), then probability that two people choose the same object is 50% $(1 - e^{\lambda})$%.

Birthday paradox - graph - I.



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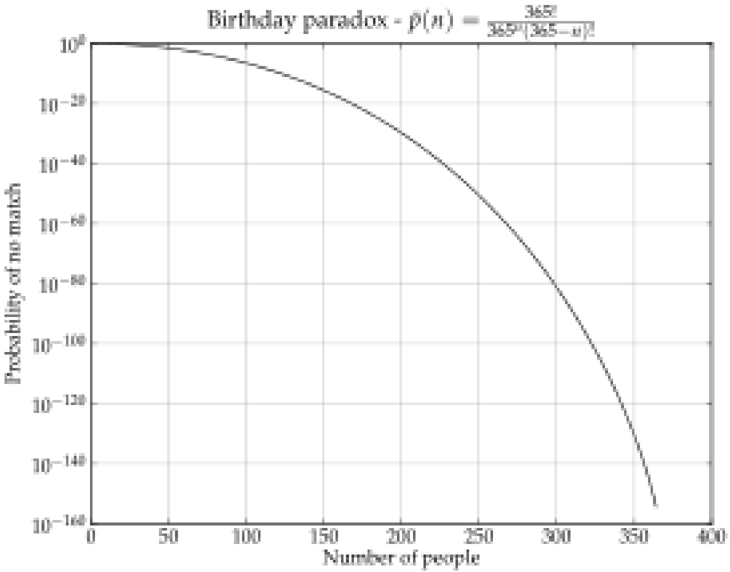

ANOTHER VERSION of THE BIRTHDAY PARADOX

Let us have n objects and two groups of r people. If $r \approx \sqrt{\lambda n}$ then the probability that someone from one group chooses the same object as someone from the other group is $1 - e^{-\lambda}$.

STABLE MARRIAGE PROBLEM - INFORMAL FORMULATION

Given is n men and n women and each of them has ranked all members of the opposite sex with a unique number between 1 and n in order to express of his/her preferences.

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If there is a no dissatisfied couple in a (group) marriage we consider the (group) marriage as stable.

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Consider a society of n men $A, B, C, \ldots$
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Example $A : abcd$ $B : bacd$ $C : adcb$ $D : dcab$
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A marriage is said to be **unstable** if there exist two married couples $X - x$, $Y - y$ such that X desires y more than x
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The task is to find a **stable marriage**. (At least one always exist!)

Example of an unstable marriage: $A - a$, $B - b$, $C - c$, $D - d$.

COMMENTS

- The stable marriage problem, and its variations, form one of the most famous and important groups of algorithmic problems with a variety of interesting and important applications.

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- Algorithms to deal with this type of problems are used, for example:
  - To assign graduates of medical schools in North America (about 30 000) to hospitals;

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- Women get their first choice and men the third one:

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Algorithm is not good because a loop can occur.

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For choices:

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EXAMPLE 3

For choices

$$M_1 : W_1 W_2 W_3 W_4 W_5 \quad M_2 : W_2 W_3 W_4 W_5 W_1 \quad M_3 : W_3 W_4 W_5 W_1 W_2$$
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we have exactly 5 stable marriages

$$\begin{array}{ccccc}
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In the x-th of the above marriages each man is married with his x-th choice.

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Assume that all men are numbered somehow.

At any step of the algorithm (due to Gale-Shapley), there will be a partial marriage, and the lowest-number unmarried man M proposes "marriage" to the most desirable women W on his list who has not rejected him yet. The woman W then decides whether to accept his proposal or to reject it.

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The algorithm repeats the process and terminates after every person has been married. It is a linear time algorithm, concerning the worst case complexity.

It is easy to see that the process terminates and resulting marriage is stable.

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The result of the men-proposal algorithm does not depend on the order men are chosen to make their proposals.

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It is said that a marriage is women-pessimal if each woman is married with her lowest ranked feasible partner.

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National residency matching program.

This program places applicants for postgraduate medical training positions into residency programs at teaching hospitals throughout US.

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In case of n men and n women, any woman has at least $(\frac{1}{2} - \epsilon) \ln n$ and at most $(1 + \epsilon) \ln n$ different stable husbands in the set of all Gale-Shapley stable matchings defined by these rankings, with probability approaching 1 as $n \to \infty$, if ϵ is any positive constant.

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There is an algorithm that outputs all stable husbands of a given women.

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Let $\mathrm{T_p}$ be the random variable that denote the number of proposals made during the execution of the *Proposal algorithm* – what is proportional to the overall time of algorithm.

Distribution of $\mathrm{T_p}$ seems to be very difficult to determine or even to analyse.

Our goal is to show that the expected value of the number of proposals is about $\mathcal{O}(n \lg n)$.

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Probability of winning our game is therefore, clearly, 1/13.

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it is sufficient that **at each step of the algorithm we fix only that random choice that needs to be revealed at that step**.

ANALYSIS of RANDOMIZED VERSION of PROPOSAL ALGORITHM 1/2

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An application to the Proposal Algorithm: We will remove dependencies by do not assuming that men have chosen their preference lists in advance.

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To eliminate the above dependency let us change the algorithm. Each time a man makes proposal he chooses randomly a woman from the set of all women. Call this new algorithm **Amnesiac Algorithm**.

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At the end we will get:

Theorem For any constant $c \in \Re$ and $m = n \ln n + cn$

$$\lim Pr[\,\mathrm{T_A} > m\,] = 1 - e^{-e^{-c}}$$

COUPON SELECTION PROBLEM

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Let $X_i, 0 \leq i < n$, be the number of trials in the i-th epoch. Then

$$X = \sum_{i=0}^{n-1} X_i$$

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By the linearity of expectations we have:

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=0}^{n-1} X_i\right] = \sum_{i=0}^{n-1} \mathbf{E}[X_i] = \sum_{i=0}^{n-1} \frac{n}{n-i} = n\sum_{i=1}^{n} \frac{1}{i} = nH_n.$$

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Since X_i are independent

$$\sigma^2_X = \sum_{i=0}^{n-1} \sigma^2_{X_i} = \sum_{i=0}^{n-1} \frac{ni}{(n-i)^2} = \sum_{i=0}^{n-1} \frac{n(n-i)}{i^2} = n^2\sum_{i=1}^{n}\frac{1}{i^2} - nH_n$$

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Since $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i^2} = \frac{\pi^2}{6}$ we have $\lim_{n\to\infty} \frac{\sigma^2_X}{n^2} = \frac{\pi^2}{6}$.

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Let ε_i^r denote the event that a coupon of type i is not collected in the first r trials.

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Therefore, for $r = \beta n \ln n$, we get

$$\mathbf{Pr}[X > r] = Pr\left[\cup_{i=1}^{n} \varepsilon_i^r\right] \leq \sum_{i=1}^{n} \mathbf{Pr}[\varepsilon_i^r] \leq \sum_{i=1}^{n} n^{-\beta} = n^{-(\beta-1)}$$

Next aim: To study the probability that X deviates from its expectation nH_n by the amount cn for any real c.

A TECHNICAL LEMMA and MAIN THEOREM

Lemma Let c be a real number and $m = n \ln n + cn$ for a positive integer n. Then, for any fixed k it holds

$$\lim_{n \to \infty} \binom{n}{k} (1 - \frac{k}{n})^m = \frac{e^{-ck}}{k!}.$$

MAIN THEOREM 1/4

Theorem Let the random variable X denote the number of trials for collecting each of the n types of coupons. Then for any $c \in \mathbf{R}$ and $m = n \ln n + cn$

$$\lim_{n\to\infty} Pr[X > m] = 1 - e^{-e^{-c}}$$

Proof Consider the event $\{X > m\} = \bigcup_{i=1}^{n} \varepsilon_i^m$. By the principle of the Inclusion-Exclusion

$$Pr\left[\bigcup_{i=1}^{n} \varepsilon_i^m\right] = \sum_{k=1}^{n} (-1)^{k+1} P_k^n \qquad (*)$$

where

$$P_k^n = \sum_{1\le i_1<\cdots<i_k\le n} Pr\left[\bigcap_{j=1}^{k} \varepsilon_{i_j}^m\right].$$

Let

$$S_k^n = P_1^n - P_2^n + P_3^n - \cdots + (-1)^{k+1} P_k^n$$

denote the partial sum formed by the first k terms in $(*)$.

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$$S_{2k}^n \le Pr\left[\bigcup_{i=1}^{n} \varepsilon_i^m\right] \le S_{2k+1}^n$$

REMAINDER

THE INCLUSION-EXCLUSION PRINCIPLE

Let $A_1, A_2, \ldots, A_n$ be events – not necessarily disjoint. The **Inclusion-Exclusion principle**, that has also a variety of applications, states that

$$
\begin{aligned}
Pr\left[\bigcup_{i=1}^{n} A_i\right] &= \sum_{i=1}^{n} Pr(A_i) - \sum_{i<j} Pr(A_i \cap A_j) + \sum_{i<j<k} Pr(A_i \cap A_j \cap A_k) - \\
&- \ \ldots \ + (-1)^{k+1} \sum_{i_1<i_2<\ldots<i_k} Pr\left[\bigcap_{j=1}^{k} A_{i_j}\right] \ \ldots \ + \\
&+ (-1)^{n+1} Pr\left[\bigcap_{i=1}^{n} A_i\right]
\end{aligned}
$$

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Another proof of Boole's inequality:

Let us define $B_i = A_i - \bigcup_{j=1}^{i-1} A_j$. Then $\bigcup A_i = \bigcup B_i$. Since B_i are disjoint and for each i we have $B_i \subset A_i$ we get

$$Pr[\bigcup A_i] = Pr[\bigcup B_i] = \sum Pr[B_i] \leq \sum Pr[A_i]$$

MAIN THEOREM 2/4 - CONTINUATION

By symmetry, all the k-wise intersections of the events ε_i^m are equally likely, and therefore

$$P_k^n = \binom{n}{k} Pr\left[\bigcap_{i=1}^{k} \varepsilon_i^m\right].$$

Probability of the intersection of k events $\varepsilon_1^m, \ldots, \varepsilon_k^m$ is the probability of not collecting any of the first k coupons in m trials, namely $(1 - \frac{k}{n})^m$. Therefore $P_k^n = \binom{n}{k}\left(1 - \frac{k}{n}\right)^m$.

By the last Lemma, for $m = n \ln n + cn$

$$lim_{n\to\infty} P_k^n = \frac{e^{-ck}}{k\,!} = P_k \text{ — \{notation\}}.$$

Let us denote also:

$$S_k = \sum_{j=1}^{k}(-1)^{j+1}P_j = \sum_{j=1}^{k}(-1)^{j+1}\frac{e^{-cj}}{j\,!}. \quad (**)$$

The right hand side of $(**)$ consists precisely of k terms of the power series expansion of $f(c) = 1 - e^{-e^{-c}}$.

Hence

MAIN THEOREM 3/4

Therefore, for all $\varepsilon > 0$ there exists $k^* > 0$ such that for any $k > k^*$

$$|S_k - f(c)| < \varepsilon.$$

Since $\lim_{n\to\infty} P_k^n = P_k$, we have $\lim_{n\to\infty} S_k^n = S_k$. Equivalently, for all $\varepsilon > 0$ and all k, for all sufficiently large. n

$$|S_k^n - S_k| < \varepsilon$$

Thus, for all $\varepsilon > 0$ any fixed $k > k^*$, and n sufficiently large

$$|S_k^n - S_k| < \varepsilon, \quad |S_k - f(c)| < \varepsilon$$

$$\Longrightarrow |S_k^n - f(c)| = |S_k^n - S_k| + |S_k - f(c)| < 2\varepsilon$$

and

$$\left|S_{2k}^n - S_{2k+1}^n\right| < 4\varepsilon.$$

As a consequence

$$\left|Pr\left[\bigcup_{i=1}^{n} \varepsilon_i^m\right] - f(c)\right| < 4\varepsilon$$

and therefore

$$\lim_{n\to\infty} Pr\left[\bigcup_{i=1}^{n} \varepsilon_i^m\right] = f(c) = 1 - e^{-e^{-c}}$$

MAIN THEOREM 4/4

what implies

$$\lim_{n\to\infty} Pr[X > n(\ln n + c)] = 1 - e^{-e^{-c}}$$

Implications With extremely high probability, the number of trials, for collecting all n coupon types, lies in a small interval centered about its expected value.

A SUMMARY of the ANALYSIS of STABLE MARRIAGE PROBLEM

In case of the stable marriage problem of n men and women we have

- The worst case complexity (of the number of proposals) in n^2,
- The average case complexity is $\mathcal{O}(n \lg n)$.
- Deviation is small from the expected case.

APPENDIX

SIMILAR PROBLEMS

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Hospital-students problems with couples Similar problem as the above one, but among students can be couples that have to be assigned either to the same hospital or to a specific pair of hospitals chosen by couples.

1. Which of the numbers e^π and π^e, is larger, for the case that e is the basis of natural logarithms

EXERCISES

1. Which of the numbers e^{π} and π^{e}, is larger, for the case that e is the basis of natural logarithms
2. Hint 1: There exists one-line proof of the correct relation.