Part I

2. Types and Basic Design Methods for Randomized Algorithms

Chapter 2. DESIGN METHODS for RANDOMIZED ALGORITHMS

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- If one takes a random variable that assigns to every computation its complexity (number of steps), then the expectation of this random variable equals the expected time complexity of A on x.

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The idea is to overcome such a situation when a set S of deterministic algorithms for P is given such that for each of them there exist bad inputs (for which the algorithm gives wrong result or computes inefficiently) and for each input there exist a lot of algorithms in S giving a correct answer efficiently).

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- Fingerprinting. For solving various problems it can be much more efficient to work with very small fingerprints (hashes) of large inputs than with such large inputs directly.

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- **An amplification** of the success probability of a randomized algorithm can be achieved by repeating independent computations on the same input (but, of course, with different random auxiliary inputs).

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Each deterministic protocol clearly requires sending at least n bits in the worth case.

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Example: For $n = 2^{64} \approx 10^{21}$ the protocol requires to send at most 256 bits.

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The so-called Prime number theorem says that there are approximately $\frac{n}{\ln n}$ primes among the first n integers.

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Probability of the error of the above equality protocol is therefore:

$$\frac{n-1}{\text{Prim}(n^2)} \leq \frac{n-1}{n^2/\ln n^2} \leq \frac{\ln n^2}{n} = \frac{2 \ln n}{n}$$

which is at most $0.369 \cdot 10^{-14}$ in case $n = 10^{16}$.

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Reminder: Probability of the correct output is at least

$$1 - \frac{2 \ln n}{n}.$$

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We show the existence of a Las Vegas algorithm to solve the above problem (that is to decide whether such an i exists) with communication complexity

$$n + \mathcal{O}(\lg n).$$

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and, finally, she sends to Bob all 20 numbers:

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- If $x_j = y_j$, Alice outputs YES (or 1);
otherwise she outputs ?? (because it may exist other k with $x_k = y_k$).

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- If $x_i \neq y_i$ for all i, then the probability that

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for all i, and therefore the probability that the above protocol produces the correct output is at least

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Moreover, it can be shown that for for sufficiently large n.

$$\left(1 - \frac{2 \ln n}{n}\right)^{10} \geq 1 - \frac{20 \ln n}{n} \geq \frac{1}{2}.$$

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Claim: Any Las Vegas algorithm A_1 can be converted to a Las Vegas algorithm A_2 that solves the same problem and never produces ??.

Construction of A_2 is simple. Each time A_1 is to produce the output ??, what can be done only with bounded probability, a new run of A_1 is initialized with the same input.

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Because of the exponential decrease of error probability using repeated applications, 1MC algorithms are very popular.

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One can show that if an $\delta > 0$ is fixed, then

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If ε and δ are considered as constant, then so is k and therefore

$$\text{Time}_{A_k}(n) = \mathcal{O}(\text{Time}_A(n)).$$

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Answer: For an UMC algorithm A it may happen that the distance between the error probability and $\frac{1}{2}$ tends to 0 with growing input size.

As a consequence, if we design, given an $\delta > 0$, an algorithm A_k, that performs k independent runs of A and

$$\text{Prob}(A_k(x)) = F(x)) > 1 - \delta$$

then running time of A_k may be exponential in the input length.

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$$b \oplus (S \oplus b) = S.$$

PERFECT MATCHING ALGORITHM - I.

Let $G = \langle V, E \rangle$ be an undirected graph. A subset $X \subseteq E$ is said to be a *matching* of G if no two edges in X have a common node.

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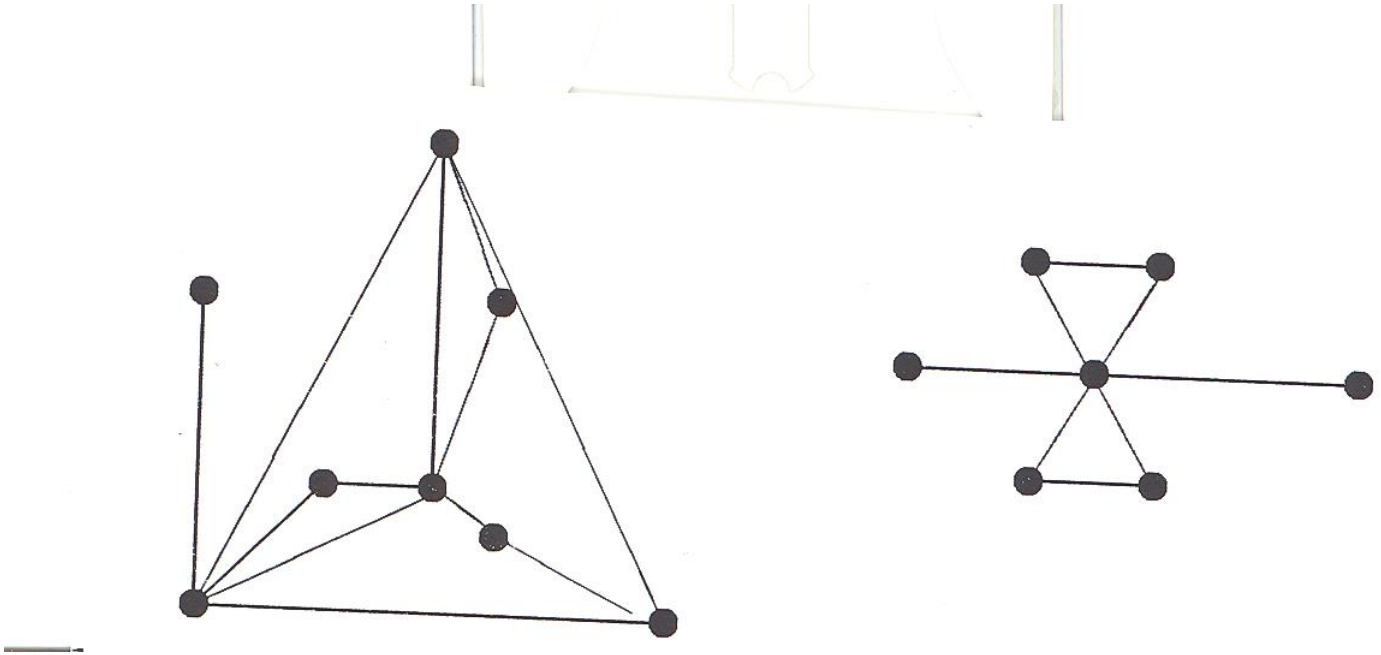

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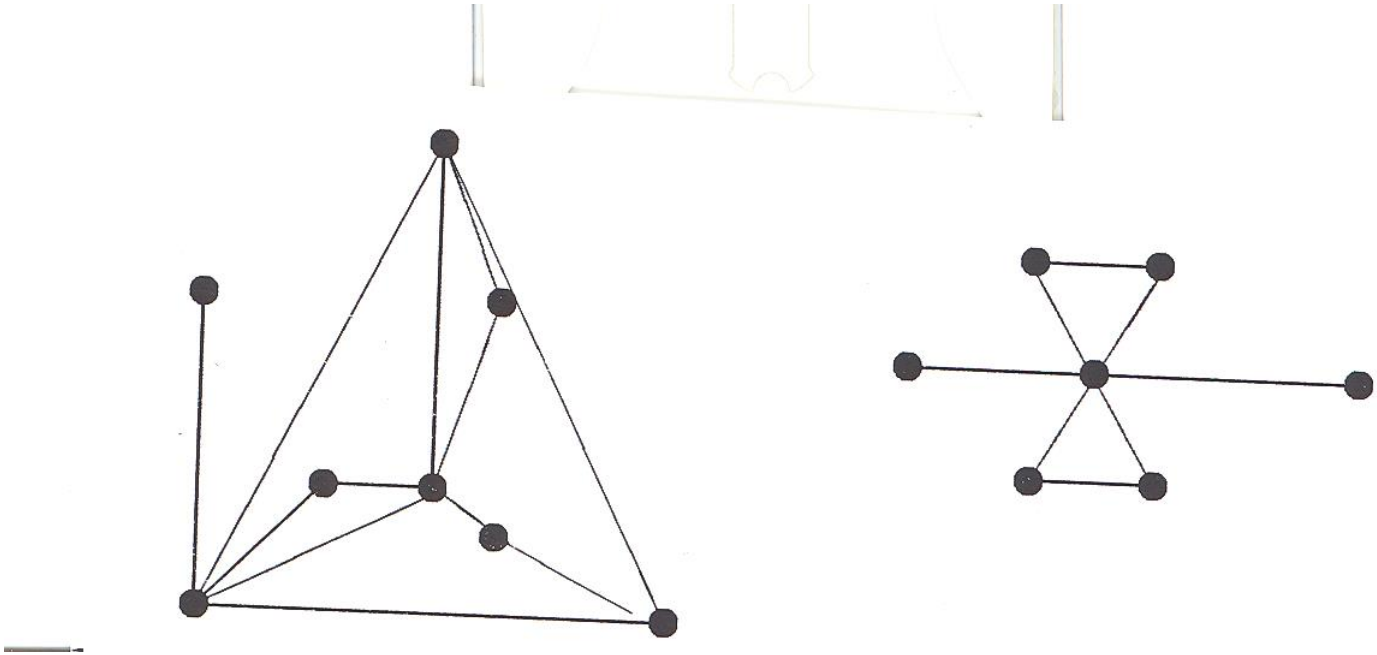

There are polynomial time algorithms to decide whether a given graph has perfect matching, but none is so simple as the randomized algorithm based on so called Tutte theorem presented bellow.

TUTTE MATRIX

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Basic concept: Tutte matrix of a graph

Let $G = \langle V, E \rangle$ be a graph with nodes $V = \{1, 2, \ldots, n\}$, $|V|$ even. The *Tutte matrix* $A_G = \{a_{ij}\}_{i,j=1}^{n}$ of G is defined by

$$a_{ij} = \left\{ \begin{array}{cl} x_{ij} & \text{if } (i,j) \in E,\ i < j; \\ -x_{ij} & \text{if } (i,j) \in E,\ i > j; \\ 0 & \text{if } (i,j) \notin E \end{array} \right\},$$

where x_{ij} are variables, all different for different i, j.

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Example: For the graph of six nodes at which node 1 is connected with nodes 4 and 5, node 2 with nodes 5 and 6 and the node 3 is connected with nodes 5 and 6 the Tutte matrix has the form:

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Proof: The determinant of A_G equals $\sum_\pi \sigma_\pi \prod_{i=1}^{n} a_{i\pi(i)}$ where π are permutations of $\{1, 2, \ldots, n\}$ and $\sigma_\pi = 1$ ($\sigma_\pi = -1$) if π is a product of an even (odd) number of transpositions.

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Notation For a perfect matching E' let $t_{E'}$ denote the product of the $a's$ corresponding to the edges of E'.

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Conclusion;

$$det(A_G) = (t_{E'_1} + \cdots + t_{E'_k})^2$$

where E'_i denotes i-th perfect matching.

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Perfect matching E' is $1-2, 3-4$, $t_{E'} = a_{12}a_{34}$ and

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$$t_E = a_{12}a_{34}, t_{E'} = a_{41}a_{23}$$

EXPLANATION

Let us illustrate claims from the previous slide for a graph with 4 nodes: 1, 2, 3 ,4

Case 1. Let us have edges 1-2, 3-4 only and permutation

$$\pi = \pi^r : 1 \to 2, 2 \to 1, 3 \to 4, 4 \to 3$$

Perfect matching E' is $1-2, 3-4$, $t_{E'} = a_{12}a_{34}$ and

$$\prod_{i=1}^{4} a_{i\pi(i)} = a_{12}a_{21}a_{34}a_{43} = x_{12}x_{12}x_{34}x_{34} = (t_{E'})^2$$

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$$\prod_{i=1}^{4} a_{i\pi(i)} + \prod_{i=1}^{4} a_{i\pi^r(i)} = a_{12}a_{23}a_{34}a_{41} + a_{12}a_{23}a_{34}a_{41} = 2t_E t_{E'}.$$

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Problem Given is a set $S = \{a_1, \ldots, a_n\}$ of $n > 0$ different numbers, and $1 \leq k \leq n$, find the k-th smallest number from S.

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   2. **if** $|S_<| \geq k$ **then** RSELECT$(S_<, k)$;
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This is clearly a Las Vegas algorithm.

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All that means that a different approach to the classification of randomized approximation algorithms is needed.

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Probability of not computing an optimal solution in one run is at most

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6. goal $\in$ (minimum, maximum) is an objective.

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If the goal is to find minimum (maximum) we talk about a minimization (maximization) problem.

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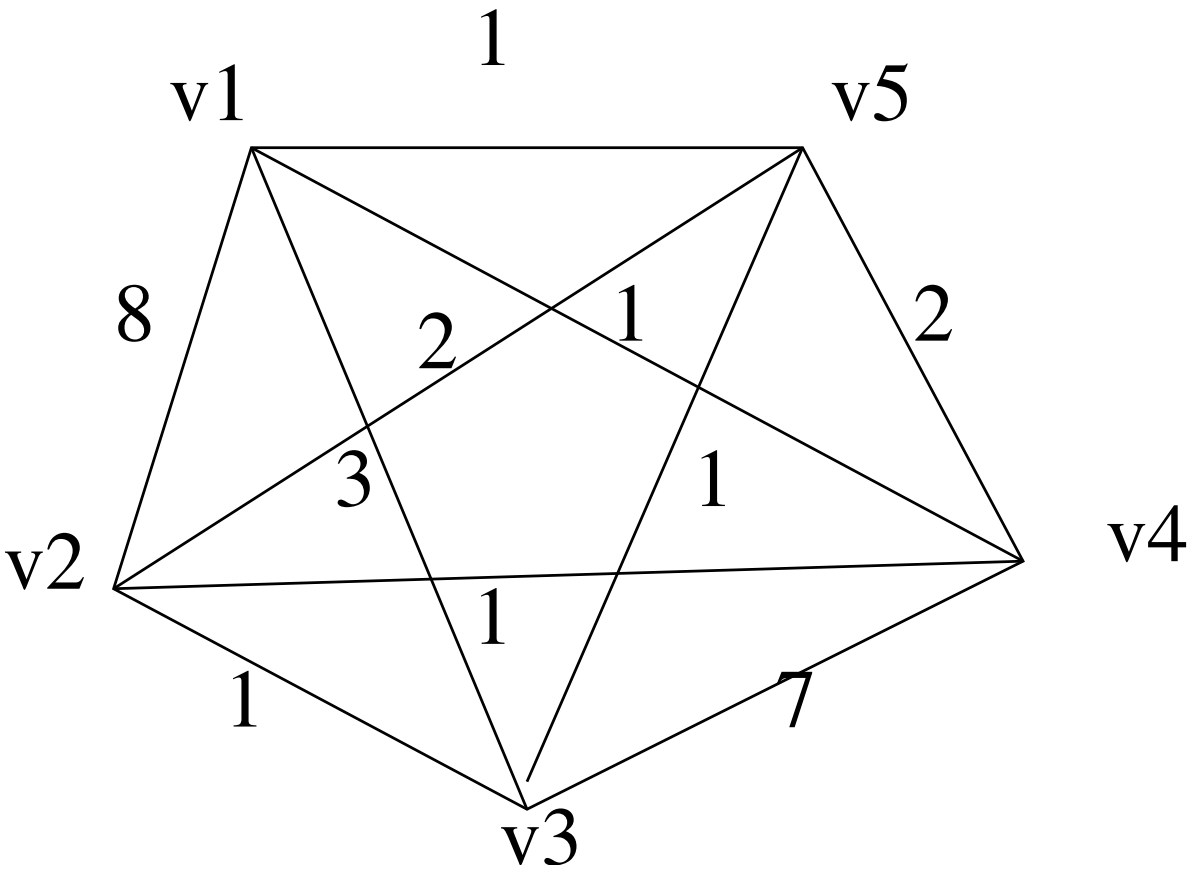
Goal: minimum - to find a Hamiltonian cycle with minimum cost.

EXAMPLE



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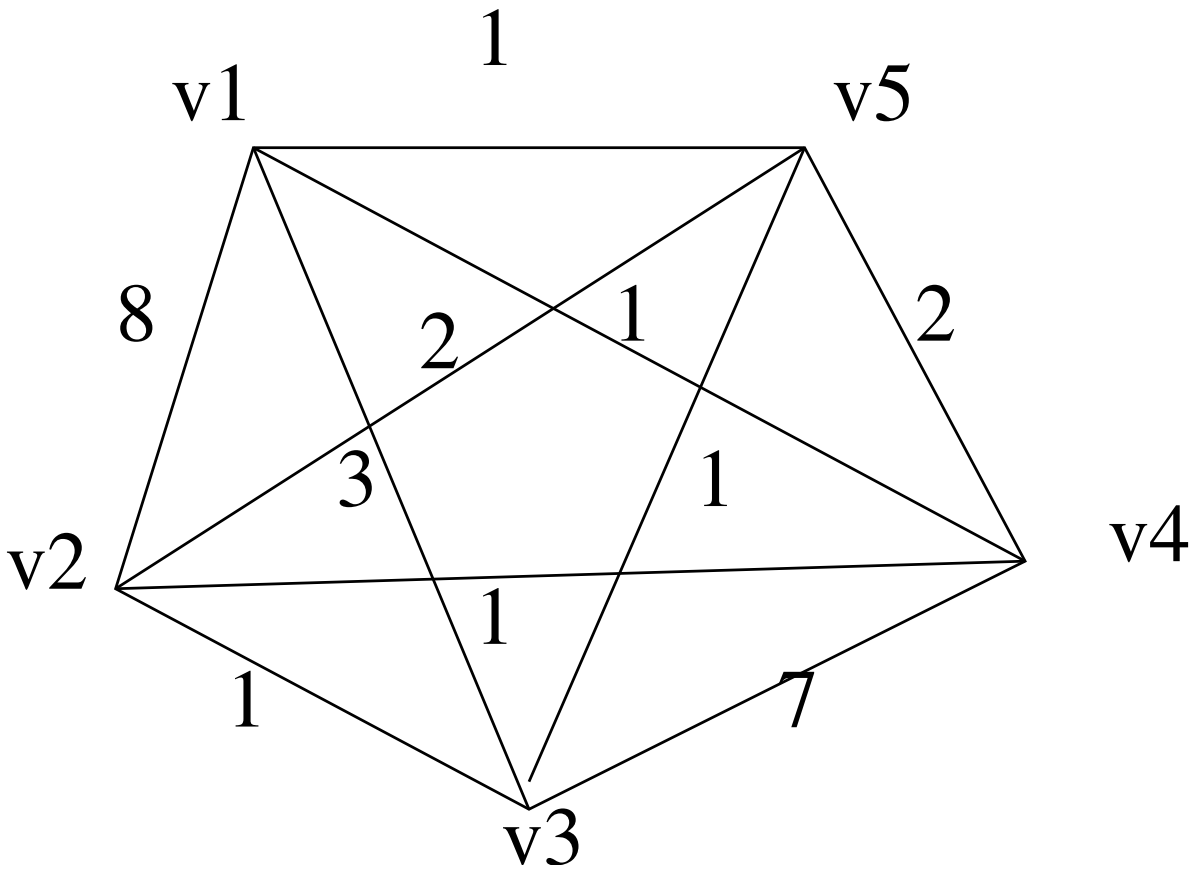

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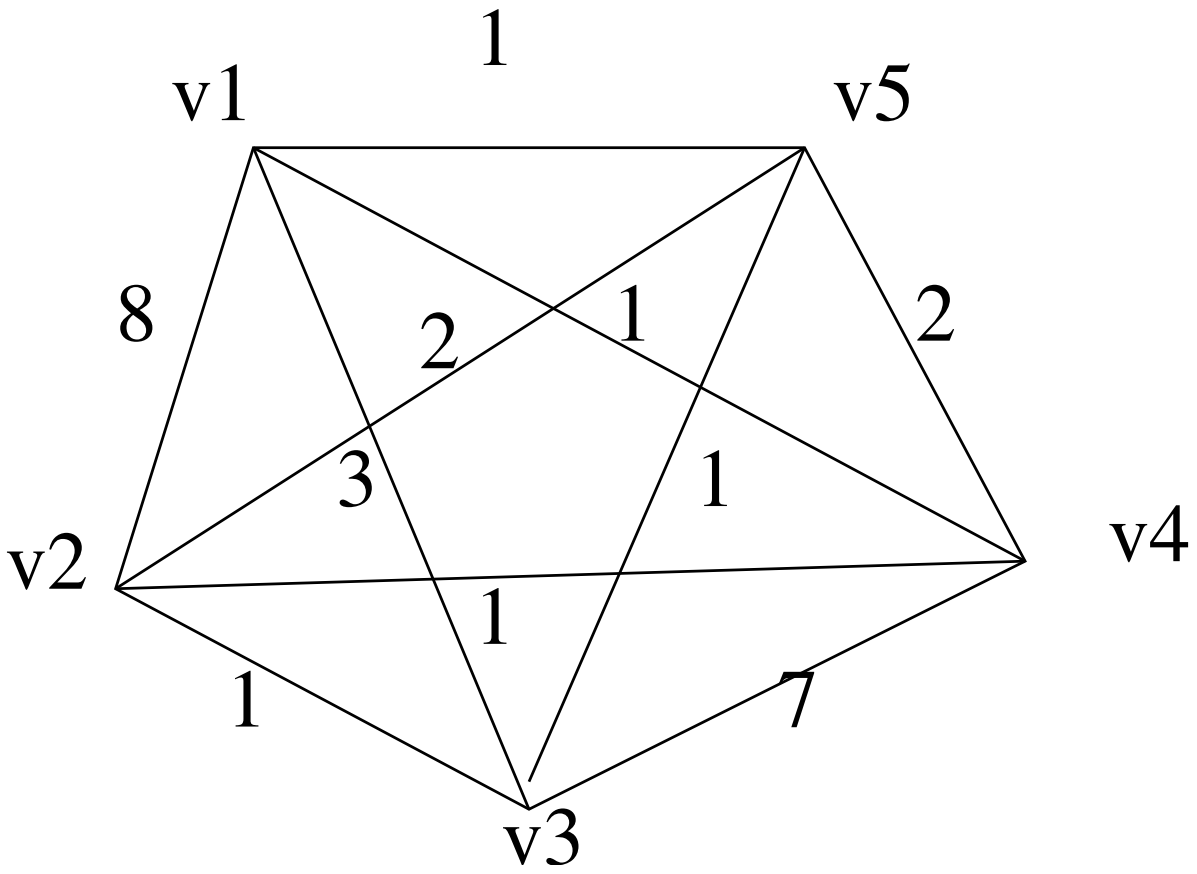

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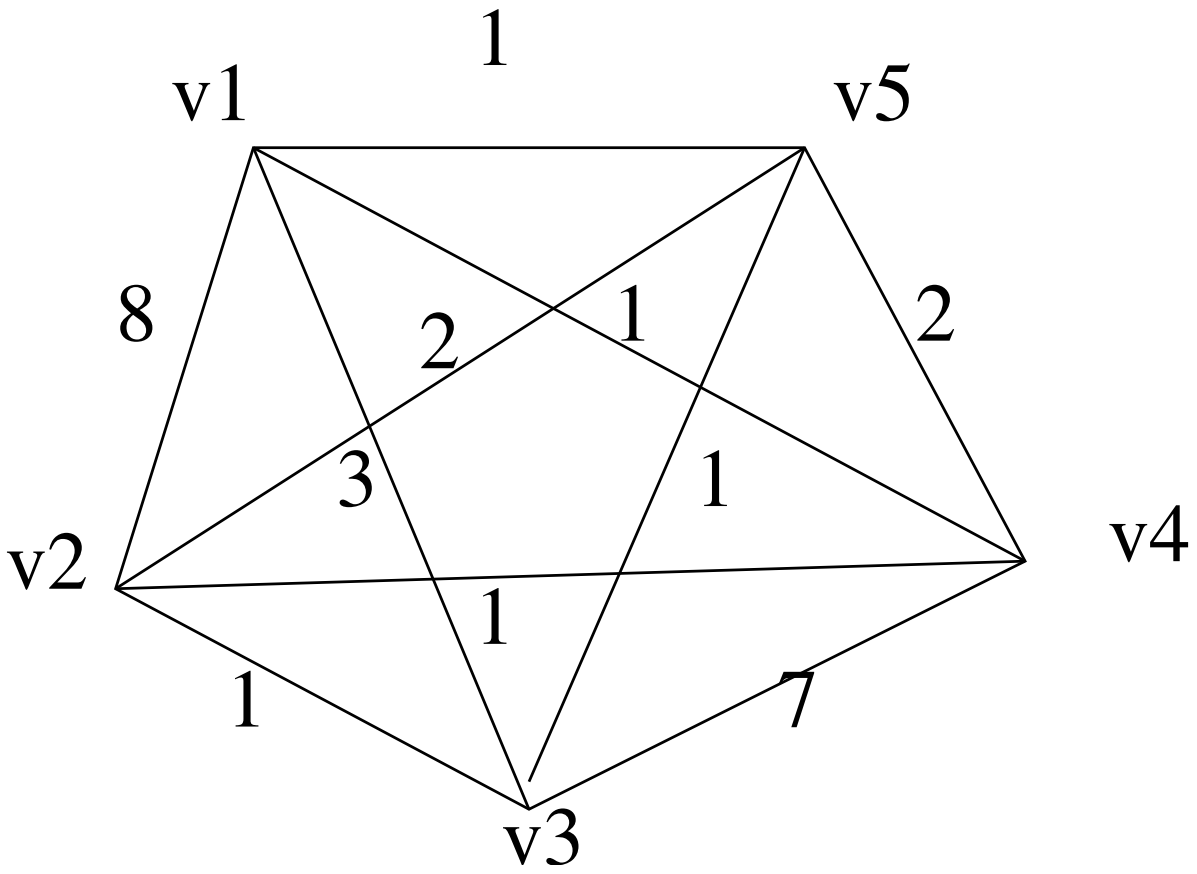

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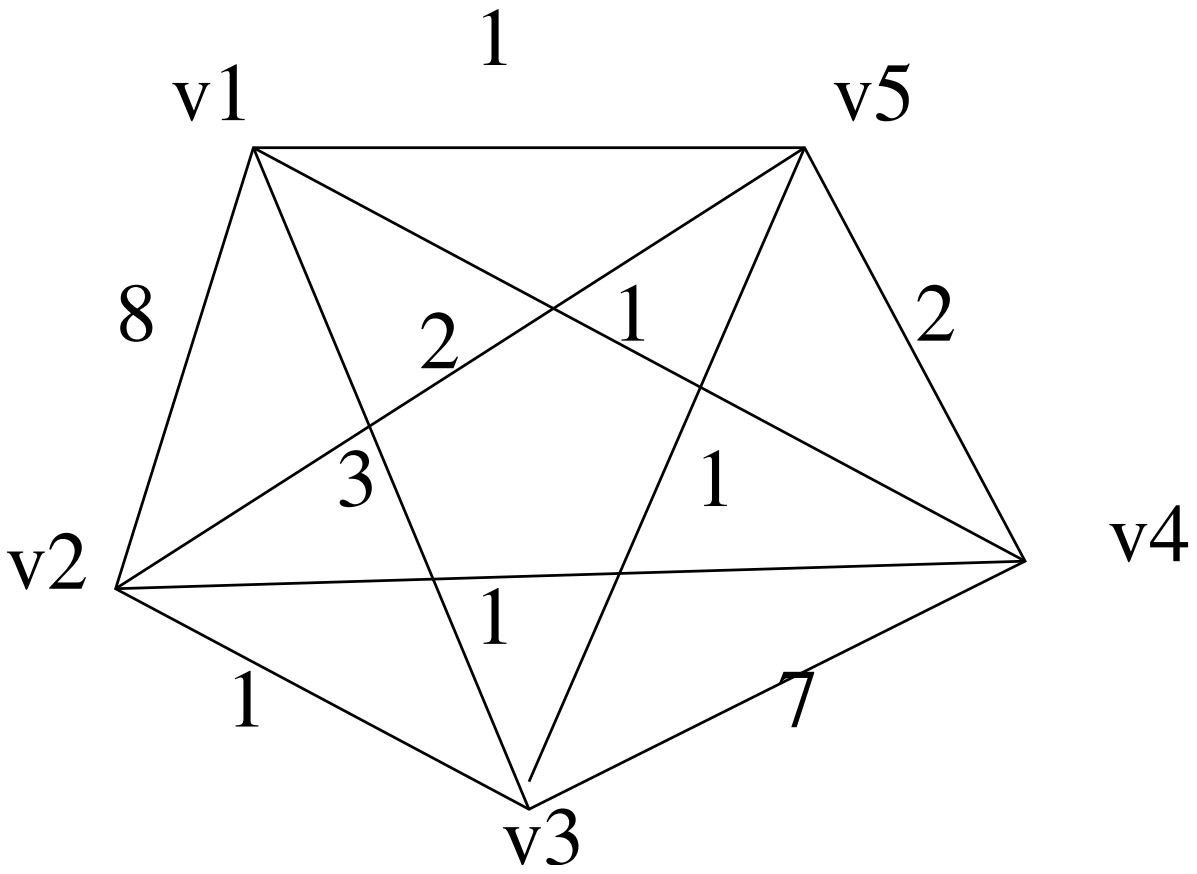

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Cost of a solution: For $X = (x_1, \ldots, x_n) \in \mathcal{M}(A, b)$

$$\text{cost}(X, c)) = c \cdot X = \sum_{i=1}^{n} c_i x_i.$$

Goal: minimum

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are chosen to correspond to our intuition and to apply simultaneously to minimization and maximization problems. Both of these bounds compare an approximation solution with the optimal one, but in two different ways.

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The ratio bound is never less than one.
An optimal algorithm has ratio bound 1.
The larger is the best possible ratio bound of an approximation algorithm, the worse is the algorithm.

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Two general problems concerning approximation of **NP**-complete problems are of special interest and importance.

The constant relative error bound problem: Given an **NP**-complete optimization problem P with a cost of solution function c and an $\varepsilon > 0$, does there exist an approximation polynomial time algorithm for P with the relative error bound ε?

The approximation scheme problem: Given an **NP**-complete problem P, does there exist for P with a cost of solution function c a polynomial time algorithm for designing, given an $\varepsilon > 0$ and an input instance x, an approximation for P and x with the relative error bound ε?

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Note that if an optimization problem P has an approximation threshold 0, this means that a (polynomial time) approximation arbitrarily close to the optimum is possible.

Note also that if P has approximation threshold 1, this means that no universal (polynomial time) approximation method is possible.

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Example 3 Unless **P** $=$ **NP**, thee approximation threshold for the **TRAVELING SALESMAN PROBLEM** is 1.

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These two different aims lead to two ways randomized approximation algorithms are defined.

Definition 1 Let $\mathcal{P} = (\Sigma_I, \Sigma_O, L, \mathcal{M}, \text{cost}, \text{goal})$ be an optimization problem. For any $\delta > 1$, a randomized algorithm A is called a **randomized E$[\delta]$-approximation algorithm** for $\mathcal{P}$ if

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Key question. How good can an online algorithm (that does not know the future) be in comparison to an algorithm that knows the whole input (the future) from the beginning?

Evaluation of online algorithms

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Evaluation of online algorithms

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Let $\delta \geq 1$ be a real. We say that an online problem P is δ-**hard** if there does not exist any d-competitive online algorithm for P with $d < \delta$.

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No extractor is currently know that has been proven to work when applied to any type of high-entropy source.

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The von Neumann extractor can be shown to produce a uniform output, even if the distribution of the input bits is not uniform, so long as each bit has the same probability of being one and there is no correlation between successive bits.

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By a probabilistic method to be discussed later one can show that there exists a (k, ε) extractor for many k and ε.

Note H_∞ stands for so-called *min-entropy*, which is a measure of the amount of randomness in the worst case.