Part I

1.Basic concepts and Examples of Randomized Algorithms

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The second aim of this chapter is to introduce main complexity classes for randomized algorithms.

Third aim is to show relations between randomized and deterministic complexity classes.

Fourth aim is to discuss in some details puzzling concept of randomness, at least in some details.

Revolution in designing algorithms

The idea that randomized algorithm can be VERY useful can be seen as the main revolutionary idea in the design of algorithms in the last 2200 years.

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5. **Randomized numerical algorithms are often better organized better to exploit parallelism of modern computer architectures**

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Randomized complexity classes offer also a plausible way to extend the very important *feasibility* concept.

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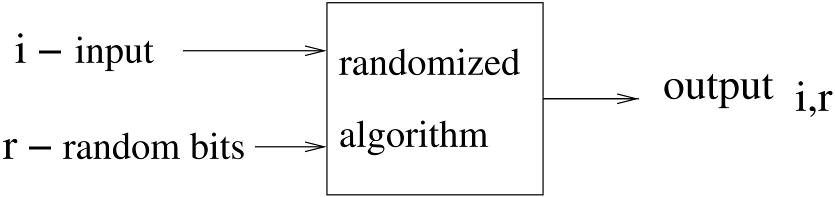

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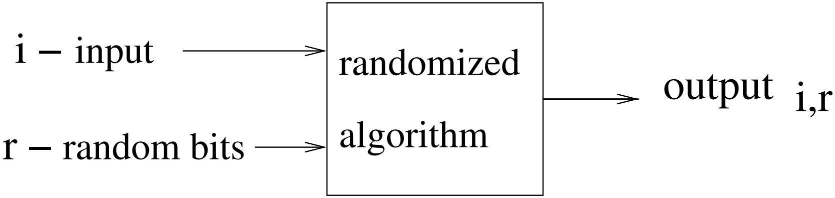

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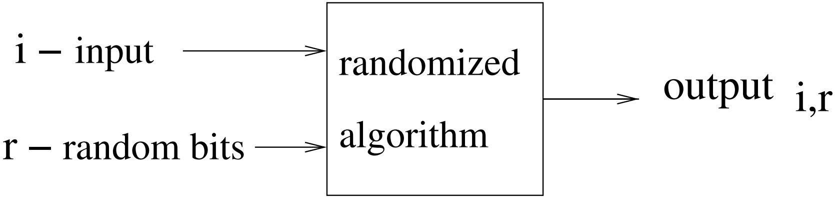

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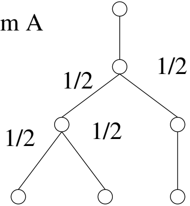
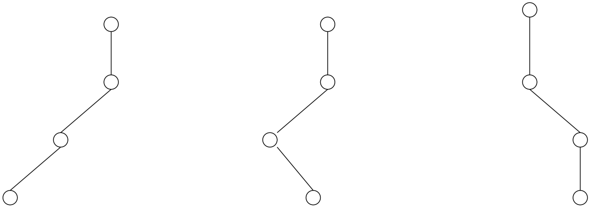
- As a nondeterministic-like algorithm which has a probability assigned to each possible transition.
- As a probability distribution on a set of deterministic algorithms - $\{\mathcal{A}_i, p_i\}_{i=1}^n$.

RANDOMIZED ALGORITHMS as PROBABILISTIC DISTRIBUTIONS on DETERMINISTIC ALGORITHMS

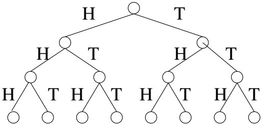
randomized algorithm A

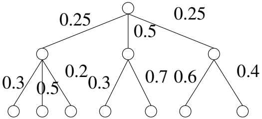


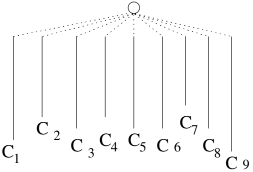
as a probabilistic distribution on three deterministic algorithms B, C, D



MODELS of RANDOMIZED ALGORITHMS II







$Pr(C_i)$

C_i are runs of dif. determ. alg.

STORY of RANDOMNESS

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There are only two possibilities, either a big chaos conquers the world, or order and law.

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Famous reply by Niels Bohr - one of the fathers of quantum mechanics.

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God is not malicious and made Nature to produce, so useful, (shared) randomness.

This is what the outcomes of the theoretical informatics imply.

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2. Quantum measurement yields, in principle, random outcomes.

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- There is no proof that perfect randomness exists in the real world.
- More exactly, there is no proof that quantum mechanical phenomena of the microworld can be exploited to provide a perfect source of randomness for the macroworld.

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- Until Kolmogorov complexity was introduced we had no meaningful way to talk about a given object being random.

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whenever you end such an iterative process, the final seed is a pseudorandom string of digits.

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Definition. Let $l(n) : N \rightarrow N$ be such that $l(n) > n$ for all n. A **(cryptographically strong) pseudorandom generator with a stretch function** l, is an efficient deterministic algorithm which on the input of a random n-bit seed outputs a $l(n)$-bit sequence which is computationally indistinguishable from any random $l(n)$-bit sequence.

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A very fundamental concept: A predicate b is a hard core predicate of the function f if b is easy to evaluate, but $b(x)$ is hard to predict from f(x). (That is, it is unfeasible, given f(x) where x is uniformly chosen, to predict $b(x)$ substantially better than with the probability $1/2$.)

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Theorem Let f be a one-way function which is length preserving and efficiently computable, and b be a hard core predicate of f, then

$$G(s) = b(s) \cdot b(f(s)) \cdots b\left(f^{l(|s|)-1}(s)\right)$$

is a (cryptographically strong) pseudorandom generator with stretch function $l(n)$.

EXAMPLES
of
RANDOMIZED ALGORITHMS

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Will the game end? If not, why? If yes, when?

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Will the game end? It can be shown that it ends almost always in approximately at most $(nw)^2$ steps.

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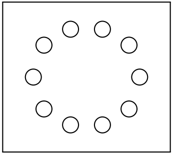
Example Let n identical processors, connected into a ring, have to choose one of them to be a "leader", under the assumption that each of the processors knows n.



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- It can be shown that this problem cannot be solved exactly and in bounded time on classical computers even in the case processors know number of nodes (n) and topology of the network.
- However, there is quantum algorithm that runs in $\mathcal{O}(n^3)$ time, its communication complexity is $\mathcal{O}(n^4)$, and it can solve this problem exactly for any network topology, provided parties are connected by quantum communication links.

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- How should they proceed to learn whether one of them paid the bill without learning which on e - for other two?

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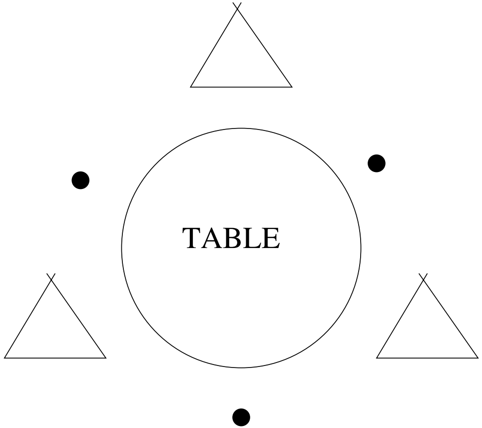
  - In a case a cryptographer paid the dinner the other two cryptographers would have no idea he did that.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In case one of them payed dinner, say Cryptographer 2, they say loudly:

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If all N values obtained are 0, then we can consider p to be identically 0. The probability of error is at most 2^{-N}.

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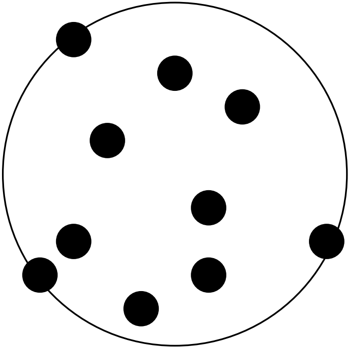

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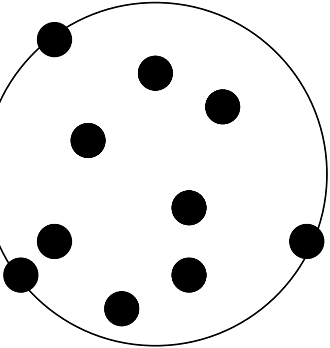

Naive solution For any three points design a disk/circle passing through them - complexity of such an algorithm is $\mathcal{O}(n^3)$

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1. Choose randomly (taking into considerations weights of points) a set of about 20 points S' and determine, somehow, $D(S')$.
2. In case there are points of S that are out of $D(S')$, then double their weights and go to Step 1. Otherwise you are done

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This way we obtain *random QUICKSORT* or RQUICKSORT.

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If p_{ij} is the probability that s_i and s_j are being compared during an execution of the algorithm, then $E[s_{ij}] = p_{ij}$.

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$$\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} \leq \sum_{i=1}^{n} \sum_{j=i}^{n} \frac{2}{j-i+1} \leq \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} \frac{2}{k} \leq$$

$$2 \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} \frac{1}{k} \leq 2nH_n = \Theta(n \log n)$$

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Theorem If $0 < \epsilon < \frac{1}{2}$, then there is a constant c such that for all but a fraction of at most $n2^n e^{-\frac{\epsilon n^2}{2}}$ of satisfiable 3-CNF Boolean formulas with n variables, the probability that the above algorithm succeeds in discovering a truth assignment in each independent trial from a random start is at least $1 - e^{-\epsilon^2 n}$.

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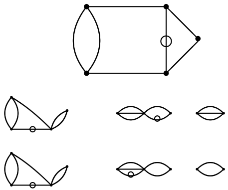

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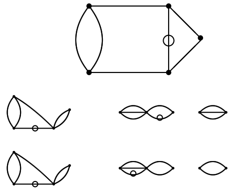

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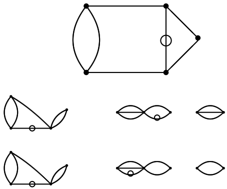

In the above example, where two options are explored in the second step, we got once the optimal result, and once a non-optimal result.

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$$Pr\left[E_i|\bigcap_{j=1}^{i-1} E_j\right] \geq 1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1}$$

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Running time of the best deterministic minimum cut algorithm is $\mathcal{O}(nm+n^2 \lg n)$, where m is number of edges and n is number of vertices.

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In other words most of the numbers between 1 and n are witnesses for composability of n.

Rabin-Miller algorithm

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LARGEST PRIME - I.

On February 3, 2016 C. Cooper from university Missouri announced a new (Mersenne) prime

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that has 5 millions more digits as previously known largest prime.

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Four research groups over the world verified after the announcement for three days that the number claimed to be a new largest prime is indeed a prime.

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Percentage of 512 bits numbers that are primes is 0.006...

RANDOMIZED COMPLEXITY CLASSES

COMPLEXITY CLASSES for DETERMINISTIC COMPUTATIONS

- **P** is the class of problems (languages) that can be solved (accepted) by deterministic algorithms running in polynomial time. (**Or P is class of problems solvable in polynomial time on deterministic Turing machines.**)
- **NP** is the class of problems solution of which can be verified in polynomial time. (**Or NP is the class of problems that can be solved in polynomial time on nondeterministic Turing machines.**)
- **co-NP** is the class of languages that are complements of languages in **NP**.
- **PSPACE** is the class of problems (languages) that can be solved (accepted) by algorithms using only polynomially large space/memory.
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- Problems: a **PP**-algorithm is free to accept with probability $1/2 + 2^{-n}$ if the answer is yes and probability $1/2 - 2^n$ if the answer is no. However how can a mortal distinguish these two cases if, for example, $n = 5000$?

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If $w \in L$, then there is at least one computation of M that accepts $w \Rightarrow$ more than half of computations of M'' accept. In addition, it holds $\mathbf{PP} \subseteq \mathbf{PSPACE}$.

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Definition A language $L \subseteq \{0,1\}^\star$ has polynomial size Boolean circuits if there is a family of Boolean circuits $G = \{C_i\}_{i=1}^{\infty}$ and a polynomial p such that size of C_n is bounded by $p(n)$, C_n has n inputs and $x \in L$ iff the output of $C_{|x|}$ is 1 if its input is x.

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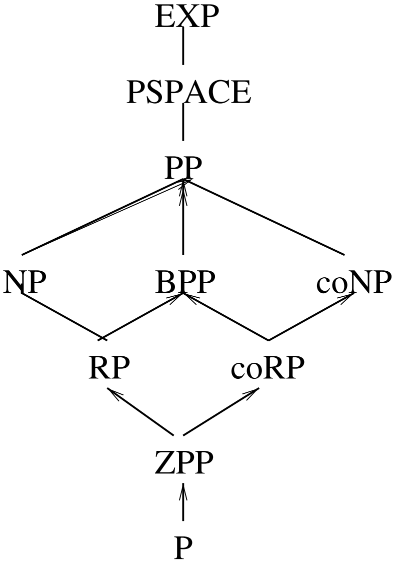
In case k is big enough, the effective error probability will be as small as we wish. This process is called **amplification of probability**.

HIERARCHY of COMPLEXITY CLASSES



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It can be shown that if $\mathbf{P} = \mathbf{BPP}$, then **MA=NP**.

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- There is a strong evidence that **P = BPP**.
- Such assumption is based on results showing that computational hardness of some problems can be used to generate pseudorandom sequences that look random to all polynomial time algorithms.
- Using such techniques Widgerson and Impagliazo showed that **P=BPP** if there is a problem computable in an exponential time that requires circuits of exponential size.

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PUZZLE

In case the set of elementary events E is infinite situation is much more complex as the following example discuss in lecture 3 illustrates.

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