Part I

Basics of Probability Theory

Chapter 3. PROBABILITY THEORY BASICS

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Key fact: Any probabilistic statement must refer to a specific underlying probability space - a space of elements to which a probability is assigned.

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to the currently acceptable axiomatic definition of probability (due to A. N. Kolmogorov in 1933).

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The fact that not all collections of events lead to well-defined probability spaces leads to the concepts presented on the next slide.

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Theorem: Law of the total probability Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ be a **partition** of a sample space Ω. Then for any event ε

$$Pr[\varepsilon] = \sum_{i=1}^{k} Pr[\varepsilon | \varepsilon_i] \cdot Pr[\varepsilon_i]$$

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2. A collection of events $\{\varepsilon_i | i \in I\}$ is **independent** if for all subsets $S \subseteq I$

$$Pr\left[\bigcap_{i \in S} \varepsilon_i\right] = \prod_{i \in S} Pr[\varepsilon_i].$$

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In modern terms the last equation says that $Pr[\varepsilon_1|\varepsilon]$, the probability of a hypothesis ε_1 (given information ε), equals $\Pr(\varepsilon_1)$, our initial estimate of its probability, times $Pr[\varepsilon|\varepsilon_1]$, the probability of each new piece of information (under the hypothesis ε_1), divided by the sum of the probabilities of data in all possible hypothesis (ε_i).

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In Bayesian interpretation, probability measures a degree of belief. Bayes' theorem then links the degree of belief in a proposition before and after receiving an additional evidence that the proposition holds.

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ε_1 be the event that the sum of both tosses is ≥ 10

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How much are: $Pr(\varepsilon_1), Pr(\varepsilon_2), Pr(\varepsilon_1|\varepsilon_2), Pr(\varepsilon_1 \cap \varepsilon_2)$?

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Therefore, the above outcome of the three coin flips increased the likelihood that the first coin is biased from $1/3$ to $2/5$

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- Assume that A and B are independent and $\Pr(B) \neq 0$. By definition we have

  $$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

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- Assume that $\Pr(A|B) = \Pr(A)$ and $\Pr(B) \neq 0$. Then

  $$\Pr(A) = \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

  and multiplying by $\Pr(B)$ we get

  $$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

  and so A and B are independent.

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- The notion of conditional probability, of A given B, was introduced in order to get an instrument for analyzing an experiment A when one has partial information B about the outcome of the experiment A before experiment has finished.
- **We say that two events A and B are independent if the probability of A is equal to the probability of A given B,**
- Other fundamental instruments for analysis of probabilistic experiments are **random variables** as functions from the sample space to $\mathbf{R}$, and **expectation** of random variables as the weighted averages of the values of random variables.

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Similarly

$$Pr[C_3|M_2] = \frac{Pr[M_2|C_3]Pr[C_3]}{Pr[M_2]} = \frac{Pr[M_2|C_3]Pr[C_3]}{Pr[M_2|C_1]Pr[C_1] + Pr[M_2|C_3]Pr[C_3]} = \frac{1/3}{1/6+1/3}$$

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The concept of random variable is one of the most important of modern science and technology.

INDEPENDENCE of RANDOM VARIABLES

Definition Two random variables X, Y are called **independent random variables** if

$$x, y \in \mathbf{R} \Rightarrow Pr_{X,Y}(x, y) = Pr[X = x] \cdot Pr[Y = y]$$

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The above relation is called **weak linearity of expectation**.

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It holds:

$$\begin{aligned} \mathbf{E}_{\mathbf{Pr}}[X_A] &= \sum_{s \in S} X_A(s) \cdot \Pr(\{s\}) \\ &= \sum_{s \in A} X_A(s) \cdot \Pr(\{s\}) + \sum_{s \in S-A} X_A(s) \cdot \Pr(\{s\}) \end{aligned}$$

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A random variable X is said to be an indicator variable if X takes on only values 1 and 0.

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$\mathbf{E}X = \mathbf{E}Y = 5.5$, $\mathbf{E}(X^2) = \frac{1}{10}\sum_{i=1}^{10} i^2 = 38.5$, $\mathbf{E}(Y^2) = 44.5$; $\mathbf{V}X = 8.25$, $\mathbf{V}Y = 14.25$

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The **mean** of a random variable X is sometimes denoted by $\mu_X = m_X^1$ and its variance by μ_X^2.

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Or none of these two strategies is better than the second one?

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For example, for the **uniform distribution** on the set $\{0, 1, \ldots, n-1\}$ the PGF has form

$$U_n(z) = \frac{1}{n}(1 + z + \ldots + z^{n-1}) = \frac{1}{n} \cdot \frac{1 - z^n}{1 - z}.$$

Problem is with the case $z = 1$.

PROPERTIES of GENERATING FUNCTIONS

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Property 1 If $X_1, \ldots, X_k$ are independent random variables with PGFs $G_1(z), \ldots, G_k(z)$, then the random variable $Y = \sum_{i=1}^{k} X_i$ has as its PGF the function

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Property 2 Let $X_1, \ldots, X_k$ be a sequence of independent random variables with the same PGF $G_X(z)$. If Y is a random variable with PGF $G_Y(z)$ and Y is independent of all X_i, then the random variable $S = X_1 + \ldots + X_Y$ has as PGF the function

$$G_S(z) = G_Y(G_X(z)).$$

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$$Pr(Y = k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k}$$

Such a probability distribution is called the **binomial distribution** and it holds

$$\mathbf{E}Y = np \qquad \mathbf{V}Y = npq \qquad G(z) = (q + pz)^n$$

and also

$$\mathbf{E}Y^2 = n(n-1)p^2 + np$$

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Let $X_1, \ldots, X_n$ be random variables having Bernoulli distribution with the common parameter p.
The random variable

$$X = X_1 + X_2 + \ldots + X_n$$

has so called binomial distribution denoted $B(n, p)$ with the density function denoted

$$B(k, n, p) = Pr(X = k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k q^{(n-k)}$$

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Property of a Poisson random variable X:

$$\mathbf{E}[X] = \lambda \qquad \mathbf{V}X = \lambda \qquad G[z] = e^{\lambda(z-1)}$$

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\mathbf{E}(S_n) &= \mathbf{E}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathbf{E}(X_i) = np \\
\mathbf{E}(S_n^2) &= \mathbf{E}((\sum_{i=1}^{n} X_i)^2) = \mathbf{E}(\sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j) = \\
&= \sum^{n} \mathbf{E}(X_i^2) + \sum \mathbf{E}(X_i X_j)
\end{aligned}$$

Hence

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and therefore, if X_i, X_j are pairwise independent, as in this case, $\mathbf{E}(X_i X_j) =$ $= \mathbf{E}(X_i)\mathbf{E}(X_j)$

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Another variant of Chebyshev's inequality:

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As a special case, namely if $g(x) = e^{tx}$, we get:

$$Pr[|X| > \lambda] \leq \frac{\mathbf{E}(e^{tX})}{e^{t\lambda}}$$ **basic Chernoff′s inequality**

Chebyshev's inequalities are used to show that values of a random variable lie close to its average with high probability. The bounds they provide are called also **concentration bounds**. Better bounds can usually be obtained using Chernoff bounds discussed in Chapter 5.

FLIPPING COINS EXAMPLES on CHEBYSHEV INEQUALITIES

Let X be a sum of n independent fair coins and let X_i be an indicator variable for the event that the i-th coin comes up heads.

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Let X be a sum of n independent fair coins and let X_i be an indicator variable for the event that the i-th coin comes up heads. Then $\mathbf{E}(X_i) = \frac{1}{2}$, $\mathbf{E}(X) = \frac{n}{2}$,

$\mathrm{Var}[X_i] = \frac{1}{4}$ and $\mathrm{Var}[X] = \sum \mathrm{Var}[X_i] = \frac{n}{4}$.

Chebyshev's inequality

$$\Pr[|X - \mathbf{E}(X)| \geq \lambda] \leq \frac{V(X)}{\lambda^2}$$

for $\lambda = \frac{n}{2}$ gives

$$Pr[X = n] \leq Pr[|X - n/2| \geq n/2] \leq \frac{n/4}{(n/2)^2} = \frac{1}{n}$$

THE INCLUSION-EXCLUSION PRINCIPLE

Let $A_1, A_2, \ldots, A_n$ be events – not necessarily disjoint. The **Inclusion-Exclusion principle**, that has also a variety of applications, states that

$$\begin{aligned} Pr\left[\bigcup_{i=1}^{n} A_i\right] &= \sum_{i=1}^{n} Pr(A_i) - \sum_{i<j} Pr(A_i \cap A_j) + \sum_{i<j<k} Pr(A_i \cap A_j \cap A_k) - \\ &- \ \ldots \ + (-1)^{k+1} \sum_{i_1<i_2<\ldots<i_k} Pr\left[\bigcap_{j=1}^{k} A_{i_j}\right] \ \ldots \ + \\ &+ (-1)^{n+1}\, Pr\left[\bigcap_{i=1}^{n} A_i\right] \end{aligned}$$

BONFERRONI'S INEQUALITIES

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the following **Bonferroni's inequalities** follow from the Inclusion-exclusion principle:

For every odd $k \leq n$

$$Pr\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{j=1}^{k} (-1)^{j+1} \sum_{i_1 < \ldots < i_j \leq n} Pr\left(\bigcap_{l=1}^{j} A_{i_l}\right)$$

For every even $k \leq n$

$$Pr\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{j=1}^{k} (-1)^{j+1} \sum_{i_1 < \ldots < i_j \leq n} Pr\left(\bigcap_{l=1}^{j} A_{i_l}\right)$$

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Another proof of Boole's inequality:

Let us define $B_i = A_i - \bigcup_{j=1}^{i-1} A_j$. Then $\bigcup A_i = \bigcup B_i$. Since B_i are disjoint and for each i we have $B_i \subset A_i$ we get

$$Pr[\bigcup A_i] = Pr[\bigcup B_i] = \sum Pr[B_i] \leq \sum Pr[A_i]$$

APPENDIX

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PUZZLE - HOMEWORK

Puzzle 1 Given a biased coin, how to use it to simulate an unbiased coin?

Puzzle 2 *n* people sit in a circle. Each person wears either red hat or a blue hat, chosen independently and uniformly at random. Each person can see the hats of all the other people, but not his/her hat. Based only upon what they see, each person votes on whether or not the total number of red hats is odd. Is there a scheme by which the outcome of the vote is correct with probability greater than 1/2.

MODERN (BAYESIAN) INTERPRETATION of BAYES RULE

Bayes rule for the process of learning from evidence has the form:

$$Pr[\varepsilon_1|\varepsilon] = \frac{Pr[\varepsilon_1 \cap \varepsilon]}{Pr[\varepsilon]} = \frac{Pr[\varepsilon|\varepsilon_1] \cdot Pr[\varepsilon_1]}{\sum_{i=1}^{k} Pr[\varepsilon|\varepsilon_i] \cdot Pr[\varepsilon_i]}.$$

In modern terms the last equation says that $Pr[\varepsilon_1|\varepsilon]$, the probability of a hypothesis ε_1 (given information ε), equals $\Pr(\varepsilon_1)$, our initial estimate of its probability, times $Pr[\varepsilon|\varepsilon_1]$, the probability of each new piece of information (under the hypothesis ε_1), divided by the sum of the probabilities of data in all possible hypothesis (ε_i).

EXAMPLE - DRUG TESTING

Suppose that a drug test will produce 99% true positive and 99% true negative results.

Suppose that 0.5% of people are drug users.

If the test of a user is positive, what is probability that such a user is a drug user?

SOLUTION

$$\Pr(\text{drg-us}|+) = \frac{\Pr(+|\text{drg-us})\Pr(\text{drg-us})}{\Pr(+|\text{drg-us})\Pr(\text{drg-us}) + \Pr(+|\text{no-drg-us})\Pr(\text{no-drg-us})}$$

$$\Pr(drg - us|+) = \frac{0.99 \times 0.005}{0.99 \times 0.005 + 0.01 \times 0.995} = \approx 33.2\%$$

BAYES' RULE INFORMALLY

Basically, Bayes' rule concerns of a broad and fundamental issue: how we analyze evidence and change our mind as we get new information, and make rational decision in the face of uncertainty.

Bayes' rule as one line theorem: by updating our initial belief about something with new objective information, we get a new and improved belief

BAYES' RULE STORY

- Reverend Thomas Bayes from England discovered the initial version of the "Bayes's law" around 1974, but soon stopped to believe in it.
- In behind were two philosophical questions
  - Can an effect determine its cause?
  - Can we determine the existence of God by observing nature?
- Bayes law was not written for long time as formula, only as the statement: **By updating our initial belief about something with objective new information, we can get a new and improved belief.**
- Bayes used a tricky thought experiment to demonstrate his law.
- Bayes' rule was later invented independently by Pierre Simon Laplace, perhaps the greatest scientist of 18th century, but at the end he also abounded it.
- Till the 20 century theoreticians considered Bayes rule as unscientific. Bayes rule had for centuries several proponents and many opponents in spite that it has turned out to be very useful in practice.
- Bayes rule was used to help to create rules of insurance industries, to develop strategy for artillery during the first and even Second World War (and also a great Russian mathematician Kolmogorov helped to develop it for this purpose).

- It was used much to decrypt ENIGMA codes during 2WW, due to Turing, and also to locate German submarines.

Part II

Basic Methods of design and Analysis of Randomized Algorithms

Chapter 4. BASIC TECHNIQUES for DESIGN and ANALYSIS

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- Game theory based lower bounds methods for randomized algorithms.

PROLOGUE

A way to see basics of deterministic, randomized and quantum computations and their differences.

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However, for any at least a bit significant task, the number of bits needed to describe such an evolution mapping, $n2^n$, is much too big. **The task of programming is then/therefore** to replace an application of such an enormously huge mapping by an application of a much shorter circuit/program.

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However, for any nontrivial problem the number 2^n is larger than the number of particles in the universe. Therefore, **the task of programming is to design a small circuit/program** that can implement such a multiplication by a matrix of an enormous size.

MATHEMATICAL VIEWS of COMPUTATION 3/3

In case of **quantum computation** on n quantum bits:

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[1] A matrix A is usually called unitary if its inverse matrix can be obtained from A by transposition around the main diagonal and replacement of each element by its complex conjugate.

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Concerning a **computation step**, this has to be again a multiplication of a vector of the probability amplitudes, representing the current state, by a very huge $2^n \times 2^n$ unitary matrix which has to be realized by a "small" quantum circuit (program).

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Example: A ship arrives at a port, and all 40 sailors on board go ashore to have fun. At night, all sailors return to the ship and, being drunk, each chooses randomly a cabin to sleep in. Now comes the **question:** *What is the expected number of sailors sleeping in their own cabins?*

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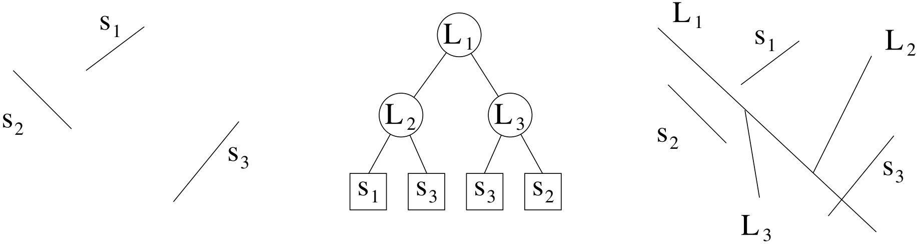

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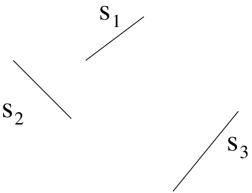

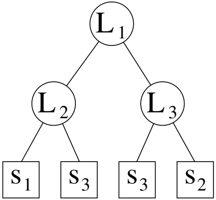

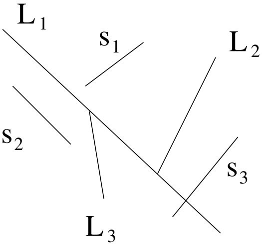

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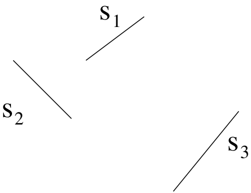

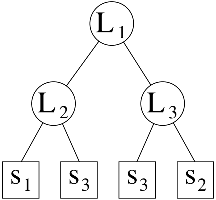

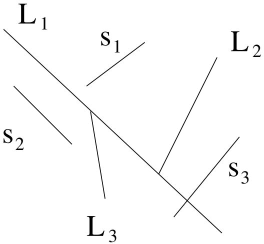

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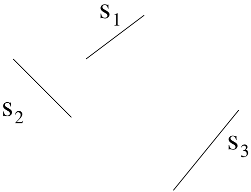

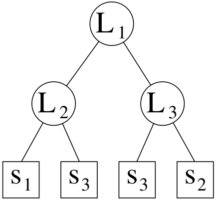

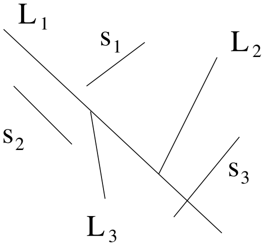

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Each line L_v will partition the region r_v into two regions $r_{l,v}$ and $r_{r,v}$ which correspond to two children of v - to the left and right one.

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$u \dashv v$ will be an event that $l(u)$ cuts v in the constructed (autopartition) tree.

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GAME TREE EVALUATION - I.

Game trees

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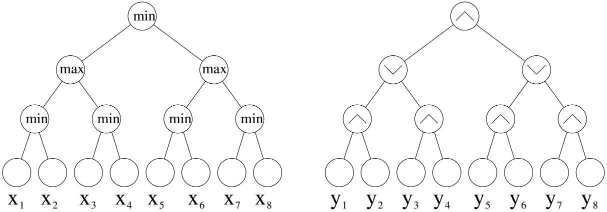

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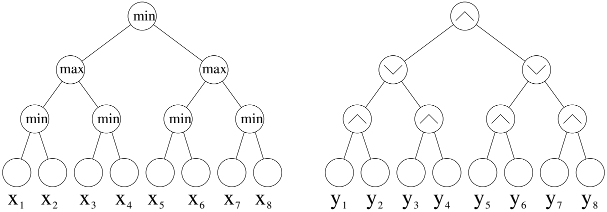

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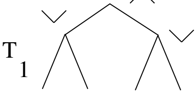

T_k – binary game tree of depth $2k$.

Goal is to evaluate the tree - the root.

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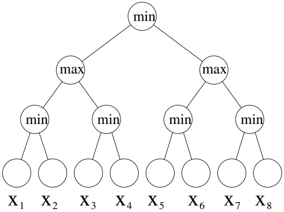

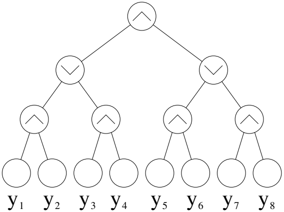

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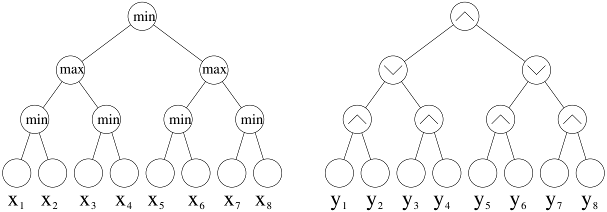

Evaluation of game trees plays a crucial role in AI, in various game playing programs.

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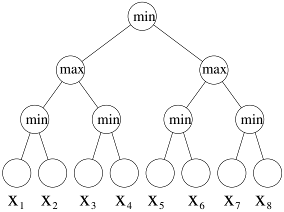

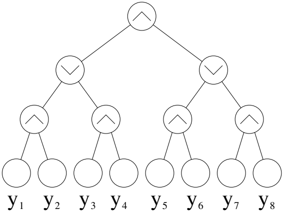

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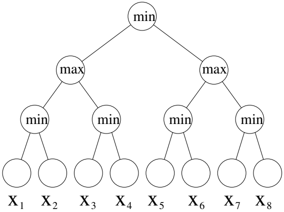

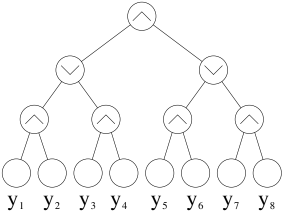

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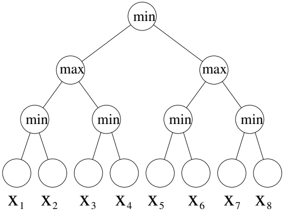

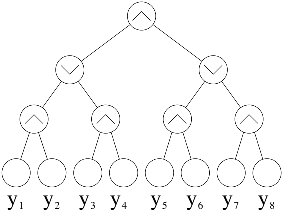

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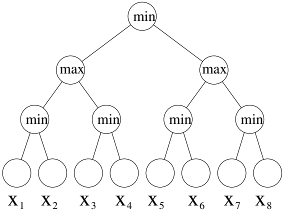

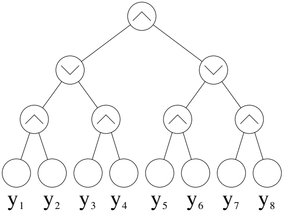

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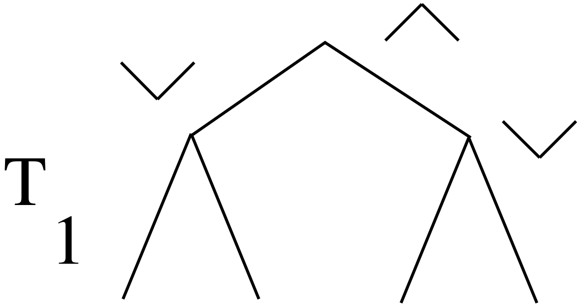

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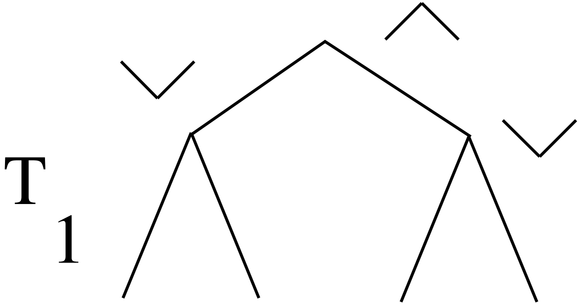

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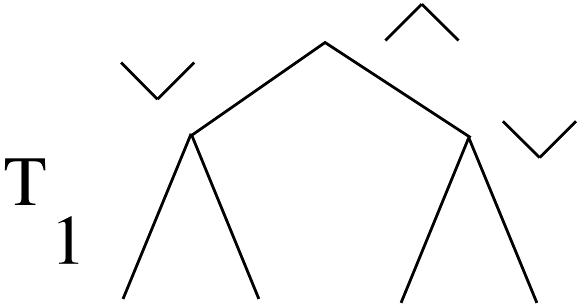

Every deterministic algorithm can be forced to inspect all leaves. The worst-case complexity of a deterministic algorithm is therefore:

$$n = 4^k = 2^{2k}.$$

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A game is called **zero-sum game** if $p_X(x, y) + p_Y(x, y) = 0$ for all $x \in X$ and $y \in Y$.

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One of the basic result of the classical game theory is that not every two-players zero-sum game has a Nash equilibrium in the set of pure strategies, but there is always a Nash equilibrium if players follow mixed strategies.

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This way, from a fair game, in which both players have the same chance to win if only classical computation and communication tools are used, an unfair game can arise, or from an unfair game a fair one.

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However, there is equilibrium if Alice chooses its strategy with probability $\frac{1}{2}$ and Bob chooses each of the four possible strategies with probability $\frac{1}{4}$.

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What is the best way for them to behave?

VERSION of PRISONERS' DILEMMA from 1992

Two members of a gang are imprisoned, each in a separate cell, without possibility to communicate. However, police has not enough evidence to convict them on the principal charge and therefore police intends to put both of them for one year to jail on a lesser charge.

Simultaneously police offer both of them so called Faustian bargain. Each prisoner gets a chance either to betray the other one by testifying that he committed the crime, or to cooperate with the other one by remaining silent. Here are payoffs they are offered:

- If both betray, they will get into jail for 2 years.
- If one betrays and second decides to cooperate, then first will get free and second will go to jail for 3 years.
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The two Nash equilibria are (O, O) and (T, T), but players are faced with tactics dilemma, because these equilibria bring them different payoffs.

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- Alice puts coins into a black box and shakes the box.
- Bob picks up one coin.
- Alice wins if coin is unfair, otherwise Bob wins

Clearly, in the classical case, the probability that Alice wins is $\frac{2}{3}$.

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This techniques can be applied to algorithms that terminate for all inputs and all random choices.

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Example - stone-scissors-paper game

PAYOFF–MATRIX

Bob

		Scissors	Paper	Stone
Alice	Scissors	0	1	-1
	Paper	-1	0	1
	Stone	1	-1	0

→ Table shows how much Bob has to pay to Alice

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ϱ and γ are so called optional strategies for Alice and Bob if

$$O_A = O_B = M_{\varrho\gamma}$$

Example of the game which has a solution ($O_A = O_B = 0$)

$$\begin{matrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{matrix}$$

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Payoff is now a random variable – **if p, q are taken as column vectors** then

$$E[\text{payoff}] = p^T Mq = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i M_{ij} q_j$$

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deterministic algorithms

$$\begin{array}{cc|c|}
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\text{T} & & \text{(i.e. used computation time)} \\
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Optimal pure strategy for Bob corresponds to a choice of an optimal deterministic algorithm.

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Loomis theorem implies that distributional complexity equals to the least possible time achievable by any randomized algorithm

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Theorem(Yao's Minimax Principle) For all distributions p over I and q over $\mathcal{A}$.

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In other words, to determine a lower bound on the performance of all randomized algorithms for a problem P, derive instead a lower bound for any deterministic algorithm for P when its inputs are drawn from a specific probability distribution (of your choice).

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In order to prove a lower bound on the randomized complexity of an algorithmic problem, it suffices to choose *any* probability distribution p on the input and prove a lower bound on the expected running time of deterministic algorithms for that distribution.

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The power of this technique lies in

1. the flexibility at the choice of p
2. the reduction of the task to determine lower bounds for randomized algorithms to the task to determine lower bounds for deterministic algorithms.

(It is important to remember that we can expect that the deterministic algorithm "knows" the chosen distribution p.)

The above discussion holds for Las Vegas algorithms only!

THE CASE OF MONTE CARLO ALGORITHMS

Let us consider Monte Carlo algorithms with error probability $0 < \varepsilon < \frac{1}{2}$.

Let us define the **distributional complexity with error ε**, notation

$$\min_{A \in \mathcal{A}} E\left[T_{\varepsilon}(I_p, A)\right],$$

to be the minimum expected time of any deterministic algorithm that errs with probability at most ε under the input I_p with distribution p.

Let us denote by

$$\max_{i \in \mathcal{I}} E\left[(T_{\varepsilon}(i, A_q)\right]$$

the expected time (under the worst input) of any randomized algorithm A_q that errs with probability at most ε.

Theorem For all distributions p over inputs and q over Algorithms, and any $\varepsilon \in [0, 1/2]$, it holds

$$\frac{1}{2}(\min_{A \in \mathcal{A}} \mathbf{E}\left[T_{2\varepsilon}(i_p, A)\right]) \leq \max_{i \in \mathcal{I}} \mathbf{E}\left[T_{\varepsilon}(i, A_q)\right]$$

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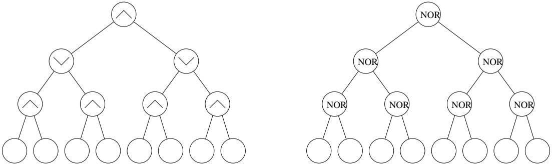
Instead of AND–OR trees of depth $2k$ **we can consider NOR–trees of depth** $2k$. Indeed, it holds:

$$(a \vee b) \wedge (c \vee d) \equiv (a \text{ NOR } b)\text{NOR}(c \text{ NOR } d)$$



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Lemma Let T be a NOR–tree each leaf of which is set to 1 with a fixed probability. Let $W(T)$ denote the minimum, over all deterministic algorithms, of the expected number of steps to evaluate T. Then there is a depth–first pruning algorithm whose expected number of steps to evaluate T is $W(T)$.

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The last lemma tells us that for the purposes of our lower bound, we may restrict our attention to the depth–first pruning algorithms.

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This implies:

Theorem The expected running time of any randomized algorithm that always evaluates an instance of T_k correctly is at least $n^{0.694}$, where $n = 2^{2k}$ is the number of leaves.

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For example, is our lower bound technique weak? ?

No, the above result just says that in order to get a better lower bound another probability distribution on inputs may be needed.

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- It has been shown that for our game tree evaluation problem the upper bound presented at the beginning is the best possible and therefore that $\theta(n^{0.79})$ is indeed the classical (query) complexity of the problem.
- It has also been shown, by Farhi et al. (2009), that the upper bound for the case quantum computation tools can be used is $O(n^{0.5})$.

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A program represented by a binary word p, is self-delimiting for a computer C, if for any input pw the computer C can recognize where p ends after reading p only..

Another way to see self-delimiting programs is to consider only such programming languages L that no program in L is a prefix of another program in L.

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Ω_C is therefore the probability that a self-delimiting computer program for C generated at random, by choosing each of its bits using an independent toss of a fair coin, will eventually halt.

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- At least n-bits long theory is needed to determine n bits of Ω_C.
- At least n bits long program is needed to determine n bits of Ω_C
- Bits of Ω can be seen as mathematical facts that are true for no reason.

- Greg Chaitin, who introduced numbers of wisdom, designed a specific universal computer C and a two hundred pages long Diophantine equation E, with 17,000 variables and with one parameter k, such that for a given k the equation E has a finite (infinite) number of solutions if and only if the k-th bit of Ω_C is 0 (is 1).

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- Knowing the value of Ω_C with n bits of precision allows to decide which programs for C with at most n bits halt.