Part I

Simple Methods of design of Randomized Algorithms

Chapter 4. SIMPLE METHODS for DESIGN of RANDOMIZED ALGORITHMS

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Especially we deal with:

- A unified approach to deterministic, randomized and quantum algorithms
- Application of the linearity of expectations method
- Design of randomized algorithms for games trees.

PROLOGUE

A way to see basics of deterministic, randomized and quantum computations and their differences.

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However, for any at least a bit significant task, the number of bits needed to describe such an evolution mapping, $n2^n$, is much too big. **The task of programming is then/therefore** to replace an application of such an enormously huge mapping by an application of a much shorter circuit/program.

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However, for any nontrivial problem the number 2^n is larger than the number of particles in the universe. Therefore, **the task of programming is to design a small circuit/program** that can implement such a multiplication by a matrix of an enormous size.

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Concerning a **computation step**, this has to be again a multiplication of a vector of the probability amplitudes, representing the current state, by a very huge $2^n \times 2^n$ unitary matrix which has to be realized by a "small" quantum circuit (program).

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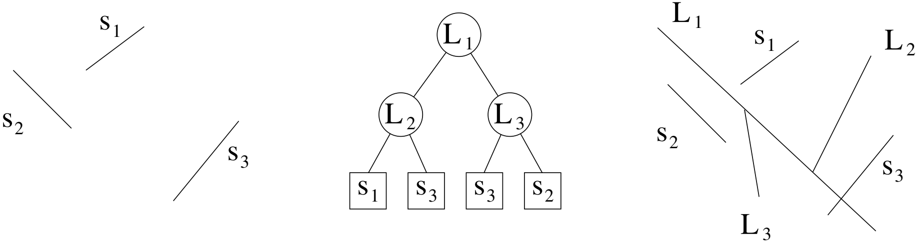

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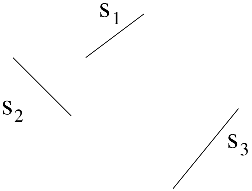

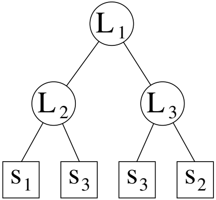

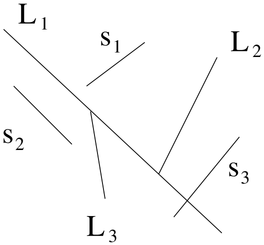

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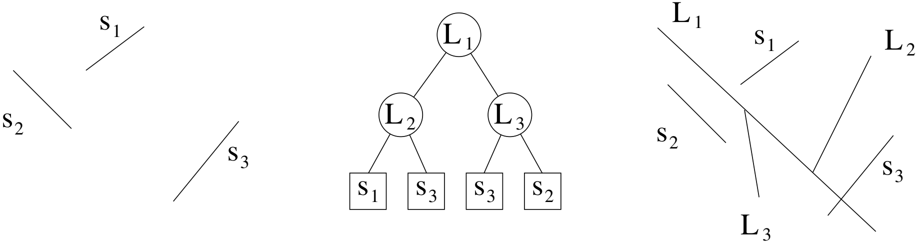

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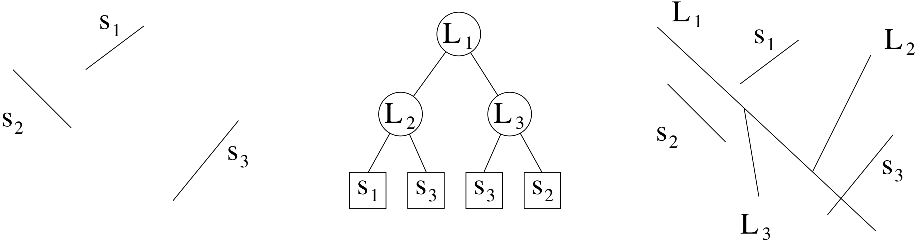

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Each line L_v will partition the region r_v into two regions $r_{l,v}$ and $r_{r,v}$ which correspond to two children of v - to the left and right one.

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$u \dashv v$ will be an event that $l(u)$ cuts v in the constructed (autopartition) tree.

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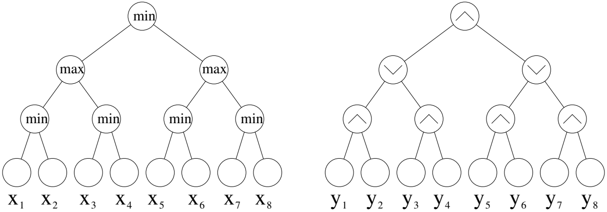
Game trees

GAME TREE EVALUATION - I.

Game trees



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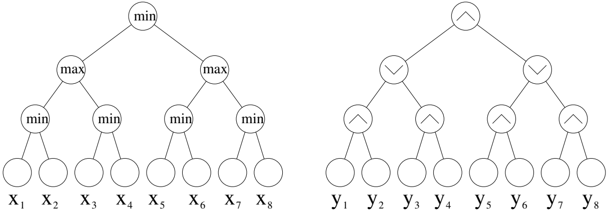

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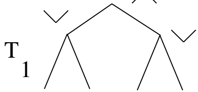

T_k – binary game tree of depth $2k$.
Goal is to evaluate the tree - the root.

GAME TREE EVALUATION - II.

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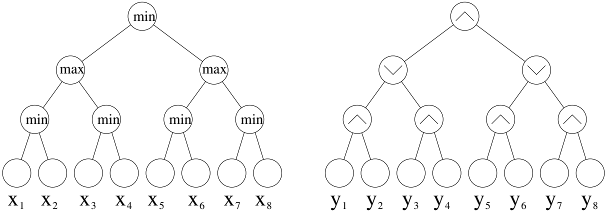

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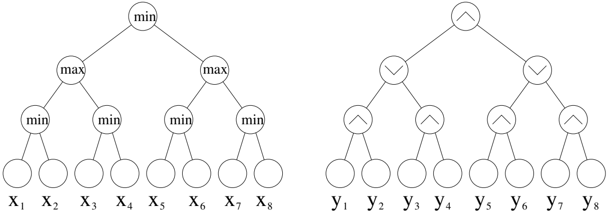

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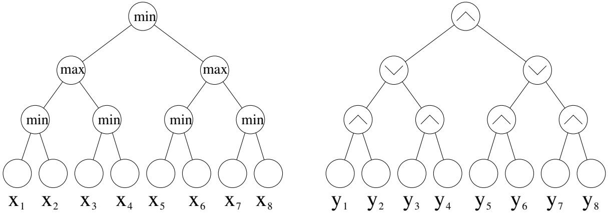

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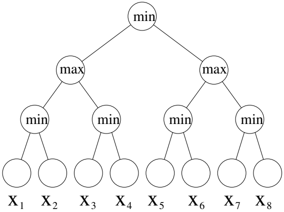

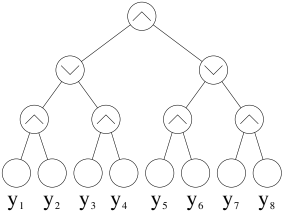

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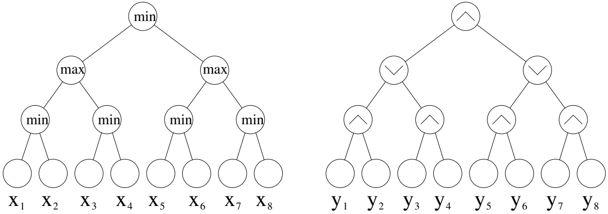

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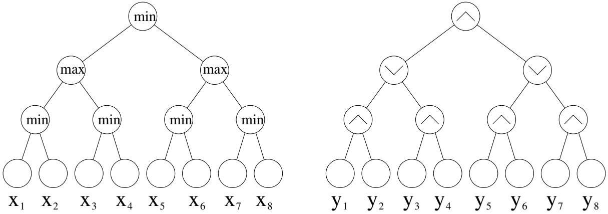

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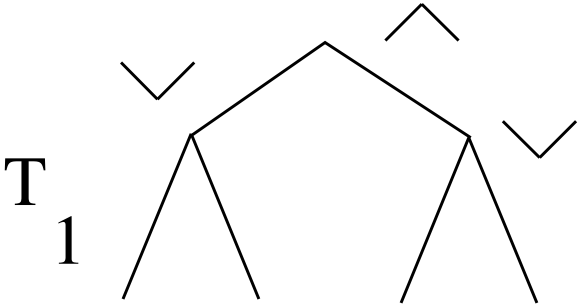

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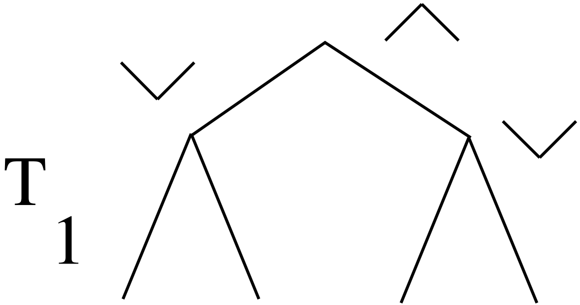

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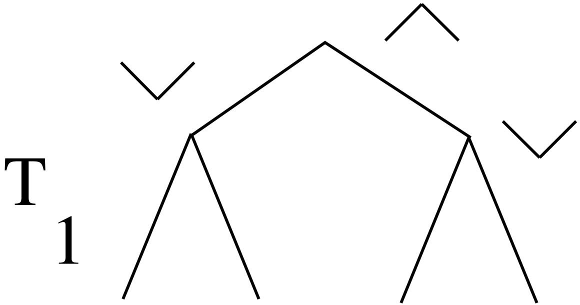

Every deterministic algorithm can be forced to inspect all leaves. The worst-case complexity of a deterministic algorithm to evaluate T_k is therefore:

$$n = 4^k = 2^{2k}.$$

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A program represented by a binary word p, is self-delimiting for a computer C, if for any input pw the computer C can recognize where p ends after reading p only..

Another way to see self-delimiting programs is to consider only such programming languages L that no program in L is a prefix of another program in L.

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Ω_C is therefore the probability that a self-delimiting computer program for C generated at random, by choosing each of its bits using an independent toss of a fair coin, will eventually halt.

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- At least n bits long program is needed to determine n bits of Ω_C
- Bits of Ω can be seen as mathematical facts that are true for no reason.

- Greg Chaitin, who introduced numbers of wisdom, designed a specific universal computer C and a two hundred pages long Diophantine equation E, with 17,000 variables and with one parameter k, such that for a given k the equation E has a finite (infinite) number of solutions if and only if the k-th bit of Ω_C is 0 (is 1).

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- Knowing the value of Ω_C with n bits of precision allows to decide which programs for C with at most n bits halt.