Part I

Games Theory and Analyses of Randomized Algorithms

CLASSICAL GAMES THEORY BRIEFLY

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An element $(x, y) \in X \times Y$ is said to be a **Nash equilibrium** of the game (X, Y, p_X, p_Y) iff $p_X(x', y) \leq p_X(x, y)$ for any $x' \in X$, and $p_Y(x, y') \leq p_Y(x, y)$ **for all** $y' \in Y$.

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A game is called **zero-sum game** if $p_X(x, y) + p_Y(x, y) = 0$ for all $x \in X$ and $y \in Y$.

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One of the basic result of the classical game theory is that not every two-players zero-sum game has a Nash equilibrium in the set of pure strategies, but there is always a Nash equilibrium if players follow mixed strategies.

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This way, from a fair game, in which both players have the same chance to win if only classical computation and communication tools are used, an unfair game can arise, or from an unfair game a fair one.

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However, there is equilibrium if Alice chooses its strategy with probability $\frac{1}{2}$ and Bob chooses each of the four possible strategies with probability $\frac{1}{4}$.

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$\frac{\text{Alice}}{\text{Bob}}$	C_A	D_A
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What is the best way for Alice and Bob to proceed in order to maximize their payoffs?

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Pay-off function is given by the matrix (columns are for Alice) (columns are for Bob)

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The two Nash equilibria are (O, O) and (T, T), but players are faced with tactics dilemma, because these equilibria bring them different payoffs.

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- Alice wins if coin is unfair, otherwise Bob wins

Clearly, in the classical case, the probability that Alice wins is $\frac{2}{3}$.

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This techniques can be applied to algorithms that terminate for all inputs and all random choices.

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Example - stone-scissors-paper game

PAYOFF–MATRIX

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Alice		Scissors	Paper	Stone
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→ Table shows how much Bob has to pay to Alice

Rules: Stone looses to paper and wins sissors.

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ϱ and γ are so called optional strategies for Alice and Bob if

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Payoff is now a random variable – **if p, q are taken as column vectors** then

$$E[\text{payoff}] = p^T Mq = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i M_{ij} q_j$$

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$$\max_p \min_q E\left[T(i_p, A_q)\right] = \min_q \max_p E\left[T(i_p, A_q)\right]$$

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Interpretation: Expected running time of the optimal deterministic algorithm for any arbitrarily chosen input distribution p for a problem Π is a lower bound on the expected running time of the optimal (Las Vegas) randomized algorithm for Π.

In other words, to determine a lower bound on the performance of all randomized algorithms for a problem P, derive instead a lower bound for any deterministic algorithm for P when its inputs are drawn from a specific probability distribution (of your choice).

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In order to prove a lower bound on the randomized complexity of an algorithmic problem, it suffices to choose *any* probability distribution p on the input and prove a lower bound on the expected running time of deterministic algorithms for that distribution.

The power of this technique lies in

1. the flexibility at the choice of p
2. the reduction of the task to determine lower bounds for randomized algorithms to the task to determine lower bounds for deterministic algorithms.

(It is important to remember that we can expect that the deterministic algorithm "knows" the chosen distribution p.)

The above discussion holds for Las Vegas algorithms only!

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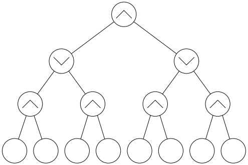
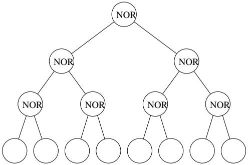
Instead of AND–OR trees of depth $2k$ we can consider NOR–trees of depth $2k$. Indeed, it holds:

$$(a \vee b) \wedge (c \vee d) \equiv (a \text{ NOR } b) \text{NOR} (c \text{ NOR } d)$$





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Lemma Let T be a NOR–tree each leaf of which is set to 1 with a fixed probability. Let $W(T)$ denote the minimum, over all deterministic algorithms, of the expected number of steps to evaluate T. Then there is a depth–first pruning algorithm whose expected number of steps to evaluate T is $W(T)$.

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The last lemma tells us that for the purposes of our lower bound, we may restrict our attention to the depth–first pruning algorithms.

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This implies:

Theorem The expected running time of any randomized algorithm that always evaluates an instance of T_k correctly is at least $n^{0.694}$, where $n = 2^{2k}$ is the number of leaves.

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No, the above result just says that in order to get a better lower bound another probability distribution on inputs may be needed.

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- It has also been shown, by Farhi et al. (2009), that the upper bound for the case quantum computation tools can be used is $O(n^{0.5})$.