# The Probabilistic Method – continuing from tutorial 7

**Task 1:** A 1959 paper of Erdős addressed the following problem in graph theory: given positive integers $g$ and $k$, does there exist a graph $G$ containing only cycles of length at least $g$, such that the chromatic number of $G$ is at least $k$? [1]

a) Let $n$ be very large and consider a random graph $G$ on $n$ vertices, where every edge in $G$ exists with probability $p = n^{(\frac{1}{g}-1)}$. Is the probability that following properties hold positive?

1. *Already done last time:* $G$ contains at most $n/2$ cycles of length less than $g$.
2. $G$ contains no independent set of size $\lceil \frac{n}{2k} \rceil)$.

b) How can we obtain a new graph $G'$ (containing only cycles of length at least $g$, such that the chromatic number of $G'$ is at least $k$) using a modification of $G$, if the graph $G$ satisfies (1.) and (2.)?

**Task 2:** Consider $m$ pairwise independent events $I_1, \ldots, I_m$. How does their dependency graph look? (Is the dependency graph for every set of events unique?)[2]

**Task 3:** Consider a graph with $n$ vertices and $nd/2$ edges. Delete each vertex (together with its incident edges) independently with probability $1 - 1/d$

a) What is the expected number of vertices and edges that remain after the deletion?

b) Prove that there is an independent set with at least $n/2d$ vertices in every graph with $n$ vertices and $nd/2$ edges (using result (a))

c) Use *Lovász Local Lemma* for proving that an independent set remains with positive probability after a similar deletion process performed on a $\sqrt{n}$-regular graph with $n$ vertices if we delete each vertex with probability $1 - \dfrac{1}{3n^{1/4}}$.

[1] Chromatic number of a graph is the minimal number of colours with which we can colour the vertices of the graph so that no two vertices of the same colour are connected with an edge. The chromatic number of a graph is lower bounded by (number of vertices)/(size of largest independent set)

[2] Recall that by a dependency graph for probabilistic events $I_1, \ldots, I_m$ we mean a directed graph that has a vertex for each $I_j$ and an event $I_j$ is mutually independent of all other events $I_k$ such that the edge $(I_j, I_i)$ is not present in the graph.