FOUNDATIONS of COMPUTING

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PREFACE

One who is serious all day will never have a good time, while one who is frivolous all day will never establish a houshold.
Ptahhotpe, 24th century BC

Science is a discipline in which even a fool of this generation should be able to go beyond the point reached by a genius of the last.
Scientific folklore, 20th century AC

It may sound surprising that in computing, in a field that develops so fast that future often turns to past before ever being presence, there is not too much more stable and more worth to learn than its foundations.

It may sound less surprising that in the field with such a revolutionary methodological impact on all sciences and technologies, and practically on all our intellectual endeavours, the importance of foundations of computing goes far beyond computing and it should be of interest to all those trying and needed to understand laws and essence of the information processing world and also to all those who want and need to have a firm ground for their life-long reeducation process—what everybody in computing has to expect.

This book presents the automata-algorithm-complexity part of foundations of computing in a new way, from several points of view, in order to meet new requirements for learning and teaching.

First of all, the book takes a broader and more coherent view of theory and its foundations than it used to be the case. It presents not only basics of automata, grammars, formal languages, universal computers, computability, computational complexity, but also of parallelism, randomization, communications, cryptography, interactive protocols, communication complexity and even of theoretical computer/communication architecture.

Secondly, the book presents foundations of computing as very rich on deep, important and exciting results that help, in an irreplacable way, to see the problems, laws, and potentials in computing and to cope with its complexity. Thirdly, the book tries to find a new balance between formal rigorousity needed to present basic concepts and results and informal motivations, illustrations and interpretations, needed to grasp their merit. Fourthly, the book tries to offer the reader or the student a systematic, complex and also up-to-date presentation of the main basic concepts, models, methods and results as well as those main new trends and results, a detailed presentation of which would require special lectures. In order to achieve that the basic concepts, models, methods and results are presented and illustrated in detail but other deep/new results, with too hard or not too illuminating proofs, are just stated, explained, interpreted and commented.

Topics covered by the book are very broad. Each chapter could be extended into a several volumes monograph/handbook. This textbook tries to concentrate only on subjects that are central for the field; on concepts, methods and models that are simple enough to present and on results that are either deep or important or useful or surprising or interesting or have several of these properties.

The aim of the book is to present foundations only to the extent that everyone should know who wants to be an expert in computing, or in areas with a deeper need for computing and its methodologies. For that reason the book tries to present only what everybody finishing university studies in computing or in related area should know from theory. Furthermore, the book is oriented to those for whom theory is only, or mainly, a tool. For those more interested in particular areas of theory the book could be a good starting point for their way through unlimited and exciting theory adventures. A detailed bibliography and historical/bibliographical notes should help those wishing to go more deeply into a subject or to find proofs and a more detailed treatment of particular subjects.

The main aim of the book is to serve as a textbook. However, because of its broad view of the field and an up-to-date presentation of the concepts, methods and results of foundations, the book can serve also as a reference book. Detailed historical and bibliographical comments at the end of each chapter, an extensive bibliography and a very detailed index are aimed to help the book to serve all its purposes.

The book is a significantly extended version of the lecture notes for a one-semester, four hours a week, course that was held at the University of Hamburg.

The book should be reasonably easy to follow and grasp for an interestyed and/or ambitious reader. It keeps formal presentation concise, it presents basic concepts, models, methods and results in a straightforward way and it devotes a large attention to examples, exercises, motivations, interpretations and explanation of connections between various approaches as well as to the impact of theory results in and also outside of computing. The book tries to demonstrate that the basic concepts, models, methods and results—products of many geniuses of the past—are, if properly seen and presented, actually very simple, with deep implications and important applications. The book tries to demonstrate that foundations of computing is an intellectually very rich, and practically very applied body of knowledge. It tries also to illustrate the ways theoretical concepts are often modified in order to get ones directly applicable. More difficult sections are marked by asterisks.

A large number of examples (235) and figures/tables (232) aim to help to understand presented concepts, models, methods, and results. The same is the rôle of a large number of exercises (1124). Quite a few of them (597) are included directly into the text to be an inherent part of it. They are mostly (very) easy or of a reasonable difficulty and should help the reader to get an immediate feedback and, at the same time, to extend knowledge and skill. Those more difficult are marked by one or two asterisks—to encourage ambitious readers with enough desire, time and energy and not to discourage the others. The rest of the exercises (512) is placed at the end of chapters. Some of them are of the same character as those in the text, only slightly different or additional ones. Other extend the subject delt with in the main text. Those that are more difficult are marked by one or two asterisks.

INTERNET suplement. The book will have an internet extension/supplement regularly upadated. It will contain three items. (i) Chapter 12: FRONTIERS, describing very recent and promising developments in computing models and modes (quantum, molecular,...); (ii) Additional examples and exercises; (iii) Corrections.

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Chapter 1

FUNDAMENTALS

INTRODUCTION

Foundations of computing is making an extensive and increasing use of a variety of very basic concepts, some very new some very old, methods and results to analyze computational problems and systems and to formulate, explore and harvest laws and limitations of information processing. This chapter therefore systematically introduces a variety of concepts, techniques and results needed for quantitative analysis in computing and for making use of randomization to increase efficiency, to extend feasibility, to innovate the concept of evidence and to secure communications. All concepts introduced are therefore of importance far beyond foundations of computing. They are needed for dealing with efficiency in and also outside of computing.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. methods to solve recurrences arising in the analysis of computing systems;
2. a powerful concept of generating functions with a variety of applications;
3. main asymptotic notations and techniques to use and to manipulate them;
4. basic concepts of number theory, especially those related to primes and congruences;
5. methods to solve various congruences;
6. intriguing problems of computing discrete square roots and discrete logarithms that play an important rôle in randomized computations and secure communications;
7. basics of discrete probability;
8. modern approaches to randomness and pseudo-random generators.

> The firm, the enduring, the simple
> and the modest are near to virtue.
>
> *Confucius (531-439 BC)*

Efficiency and inherent complexity play a key rôle in computing and are of growing importance also outside of computing. They provide both practically important quantitative evaluations and benchmarks as well as theoretically deep insights into the nature of computing and communication. Their importance grows with maturing of the discipline and also with the increase of performance of computing and communication systems. Therefore the main concepts, tools, methods and results of complexity analysis belong to the very basic body of knowledge and techniques in computing. They are also natural subjects to begin with in a textbook on foundations of computing, because of their importance throughout the book. Moreover, their simplicity and elegance provides a natural basis to start in order to present, demonstrate and exercise the richness and power of the concepts and methods of foundations of computing.

The following three important approaches concerning complexity issues in designs and performance analysis of computing systems are considered in this chapter: recursion, (asymptotic) estimations, and randomization.

Complex systems that we are able to design, describe or understand are often recursive by nature or designs. Their complexity analysis leads therefore naturally to recurrences. That is, why we start this chapter with methods to solve recurrences.

In the analysis of complex computational systems we are generally unable to determine exactly resources needed. For example, the exact number of computer operations to solve a problem. Fortunately, we mostly actually do not need to do that. Simple, but good enough, asymptotic estimations, that provide robust and a particular computer independent results, are in most of the cases not only quite satisfactory, but often much more useful. Methods how to handle, in a simple but precise way, asymptotic characterizations of functions are therefore of key importance for analysis of computing systems and are also treated in detail in this chapter.

The discovery that randomness is an important resource to manage complexity is one of the most important outcomes of foundations of computing in the last years. The understanding has developed already for a longer time that the analysis of algorithms with respect to a random distribution of input data may provide more realistic results. A new, and actually currently the main use of randomness, is in randomized algorithms, communication protocols, designs, proofs, etc. Coin-tossinging techniques allow to manage in many cases the complexity surprisingly well. Elements of the probability theory and of randomness are therefore also included into this introductory chapter and will be needed throughout the book. Interesting enough, these very modern approaches to use randomness to provide security are often based on old and basic concepts, methods, and results of the number theory, which will be also introduced in this chapter.

1.1 Examples -November 8, 2005

Quantitative analysis of computational resources—time, storage, processors, program, communication, randomness, interactions, knowledge, or of the size of computing systems: circuits, networks, automata, grammars, computers, algorithms or protocols—is of great importance. It may provide irreplaceable information how good a particular system is, and also deep insights into the nature of the underlying computational and communication problems.

Large and/or complex computing systems are often designed or described recursively. Their quantitative analysis leads naturally to recurrences. A **recurrence** is a system of equations or inequalities that describes a function in terms of its values for smaller inputs.

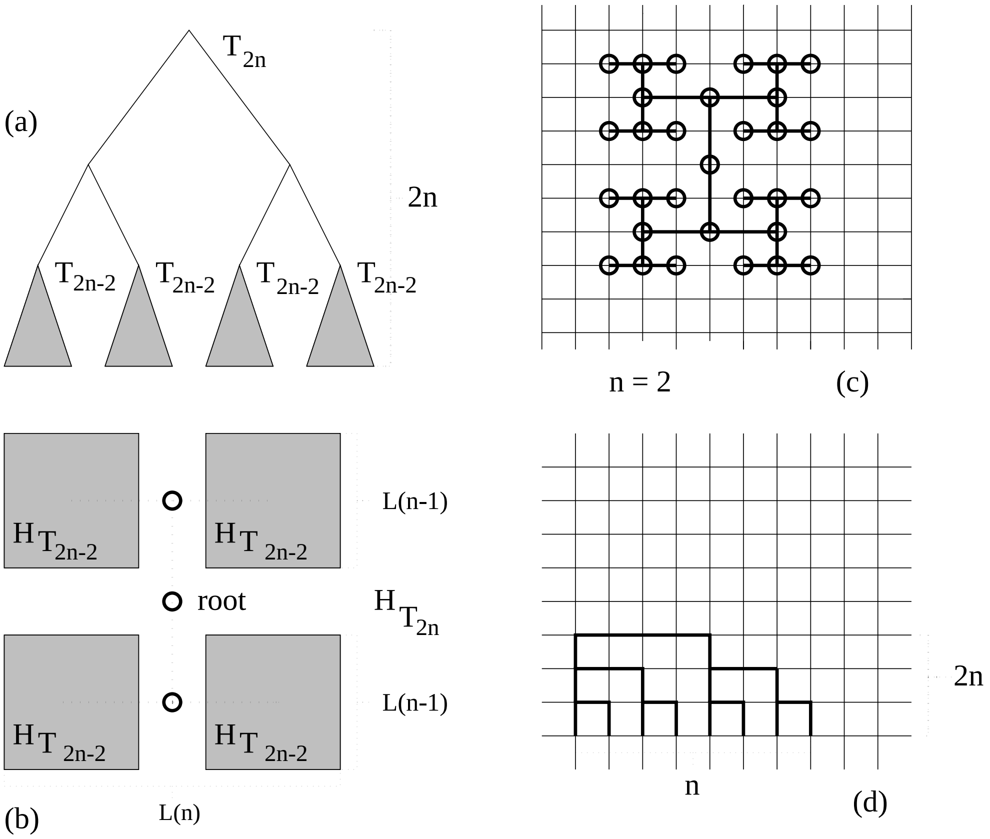

Figure 1.1: H-layout of complete binary trees 70%

Example 1.1.1 (H-layout of binary trees) *A layout of a graph G into the two-dimensional grid is a mapping of different nodes of G into different nodes of the grid and edges (u, v) of G into non-overlapping paths, along the grid lines, between the images of nodes u and v in the grid.*

The so-called H-layout $H_{T_{2n}}$ of a complete binary tree T_{2n} of depth $2n$, $n \geq 0$ (see Figure 1.1a for T_{2n} and its subtrees T_{2n-2}), is described recursively in Figure 1.1c.

A more detailed treatment of such layouts is in Section 10.6. For the moment it is for us of importance only the fact that for the length $L(n)$ of the side of the layout E_n we get the recurrence

$$L(n) = \begin{cases} 2, & \text{if } n = 1; \\ 2L(n-1)+2, & \text{if } n > 1. \end{cases}$$

As we shall see later, $L(n) = 2^{n+1} - 2$. A complete binary tree of depth $2n$ has $2^{2n+1} - 1$ nodes. The total area $A(m)$ of the H-layout of a complete binary tree with m nodes is therefore proportional to the number of nodes of the tree.[1] Observe that in the "natural layout of the binary tree", shown in Figure 1.1d, the area of the smallest rectangle that contains the layout is proportional to $m \log m$. To express this concisely we will use notation $A(m) = \Theta(m)$ in the first case and $A(m) = \Theta(m \lg m)$ in the second case. The notation $f(n) = \Theta(g(n))$—that means that "$f(n)$ grows proportionally to $g(n)$"[2]—is discussed in detail and formally in Section 1.5.

Algorithmic problems have often a recursive solution even if their usual formulation does not indicate that.

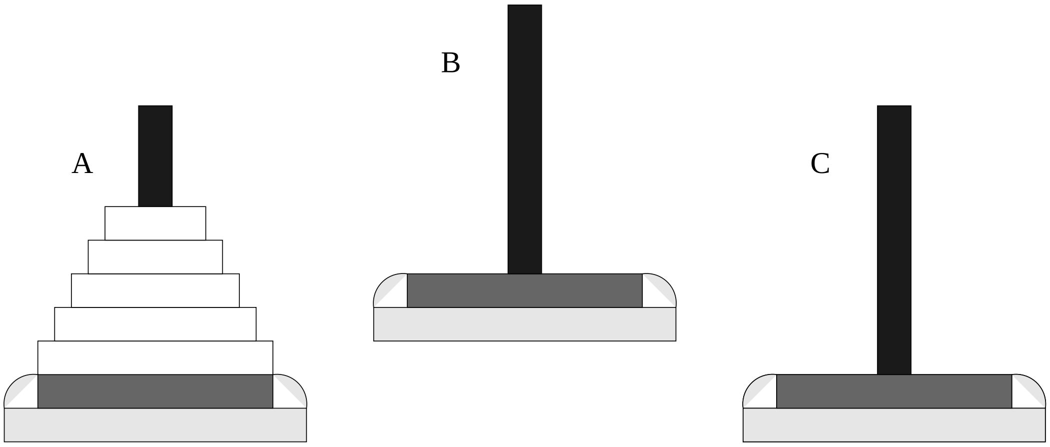

Figure 1.2: Towers of Hanoi-70%

Example 1.1.2 (Towers of Hanoi Problem) *Suppose we are given three rods A, B, C, and n rings piled in the descending order of magnitude on A while the other rods are empty—see Figure 1.2 for $n = 5$. The task is to move rings from A to B, using perhaps C in this process, in such a way that in one step only one ring is moved, and at no instant a ring is placed atop of a smaller one.*

There is a simple recursive algorithm to solve the problem:

Algorithm 1.1.3 (Towers of Hanoi—a recursive algorithm)

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[1] The task to design layouts of various graphs into the two-dimensional grid, with as small area as possible, is of importance for VLSI designs. For more on layouts see Section 10.6.

[2] Or, more exactly, that there are constants $c_1, c_2 > 0$ such that $c_1|g(n)| \leq f(n) \leq c_2|g(n)|$ for all but finitely many n.

1. Move $n-1$ top rings from A to C;
2. Move the largest ring from A to B;
3. Move all $n-1$ rings from C to B.

The correctness of this algorithm is obvious. It is also clear that the number $T(n)$ of ring moves satisfies the equations:

$$T(n) = \begin{cases} 1, & \text{if} \quad n = 1; \\ 2T(n-1)+1, & \text{if} \quad n > 1. \end{cases} \tag{1.1}$$

In spite of the simplicity of the above algorithm, it is natural to ask whether there exist a faster algorithm—one that performs less ring moves. It is actually a simple task to show that such an algorithm does not exist. Indeed, denote by $T_{min}(n)$ the minimal number of moves needed to solve the task. Clearly, $T_{min}(n) \geq 2T_{min}(n-1)+1$, because in order to remove all rings from the rod A to B we have first to move the top $n-1$ of them to C, then the largest one to B, and finally the remaining ones to B. This implies that our solution is the best possible.

Algorithm 1.1.3 is very simple. However, it is not so easy to be performed "by hand", because of the need to keep a track of many levels of recursion. The second, "iterative" algorithm presented bellow is from this point of view much simpler; even a child can follow it. (Try to apply both algorithms for $n = 4$).

Algorithm 1.1.4 (Towers of Hanoi—an iterative algorithm)

Do the following alternating steps, starting with step 1, until all rings are properly transferred:

1. Move the smallest top ring in the clockwise order $(A \to B \to C \to A)$ if the number of rings is odd; and in anti-clockwise order if the number of rings is even.
2. Make the only possible move that does not involve the smallest top ring.

In spite of the simplicity of Algorithm 1.1.4, it is far from obvious that this algorithm is correct. It is also far from obvious how to determine the number of its ring moves—until one shows, what can be done by induction, that both algorithms perform exactly the same sequence of ring moves.

Consider now the following modification of the Towers of Hanoi problem: The goal is the same, but it is never allowed to move rings from A on B or from B on A. It is easy to show that also in this case there is a similar simple recursive algorithm to solve the problem and for its number $T'(n)$ of ring moves we have $T'(1) = 2$ and

$$T'(n) = 3T'(n-1)+2, \quad \text{for} \quad n > 1. \tag{1.2}$$

There is a modern legend saying that after creating the world, Brahma also designed three rods made of diamond and 64 golden rings on one of them near a monastery in Tibet and ordered monks to transfer rings, following the rules described above. According to the legend, the world would come to the end when monks finish their task.[3]

Exercise 1.1.5 *Use both algorithms for the Towers of Hanoi Problem to solve the cases: (a) $n = 3$; (b) $n = 5$; (c)* $n = 6$.*

Exercise 1.1.6 **(Parallel version of the Towers of Hanoi problem) Assume that in each step more than one ring can be moved with the following restriction: in each step from each rod at most one ring is removed and to each rod at most one ring is added. Determine the recurrence for the minimal number $T_p(n)$ of parallel moves needed to solve the parallel version of the Towers of Hanoi problem. (Hint: Determine $T_p(1)$, $T_p(2)$ and $T_p(3)$ and express $T_p(n)$ using $T_p(n-2)$.)*

Two previous examples are not singular. Complexity analysis leads to recurrences whenever algorithms or systems are designed using one of the most powerful design method—**divide-and-conquer**.

Example 1.1.7 *We can often easily and efficiently solve an algorithmic problem P of the size $n = c^i$, where c, i are integers, using the following recursive method, where b_1, b_2 and d are constants—see Figure 1.3:*

1. *Decompose P, in time $b_1 n$, to a subproblems of the same type and size $\frac{n}{c}$.*
2. *Solve all subproblems recursively—using the same method.*
3. *Compose, in time $b_2 n$, the solution of P from solutions of all its a subproblems.*

For the time complexity $T(n)$ of the resulting algorithm we have the recurrence:

$$T(n) = \begin{cases} d, & \text{if } n = 1; \\ aT(\frac{n}{c}) + b_1 n + b_2 n, & \text{if } n > 1. \end{cases} \tag{1.3}$$

As an illustration, we present the well known recursive algorithm for sorting a sequence of $n = 2^k$ numbers:

Algorithm 1.1.8 (MERGESORT)

1. Divide the sequence in the middle into two subsequences.
2. Sort recursively both subsequences.
3. Merge both—already sorted—subsequences.

If arrays are used to represent sequences, steps (1) and (3) can be performed in the time proportional to n.

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[3] Such a prophecy is not unreasonable. Since $T(n) = 2^n - 1$, as it will be seen soon, it would take more than 500 000 years to finish the task if monks move one ring per second.

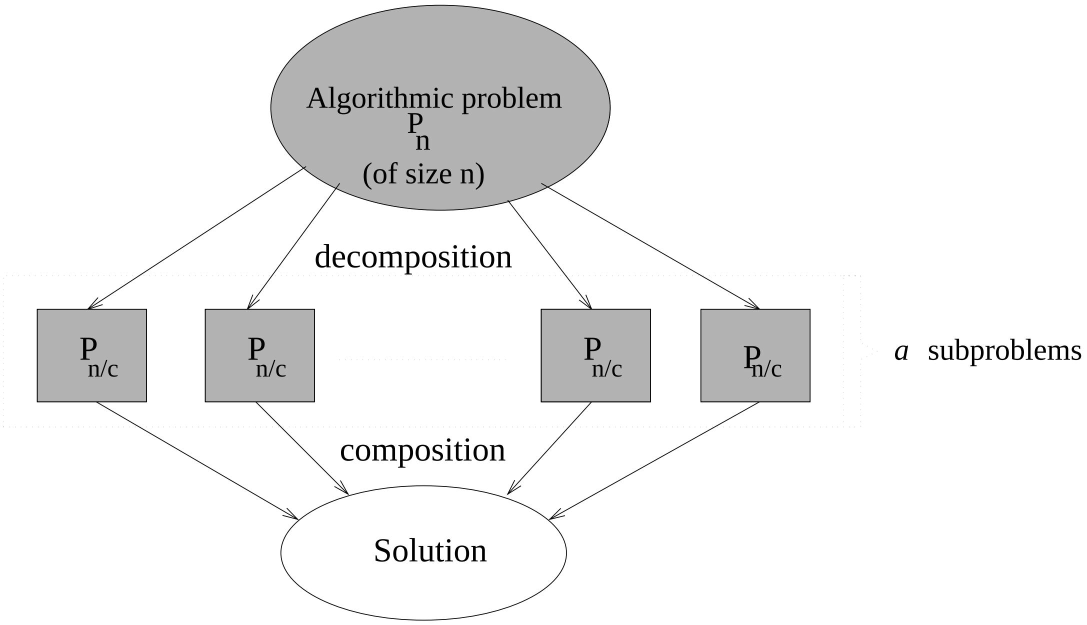

Figure 1.3: Divide-and-conquer method-70%

Remark 1.1.9 Note that we have derived the recurrence (1.3) without knowing the nature of the problem P and the computational model to be used. The only information we have utilized is that both decomposition and composition can be performed in the time proportional to the size of problem.

Exercise 1.1.10 *Suppose that n circles are drawn in the plane in such a way that no three circles meet in a point and each pair of circles intersects in exactly two points. Determine the recurrence for the number of distinct regions of the plane created by such n circles?*

An analysis of computational complexity of algorithms depends often quite significantly on the underlying model of computation and the exact analysis is often impossible; either because of the complexity of the algorithm or of the computational model (device) that is used. Fortunately, the exact analysis is most of the time not only unnecessary; it is often superfluous. Simple, the so-called asymptotic estimations provide not only more insights but they can also be to a large degree independent of a particular computing model/device.

Example 1.1.11 (Matrix multiplication) *Multiplication of two matrices $A = \{a_{ij}\}_{i,j=1}^n$, $B = \{b_{ij}\}_{i,j=1}^n$, of degree n, with the resulting matrix $C = AB = \{c_{ij}\}_{i,j=1}^n$, using the well known relation*

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \tag{1.4}$$

requires $T(n) = 2n^3 - n^2$ *arithmetical operations to perform.*

It is again simpler to say, and mostly sufficiently informative, that $T(n) = \Theta(n^3)$ than to write exactly $T(n) = 2n^3 - n^2$. If a program to compute c_{ij} using the formula (4.3.3) is written in a natural way in a high-level programming language and implemented on a usual sequential computer, then the exact analysis of the number of computer instructions to perform, or of the time $T(n)$ needed, is practically impossible, because it depends also on the available compiler, operating system, computer, and so on. Nevertheless, the basic claim $T(n) = \Theta(n^3)$ remains valid, in any case—provided we assume that each arithmetical operation takes a unit of time.

Remark 1.1.12 If, on the other hand, parallel computations are allowed, quite different results concerning the number of steps needed to multiply two matrices are obtained. Using n^3 processors all multiplications in (4.3.3) can be performed in one parallel step. Since any sum of n numbers $x_1 + \ldots + x_n$ can be computed with $\frac{n}{2}$ processors using the so-called **recursive doubling technique**[4] in $\lceil \log_2 n \rceil$ steps; in order to compute all c_{ij} in (4.3.3) by the above method we need $\Theta(n^3)$ processors and $\Theta(\log n)$ parallel steps.

Example 1.1.13 (Exponentiation) *Let* $b_{k-1} \ldots b_0$ *be the binary representation of an integer* n *with* b_0 *as the least significant bit and* $b_{k-1} = 1$. *Exponentiation* $e = a^n$ *can be done in* $k = \lceil \log_2(n+1) \rceil$ *steps using the following so-called* **repeated squaring method** *based on the equalities:*

$$e = a^{\sum_{i=0}^{k-1} b_i 2^i} = \prod_{i=0}^{k-1} a^{b_i 2^i} = \prod_{i=0}^{k-1} (a^{2^i})^{b_i}.$$

Algorithm 1.1.14 (Exponentiation)

```
begin $e \leftarrow 1; p \leftarrow a$;
      for $i \leftarrow 0$ to $k-1$
          do if $b_i = 1$ then $e \leftarrow e \cdot p$;
             $p \leftarrow p \cdot p$
          od
end
```

Exercise 1.1.15 *Determine exactly the number of multiplications Algorithm 1.1.14 performs.*

Remark 1.1.16 The term recurrence is sometimes used to denote only the equation in which the inductive definition is made. This terminology is used often explicitely in cases when the specific value of initial conditions is not important.

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[4] For example, to get $x_1 + \ldots + x_8$, we compute in the first step $z_1 = x_1 + x_2, z_2 = x_3 + x_4, z_3 = x_5 + x_6, z_4 = x_7 + x_8$, in the second step $z_5 = z_1 + z_2, z_6 = z_3 + z_4$, and in the last step $z_7 = z_5 + z_6$.

1.2 Solution of Recurrences—Basic Methods

Several basic methods to solve recurrences are presented in this chapter. It is not always easy to decide which one to try first. However, in any case it is a good practice to start with computing some of the values of the unknown function; for several small arguments. This often helps:

1. to guess the solution;
2. to verify a to-be-solution.

Example 1.2.1 *For small values of n the unknown functions $T(n)$ and $T'(n)$ from the recurrences (1.1) and (1.2) have the following values:*

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	1	3	7	15	31	63	127	255	511	1023
$T'(n)$	2	8	26	80	242	728	2186	6560	19682	59049

and from this table we can easily guess that $T(n) = 2^n - 1$ and $T'(n) = 3^n - 1$. Such guesses have then to be verified, for example by induction, as we shall do later for $T(n)$ and $T'(n)$.

Example 1.2.2 *The recurrence*

$$Q_n = \begin{cases} \alpha, & \textit{if} \quad n = 0; \\ \beta, & \textit{if} \quad n = 1; \\ \frac{\alpha\beta + \beta^2 - \beta Q_{n-1}}{Q_{n-2}}, & \textit{if} \quad n > 1; \end{cases}$$

where $\alpha, \beta > 0$, looks quite complicated. However, it is easy to determine that $Q_2 = \beta, Q_3 = \alpha, Q_4 = \beta$. Hence

$$Q_n = \begin{cases} \alpha, & \textit{if } n = 3k \textit{ for some } k; \\ \beta, & \textit{otherwise.} \end{cases}$$

1.2.1 Substitution Method

Once we have guessed the solution of a recurrence, induction is often a good method to verify the correctness of the guess.

Example 1.2.3 (Tower of Hanoi problem) *We show by induction that our guess $T(n) = 2^n - 1$ is correct. Since $T(1) = 2^1 - 1 = 1$, the initial case $n = 1$ is verified. From the inductive assumption $T(n) = 2^n - 1$ and the recurrence (1.1), we get, for $n > 1$,*

$$T(n+1) = 2T(n) + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1.$$

This completes the induction step.

Similarly, we can show that $T'(n) = 3^n - 1$ is the correct solution of the modified Tower of Hanoi problem, and $L(n) = 2^{n+1} - 2$ is the length of the side of the H-layoutin Example 1.1.1. The inductive step in the last case is: $L(n+1) = 2L(n) + 2 = 2(2^{n+1} - 2) + 2 = 2^{n+2} - 2$.

1.2.2 Iteration Method

Using an **iteration (unrolling)** of a recurrence, we can often reduce the recurrence to a summation that may be easier to compute or estimate.

Example 1.2.4 *For the recurrence (1.2), of the modified Towers of Hanoi problem, we get by an unrolling*

$$\begin{aligned} T'(n) &= 3T'(n-1)+2 = 3(3T'(n-2)+2)+2 = 9T'(n-2)+6+2 \\ &= 9(3T'(n-3)+2)+6+2 = 3^3T'(n-3)+2\times 3^2+2\times 3+2 \\ &\vdots \\ &= \sum_{i=0}^{n-1} 3^i \times 2 = 2\sum_{i=0}^{n-1} 3^i = 2\frac{3^n-1}{3-1} = 3^n - 1. \end{aligned}$$

Example 1.2.5 *For the recurrence* $T(1) = g(1)$ *and* $T(n) = T(n-1) + g(n)$*, for* $n > 1$*, the unrolling yields:*

$$T(n) = \sum_{i=1}^{n} g(i).$$

Example 1.2.6 *By an unrolling of the recurrence*

$$T(n) = \begin{cases} b, & \text{if } n = 1; \\ aT(\frac{n}{c}) + bn, & \text{if } n = c^i > 1; \end{cases}$$

obtained at the analysis of divide-and-conquer algorithms, we get

$$\begin{aligned} T(n) &= aT\left(\frac{n}{c}\right) + bn = a\left(aT\left(\frac{n}{c^2}\right) + b\frac{n}{c}\right) + bn = a^2T\left(\frac{n}{c^2}\right) + bn\frac{a}{c} + bn \\ &= a^2\left(aT\left(\frac{n}{c^3}\right) + b\frac{n}{c^2}\right) + bn\frac{a}{c} + bn = a^3T\left(\frac{n}{c^3}\right) + bn\frac{a^2}{c^2} + bn\frac{a}{c} + bn \\ &\vdots \\ &= bn\sum_{j=0}^{\log_c n}\left(\frac{a}{c}\right)^j. \end{aligned}$$

Therefore:

- *Case 1,* $a < c$*:* $T(n) = \Theta(n)$*, because the sum* $\sum_{i=0}^{\infty}\left(\frac{a}{c}\right)^i$ *converges.*
- *Case 2,* $a = c$*:* $T(n) = \Theta(nlogn)$.
- *Case 3,* $a > c$*:* $T(n) = \Theta(n^{\log_c a})$.

Indeed, in Case 3 we get

$$\begin{aligned} T(n) &= bn \sum_{i=0}^{\log_c n} \left(\frac{a}{c}\right)^i = bn \frac{\left(\frac{a}{c}\right)^{\log_c n+1} - 1}{\frac{a}{c} - 1} \\ &\approx bn \left(\frac{a}{c}\right)^{\log_c n} \\ &= bn \frac{a^{\log_c n}}{n} = ba^{\log_c n} = bn^{\log_c a}, \end{aligned}$$

using the identity $a^{\log_c n} = n^{\log_c a}$.

Observe that the time complexity of a *divide-and-conquer* algorithm depends only on the ratio $\frac{a}{c}$ and neither on the problem being solved, nor on the computing model (device) being used; provided that the decomposition and composition require only linear time.

Exercise 1.2.7 *Solve recurrences obtained by solving Exercises 1.1.6 and 1.1.10.*

Exercise 1.2.8 *Solve the following recurrence using the iteration method:* $T(n) = 3T(\frac{n}{4}) + n$, *for* $n = 4^k > 1$.

Exercise 1.2.9 *Determine* g_n, n *a power of 2, defined by the recurrence*

$$g_1 = 3 \text{ and } g_n = (2^{\frac{n}{2}} + 1)g_{\frac{n}{2}} \text{ for } n \geq 2.$$

Exercise 1.2.10 *Express* $T(n)$ *in terms of the function* g *for the recurrence:* $T(1) = a$, $T(n) = 2^p T(n/2) + n^p g(n)$, *where* p *is an integer,* $n = 2^k$, $k > 0$ *and* a *is a constant.*

1.2.3 Reduction on Algebraic Equations

A large class of recurrences, the so-called **homogeneous linear recurrences**, can be solved by a reduction to algebraic equations. Before we present the general method, let us demonstrate its basic idea on an example.

Example 1.2.11 (Fibonacci numbers) *Leonardo Fibonacci*[5] *introduced in 1202 a se-*

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[5] Leonardo of Pisa (1170-1250), known also as Fibonacci, perhaps the most influential mathematician of the medieval Christian world. Educated in Africa, by a Muslim teacher, he was famous for his possesion of the mathematical knowledge of his own and the preceeding genarations. In his celebrated and influential classics *Liber Abachi* (that appeared in print only in 19th century) he introduced to the Latin world Arabic positional system and Hindu methods of calculations with fractions, square roots, cube roots,... The following problem from *Liber Abachi* led to the Fibonacci numbers: *How many pair of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on.*

quence of numbers defined by the recurrence

$$F_0 = 0, \quad F_1 = 1, \qquad \textit{(the initial conditions);} \tag{1.5}$$

$$F_n = F_{n-1} + F_{n-2}, \quad \textit{if} \quad n > 1 \qquad \textit{(the inductive equation).} \tag{1.6}$$

Fibonacci numbers form one of the most interesting sequences of natural numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \dots$$

Exercise 1.2.12 *Explore the beauty of Fibonacci numbers: (a) find all n such that $F_n = n$ and all n such that $F_n = n^2$; (b) determine $\sum_{i=0}^{k} F_i$; (c) show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all n; (d) show $F_{2n+1} = F_n^2 + F_{n+1}^2$ for all n; (e) compute $F_{16}, \dots, F_{49}$ ($F_{50} = 12586269025$).*

It is natural to ask whether, given an integer n, we can determine F_n without computing F_i for all $i < n$. More exactly, can we find an explicit formula for F_n?

Let us first try to find a solution of the inductive equation (1.6) in the form $F_n = r^n$, where r is, so far, an unknown constant. Suppose r^n is a solution of (1.6), then

$$r^n = r^{n-1} + r^{n-2},$$

has to hold for all $n > 1$, and therefore either $r = 0$, which is an uninteresting case, or $r^2 = r + 1$. The last equation has two roots:

$$r_1 = \frac{1+\sqrt{5}}{2}, \qquad r_2 = \frac{1-\sqrt{5}}{2}.$$

Unfortunately, none of the functions r_1^n, r_2^n satisfies initial conditions in (1.5). We are therefore not ready yet. Fortunately, each linear combination $\lambda r_1^n + \mu r_2^n$ also satisfies the inductive equation (1.6). Therefore, if λ, μ are chosen in such a way that also the initial conditions (1.5) are met; that is if

$$\lambda r_1^0 + \mu r_2^0 = F_0 = 0, \qquad \lambda r_1^1 + \mu r_2^1 = F_1 = 1, \tag{1.7}$$

then $F_n = \lambda r_1^n + \mu r_2^n$ is the solution of recurrences (1.5) and (1.6). From (1.7) we get:

$$\lambda = -\mu = \frac{1}{\sqrt{5}}$$

and thus

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Since $\lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n = 0$, we get also a simpler, approximate, expression for F_n in the form

$$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n, \qquad \text{for } n \to \infty.$$

$\square$

The method used in the previous example will be now generalized. Let us consider a **homogeneous linear recurrence**; that is, the recurrence where the value of the unknown function is expressed as a linear combination of a fixed number of its values for smaller arguments:

$$u_n = a_1u_{n-1} + a_2u_{n-2} + \cdots + a_ku_{n-k}, \qquad \text{if } n \geq k, \qquad \text{(the inductive equation)} \quad (1.8)$$

$$u_i = b_i, \qquad \text{if } 0 \leq i < k, \qquad \text{(the initial conditions)} \quad (1.9)$$

where $a_1, \ldots, a_k$ and $b_0, \ldots, b_{k-1}$ are constants and let

$$P(r) = r^k - \sum_{j=1}^{k} a_j r^{k-j} \tag{1.10}$$

be the so-called **characteristic polynomial** of the inductive equation (1.8) and $P(r) = 0$ its **characteristic equation**. The roots of the polynomial (1.10) are called **characteristic roots** of the inductive equation (1.8). The following theorem says that we can always find a solution of a homogeneous linear recurrence when the roots of its characteristic polynomial are known.

Theorem 1.2.13 *(1) If the characteristic equation $P(r) = 0$ has k different roots $r_1, \ldots r_k$, then the recurrence (1.8) with the initial conditions (1.9) has the solution*

$$u_n = \sum_{j=1}^{k} \lambda_j r_j^n, \tag{1.11}$$

where λ_j are solutions of the system of linear equations

$$b_i = \sum_{j=1}^{k} \lambda_j r_j^i, \quad 0 \leq i < k. \tag{1.12}$$

(2) If the characteristic equation $P(r) = 0$ has p different roots, $r_1, \ldots, r_p$, $p < k$, and the root r_j, $1 \leq j \leq p$, has the multiplicity $m_j \geq 1$, then also $r_j^n, nr_j^n, n^2r_j^n, \ldots, n^{m_j-1}r_j^n$, are solutions of the inductive equation (1.8), and there is the solution of (1.8), satisfying the initial conditions (1.9), in the form $u_n = \sum_{j=1}^{p} P_j(n)r_j^n$, where each $P_j(n)$ is a polynomial of the degree $m_j - 1$, the coefficient of which can be obtained as the unique solution of the system of linear equations $b_i = \sum_{j=1}^{p} P_j(i)r_j^i$, $1 \leq i \leq k$.

Proof. (1) Since the inductive equation (1.8) is satisfied with $u_n = r_j^n$ for $1 \leq j \leq k$, it is satisfied also with an arbitrary linear combination $\sum_{j=1}^{k} a_j r_j^n$. In order to prove the first part of the theorem, it is therefore sufficient to show that the system of linear equations (1.12) has a unique solution. This is the case when the determinant of the matrix of the system does not equal zero. However, this is the well known result from linear algebra, because the corresponding (Vandermond) matrix and determinant have the form:

$$det \begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_k \\ \vdots & \vdots & & \vdots \\ r_1^{k-1} & r_2^{k-1} & \ldots & r_k^{k-1} \end{vmatrix} = \prod_{j>i} (r_i - r_j) \neq 0.$$

(2) A detailed proof of the second part of the theorem is quite technical and we present here only its basic idea.

We have first to show that if an r_j is a root of the equation $P(r) = 0$ of a multiplicity $m_j > 1$, then all functions $u_n = r_j^n$, $u_n = nr_j^n$, $u_n = n^2 r_j^n, \dots, u_n = n^{m_j-1} r_j^n$ satisfy the inductive equation (1.8). In order to prove that, we can make use of the well-known fact from calculus, namely that if r_j is a root of multiplicity $m_j > 1$ of the equation $P(r) = 0$, then r_j is also the root of the equations $P^{(j)}(r) = 0$, $1 \leq j < m_j$, where $P^{(j)}(r)$ is the j-th derivative of $P(r)$.

Let us consider the polynomial

$$Q(r) = r \cdot (r^{n-k} P(r))' = r \cdot [(n-k) r^{n-k-1} P(r) + r^{n-k} P'(r)].$$

Since $P(r_j) = P'(r_j) = 0$, we have $Q(r_j) = 0$. However,

$$\begin{aligned} Q(r) &= r[r^n - a_1 r^{n-1} - \cdots - a_k r^{n-k}]' \\ &= nr^n - a_1(n-1)r^{n-1} - \cdots - a_{k-1}(n-k+1)r^{n-k+1} - a_k(n-k)r^{n-k} \end{aligned}$$

and, since $Q(r_j) = 0$, we have that $u_n = nr_j^n$ is the solution of the inductive equation (1.8).

In a similar way we can show by induction that all $u_n = n^s r_j^n$, $1 < s < m_j$, are solutions of (1.8), by considering the following sequence of polynomials: $Q_1(r) = Q(r)$, $Q_2(r) = rQ_1'(r), \dots, Q_s(r) = rQ_{s-1}'(r)$.

It remains then to show that the matrix of the system of linear equations $b_i = \sum_{j=1}^{p} P_j(i) r_j^i$, $1 \leq i \leq k$, is nonsingular. This is an (nontrivial) exercise from linear algebra. □

Example 1.2.14 *The recurrence*

$$\begin{aligned} u_n &= 3u_{n-1} - 2u_{n-2}, \quad n > 2; \\ u_0 &= 0, \quad u_1 = 1, \end{aligned}$$

has the characteristic equation $r^2 = 3r - 2$ *with two roots:* $r_1 = 2, r_2 = 1$. *Hence* $u_n = \lambda_1 2^n + \lambda_2$, *where* $\lambda_1 = 1$ *and* $\lambda_2 = -1$ *are solutions of the system of equations:* $0 = \lambda_1 2^0 + \lambda_2 1^0$ *and* $1 = \lambda_1 2^1 + \lambda_2 1^1$.

Example 1.2.15 *The recurrence*

$$\begin{aligned} u_n &= 5u_{n-1} - 8u_{n-2} + 4u_{n-3}, \quad n \geq 3; \\ u_0 &= 0, \; u_1 = -1, \; u_2 = 2, \end{aligned}$$

has the characteristic equation $r^3 = 5r^2 - 8r + 4$, *which has one simple root,* $r_1 = 1$, *and one root of multiplicity 2,* $r_2 = 2$. *The recurrence has therefore the solution* $u_n = a + (b + cn)2^n$, *where* a, b, c *satisfy equations:*

$$0 = a + (b + c \cdot 0)2^0, \quad -1 = a + (b + c \cdot 1)2^1, \quad 2 = a + (b + c \cdot 2)2^2.$$

Example 1.2.16 *(a) The recurrence* $u_0 = 3, u_1 = 5$ *and* $u_n = u_{n-1} - u_{n-2}$*, for* $n > 2$*, has two roots:* $x_1 = \frac{1+i\sqrt{3}}{2}$ *and* $x_2 = \frac{1-i\sqrt{3}}{2}$ *and the solution* $u_n = (\frac{3}{2} - \frac{7}{6}i\sqrt{3})x_1^n + (\frac{3}{2} + \frac{7}{6}i\sqrt{3})x_2^n$*. (Verify that all* u_n *are integers!)*

(b) For the recurrence $u_0 = 0, u_1 = 1$ *and* $u_n = 2u_{n-1} - 2u_{n-2}$*, for* $n \geq 2$*, the characteristic equation has two roots:* $x_1 = (1+i)$ *and* $x_2 = (1-i)$*, and we get*

$$u_n = \frac{1}{2}i((1-i)^n - (1+i)^n) = 2^{\frac{n}{2}} \sin(\frac{n\pi}{4}),$$

using a well-known identity from calculus.

Exercise 1.2.17 *Solve the recurrences: (a)* $u_0 = 6$, $u_1 = 8$, $u_n = 4u_{n-1} - 4u_{n-2}$, $n \geq 2$*; (b)* $u_0 = 1$, $u_1 = 0$, $u_n = 5u_{n-1} - 6u_{n-2}$, $n \geq 2$*; (c)* $u_0 = 4$, $u_1 = 10$, $u_n = 6u_{n-1} - 8u_{n-2}$, $n \geq 2$.

Exercise 1.2.18 *Solve the recurrences: (a)* $u_0 = 0$, $u_1 = 1$, $u_2 = 1$, $u_n = 2u_{n-2} + u_{n-3}$, $n \geq 3$*; (b)* $u_0 = 7$, $u_1 = -4$, $u_2 = 8$, $u_n = 2u_{n-1} + 5u_{n-2} - 6u_{n-3}$, $n \geq 3$*; (c)* $u_0 = 1$, $u_1 = 2$, $u_2 = 3$, $u_n = 6u_{n-1} - 11u_{n-2} + 6u_{n-3}$, $n \geq 3$.

Exercise 1.2.19 * *Using some substitutions of variables, transform the following recurrences to the cases dealt with in this section and in this way solve the recurrences: (a)* $u_1 = 1$, $u_n = u_{n-1} - u_n u_{n-1}$, $n \geq 2$*; (b)* $u_1 = 0$, $u_n = n(u_{n/2})^2$, n *is a power of* 2*; (c)* $u_0 = 1$, $u_1 = 2$, $u_n = \sqrt{u_{n-1}u_{n-2}}$, $n \geq 2$.

Finally, we present an interesting open problem due to Lothar Collatz (1930)—a class of recurrences that look like linear but their solution is not known.

For any positive integer i we define the so-called $(3x+1)$-recurrence by (a Collatz' process) $u_0^{(i)} = i$ and for $n > 0$

$$u_{n+1}^{(i)} = \begin{cases} \frac{u_n^{(i)}}{2}, & \text{if } u_n^{(i)} \text{ is even;} \\ 3u_n^{(i)} + 1, & \text{if } u_n^{(i)} \text{ is odd.} \end{cases}$$

It was verified that for any $i < 2^{40}$ there exists an integer n_i such that $u_{n_i}^{(i)} = 1$ (and therefore $u_{n_i+1}^{(i)} = 4, u_{n_i+2}^{(i)} = 2, u_{n_i+3}^{(i)} = 1, \ldots$). However, it is an open problem, since the early 1950's, the so-called the Collatz' problem, whether this is true for all i.

Exercise 1.2.20 *Denote by* $\sigma(n)$ *the smallest* i *such that* $u_n^{(i)} < n$*. Determine: (a)* $\sigma(26)$, $\sigma(27)$, $\sigma(28)$*; (b)** $\sigma(2^{50}-1)$, $\sigma(2^{50}+1)$, $\sigma(2^{500}-1)$, $\sigma(2^{500}+1)$.

1.3 Special Functions

There are several simple functions that are often used in the design and analysis of computing systems. In this section we deal with some of them: ceiling and floor functions for reals-to-integers conversions, logarithms, and binomial functions.

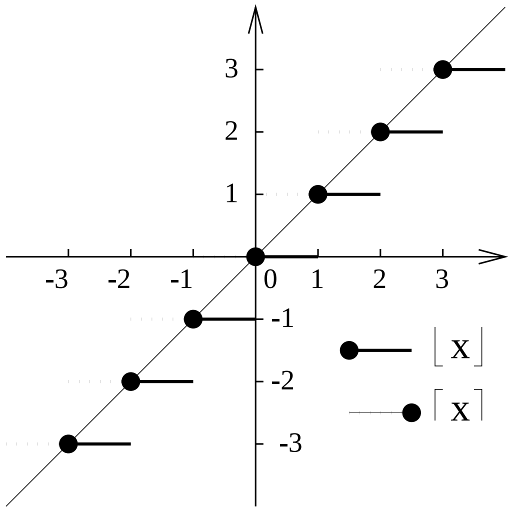

Figure 1.4: Ceiling and floor-65%

In spite of their apparent simplicity these functions have various interesting properties and, as discussed later, also a surprising computational power in case of ceiling and floor functions.

1.3.1 Ceiling and Floor

Integers play an important rôle in computing and communication and the same is true also for two basic reals-to-integers conversion functions.

Floor: $\lfloor x \rfloor$ – the largest integer $\leq x$
Ceiling: $\lceil x \rceil$ – the smallest integer $\geq x$

For Example:

$$\begin{array}{lcllcl} \lfloor 3.14 \rfloor = & 3 & = \lfloor 3.75 \rfloor, & \qquad \lfloor -3.14 \rfloor = & -4 & = \lfloor -3.75 \rfloor; \\ \lceil 3.14 \rceil = & 4 & = \lceil 3.75 \rceil, & \qquad \lceil -3.14 \rceil = & -3 & = \lceil -3.75 \rceil. \end{array}$$

The following basic properties of the floor and ceiling functions are easy to verify:

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n \text{ and } \lceil x + n \rceil = \lceil x \rceil + n, \quad \text{if } n \text{ is an integer;}$$

$$\lfloor x \rfloor = x \Leftrightarrow x \text{ is an integer } \Leftrightarrow \lceil x \rceil = x;$$

$$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1, \quad \text{for any } x;$$

$$\lfloor -x \rfloor = -\lceil x \rceil \text{ and } \lceil -x \rceil = -\lfloor x \rfloor, \quad \text{for any } x;$$

$$\lfloor x \rfloor \le \left\lceil \frac{x}{2} \right\rceil + \left\lfloor \frac{x}{2} \right\rfloor \le \lceil x \rceil;$$

$$\left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{x+1}{2} \right\rfloor, \text{ for any } x;$$

$$\lceil x \rceil - \lfloor x \rfloor = \begin{cases} 0, & \text{if } x \text{ is an integer;} \\ 1, & \text{otherwise.} \end{cases}$$

In spite of the simplicity of ceiling and floor functions, expressions that involve more occurrences of these functions are usually not easy to handle. It is therefore natural to ask when we can remove some of these functions from formulas without effecting their values. The case $\lfloor \lceil x \rceil \rfloor$ is trivial. It is less obvious that we can do that also in such expressions as $\lfloor \sqrt{\lfloor x \rfloor} \rfloor$ and $\lceil \log \lceil x \rceil \rceil$. This follows from the following general result.

Lemma 1.3.1 *Let $f(x)$ be a continuous and strictly monotonically increasing function such that if $f(x)$ is an integer, then so is x. Then*

$$\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor \quad \textit{and} \quad \lceil f(\lceil x \rceil) \rceil = \lceil f(x) \rceil.$$

Proof. We show the result for the floor function. The proof for the ceiling is similar. If $\lfloor x \rfloor = x$, then there is nothing to prove. Therefore, let $\lfloor x \rfloor < x$. Then $f(\lfloor x \rfloor) < f(x)$, since f is increasing. Thus $\lfloor f(\lfloor x \rfloor) \rfloor \le \lfloor f(x) \rfloor$. If $\lfloor f(\lfloor x \rfloor) \rfloor < \lfloor f(x) \rfloor$, there must exist a number y such that $\lfloor x \rfloor < y \le x$, $f(y) = \lfloor f(x) \rfloor$, because f is continuous. This means that $f(y)$ is an integer and, because of the special property of f, so must be y. This is a contradiction—there is no integer between $\lfloor x \rfloor$ and x. Thus $\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$. □

Exercise 1.3.2 *Draw the graphs of functions: (a) $f(x) = \lfloor 2x \rfloor$; (b) $f(x) = \lfloor x/2 \rfloor$; (c) $f(x) = \lfloor x^2 \rfloor$.*

Exercise 1.3.3 *Show: (a) $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$, for any real x;*
(b) $\lceil n/k \rceil = \lfloor (n-1)/k \rfloor + 1$, for any integers $n, k \in \mathbf{N}^+$;
(c) $\lfloor \sqrt{2n(n+1)} \rfloor = \lfloor \sqrt{2}(n + \frac{1}{2}) \rfloor$.

The ceiling and the floor functions often occur in recurrences obtained from an analysis of divide-and-conquer algorithms. For example, for the number of comparisons of the MERGESORT we get:

$$T(1) = 0; \tag{1.13}$$

$$T(n) = n - 1 + T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) \tag{1.14}$$

because the most balanced partition of a sequence of n elements is to $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ elements. With this type of recurrences we deal in Section 1.6.2.

The following two identities indicate that sums of floors and also ceilings, that look complicated, have simple solutions. For integers m, n and k we have:

$$\sum_{k=0}^{m-1} \left\lfloor \frac{n+k}{m} \right\rfloor = n;$$

$$\sum_{k=0}^{n-1} \lfloor \sqrt{k} \rfloor = na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a, \text{ where } a = \lfloor \sqrt{n} \rfloor.$$

1.3.2 Logarithms

The logarithmic function $\log_a x$, $a > 1$, is continuous, and monotonously increasing for $x \geq 1$. If a is an integer, then $\log_a x$ acquires integer values only at integer points. By Lemma 1.3.1 we therefore get $\lceil \log_a \lceil x \rceil \rceil = \lceil \log_a x \rceil$ and $\lfloor \log_a \lfloor x \rfloor \rfloor = \lfloor \log_a x \rfloor$ for $x \geq 1$. Logarithmic functions with the bases $2, e,$ and 10 are so often used that a special notation has been developed for them:

$$\begin{aligned} \lg n &= \log_2 n && \{\textit{binary logarithm}\}; \\ \ln n &= \log_e n && \{\textit{natural logarithm}\}; \\ \log n &= \log_{10} n && \{\textit{decimal logarithm}\}. \end{aligned}$$

Since $\log_a b = \frac{\log_c b}{\log_c a}$ we have the following relations between the main logarithms:

$$\lg x \approx 1.4426952 \ln x \approx 3.3219286 \log x.$$

There are various interpretations of binary logarithms worth to notice: $\lceil \lg(n+1) \rceil$ is the number of bits in the binary representation of the integer n. $\lceil \lg n \rceil$ is the minimal depth of the binary tree with n leaves and also the minimal number of parallel steps to compute $x_1 \circ x_2 \circ \cdots \circ x_n$, where $\circ$ is an associative binary operation.

Since $\ln x = \ln 2 + \ln \frac{x}{2}$, of a special importance are properties of the function $\ln(x+1)$, for $|x| < 1$. Two of them, derived in calculus:

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots, \quad \text{for } |x| < 1;$$

$$\frac{x}{1+x} \leq \ln(1+x) \leq x.$$

In the analysis of computations we often encounter powers and compositions of logarithmic functions. The following notation has been developed for them:

$$\begin{aligned} \lg^k n &= (\lg n)^k, \quad k \text{ is an integer}; \\ \lg \lg n &= \lg(\lg n); \\ \lg \lg \lg n &= \lg(\lg(\lg n)); \\ \lg^{(k)} n &= \underbrace{\lg \lg \ldots \lg}_{k \textit{ times}} n, \quad \text{if } \lg^{(k-1)} n > 0\ . \end{aligned}$$

Another related function, that also occurs often in the complexity analysis, is the so-called **iterated logarithm**:

$$\lg^* n = \min\{i \geq 0 \mid \lg^{(i)} n \leq 1\}.$$

This function grows so slowly that its value is at most 5 for all "feasible" arguments. Indeed,

$$\lg^* 2 = 1, \qquad \lg^* 4 = 2, \qquad \lg^* 16 = 3;$$
$$\lg^* 65536 = 4, \qquad \lg^*(2^{65536}) = 5.$$

(Observe that 2^{65536} is much greater than most of the current estimates for the number of atoms in the universe.) However, also the functions $\lg\lg n$ and $\lg\lg\lg n$ grow very slowly. Indeed,

$$\lg\lg 2^{65536} = 16, \quad \lg\lg\lg 2^{65536} = 4.$$

This shows that doubly logarithmic and iterated logarithmic functions, which occur often in the analysis of algorithms, actually contribute to the overall complexity, as an additive or a multiplicative factor, only by a small amount.

The inverse to the natural logarithm function is e^x—$exp(x)$ is another notation sometimes more convinient.

1.3.3 Binomial Functions—Coefficients

The binomial function, or the binomial coefficient—what is a more common terminology—$\binom{r}{k}$ is defined for an arbitrary real r and an integer k as follows:

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots 1}, & k > 0; \\ 1, & k = 0; \\ 0, & k < 0. \end{cases} \tag{1.15}$$

Observe that if $r > k > 0$ are integers, then $\binom{r}{k}$ is the number of possibilities to choose k elements from r elements.

There are various reasons why binomial coefficients occur often in discrete computations. One of them is the identity:

Binomial theorem: $(x+y)^n = \sum_{0 \leq k \leq n} \binom{n}{k} x^k y^{n-k}.$

Another reason, of immediate importance for us, is that binomial coefficients occur often as coefficients of generating functions—a useful tool, as we shall see in the next section, to solve recurrences.

Some important identities for binomial coefficients are listed below. They can be easily proved using the basic definition. The acquaintance with them may help to find simple solutions for quite complicated expressions and sums.

$$\begin{array}{rclcrcll}
\binom{n}{k} & = & \frac{n!}{k!(n-k)!}, & n \geq k \geq 0 & \binom{n}{k} & = & \binom{n}{n-k}, & n \geq 0, \\
\binom{r}{k} & = & \frac{r}{k}\binom{r-1}{k-1}, & k \neq 0 & \binom{r}{k} & = & \frac{r}{r-k}\binom{r-1}{k}, & r \neq k, \\
\binom{r}{k} & = & \binom{r-1}{k} + \binom{r-1}{k-1}, & (*) & \binom{r}{m}\binom{m}{k} & = & \binom{r}{k}\binom{r-k}{m-k} & (*)
\end{array}$$

Exercise 1.3.4 *Prove the identities listed above using the basic definition of* $\binom{n}{k}$.

Example 1.3.5 *If* $n \geq r \geq 0$ *are integers, then we can use the first identity in* $(*)$ *to compute the following sum:*

$$\sum_{0 \leq k \leq n} \binom{r+k}{k} = \binom{r}{0} + \binom{r+1}{1} + \binom{r+2}{2} + \cdots + \binom{r+n}{n} = \binom{r+n+1}{n} \tag{1.16}$$

Indeed, it holds

$$\begin{array}{llllllllll}
\binom{r}{0} = & \binom{r+1}{0} \\
& + & \binom{r+1}{1} \underbrace{=}_{by(*)} & \binom{r+2}{1} \\
& & & + & \binom{r+2}{2} \underbrace{=}_{by(*)} & \binom{r+3}{2} \\
& & & & & + & \binom{r+3}{3} \underbrace{=}_{by(*)} & \binom{r+4}{3} \\
& & & & & & & + & \cdots \underbrace{=}_{by(*)} & \binom{r+n+1}{n}
\end{array}$$

and using this idea we can easily proove (1.16) by induction.

Example 1.3.6 *Let* $n \geq m \geq 0$ *be integers. Compute*

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}}.$$

Using the second identity in $(*)$ *we get:*

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = \sum_{k=0}^{m} \frac{\binom{n-k}{m-k}}{\binom{n}{m}} = \frac{1}{\binom{n}{m}} \sum_{k=0}^{m} \binom{n-k}{m-k}.$$

To solve the problem, we now need to compute the last sum. If we replace in that sum k *by* $m-k$*, and then use the result of the previous example, we get*

$$
\begin{aligned}
\sum_{k=0}^{m} \binom{n-k}{m-k} &= \sum_{k=0}^{m} \binom{n-(m-k)}{m-(m-k)} && (1.17)\\
&= \sum_{k=0}^{m} \binom{n-m+k}{k} = \binom{n+1}{m}. && (1.18)
\end{aligned}
$$

The overall result is therefore $\frac{n+1}{n-m+1}$.

Exercise 1.3.7 * *Show the following identities for all natural numbers* a, b *and* $n > 1$*:*

(a) $\binom{a+b}{n} = \sum_{k=0}^{\min(n,a)} \binom{a}{k}\binom{b}{n-k}$; *(b)* $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$;

(c) $\binom{3n}{n} = \sum_{k=0}^{n} \sum_{i=0}^{n-k} \binom{n}{k}\binom{n}{i}\binom{n}{i+k}$.

Exercise 1.3.8 * *Show:* $\sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$.

1.4 Solution of Recurrences—Generating Function Method

The concept of **generating functions** is fundamental and represents an important methodology with numerous applications. In this chapter we describe and illustrate two of them. The first one is to solve recurrences.

The essence of the power of generating functions, as a methodology, is that they allow to reduce complex manipulations with infinite objects (sequences) onto easy operations with finite objects; for example, with rational functions. This often allows to solve quite complicated problems in a surprisingly simple way.

1.4.1 Generating Functions

With any infinite sequence $\langle a_0, a_1, a_2, \ldots \rangle$ of numbers we associate the following **generating function—formal power series:**

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k \geq 0} a_k z^k = \sum a_k z^k, \tag{1.19}$$

where the use of the symbol 'z' for a variable indicates that we may deal with complex numbers; (even in cases we are interested only in natural numbers). The word **formal** emphasizes that the role of powers z^n in (1.19) is mostly that of **position holders** for elements a_n. (The question of convergences will not be of importance for us and will be discussed later.) Concerning coefficients it is sometimes convenient to assume that $a_k = 0$ for $k < 0$ and to use the notation $\sum a_k z^k$, see (1.19), with k running through all integers. Observe also that some a_k in (1.19) may be equal to zero. Therefore also finite sequences can be represented by generating functions. For coefficient of z^n in a generating function $A(z)$ we use the notation:

$$[z^n]A(z) = a_n \tag{1.20}$$

The main reason why generating functions are so important is that they—as functions of complex variables—may have simple, closed-form expressions that represent the whole (infinite or finite), sequence, as the following example illustrates.

Example 1.4.1 (1) $(1+z)^r$ *is the* generating function *for the sequence* $\langle\binom{r}{0},\binom{r}{1},\binom{r}{2},\binom{r}{3},\ldots\rangle$. *Indeed, by the binomial theorem and the fact that* $\binom{r}{k} = 0$ *for* $k > r$, *we get*

$$(1+z)^r = \sum_{k\geq 0}\binom{r}{k}z^k.$$

(2) $\frac{1}{1-z}$ *is the generating function for the power series* $\sum_{n\geq 0} z^n$, *because*

$$(1-z)(\sum_{n\geq 0} z^n) = 1. \tag{1.21}$$

Such basic operations as addition, subtraction, multiplication, inverse of a function (if $a_0 \neq 0$), special division by z^n, derivation, and integration can be performed on generating functions "component-wise" . Table 1.1, in which

$$F(z) = \sum_n f_n z^n \qquad \text{and} \qquad G(z) = \sum_n g_n z^n,$$

summarizes some basic operations on generating functions and on the corresponding formal power series. These identities can be derived in a straightforward way. Some examples:

Linear combination:

$$\alpha F(z) + \beta G(z) = \alpha\sum_n f_n z^n + \beta\sum_n g_n z^n = \sum_n (\alpha f_n + \beta g_n)\, z^n.$$

Multiplication with z^m $(m \geq 0)$:

$$z^m G(z) = \sum_n g_n z^{n+m} = \sum_n g_{n-m} z^n.$$

Derivation

$$G'(z) = g_1 + 2g_2 z + 3g_3 z^2 + \cdots = \sum_n (n+1)g_{n+1}z^n$$

and therefore $G'(z)$ is the generating function for the sequence $\langle(n+1)g_{n+1}\rangle = \langle g_1, 2g_2, \ldots\rangle$ and $zG'(z)$ for the sequence $\langle ng_n\rangle = \langle 0, g_1, 2g_2, \ldots\rangle$.

$$
\begin{aligned}
\alpha F(z)+\beta G(z) &= \textstyle\sum_n \left(\alpha f_n+\beta g_n\right) z^n \\
F(z)G(z) &= \textstyle\sum_n \left(\sum_{k=0}^n f_k g_{n-k}\right) z^n \\
\tfrac{1}{1-z}G(z) &= \textstyle\sum_n \left(\sum_{k=0}^n g_k\right) z^n \quad \{\text{summation rule}\} \\
F^{-1}(z) &= \textstyle\sum_{n\ge 0} b_n z^n \quad \text{where } b_0=f_0^{-1} \\
& \ \ b_n=-f_0^{-1}\textstyle\sum_{k=1}^n f_k b_{n-k} \\
z^m G(z) &= \textstyle\sum_n g_{n-m} z^n \qquad m\ge 0 \\
\tfrac{G(z)-g_0-g_1 z-\cdots-g_{m-1}z^{m-1}}{z^m} &= \textstyle\sum_{n\ge 0} g_{n+m} z^n \\
G'(z) &= \textstyle\sum_n (n+1) g_{n+1} z^n \\
zG'(z) &= \textstyle\sum_n n g_n z^n \\
\textstyle\int_0^z G(t)dt &= \textstyle\sum_{n\ge 1} \tfrac{1}{n} g_{n-1} z^n
\end{aligned}
$$

Table 1.1: Operations on generating functions and the corresponding sequences

For **multiplication** we get:

$$
\begin{aligned}
F(z)G(z) &= (f_0+f_1z+f_2z^2+\cdots)(g_0+g_1z+g_2z^2+\cdots) \\
&= f_0g_0+(f_0g_1+f_1g_0)z+(f_0g_2+f_1g_1+f_2g_0)z^2+\cdots
\end{aligned}
$$

and therefore

$$[z^n]F(z)G(z)=(f_0g_n+f_1g_{n-1}+\cdots+f_ng_0)=\sum_{k=0}^n f_k g_{n-k}.$$

The product $F(z)G(z)$ where $[z^n]A(n)B(n)=\sum_{k=0}^n f_k g_{n-k}$ is called the **discrete convolution** of $F(z)$ and $G(z)$.

Summation rule from Table 1.1is a special case of multiplication with $F(z)=\frac{1}{1-z}$.

Remark 1.4.2 In case of generating functions we often do not care about convergence of the corresponding power series. It is therefore natural to ask whether our manipulations with infinite sums, as for example in (1.21), are correct. There are two reasons not to worry. First of all, one can show formally that all operations mentioned above are correct. The second reason is quite different. It is often not very important to care whether all operation we perform on generating functions are indeed correct. Why? Once we get, using these operations, some results, we can use other methods, for example an induction, to show their correctness.

Let us illustrate this approach on an example:

Example 1.4.3 *The functions* $(1+z)^r$ *and* $(1+z)^s$ *are generating function for sequences* $\langle \binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \ldots \rangle$ *and* $\langle \binom{s}{0}, \binom{s}{1}, \binom{s}{2}, \ldots \rangle$. *Because* $(1+z)^r(1+z)^s = (1+z)^{r+s}$ *we have*

$$[z^n](1+z)^r(1+z)^s = \sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n} = [z^n](1+z)^{r+s}.$$

In a similar way we can show, using the identity $(1-z)^r(1+z)^r = \left(1-z^2\right)^r$, that

$$\sum_{k=0}^{n} \binom{r}{k}\binom{r}{n-k}(-1)^k = (-1)^{\frac{n}{2}}\binom{r}{\frac{n}{2}}, \quad \text{if } n \text{ is even.}$$

In this way we have easily obtained two far from obvious identities for sums of binomial coefficients. Their correctness can now be verified by induction.

Generating functions for some important sequences are listed in the Table 1.2. Some of the results in the table follow in a straightforward way from the rules in the Table 1.1. The generating function for the sequence $\langle 1, 2, 3, \ldots \rangle$ can be obtained using the summation rule with $G(z) = \frac{1}{1-z}$. Since the sequence $\langle 1, 2, 3, \ldots \rangle$ has also the form $\langle 1, 2, \binom{3}{2}, \binom{4}{3}, \binom{5}{4}, \ldots \rangle$ we get, using the summation rule and the identity from Example 1.3.5, that $\frac{1}{(1-z)^3}$ is the generating function for the sequence $\langle 1, 3, \binom{4}{2}, \binom{5}{3}, \binom{6}{4}, \ldots \rangle$. By induction we can then show, using again the summation rule and the identity from Example 1.3.5, that, for an arbitrary integer c, $\frac{1}{(1-z)^c}$ is the generating function for the sequence $\langle 1, c, \binom{c+1}{2}, \binom{c+2}{3}, \ldots \rangle$. Generating functions for the last three sequences in Table 1.1 are well known from calculus.

Exercise 1.4.4 *Find a closed form of the generating function for the sequences:*

$$(a)\ a_n = 3^n + 5^n + n, n \geq 1;\ \ (b)\ \langle 0, 2, 0, \frac{2}{3}, 0, \frac{2}{5}, 0, \frac{2}{7}, \ldots \rangle.$$

Exercise 1.4.5 * *Find a generating function* $F(z)$ *such that* $[z^n]F(z) = \sum_{i=0} \binom{n}{i}\binom{n}{n-2i}$, *for* $n \geq 1$.

**

Exercise 1.4.6 *Use generating functions to show that* $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$.

sequence	generating function	closed form
$\langle 1, 1, 1, \ldots \rangle$	$\sum_{n\geq 0} z^n$	$\frac{1}{1-z}$
$\langle 1, -1, 1, -1, \ldots \rangle$	$\sum_{n\geq 0} (-1)^n z^n$	$\frac{1}{1+z}$
$\langle \underbrace{1, 0, 0, \ldots,}_{m} \underbrace{1, 0, 0, \ldots,}_{m} 1 \ldots \rangle$	$\sum_{n\geq 0} [m \setminus n] z^n$	$\frac{1}{1-z^m}$
$\langle 1, 2, 3, 4, \ldots \rangle$	$\sum_{n\geq 0} (n+1) z^n$	$\frac{1}{(1-z)^2}$
$\langle 1, c, \binom{c}{2}, \binom{c}{3}, \ldots \rangle$	$\sum_{n\geq 0} \binom{c}{n} z^n$	$(1+z)^c$
$\langle 1, c, \binom{c+1}{2}, \binom{c+2}{3}, \ldots \rangle$	$\sum_{n\geq 0} \binom{c+n-1}{n} z^n$	$\frac{1}{(1-z)^c}$
$\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$	$\sum_{n\geq 1} \frac{1}{n} z^n$	$\ln \frac{1}{1-z}$
$\langle 0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots \rangle$	$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} z^n$	$\ln(1+z)$
$\langle 1, 1, \frac{1}{2!}, \frac{1}{3!}, \ldots \rangle$	$\sum_{n>0} \frac{1}{n!} z^n$	e^z

Table 1.2: Generating functions for some sequences and their closed forms

1.4.2 Solution of Recurrences

The following general method can be often useful to find a closed form for elements of a sequence $\langle g_n \rangle$ the elements of which are defined through a recurrence:

Step 1 Form a single equation in which g_n is expressed through other elements of the sequence. It is important that this equation holds for any n; also for those n for which g_n is defined through the initial values, and also for $n < 0$ (assuming $g_n = 0$).

Step 2 Multiply both sides of the resulting equation by z^n and sum over all n. This gives on the left-hand side $G(z) = \sum g_n z^n$—the generating function for $\langle g_n \rangle$. Arrange the right-hand side in such a way that an expression in terms of $G(z)$ is obtained.

Step 3 Solve the equation to get a closed form for $G(z)$.

Step 4 Expand $G(z)$ into a power series. The coefficient at z^n is a closed form for g_n.

Examples

In the following three examples we show how to perform the first three steps of the above method. Later we present a method how to perform Step 4—usually the most difficult one. This method will then be applied to finish the examples.

In the examples, and also in the rest of the book , we use the following mapping of the truth values of predicates $P(n)$ onto integers:

$$[P(n)] = \begin{cases} 1, & \text{if } P(n) \text{ is true;} \\ 0, & \text{if } P(n) \text{ is false.} \end{cases}$$

Example 1.4.7 *Let us apply the above method on the recurrences (1.5) and (1.6) for Fibonacci numbers with the initial conditions* $f_0 = 0, f_1 = 1$ *and the inductive equation*

$$f_n = f_{n-1} + f_{n-2}, \quad n > 1.$$

Step 1: The single equation capturing both the inductive step and the initial conditions has the form

$$f_n = f_{n-1} + f_{n-2} + [n = 1].$$

(Observe, and this is important, that the equation is valid also for $n \le 1$, because $f_n = 0$ for $n < 0$.)

Step 2: A multiplication by z^n and a summation produce:

$$\begin{aligned} F(z) = \sum_n f_n z^n &= \sum_n \left(f_{n-1} + f_{n-2} + [n = 1]\right) z^n \\ &= \sum_n f_{n-1} z^n + \sum_n f_{n-2} z^n + \sum_n [n = 1] z^n \\ &= zF(z) + z^2 F(z) + z. \end{aligned}$$

Step 3 From the previous equation we get

$$F(z) = \frac{z}{1 - z - z^2}.$$

Example 1.4.8 *Solve the recurrence:*

$$g_n = \begin{cases} 1, & \text{if } n = 0; \\ 2, & \text{if } n = 1; \\ 2g_{n-1} + 3g_{n-2} + (-1)^n, & \text{if } n > 1. \end{cases}$$

Step 1: A single equation for g_n has the form:

$$g_n = 2g_{n-1} + 3g_{n-2} + (-1)^n [n \ge 0] + [n = 1].$$

Step 2: Multiplication by z^n and summation give:

$$\begin{aligned} G(z) = \sum_n g_n z^n &= \sum_n \left(2g_{n-1} + 3g_{n-2} + (-1)^n [n \ge 0] + [n = 1]\right) z^n \\ &= \sum_n 2g_{n-1} z^n + 3 \sum_n g_{n-2} z^n + \sum_{n \ge 0} (-1)^n z^n + \sum_{n=1}^{1} z^n \\ &= 2zG(z) + 3z^2 G(z) + \frac{1}{1+z} + z. \end{aligned}$$

Step 3 By solving the last equation for $G(z)$ we get

$$G(z) = \frac{z^2+z+1}{(1+z)^2(1-3z)}.$$

As it is illustrated by the following example, the generating function method can be used also to solve recurrences with two unknown functions. It also shows that such recurrences can arise in a natural way even in the case when the task is to determine only one unknown function.

Example 1.4.9 (Domino-Problem) *Determine the number* u_n *of ways one can cover a* $3 \times n$ *rectangle with the identical dominoes of size* 1×2.

Clearly $u_n = 0$ for $n = 1, 3$ and $u_2 = 3$. To deal with the general case let us introduce a new variable v_n to denote the number of ways we can cover a $3 \times n$ with-a-corner-rectangle (see Figure 1.5b) with such dominoes. For the case $n = 0$ we have exactly one possibility to cover a $3 \times n$ rectangle—to use no domino. We get therefore the recurrencies:

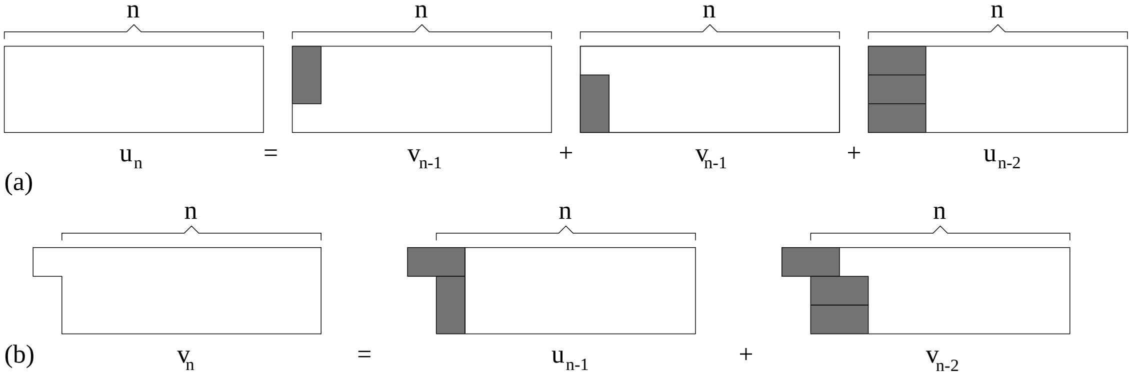

Figure 1.5: Recurrences for tiling by dominoes-60%

$$\begin{aligned} &u_0 = 1, \quad u_1 = 0; && v_0 = 0, \quad v_1 = 1; \\ &u_n = 2v_{n-1} + u_{n-2}; && v_n = u_{n-1} + v_{n-2}, \qquad n \geq 2. \end{aligned}$$

Let us now perform Steps 1 to 3 of the above method:

Step 1: $u_n = 2v_{n-1} + u_{n-2} + [n = 0], \qquad v_n = u_{n-1} + v_{n-2}.$

Step 2: $U(z) = 2zV(z) + z^2U(z) + 1, \qquad V(z) = zU(z) + z^2V(z).$

Step 3: The solution of this system of two equations in two unknown functions has the form:

$$U(z) = \frac{1-z^2}{1-4z^2+z^4}, \qquad V(z) = \frac{z}{1-4z^2+z^4}.$$

A general method to perform step 4

In the last three examples, the task in Step 4 was to determine the coefficients $[z^n]R(z)$ of a rational function $R(z) = P(z)/Q(z)$. This can be done using the following general method.

If the degree of the polynomial $P(z)$ is greater than or equal to the degree of $Q(z)$, then by dividing $P(z)$ with $Q(z)$ we can express $R(z)$ in the form $T(z) + S(z)$, where $T(z)$ is a polynomial and $S(z) = P_1(z)/Q(z)$ is a rational function with the degree of $P_1(z)$ smaller than that of $Q(z)$. Since $[z^n]R(z) = [z^n]T(z) + [z^n]S(z)$, the task has been reduced to that of finding $[z^n]S(z)$.

From the sixth row of the Table 1.2 we find that

$$\frac{a}{(1-\rho z)^{m+1}} = \sum_{n\geq 0} \binom{m+n}{n} a\rho^n z^n$$

and therefore we are easily done with finding the coefficient $[z^n]S(z)$ in case that $S(z)$ has the form

$$\sum_{i=1}^{m} \frac{a_i}{(1-\rho_i z)^{m_i-1}} \tag{1.22}$$

for some constants a_i, ρ_i and $m_i, 1 \leq i \leq m$. This implies that in order to dvelope a methodology to solve Step 4 of the above method, it issufficient to show that $S(z)$ can always be transformed either to the above form or to a similar one.

In order to transform $Q(z) = q_0 + q_1 z + \cdots q_m z^m$ into the form $Q(z) = q_0(1-\rho_1 z)(1-\rho_2 z)\ldots(1-\rho_m z)$ we need to determine the roots $\frac{1}{\rho_1}, \ldots, \frac{1}{\rho_m}$ of $Q(z)$. Once these roots are found, one can use one of the following theorems to perform Step 4:

Theorem 1.4.10 *If* $S(z) = \frac{P_1(z)}{Q(z)}$, *where* $Q(z) = q_0(1-\rho_1 z)\ldots(1-\rho_m z)$, *the numbers* $\rho_1, \ldots, \rho_m$ *are distinct, and the degree of* $P_1(z)$ *is smaller than that of* $Q(z)$, *then*

$$[z^n]S(z) = a_1{\rho_1}^n + \cdots + a_l{\rho_m}^n,$$

where

$$a_k = \frac{-\rho_k P_1(\frac{1}{\rho_k})}{Q'(\frac{1}{\rho_k})}.$$

Proof. It is a well known fact from calculus that in case all ρ_i are different there exists a decomposition

$$S(z) = \frac{a_1}{(1-\rho_1 z)} + \cdots + \frac{a_l}{(1-\rho_l z)}$$

where $a_1, \ldots, a_l$ are constants and thus

$$[z^n]S(z) = a_1\rho_1^n + \cdots + a_l\rho_l^n.$$

Therefore, for $i = 1, \ldots, l$,

$$a_i = \lim_{z\to 1/\rho_i} (1-\rho_i z)R(z)$$

and using l'Hospital's rule we obtain:

$$a_i = \frac{-\rho_i P_1(\frac{1}{\rho_i})}{Q'(\frac{1}{\rho_i})},$$

where Q' is the derivative of the polynomial Q. ▯

The second theorem deals with the case of multiple roots of the denominator. For the proof that is more technical see bibliographical references.

Theorem 1.4.11 *Let* $R(z) = \frac{P(z)}{Q(z)}$, *where* $Q(z) = q_0(1-\rho_1 z)^{d_1} \dots (1-\rho_l z)^{d_l}$, ρ_i *are distinct, and the degree of* $P(z)$ *is smaller than that of* $Q(z)$; *then*

$$[z^n]R(z) = f_1(n)\rho_1^n + \cdots f_l(n)\rho_l^n, \quad n \geq 0,$$

where each $f_i(n)$ *is a polynomial of the degree* $d_i - 1$, *the main coefficient of which is*

$$a_i = \frac{(-\rho_i)^{d_i} P(\frac{1}{\rho_i}) d_i}{Q^{(d_i)}(\frac{1}{\rho_i})},$$

where $Q^{(d_i)}$ *is the i-th derivative of* Q.

To apply Theorems 1.4.10 and 1.4.11 to a rational function $\frac{P(z)}{Q(z)}$ with $Q(z) = q_0 + q_1 z + \cdots + q_m z^m$ we have to express $Q(z)$ in the form $Q(z) = q_0(1-\rho_1 z)^{d_1} \dots (1-\rho_m z)^{d_m}$. The numbers $\frac{1}{\rho_i}$ are clearly roots of $Q(z)$. Applying the transformation $y = \frac{1}{z}$ and then replacing y by z we get that ρ_i are roots of the "reflected" polynomial

$$Q^R(z) = q_m + q_{m-1}z + \cdots + q_0 z^m$$

and this polynomial is sometimes more easy to handle.

Examples—continuation

Let us now apply Theorems 1.4.10 and 1.4.11 to finish Examples 1.4.7, 1.4.8, and 1.4.9. In Example 1.4.7 it remains to determine

$$[z^n]\frac{z}{1-z-z^2}.$$

The reflected polynomial $z^2 - z - 1$ has two roots

$$\phi_1 = \phi = \frac{1+\sqrt{5}}{2}, \ \phi_2 = \bar{\phi} = \frac{1-\sqrt{5}}{2}.$$

Theorem 1.4.10 therefore yields

$$F_n = [z^n]\frac{z}{1-z-z^2} = a_1\phi^n + a_2\bar{\phi}^n,$$

where $a_1 = \frac{1}{\sqrt{5}}, a_2 = -\frac{1}{\sqrt{5}}$.

To finish Example 1.4.8 we have to determine

$$g_n = [z^n]\frac{1+z+z^2}{(1-3z)(1+z)^2}.$$

The denominator already has the required form. Since one root has multiplicity 2, we need to use Theorem 1.4.11. Calculations yield

$$g_n = (\frac{1}{4}n + c)(-1)^n + \frac{13}{16}3^n.$$

The constant $c = \frac{3}{16}$ can be determined using the equation $1 = g_0 = c + \frac{13}{16}$. Finally, in Example 1.4.9, the task remained to determine:

$$[z^n]U(z) = [z^n]\frac{1-z^2}{1-4z^2+z^4} \qquad \text{and} \qquad [z^n]V(z) = [z^n]\frac{z}{1-4z^2+z^4}. \tag{1.23}$$

In order to apply our method directly, we would need to find roots of a polynomial of degree 4. This, but also the whole task, can be simplified by realizing that all powers in (1.23) are even. Indeed, if we define

$$W(z) = \frac{1}{1-4z+z^2}$$

then

$$U(z) = (1-z^2)W(z^2) \text{ and } V(z) = zW(z^2).$$

Therefore

$$\begin{array}{rclcl@{\qquad}rcl} U_{2n+1} & = & [z^{2n+1}]U(z) & = & 0, & U_{2n} & = & W_n - W_{n-1}; \\ V_{2n} & = & [z^{2n}]V(z) & = & 0, & V_{2n+1} & = & W_n; \end{array}$$

where

$$W_n = [z^n]\frac{1}{1-4z-z^2}$$

what is more easy to determine.

Exercise 1.4.12 *Use the generating function method to solve recurrences from Exercises 1.2.17 and 1.2.18.*

Exercise 1.4.13 *Use the generating function method to solve recurrences:*

(a) $u_0 = 0$, $u_1 = 1$, $u_n = u - n - 1 + u_{n-2} + (-1)^n$, $n \geq 2$*;*

(b) $g_n = 0$, *if* $n < 0$, $g_0 = 1$ *and* $g_n = g_{n-1} + 2g_{n-2} + \ldots + ng_0$ *for* $n > 0$.

Exercise 1.4.14 *Use the generating function method to solve the system of recurrences:* $a_0 = 1$, $b_0 = 0$*;* $a_n = 5a_{n-1} + 12b_{n-1}$, $b_n = 2a_{n-1} + 5b_{n-1}$, $n \geq 1$.

Remark 1.4.15 In all previous methods for solving recurrences it has been assumed that all components—constants and functions—are fully determined. This is, however, not always the case in practice. Mostly only some estimations of them are available. In Section 1.6.2 we show how to deal also with such cases. Before doing that we switch to a detailed treatment of asymptotic estimations.

1.5 Asymptotics

Asymptotic estimations allow to produce often surprisingly simple, deep, powerful, useful and technology independent analysis of the performance or of the size of computing systems. Their existence has been an important factor contributing much to the fast development of a deep and practically relevant theory of computing.

In the asymptotical analysis of a function $T(n)$ (from integers to reals) or $A(x)$ (from reals to reals), the task is to determine an "**estimation in limit**" of $T(n)$ for $n \rightarrow \infty$ or of $A(x)$, for $x \rightarrow a$, where a is a real. The aim is to determine an estimation as good as possible, or at least good lower and upper bounds for it.

The key underlying problem is how to compare 'in a limit' the growth of two functions. The main approaches to this problem and relations between them will now be discussed. Especially important rôle play here the $\mathcal{O}$-, $\Omega-$ and $\Theta-$ notation s and we shall discuss in detail ways how to handle them.

Because of special importance and peculiarities of asymptotic estimations a discussion of their merits seems to be appropriate.

There is a quite widespread illusion that in science and technology the exact solutions, analyses, and so on, are what is really needed and to be aimed for. Estimations are seen often as substitutes, when exactness is not available or achievable. However, this does not apply to the analysis of computing systems. Simple and good estimations are what is really needed. There are several reasons for that.

Feasibility. Exact analyses are often impossible, even for apparently simple systems. There are often too many factors of enormous complexity involved. For example, for a really detailed time analysis of even a simple program one would mostly need to study complicated compilers, operating systems, computers, and in case of multiuser systems also the patterns of their interactions.

Usefulness. An exact analysis could easily be many pages long and therefore incomprehensible. Moreover, as the results of asymptotic analysis indicate, most of that would be of negligible importance. In addition, what we really need, are such results of an analysis of computing systems that are independent of a particular computer and, in general, of the underlying hardware and software technology. What we need are estimations that are some kinds of **invariants** of computing technologies. Various constant factors, that reflect these technologies are therefore not of prime interest. Finally, what is mostly needed, is not the performance of particular systems for particular data, but mainly the knowledge about the growth of the performance of systems as a function of the growth of the size of their input data. Again, factors with a negligeable growth and constant factors are not of prime importance for asymptotic analysis—even though they may be of large importance for applications.

Example 1.5.1 *How much time is needed to multiply two n-digit integers (by a person or by a computer) when a classical school algorithm is used?*

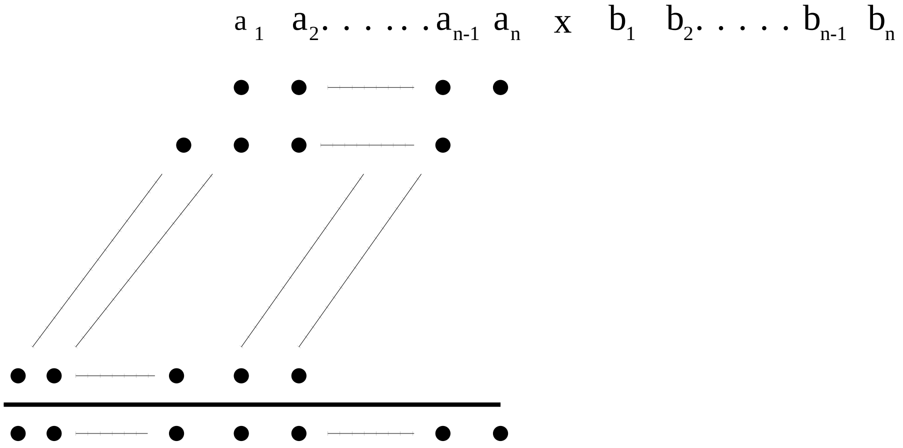

Figure 1.6: Integer multiplication-60%.

The exact analysis may be quite complicated. It depends also on many factors: which of many variants of the algorithm is used (see one in Figure 1.6); who is to execute it; or on a way it is programmed, and on the computer on which it is run. However, all these cases have, however, one thing in common. k^2-times more time is needed to multiply k-times larger integers. We can therefore say, simply and in a full generality, that the time to multiply two integers by a school algorithm is $\Theta(n^2)$. Observe also that this result holds no matter which kind of positional number system is used to represent integers: binary, ternary, decimal, and so on.

It is also important to realize that simple and well understood estimations are of large importance even in cases the exact solutions are available. Some examples:

Example 1.5.2 *At the analysis of algorithms one encounters often the so-called* **harmonic numbers:**

$$H_n \;=\; 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \;=\; \sum_{i=1}^{n}\frac{1}{i}. \tag{1.24}$$

Using definition (1.24) we can exactly determine H_n for any given integer n. This is, however, not always enough. Mostly we need to know how big H_n is in general, as a function of n, and not for a particular n. Unfortunately, no closed form for H_n is known. Therefore, good approximations for H_n are much needed. For example:

$$\ln n < H_n < \ln n + 1, \qquad \text{for } n > 1.$$

This is often enough, although sometimes a better approximation is required. For example,

$$H_n = \ln n + 0.5772156649 + \frac{1}{2n} - \frac{1}{12n^2} + \Theta(n^{-4}).$$

Example 1.5.3 *The factorial $n! = 1 \cdot 2 \cdot \ \ldots \ \cdot n$ is another function of importance for the analysis of algorithmsThe fact that we can determine $n!$ exactly is not always good enough for complexity analysis. The following approximation, due to James Stirling (1692-1770),*

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

may be much more useful. For example, this approximation yields:

$$\lg n! = \Theta(n \log n).$$

1.5.1 An Asymptotic Hierarchy

An important formalization of the intuitive idea that one function grows essentially faster than another function is captured by the relation $\prec$ defined by

$$f(n) \prec g(n) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

Basic properties of this relation can be summarized as follows:

$$\begin{array}{rcll}
f(n) \prec g(n), g(n) \prec h(n) & \Rightarrow & f(n) \prec h(n), & \text{(transitivity)}; \qquad (1.25) \\
f(n) \prec g(n) & \Leftrightarrow & \dfrac{1}{g(n)} \prec \dfrac{1}{f(n)}, & \text{(for } f, g \text{ never equal zero)}; \qquad (1.26) \\
1 \prec f(n) \prec g(n) & \Rightarrow & c^{|f(n)|} \prec c^{|g(n)|}, & \text{(if } c > 1 \text{ is a constant)}; \qquad (1.27) \\
n^{\alpha} \prec n^{\beta} & \Leftrightarrow & \alpha < \beta, & \text{if } (\alpha, \beta \text{ are positive reals}). \qquad (1.28)
\end{array}$$

These properties, as well as the following hierarchy (with $0 < \varepsilon < 1 < c$), can be derived from the basic definition of the relation $\prec$ using elementary methods of calculus.

$$1 \prec \lg^{\star} n \prec \lg \lg n \prec \lg n \prec n^{\varepsilon} \prec n^{c} \prec n^{\lg n} \prec c^{n} \prec n^{n} \prec c^{c^{n}}. \qquad (1.29)$$

These results remain valid when lg is replaced by ln or log.

Example 1.5.4 *Where does in this hierarchy the function $2^{\sqrt[m]{\lg n}}$, $m > 1$, lie?*

Clearly, $\lg \lg n \prec \sqrt[m]{\lg n} \prec \varepsilon \lg n$, for any $\varepsilon > 0$, and therefore, by (1.27), $2^{\lg \lg n} \prec 2^{\sqrt[m]{\lg n}} \prec 2^{\varepsilon \lg n}$. Since $2^{\lg \lg n} = \lg n$ and $2^{\varepsilon \lg n} = n^{\varepsilon}$, we get

$$\lg n \prec 2^{\sqrt[m]{\log n}} \prec n^{\varepsilon}.$$

Exercise 1.5.5 *Which of the following functions grow faster:*
(a) $n^{(\ln n)}$ *or* $(\ln n)^n$*; (b)* $(\ln n)!$ *or* $n^{\ln n \ln \ln n}$ *?*

In a similar way, we can formalize the intuitive idea that two functions $f(n)$ and $g(n)$ have the same rate of growth as follows:

$$f(n) \sim g(n) \Leftrightarrow \lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = 1. \tag{1.30}$$

It would be nice could we say that for any two functions $f(n)$ and $g(n)$ one of the relations $f(n) \prec g(n)$, $f(n) \sim g(n)$, or $g(n) \prec f(n)$ holds. This is, however, not the case. For example, for the pairs $f(n) = n$, $g(n) = 2n$ and $f(n) = 1$, $g(n) = \sin n$ none of these relations holds. The relation $\sim$ is simply too strong.

There is another formalization of the intuitive idea that two functions have the same rate of growth:

$$\begin{aligned} f(n) \asymp g(n) \quad \Leftrightarrow \quad & |f(n)| \leq c|g(n)| \quad \text{and} \quad |g(n)| \leq c|f(n)| \\ & \text{for a constant } c \text{ and every large enough } n, \end{aligned} \tag{1.31}$$

that has a nice and important property that for any two the so-called **logarithmico-exponential functions** $f(n)$ and $g(n)$ exactly one of the relations $f(n) \prec g(n)$,$f(n) \asymp g(n)$ or $g(n) \prec f(n)$ holds.

The family of logarithmico-exponential functions contains practically all functions one encounters in the asymptotic analysis. It has been defined by G. H. Hardy (1910), as the smallest family $\mathcal{L}$ of functions satisfying the following properties.

1. Functions $f(n) = n$ and $f(n) = c$ (a constant) are in $\mathcal{L}$.
2. If $f(n), g(n)$ are in $\mathcal{L}$, then so are functions: $f(n) + g(n)$, $f(n) - g(n)$, $f(n) \cdot g(n)$, $e^{f(n)}$, $g(n)/f(n)$ and $\log f(n)$—if $f(n) > 0$ for almost all n.

Asymptotic relations such as $\prec$ and the ones defined later, that capture 'in limit' our intuition about the growths of functions require to stretch our imagination and to **THINK BIG**. For example, if $f(n) \prec g(n)$, then this does mean that eventually, for all $n \geq n_0$ for some n_0, $f(n) < g(n)$. However, such an n_0 can be very large! For example,

$$\log n \prec n^{0.0001}$$

but the smallest n_0 such that $\log n < n^{0.0001}$ for all $n \geq n_0$ is between 10^{10^3} and 10^{10^4}.

In general, the asymptotic analysis results are more relevant larger arguments we have. This also implies that practical importance of the asymptotic analysis results grows with the growth of the performance of computers and of the size of problems that need to be solved.

1.5.2 $\mathcal{O}$-, Θ-, and Ω-notation

The main concepts of the asymptotic analysis: 'to grow as fast as', 'to grow not faster than' and 'to grow at least as fast as' seem to be captured in the best way by the following $\mathcal{O}-, \Theta-$, and Ω-notation; with a function $g(n)$ as an argument.

$$\boxed{\begin{aligned}
\Theta(g(n)) &= \{f(n) \mid \exists c_1, c_2 : c_1|g(n)| \le |f(n)| \le c_2|g(n)|\}.\\
\mathcal{O}(g(n)) &= \{f(n) \mid \exists c : |f(n)| \le c|g(n)|\}.\\
\Omega(g(n)) &= \{f(n) \mid \exists c > 0 : c|g(n)| \le |f(n)|\}.
\end{aligned}}$$

$\mathcal{O}$-notation was introduced in 1892 by a German mathematician G. H. Bachmann (1837-1890). It got more widely known through the work of another German mathematician Edmund Landau (1877-1938) and it has been known as Landau notation. However, it was actually D. E. Knuth who introduced Ω- and Θ-notation and much popularized all these notations.

$\Theta(g(n))$, $\mathcal{O}(g(n))$, $\Omega(g(n))$ are sets of functions. However,

$$\begin{array}{rrclrclrcl}
\text{instead of} & f(n) & \in & \Theta(g(n)), & f(n) & \in & \mathcal{O}(g(n)), & f(n) & \in & \Omega(g(n)),\\
\text{we usually write:} & f(n) & = & \Theta(g(n)), & f(n) & = & \mathcal{O}(g(n)), & f(n) & = & \Omega(g(n)).
\end{array}$$

There are two reasons for using this notation with the "equal" sign. The first one is a tradition. The $\mathcal{O}$-notation with the equals sign has been well established in mathematics, especially in number theory. Moreover, we often read "=" as "is". For example, we read $H_n = \Theta(\lg n)$ as "H_n is a big theta of $\lg n$". The second reason is that in asymptotic calculations, as illustrated later, we often need to use this notation in the middle of expressions. Our intuition is then better satisfied if we interpret the equals sign as an equality rather than as an inclusion, as discussed later.

Relations between functions $f(n)$ and $g(n)$ such that $f(n) = \Theta(g(n))$, or $f(n) = \Omega(g(n))$, or $f(n) = \mathcal{O}(g(n))$ are illustrated in Figure 1.7:

Transitivity is one of the basic properties for $\mathcal{O}$-, Θ- and Ω-notation. For example,

$$f(n) = \mathcal{O}(g(n)) \text{ and } g(n) = \mathcal{O}(h(n)) \Rightarrow f(n) = \mathcal{O}(h(n)).$$

The $\mathcal{O}$-, Θ-, and Ω-notation is used also with various restrictions on variables. For example, the notation

$$f(n) = \Theta(g(n)), \qquad \text{for } n \to \infty \tag{1.32}$$

means that there are $c_1, c_2 > 0$, and n_0 such that $c_1|g(n)| \le |f(n)| \le c_2|g(n)| \quad \text{for } n \ge n_0$.

Note, that notation $\Theta(f(n)$, and also $\mathcal{O}$- and Ω-notations are very often used as in (1.32), without writing $n \to \infty$ explicitly.

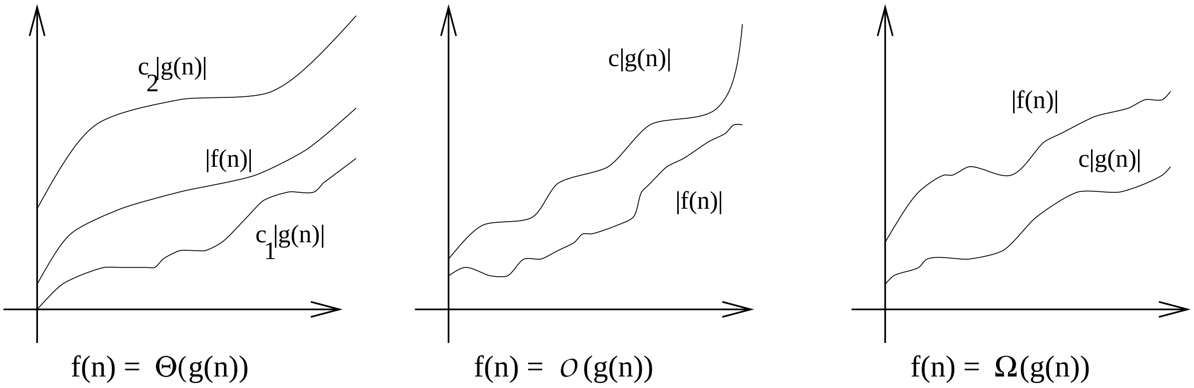

Figure 1.7: Asymptotic relations between functions-75%

Example 1.5.6 *In order to show* $(n+1)^2 = \Theta(n^2)$ *we look for* $c_1, c_2 > 0$ *such that*

$$c_1 n^2 \leq |n^2 + 2n + 1| \leq c_2 n^2, \quad \textit{for } n > 0,$$

or, after dividing by n^2, $c_1 \leq 1 + \frac{2}{n} + \frac{1}{n^2} \leq c_2$. *This inequality is satisfied, for example, with* $c_1 = 1$ *and* $c_2 = 4$.

Example 1.5.7 *To show* $\frac{n^2-1}{n+1} = \Theta(n)$, *for* $n \to \infty$, *we need constants* $c_1, c_2 > 0$ *such that*

$$c_1 n \leq \left| \frac{n^2 - 1}{n+1} \right| \leq c_2 n, \quad \textit{for} \quad n \geq n_0.$$

For $n > 1$, *this is equivalent to finding* c_1, c_2 *such that* $c_1 n \leq |n - 1| \leq c_2 n$, *or, equivalently,*

$$c_1 \leq |1 - \frac{1}{n}| \leq c_2.$$

This inequality is satisfied, for $n > 1$, *with* $c_1 = \frac{1}{2}$ *and* $c_2 = 1$.

Example 1.5.8 *Since*

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k$$

we get

$$\sum_{i=1}^{n} i^k = \mathcal{O}(n^{k+1}).$$

Example 1.5.9 *To prove* $4n^3 \neq \mathcal{O}(n^2)$, *for* $n \to \infty$, *let us assume that there are* c_1 *and* n_0 *such that* $4n^3 \leq c_1 n^2$, *for* $n \geq n_0$. *This would imply* $n \leq c_1/4$, *for all* $n \geq n_0$*—a contradiction.*

The Θ-, $\mathcal{O}$-, and Ω-notation is also used for functions with variables over reals and for convergence to a real. For example, the notation

$$f(x) = \mathcal{O}(g(x)), \qquad \text{for } x \to 0,$$

means that there are constants $c, \varepsilon > 0$ such that

$$|f(x)| \leq c|g(x)|, \qquad \text{for } 0 < |x| \leq \varepsilon.$$

Example 1.5.10 *$x^2 = \mathcal{O}(x)$, for $x \to 0$.*

Exercise 1.5.11 *Show that (a) $\lfloor x \rfloor \lceil x \rceil = \Theta(x^2)$; (b) $(x^2+1)/(x+1) = \Theta(x)$.*

Exercise 1.5.12 *Give an $\mathcal{O}$-estimation, as good as possible, for the following functions:*
(a) $(n! + 3^n)(n^2 + \log(n^3+1))$; (b) $(2^n + n^2)(n^3 + 5^n)$; (c) $n^{2^n} + n^{n^2}$.

The $\mathcal{O}$-notation has its peculiarities which we shall now discuss in more detail. The expressions of the type $f(n) = \mathcal{O}(g(n))$, for example

$$\frac{1}{2}n^3 - 2n^2 + 3n - 4 = \mathcal{O}(n^3),$$

should be seen as a **one-way-equality**,and should never been written with the sides reversed. For example, we should not write $\mathcal{O}(n^3) = \frac{1}{2}n^3 - 2n^2 + 3n - 4$! This could lead to incorrect conclusions. For example, from $n = \mathcal{O}(n^3)$ and $\mathcal{O}(n^3) = n^2$ one could be inclined to conclude that $n = \mathcal{O}(n^3) = n^2$. This does not mean that the $\mathcal{O}$-notation can not be used on the left side of an equation. It has only to be dealt with properly and interpreted consistently as a set of functions. One can also write $\mathcal{O}(g_1(n)) + \mathcal{O}(g_2(n))$ or $\mathcal{O}(g_1(n)) \cdot \mathcal{O}(g_2(n))$, and so on, with the usual interpretation of operations on sets. For example,

$$\mathcal{O}(g_1(n)) + \mathcal{O}(g_2(n)) = \{h(n) \mid h(n) = h_1(n) + h_2(n), h_1(n) = \mathcal{O}(g_1(n)), h_2(n) = \mathcal{O}(g_2(n))\}.$$

We can therefore write

$$2n + \mathcal{O}(n^2) + \mathcal{O}(n^3) = \mathcal{O}(n^3)$$

what actually means that $2n + \mathcal{O}(n^2) + \mathcal{O}(n^3) \subseteq \mathcal{O}(n^3)$.

If the $\mathcal{O}$-notation is used in an environment it actually represents a set of functions over all variables that are 'free' in that environment. Let us illustrate this on the use of $\mathcal{O}$-notation within a sum—what happens often in the analysis of algorithms. We show the identity

$$\sum_{k=0}^{n} \Big(2k^2 + \mathcal{O}(k)\Big) = \frac{1}{3}n^3 + \mathcal{O}(n^2).$$

The expression $2k^2 + \mathcal{O}(k)$ represents, in this context, a set of functions of the form $2k^2 + f(k,n)$, for which there is a constant c such that $f(n,k) \leq ck$, for $0 \leq k \leq n$. Therefore

$$\begin{aligned} \sum_{k=0}^{n} 2k^2 + f(k,n) \leq 2\sum_{k=0}^{n} k^2 + c\sum_{k=0}^{n} k &= 2(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}) + c\frac{n^2}{2} + c\frac{n}{2} && (1.33) \\ &= \frac{2}{3}n^3 + (1 + \frac{c}{2})n^2 + (\frac{1}{3} + \frac{c}{2})n && (1.34) \\ &\leq \frac{2}{3}n^3 + c_1 n^2, \text{ for } c_1 = c + 2. && (1.35) \end{aligned}$$

During a complexity analysis it often happens that estimations depend on more than one parameter. For example, the complexity of a graph algorithm may depend on the number of nodes and also on the number of edges. To deal with such cases, the $\mathcal{O}$-,Θ- and Ω-notation is generalized in a natural way. For example,

$$\mathcal{O}(f(m,n)) = \{g(m,n) \,|\, \exists c, n_0 : \ |f(m,n)| \leq c|g(m,n)|, \text{ for all } n \geq n_0, m \geq n_0\}.$$

Notation $\mathcal{O}$ is sometimes used also to relate functions $f, g : \Gamma \rightarrow \mathbf{N}$, where Γ is an arbitrary infinite set. $f(x) = \mathcal{O}(g(x))$ then means that there exists a constant c such that $f(x) \leq cg(x)$ for almost all $x \in \Gamma$.

Exercise 1.5.13 *Show: (a)* $n^a = \mathcal{O}(n^b)$ *if* $0 \leq a \leq b$*; (b)* $a^n = \mathcal{O}(b^n)$ *if* $1 \leq a \leq b$*; (c)* $n^a = \mathcal{O}(b^n)$*, for any* $a > 0, b > 1$.

Exercise 1.5.14 *Show: (a)* $n!$ *is not* $\mathcal{O}(2^n)$*; (b)* n^n *is not* $\mathcal{O}(n!)$.

Exercise 1.5.15 *Show that* $f(n) = \mathcal{O}(n^k)$ *for some* k *if and only if* $f(n) \leq kn^k$ *for some* $k > 0$.

Remark 1.5.16 *One of the main use of* $\Theta-$*,* Ω*- and* $\mathcal{O}$*-notation is in the computational analysis of algorithms. For example, in case of the running time* $T(n)$ *of an algorithm the notation* $T(n) = \Theta(f(n))$ *means that* $f(n)$ *is an asymptotically tight bound; notation* $T(n) = \Omega(f(n)$ *means that* $f(n)$ *is an asymptotical lower bound and, finally, notation* $T(n) = \mathcal{O}(f(n))$ *demonstrate that* $f(n)$ *is an asymptotical upper bound.*

1.5.3 Relations between Asymptotic Notations

The following relations between the $\mathcal{O}$-, Θ- and Ω-notations follow directly from the basic definition:

$$\begin{aligned} f(n) = \mathcal{O}(g(n)) &\Leftrightarrow g(n) = \Omega(f(n)), \\ f(n) = \Theta(g(n)) &\Leftrightarrow f(n) = \mathcal{O}(g(n)) \text{ and } f(n) = \Omega(g(n)). \end{aligned}$$

In addition,

$$f(n) = \Theta(g(n)) \Leftrightarrow f(n) \asymp g(n).$$

The following 'little oh' notation

$$f(n) = o(g(n)) \Leftrightarrow |f(n)| \leq \varepsilon |g(n)| \text{ for all } \varepsilon > 0,\ n > n_0(\varepsilon)$$

and its 'inverse' notation

$$f(n) = \omega(g(n)) \Leftrightarrow g(n) = o(f(n))$$

are also sometimes used. Between the 'little oh' and $\asymp$ notation, there is the relation:

$$f(n) \asymp g(n) \Leftrightarrow f(n) = g(n) + o(g(n)).$$

Exercise 1.5.17 *Show (a)* $x^3 = o(x^4)$*; (b)* $x \lg x = o(x^2)$*; (c)* $x = o(x^{1+\varepsilon})$ *if* $\varepsilon > 0$*; (d)* $x^2 = o(2^x)$*.*

Given two functions $f(n)$ and $g(n)$, it may not be obvious which of the asymptotic relations hold between them. The following theorem contains a useful sufficient condition to determine that.

Theorem 1.5.18 *If* $f(n), g(n) > 0$*, for all* $n > 0$*, then*

(1) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = a \neq 0 \quad \Rightarrow \quad f(n) = \Theta(g(n));$

(2) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \quad \Rightarrow \quad f(n) = \mathcal{O}(g(n)),\ f(n) = o(g(n))$ *but not* $f(n) = \Theta(g(n));$

(3) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \quad \Rightarrow \quad g(n) = \mathcal{O}(f(n))$ *and not* $g(n) = \Theta f(n).$

Proof. Let $\lim_{n\to\infty} \frac{f(n)}{g(n)} = a \neq 0$. Then there are $\varepsilon \geq 0$ and an integer n_0 such that, for all $n > n_0$,

$$\left| \frac{f(n)}{g(n)} - a \right| \leq \varepsilon.$$

This implies $(a - \varepsilon)g(n) \leq f(n) \leq (a + \varepsilon)g(n)$. Therefore $f(n) = \Theta(g(n))$. Proofs of (2) and (3) are left as exercises.

Example 1.5.19 *For* $a, b > 1$, $\lim_{n\to\infty} \frac{(\log n)^a}{n^b} = 0$ *and therefore* $(\log n)^a = \mathcal{O}(n^b)$*.*

Exercise 1.5.20 *Fill an entry in the following table by a cross whenever the pair of functions in that row is in the relation $A = \psi(B)$, where ψ is the symbol shown by the column head. In the table we use integer constants $k, c \geq 1$ and $\varepsilon > 0$.*

A	B	$\mathcal{O}$	o	Ω	ω	Θ
$\ln^k n$	n^ε					
n^k	c^n					
$n \lg n$	n^{cosn}					
2^n	$3^{\frac{3}{2}}$					
$n^{\log m}$	$m^{\log n}$					
$\lg n!$	$\lg n^n$					

1.5.4 Manipulations with the $\mathcal{O}$-notation

There are several simple rules how to manipulate $\mathcal{O}$-expressions that follow easily from the basic definition:

$$
\begin{aligned}
n^m &= \mathcal{O}(n^{m'}), \quad \text{if } m \leq m'; && (1.36)\\
f(n) &= \mathcal{O}(f(n)); && (1.37)\\
c\mathcal{O}(f(n)) &= \mathcal{O}(cf(n)) \quad = \quad \mathcal{O}(f(n)); && (1.38)\\
\mathcal{O}(\mathcal{O}(f(n))) &= \mathcal{O}(f(n)); && (1.39)\\
\mathcal{O}(f(n)) + \mathcal{O}(g(n)) &= \mathcal{O}(\max\{|f(n)|, |g(n)|\}); && (1.40)\\
\mathcal{O}(f(n))\mathcal{O}(g(n)) &= \mathcal{O}(f(n)g(n)); && (1.41)\\
\mathcal{O}(f(n)g(n)) &= f(n)\mathcal{O}(g(n)). && (1.42)
\end{aligned}
$$

For example, from (1.36), (1.38) and (1.40) we get for a fixed m:

$$\sum_{i=0}^{m} a_i n^{m-i} = \sum_{i=0}^{m} \mathcal{O}(n^m) = \mathcal{O}(mn^m) = \mathcal{O}(n^m).$$

The rule (1.40) is often used in the analysis of algorithms to get the overall $\mathcal{O}$-estimation of time complexity of an algorithm from the estimations of complexity of its parts.

Manipulations with power series aimed to get better and better estimations play also an important rôle in the complexity analysis: For example, if the power series

$$S(z) = \sum_{n \geq 0} a_n z^n$$

converges absolutely for a complex number z_0, then $S(z) = \mathcal{O}(1)$ for all $|z| \leq |z_0|$ because

$$|S(z)| \leq \sum_{n \geq 0} |a_n||z|^n \leq \sum_{n \geq 0} |a_n||z_0|^n = c < \infty,$$

for a constant c. This implies that we can truncate the power series after each term and to estimate the remainder with $\mathcal{O}$ as follows:

$$\begin{array}{rcl} S(z) & = & a_0 + \mathcal{O}(z); \\ S(z) & = & a_0 + a_1 z + \mathcal{O}(z^2); \\ \vdots & & \\ S(z) & = & \sum_{i=0}^{k} a_i z^i + \mathcal{O}(z^{k+1}). \end{array}$$

The following power series are of a special interest in the asymptotic analysis:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \mathcal{O}(z^5); \quad (1.43)$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \mathcal{O}(z^5). \quad (1.44)$$

With the truncation method one can use these power series to show the correctness of the following rules:

$$\ln(1 + \mathcal{O}(f(n))) = \mathcal{O}(f(n)), \quad \text{if } f(n) \prec 1; \quad (1.45)$$

$$e^{\mathcal{O}(f(n))} = 1 + \mathcal{O}(f(n)), \quad \text{if } f(n) = \mathcal{O}(1); \quad (1.46)$$

$$(1 + \mathcal{O}(f(n)))^{\mathcal{O}(g(n))} = 1 + \mathcal{O}(f(n)g(n)), \quad \text{if } f(n) \prec 1, f(n)g(n) = \mathcal{O}(1). \quad (1.47)$$

Proof of (1.45): Since $f(n) \prec 1$, whenever $g(n) \in \mathcal{O}(f(n))$ there are constants C, c such that $|g(n)| \leq c|f(n)| \leq C < 1$, for $n \geq n_0$. Hence

$$\ln(1 + g(n)) = g(n)\left(1 - \frac{1}{2}g(n) + \frac{1}{3}g^2(n) - \cdots\right) \quad (1.48)$$

$$\leq g(n)\left(1 + \frac{1}{2}C + \frac{1}{3}C^2 + \cdots\right) \leq (\frac{1}{1-C})g(n) = \mathcal{O}(f(n)) \quad (1.49)$$

and therefore $\ln(1 + \mathcal{O}(f(n))) = \mathcal{O}(f(n))$. In a similar way we can show (1.46). It also holds

$$\begin{array}{rclcl} (1 + \mathcal{O}(f(n)))^{\mathcal{O}(g(n))} & = & e^{\mathcal{O}(g(n))\ln(1+\mathcal{O}(f(n)))} & = & e^{\mathcal{O}(g(n))\mathcal{O}(f(n))} \\ & = & e^{\mathcal{O}(f(n)g(n))} & = & 1 + \mathcal{O}(f(n)g(n)). \end{array}$$

Exercise 1.5.21 *Determine* $\mathcal{O}(f(n))$ *for: (a)* $f(n) = (1+n^2)^{\frac{1}{n^3}}$*; (b)* $f(n) = e^{\frac{1}{\sqrt{n}}}$.

1.5.5 Asymptotic Notation—Summary

In Table 1.3 we present simpler definitions for the $\mathcal{O}$-,Θ-, and Ω-notation which are often used and are mostly sufficient. We also summarize definitions for the o- and ω-notation. The notation $\overset{\infty}{\underset{n}{\forall}}$ used there means "for almost all n"; $f(n)$ and $g(n)$ are non-negative functions from integers to integers.

$f(n) = \mathcal{O}(g(n))$	$\Leftrightarrow$	$\exists c \in \mathbf{N} \overset{\infty}{\underset{n}{\forall}} f(n) \leq cg(n)$
$f(n) = \Omega(g(n))$	$\Leftrightarrow$	$g(n) = \mathcal{O}(f(n))$
	$\Leftrightarrow$	$\exists c \in \mathbf{N} \overset{\infty}{\underset{n}{\forall}} f(n) \geq \frac{1}{c}g(n)$
$f(n) = o(g(n))$	$\Leftrightarrow$	$\forall c \in \mathbf{N} \overset{\infty}{\underset{n}{\forall}} f(n) \leq \frac{1}{c}g(n)$
$f(n) = \omega(g(n))$	$\Leftrightarrow$	$g(n) = o(f(n))$
	$\Leftrightarrow$	$\forall c \in \mathbf{N} \overset{\infty}{\underset{n}{\forall}} f(n) \geq cg(n)$
$f(n) = \Theta(g(n))$	$\Leftrightarrow$	$f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$

Table 1.3: Summary of asymptotic notations

In the following we say that an algorithm is a **polynomial time algorithm** if it runs in time $\mathcal{O}(n^k)$ for some constant k on a typical sequential computer. Such an algorithm will be also called **feasible** or even efficient. An algorithmic problem is called feasible or tractable if it can be solved by an algorithm running in polynomial time. Otherwise it is called **unfeasible** or **intractable**. This will be discussed more in Section 1.10 and especially in Chapters 4 and 5.

1.6 Asymptotics and Recurrences

Two basic problems concerning the use of asymptotics in solving recurrences and methods how to solve them will now be considered. First, we show a method how to use asymptotics to find asymptotical solutions of recurrences. Second, we show how to solve recurrences some terms of which are known only asymptotically. This is actually the most frequent case we find in practice when attempting to analyze the efficiency of complex recursive systems.

1.6.1 Bootstrapping

This is a quite general method how to find better and better approximations for terms of sequences defined by recurrences. The basic idea is simple. One first guesses an approximation (after computing some of the first values), then substitutes this approximation in the recurrence to obtain a better approximation. This process terminates when it does not seem to provide an improvement any more.

Example 1.6.1 *Find, for $n \to \infty$, an approximation of the coefficients g_n of the generating function*

$$G(z) = exp(\sum_{k=1}^{\infty} \frac{z^k}{k^2}) = \sum_{n=0}^{\infty} g_n z^n. \tag{1.50}$$

Differentiation of $G(z)$, according to z, yields

$$G'(z) = \sum_{n=0}^{\infty} n g_n z^{n-1} = \left(\sum_{k=1}^{\infty} \frac{z^{k-1}}{k} \right) G(z)$$

and if we compare the coefficients by z^{n-1} on both sides of the equation we get

$$n g_n = \sum_{k=0}^{n-1} \frac{g_k}{n-k}. \tag{1.51}$$

It is easy to see, from (1.50), that $g_0 = 1$ and from (1.51), that $g_1 = 1$. Moreover, by induction we can prove that $0 < g_n \leq 1$, for all n. We can therefore start with the approximation $g_n = \mathcal{O}(1)$. From (1.51) we get

$$n g_n = \sum_{k=0}^{n-1} \frac{\mathcal{O}(1)}{n-k} = \left(\sum_{k=0}^{n-1} \frac{1}{n-k} \right) \mathcal{O}(1) = H_n \mathcal{O}(1) = \mathcal{O}(\log n)$$

and we have the second approximation

$$g_n = \mathcal{O}\left(\frac{\log n}{n} \right), \qquad \text{for } n > 1.$$

After the next step of bootstrapping

$$\begin{array}{rclcl}
n g_n & = & \frac{1}{n} + \sum\limits_{0<k<n} \frac{\mathcal{O}((\log k)/k)}{n-k} & = & \frac{1}{n} + \sum\limits_{0<k<n} \frac{\mathcal{O}(\log n)}{k(n-k)} \\
 & = & \frac{1}{n} + \sum\limits_{0<k<n} \left(\frac{1}{k} + \frac{1}{n-k} \right) \frac{\mathcal{O}(\log n)}{n} & = & \frac{1}{n} + \frac{2}{n} H_{n-1} \mathcal{O}(\log n) \\
 & = & \frac{1}{n} \mathcal{O}(\log n) \mathcal{O}(\log n) & = & \frac{1}{n} \mathcal{O}(\log n)^2,
\end{array}$$

and we get the third approximation

$$g_n = \mathcal{O}\left(\frac{\log n}{n} \right)^2.$$

One could continue with bootstrapping but the next step does not lead to an improvement.

Exercise 1.6.2 *Use the bootstrapping to solve the recurrence*

$u_0 = 1$, $u_n = n + u_{n-1}$, $n \geq 1$.

1.6.2 Analysis of Divide-and-conquer Algorithms

We present now a general method how to solve recurrences arising from analysis of algorithms and systems that are designed using the divide-and-conquer method. They are recurrences

$$T(n) = aT(\frac{n}{c}) + f(n) \tag{1.52}$$

where for $f(n)$ only an asymptotic estimation is known. The following theorem shows how to determine $T(n)$ for most of the cases.

Theorem 1.6.3 *Let* $T(n) = aT(\frac{n}{c}) + f(n)$ *for all sufficiently large* n *and constants* $a \geq 1$, $c > 1$. *Let* $\frac{n}{c}$ *mean here either* $\lfloor \frac{n}{c} \rfloor$ *or* $\lceil \frac{n}{c} \rceil$. *Then*

1. *If* $f(n) = \mathcal{O}(n^{(\log_c a)-\varepsilon}), \varepsilon > 0,$ *then* $T(n) = \Theta(n^{\log_c a})$;
2. *If* $f(n) = \Theta(n^{\log_c a}),$ *then* $T(n) = \Theta(n^{\log_c a} \log n)$;
3. *If* $f(n) = \Omega(n^{(\log_c a)+\varepsilon}), \varepsilon > 0,$ $af(\frac{n}{c}) \leq bf(n)$ *for almost all* n *and some* $b > 1$, *then* $T(n) = \Theta(f(n))$.

The proof of this "master theorem" for the asymptotic analysis is very technical and it can be found in Cormen, Leiserson and Rivest (1990), pages 62-72. We present here only some remarks concerning the theorem and its applications.

1. It is important to see that one does not have to know $f(n)$ exactly in order to be able to determine $T(n)$ asymptotically exactly. For more complex systems an exact determination of $f(n)$ is practically impossible anyway.

2. From the asymptotical point of view $T(n)$ equals the maximum of $\Theta(f(n))$ and $\Theta(n^{\log_c a})$, unless both terms are equal. In such a case $T(n) = \Theta(n^{\log_c a} \log n)$.

3. In order to apply Case 1 it is necessary that $f(n)$ is not only asymptotically smaller than $n^{\log_c a}$; it must be smaller by a polynomial factor—this is captured by the "$-\varepsilon$" in the exponent. Similarly, in Case 3, $f(n)$ has to be larger by a polynomial factor, again captured by "$+\varepsilon$" in the exponent.

4. Observe that the theorem does not deal with all possible cases for $f(n)$.

Example 1.6.4 *Consider the recurrence* $T(n) = T(\frac{2n}{3}) + 1$. *We have* $a = 1, c = \frac{3}{2}, n^{\log_c a} = n^0 = 1$. *Therefore Case 2 applies and yields* $T(n) = \Theta(\log n)$.

Example 1.6.5 *Consider the recurrence* $T(n) = 3T(\frac{n}{4}) + n \log n$. *We have* $a = 3, c = 4,$ $n^{\log_c a} = \mathcal{O}(n^{0.793})$, *and therefore* $f(n) = \Omega(n^{\log_4 3+\varepsilon})$, *for* $\varepsilon = 0.2$. *Moreover,* $af(\frac{n}{c}) = 3\frac{n}{4}\log(\frac{n}{4}) \leq \frac{3}{4} n \log n$. *Therefore, Case 3 of theorem applies and* $T(n) = \Theta(n \log n)$.

Example 1.6.6 (Integer multiplication) *Consider the following divide-and-conquer algorithm for an integer multiplication.*

Let x *and* y *be two* n*-bit integers, where* n *is even. Then* $x = x_1 2^{\frac{n}{2}} + x_2$, $y = y_1 2^{\frac{n}{2}} + y_2$ *where* x_1, x_2, y_1, y_2 *are* $\frac{n}{2}$*-bit integers. Then*

$$xy = x_1 y_1 2^n + (x_1 y_2 + x_2 y_1) 2^{\frac{n}{2}} + x_2 y_2.$$

This seems to mean that in order to compute $x \cdot y$ *one needs to perform four multiplications of* $\frac{n}{2}$*-bit integers, two additions and two shifts. There is, however, another method for computing* x_1y_1, $x_1y_2 + x_2y_1$ *and* x_2y_2. *We first compute* x_1y_1, x_2y_2, *what needs three multiplications, and then*

$$z_1 = (x_1 + x_2)(y_1 + y_2) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$$

what requires the third multiplication. Finally, we compute $z_1 - x_1y_1 - x_2y_2 = x_1y_2 + x_2y_1$ *what requires only two subtractions.*

This means that the problem of multiplication of two n-bit integers can be reduced to the problem of three multiplications of $\frac{n}{2}$-bit integers and some additions, subtractions and shifts—what requires an amount of time proportional to n. The method can be easily adjusted for the case when n is odd. Since in this algorithm $a = 3$, $c = 2$ we have the Case 1 of Theorem 1.6.3. The algorithm requires therefore time $\Theta(n^{\log_2 3}) = \Theta(n^{1.73})$.

Example 1.6.7 (MERGESORT) *Concerning the analysis of the number of comparisons of MERGESORT, Theorem 1.6.3 yields, on the basis of the recurrence (1.14),* $T(n) = \Theta(n \lg n)$.

Exercise 1.6.8 *Find asymptotic solutions for the following recurrences:*
(a) $T(n) = 7T(\frac{n}{3}) + n^2$; *(b)* $T(n) = 7T(\frac{n}{2}) + n^2$; *(c)* $T(n) = 2T(\frac{n}{4}) + \sqrt{n}$.

1.7 Primes and Congruences

Primes and congruences induced by modulo operation on integers have been playing an important rôle for more than two thousands years in perhaps the oldest mature science—number theory. Nowadays they are the key concepts and tools in many theoretically advanced and also practically important areas of computing, especially in randomized computations, secure communications, and in the analysis of algorithms.

We start with the introduction and analysis of perhaps the oldest algorithm that still plays an important rôle in modern computing and its foundations.

1.7.1 Euclid's Algorithm

If n, m are integers, then the **quotient of** n **divided by** m is $\lfloor n/m \rfloor$. For the **remainder** the notation 'n mod m' is used—m is called **modulus**. This motivation lies in the background of the following definition in which n and m are arbitrary integers:

$$n \bmod m = \begin{cases} n - m\lfloor n/m \rfloor, & \text{for } m \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

For example:

$$7 \bmod 5 = 2, \qquad 7 \bmod -5 = -3;$$
$$-7 \bmod 5 = 3, \qquad -7 \bmod -5 = -2.$$

The basic concepts of divisibility are closely related. We say that an integer m *divides an integer* n (notation $m \setminus n$) if n/m is an integer, i.e.:

$$m \setminus n \Leftrightarrow (n \bmod m = 0). \tag{1.53}$$

A similar relation is '*n is a multiple of m*'. Related concepts are those of the **greatest common divisor** of integers m and n—$\gcd(m,n)$—and of the **least common multiple** of m and n—$\mathrm{lcm}(m,n)$, that are defined for $m+n>0$, $m \geq 0$, $n \geq 0$:

$$\begin{aligned} \gcd(m,n) &= \max\{k \mid k \setminus m \text{ and } k \setminus n\}; & (1.54)\\ lcm(m,n) &= \min\{k \mid k>0, m \setminus k \text{ and } n \setminus k\}. & (1.55) \end{aligned}$$

To compute $\gcd(m,n)$, $0 \leq m < n$, we can use the following, more than 2300 years old, algorithm—a recurrence:

Algorithm 1.7.1 (Euclid's algorithm) *For* $0 \leq m < n$,

$$\begin{aligned} \gcd(0,n) &= n;\\ \gcd(m,n) &= \gcd(n \bmod m, m), \quad \textit{for} \ \ m>0. \end{aligned}$$

For example, $\gcd(27,36) = \gcd(9,27) = \gcd(0,9) = 9$; $gcd(214,352) = gcd(138,214) = gcd(76,138) = gcd(62,76) = gcd(14,62) = gcd(6,14) = gcd(2,6) = gcd(0,3) = 3$.

Euclid's algorithm can also be used to compute, given $m \leq n$, integers n' and m' such that

$$m'm + n'n = \gcd(m,n)$$

and this is one of its most important applications. Indeed, if $m=0$, then $m'=0$ and $n'=1$ will do. Otherwise, take $r = n \bmod m$, and compute recursively r'', m'' such that $r''r + m''m = \gcd(r,m)$. Since $r = n - \lfloor n/m \rfloor m$ and $\gcd(r,m) = \gcd(m,n)$, we get:

$$r''(n - \lfloor n/m \rfloor m) + m''m = \gcd(m,n) = (m'' - r''\lfloor n/m \rfloor)m + r''n.$$

Example 1.7.2 *For* $m = 57, n = 237$ *we have* $gcd(57,237) = gcd(9,57) = gcd(3,9) = 3$. *Thus*

$$\begin{aligned} 237 &= 4 \cdot 57 + 9,\\ 57 &= 6 \cdot 9 + 3, \end{aligned}$$

and therefore

$$3 = 57 - 6 \cdot 9 = 57 - 6 \cdot (237 - 4 \cdot 57) = 25 \cdot 57 - 6 \cdot 237.$$

If $\gcd(m,n) = 1$, we say that the numbers n and m are **relatively prime**—notation $n \perp m$. The above result therefore implies that if m, n are relatively prime, then we can find, using the Euclid's algorithm, an integer denoted by $m^{-1} \bmod n$, and called the **multiplicative inverse** of m modulo n, such that $m(m^{-1} \bmod n) \equiv 1 \pmod{n}$.

Exercise 1.7.3 *Compute* a, b *such that* $ax+by = gcd(x,y)$ *for the following pairs* x, y*: (a)* $(34, 51)$*; (b)* $(315, 53)$*; (c)* $(17, 71)$*.*

Exercise 1.7.4 *Compute: (a)* $17^{-1} \bmod 13$*; (b)* $7^{-1} \bmod 19$*; (c)* $37^{-1} \bmod 97$*.*

Analysis of the Euclid's Algorithm

Let us now turn to the complexity analysis of the Euclid's algorithm.

In spite of the fact that we have presented a variety of methods for complexity analysis they are far from covering all cases. Complexity analysis of many algorithms requires a specific approach. The Euclid's algorithm is one of them.

The basic recurrence has the form, for $0 < m \leq n$,

$$gcd(m, n) = gcd(n \bmod m, m).$$

This means that after the first recursive step, the new arguments are (n_1, m), with $n_1 = n \bmod m$, and after the second step, the arguments are (n_1, m_1), with $m_1 = m \bmod n_1$. Since $a \bmod b < \frac{a}{2}$, for any $0 < b < a$ (see Exercise 49), we have $m_1 \leq \frac{m}{2}, n_1 \leq \frac{n}{2}$. This means that after two recursion steps of the Euclid's algorithm, both arguments have at most half of the original value. Hence $T(n) = \mathcal{O}(\lg n)$ for the number of steps of the Euclid's algorithm in case n is the largest argument.

This analysis was made more precise by E. Lucas (1884) and Lamé (1884), in perhaps the first deeper analysis of algorithms.

It is easy to see that if F_n is the n-th Fibonacci number, then after the first recursive step with arguments (F_n, F_{n-1}) we get arguments (F_{n-1}, F_{n-2}). This implies that for arguments (F_n, F_{n-1}) the Euclid's algorithm performs $n-2$ recursive steps.

There were established even deeper relations between Euclid's algorithm and Fibonacci numbers. They are summarized in the following theorem. The first part of the theorem is easy to prove by induction using the fact that if $m \geq F_{k+1}, n \bmod m \geq F_k$, then $n \geq m + (n \bmod m) \geq F_{k+1} + F_k = F_{k+2}$. The second part of theorem follows from the first part.

Theorem 1.7.5 *(1) If* $n > m \geq 0$*, and the application of Euclid's algorithm to arguments* n, m *results in* k *recursive steps, then* $n \geq F_{k+2}, m \geq F_{k+1}$*.*

(2) If $n > m \geq 0, m < F_{k+1}$*, then the application of Euclid's algorithm to the arguments* n, m *requires less than* k *recursive steps.*

Remark 1.7.6 It is natural to ask whether the Euclid's algorithm is the fastest way to compute the greatest common divisor. This problem was open till 1989 and it is discussed in more detail in Section 4.2.4.

1.7.2 Primes

A positive integer $p > 1$ is called **prime** if it has just two divisors: 1 and p; otherwise it is called **composite**. The first 25 primes are the following ones:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.$$

Primes play a central rôle among integers and also in computing. This will be demonstrated especially in the chapter on cryptography. The following easy to show theorem is the first reason for that.

Theorem 1.7.7 (Fundamental theorem of arithmetic) *Each integer n has a unique prime decomposition in the form $n = \prod_{i=1}^{k} p_i^{e_i}$, where $p_i < p_{i+1}, i = 1, \ldots, k-1$, are primes and e_i are integers.*

There exist infinitely many primes. This can be deduced easily from the observation that if we take primes $p_1, \ldots, p_k$, none of them divides $p_1 \cdot p_2 \cdot \ldots \cdot p_k + 1$. There are even infinitely many primes of special forms. For example

Theorem 1.7.8 *There exist infinitely many primes of the form $4k + 3$.*

Proof. Suppose there exist only finitely many primes $p_1, p_2, \ldots, p_s$ of the form $4k + 3$, i.e., $p_i \bmod 4 = 3$, $1 \leq i \leq s$, and take $N = 4 \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_s - 1$. Clearly, $N \bmod 4 = 3$. Since $N > p_i, 1 \leq i \leq s$, N can not be a prime of the form $4k + 3$ and cannot be divided by a prime of such a form. Moreover, since N is odd, N is also not divisible by a number of the type $4k + 2$ or $4k$. Hence N has to be a product of primes of the type $4k + 1$. However, this is also impossible. Indeed, $(4k + 1)(4l + 1) = 4(kl + k + l) + 1$, for any integers k, l, and therefore any product of primes of the form $4k + 1$ is again a number of such a form—but N is of the form $4k + 3$. This way we have ruled out all possibilities for N, and therefore our assumption, namely that the number of primes of the form $4k + 3$ is finite, has to be wrong. $\Box$

Discovery of as large primes as possible is an old problem. All primes up to 10^7 were computed already in 1909. The largest discovered prime at the time this book went to print is due to D. Slowinski and Gage in 1994, using computer Cray: $2^{859433} - 1$ and has 258 716 digits.[6]

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[6] Finding as large primes as possible is an old and big challenge for both scientists and technology. The first recorded method is the "sieve method" of Eratosthenes of Cyrene (284-202 BC). Fibonacci (around 1200) improved the method by observing that sieving can be stopped when the square root of the number to be tested is reached. This was the fastest general method up to 1974, when R. S. Lehman showed that primality testing of an integer n can be done in $\mathcal{O}(n^{\frac{1}{3}})$ time.

The previous "world records" were $2^{756839} - 1$ (1992) and $2^{216091} - 1$ (1985)—all due to D. Slowinski. The largest prime from the pre-computer era was $2^{127} - 1$ (1876). This record lasted 76 years. Next world records were from 1952: $2^{521} - 1$, $2^{607} - 1$, $2^{1279} - 1$, $2^{2203} - 1$, $2^{2281} - 1$. All known very large primes have the form $2^p - 1$, where p is a prime; they are called Mersenne primes. It is certainly a challenge to find new very large Mersenne primes; especially because all numbers smaller as 2^{350000} and also in the range $2^{430000} - 2^{520000}$ have already been checked for primality, and it is not known whether there are infinitely many Mersenne primes.

Another important question is how many primes are among the first $n$positive integers; for this number the notation $\pi(n)$ is used. The basic estimation $\pi(n) = \Theta(\frac{n}{\ln n})$ was guessed already by Gauss[7] at the age of 15. Better estimations are

$$\frac{n}{\ln n} < \pi(n) \leq \frac{5}{4}\frac{n}{\ln n}, \qquad \text{for } n \geq 114$$

and

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + \frac{3!n}{(\ln n)^4} + \frac{4!n}{(\ln n)^5} + \Theta\left(\frac{n}{(\ln n)^6}\right). \qquad (1.56)$$

Additional information about distribution of primes is given in the following theorem in which ϕ is the so-called **Euler's phi function**; $\phi(n)$ is the number of positive integers smaller than n that are relatively prime to n—for example $\phi(p) = p - 1$ and $\phi(pq) = (p-1)(q-1)$ if p, q are primes.

Theorem 1.7.9 *(Prime Number Theorem)*[8] *If gcd$(b, c) = 1$, then for the number $\pi_{b,c}(n)$ of primes of the form $bk + c$ we have:*

$$\pi_{b,c}(n) \sim \frac{1}{\phi(b)}\frac{n}{\ln n}.$$

The following table shows how good is the estimation $\pi(n) = n/\ln n$.

n	10^4	10^7	10^{10}
$\pi(n)$	1,229	664,579	455,052,511
$n/\ln n$	1,089	621,118	434,782,650
$\pi(n)/(n/\ln n)$	1.128	1.070	1.046

The largest computed value of $\pi(x)$ is $\pi(10^{18}) = 24{,}739{,}954{,}287{,}740{,}860$ by Deliglise and Rivat in 1994.

We deal with the problem how to determine whether a given integer is prime in Section 5.5.2 and with the problem of finding large primes in Section 8.3.4. The importance of primes in cryptography is due to the fact that we can find efficiently large primes, but we are not able to factorize efficiently large products of primes. Moreover, some important computations can be done in polynomial time if an argument is prime but seem to be unfeasible if this argument is an arbitrary integer.

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[7] Karl Frederic Gauss (1752-1833)—a German mathematician, physicist, and astronomer considered as the greatest mathematician of his time, made fundamental contributions in algebra, number theory, complex variable, differential geometry, approximation theory, calculation of orbits of planets and comets, electro- and geomagnetism. Gauss developed foundations for absolute metric system and with W. Weber, invented electrical telegraph.

[8] The term "Prime Number Theorem" is used also for the Gauss estimation for $\pi(n)$ or for the estimation (1.56).

Exercise 1.7.10 *Show that if n is composite, then so is $2^n - 1$.*

Exercise 1.7.11 ** *Show that there exist infinitely many primes of the type $6k+5$.*

1.7.3 Congruence Arithmetic

The modulo operation and the corresponding congruence relation

$$a \equiv b \pmod m \Leftrightarrow a \bmod m = b \bmod m \tag{1.57}$$

defined for arbitrary integers a, b and $m > 0$, play an important rôle in producing (pseudo-)randomness and in randomized computations and communications. We read '$a \equiv b \pmod m$' as 'a **is congruent to** b **modulo** m'. From (1.57) we also get that $a \equiv b \pmod m$ if and only if $a - b$ is a multiple of m. This congruence defines an equivalence relation on $\mathbf{Z}$ and its equivalence classes are called **residue classes modulo n**. $\mathbf{Z}_n$ is used to denote the set of all such residue classes and $\mathbf{Z}_n^*$ its subset consisting of those classes elements of which are relatively prime to n.

The following properties of congruence can be verified using the definition (1.57):

$$a \equiv b \text{ and } c \equiv d \pmod m \;\Rightarrow\; a + c \equiv b + d \pmod m; \tag{1.58}$$

$$a \equiv b \text{ and } c \equiv d \pmod m \;\Rightarrow\; a - c \equiv b - d \pmod m; \tag{1.59}$$

$$a \equiv b \text{ and } c \equiv d \pmod m \;\Rightarrow\; ac \equiv bd \pmod m; \tag{1.60}$$

$$ad \equiv bd \pmod m \;\Leftrightarrow\; a \equiv b \pmod m \text{ for } d \perp m; \tag{1.61}$$

$$ad \equiv bd \pmod{md} \;\Leftrightarrow\; a \equiv b \pmod m \text{ for } d \neq 0; \tag{1.62}$$

$$a \equiv b \pmod{mn} \;\Leftrightarrow\; a \equiv b \pmod m \text{ and } a \equiv b \pmod n \text{ if } m \perp n. \tag{1.63}$$

The property (1.63) can be used to simplify the computation of congruences as follows: If $\prod_{i=1}^{k} p_i^{e_i}$ is the prime decomposition of m, then

$$a \equiv b \pmod m \Leftrightarrow a \equiv b \pmod{p_i^{e_i}}, \text{ for } 1 \leq i \leq k. \tag{1.64}$$

Congruences modulo powers of primes are therefore building blocks for all congruences modulo integers.

Exercise 1.7.12 *Show that $((a \bmod n)(b \bmod n) \bmod n) = ab \bmod n$, for any integers a, b, r.*

Exercise 1.7.13 *Show: $2^{xy^2} \bmod (2^{xy} - x) = x^y$.*

One of the main use of the modulo operation in randomization is related to the fact that solving "nonlinear" congruence equations is in general computationally intractable task.

On the other hand the linear congruence equations

$$cx \equiv d \pmod{m}$$

are easy to deal with. Indeed, we have

Theorem 1.7.14 *A linear congruence* $cx \equiv d \pmod{m}$ *has a solution if and only if* d *is a multiple of* $gcd(c,m)$*. In this case, the equation has exactly* $k = gcd(c,m)$ *distinct integer solutions*

$$x_0\frac{d}{k}, x_0\frac{d}{k} + \frac{m}{k}, \ldots, x_0\frac{d}{k} + (k-1)\frac{m}{k},$$

in the interval $(0,m)$*, where* x_0 *is the unique integer solution of the equation* $cx + ym = gcd(c,m)$*—that can be found using the Euclid's algorithm.*

The proof is easy once we realize that the problem to solve the equation $cx \equiv d \pmod{m}$ is equivalent to the one to find integer solutions of the equation $cx + ym = d$.

Another useful fact is that if $cx \equiv d \pmod{m}$, then also $c(x+m) \equiv d \pmod{m}$.

Example 1.7.15 *For the congruence* $27x \equiv 1 \pmod{47}$ *we have* $gcd(27,47) = 1$ *and* $7 \cdot 27 - 4 \cdot 47 = 1$*. Hence* $x = 7$ *is the unique solution.*

Example 1.7.16 *For the congruence* $51x \equiv 9 \pmod{69}$ *we have* $gcd(51,69) = 3$ *and* $-4 \cdot 51 + 3 \cdot 69 = 3$*. Hence* $x = -12, 11, 34$ *or, expressed using only positive integers,*

$$x = 11, 34, 57.$$

Exercise 1.7.17 *Solve the linear congruence equations: (a)* $4x \equiv 5 \bmod 9$*; (b)* $2x \equiv 17 \bmod 19$.

There is also an old method to solve special systems of linear congruences. The following result, easy to verify, has been attributed to Sun Tsŭ from China—around 350 AD.

Theorem 1.7.18 (Chinese Remainder Theorem) *Let* $m_1, \ldots, m_t$ *be integers,* $m_i \perp m_j$ *for* $i \neq j$*, and* $a_1, \ldots, a_t$ *be integers with* $0 < a_i < m_i$*. Then the system of congruences*

$$x \equiv a_i \pmod{m_i}, \quad \text{for } i = 1, \ldots, t,$$

possesses the solution (what is straightforward to verify),

$$x = \sum_{i=1}^{t} a_i M_i N_i, \tag{1.65}$$

where $M = \prod_{i=1}^{t} m_i, M_i = M/m_i$ *and* $N_i = M_i^{-1} \bmod m_i, 1 \leq i \leq t$*. Moreover, the solution (1.65) is unique up to the congruence modulo* M*, i.e.,* $z \equiv x \pmod{M}$ *for any other solution* z*.*

The Chinese Remainder Theorem has numerous applications. One of them is the following modular representation of integers:

Let $m_1, ..., m_n$ be pairwise relatively prime and M their product. It follows from Theorem 1.7.18 that each integer $0 \leq x < M$ is uniquely represented by the n-tuple

$$(x \bmod m_1, \ldots, x \bmod m_n).$$

For example, if $m_1 = 2, m_2 = 3, m_3 = 5$, then $(1, 0, 2)$ represents 27. Such a modular representation of integers may look artificial. However, it has its advantages because it allows parallel, component-wise, performance of basic arithmetic operations.

Exercise 1.7.19 *(a) Find modular representations of integers* $7, 13, 20, 6$ *and* 91 *with respect to the 4-tuple of integers* $(2, 5, 11, 19)$*;*

(b) Show that if $(x_1, \ldots, x_n)$ *is a modular representation of the integer* x *and* $(y_1, \ldots, y_n)$ *of the number* y*, both with respect to pairwise primes* $(m_1, \ldots, m_n)$*, and* $\circ$ *is one of the operations: addition, subtraction or multiplication, then*

$$((x_1 \circ y_1) \bmod m_1, \ldots, (x_n \circ y_n) \bmod m_n)$$

represents the number $x \circ y$*—provided this number is smaller than* m*.*

In cryptographical applications one often needs to compute **modular exponentiation** $a^b \bmod n$, where a, b and n can be very large; n may have between 512 and 1024 bits and a has also about that number of bits. Moreover, b can have easily several hundred of bits. In such a case a^b would have more than 10^{30} bits. To compute such numbers seems to be difficult. Fortunately, congruences have various properties that allow to simplify computation of modular exponentiations essentially. For example,

$$a^2 \equiv (a \bmod n)^2 \pmod{n}$$

for any a and b. This allows, together with the exponentiation Algorithm 1.1.14, to compute $a^b \bmod n$ in such a way that one has never to work with numbers that have more than twice more bits than numbers a, b, n. For example

$$a^8 \bmod n \equiv (((a \bmod n)^2 \bmod n)^2 \bmod n)^2 \bmod n.$$

Efficient modular exponentiation is also in the heart of the efficient primality testing as shown in Section 5.5.2. To simplify modular exponentiation one can also use the following theorem and its subsequent generalization. Both of these results play an important rôle also in cryptography:

Theorem 1.7.20 (Fermat's[9] Little Theorem—1640) *If* p *is a prime,* $n \in \mathbf{N}^+$*,* $a \in \mathbf{N}$*, then*

$$a^p \equiv a \pmod{p} \tag{1.66}$$

and if a is not divisible by p, then

$$a^{p-1} \equiv 1 \pmod{p}. \tag{1.67}$$

Proof. It is true for $a = 1$. Assuming that it is true for a, then by induction $(a+1)^p \equiv a^p + 1 \equiv a + 1 \pmod{p}$, because $\binom{p}{k} \equiv 0 \pmod{p}$ for $0 < k < p$. So Theorem 1.7.20 holds for all $a \in \mathbf{N}$. □

A more general version of this theorem, that needs quite a different proof, the so-called **Euler's Totient Theorem**, has the form

$$n^{\phi(m)} \equiv 1 \pmod{m} \ \text{ if } n \perp m. \tag{1.68}$$

Example 1.7.21 *In order to compute* $3^{1000} \bmod 19$ *we can use the fact that by Fermat's Little Theorem,* $3^{18} \equiv 1 \bmod 19$*. Since* $1000 = 18 \cdot 55 + 10$*, we get*

$$3^{1000} = 3^{18 \cdot 55 + 10} = (3^{18})^{55}(3^5)^2 \equiv (3^5 \bmod 19)^2 \equiv (15)^2 \equiv 225 \equiv 16 \pmod{19}.$$

Exercise 1.7.22 *Compute: (a)* $2^{340} \bmod 11$*; (b)* $3^{100} \bmod 79$*; (c)* $5^{10000} \bmod 13$*.*

Exercise 1.7.23 *Show that* $\sum_{i=1}^{p-1} i^{p-1} \equiv -1 \pmod{p}$*.*

1.8 Discrete Square Roots and Logarithms*

In these section we deal with two interesting and computationally intriguing problems of large importance especially for modern cryptography.

1.8.1 Discrete Square Roots

The problem of solving quadratic congruence equations

$$x^2 \equiv a \pmod{m}$$

or, in other words, computing "discrete" square roots modulo m, is intriguing and of importance for various applications.

For an integer m denote

$$\begin{aligned} \mathbf{Z}_m &= \{0, 1, \ldots, m-1\} \\ \mathbf{Z}_m^* &= \{a \mid a \in \mathbf{Z}_m, gcd(a, m) = 1\} \end{aligned}$$

and let their elements to denote the corresponding residue classes. Observe that $\mathbf{Z}_n^*$ has $\phi(n)$ elements.

Example 1.8.1 $\mathbf{Z}_{10}^* = \{1,3,7,9\}, \mathbf{Z}_{11}^* = \{1,2,3,4,5,6,7,8,9,10\}$.

An integer $x \in \mathbf{Z}_m^*$ is called a **quadratic residue modulo** m if $x \equiv y^2 \pmod m$ for some $y \in \mathbf{Z}_m^*$; otherwise x is called a **quadratic nonresidue modulo** m. Notation:

QR_m — the set of all quadratic residues modulo m,
QNR_m — the set of all quadratic nonresidues modulo m.

Exercise 1.8.2 *Show that if p is prime, then each quadratic residue modulo p has exactly two distinct square roots.*

Exercise 1.8.3 **Explain why exactly half of the numbers $1,\ldots,p-1$ are quadratic residues modulo p.*

Exercise 1.8.4 * *Find all square roots of* 64 *modulo* 105.

To deal with quadratic residues it is useful to use the following notation defined for any integer m and $x \in \mathbf{Z}_m^*$:

$$(x|m) = \begin{cases} 1, & \text{if } x \in QR_m \quad \text{and } m \text{ is a prime;} \\ -1, & \text{if } x \in QNR_m \quad \text{and } m \text{ is a prime;} \\ \prod_{i=1}^n (x|p_i), & \text{if } m = \prod_{i=1}^n p_i, x \perp m \text{ and } p_i \text{ are primes.} \end{cases}$$

$(x|m)$ is called **Legendre symbol** if m is prime and **Legendre-Jacobi**[10] **symbol** if m is composite. It is easy to determine whether an $x \in \mathbf{Z}_m^*$ is a quadratic residue modulo m if m is a prime—one needs only to compute $(x|m)$. This can be done in $\mathcal{O}(\lg m)$ time using the following identities:

Theorem 1.8.5 *Let $x, y \in \mathbf{Z}_m^*$.*

1. [11] $x^{(p-1)/2} \equiv (x|p) \pmod p$ *for any prime $p > 2$ and $x \in Z_p^*$.*
2. *If $x \equiv y \pmod m$, then $(x|m) = (y|m)$.*
3. $(x|m) \cdot (y|m) = (x \cdot y|m)$.
4. $(-1|m) = (-1)^{(m-1)/2}$, *if m is odd.*
5. $(2|m) = (-1)^{(m^2-1)/8}$, *if m is odd.*
6. [12] *If $m \perp n$ and m, n are odd, then $(n|m)(m|n) = (-1)^{(m-1)(n-1)/4}$.*

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[10] Adrien Marie Legendre (1752-1833)—a French mathematician; Carl Gustav Jacobi (1804-1851)—a German mathematician.

[11] This claim is also called the Euler's criterion.

[12] This assertion, due to F. L. Gauss, is known as the Law of Quadratic Reciprocity. It plays an important rôle in number theory and, by Gesternhaber (1963), at least 152 different proofs of this "law" are known.

Example 1.8.6

$$\begin{aligned}(28|97) &= (2|97)\cdot(2|97)\cdot(7|97) = (7|97)\\ &= (97|7)\cdot(-1)^{(97-1)(7-1)/4} = (6|7)\\ &= (2|7)\cdot(3|7) = (-1)^6(3|7) = (7|3)\cdot(-1)^3\\ &= -(1|3) = -1.\end{aligned}$$

Exercise 1.8.7 *Compute: (a)* $(32|57)$*; (b)* $(132|37)$*; (c)* $(47,53)$*; (d)* $(3|p)$*, where* p *is prime.*

It is straightforward to see from the Euleur's criterion in Theorem 1.8.5 that if p is a prime of the form $4k+3$ and $x \in QR_p$, then $\pm x^{(p+1)/4} \bmod p$ are two square roots of x modulo p. For such a p and x, one can therefore efficiently compute square roots. On the other hand no efficient deterministic algorithm for computing square roots is known when p is a prime of the form $4k+1$. However, we show now that even in such a case there is an efficient randomized algorithm to do that.

Informally, an algorithm is called random if its behaviour is determined not only by the input but also by the values produced by a random number generator. For example, if in a computation step a number is randomly choosen (or produced by a random number generator). Informally again, an algorithm is called polynomial time algorithm if it can be realised on a typical sequential computer in such a way that time of computation grows polynomially, with respect to the size of the input data.

The concepts of "a randomized algorithm" and a "polynomial time" algorithm belong to the most basic ones investigated in foundation of computing and one of the main outcomes is, as we shall see later, that these two concepts are very robust and practically independent for from the choice of a computer model or a source of randomness. This also justifies that we can start to use them now on the base of the above informal description. A formal description provided later will be necessary once we try to show properties of these concepts or even non-existence of such algorithms for some problems.

Theorem 1.8.8 (Adleman-Manders-Miller's Theorem) *There exists a randomized polynomial time algorithm to compute the square root of* a *modulo* p*, where* $a \in QR_p$*, and* p *is a prime.*

Proof. Let us consider a decomposition $p-1 = 2^e P$, where P is odd. Choose randomly[13] $b \in QNR_p$ and let us define the sequence $a_1, a_2, \ldots$ of elements from QR_p and of integers $e \geq k_1 > k_2 > \ldots > k_i > \ldots$ inductively as follows:

- $a_1 = a$;
- $k_i =$ the smallest $k \geq 0$ such that $a_i^{2^k P} \equiv 1 \pmod p$, for $i \geq 1$;

---

[13] This means one chooses randomly $b \in \mathbf{Z}_p$ and verifies whether $b \in QR_p$. If not, one chooses another b until a $b \in QNR_p$ is found. Due to Exercise 1.8.3, this can be done with high probability efficiently.

- $a_i = a_{i-1}b^{2^{e-k_{i-1}}} \bmod p$, for $i > 1$.

We show now that $k_i < k_{i-1}$ for all $i > 1$. In doing so we make use of the minimality of k_i and of the fact that $b^{2^{e-1}P} = b^{(p-1)/2} \equiv (b|p) \equiv -1 \pmod p$, by (6) of Theorem 1.8.5. Since

$$a_i^{2^{k_{i-1}-1}P} \equiv (a_{i-1}b^{2^{e-k_{i-1}}})^{2^{k_{i-1}-1}P} \equiv a_{i-1}^{2^{k_{i-1}-1}P} b^{2^{e-1}P} \equiv (-1)(-1) \equiv 1 \pmod p,$$

k_i has to be smaller than k_{i-1}. Therefore there has to exist an $n < e$ such that $k_n = 0$. For such an n we have $a_n^{p+1} \equiv a_n \bmod p$, what implies that $a_n^{(p+1)/2}$ is a square root of a_n.

Let us now define, by the reverse induction, the sequence $r_n, r_{n-1}, \ldots, r_1$ as follows:

- $r_n = a^{(P+1)/2} \bmod p$,
- $r_i = r_{i+1}(b^{2^{e-k_i-1}})^{-1} \bmod p$, for $i < n$.

It is easy to verify that $a_i = r_i^2 \bmod p$ and therefore $a = r_1^2 \bmod p$.

Clearly, $n < \lg p$ and therefore the algorithm requires polynomial time in length of p and a—plus time to choose randomly a b such that $(b|p) = -1$. $\square$

There is a conceptually much simpler algorithm to compute square roots. However, it requires to work with congruences on polynomials and to apply Euclid's algorithm also to find the greatest common divisor of two polynomials—what can be done in quite a natural way. In other words a little bit more sophisticated mathematics has to be used.

Suppose that a is a quadratic residue in $\mathbf{Z}_p^*$ and we want to find its square roots. Observe first that the problem of finding an x such that $x^2 \equiv a \pmod p$ is equivalent to the problem, for an arbitrary $c \in \mathbf{Z}_p^*$, to find an x such that $(x-c)^2 \equiv a \pmod p$—in order to solve the original problem only a shift of roots is required. Suppose now that $(x-c)^2 - a \equiv (x-r)(x-s) \pmod p$. In such a case $rs \equiv c^2 - a \pmod p$ and, by (2) in Theorem 1.8.5, $((c^2-a)|p) = (r|p)(s|p)$. So if $((c^2-a)|p) = -1$, then one of r and s is a quadratic residue. On the other hand, it follows from (Euler's criterion) of Theorem 1.8.5, that all quadratic residues in $\mathbf{Z}_p^*$ are roots of the polynomial $x^{(p-1)/2} - 1$. This implies that the greatest common divisor of the polynomials $(x-c)^2 - a$ and $x^{(p-1)/2} - 1$ is the first degree polynomial whose root is that root of $(x-c)^2 - a$ which is the quadratic residue. This leads to our second randomized algorithm.

Algorithm 1.8.9

- *Choose randomly* $c \in \mathbf{Z}_p^*$.
- *If* $((c^2-a)|p) = -1$, *then compute* $gcd(x^{(p-1)/2} - 1, (x-c)^2 - a) = \alpha x - \beta$.
- *Output* $\pm(c + \alpha^{-1}\beta)$ *as the square root of* a *modulo* p.

The efficiency of the algorithm is based on the following fundamental result from the number theory: if a is a quadratic residue from $\mathbf{Z}_p^*$ and c is chosen randomly from $\mathbf{Z}_p^*$, then with the probability larger than $\frac{1}{2}$ we have $((c^2-a)|p) = -1$.

Another important fact is that there is an effective way to compute square roots modulo n even in the case n is composite—if prime factors of n are known.

Theorem 1.8.10 *If* $p, q > 2$ *are primes, then* $x \in QR_{pq} \Leftrightarrow x \in QR_p \wedge x \in QR_q$. *Moreover, there is a polynomial time algorithm which, given as inputs* x, u, v, p, q *such that* $x \equiv u^2 \bmod p$, $x \equiv v^2 \bmod q$, *computes* w *such that* $x \equiv w^2 \bmod pq$.

Proof. The first claim follows directly from (3) of Theorem 1.8.5. To prove the rest of the theorem, let us assume that x, u, y, v, p and q satisfy the hypothesis. Using Euclid's algorithm we can compute a, b such that $ap + bq = 1$. If we now denote $c = bq = 1 - ap$ and $d = ap = 1 - bq$, then

$$c \equiv 0 \bmod q, \quad d \equiv 0 \bmod p, \quad c \equiv 1 \bmod p, \quad d \equiv 1 \bmod q. \tag{1.69}$$

This will now be used to show that for $w = cu + dv$ we have $x = w^2 \bmod pq$. In order to do that it is enough, due to the first part of the theorem, to prove that $x = w^2 \bmod p$ and $x = w^2 \bmod q$. We do that only for p; the other case can be treated similarly. By (1.69)

$$w^2 = (cu + dv)^2 = c^2u^2 + 2cduv + d^2v^2 \equiv u^2 \equiv x \pmod{p}.$$

□

On the other hand, no effective algorithm is known, given an integer n and an $a \in QR_n$, to compute a square root of a modulo n. As shown in Theorem 1.8.15, this problem is as difficult as the problem of factorization of integers—the problem on intractability of which many modern cryptographical techniques are based—see Chapters 8 and 9.

Before presenting the next theorem let us look deeper into the problem of finding square roots modulo a composite integer n.

Lemma 1.8.11 *Any quadratic residue* $a \in QR_n$, $n = pq$, p *and* q *are odd primes, has four square roots modulo* n.

Proof. Let $a \in QR_n$ and $a \equiv a_1^2 \pmod{n}$. By Chinese Remainder Theorem there are integers u, v such that

$$u \equiv a_1 \bmod p, \qquad u \equiv -a_1 \bmod q;$$

$$v \equiv a_1 \bmod q, \qquad v \equiv -a_1 \bmod p.$$

Since p, q are odd, u, v, a_1 and $-a_1$ have to be distinct. Moreover, $u^2 \equiv v^2 \equiv_1^2 \bmod pq$ and therefore $u, v, a_1, -a_1$ are four distinct square roots of a. □

Exercise 1.8.12 * *Find all four solutions of congruences: (a)* $x^2 \equiv 25 \pmod{33}$;

(b) $x^2 \equiv 11 \pmod{p}$.

Exercise 1.8.13 *Determine in general the number of square roots modulo* n *for an* $a \in QR_n$.

Of a special interest is the case when $n = pq$ and $p \equiv q \equiv 3 \pmod 4$—such integers are called **Blum integers**. In such a case by Theorem 1.8.5:

$$(-x|n) = (-x|p)(-x|q) = (x|p)(-1)^{(p-1)/2}(x|q)(-1)^{(q-1)/2} = (x|p)(x|q) = (x|n).$$

Moreover, it holds

Theorem 1.8.14 *(1) If* $x^2 \equiv y^2 \bmod n$ *and* $x, y, -x, -y$ *are distinct modulo* n*, then* $(x|n) = -(y|n)$.

(2) If $n = pq$ *is a Blum integer, then the mapping*

$$x \to x^2 \bmod n$$

is a permutation on QR_n*. In other words, each quadratic residue has a unique square root that is also a quadratic residue and it is called its* **principal square root.**

Proof. (1) Since pq divides $(x^2 - y^2) = (x + y)(x - y)$ and $x, y, -x, -y$ are distinct modulo n, neither $x + y$ nor $x - y$ can be divided by both p and q. Without loss of generality assume that p divides $(x - y)$ and q divides $(x + y)$; the other case can be dealt with similarly. Then $x \equiv y \bmod p, x \equiv -y \bmod q$ and therefore $(x|p) = (y|p), (x|q) = -(y|q)$. Thus $(x|n) = -(y|n)$.

(2) Let a be any quadratic residue modulo n. By Lemma 1.8.11, a has exactly 4 roots; say $x, -x, y, -y$. By (1) $(x|n) = -(y|n)$. Let x be such a square root that $(x|p) = 1$. Then either $(x|p) = (x|q) = 1$ or $(-x|p) = (-x|q) = 1$. Hence either x or $-x$ is a quadratic residue modulo n. □

For Blum integers two key algorithmic problems of modern cryptography are computationally equivalent, with respect to the existence of polynomial time randomized algorithm.

Theorem 1.8.15 (Rabin‘s Theorem) *The following statements are equivalent:*

(1) There is a polynomial time randomized algorithm to factor Blum integers.

(2) There is a polynomial time randomized algorithm to compute the principal square root for $x \in QR_n$*, if* n *is a Blum integer.*

Proof. (1) Assume that a polynomial time randomizd algorithm $\mathcal{A}$ to compute square roots modulo Blum integers is given. A Blum integer n can be factored as follows: :

1. Chose a random y such that $(y|n) = -1$.
2. Compute $x \equiv y^2 \bmod n$.
3. Find, using $\mathcal{A}$, an $z \in QR_n$ such that $x \equiv z^2 \bmod n$.

We show now that $gcd(y + z, n)$ is a prime factor of n. Clearly, pq divides $(y - z)(y + z)$. Since $(-z|n) = (-1|n)(z|n) = (-1)^{\frac{p-1}{2}}(-1)^{\frac{q-1}{2}}(z|n) = 1$, we have $y \not\equiv -z \bmod n$ and therefore $gcd(y + z, n)$ has to be one of the prime factors of n.

(2) Assume now that we can efficiently factor n to get p, q such that $n = pq$. We show now how to compute principal square roots modulo n. Let $x \in QR_n$

- Using the Adleman-Menders-Miller's algorithm compute $u \in QR_p$ and $v \in QR_q$ such that $x \equiv u^2 \bmod p, x \equiv v^2 \bmod q$.
- Using the Euclid's algorithm compute a, b such that $ap + bq = 1$.
- Compute $c = bq, d = ap$.

We show now that $w = cu + dv$ is in QR_n and it is a square root of a. Indeed, since $c \equiv 1 \bmod p$ and $d \equiv 1 \bmod q$, we have $w^2 \equiv u^2 \equiv x \bmod p, w^2 \equiv v^2 \equiv x \bmod q$, and by (1.63), $w^2 \equiv x \bmod n$. To show that $w \in QR_n$ we proceed as follows:

Since $c \equiv 0 \bmod q, d \equiv 0 \bmod p$, we get $(w|p) = (u|p) = 1, (w|q) = (v,q) = 1$ and therefore $(w|pq) = (w|p)(w|q) = 1$. $\square$

1.8.2 Discrete Logarithm Problem

It is the problem to determine, given integers a, n, x, an integer m such that $a^m \equiv x \bmod n$—if such an m exists.

It may happen that there are two such m—for example, $m = 10$ and $m = 4$ for the equation $5^x \equiv 16 \bmod 21$; or none, for example, for the equation $5^x \equiv 3 \bmod 21$. An important case is when g is a **generator** or a **principal root** of $\mathbf{Z}_n^*$, that is if

$$\mathbf{Z}_n^* = \{g^i \bmod n \mid 0 \leq i \leq \phi(n)\}.$$

In such a case to any $x \in \mathbf{Z}_n^*$ there is a unique $m < \phi(n)$ such that $x \equiv g^m \bmod n$. Such an m is called the **discrete logarithm** or **index** of x with respect to n and g; in short $index_{n,g}(x)$.

If $\mathbf{Z}_n^*$ has a principal root, then it is called cyclic. It has been known already to F. L. Gauss that $\mathbf{Z}_n^*$ is cyclic if and only if n is one of the numbers $2, 4, p^i, 2p^i$, where $p > 2$ is a prime and i is a positive integer.

Example 1.8.16 *The table of indices, or discrete logarithms for $\mathbf{Z}_{13}^*$ and the generator 2:*

x	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
2	*0*	*1*	*4*	*2*	*9*	*5*	*11*	*3*	*8*	*10*	*7*	*6*

There is no efficient deterministic algorithm known to compute, given a, n and x, the discrete logarithm m such that $a^m \equiv x \bmod n$. This fact plays an important rôle in cryptography and its applications and also for (perfect) random number generators—see next section. An exception is the case that p is a prime and factors of $p - 1$ are small—of the order $\mathcal{O}(\log p)$.[14]

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[14] In the general case of a prime p the fastest algorithm, due to Adleman (1980), runs in time $\Theta(2^{\sqrt{\lg p \lg \lg p}})$. However, there is a polynomial time randomized algorithm for (potential) quantum computer to compute the discrete logarithm due to Shor (1994).

Exercise 1.8.17 *Find all principal roots in* $\mathbf{Z}_{11}^*$ *and compute all discrete logarithms of elements in* $\mathbf{Z}_{11}^*$*, with respect to these principal roots.*

Exercise 1.8.18 * *Let* g *be a principal root modulo the prime* $p > 2$*. Show that for all* $a, b \in \mathbf{Z}_p^*$*: (a)* $index_{p,g}(ab) \equiv index_{p,g}(a) + index_{p,g}(b) \pmod{p-1}$*;*

(b) $index_{p,g}(a^n) \equiv n \cdot index_{p,g}(a) \pmod{p-1}$*.*

1.9 Probability and Randomness

In the design of algorithms, communication protocols or even networks, an increasingly important rôle is played by randomized methods, where random bits, or, less formally, coin-tossing, are used to make decisions. The same is true for randomized methods in the analysis of computing systems—both deterministic and randomized. Basic concepts and methods of discrete probability and randomness are therefore introduced in this section.

1.9.1 Discrete Probability

A **probability** (or **sample) space** is a set Ω (of all possible things that can happen), together with a **probability distribution** Pr—a mapping that maps each element of Ω onto a nonnegative real such that

$$\sum_{\omega \in \Omega} Pr(\omega) = 1.$$

A subset $E \subseteq \Omega$ is called an **event**; its probability is defined as $Pr(E) = \sum_{\omega \in E} Pr(\omega)$. Elements of Ω are called **elementary events**. If all elementary events have the same probability we talk about a **uniform probability distribution**.

For example, let Ω_0 be the set of all possible outcomes of throwing simultaneously three dice. $|\Omega_0| = 216$ and if all dice are perfect, then each elementary event has the probability $\frac{1}{216}$.

From the definition of a probability distribution the following identities, in which A, B are events, follow easily:

$$Pr(\bar{A}) = 1 - Pr(A) \qquad Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B).$$

Exercise 1.9.1 *Let* $E_1, \ldots, E_n$ *be events. Show: (a)* **Bonferroni's inequality***:*
$Pr(E_1 \cap E_2 \cap \ldots \cap E_n) \geq Pr(E_1) + Pr(E_2) + \cdots + Pr(E_n) - (n-1)$*;*
(b) **Bode's inequality***:*$Pr(E_1 \cup E_2 \cup \ldots \cup E_n) \leq Pr(E_1) + \cdots + Pr(E_n)$*.*

The **conditional probability** of an event A, given that another event B occurs, is defined by

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}. \tag{1.70}$$

This formalizes an intuitive idea of having a priory partial knowledge of the outcome of an experiment.

Comparing $Pr(A|B)$ and $Pr(B|A)$, expressed by (1.70), we get:

Theorem 1.9.2 (Bayes Theorem)

$$Pr(A)Pr(B|A) = Pr(B)Pr(A|B).$$

Two events A, B are called **independent** if $Pr(A \cap B) = Pr(A) \cdot Pr(B)$.

Exercise 1.9.3 *What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive* 0*'s assuming that probabilities of* 0 *and* 1 *are equal and the first bit is* 1*?*

A **random variable** $X$is any function from a sample space Ω to reals. The function $F_X(x) = Pr(X = x)$ is called the **probability density function** of X.

For example, if $X(\omega)$ ($\omega \in \Omega_0$), is the sum of numbers on dices, then its probability density function has the form:

X	3	4	5	6	7	8	9	...
$f_X(x)$	$\frac{1}{216}$	$\frac{3}{216}$	$\frac{6}{216}$	$\frac{10}{216}$	$\frac{12}{216}$	$\frac{21}{216}$	$\frac{25}{216}$	...

Two random variables X and Y over the same sample space Ω are called **independent** if for any $x, y \in \Omega$:

$$Pr(X = x \text{ and } Y = y) = Pr(X = x)Pr(Y = y).$$

Exercise 1.9.4 ***[Schwartz' Lemma] Let* p *be a polynomial with* n *variables that has degree at most* k *in each of its variables. Show that if* p *is not identically* 0 *and the values* α_i, $i = 1, 2, \ldots, n$, *are chosen in the interval* $[0, N-1]$ *independently of each other according to the uniform distribution, then* $Pr(p(\alpha_1, \ldots, \alpha_n) = 0) \leq \frac{kn}{N}$. *(Use induction and the decomposition* $p = p_1x_1 + p_2x_1^2 + \ldots + p_tx_1^t$, *where* $p_1, \ldots, p_t$ *are polynomials of variables* $x_2, \ldots, x_n$.*)*

An intuitive concept of an average value of a random variable X on a probability space Ω is defined formally as the **mean** or **expected value**

$$EX = \sum_{\omega \in \Omega} X(\omega)Pr(\omega) \tag{1.71}$$

provided this potentially infinite sum exists.

If X, Y are random variables, then so are $X+Y$, cX, $X \cdot Y$, where c is a constant. Directly from (1.71) we easily get:

$$\begin{aligned} E(X+Y) &= EX+EY && (1.72)\\ E(cX) &= cE(X) && (1.73)\\ E(X\cdot Y) &= EX\cdot EY, \qquad \text{if } X, Y \text{ are independent.} && (1.74) \end{aligned}$$

Exercise 1.9.5 *A ship with crew of 77 sailors sleeping in 77 cabins, each for 77 sailors, arrives at a port and sailors go out to have fun. Later at night they return and being in the state of inebriation they choose randomly a cabin to sleep in. What is the epected number of sailors sleeping in their own cabins? (Hint: Consider random variables* X_i *the value of which is* 1 *if* i*-th sailor sleeps in his own cabin and 0 otherwise. Compute* $E[\sum_{i=1}^{77} X_i]$*.)*

Other important attributes of a random variable X are its **variance** VX and **standard deviation** $\sigma X = \sqrt{VX}$, where

$$VX = E((X-EX)^2).$$

Since

$$\begin{aligned} E((X^2-EX)^2) &= E(X^2-2X(EX)+(EX)^2) && (1.75)\\ &= E(X^2)-2E(X)E(X) = (EX)^2 && (1.76) \end{aligned}$$

we get another formula for VX:

$$VX = E(X^2) - (EX)^2. \qquad (1.77)$$

The variance captures how much are values of a variable spread around the expected value. The deviation just scales down the variance that may take very large values otherwise.

Example 1.9.6 *Let* X, Y *be two random variables on the sample space* $\Omega = \{1, 2, \ldots, 10\}$, *where all elementary events have the same probability, and* $X(i) = i$, $Y(i) = i-1$. *For* $i \leq 5$, $Y(i) = i+1$, *for* $i > 5$. *It is easy to check that* $EX = EY = 5.5, E(X^2) = \frac{1}{10}\sum_{i=1}^{10} i^2 = 38.5, E(Y^2) = 44.5$ *and therefore* $VX = 8.25, VY = 14.25$.

The probability density function of a random variable X whose values are natural numbers can be represented by the following **probability generating function**

$$G_X(z) = \sum_{k\geq 0} Pr(X=k)z^k.$$

Since $\sum_{k\geq 0} Pr(X=k)=1$, we get $G_X(1)=1$. Probability generating functions allow often to compute quite easily the mean and the variance. Indeed,

$$\begin{aligned} EX &= \sum_{k\geq 0} kPr(X=k) = \sum_{k\geq 0} Pr(X=k)(k1^{k-1}) && (1.78)\\ &= G'_X(1) && (1.79) \end{aligned}$$

and since

$$\begin{aligned} E(X^2) &= \sum_{k\geq 0} k^2 Pr(X=k) = \sum_{k\geq 0} Pr(X=k)(k(k-1)1^{k-2}+k1^{k-1}) && (1.80)\\ &= G''_X(1)+G'_X(1), && (1.81) \end{aligned}$$

we get from (1.77):

$$VX = G''_X(1)+G'_X(1)-G'_X(1)^2.$$

Two important distributions are connected with experiments called **Bernoulli trials**; the experiments with two possible outcomes: **success** with the probability p and **failure** with the probability $q=1-p$. Coin-tossing is an example of a Bernoulli trial experiment.

Let the random variable X be the number of trials needed to obtain a success. Then X has values in the range $\mathbf{N}$ and it clearly holds $Pr(X=k)=q^{k-1}p$. The probability distribution X on $\mathbf{N}$ with $Pr_X(k)=q^{k-1}p$ is called the **geometric distribution**.

Exercise 1.9.7 *Show that for the geometric distribution*

$$EX = \frac{1}{p}, \qquad VX = \frac{q}{p^2}. \tag{1.82}$$

Let the random variable Y express the number of successes in n trials. Then Y has values in the range $\{0,1,2,\dots n\}$ and we have

$$Pr(Y=k) = \binom{n}{k} p^k q^{n-k}.$$

The probability distribution Y on the set $\{1,2,\dots,n\}$ with $Pr(Y=k)=\binom{n}{k}p^kq^{n-k}$ is called **the binomial distribution**.

Exercise 1.9.8 *Show that for the binomial distribution*

$$EY = np \qquad VY = npq. \tag{1.83}$$

Geometric and binomial distributions are illustrated for $p=0.35$ and $n=14$ in Figure 1.8.

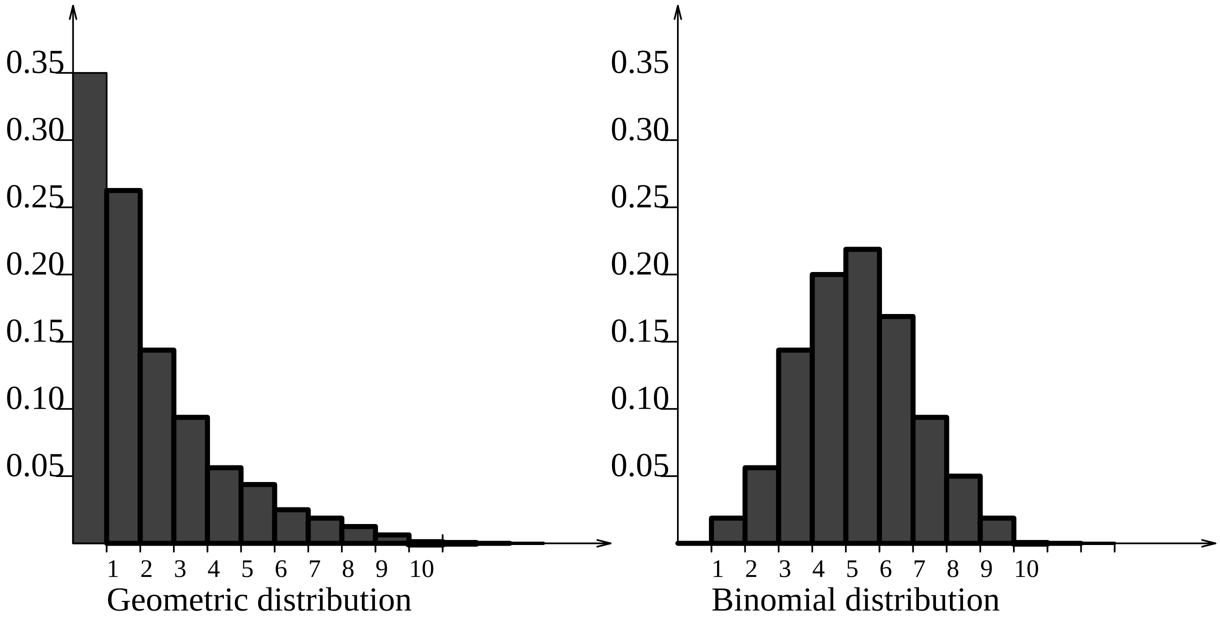

Figure 1.8: Geometric and binomial distributions.

Exercise 1.9.9 **[Balls and Bins] Consider the process of tossing randomly balls into b bins in such a way that at each toss the probability that a tossed ball falls in any given bin is* $\frac{1}{b}$. *Answer the following questions about this process.*

1. *How many balls fall on the average into a given bin at n tosses?*
2. *How many balls must one toss, on the average, until a given bin contains a ball?*
3. *How many balls must one toss, on the average, until every bin contains a ball?*

The following example illustrates a probabilistic average-case analysis of algorithms. By that we mean the following. For an algorithm $\mathcal{A}$ let $T_{\mathcal{A}}(x)$ denote the computation time of $\mathcal{A}$ for an input x and let Pr_n be, for all integer n, a probability distribution on the set of all inputs of $\mathcal{A}$ of length n. By the average-case complexity of $\mathcal{A}$—$ET_{\mathcal{A}}(w)$ we then mean the function

$$ET_{\mathcal{A}}(n) = \sum_{|x|=n} Pr_n(x) T_{\mathcal{A}}(x).$$

Example 1.9.10 *Determine the average-time complexity of the Algorithm 1.9.11 for the following problem: given an array* $X[1], X[2], \ldots, X[n]$ *of distinct elements, determine the maximal* j *such that* $X[j] = max\{X[i] \,|\, 1 \leq i \leq n\}$

Algorithm 1.9.11 (Finding the last maximum)

```
begin $j \leftarrow n; m \leftarrow X[n]$;
    for $k \leftarrow n-1$ downto 1 do if $X[k] > m$
                                  then $j \leftarrow k; m \leftarrow X[k]$
end
```

The time complexity of this algorithm for a conventional sequential computer is $T(n) = k_1 n + k_2 A + k_3$, where k_1, k_2, k_3 are constants and A equals the number of times the algorithm executes the statements $j \leftarrow k$; $m \leftarrow X[k]$. The term $k_1 n + k_2$ captures here for $n-1$ decrement and comparison operations. The value of A clearly does not depend on particular values in the array X, only on the permutation represented by the sequence $X[1], \ldots, X[n]$.

If we assume that all permutations of data in X have the same probability, then the average-time complexity of the above algorithm depends on the average value A_n of A.

Let p_{nk}, for $0 \leq k < n$, be the probability that $A = k$. Then

$$p_{nk} = \frac{\text{number of permutations of } n \text{ elements such that } A = k}{n!}$$

and the following fact clearly holds:

$$A_n = \sum_{k=0}^{n-1} k p_{nk}.$$

Our task now is to determine p_{nk}. Without loss of generality we can assume that data in the array form a permutation $x_1, \ldots, x_n$ of $\{1, 2, \ldots, n\}$ and we need to determine the value of A for such a permutation. If $x_1 = n$, then the value of A is 1 higher than that for $x_2, \ldots, x_n$—in this case $X[1] \geq m$ in the algorithm. If $x_1 \neq n$, then the value of A is the same as for $x_2, \ldots, x_n$—in this case $X[1] < m$. Since the probability that $x_1 = n$ is $\frac{1}{n}$, and the probability that $x_1 \neq n$ is $\frac{n-1}{n}$, we get the following recurrence for p_{nk}

$$p_{nk} = \frac{1}{n} p_{(n-1)(k-1)} + \frac{n-1}{n} p_{(n-1)k} \tag{1.84}$$

with the initial conditions

$$p_{10} = 1, \; p_{1k} = 0, \quad \text{for} \quad k > 0; \; p_{nk} = 0 \text{ for } k < 0.$$

In order to determine A_n let us consider the generating function

$$G_n(z) = \sum_{k=0}^{n} p_{nk} z^k.$$

Clearly $G_n(1) = 1$ and from (1.84) we get

$$G_n(z) = \frac{z}{n} G_{n-1}(z) + \frac{n-1}{n} G_{n-1}(z) = \frac{z+n-1}{n} G_{n-1}(z). \tag{1.85}$$

We know, see (1.79), that $A_n = G'_n(1)$. For $G'_n(z)$ we get from (1.85):

$$G'_n(z) = \frac{1}{n} G_{n-1}(z) + \frac{z+n-1}{n} G'_{n-1}(z)$$

and therefore

$$G'_n(1) = \frac{1}{n} + G'_{n-1}(1).$$

Thus

$$A_n = \sum_{i=2}^{n} \frac{1}{i} = H_n - 1$$

where H_n is the n-th Harmonic number. ▯

1.9.2 Bounds on Tails of Binomial Distributions*

It is especially the binomial distribution—expressing the probability of k successes during n Bernoulli trails (coin tossings)—that plays an important rôle by randomization in computing. For the analysis of randomized computing of importance is often to know how large are **tails of binomial distributions**—regions of the distribution far from the mean.

Several bounds are known for tails. For the case of one random variable X with the probability of success p and of failure $q = 1 - p$, the following basic bounds for $Pr(X \geq k)$ and for $Pr(X \leq l)$ can be derived by making careful estimations on binomial coefficients. The first two bounds are general, the last two are for tails far from the mean.

For any $0 \leq k, l \leq n$

$$Pr(X \geq k) \leq \binom{n}{k} p^k, \qquad Pr(X \leq l) \leq \binom{n}{l} q^{n-l} \tag{1.86}$$

and for $n > k > np > l > 0$:

$$Pr(X > k) < \binom{n}{k} \frac{n-k}{k-np} p^{k+1} q^{n-k} \qquad Pr(X < l) < \binom{n}{l} \frac{kq}{np-k} p^k q^{n-k+1}. \tag{1.87}$$

Also the following bound is often used:

Lemma 1.9.12 (Markov's Inequality) $Pr(X \geq kEX) \leq \frac{1}{k}$ *for any random variable* X *acquiring non-negative integer values and* $k > 0$.

Proof. The Lemma follows from the following inequality

$$E(X) = \sum_{i \in \mathbf{N}} iPr(X = i) = \sum_{i < kE(X)} iPr(X = i) + \sum_{i \geq kE(X)} iPr(X = i) \geq kE(X)Pr(X \geq kE(X)).$$

▯

In order to motivate the next bound, that will play an important rôle later, especially in Chapter 9, let us assume that we have a biased coin-tossing of one side has the probability $\frac{1}{2} + \varepsilon$ and of other side $\frac{1}{2} - \varepsilon$—but we do not know which one. How to find out this? The basic idea is simple. Toss the coin many times and pick up the side that comes most of the time as the one with probability $\frac{1}{2} + \varepsilon$. However, how many times one has to toss the coin in order to make a correct guess with a high probability?

Lemma 1.9.13 (Chernoff's Bound) *Suppose* $X_1, \dots, X_n$ *are independent random variables that acquire values 1 and 0 with probabilities* p *and* $1-p$*, respectively, and consider their sum* $X = \sum_{i=1}^{n} X_i$*. Then for all* $0 \leq \theta \leq 1$*,*

$$Pr(X \geq (1+\theta)pn) \leq e^{-\frac{\theta^2}{3}pn}.$$

Proof. Since $Pr(X \geq (1+\varepsilon)pn) = Pr(e^{tX} \geq e^{t(1+\varepsilon)pn})$, for any $t \geq o$ and $\varepsilon > 0$, Markov's Inequality yields

$$Pr(X \geq (1+\varepsilon)pn) = e^{-t(1+\varepsilon)pn} e^{t(1+\varepsilon)pn} Pr(e^{tX} \geq e^{t(1+\varepsilon)pn}) \leq e^{-t(1+\varepsilon)pn} E(e^{tX}).$$

Since $X = \sum_{i=1}^{n} X_i$ and X_i are independent, we get, making a use of the inequality $(1+a)^n \leq e^{an}$, that holds for any $a \in \mathbf{R}, n \in \mathbf{N}$,

$$E(e^{tX}) = (E(e^{tX_1}))^n = ((pe^t + (1-p))^n = (1 + p(e^t - 1))^n \leq e^{pn(e^t-1)},$$

Putting $t = \ln(1+\varepsilon)$ yields

$$Pr(X \geq (1+\varepsilon)pn) \leq (1+\varepsilon)^{-(1+\varepsilon)pn} e^{\varepsilon pn}$$

and therefore

$$Pr(X \geq (1+\varepsilon)pn) \leq \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right)^{pn}.$$

Since $\varepsilon - (1+\varepsilon)\ln(1+\varepsilon) = -\frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \ldots \leq -\frac{1}{3}\varepsilon^2$, for $0 \leq \varepsilon \leq 1$, we get the inequality **??**.
□

Exercise 1.9.14 *Show: (a)* $Pr(X \geq (1+\varepsilon)pn) \leq \left(\frac{e}{1+\varepsilon}\right)^{(1+\varepsilon)pn}$*;*
(b) $Pr(X \geq r) \leq 2^{-r}$*, for* $r \geq 6pn$*.*

Exercise 1.9.15 *Show:* $Pr(X \leq (1-\varepsilon)pn) \leq e^{-\frac{\varepsilon^2}{2}pn} \leq \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right)^{pn}$*, if* $0 \leq \varepsilon \leq 1$*.*

In other words, Chernoff's bound says that a big difference from the expected value is unlikely.

1.9.3 Randomness and Pseudo-random Generators

A **random number generator** can be seen as a stochastic process RANDOM(a,b) which, each time it is activated, produces an integer between a and b, inclusive. With each such an integer being equally likely, and with each new integer produced by the process RANDOM(a,b) as being independent from integers returned on previous calls of RANDOM(a,b). For example, RANDOM(0, 1) produces 0 or 1 both with probability $\frac{1}{2}$. One can imagine RANDOM(a, b) as rolling a perfect $(b - a + 1)$-sided dice to obtain its output.

The basic problem is how to realize such a random number generator. There are quantum-mechanical processes that produce true randomness. Of course, they are not cheap. It is therefore natural to try to use deterministic algorithms that could produce long sequences of pseudo-random bits, if they get as input a few of really random bits. That would be sufficiently good for many applications.

Intuitively, a sequence of bits

01010101010101010101010101010101010101010

is less random than the sequence

100101100010111101100010011111010001110101010111

but how to formalize this intuition? A classical way to deal with this problem is through **statistical tests** (e.g. the so-called the χ^2-test, Kolmogorov-Smirnov's test) or empirical tests on frequency, uniformity, permutations, subsequences and so on.

An infinite sequence of bits $b = (b_1, b_2, \ldots)$ is said to be random with respect to a set $\mathcal{S}$ of (statistical and other) tests of randomness if b passes all tests from $\mathcal{S}$. There are many tests available.

Some examples of tests: A random infinite sequence of bits has to be **normal**— for each integer k it has to hold that the frequency of occurrences of each particular subword of length k in the prefix of length n has to go to the limit 2^{-k} for $n \to \infty$. The number of 1's minus the number of 0's in the prefix of length n has to be positive for infinitely many n and also negative for infinitely many n.

The view of randomness through **polynomial time indistinguishability** s discussed below seems to be better suited for computing. Still another approach to randomness is discussed in Section 6.5.

Deterministic algorithms that take as an input a short random sequence and produce longer "almost random" sequence of bits are called pseudo-random generators. They play, surprisingly, as we shall see, the fundamental rôle in theory and practice of computing.

More formally: A **pseudo-random generator** is a deterministic polynomial time algorithm which expands short random sequences (called **seeds**) into longer bit sequences (each with certain probability) such that the resulting probability distribution is **in polynomial time indistinguishable** from the uniform probability distribution.

More specifically, the pseudo-random generator, say G, expands an n-bit sequence into a longer, say n^k-bit sequence, for some $k \in \mathcal{N}$, such that for every polynomial time algorithm with 0-1-output—called **test**—T, for any $\varepsilon > 0$, and for sufficiently large n

$$|Pr(T(G(X_n)) = 1) - Pr(T(X_{n^k}) = 1)| \leq n^{-\varepsilon}$$

where X_m is a random variable obtaining as values strings of length m, with uniform probability distribution. (One says also that a generator of pseudo-random numbers can **fool** any polynomial time algorithm T trying to check whether its outcomes are really random numbers. Observe also, that what we can do with nonpolynomial time algorithms is not of interest to us because they can not finish their work in a reasonable time.)

A classical proposal for a pseudo-random generator is the **linear congruential generator** from D. H. Lehmer (1948). One chooses four n-bit numbers m, a, b, X_0 and generate an n^3 bit sequence $X_1 X_2 \ldots X_{n^2}$ of n-bit numbers by the iterative process

$$X_{i+1} = (aX_i + b) \bmod m.$$

The choice of parameters m, a, b and X_0 is crucial. Although the resulting sequence $X_0, X_1, X_2, \ldots$ is eventually periodic, it is desirable that the period is as large as possible. For example, for $a = b = 7$, $m = 10$ and $X_0 = 7$ we get $7, 6, 9, 0, 7, \ldots$.[15]

Exercise 1.9.16 *Show that for the linear congruential generator:*

(a) $x_n \equiv a^n x_0 + (a^{n-1} + a^{n-2} + \cdots + a + 1) \pmod m$*;*

(b) if $gcd(a,m) = 1$*, then* $(a-1) \setminus b \Rightarrow x_n = x_{\phi(m)+n}$.

It was shown that Lehmer's pseudo-random generator does not satisfy strong requirements stated above. For example, there is a polynomial time algorithm (with respect to $\lg m$), such that given a sufficiently long sequence $X_0, X_1, \ldots X_t$ produced by a linear congruential generator, the algorithm determines a, b and m used to produce this sequence. However, there are no proofs yet but there is already strong evidence that pseudo-random generators satisfying strong requirements formulated above do exist.

The existing candidates are based on the fact that there are **intractable algorithmic problems** for which no polynomial time algorithms exist. For example, the existence of pseudo-random generators satisfying such strong requirements has been proven under the assumption that factorization of integers is intractable. One such candidate is called **Blum-Micali pseudo-random generator** or **index generator**, and it is based on the intractability of the discrete logarithm problem. It is defined as follows:

Let p be a prime such that $p - 1$ has only small factors (smaller than $\lg p$), and let g be a principal root of Z_p^*. Blum-Micali pseudo-random generator is based on the iteration

$$X_{i+1} = g^{X_i} \bmod p$$

and producing the sequence of bits

$$b_1, b_2, b_3, \ldots$$

where

$$b_i = \begin{cases} 1, & X_i \leq \frac{p-1}{2}; \\ 0, & \text{otherwise}, \end{cases}$$

satisfies the condition on polynomial time indistinguishability provided computing discrete logarithm is intractable.

---

[15] It has turned out that for 32-bit arithmetic an appropriate choice of parameters for linear congruential generator is $m = 2^{31} - 1$, $b = 0$ $a = 7^5 = 16807$.

Another example of a simple pseudo-random generator that satisfies the requirement of polynomial time indistinguishibility under the assumption that factoring of integers is intractable is the so-called BBS pseudo-random generator. It is based on the iteration

$$x_{i+1} = x_i^2 \bmod n,$$

where $x_0 \in QR_n$ is randomly chosen and n is a Blum integer, and produces the pseudo-random sequence $b_0, b_1, b_2, \ldots$, where $b_i = x_i \bmod 2$. In more detail we discuss the design of pseudo-random generators in Chapter 8.

The fact that computer time is much cheaper for producing pseudo-randomness than quantum mechanical processes to produce true randomness is not the only reason for using pseudo-random generators. We often need to repeat randomized computational processes for various reasons, including error checking. In such a case, if our source of randomness is a real one, then the only way to use random bits again is to store them. Moreover, in many applications we actually do not need truly random bits. For example, in cryptography pseudo-random bits are fully sufficient.

Randomness has become a computational resource. The question of how much and how good randomness we really need for specific tasks, is one of the key problems of computing. Also the task to economize the amount of randomness needed for particular applications is of importance. The concept of randomness presented above for sequences can be transferred to other objects. For example, an integer is said to be random if its binary representation is a random string (sequence). To generate a random integer of n bits therefore means to generate an n-bit binary sequence.

A more general random generator can be used. For example, a process can be used that for any two real numbers $a < b$, as parameters, returns a real number x randomly, uniformly and independently chosen in the interval $a \leq x \leq b$. Similarly, RANDOM(X), for a set X, is expected to return an element chosen randomly, uniformly and independently, among the elements of X. Such random number generators can often be constructed using a $RANDOM(0,1)$ generator as the basic source of randomness.

A theory of pseudo-randomness is being developed that is aimed at understanding the minimum amount of randomness that a randomized method of computation really needs.

1.9.4 Probabilistic Recurrences*

If efficiency of recursive randomized algorithms or designs is analyzed, a special type of recurrences, the so-called probabilistic recurrences, with random variables arise.

For example, consider a randomized algorithm $\mathcal{A}$ that generates, on an input x, in time $a(x)$ a subproblem of size $S(x)$, where $S(x)$ is a random variable with values in $[0, x]$, whose probability distribution depends on $\mathcal{A}$—and then solves the subproblem recursively. The time complexity of the algorithm is then described by the **(probabilistic) recurrence**:

$$T(x) = a(x) + T(S(x)). \tag{1.88}$$

$T(x)$ is therefore a random variable whose distribution depends on the distribution of $S(x)$.

The performance of such a randomized algorithm can be charaed by various statements on the distribution of the random variable T. For example, by information on the tail of the distribution of T. We discuss two such bounds: They have quite general character and are easy to apply.

Consider again the recurrence (1.88), where $E(S(x)) \leq m(x)$ for a fixed function m such that $0 \leq m(x) \leq x$ and where a is a nondecreasing function. The recurrence

$$u(x) = a(x) + u(m(x)) \tag{1.89}$$

is regarded as the deterministic counterpart of the probabilistic recurrence (1.88).

Using the iteration method to solve recurrences, we get that whenever the recurrence 1.89 has a solution, then this solution has the form $u(x) = \sum_{i \geq 0} a(m^{(i)}(x))$, where $m^{(0)}(x) = x$ and $m^{(i+1)}(x) = m(m^{(i)}(x))$ for $i \geq 0$.

We are now in the position to formulate two bounds on tails of T.

Theorem 1.9.17 (Karp's bound) *If $m(x)$ and $a(x)$ are continuous functions such that $m(x)/x$ is nondecreasing and $a(x)$ is increasing on $\{x \,|\, a(x) > 0\}$, then for every $x \in \mathbf{R}^+$ and $w \in \mathbf{N}^+$*

$$Pr[T(x) \geq u(x) + wa(x)] \leq \left(\frac{m(x)}{x}\right)^w,$$

where u is the solution of the deterministic recurrence (1.89) and T of the probabilistic recurrence (1.88).

The second bound, is weaker but easier to apply.

Theorem 1.9.18 *If m, a, u and T are as in Theorem 1.9.17, then for any sufficiently large $k \in \mathbf{N}$*

$$Pr[T(x) \geq ku(x)] \leq \left(\frac{m(x)}{x}\right)^{\frac{k}{2}-1}.$$

Proofs of both theorems are beyond the scope of this book even though the proof of the second one is based on Markov's inequality and a variation of Chernoff's bound—see references.

Example 1.9.19 (Maximal clique of the graph) *A clique of a graph G is its subgraph which is a complete graph—each two of its nodes are connected by an edge. There is no polynomial time algorithm known to find a maximal clique in a graph. However, there is a very simple randomized algorithm to do it—and this is, surprisingly, the fastest known randomized algorithm for this problem.*

Algorithm 1.9.20 *Start with the given graph G and an empty clique $\mathcal{C}$ and repeat the following step until the set of nodes of the considered graph is empty.*

- *Choose randomly a node from the current graph, add it to the clique $\mathcal{C}$, remove from the considered graph the chosen node and also all nodes that are not adjacent to it.*

In order to analyze the algorithm we need to consider its application to a random graph $G_{n,p}$ with n nodes in which each edge is presented with the probability p. At a step of algorithm, when the set of vertices has size m, the expected number of vertices that do not get deleted is $p(n-1)$. Denote by $C(n)$ the size of the clique when the algorithm is applied to $G_{n,p}$ for any fixed p. Then

$$C(n) = 1 + C(S(n)),$$

where $E(S(n)) = p(n-1) \leq pn$. The corresponding deterministic recurrence is therefore $u(n) = 1 + u(pn)$. Using methods of Section 1.2 we get $u(n) = \lg n / \lg(1/p)$. An application of Theorem **??** gives, for sufficiently large k, the bound

$$Pr[C(n) > \frac{k \lg n}{\lg(1/p)}] < p^{\frac{k}{2}-1}$$

and an application of Theorem 1.9.17 yields

$$Pr[C(n) > \frac{\lg n}{\lg(1/p)} + w] < p^{w}.$$

Exercise 1.9.21 * *If $C'(n)$ denotes the number of adjancency checks of the above randomized algorithm, then for $C'(n)$ we have the recurrence $C'(n) = n - 1 + C'(S(n))$. Get bounds for the tails of $C'(n)$.*

1.10 Asymptotic Complexity Analysis

A variety of concepts, methods and results of fundamental importance for complexity analysis were formally introduced and demonstrated in previous sections. Now we discuss informally global problems, paradigms, tools, methods, but also pitfalls of complexity investigations and interpretations. In doing so we have three main aims in mind:

1. To complement the presentation of technical tools introduced so far with a global view of complexity analysis, its aims, problems, methods and merits.
2. To introduce informally concepts vital for complexity considerations that will be formally defined and investigated later in this book, but will be used already starting with the next chapter.
3. To provide a motivation for the contents of the book and a summary of its main aims and topics.

1.10.1 Tasks of Complexity Analysis.

The main goal of complexity analysis is to determine how good are, asymptotically, the best possible solutions of important problems in computing and communication. For example, problems to design algorithms, automata, computers, networks or circuits to perform certain tasks—with respect to various computational or communication models and computational resources. In other words, to determine the inherent complexity of computational or communication problems. For example, the minimum number of arithmetical operations needed to multiply two matrices of degree n.

To establish the inherent complexity of computational problems is of large practical and theoretical importance. If it is known how good is the best possible solution, then a **benchmark** is established with which any specific solution can be compared. Moreover, it turned out that the study of inherent complexity of computational, communication, and design problems brings deep insights into the nature of computing.

There are two basic approaches how to determine the inherent complexity. First, a direct one, where the inherent complexity of a specific problem is explicitly determined. For example, $\Theta(n \lg n)$ is the inherent complexity of sorting of n numbers with respect to the number of comparisons. Second, an indirect one, in which relations between complexities of various problems are established. For example, it is shown that the complexity of one problem is not larger than of another one, or that two problems have asymptotically the same complexity. Another possibility is to show that a problem is feasible if and only if another one is such. For example, matrix multiplication has been shown to be of the same complexity, with respect to the number of arithmetical operations, as matrix inversion; integer factorization was shown, Section 1.7, feasible if and only if computing square roots of modulo an arbitrary integer is such. Methods of this chapter are of importance for both of these approaches. In order to develop the indirect approach, a deeper knowledge of the structure of complexity classes and reduction methods is needed—with this we will deal in Chapter 5.

There are two basic types of computational problems—those with complete information about input and those with incomplete or contaminated information. Problems of the first type, for example computational problems on graphs, are typical for computing with discrete objects and they are of main interest in this book. Their solutions and asymptotic complexity analyses require to use methods of discrete mathematics as discussed in this chapter. Problems of the second type, for example computing an integral of a function given by its values in certain points only, are typical for computing with continuous objects and their solutions. Asymptotical complexity analysis requires to use in such cases also deep methods of calculus. With these problems we deal not in this book. They are also of large importance for computing but would require to extend the scope of the book essentially.

There are also two basic types of computational and communication paradigms—determinism and randomization. Randomized algorithms are those that make random choices during their execution. Different methods are used for complexity analysis of deterministic and randomized systems. Deterministic algorithms are further divided onto those providing exact and those providing approximate solutions. Complexity analysis of approximation algorithms requires also special approaches and tools.

Randomization seems to be very powerful. For many problems we have fast randomized algorithms, but no fast deterministic one. Computation of square roots modulo a prime is such an example. On the other hand, we still miss proofs that polynomial time randomized computing is essentially more powerful than deterministic —this is dealt with in Chapter 5.

There are several basic types of randomized algorithms and communication protocols:

- **Las Vegas**—results are always correct, but it may happen, with small probability, that no result is produced.

- **Monte Carlo**—results of certain type are always correct; others may be wrong, but probability of that is small—we talk also about **one-sided Monte Carlo** algorithms. (For example a Monte Carlo algorithm may test primality in such a way that if it says that a given integer is composite, then this is 100% true; however, if it says that it is a prime this may be, with a small probability, wrong.) (
- **BPP (Bounded Error Probability)**—no output is 100% correct, but in any case the probability of error is small and bounded—we talk also about **two-sided Monte Carlo** algorithms.

Power and methods of randomization are discussed intensively in this book. Probabilistic finite automata are considered in Chapter 3, randomized computations on universal computers in Chapter 4, randomized complexity classes in Chapter 5, limitations of randomness in Chapter 6. An intensive use of randomness is in cryptography (Chapter 8), interactive protocols (Chapter 9) and in communications (Chapter 11). Randomization plays an important rôle also in futuristic approaches to computing. For example, quantum computing is in principle randomized.

It is natural to ask what makes randomization so powerful. The explanation is simple. It is sometimes preferable to choose, during a computation, a course of actions at random than to spent time to find out which alternative is better or to try all of them. Such situations arise when the time needed to determine the optimal choice is too large, compared to the time saved on average by making the optimal choice.

1.10.2 Methods of Complexity Analysis

The complexity analysis of computational, communication and design problems and systems depends on several factors:

Inputs — the way inputs are presented and their size measured;

Computer and communication models — models of sequential computers (Turing machines, RAM—a model of a conventional sequential computer,...), models of parallel computers (PRAM—a parallel version of RAM,...), automata, circuits, networks, ...

Computing and communication modes —deterministic, nondeterministic, randomized, ...

Complexity analysis modes — worst-case, average-case, expected-case,...

Computation resources — time, storage, program, processors, communication, randomness, ...

All these problems are considered in this book. Various representations of integers, graphs and other objects are considered in Chapter 2; several types of automata in Chapters 3 and 7; models of universal computers in Chapter 4; networks in Chapter 10, circuits in Chapters 2, 4, 5 and 11; interactive protocols in Chapter 9, communication models in Chapter 11.

Computational analysis is performed with respect to the size of input data as the main parameter. Since there are usually many inputs of the same size, two basic approaches are used to deal with this problem: to determine the so-called worst-case complexity or to determine the average-case complexity for inputs of a given size.

The worst-case complexity analysis is in general simpler. It is of main importance for critical real-time applications and it has been very successful in providing insights into the nature of computing. The worst-case complexity analysis results may be, however, quite irrelevant for some applications—or at least questionable. For example, for cryptography. This is due to the fact that the worst case may be achieved only for pathologically "bad" input data that do not normally, or often, occur—it may happen that these "bad" data are actually "nice" ones. For example, QUICKSORT performs at worst for already sorted inputs.

The average-case complexity analysis seems to be producing results that are more realistic for many applications. The difficulty here lies in the fact that one has to know the probability distribution of the input data—and this is rarely easily available. The usual assumption, namely that all data of a given size have the same probability, is often easy to deal with, but rarely realistic. Moreover, it can be shown that for an important probability distribution, the so-called apriori probability distribution - see Section 6.5.3 - the average-case complexity actually equals the worst case complexity. Finally, for some important algorithmic problems no good average-case analysis is known and for some others there is no big difference between the worst-case and the average-case results. For example, for HEAPSORT the worst-case and the average-case complexity are the same—$\Theta(n \lg n)$.

The aim is to get asymptotic analysis results in terms of Θ-notation. This is in general not easily achievable. The task is then to find the smallest possible upper bound and the largest possible lower bound. To get a better upper bound a better system, for example a faster algorithm, is needed. To improve a lower bound a better proof technique is needed.

In case of randomized algorithms we also speak of **expected-time** performance. It is the mean time of solving a problem for the same input. This is essentially different from the concept of the average-case complexity that refers to deterministic algorithms and the average peformance when all inputs are considered—each with certain probability. It is therefore meaningful to speak about the average-case expected time and the worst-case expected time.

Asymptotic complexity analysis results are robust. In order to get them one usually ignores details of the algorithm and concentrates on their loop and data structures. This robustness is the strength and but also the weakness of the asymptotic analysis. In any case, however, this robustness, coupled with the fact that asymptotically better systems are mostly also better in practice, makes the asymptotic complexity analysis so interesting and important.

1.10.3 Efficiency and Feasibility

Table 1.4 shows time needed by algorithms of different complexities to solve problems with a different sizes of inputs.[16] The table demonstrates that algorithms whose time complexity

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[16] Some numbers, according to Dyson (1979), that help to get a more real feeling how large are some numbers from Table 1.4: time to the next ice age—2^{28} sec.; the age of the universe—2^{48}sec.; total life time of the universe (if the universe is closed)—2^{61} sec.; time until all matter is liquid at zero temperature—2^{240};

grows exponentially cannot be used to solve larger problems; no matter how fast computers are available.

size of inputs / function	2^3	2^5	2^7	2^9	2^{11}
$\lg^* n$	2	2	2	3	3
$\lg n$	3	5	7	9	11
$8n$	2^6	2^8	2^{10}	2^{12}	2^{14}
$n \lg n$	$3 \cdot 2^3$	$5 \cdot 2^5$	$7 \cdot 2^7$	$9 \cdot 2^9$	11.2^{11}
n^2	2^6	2^{10}	2^{14}	2^{18}	2^{22}
2^n	2^8	2^{32}	2^{128}	2^{512}	2^{2048}
$n!$	2^{15}	2^{118}	717	2^{3876}	2^{19581}
n^n	2^{24}	2^{160}	2^{896}	2^{4608}	2^{22528}

Table 1.4: Growth of some functions—results for $n!$ are shown as 2^x, where $2^x < n! \leq 2^{x+1}$.

Unfortunately, for too many practically important problems we know only algorithms with exponential time complexity. Three of them — **travelling salesman problem, graph isomorphism problem**, and **integer factorization**, will be discussed from several points of view uter foreseeable that could find the shortest route for a traveler who wants to visit 10000 cities or to decompose a 1000 bit long composite integer. For many of these problems there is still a very large gap, actually exponential, between the best known upper bound (exponential) and lower bound (sometimes even linear).

There are several ways how to deal with the fact that for some practically very important computational problems no polynomial time sequantial algorithm—with respect to the worst-case complexity—is known.

1. To design (heuristic) algorithms that have exponential worst-case complexity but their average-case complexity is polynomial; or algorithms that have been experimentally verified to be fast on data occurring in real applications.
2. In case of optimization algorithms to develop, if possible, fast and good approximation algorithm. This is an intriguing problem discussed in Chapter 5.
3. To develop fast randomized algorithms, provided they exist. Computational hardness can often be trade-off for randomness and randomness becomes precious computational resource.

The development of an understanding when to consider an algorithmic problem or an algorithm as tractable (feasible) is one of central tasks of foundations of computing discuss intensively in Chapters 4, 5, 9. There is a quite good consensus that a problem should be called tractable if it can be solved in polynomial time. However, views differ which computational modes to consider: deterministic, randomized, interactive,... All that may provide very substantial difference, as shown in Chapter 9.

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number of atoms in universe (black matter excluded)—2^{265}; volume of the universe 2^{280} cm^3; time until all matter collapses to black holes—$10^{10^{76}}$.

It is now undoubtedly that consideration of sequential deterministic polynomial time computations as the ones representing feasible computations has been very successful with respect to the development of efficient computing systems and that it has also brought deep insights into the nature of computing. On the other hand, this standpoint is not without serious problems. Why to consider algorithms time complexity of which is proportional to a very high degree polynomial as feasible? For example, one can certainly hardly consider as practical an algorithm time complexity of which is n^{10000}. Fortunately, it has turned out that if there is a polynomial time algorithm for a reasonable problem that has a simple, understandable formulation, then the degree of such a polynomial is reasonable small.

A polynomial time algorithm is in general considered as an acceptable solution of a problem. However, as really fast we consider only algorithms of complexity $O(n)$ or $O(n \lg n)$, in case of sequential computing, and of complexity $O(\lg n)$ or $O(\lg^2 n)$ for parallel computing.

Algorithms to solve a problem may differ dramatically in their efficiency. This is clear for the case that there is an exponential time and also a polynomial time algorithm for the same problem. It seems to be less obvious that even apparently very small time complexity differences, for example $\Theta(n^2)$ and $\Theta(n \lg n)$, between algorithms, can produce practically quite dramatic difference.

Assume, for example, that an $\Theta(n^2)$ algorithm A_1 has been cleverly programmed for a supercomputer, with 10^8 operations per second, to run in time $T(n) = 2n^2$, and that another algorithm, A_2, for the same problem, with the asymptotic complexity $\Theta(n \lg n)$, is badly programmed for a PC, with 10^6 operations per second, to run in time $50n \log n$. For $n = 10^6$, A_1 needs 2000 seconds and A_2 needs 100 seconds!

We see that even small asymptotic differences in time performance of algorithms may have a larger practical impact than the difference between performance of a personal computer and supercomputer. Observe also that in the above example, even supercomputers with $\Theta(n^2)$ algorithm could not beat a PC with a $\Theta(n \lg n)$ algorithm. Algorithms, networks,... should therefore be seen as an important technology and progress in this technology can be matched with that in hardware. Asymptotic complexity analysis with its benchmarks and performance evaluation is an important factor contributing to the fast development of this technology.

1.10.4 Complexity Classes and Complete Problems

One of the important tasks of foundation of computing is to develop understanding of the power and structure of main computational and communication complexity classes. What kind of problems can be solved using a particular computing model and a specific amount of computational resources? Which relations hold between various complexity classes and what kind of structure do these classes have?

Table 1.5 summarizes main resource bounds used in complexity analysis and the following list contains main complexity classes.

NC Class of problems solvable by circuits of polynomial size and polylogarithmic depth; or, alternatively, the class of problems solvable by PRAM in polynomial time with polylog number of processors.

constant	$\mathcal{O}(1)$
almost constant	$\mathcal{O}(\lg^* n)$
polylog	$\mathcal{O}(\lg^{\mathcal{O}(1)})$
subpolynomial	$n^{o(1)}$
polynomial	$n^{\mathcal{O}(1)}$
superpolynomial	$n^{polylog}$
subexponential	$2^{n^{o(1)}}$
exponential	$2^{n^{\mathcal{O}(1)}}$

Table 1.5: Main resource bounds.

P Class of problems solvable in polynomial time by a sequential machine (RAM, Turing machine,...).[17]

NP Class of problems that have solutions verifiable in polynomial time on sequential computers.

PSPACE Class of problems solvable in polynomial space by a sequential computer or in polynomial time by a parallel computer or by an interactive protocol.

One of the main outcomes of foundations of computing is that all above listed complexity classes are very robust and to a large degree independent on details of a specific computer model. Robustness of these concepts also justifies their use in this book even before they are formally introduced.

Another robust and key concept is that of **complete problems**—for a complexity class and a type of reductions. Informally, complete problems of a complexity class are the hardest problems of that class. All other problems in that class can be reduced to them using an "easy" reduction. The so-called **NP**-complete problems are of a special importance. They are those problems in **NP** that any other problem in **NP** can be reduced to them in polynomial time. Several thousands of (important) algorithmic problems have already been identified to be **NP**-complete. For none of these problems a polynomial time deterministic algorithm is known, and also a proof that no such algorithm exists is missing. Their importance lies especially in the fact that finding a polynomial time deterministic algorithm for one of them would provide polynomial time deterministic algorithms for all of them.

1.10.5 Pitfalls

Asymptotic complexity analysis has many pitfalls and it is important to be aware of them. Let us discuss some of them.

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[17] The distinguished rôle of polynomial time algorithms is underscored by the fact that some natural syntactical conditions can be imposed on algorithms that are equivalent to the requirement that they run in polynomial time. For example, an algorithm runs in a polynomial time if and only if it can be programmed in Pascal without constructs **goto**, **repeat** and **while** and with upper bounds in the size of input for **for** cycles.

1. As already stated, the real meaning of complexity analysis results presented in terms of $\mathcal{O}$-, o-, Ω-, ω-, Θ-notation may depend much on the way inputs are presented, their size counted, a computing model used, complexity measures considered,… . These dependences are not always clearly stated in papers and books.

For example, it is usually stated that the time complexity of MERGESORT is $O(n \lg n)$. This is true in the sense that there is a constant c such that for any n elements drawn from a totally ordered set, at most $cn \lg n$ comparisons are needed to produce a sorted sequence. This means that nothing else is counted except the number of comparisons; each comparison is considered to take one time unit. Other time consuming operations, as transportation of elements, are ignored as well as the size of elements.

2. Constant factors, that are for good reasons ignored by the asymptotic complexity analysis, may be sometimes prohibitively large. Because of that it may turn out that an algorithm with larger asymptotic complexity is more efficient for practical purposes.

3. A care is also needed by the interpretation of complexity results in cases when the complexity depends, apart from the size of input, also on particular input data. For example, for complexity $T(n)$ of INSERTSORT we get $T(n) = \Omega(n)$ and $T(n) = O(n^2)$. Neither the lower bound nor the upper bound can be improved, because there are inputs requiring quadratic time and inputs for which linear time is enough. However, if the worst-case complexity is considered, then $T_w(n) = \Theta(n^2)$.

4. There is often a confusion between $\mathcal{O}$-notation and Θ-notation; $\mathcal{O}$-notation is often used where actually Θ-notation is more appropriate because the indicated upper bound is asymptotically also a lower bound.

Moral. The analysis of even simple computing systems can be a challenge. Complexity analysis is an art, science and an engineering task. A good rule of thumb by complexity analysis is therefore, as in life, to use common sense, exercise good taste, and to listen to your conscience.

1.11 Exercises

1. Denote by u_n the number of binary strings of length n that do not contain two consecutive 0's. We have $u_1 = 3, u_2 = 8, u_3 = 22$. Construct a recurrence and determine u_n, for any n.

2. Find a recurrence relation, including the initial conditions, for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

3. Find a recurrence relation, including the initial conditions, for the number of binary strings of length n containing a pair of consecutive 0's.

4. Find a recurrence for the number of bit strings of length n containing neither three consecutive 0's nor three consecutive 1's.

5. Express $T(n)$ from the recurrence:

$$T(1) = a, \qquad T(n) = kT(n/2) + g(n)$$

as a function of g—for example, by a reduction to the problem in Exercise 1.2.10 in the text.

6. Find a recurrence for the number of distinct binary trees with n nodes.

7. Solve the recurrence: $u_1 = 1, u_n = au_{n/b} + d(n)$, where $d(n)$ is a multiplicative function and b is a constant, i.e., $d(xy) = d(x)d(y)$ for any x, y—as a consequence: $d(b^i) = (d(b))^i$ for any $i \in \mathbf{N}$.

8. Solve the recurrences with $u_1 = 1$ and for every $n > 1$: (a) $u_n = 4u_{n/2} + n$; (b) $u_n = 4u_{n/2} + n^2$; (c) $u_n = 4u_{n/2} + n^3$.

9. Solve the recurrence: $u_0 = a, u_n = 3\sum_{k=0}^{n-1} u_k + 1$ for $n \geq 1$. (There is a very easy solution.)

10. Show the identities: (a) $F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}$; (b) $F_{2n} = \sum_{k=0}^{n} F_k$.

11. Show the following properties of Fibonacci numbers:

    (a) $F_{n+m} = F_m F_{n+1} + F_{m+1} F_n$; (b) $F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}$;

    (c) $F_{2n} = \sum_{k=1}^{n} F_{2k-1}$; (d) $\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$.

12. The **Lucas numbers** satisfy the recurrence: $L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$. (a) Determine L_n; (b) show that $L_{n+1} = F_n + F_{n+1}$; (c) show that $\sum_{i=0}^{n} L_i^2 = L_n L_{n+1} + 2$.

13. Solve the recurrence: $u_1 = a, u_{2n} = 2u_n + 3, u_{2n+1} = 2u_n + 8$ for $n > 1$.

14. * Show that for the recurrence

    $$u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k} + b^n p(n), \tag{1.90}$$

    where $b, a_1, \ldots, a_k$ are constants and $p(n)$ is a polynomial of degree d, the corresponding characteristic equation has the form:

    $$(x^k - a_1 x^{k-1} - \cdots - a_{k-1} x - a_k)(x - b)^{d+1} = 0$$

    and once this equation is solved we can find the solution of the recurrence (1.90) in a similar way as in the homogeneous case (when $p(n) = 0$).

15. Solve the recurrence:

    (a) $u_1 = 1, u_n = u_{n-1} + 4^n$ for $n > 1$; (b) $u_1 = 1, u_n = 3u_{n-1} + n$, for $n > 1$.

16. Show that $\lceil \frac{n}{m} \rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor$ for each $m \in \mathbf{N}^+$ and $n \in \mathbf{N}$.

17. Show, for any integer n and reals $a, b > 0$: (a) $\lceil \lceil n/a \rceil / b \rceil = \lceil n/ab \rceil$;

    (b)$\lfloor \lfloor n/a \rfloor / b \rfloor = \lfloor n/ab \rfloor$.

18. Let $f : R \to R$ be a monotonically increasing function such that $0 < f(x) < x$, for $x > 0$. Define $f^{(0)}(x) = x, f^{(i+1)}(x) = f(f^{(i)})(x)$ and for a $c > 0$ $f_c^*(x) = \min\{i \geq 0, f^{(i)}(x) < c\}$. Determine $f_c^*(x)$ for: (a) $f(x) = \frac{x}{2}, c = 1$; (b)$f(x) = \sqrt{x}, c = 1$.

19. ** Show the inequalities: (a) $n! > ((\frac{n}{2})!)^2$, if n is even; (b) $n! \geq (\frac{n+1}{2})!(\frac{n-1}{2})!$, if n is odd; (c) $(\frac{n}{3})^n < n! < (\frac{n}{2})^n$, for all $n \geq 6$.

20. Show the following identities, where n, k are integers and r is a real:
    (a) $\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}$; (b) $\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$;
    (c) $\sum_{k=0}^{n} (-1) k \binom{n}{k} = 0, n \geq 1$.

21. * Show the following identities: (a) $\binom{n}{r-1} = \binom{n+2}{r+1} - 2\binom{n+1}{r+1} + \binom{n}{r+1}$; (b) $\sum_{j=1}^{n} \binom{j}{2} = \binom{n+1}{3}$.

22. * Prove the following identities for binomial coefficients: (a) $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$;
    (b)$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.

23. Show the inequalities: (a) $\binom{k}{l} \leq \left(\frac{k \cdot e}{l}\right)^l$; (b) $\binom{b}{r}\left(\frac{1}{n}\right)^r \leq (\frac{e}{r})^r$, where e is the base of natural logarithms.

24. Find the generating function for the sequence $\langle F_{2i} \rangle_{i=0}^{\infty}$.

25. * Show, e.g., by expressing the generating function for Fibonacci numbers properly, that
$$\sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5}(2nF_{n+1} - (n+1)F_n).$$

26. In how many ways can one tile a 2 × n rectangle with 2 × 1 and 2 × 2 “dominoes”?

27. Use the concept of generating functions to solve the following problem: determine the number of ways one can pay n dollars with dimes, quarters and half dollars.

28. Show: (a) $\sqrt{4 + \sqrt{3x}} = \Theta(x^{\frac{1}{2}})$; (b) $(1 + \frac{2}{x})^x = \Theta(1)$.

29. Show: (a) $x^2 + 3x \sim x^2$; (b) $\sin \frac{1}{x} \sim \frac{1}{x}$.

30. Find two functions $f(n)$ and $g(n)$ such that neither of the relations $f(n) = \mathcal{O}(g(n))$ nor $g(n) = \mathcal{O}(f(n))$ hold.

31. Find an $\mathcal{O}$-estimation for $\sum_{j=1}^{n} j(j+1)(j+2)$.

32. Show that if $f(n) = a, f(n) = cf(n-1) + p(n)$ for $n > 1$, where $p(n)$ is a polynomial and c is a constant, then $f(n) = \Theta(n)$.

33. Show $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$.

34. Which of the following statements are true? (a) $(n^2 + 2n + 2)^3 \sim n^6$; (b) $n^3(\lg\lg n)^2 = o(n^3 \lg n)$; (c) $\sin x = \Omega(1)$; (d) $\sqrt{\lg n + 2} = \Omega(\lg\lg n)$.

35. Find a function f such that $f(x) = \mathcal{O}(x^{1+\varepsilon})$ is true for every $\varepsilon > 0$ but $f(x) = \mathcal{O}(x)$ is not true.

36. * Order the following functions according to their asymptotic growths, i.e., find an ordering $f_1(n), ..., f_{35}(n)$ of the following functions such that $f_{i+1}(n) = \Omega(f_i(n))$.

$$\begin{array}{ccccc}
\lg(\lg^* n) & 2^{\lg^* n} & F_n & (\sqrt{2})^{\lg n} & n^2 \\
n! & (\lg n)!(4/3)^n & n^3 & 2^{\lg^8 n} & e^{\lg n^3} \\
\lg^2 n & \lg(n!) & 2^{2^n} & n^{1/\ln n} & \ln\ln n \\
\lg^* n & n \cdot 2^n & n^{\lg\lg n} & \ln n & H_n \\
1 & \pi(n) & 2^{\lg n} & (\ln n)^{\ln n} & e^n \\
8^{\lg n} & (n+1)! & \sqrt{\ln n} & \lg^*(\lg n) & \lg^* \lg^* n \\
2^{\sqrt{3}\lg n} & n & 2^n & n\lg n & 2^{2^{n+2}}
\end{array}$$

37. Suppose that $f(x) = \mathcal{O}(g(x))$. Does it imply that (a) $2^{f(x)} = \mathcal{O}(2^{g(x)})$; (b) $\lg f(x) = \mathcal{O}(\lg g(x))$; (c) $f^k(x) = \mathcal{O}(g^k(x))$, for $k \in \mathbf{N}$?

38. Show that if $f_1(x) = o(g(x)), f_2(x) = o(g(x))$, then $f_1(x) + f_2(x) = o(g(x))$.

39. Show: (a) $\sin x = o(x)$; (b) $\frac{1}{x} = o(1)$; (c) $100 \lg x = o(x^{0.3})$.

40. Show that: (a) $\frac{1}{1+x^2} = \mathcal{O}(1)$; (b) $\frac{1}{1+x^2} = o(1)$.

41. Does $f(x) = o(g(x))$ imply $2^{f(x)} = o(2^{g(x)})$?

42. Show that $f(x) = o(g(x)) \Rightarrow f(x) = \mathcal{O}(g(x))$ but not necessarily $f(x) = \mathcal{O}(g(x)) \Rightarrow f(x) = o(g(x))$.

43. Show that if $f_1(x) = \mathcal{O}(g(x)), f_2(x) = o(g(x))$, then $f_1(x) + f_2(x) = \mathcal{O}(g(x))$.

44. Show that $o(g(n)) \cap \omega(g(n))$ is the empty set for any function $g(n)$.

45. What is wrong with the following deduction? Let $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$, $T(1) = 0$. We assume inductively that $T(\lfloor \frac{n}{2} \rfloor) = \mathcal{O}(\lfloor \frac{n}{2} \rfloor)$ and $T(\lfloor \frac{n}{2} \rfloor) \leq c\lfloor \frac{n}{2} \rfloor$. Then $T(n) \leq 2c\lfloor \frac{n}{2} \rfloor + n \leq (c+1)n = \mathcal{O}(n)$?

46. Solve recurrences:

    (a) $T(1) = a, T(n) = 2T(n/2) + n \lg n$;

    (b) $T(1) = a, T(n) = 7T(n/2) + \mathcal{O}(n^2)$.

47. Let $T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \lg n$. Show (i.e., using the substitution $n = 2^m$), that $T(n) = \mathcal{O}(\lg n \lg\lg n)$.

48. Show that $u_n = \mathcal{O}(n!)$ if u_n is defined by recurrence $u_n = 1, u_n = nu_{n-1} + bn^2$. Can you find a better estimation for u_n?

49. Show that if $n > m$, then $n \bmod m < \frac{n}{2}$.

50. Show that $d \setminus n \Rightarrow F_d \setminus F_n$

51. Compute the greatest common divisor for pairs of numbers: (a) $(325, 53)$; (b) $(2002, 2339)$; (c) $(3457, 4669)$; (d) $(143, 1326)$; (e) $(585, 3660)$.

52. Express the greatest common divisor of each of the following pairs of integers as a linear combination of these integers: (a) (117, 213); (b) (3454, 4666); (c) (21, 55); (d) (10001, 13422); (e) (10233, 33341).

53. Show, by induction, that the number of steps to compute $\gcd(n, m), n > m$ is smaller than $\log_r n$ where $r = (1 + \sqrt{5})/2$.

54. Find a prime n such that $2^n - 1$ is not prime.

55. Show that the n-th prime is smaller than $2^{2^{n+1}}$.

56. Show that an integer n is prime iff there exists an a $(0 < a < n)$ such that $a^{n-1} \equiv 1 \pmod{n}$, and for each prime q dividing $n - 1$ we have $a^{\frac{n-1}{q}} \not\equiv 1(\mathrm{mod} n)$.

57. (Wilson's prime number test.) Show that $(p - 1)! \equiv -1 \bmod p$ iff p is a prime.

58. An integer is called perfect if it equals the sum of its proper divisors. (For example 6 and 28 are perfect integers.) (a) Find the first 10 perfect numbers; (b) Show that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect.

59. Compute; (a) $5^{11}3^{14} \bmod 26$; (b) $4^{80} \bmod 65$; (c) $3^{100} \bmod 79$; $3^{1000} \bmod 17$; (d) $5^{11}3^{14} \bmod 26$; (e) $4^{80} \bmod 65$; (f) $2^{34} \bmod 341$.

60. Show that if $a, b \in \mathbf{N}^+$, then: (a) $(2^a - 1) \bmod (2^b - 1) = 2^{a \bmod b} - 1$; (b) $gcd(2^a - 1, 2^b - 1) = 2^{gcd(a,b)} - 1$.

61. Compute: $(\sum_{i=1}^{100} i!) \bmod 12$; (b) $(\sum_{i=1}^{90} i^5) \bmod 4$.

62. Solve congruences (a) $32x \equiv 6 \pmod{70}$; (b) $7x \equiv 3 \pmod{24}$;

    (c) $32x \equiv 1 \pmod{45}$; (d) $14x \equiv 5 \pmod{54}$.

63. Determine inverses: (a) $4^{-1} \bmod 9$; (b) $7^{-1} \bmod 17$; (c) $21^{-1} \bmod 143$.

64. Show that if a is odd, then $a^{2^{n-2}} \equiv 1 \pmod{2^n}$ for each $n \geq 3$.

65. Let n be an integer, $x < 2^n$ a prime, $y < 2^n$ a composite number. Show, i.e., by using the Chinese Remainder Theorem, that there is an integer $p < 2n$ such that $x \not\equiv y \bmod p$.

66. Design the multiplication table for (a) $\mathbf{Z}_9^*$; (b) $\mathbf{Z}_{11}^*$.

67. Let $p > 2$ be a prime and g a principal root of $\mathbf{Z}_p^*$. Then, for any $x \in \mathbf{Z}_p^*, x \in QR_p \Leftrightarrow index_{p,g}(x)$ is even.

68. If p is a prime, $e \in \mathbf{N}$, then the equation $x^2 \equiv 1 \pmod{p^e}$ has only two solutions: $x = 1$ and $x = -1$.

69. Show that if g is a principal root of Z_n^*, then the equality $g^x \equiv g^y \pmod{n}$ holds iff $x \equiv y \pmod{\phi(n)}$.

70. Show, for any constant a, b and a random variable X: (a) $E(aX + b) = aEX + b$; (b) $V(aX + b) = a^2VX$.

71. (Variance Theorem.) Show, for random variables X, Y and reals a, b: (a) $V(aX+bY) = a^2VX + b^2VY + 2abE((X - EX)(Y - EY))$; (b) $V(X + Y) = VX + VY$.

72. Find the probability that a family of four children does not have a girl if the sexes of children are independent and if (a) the probability of a girl is 51%; (b) boys and girls are equally likely.

73. Find the following probabilities when n independent Bernoulli's trials are carried out with probability p of success: (a) the probability of no failure; (b) of at least one failure; (c) of at most one failure; (d) of at most two failures.

74. Determine the probability that exactly seven 0's are generated when 10 bits are generated and the probability that 0 is generated is 0.8, the probability that 1 is generated is 0.2 and bits are generated independently.

75. (Birthday paradox) Birthdays are important days. (a) Determine the probability that in a group of n persons there are at least two with the same birthday (assume that all 366 days are equally likely as birthdays; (b) How many people are needed to make the probability that two people have the same birthday greater than 1/2.

76. There are many variants of the Chernoff's bound. Show correctness of the following one: If $X_1, \ldots, X_n$ are independent random variables and $Pr(X_i = 1) = p$, for all $1 \leq i \leq n$, then $Pr(\sum_{i=1}^{n} X_i \geq pn + a) \leq e^{\frac{-2a^2}{n}}$.

77. Show **Chebyshev's inequality**: If X is a random variable, then $Pr((X - EX) > \alpha) < \frac{VX}{\alpha^2}$, for all $\alpha > 0$.

78. Let $p(x_1, \ldots, p_n)$ be a polynomial with rational coefficients of the degree d and not identically 0. Show that $p(\alpha) \neq 0$ at least for half of all vectors $\alpha \in \{-nd, \ldots, 0, \ldots, nd\}^n$.

79. Show, for the linear congruential generator $x_{i+1} = (ax_i) \bmod m$, that if m is prime and a choice of a yields the maximum period (of $m - 1$), than also a^k yields the maximum period provided $k < m$ and $(m - 1)$ is not divisible by k.

80. Suppose that $m, n \in \mathbf{N}^+$. What is the probability that randomly chosen positive integer smaller than mn is not divisible by either m or n?

81. Show **Chebyshev's inequality**: If X is a random variable, then $Pr((X - EX) > \alpha) < \frac{VX}{\alpha^2}$, for all $\alpha > 0$.

82. * There are many variants of Chernoff's bound. Show correctness of the following one: If $X_1, \ldots, X_n$ are independent random variables and $Pr(X_i = 1) = p$, for all $1 \leq i \leq n$, then $Pr(\sum_{i=1}^{n} X_i \geq pn + a) \leq e^{\frac{-2a^2}{n}}$.

83. Show, for the iteration $x_{i+1} = (ax_i + b) \bmod m, i \geq 0$: if $a \perp m$ then x_0 repeats periodically in $x_1, x_2, \ldots$.

84. Show, for the linear congruential generator and $X'_n = X_{n+1} - X_n$:
(a) $X'_n \equiv a^{n-1}X'_1 \pmod{m}$; (b) $X'_n = X'_{\phi(m)+n}$.

QUESTIONS

1. Is F_n larger than $\left(\frac{1+\sqrt{5}}{2}\right)^n$?
2. Is it true that $3^{n+2} = \mathcal{O}(3^n)$ and $3^{2n} = \mathcal{O}(3^n)$?
3. We know that $n! \prec n^n$. Which of the relations $n^n \prec (n!)^2$ and $(n!)^2 \prec n^n$ holds?
4. Are the functions $\lceil \lg n \rceil!$ and $\lceil \lg \lg n \rceil!$ polynomially bounded?
5. Is it true that $\Theta(f(n) + g(n)) = \Theta(\max\{f(n), g(n)\})$?
6. How many primes do we have of a given length n?
7. How do you interpret the equation $3n^3 + \Theta(n^2) = \Theta(n^3)$?
8. Which properties have Blum integers?
9. In which cases is it better to use pseudo-random generators than a true random generator?
10. Why are constant factors at the analysis of algorithms not of importance and why they are of importance?

1.12 Historical and Bibliographical References

The history of basic concepts, methods and results used in the analysis of algorithms is deeply rooted in the history of mathematics. The most important contribution for their introduction and further development for computing was made by Donald E. Knuth—who received Turing award for 1974. His monumental work and pioneering books [Knuth (1968); Knuth (1969); Knuth (1973); Greene and Knuth (1981); Graham, Knuth and Patashnik (1989)] are also the main sources of detailed historical and bibliographical references. Moreover, the last of these books represents as a textbook a masterpiece of hardly matched quality in computing.

Concrete Mathematics of Graham, Knuth and Patashnik (1989) is not only the main textbook to recommend for a more detailed and deeper treatment of the subject but it has also influenced the choice of material and presentation for a large portion of this chapter. Additional books used and to be recommended are by Kranakis (1986); Harel (1987); Brassard and Brattey (1988); Cormen, Leiserson and Rivest (1990); Aigner (1993); Arnold and Guessarian (1996). An extensive introduction to the analysis of algorithms is due to Sedgewick and Flayolet (1996) and for more advanced tools see a comprehensive book of Hofri (1995).

With layouts of graphs and especially trees we deal in some detail in Section 10.6. The Towers of Hanoi Problem and its legend were introduced by Lucas (1894), according to

Graham, Knuth and Patashnik (1989), where the problem and several of its modifications are analyzed. Its generalization to 4 rods is considered by Gedeon (1992). Methods to solve recurrences are discussed by Bentley, Haken and Saxe (1978), Luecker (1980), Greene and Knuth (1981), Aho, Hopcroft and Ullman (1983), Purdon and Brown (1985), Brassard and Brattey (1988), Graham, Knuth and Patashnik (1989), Cormen, Leiserson and Rivest (1990), Aigner (1993) and Arnold and Guessarian (1996).

The generating function concept has been intensively used since L. Euler, 1741, and L. Laplace, 1812—their application to solving recurrences is elaborated by Graham, Knuth and Patashnik (1989). From there also the domino problem and the bootstraping method come. A detailed proof of Theorem 1.4.11 can be found in Aigner (1993). A survey on $3x + 1$ problem is due to Lagarias (1985).

Names and notation for the ceiling and floor functions are due to Iverson (1960); Lemma 1.3.1 is due to Graham, Knuth and Patashnik (1989), where also binomial coefficients are discussed in detail.

The amazing story of primes and records on primes is nicely presented in the book by Ribenboim (1988) and in the expository paper by Zagier (1977). Estimations for $\pi(n)$ can be found in Rosser and Schoenfeld (1962). Prime Number Theorem is due to Dirichlet, Hadamard and de la Vallée Poussin; a proof can be found in Landau (1953).

Quadratic residues, computing of discrete square roots, discrete logarithms and their applications in pseudo-random generators and in cryptography are well presented by Kranakis (1986)—Section 1.7 much profited from it and also the proof of Theorem 1.8.5 can be found there. A polynomial time algorithm for computing square roots modulo prime was first designed by Berlekamp (1970). An elegant Las Vegas algorithm to compute square roots modulo a prime is from Karp (1991) and is credited to Lehmer (1951). A polynomial time algorithm for computing discrete logarithm modulo a prime p such that $p - 1$ has only small factors is due to Pohlig and Hellman (1978). Theorem 1.8.15 is, as many basic results in foundations of computing, due to Michael Rabin (1979)—he received the Turing award in 1976. Gauss' characterizations of those n that Z_N^* is cyclic can be found in Kranakis (1986). For a survey of algorithmic number theory see Adleman (1994).

There are many excellent books on probability; for example by Feller (1964) and Billingsley (1986). Execise 1.9.5 follows Motwavi, Raghavan (1995). Classical approaches to pseudo-random generators are treated exhaustively by Knuth (1969) and the complexity based approaches are well presented in Kranakis (1986). For proofs of inequalities (1.86) and (1.87) see, for example, Cormen, Rivest and Leiserson (1990). For Chernoff's bounds see Hagerup and Rüb (1989).The concept of polynomial time indistinguishibility is due to Yao (1982) and Blum and Micali (1982). The BBS pseudo-random generator is due to Blum, Blum and Shub (1986); Blum-Micali pseudo-random generator was introduced by Blum and Micali (1982). The result on predictability of linear congruential generator is due to Plumstead (1982). Methods to solve probabilistic recurrences have been worked out by Richard M. Karp—Turing award for 1985—who has contributed significantly to the development of randomized computing. Theorem 1.9.17 is due to Karp (1991) and Theorem 1.9.18 is due to Chaudhuri and Dubhashi (1995).

D. Harel's Algorithmics (1987) is another masterpiece in providing not too formal but deep insights into computing including complexity analysis and an excellent reference for additional insights about aims, methods, results and merits of complexity analysis discussed in Section 1.10.

Chapter 2

FOUNDATIONS

INTRODUCTION

Foundations of computing are deeply rooted in mathematics and fluency in the basic mathematical culture is a must for the computer science. In addition, computing itself has strong impact on this culture. The chapter serves therefore several aims. First, to summarize in a concise way foundational concepts of mathematics needed in computing and to set up terminology and notational conventions—it therefore serves as a reference source. Secondly, and mainly, it presents, illustrates, and discusses new paradigms computing brought into these foundational areas of mathematics that are of key importance also for computing. In doing so some basic problems are introduced with which this book deals with as well as basic method to solve them. This way the chapter serves as a motivation and introduction to various topics of the book. Thirdly, special concepts and methods to be used later are introduced and illustrated.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. basic concepts, terminology, and notations concerning sets, relations, functions, graphs, languages, and algebras;
2. basic methods of representation of sets, relations, and graphs;
3. new paradigms the computing brought into the foundational areas of mathematics, that are of importance for science in general, and computing science in particular;
4. several methods to visualize words and languages and to represent pictures and images by words and languages.

> If in other sciences we should arrive at certainty without doubt and truth without error, it behooves us to place the foundations of knowledge on mathematics.
>
> *Roger Bacon*[1] *(1214-1294)*

Basic mathematical structures, as sets, relations, functions, graphs, algebras and formal languages, play an important rôle in foundations of any mature science. This alone is a sufficient reason to summarize basic concepts and results of these areas, as well as to fix terminology and notation, in an introductory chapter of a book on foundations of computing. However, the aim of this chapter goes further. Foundations of computing bring also into these foundational areas of modern science new deep paradigms, concepts, methods and problems that are then, in turn, of large importance for the whole computing. The second aim of this chapter is therefore to point out, motivate and illustrate these new viewpoints, concepts, problems and methods. At the same time this provides basic motivations for many subjects the book deals with. Complexity standpoints and asymptotic estimations play an unreplaceable rôle here—they provide guidance and evaluations. This is why this chapter follows, and not precedes, that on fundamentals of complexity analysis and randomness.

2.1 Sets-November 8, 2005

The simplest and at the same time perhaps the most complex and richest mathematical objects are sets. Their position and rôle in computing is the same. They are fundamental (discrete) structures on which other (discrete) structures are built. Many key problems of foundation of computing can be seen as problems concerning sets.

2.1.1 Basic Concepts

A **set** is a collection of distinguishable objects (elements). The **empty set** $\emptyset$ has no objects. If A is a set, then $x \in A$ ($x \notin A$) denotes that x is (is not) an object of A. Notation $A \subseteq B$ expresses that the set A is a **subset** of B—each object of A is in B. Notation $A \subset B$ (or $A \subsetneq B$) means that A is a **proper subset** of B. Sets A, B are **equal**, notation $A = B$, if $A \subseteq B$ and $B \subseteq A$. Two sets are said to be **disjoint** if no object is in both of them. If elements of a set $\mathcal{A}$ are sets we use also term a **family of sets** for $\mathcal{A}$. Notation $x \in_{\mathcal{D}} A$ starts to be used to denote that x is an element of the set A randomly chosen with respect to the probability distribution $\mathcal{D}$. Notation $x \in_{\mathcal{U}} A$ is reserved for choosing an element of A with respect to the uniform distribution.

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[1] An English philosopher and natural scientists,considered sometimes as the first theoretician of experimental sciences.

Exercise 2.1.1 *Find two sets* A, B *such that* $A \in B$ *and* $A \subseteq B$.

It is often assumed that all sets we deal with are subsets of a **universal set** U. With each $A \subseteq U$ we can then associate a **characteristic function** (or a **membership predicate**) $f_A : U \to \{0,1\}$, defined by $f_A(x) = [x \in A]$.

Binary operations on sets:

$$
\begin{array}{rcl}
\textbf{union } A \cup B & = & \{x \mid x \in A \text{ or } x \in B\}; \\
\textbf{intersection } A \cap B & = & \{x \mid x \in A \text{ and } x \in B\}; \\
\textbf{difference } A - B & = & \{x \mid x \in A \text{ and } x \notin B\}; \\
\textbf{symmetric difference} A = B \bigtriangleup C & = & A - B \cup B - a; \\
\textbf{Cartesian product } A \times B & = & \{(a,b) \mid a \in A, b \in B\}.
\end{array}
$$

The concept of the Cartesian product can be generalized, in a natural way, to more than two sets. Moreover, since the union and intersection are commutative and associative operations, we consider also

$$
\begin{array}{rl}
\textbf{generalized union} & \bigcup_{i \in I} A_i = \{x \mid \exists i \in I, x \in A_i\}; \\
\textbf{generalized intersection} & \bigcap_{i \in I} A_i = \{x \mid x \in A_i \forall i \in I\}; \\
\textbf{generalized Cartesian product} & \bigotimes_{i=1}^{n} A_i = \{x \mid x = (x_1, \ldots, x_n), x_i \in A_i\}
\end{array}
$$

where $I \neq \emptyset$ is an index set. For a Cartesian product of a set A with itself we use the notation $A^i = \bigotimes_{j=1}^{i} A$.

Two special versions of union and difference operations are of interest in computing: **insertion** $A \cup \{x\}$, and **deletion** $A - \{x\}$, where A is a set and x an object. If the underlying set is clear from the context, a simpler notation INSERT(x) and DELETE(x) is often used.

Unary operations on sets:

$$
\begin{array}{rcl}
\textbf{complement } A^c & = & U - A, \text{ another notation: } \overline{A}; \\
\textbf{power set } 2^A & = & \{B \mid B \subseteq A\}; \\
\textbf{iterations } A^*(A^\omega)[A^{\omega\omega}] & - & \text{a set of all finite (infinite) [bi-infinite] sequences} \\
 & & \text{of elements of } A.
\end{array}
$$

Exercise 2.1.2 *Let* A, B *be sets. Do the following implications hold?*
(a) $A \cap B = \emptyset \Rightarrow 2^A \cap 2^B = \emptyset$; *(b)* $2^A = 2^B \Rightarrow A = B$.

Exercise 2.1.3 *Which of the following relations hold?*
(a) $2^A \cap 2^B = 2^{A \cap B}$; *(b)* $2^A \cup 2^B = 2^{A \cup B}$.

Partition of a set A is a family, (finite or infinite), of mutually disjoint subsets $B_i, i \in I$, of A such that $A = \bigcup_{i \in I} B_i$. Notation $C = A \dot{\cup} B$ will be used to express that the set C is partitioned into two disjoint subsets A and B.

Important number sets: $\emptyset$ (the empty set), $\mathbf{N}$ (nonnegative integers), $\mathbf{N}^+$ (positive integers), $\mathbf{Z}$ (all integers), $\mathbf{Q}$ (rational numbers), $\mathbf{R}$ (real numbers), $\mathbf{R}^+$ (positive real numbers), $\mathbf{C}$ (complex numbers), $\mathbf{Z}_n$ (the set of residue classes modulo n), and $\mathbf{Z}_n^*$ (those residue classes from $\mathbf{Z}_n$ that contain elements relatively prime to n). Moreover, if $n \in \mathbf{N}$, then $[n] = \{0, 1, \ldots, n-1\}$. In particular, $[2] = \{0, 1\}$.

The term **alphabet** will be often used to denote an arbitrary finite nonempty set A. In such a case elements of A are called symbols, elements of A^* strings (or words), and elements of A^ω ω-strings. Subsets of A^* are called languages and subsets of A^ω are ω-languages. The elements of the set $\mathbf{B} = \{0, 1\}$ will be interpreted, depending on the context, either as integers, or (binary) symbols—bits—or Boolean (truth) values. Elements of the set $\mathbf{B}^*$ are called often **binary strings** and elements of the set $\mathbf{B}^\omega$ are called **infinite binary strings**. The set $\mathbf{B}$ is called also the **binary alphabet**.

Exercise 2.1.4 *A* **multiset** *is a collection of elements, where an element can occur more than once. Let* $\{m_1.a_1, \ldots, m_k.a_k\}$ *denote the multiset in which the element* a_i *occurs* m_i*-times. Define in a natural way operations of union, intersection and difference for multisets, and determine, for* $A = \{2.a, 4.b, 3.c\}, B = \{1.a, 2.b, 3.c\}$, $A \cup B, A \cap B, A - B$.

Cardinality: $|A|$ denotes the cardinality of the set A. If A is finite, $|A|$ is the number of its elements. For infinite sets cardinality is defined indirectly. Two infinite sets A, B are said to have the same cardinality if there is a bijection between them. Sets that have the same cardinality as $\mathbf{N}$ are called **countable**. The set $\mathbf{R}$ is said to have cardinality of **continuum**.

Power set and iteration operations can much increase cardinality. If A is a finite set, then clearly

$$|2^A| = 2^{|A|}, \qquad |A^*| = |\mathbf{N}|, \qquad |A^\omega| = |\mathbf{R}|. \tag{2.1}$$

In addition, $|2^{\mathbf{N}}| = |\mathbf{R}|$.

Remark 2.1.5 The description of a set as a collection of objects, based on an intuitive notion of object, is due to G. Cantor (1895). Resulting set theory is called a **naive set theory** and this is what we will use. The discovery by Russel[2] (1902) that this theory leads to **paradoxes**, or **logical inconsistencies**, has first caused nightmare in mathematics. This has been later resolved through the development of an **axiomatic set theory**.

> **Russel's paradox**: Let S be the set that contains a set X if and only if X does not belong to itself—in other words let $S = \{x \mid x \notin x\}$. Is $S \in S$? If yes, then necessarily $S \notin S$. If not, then necessarily $S \in S$. A contradiction in both cases.

---

[2] Bertrand Arthur William Russel (1872-1970), an English philosopher, logician, mathematician and social reformers. His "Principles of Mathematics" and "Principia Mathematica" (with A. N. Whithead) much influenced modern logic. Wrote more than 68 books. Several times imprisoned for his articles and anti-military activities. Got Nobel price for literature.

The existence of paradoxes looks as a negative phenomenon. However, the paradoxes play, as many other negative results as we shall see later, an important positive rôle in foundation of computing. For example, they were used to show several important results concerning the limitations of computing and formalization. This will be discussed in detail in Chapter 6.

2.1.2 Representation of Objects by Words and Sets by Languages

By a representation of objects of a set A by objects of a set C—the term **encoding** is often used for that—a mapping $c : C \rightarrow A$ is understood that maps uniquely objects of C into objects of A. $c(a)$ is then the object represented by the "code" a.

It follows from (2.1) that the sets of binary strings and infinite binary strings have enough elements to encode uniquely objects of any set we practically need to deal with in computing. Not only that. For almost all sets we deal with, there is a natural encoding of their elements by binary strings. Moreover, most of the basic problems in computing have a natural formulation in terms of sets (of strings). This is why in foundations of computing it is to a large degree sufficient to deal mainly with sets of (binary) strings—called languages.

Example 2.1.6 (Binary number representations) *There are several ways binary strings can be seen as representing integers. Each finite binary word $b = b_1 \dots b_n$ and each infinite binary word $c = c_1 c_2 \dots$ can be seen as representing an integer, notation $bin(b)$, and a real number in $[0, 1]$, notation $bre(c)$, defined by*

$$bin(b) = \sum_{i=1}^{n} b_i 2^{n-i}, \qquad bre(c) = \sum_{i=1}^{\infty} c_i 2^{-i}. \tag{2.2}$$

Observe that the mapping $bre : \mathbf{B}^\omega \rightarrow [0, 1]$ is one-to-one if we restrict it to the set $\mathbf{B}^\omega - \mathbf{B}^*\{1\}^\omega$. If $x < 2^n$, $x, n \in \mathbf{N}$, then x can be uniquely represented by an n-bit string $b = b_1 \dots b_n$, notation $bin_n^{-1}(x)$, such that $bin(bin_n^{-1}(x)) = x$. (Observe that $b_i = \lfloor \frac{x}{2^{n-i}} \rfloor \bmod 2$.) Notation $bin^{-1}(n)$ will be used to denote the binary representation of n with the leftmost bit equal 1 if $n \neq 0$ and $bin^{-1}(0) = 0$, otherwise.

Exercise 2.1.7 (Dyadic representation of integers.) *If $w \in \{1, 2\}^*$, then define $dya(\varepsilon) = 0$ and for $w = a_{n-1} \dots a_0, a_i \in \{1, 2\}$, $dya(w) = \sum_{i=0}^{n-1} a_i 2^i$. (a) Show that each integer has a unique dyadic representation; (b) show dyadic representations for numbers $7, 77, 777, 7777, 777777$; (c) design an algorithm to add two integers given by their dyadic representations.*

Exercise 2.1.8 *(Fibonacci number representation) (a)* Show that each non-negative integer has a unique "Fibonacci representation" $n = \sum_{i=1}^{k} F_{m_i}$, where $m_i \geq m_{i+1} + 2$, for $1 \leq i < k$, and either $n = 0$ or $m_k \geq 2$; (b) determine the Fibonacci number representations of integers $2^i, 1 \leq i \leq 10$.*

Example 2.1.9 (Universality of binary representations) *Let* $A = \{a_1, \ldots, a_n\}$ *be a finite set—the alphabet—and* $2^{m-1} \leq n < 2^m$ *for some integer* m*. Each element of* A *can be uniquely encoded by a binary string of length* m*. This can in turn be used to encode each element of* A^* *and of* A^ω *by a finite, or an infinite binary string—each symbol from* A *is simply replaced by its binary code. This simple encoding of strings from a finite alphabet by binary strings allows often to deal, without loss of generality, only with sets of binary strings.*

Example 2.1.10 (Binary string representation of matrices) *Let* $M = \{a_{ij}\}$ *be an* $n \times n$ *matrix of nonnegative integers.* M *can be represented by the following word over the alphabet* $\Sigma = \{0, 1, \#\}$*:*

$$\mathit{bin}^{-1}(a_{11})\#\mathit{bin}^{-1}(a_{12})\#\ldots\#\mathit{bin}^{-1}(a_{1n})\#\#\mathit{bin}^{-1}(a_{21})\#\ldots\#\mathit{bin}^{-1}(a_{2n})\#\#\ldots\#\#\mathit{bin}^{-1}(a_{n1})\#\ldots\#\mathit{bin}^{-1}(a_{nn})$$

We can then use, for example, the encoding $0 \rightarrow 00$, $1 \rightarrow 11$ *and* $\# \rightarrow 01$ *to get a completely binary encoding of* M*.*

In special cases much simpler encoding of matrices by binary strings is available. For example, an $n \times n$ *symmetric Boolean matrix can be encoded by a binary string of length* $n(n+1)/2$ *by simply listing, one after another, all rows of the matrix above the main diagonal.*

For each set there are infinitely many ways its objects can be encoded by strings over an alphabet. Which of them is the best depends on the way the set is going to be dealt with.

There are also infinitely many ways strings and sets of strings can be interpreted. It may also happen, as the following example indicates, that very simple sets of strings represent quite complex objects.

Example 2.1.11 *Each string* $(a_1, b_1), \ldots, (a_n, b_n)$ *over the alphabet* $A = \{(0,0), (0,1), (1,0)\}$ *can be interpreted as a point of the plane with coordinates* $(0.a_1 \ldots a_n,\ 0.b_1 \ldots b_n)$*. At this interpretation* A^* *represents the set of points of the fractal structure called the Sierpiński triangle and shown in Figure 2.1a.*

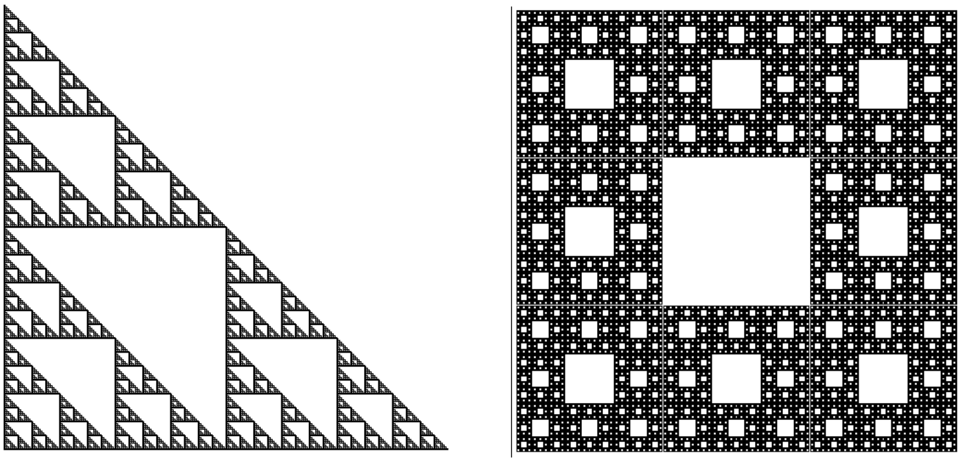
Exercise 2.1.12 *Find a finite set* A *and an interpretation of strings over* A *such that* A^* *represents the fractal structure called the Cantor carpet and shown in Figure 2.1b.*

Exercise 2.1.13 *Find a nice and simple set description of the Sierpiński tetrahedron whose construction is shown in Figure 2.2.*

2.1.3 Specifications of Sets—Generators, Recognizers and Acceptors

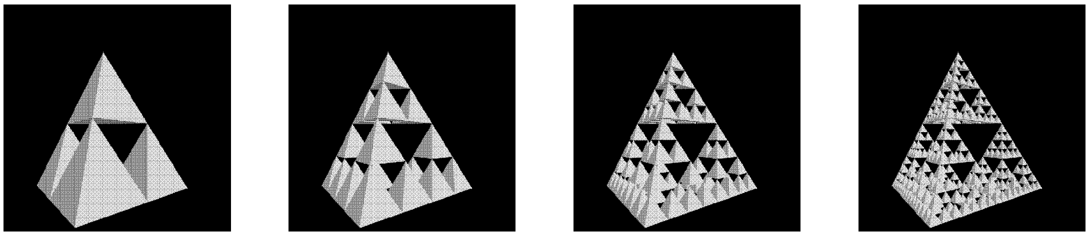
Basic problems concerning set specifications (descriptions) can be summarized as follows.

1. How to specify a set—in a simple and useful way? For example, it is certainly not obvious that the sets of points shown in Figure 2.1 have such simple descriptions.



(a) (b)

Figure 2.1: Sierpiński triangle and Cantor carpet.



(a) (b) (c) (d)

Figure 2.2: A stepwise design of the Sierpińskitetrahedron—

2. How to determine, given a set description, which set it describes? How to determine which objects are in the specified set? For example, which natural numbers are in the set described by formula (2.3) on page 96? In particular, how to find out, given a set specification and an object, whether this object is in the specified set?
3. How to specify important families of sets? For example, those that can be accepted by automata of a certain type.
4. Which properties have sets specified using some particular specification languages? For example, logic formulas of a certain type, programs, automata or grammars of a certain type?
5. Which sets have specifications that allow to find efficiently answers to various basic questions. For example, whether a given element is in the set, or whether the set is empty (infinite)?

All these problems belong to the most fundamental ones in computing and we will deal with them intensively in this book.

There are several basic ways how to specify sets.

1. **Enumeration.** Enumerating elements of a finite set. For example,

$$\begin{aligned} P \;=\; &\{19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689\\ &9941, 11213, 19937, 21701, 23209, 44497, 86243, 216091, 756839, 859433\}\end{aligned}$$

is the set of all primes p such that $2^p - 1$ used to be, starting from 1589, the largest known prime. Enumeration clearly does not work if the set is too big and even if it is not, enumeration can be far from the most effective way of describing a set. For example, *the set of the first* 2^{65536} *primes*—that has such a short description—can not be practically enumerated. On the other hand, enumeration is not to forget as the way to describe sets because, as shown later, most of finite sets are so random that there is no shorter way to describe them.

2. **Binary strings representation.** If the universal set U is finite and has n elements $u_1, \ldots, u_n$, then each subset $A \subseteq U$ can be uniquely represented by the binary word $b = b_1 \ldots b_n$, where $b_i = 1 \Leftrightarrow u_i \in A$. Observe that with this representation of subsets of U it is possible to perform, in one parallel step, set union, intersection and subtraction! If U is countable, then any subset of U can be represented by an ω-string.

3. **Logical formulas.** A formal or informal logical formula can be used to describe a set by specifying properties of its objects. For example:

$$F = \{n \,|\, n \in N, n > 2, \exists \text{ integers } x, y, z > 0, x^n + y^n = z^n\} \tag{2.3}$$

used to be perhaps the most famous description of a set in mathematics.

4. **Generators.** A generator that produces all elements of a set A, and no others, can be seen as specifying the set A. There are many forms a generator may have. It can be a deterministic automaton—see Figure 2.3a for a generator that generates a set of integers—or a randomized automaton—see Figure 2.3b for an automaton that can be seen as generating, given three points A, B and C of the plane forming nodes of an equilateral triangle, Sierpiński triangle with nodes in points A, B and C—with the exception of few initial points.

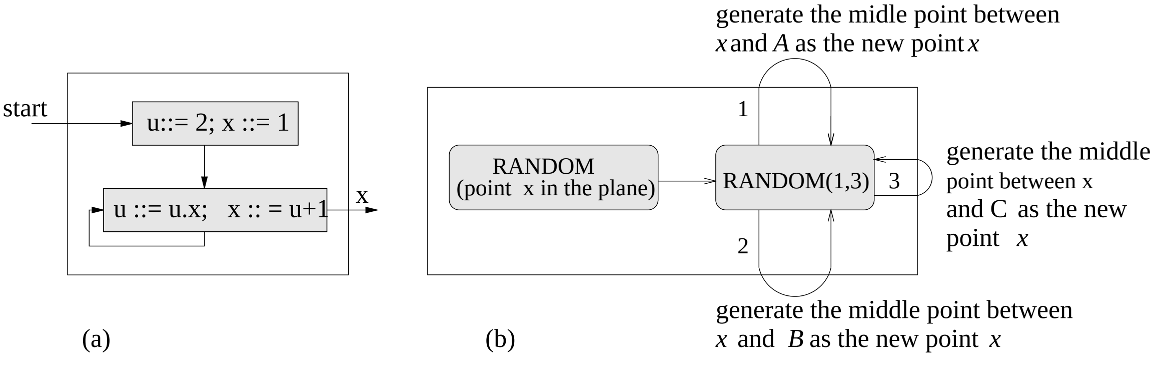


Figure 2.3: Deterministic and randomized generators—gen % 65

A generator can also have the form of a so-called rewriting system. For example, with the (rewriting) rules

$$A \rightarrow aAb, \qquad A \rightarrow ab \tag{2.4}$$

that are used as follows: The generator starts with the symbol A and in each generation step replaces A, in the string generated in the last step—if there is one A there—by the string aAb or by the string ab. If the string obtained at the last step does not contain A, generation terminates. In this way the rewriting system in (2.4) generates the set of strings $\{a^nb^n \mid n \geq 1\}$.

Exercise 2.1.14 *Are all integers generated by the generator from Figure 2.3a primes?*

Exercise 2.1.15 *Design rewriting systems generating: (a) the set of strings* $\{a^nb^na^mb^m \mid n,m \geq 1\}$*; (b) the set of all palindromes in the English alphabet (or in some other alphabet)—that is those words that are read the same from the left and the right.*

A generator can be specified also by some other processes. For instance, through an recurrence—the recurrence (1.6) in Section 1.2 can be seen as a process (generator) generating Fibonacci numbers.

Example 2.1.16 (Cantor set) *The sequence of sets—see Figure 2.4—defined by the recurrence*

$$\begin{array}{lcl} C_0 & = & \{x \mid 0 \leq x < 1\} = [0,1), \\ C_i & & i > 0, \textit{is obtained from } C_{i-1} \textit{ by deleting, in each maximal subinterval } [a,b) \textit{ of } C_{i-1}, \\ & & \textit{its middle open subinterval } [a + \frac{1}{3}(b-a), a + \frac{2}{3}(b-a)), \end{array}$$

is decreasing and its intersection $C = \bigcap_{i=0}^{\infty} C_i$ *is called the* **Cantor set**[3]. *(Observe that the Cantor set can be seen as a diagonal of the Cantor carpet.)*

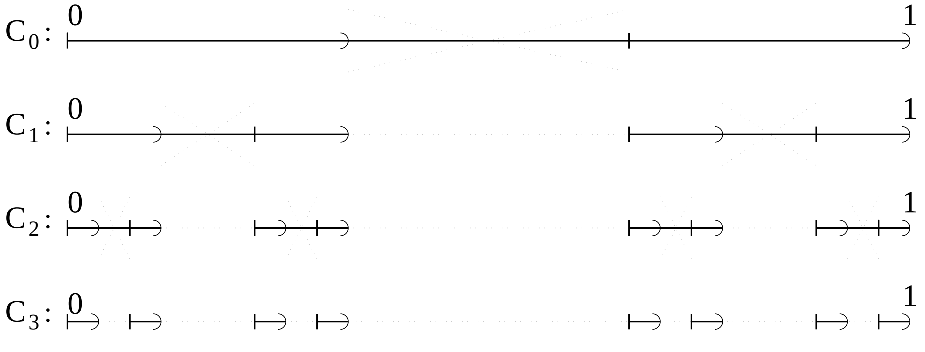


Figure 2.4: Cantor set- 65%

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[3] Georg Cantor (1845-1918), a German mathematician of the Russian origin, one of the founders of the modern set theory, introduced his set, today named after him, in 1889 as an example of an exceptional (monster) set.

Exercise 2.1.17 *Determine the cardinality of the sets: (a) a set of points of the Sierpiński triangle; (b)* a set of numbers of the Cantor set.*

Example 2.1.18 *Consider the process that starts with the curve shown in Figure 2.5a and in each step all edges of the curve obtained at the last step are replaced by the curve of the type shown in Figure 2.5a—with each edge of the length one third of the length of the former one. The resulting set contains the so-called* **Koch curves.** *See Figures 2.5b, c, d for the first three of them.*

(a) (b)

(c) (d)

Figure 2.5: Koch curves.

Exercise 2.1.19 *Consider the same process as in the Example 2.1.18 but starting with the equilateral triangle. The curves you get are called* **Koch snow-flakes**. *Generate few of them. Another possibility is to start with a square. What do you get "in the limit" in such a case?*

Example 2.1.20 *The proces defined by the mapping*

$$p_c^1(z) = z^2 + c,$$

and, for $i > 1$,

$$p_c^i(z) = (p_c^{i-1})^2 + c,$$

where c *is a complex number, defines the so-called* **Mandelbrot set**

$$M = \{c \in C \mid \lim_{n\to\infty} |p_c^n(0)| \text{ is not } \infty\},$$

where $p_c^n(0)$ *is the result of the* n*-th iteration of the process with* 0 *being the initial value.*

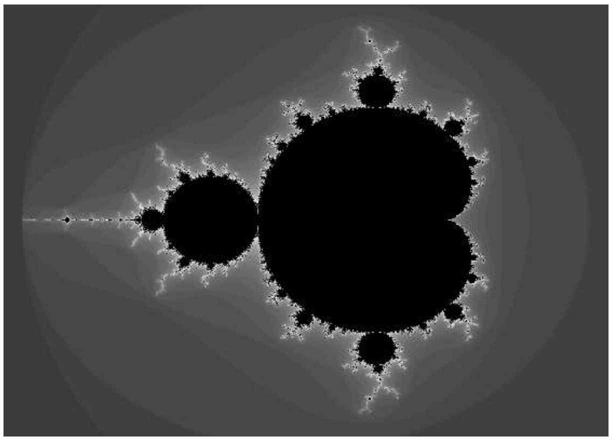

Figure 2.6: Mandelbrot set

The resulting set of (very black) points is shown in Figure 2.6.

5. **Recognizers and acceptors.** A **deterministic recognizer**—see Figure 2.7a—for a subset S of a universal set U is an automaton $\mathcal{A}$ that stops for any input $x \in U$ and says

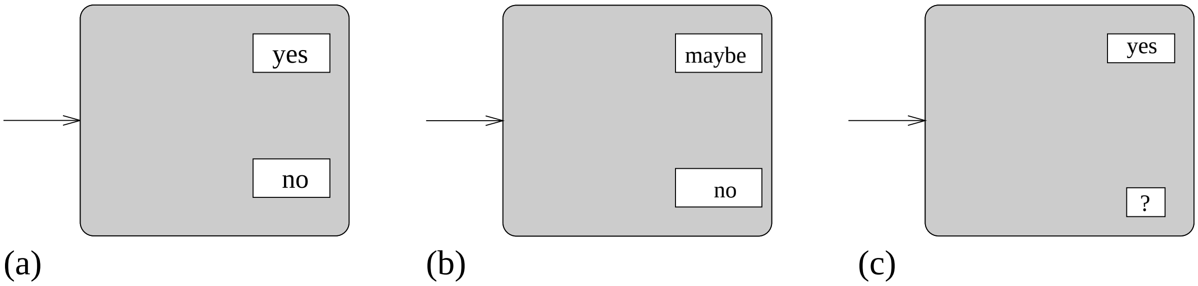

Figure 2.7: Automata: a recognizer, a Monte Carlo randomized recognizer, and an acceptor-80%

"yes" ("no") if and only if $x \in S$ ($x \notin S$). This way $\mathcal{A}$ describes, or defines, S. For example, Figure 2.8a shows a recognizer that has been used to find out very large Mersenne primes. It realizes the so-called Lucas-Lehmer's test to decide whether a given prime p is in the set $\{p \,|\, 2^p - 1 \text{ is a prime}\}$.

A **Monte Carlo randomized recognizer**—see Figure 2.7b—for a set $S \subseteq U$ is an automaton that uses a random number generator to compute and that stops for any input $x \in U$ and either says "no", and then surely $x \notin S$, or says "maybe", which should mean that $x \in S$ but in the later case an error is possible though the probability of an error is smaller than $\frac{1}{2}$.

Is such an recognizer useful when we cannot be sure whether its outcome is correct? Can it be seen as a description of a set? It can, because we can find out, with as high probability as we wish, whether a given input is in the set specified by the acceptor. Indeed, let us take the same input more times, say 100 times. If the answer is no at least once, we know that

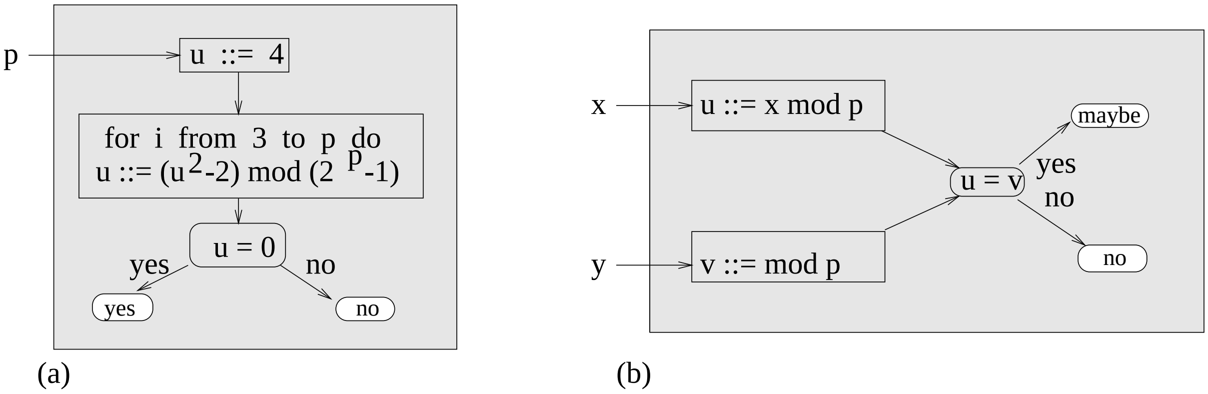

Figure 2.8: Deterministic and randomized recognizers 70%

the input is not in the set specified by the acceptor. If we get 100 times answer yes, then the probability that the input is not in the set is smaller than $\frac{1}{2^{100}}$, what is practically 0.

For example, Figure 2.8b shows a recognizer that recognizes whether two input numbers x and y are the same, by choosing first randomly a prime p and then computing as shown in Figure 2.8b. (This may be seen as a very complicated way of comparing two numbers but assume, for a moment, that x and y are given by very long strings, their sources are far away from each other and sending x and y to the place where comparison is done is very costly. In Chapter 11 we shall see another interpretation where this setting has a sense, and there it will be shown that if p is properly chosen, then in this way we get really a good Monte Carlo recognizer.)

An **acceptor**—see Figure 2.7c—for a set $S \subseteq U$ is an automaton that for an input $x \in U$ may stop and it surely stops when $x \in S$ and reports "yes" in such a case. If $x \notin S$, the acceptor may stop and report "no" or it may not stop at all. This means that if the automaton "keeps running", then one has no idea whether it will eventually stop and reports something or not.

For example, let us imagine an automaton that for an input $x \in \mathbf{N}$ performs the Collatz' process described on page 15 and stops when it gets 1 as outcome. According to our current knowledge this is an acceptor that accepts the set of all those integers for which the Collatz process stops.

Exercise 2.1.21 *Design a prime recognizer.*

Exercise 2.1.22 * *Design an acceptor for the set* $\{n \mid \exists p > n, 2^p - 1 \text{ is a prime}\}$.

2.1.4 Decision and Search Problems.

Computational problems can be roughly divided into two groups. A **decision problem**, for a set $S \subseteq U$ and an element $x \in U$, is the problem to decide whether $x \in S$. A **search**

problem for a relation $R \subseteq U \times U$ and an element $x \in U$ is the problem to find (search) an $y \in U$, if it exists, such that $(x, y) \in R$.

For example, the problem to decide whether a given integer is a prime is a decision problem. The problem to find to a Mersenne prime that is larger than a given integer is a search problem.

Exercise 2.1.23 *The famous Goldbach's conjecture says that any even positive integer can be written as the sum of two primes. Design an automaton that for any integer n finds two prime p_n and p'_n, if they exist, such that $n = p_n + p'_n$.*

There are three basic decision problems concerning sets with which we deal repeatedly in this book: In all three cases at least one of the inputs is a description of a set.

Emptiness problem. Given a description of a set. Does it describe the empty set?

Membership problem. Given a description of a set and an element a. Is a in the set?

Equivalence problem Given two descriptions of sets. Do they describe the same set?

It does not seem at the first glance that the emptiness problem can be a big deal. Actually, it is, and some of the most important problems in computing, and not only in computing, are of this type. For example, perhaps the most famous problem of mathematics for the last two hundred years was the problem to find out whether the **Fermat's Last Theorem**[4] holds. This theorem actually claims that the set specified by formula (2.3) is empty. Moreover, the equivalence problem for two sets A, B can be reduced to the emptiness problem for the sets $A - B$ and $B - A$. It is clearly often of importance to find out whether two sets are equal. For example, currently the most important problem of foundations of computing and currently perhaps also one of the most important problems in science in general, is the problem to determine whether $\mathbf{P} = \mathbf{NP}$.

The most interesting variants of decision and search problems occurs when the computational complexity questions start to be of importance. Is there a (feasible) [fast] algorithm to decide, given a set description of a certain type, whether the specified set is empty? Or is there an (feasible)[fast] algorithm to decide, given two descriptions of sets from a certain set of set descriptions, whether they describe the same set? The same for the set membership problem.

2.1.5 Data Structures and Data Types

In computing any manipulation with a set, or with elements of a set, even a "look from one element of a set to another one", costs something. This is a new factor, not considered in the classical set theory, that initiated the development of a new theory, practically important

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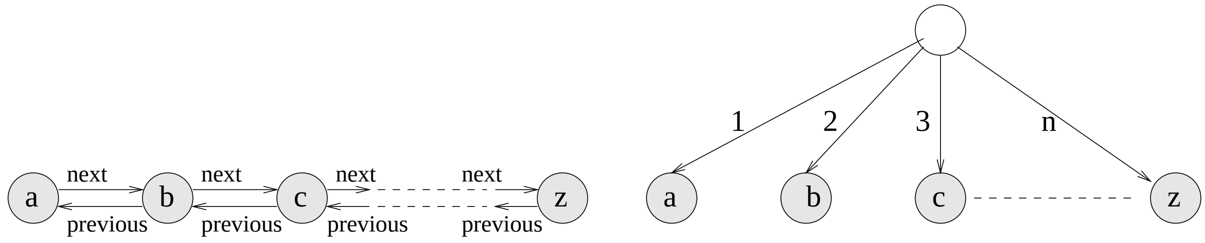
[4] Fermat wrote on the margin of the latin translation of Diophantus book *Arithmetica* that he has a truly marvelous demonstration of the statement. The proof was not found and numerous attempts to prove it failed—until June 1993. It is now believed by the experts that Andrew Wiles has proven Fermat's Last Theorem. The proof (more than 200 pages) is too big to fill in this margin.

and theoretically interesting, of efficient representations of sets and multisets, and of efficient implementations of operations and predicates on them.

A general scheme how to represent sets by graphs of a certain type whose nodes are used "to store" set elements and whose edges represent access paths to elements of the set and among these elements, is called a **data structure**. For example, two basic data structures to represent sets are **lists** and (sorted or unsorted) **linear arrays**—see Figure 2.9. With various more sophisticated data structures to represent sets we deal in Sections 2.4, 2.4.1, 4.3.24, and 10.1.



(a) list (b) array

Figure 2.9: A list and an array as data structures 70%

Observe also that the aim of some frequently used algorithms is only to change one representation of a set, or a multiset, to another one—better in some sense. For example, sorting and merging algorithms are of such a type.

There are many important set operations and predicates on sets. However, any particular algorithm uses only few of them—though usually many times. The most basic operations are **INSERT, DELETE, MEMBER**. (**MEMBER**$(a, A) = [a \in A]$—in case the underlying set A is clear we use the notation **MEMBER**(a).)

A set and a collection of set operations and predicates is called a **data type**. For example, a set with the predicate **MEMBER** and operations **INSERT, DELETE** forms a data type called **dictionary**. If a data type is defined in an implementation independent way, we speak about an **abstract data type**.

One of the important tasks of computational set theory is to understand the complexity of implementations of frequently used data types and to develop the best possible implementations for them.

As an illustration, two simple sequential and one simple parallel implementation of the data type dictionary will be discussed. With the third one, with pseudorandom features, we deal in Section 2.3.4.

Example 2.1.24 (Dictionary Implementations)

In the following table the worst-case complexity of dictionary operations is shown for the cases that sorted or unsorted arrays are used to represent the set. (To simplify the discussion we consider also **MEMBER** as an operation.) n denotes the number of elements of the underlying set. Observe that linear time complexity of the operations **INSERT** and **DELETE**, for the case a sorted array is used, is due to the need to shift a part of the array when performing insertion or deletion. In the unsorted case, linear time is needed to find out whether an element, to delete or to insert, is actually in the array.

set representation	INSERT	DELETE	MEMBER
sorted array	$\Theta(n)$	$\Theta(n)$	$\Theta(\lg n)$
unsorted array	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$

Using a complete tree representation for the underlying set, a $\Theta(\lg n)$ performance can be achieved for all three dictionary operations on a usual sequential computer.

We show now that by a simple parallel implementation of dictionary operation we can achieve not only $\Theta(\lg n)$ performance for all dictionary operations, but also only $\mathcal{O}(1)$ **period** of computation.[5] This will now be shown under the assumption that the underlying set never has more than n elements, and that one never tries to add (delete) an element that is (is not) in the underlying set. For simplicity we assume that $n = 2^k$, for some k.

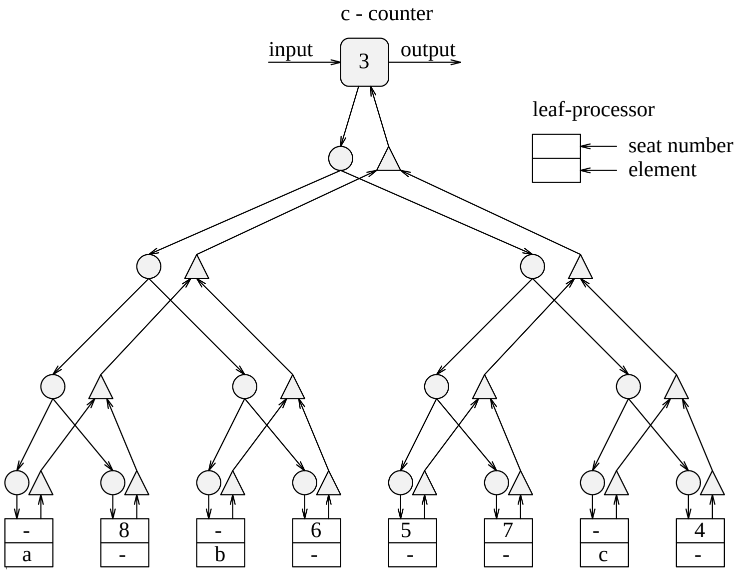


Figure 2.10: A binary tree implementation of dictionary data type–60%

We use a complete binary tree network of processors with n leaf-processors—see Figure 2.10. Each leaf-processor can be seen as having a "seat" that is either occupied, by an element of the to-be-represented set, or that is empty. At any moment all empty seats are numbered, by consecutive integers $1, 2, \ldots, k$, as their "seat numbers", and the number k is stored in the counter of the root-processor, which is both the input and the output processor. All other internal processors can be seen as consisting of two subprocessors. ◯-processors are used to transmit information from the root-processor to leaf-processors. △-processors process information obtained from their children and send results to their parent. Dictionary operations are implemented as follows:

MEMBER(a) The request is sent to all leaf-processors and their responses are processed

---

[5] Period of a parallel computation is time needed to wait after an input starts to be processed until the processing of the next input can begin.

("OR-ed") by $\triangle$-processors until, in time $2\lg n$, the final answer is assembled by the root-processor.

INSERT(a) The triple (i, a, k), indicating that the operation is insert, the element is a, and the contents of the counter in the root-processor is k, is transmitted from the root to all leaf-processors. Once this information leaves the root the contents of its counter is decreased by 1. Once the triple (i, a, k) reaches leaves, then a is stored "on the seat" of that processor that has a free seat with k as the seat number.

DELETE(a) The contents of the counter in the root-processor is increased by 1 and the triple (d, a, k) is sent to all leaf-processors. The leaf-processor that contains a removes a and labels its now free seat by k.

Observe that the root-processor can start each time to process a new operation. It initiates a flow of data from the root-processor to the leaf-processor and back. The key point is that in no moment a subprocessor of an internal node has to handle more than two flows of data. (Explain why.) Therefore, up to $2\lg n$ operations can be simultaneously processed by the system. Moreover, if handling of the root-counter is done in the way described above, then all operations are implemented correctly.

Exercise 2.1.25 * *The data type called* **priority queue** *has the predicate MEMBER(a) as well as the operations INSERT(a) and DELETEMIN—to delete the smallest element for the set of concern. Find an efficient implementation for this data type.*

2.2 Relations

The intuitive concept of a relationship between objects is captured by the mathematical concept of a relation. Its applications in computing are limitless. Relations are used to describe the structure of complex objects. Specific for computing are relations of steps of (computational) processes.

2.2.1 Basic Concepts

Let $S_1, \ldots, S_n$ be sets. Any subset $R \subseteq S_1 \times \cdots \times S_n$ is called an ***n*-ary relation** on $S_1, \times \ldots, \times S_n$. If $n = 2$ we speak about a **binary relation**.

The concept of an n-ary relation is needed in some areas of computing in its full generality; for example in data bases. Binary relations are, however, the basic ones.

For a binary relation $R \subseteq A \times B$ we define

$$\begin{aligned} \text{domain}(R) &= \{a \mid \exists b \in B, (a, b) \in R\}; \\ \text{range}(R) &= \{b \mid \exists a \in A, (a, b) \in R\}. \end{aligned}$$

Two basic unary operations on relations are:

$$\begin{array}{rcll} R^{-1} & = & \{(b,a) \mid (a,b) \in R\}, & \text{the } \textbf{inverse relation} \text{ to } R; \\ R^{c} & = & A \times B - R, & \text{the } \textbf{complement relation} \text{ to } R. \end{array}$$

Exercise 2.2.1 *Let R, R_1, R_2 be relations on a set A. Show:*
(a) $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$; (b) $(R^c)^{-1} = (R^{-1})^c$;
(c) $R_1 \subseteq R_2 \Rightarrow R_1^{-1} \subseteq R_2^{-1}$.

The most important binary operation on relations is the **composition** $R_1 \circ R_2$, in short R_1R_2, defined, for the case that range$(R_1) \subseteq$ domain(R_2), by:

$$R_1R_2 = \{(x,z) \mid \exists y \ \ (x,y) \in R_1, (y,z) \in R_2\}.$$

If $R \subseteq S \times S$ for a set S, then R is said to be a relation on S. The identity relation $I_S = \{(a,a) \mid a \in S\}$ is such an relation. (If S is clear from the context we write the identity relation on S simply as I.) For a relation R on a set S we define its **powers** R^i, **transitive closure** R^+, and **transitive and reflexive closure** R^* by

$$R^0 = I, \quad R^{i+1} = RR^i, \quad i \geq 0;$$

$$R^+ = \bigcup_{i=1}^{\infty} R^i, \qquad R^* = \bigcup_{i=0}^{\infty} R^i.$$

Special properties of relations. A binary relation $R \subseteq S \times S$ is called:

reflexive	if	$a \in S \Rightarrow (a,a) \in R$,
symmetric	if	$(a,b) \in R \Rightarrow (b,a) \in R$,
antisymmetric	if	$(a,b) \in R \Rightarrow (b,a) \notin R$,
weakly antisymmetric	if	$(a,b) \in R, a \neq b \Rightarrow (b,a) \notin R$,
transitive	if	$(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$,
partial function	if	$(a,b) \in R, (a,c) \in R \Rightarrow b = c$.

Exercise 2.2.2 *Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric or transitive, where $(x,z) \in R$ if and only if (a) $x \neq y$; (b) $xy \geq 1$; (c) x is a multiple of y; (d) $x \geq y^2$.*

In addition,

R is an **equivalence**	if	R is reflexive, symmetric, and transitive;
R is a **partial order**	if	R is reflexive, weakly antisymmetric and transitive;
R is a **total order (ordering)**	if	R is a partial order and, for every $a, b \in S$, either $(a,b) \in R$ or $(b,a) \in R$.

If R is an equivalence on S and $a \in S$, then the set $[a]_R = \{b \mid (a,b) \in R\}$ is called an **equivalence class** on S with respect to R. This definition yields

Lemma 2.2.3 *If R is an equivalence on a set S and $a, b \in S$, then the following statements are equivalent:*

$$(a)\ (a,b) \in R, \qquad (b)\ [a]_R = [b]_R, \qquad (c)\ [a]_R \cap [b]_R \neq \emptyset.$$

This implies that any equivalence R on a set S defines a partition on S such that two elements a, b of S are in the same set of the partition if and only if $(a,b) \in R$. Analogically, each partition of the set S defines an equivalence relation on S—two elements are equivalent if and only if they belong to the same set of the partition.

Example 2.2.4 *For any integer n, $R_n = \{(a,b) \mid a \equiv b \pmod{n}\}$ is an equivalence on $\mathbf{N}$. This follows from the properties of the congruence relation shown in Section 1.7.*

Exercise 2.2.5 *Which of the following relations on the set of all people is an equivalence?*

(a) $\{(a,b) \mid a$ and b have common parents$\}$;

(b) $\{(a,b) \mid a$ and b share a common parent$\}$.

Exercise 2.2.6 *Which of the following relations on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$ is an equivalence?*

(a) $\{(f,g) \mid f(0) = g(0)$ or $f(1) = g(1)\}$;

(b) $\{(f,g) \mid f(0) = g(1)$ and $f(1) = g(0)\}$.

Two important types of total order are the so-called **lexicographical ordering** on a Cartesian product of sets and a **strict ordering** on sets A^*, where A is an alphabet (endowed with a total order).

Let $(A_1, \preceq_1), (A_2, \preceq_2), \ldots, (A_n, \preceq_n)$ be totally ordered sets. A lexicographical ordering $\preceq$ on the Cartesian product $A_1 \times \cdots \times A_n$ is defined as follows: $(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n)$ if and only if either $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ or $a_i \preceq_i b_i$ for the smallest i such that $a_i \neq b_i$.

A strict ordering on a set A^*, induced by a total order $(A, \preceq)$, where A is an alphabet, is defined as follows: If a string s is shorter than a string u then $s \preceq u$. If they have the same length then $s \preceq u$ if and only if either they are the same or $s_i \preceq u_i$ for the smallest i such that the i-th symbol of s, s_i is different from the i-th symbol, u_i, of u.

For example, for the alphabet $A = \{0, 1, 2\}$ with the total order $0 \preceq 1 \preceq 2$, we get the following strict ordering of strings on A^*

$$\varepsilon, 0, 1, 2, 00, 01, 02, 10, 11, 12, 20, 21, 22, 000, 001, 002, 010, 011, 012, 020, 021, 022, 100, \ldots.$$

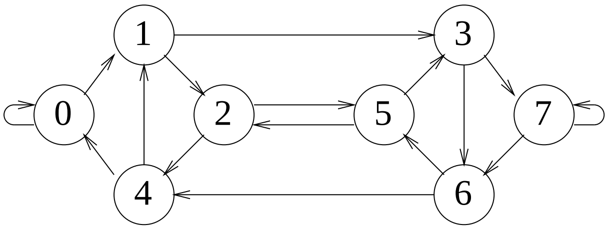
There is a close relationship between relations and functions. To any relation $R \subseteq A \times B$ we can associate a function $f_R : A \times B \to \{0,1\}$—the so-called characteristic function of R defined by $f(a,b) = 1$ if and only if $(a,b) \in R$. Similarly, to any function $f : A \times B \to \{0,1\}$ we can associate the relation R_f such that $(a,b) \in R_f$ if and only if $f(a,b) = 1$.

2.2.2 Representations of Relations

Two most important representations of binary relations are by **Boolean matrices** and **directed graphs**.

	0	1	2	3	4	5	6	7
0	1	1	0	0	0	0	0	0
1	0	0	1	1	0	0	0	0
2	0	0	0	0	1	1	0	0
3	0	0	0	0	0	0	1	1
4	1	1	0	0	0	0	0	0
5	0	0	1	1	0	0	0	0
6	0	0	0	0	1	1	0	0
7	0	0	0	0	0	0	1	1

(a)



(b)

Figure 2.11: A matrix and a graph representations of a binary relation

A binary relation $R \subseteq S \times S$, with $|S| = n$, can be represented by an $n \times n$ Boolean matrix M_R, rows and columns of which are labeled by elements of S, such that there is 1 in the entry for a row a and a column b if and only if $(a,b) \in R$. See, for example, the representation of the relation

$$R = \{(0,0),(0,1),(1,2),(1,3),(2,4),(2,5),(3,6),(3,7),(4,0),(4,1),(5,2),(5,3),(6,4),(6,5),(7,6),(7,7)\}$$

by the matrix in Figure 2.11a.

Similarly, any binary relation $R \subseteq S \times S$ can be represented by a directed graph $G_R = \langle V, E \rangle$, where $V = \text{domain}(R) \cup \text{range}(R)$ and $E = \{(a,b) \mid (a,b) \in R\}$—see the representation of the relation from Figure 2.11a in Figure 2.11b.

There is clearly a one-to-one correspondence, up to the notation of elements, between binary relations, Boolean matrices and directed graphs. Moreover, one can easily, in low polynomial time, with respect to the size of the relation , i.e. $|\{(a,b) \mid aRb\}|$, transfer one of these representations to another. n-ary relations for $n > 2$, on the other hand, are represented by **hypergraphs**—see Section 2.4.

Both representation of binary relations, by Boolean matrices and directed graphs, have their advantages.

If $M_{R_i}, i = 1,2$, is a Boolean matrix representation of a relation R_i, then for the matrix representations of the union and the intersection of these relations we get

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}, \qquad M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2},$$

where $\vee$ and $\wedge$ are component-wise disjunction and conjunction operations on the elements of Boolean matrices.

On the other hand, if a binary relation $R \subseteq S \times S$ is represented by a directed graph G_R, then $(a,b) \in R^i$ if and only if there is a path of length at most i in G_R from the node a to the node b. Similarly, $(a,b) \in R^*$ if and only if there is a path in G_R from the node a to b. Using these facts one can in principle easily construct from the graph G_R the graphs representing the relations $R^i, i > 1$, R^+ and R^*. Moreover, if $|S| = n$, then there is a path in G_R from a node a to a node b only if there is a path from a to b of length at most n. This implies that the relations R^+ and R^* can be expressed using finite unions as follows:

$$R^+ = \bigcup_{i=1}^{n} R^i \quad R^* = \bigcup_{i=0}^{n} R^i.$$

Exercise 2.2.7 *Design a matrix and a graph representation of the relation*

$$R = \{(i, (2i) \bmod 16), (i, (2i+1) \bmod 16) \,|\, i \in [16]\}.$$

2.2.3 Transitive and Reflexive Closure

The concept of a process as a sequence of elementary steps is crucial for computing. An elementary step is often specified by a binary relation R on the set of the so-called configurations of the process. $(a, b) \in R^*$ then means that one can get from a configuration a to b after a finite number of steps. This is one reason why a computation of the transitive and reflexive closure of binary relations is of such an importance in computing. In addition, this problem allows us to demonstrate several techniques of the design and analysis of algorithms.

If $R \subseteq S \times S$ is a relation, $|S| = n$, and M_R the Boolean matrix representing R, then it clearly holds

$$M_{R^*} = \bigvee_{i=0}^{n} M_R^i,$$

where $M_R^0 = I, M_R^{i+1} = M_R \vee M_R^i$, for $i > 0$, and therefore, in order to compute the transitive and reflexive closure of R, it is sufficient to compute the transitive and reflexive closure of the Boolean matrix M_R that is equal to $\bigvee_{i=0}^{n} M_R^i$. We present three methods how to do it. The most classical one is the so-called Warshall's algorithm.

Let $M = \{a_{ij}\}$, $1 \leq i, j \leq n$, $a_{ij} \in \{0, 1\}$, be a Boolean matrix and G_M the directed graph representing the relation defined by M, with nodes labeled by integers $1, 2, \ldots, n$. The following algorithm computes elements c_{ij} of the matrix $C = M^*$

Algorithm 2.2.8 (Warshall's algorithm)

```
begin for $i \leftarrow 1$ to $n$ do $c^0_{ii} \leftarrow 1$;
      for $1 \leq i, j \leq n, i \neq j$ do $c^0_{ij} \leftarrow a_{ij}$;
      for $k \leftarrow 1$ to $n$ do
        for $1 \leq i, j \leq n$ do $c^k_{ij} \leftarrow c^{k-1}_{ij} \vee (c^{k-1}_{ik} \wedge c^{k-1}_{kj})$;
      for $1 \leq i, j \leq n$ do $c_{ij} \leftarrow c^n_{ij}$
end
```

In order to demonstrate the correctness of this algorithm it is sufficient to show, that

$c_{ij}^k = 1$ if and only if there is a path in the graph G_M from the node i to the node j that passes only through nodes of the set $\{1, 2, \ldots, k\}$.

what can be done easily by induction on k.

Time complexity of this algorithm is $\Theta(n^3)$. Indeed, for any $1 \leq k \leq n$, the algorithm performs n^2 times statements updating c_{ij}^k.

The second method to compute M^* is based on the equality

$$M^* = (I \vee M)^n$$

as it is easy to verify using the binomial theorem, and the fact that $A \cup A = A$ for any set A.

Let $m(n) = \Omega(n^2)$ be the time complexity of the multiplication of Boolean matrices of degree n. Using the repeated squaring method—see Algorithm 1.1.14—we can compute $(I \cup M)^n$ with $\lg n$ Boolean matrix multiplications and at most the same number of Boolean matrix additions (each addition can be performed in $\Theta(n^2)$ time). The overall complexity is therefore $\mathcal{O}(m(n) \lg n)$. It has been shown that $m(n) = \mathcal{O}(n^{2.376})$, see also page 282, if time is counted by the number of arithmetical operations needed and $m(n) = \mathcal{O}(n^{2.376} \lg n \lg\lg n \lg\lg\lg n)$ if only bit operations are counted, and therefore the second algorithm is asymptotically faster than the first one.

The third algorithm, asymptotically even better than the second one, is based on the divide-and-conquer method.

Algorithm 2.2.9 (Divide-and-conquer algorithm for transitive closure)

1. Divide M into 4 submatrices A, B, C, D, as shown below, where A is an $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$ matrix and D is an $\lceil \frac{n}{2} \rceil \times \lceil \frac{n}{2} \rceil$ matrix

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

2. Recursively compute D^*.
3. Compute $F = A + BD^*C$.
4. Recursively compute F^*.
5. Set

$$M^* = \left[\begin{array}{c|c} F^* & F^*BD^* \\ \hline D^*CF^* & D^* + D^*CF^*BD^* \end{array}\right].$$

Correctness of this algorithm can be shown informally by the following argument. Let us assume that nodes of the graph G_M are partitioned into two sets N_A and N_D in such a way that A describes edges between the nodes of N_A, B edges from nodes in N_A to nodes in N_D, C edges from nodes in N_D to nodes in N_A, and finally D edges between nodes of N_D. Then F^* clearly determines all paths from nodes in N_A to nodes in N_A in G_M. F^*BD^* determines all paths that start from a node in N_A and go to a node in N_D. Similarly, for other matrix expressions in the formula for M^* in the item 5 of the Algorithm 2.2.9.

For the complexity $T(n)$ of the algorithm we get the recurrence

$$T(n) = 2T(\frac{n}{2}) + cm(\frac{n}{2}) + d(\frac{n}{2})^2,$$

where c, d are constants. Since $m(n) = \Omega(n^2)$ we can assume that $m(n) \leq \frac{1}{4}m(2n)$ and therefore we can use Case 3 of the Theorem 1.6.3 to get $T(n) = \Theta(m(n))$.

Similarly, we can show that if there is an algorithm to calculate M^* in time $T(n)$ then there is an algorithm to multiply two Boolean matrices of the degree n in time $\mathcal{O}(T(3n))$. Indeed, if we put two Boolean matrices A, B, of the degree n, in a proper way as parts of a $3n \times 3n$ Boolean matrix, we get:

$$\begin{bmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{bmatrix}^* = \begin{bmatrix} I & A & AB \\ 0 & I & B \\ 0 & 0 & I \end{bmatrix}.$$

Exercise 2.2.10 *Compute* R^2, R^3, R^4, R^* *for*

$R = \{(1,3),(2,4),(3,1),(3,5),(5,1),(5,2),(5,4),(2,6),(5,6),(6,3),(6,1)\}$.

Exercise 2.2.11 *Determine transitive closure of the following relations*

(a) $\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$;

(b) $\{(a,b),(a,c),(a,e),(b,a),(b,c),(c,a),(d,c),(e,d)\}$;

(c) $\{(1,5),(2,1),(3,4),(4,1),(5,2),(5,3)\}$.

Exercise 2.2.12 *Compute* a_n, b_n, c_n, d_n *defined by*

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$$

with the matrix multiplication being: (a) ordinary matrix multiplication;

(b) Boolean matrix multiplication.

2.2.4 Posets

A set S together with a partial order relation R on S is called a **partially ordered set** or a **poset**, and is denoted by (S, R). In case of posets one usually uses notation aRb to denote that $(a, b) \in R$.

Example 2.2.13 *If* $\backslash$ *denotes the divisibility relation among integers, then* $(\mathbf{N}, \backslash)$ *is a poset but not a totally ordered set.* $(\mathbf{Z}, \leq), (\mathbf{Q}, \leq)$, *and* $(\mathbf{R}, \leq)$ *are totally ordered sets with respect to the relation "smaller than or equal to". If* A *is any set, then* $(2^A, \subseteq)$ *is also a poset.*

Posets can be represented graphically more economically by the so-called **Hasse diagrams**. This representation is based on the fact that if aRb, and bRc in a poset (S, R), then certainly aRc and therefore this particular relationship does not have to be explicitly shown.

Let (S, R) be a poset. Denote $R_H \subseteq R$ the relation defined as follows: $aR_H b$ if and only if aRb and there is no $a \neq c \neq b$ such that aRc and cRb. In other words, R_H is the smallest subset of R such that $R_H^* = R$. The graph G_{R_H} is called the Hasse diagram for R.

For example, the Hasse diagram for the poset $(2^{\{0,1,2\}}, \subseteq)$ is shown in Figure 2.12a and the Hasse diagram for the relation

$$R = \{(0,1), (0,2), (0,3), (0,4), (0,5), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5)\}$$

is shown in the Figure 2.12b.[6]

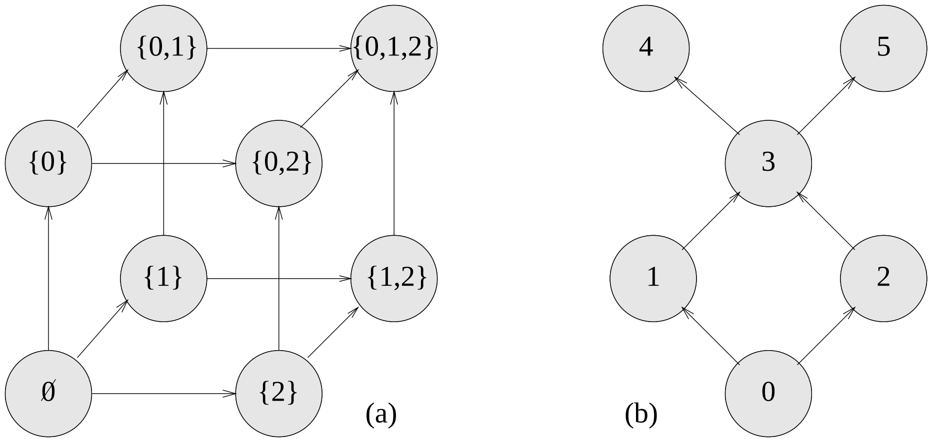


Figure 2.12: Hasse diagrams—60%.

The Hasse diagram for a relation R represented by a graph G_R can be clearly obtained from G_R by a process at which in each step one removes an edge the existence of which can be deduced from the transitivity or reflexivity of the relation of partial order—if there is such an edge; otherwise the process halts.

The economy and transparency of descriptions are the main advantages of the Hasse diagrams.

Exercise 2.2.14 *Draw the Hasse diagram for the divisibility relation on the set* $\{1, 2, 3, 6, 12, 24, 36, 48\}$.

Exercise 2.2.15 ** *Show that for every* $n \in \mathbf{N}$ *there is a relation which is represented by a graph with* n *edges and the Hasse diagram for this relation has* $\lfloor \sqrt{n} \rfloor$ *edges.*

---

[6] Observe that if a set A has n elements then the Hasse diagram for the poset $(2^A, \subseteq)$ is actually the n-dimensional hypercube—a graph treated in detail in Section 10.1.

2.3 Functions

The modern concept of a function, as a special type of a relation, is an abstraction and a significant generalization of the old intuitive concept of a mapping provided by a computation. To develop a deeper understanding which functions are computable (theoreticly, practically, fast, or even very fast) is one of the main goals in foundations of computing. Some classes of functions play by that a special rôle; for example the Boolean functions.

Of increasing importance in computing are functions that exhibit randomized or chaotic behavior, with respect to the input-output relation they represent, and thereby can help to utilize the power of randomization. These functions are actually not really new, only their potential for creating randomness has been recognized only recently—for example, one-way functions and hash functions.

2.3.1 Basic Concepts

Intuitively, a (total) **function** $f : A \rightarrow B$ assigns to any object from the set A an object from the set B. Formally, f is a relation on $A \times B$ such that to any $a \in A$ there is exactly one $b \in B$ such that $(a,b) \in f$. Instead of $(a,b) \in f$, we usually write $f(a) = b$, to emphasize the assignment. A is called the **domain** of f, B its **codomain** and the set $\{b \mid b \in B, \exists a \in A, f(a) = b\}$ is the **range** of f. If $f(a) = b$, then b is called the **image** of a and a is a **preimage** of b. If $A_0 \subset A$, then $f(A_0) = \{b \mid b = f(a), a \in A_0\}$. The set of all functions from A to B is denoted by B^A. The value of a function $f : A \rightarrow B$ for an $a \in A$ is denoted by $f(a)$ or $f(x)|_{x=a}$.

Exercise 2.3.1 *Show that if A and B are finite sets, then $|B^A| = |B|^{|A|}$.*

A partial function $f : A \rightarrow B$ is a mapping that assigns to some elements of A elements from B. $\{a \mid a \in A, f(a) \text{ is defined}\}$ is called the domain of f.

Example 2.3.2 *Observe that*

$$f = \{(a,b) \mid a, b \in \mathbf{N}, b = a \bmod 2\}$$

is a function, but this is not the case for the relation

$$g = \{(a,b) \mid a, b \in \mathbf{N}, b \equiv a \pmod 2\}.$$

A function $f : A \rightarrow B$ is called

surjective (resp. a surjection),	if range$(f) = B$;
injective (resp. an injection),	if $f(a) = f(b) \Rightarrow a = b$;
bijective (resp. a bijection),	if f is surjective and injective.

A surjection is sometimes called a **mapping of A onto B**; an injection a **one-to-one mapping**; bijection a **one-to-one correspondence** between A and B.

The **inverse** f^{-1} of a function f is the relation $f^{-1} = \{(b,a) \mid f(a) = b\}$. If f is an injection, then f^{-1} is also a (partial) function.

For example, observe that the inverse of the function $f : \mathbf{N}^+ \to \mathbf{Z}$ with $f(n) = (-1)^n \lfloor \frac{n}{2} \rfloor$ is a function, but the inverses of $g(n) = \lfloor \frac{n}{2} \rfloor$ and of $h(n) = (-1)^{-n} \lfloor \frac{n}{3} \rfloor$ are not.

Two (partial) functions f_1 and f_2 are called **equal** if they have the same domain and codomain and $f_1(a) = f_2(a)$ for all a in the domain. The **composition** $f_1 \circ f_2$ of functions $f_1 : A \to B$ and $f_2 : B \to C$ is a function from A to C defined by $f_1 \circ f_2(a) = f_2(f_1(a))$, for any $a \in A$.

Exercise 2.3.3 *(Pigeonhole principle.) Show that if* A *and* B *are finite sets such that* $|A| > |B|$ *and* $f : A \to B$*, then* f *is not bijective.*

Exercise 2.3.4 *Let* $f : A \to B$ *be a function. Characterize properties "*f *is injective", "*f *is surjective" and "*f *is bijective" in terms of* $f^{-1}(\{y\})$*, where* $y \in B$.

Exercise 2.3.5 *Show, for a function* $f : A \to B$*:*

(a) f *is injective if and only if* $\forall X \subseteq A, f^{-1}(f(X)) = X$;

(b) f *is surjective if and only if* $\forall Y \subseteq B, f(f^{-1}(Y)) = Y$;

(c) f *is bijective if and only if* $\forall X, Y \subset A, f(X \cap B) = f(X) \cap f(Y)$.

The **graph of a function** $f : X \to Y$ is the set $\{(x, f(x) \mid x$ is in the domein of $f\}$

If $f : A \to A$ is a function, then any $x \in A$ such that $f(x) = x$ is called a **fixedpoint** of f. Any subset A_0 of A such that $f(A_0) = A_0$ is called an **invariant** of f. For example: the mapping $f(x) = x^3 - 6x^2 + 12x - 6$ has three fixedpoints: $1, 2, 3$.

Exercise 2.3.6 ** *Let nodes of the Sierpiński triangle, see Figure 2.1, be arbitrarily denoted as* $1, 2, 3$*, and for* $i = 1, 2, 3$ *the mapping* f_i *be defined on the plane as mapping of any point* x *to the midlepoint point of the line connecting* x *and the node* i *of the triangle. Show that for all these three mappings the set of points of the Sierpiński triangle is an invariant.*

Iterations $f^{(i)}, i \geq 0$ of a function $f : X \to X$ are defined by $f^{(0)}(x) = x$ and $f^{(i+1)}(x) = f(f^{(i)}(x))$, for $i > 0$.

A function $f : \{1, \ldots, n\} \to A$ is called a **finite sequence**; a function $f : \mathbf{N} \to A$ an **infinite sequence** and a function $f : \mathbf{Z} \to A$ a **doubly infinite sequence**.

When the domain of a function is a Cartesian product, say $f : A_1 \times A_2 \times \ldots \times A_n \to B$, then the extra parentheses surrounding n arguments are usually omitted and we write simply $f(a_1, \ldots, a_n)$ instead of $f((a_1, \ldots, a_n))$.

Two case studies in the rest of this subsection will illustrate basic concepts just summarized and to introduce important functions and concepts we will deal with later.

Case study 1—permutations

A bijection $f : S \to S$ is often called a **permutation**. A permutation of a finite set S can be seen as an ordering of elements of S into a sequence—with each element appearing exactly once. Examples of permutations of the set $\{1, 2, 3, 4\}$ are: $(1, 2, 3, 4), (2, 4, 3, 1), (4, 3, 2, 1)$. If S is a finite set, then the number of its permutations is $|S|!$.

Since elements of any finite set can be numbered by consecutive integers, it is sufficient to consider only permutations on sets $\mathbf{N}_n = \{1, 2, \dots, n\}, n \in \mathbf{N}^+$. A permutation π is then a bijection $\pi : \mathbf{N}_n \to \mathbf{N}_n$. Two basic notations are used for permutations:

enumeration of elements	$\pi = (a_1, \dots, a_n)$ such that $\pi(i) = a_i, 1 \le i \le n$. Example: $\pi = (3, 4, 1, 5, 2, 6)$.
enumeration of cycles	$\pi = c_1 c_2 \dots c_k, c_i = \langle b_0, \dots, b_s \rangle, 1 \le i \le k$, such that $\pi(b_j) = b_{(j+1) \bmod (s+1)}, 0 \le j \le s$. Example: $\pi = \langle 1, 3 \rangle \langle 2, 4, 5 \rangle \langle 6 \rangle$; that is $\pi(1) = 3, \pi(3) = 1, \pi(2) = 4, \pi(4) = 5, \pi(5) = 2, \pi(6) = 6$.

Special permutations: **identity permutation** $id = (1, 2, \dots, n)$, i.e. $id(i) = i$ for $1 \le i \le n$; **transposition** $\pi = [i_0, j_0]$, where $1 \le i_0, j_0 \le n$, (that is $\pi(i_0) = j_0$, $\pi(j_0) = i_0$ and $\pi(i) = i$, otherwise). For example, $\pi = [2, 4] = (1, 4, 3, 2, 5, 6)$.

The **inverse permutation** π^{-1} to a permutation π is defined by $\pi^{-1}(i) = j \Leftrightarrow \pi(j) = i$. For example, if $\pi = (3, 4, 1, 5, 2, 6)$, then $\pi^{-1} = (3, 5, 1, 2, 4, 6)$.

Composition of permutations π_1 and π_2 is the permutation $\pi = \pi_1 \circ \pi_2$, where $\pi(i) = \pi_2(\pi_1(i))$, for $1 \le i \le n$. For example, $(3, 5, 1, 2, 6, 4) \circ (2, 6, 1, 3, 4, 5) = (1, 4, 2, 6, 5, 3)$. Clearly, $\pi \circ \pi^{-1} = id$ for any permutation π.

Powers of permutations: $\pi^1 = \pi, \pi^{i+1} = \pi \circ \pi^i, i \ge 1$. For example,

$$(3, 5, 1, 2, 6, 4)^2 = (1, 6, 3, 5, 4, 2) \quad (3, 5, 1, 2, 6, 4)^4 = (1, 2, 3, 4, 5, 6).$$

An **inversion** a permutation π on $\{1, \dots, n\}$ is any pair $1 \le i < j \le n$, such that $\pi(j) < \pi(i)$.

As the following lemma indicates powers of a permutation always lead to the identity permutation.

Lemma 2.3.7 *For any permutation π of a finite set there is an integer k (the so-called* **degree** *of π) such that $\pi^k = id$.*

Proof. Clearly, there are $i < j$ such that $\pi^i = \pi^j$. Then $id = \pi^i \circ \pi^{-i} = \pi^j \circ \pi^{-i} = \pi^{j-i}$. □

Exercise 2.3.8 *Determine the degree of the following permutations:* *(a)* $(2,3,1,8,5,6,7,4)$*; (b)* $(8,7,6,5,4,3,2,1)$*; (c)* $(2,4,5,8,1,3,6,7)$.

Exercise 2.3.9 * *Determine the number of permutations* $\pi : \{1,\ldots,n\} \rightarrow \{1,\ldots,n\}$ *such that* $\pi(i) \neq i$ *for all* i.

Case study 2—cellular automata mappings

Informally, a one-dimensional cellular automaton $\mathcal{A}$ is a doubly infinite sequence of processors; see Figure 2.3.11:

$$\ldots, P_{-i}, P_{-i+1}, \ldots, P_{-1}, P_0, P_1, \ldots, P_{i-1}, P_i, \ldots$$

and at each moment of the discrete time each processor is in one of the states from a finite set of states Q. Processors of $\mathcal{A}$ work in parallel in discrete time steps. At each moment of the discrete time each processor changes its state according to the local transition function that takes as arguments its current state and states of its k neighbours from the left and also k neighbours from the right, for a fixed k.

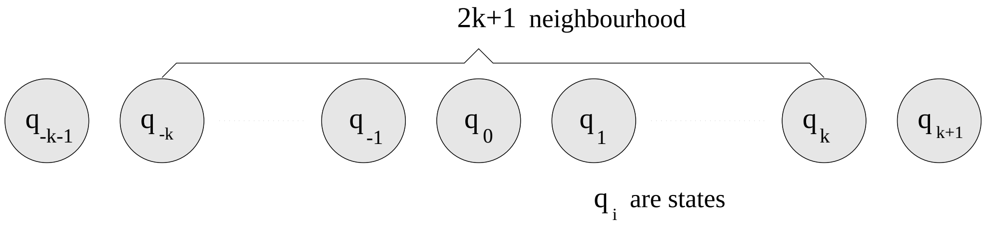


Figure 2.13: One-dimensional cellular automaton-60%

Formally, a **one-dimensional cellular automaton** $\mathcal{A} = \langle Q, k, \delta \rangle$ is defined by a finite set Q of **states**, an integer $k \in \mathbf{N}$—the size of the **neighborhood**—and a **local transition function** $\delta : Q^{2k+1} \rightarrow Q$.

A mapping $c : \mathbf{Z} \rightarrow Q$ is called a **configuration** of $\mathcal{A}$. The so-called **global transition function** $G_{\mathcal{A}}$ maps the set $Q^{\mathbf{Z}}$ of all configurations of $\mathcal{A}$ into itself and is defined by

$$G_{\mathcal{A}}(c) = c' \text{ where } c'(i) = \delta(c(i-k), c(i-k+1), \ldots, c(i+k-1), c(i+k)) \text{ for all } i \in \mathbf{Z}.$$

Moreover, a cellular automaton $\mathcal{A}$ is called **reversible** (or its global transition function is called reversible), if there is another cellular automaton $\mathcal{A}' = \langle Q, k', g' \rangle$ such that for any configuration $c \in Q^{\mathbf{Z}}$ we have $G_{\mathcal{A}'}(G_{\mathcal{A}}(c)) = c$.

Exercise 2.3.10 * *Show that the following one-dimensional cellular automaton* $\mathcal{A} = \langle\{0,1\}, 2, g\rangle$, *is reversible, where the local transition function* $g : \{0,1\}^5 \rightarrow \{0,1\}$ *is defined as follows:*

$$\begin{array}{llll}
00000 \rightarrow 0 & 00100 \rightarrow 1 & 01000 \rightarrow 0 & 01100 \rightarrow 1 \\
10000 \rightarrow 0 & 10100 \rightarrow 1 & 11000 \rightarrow 0 & 11100 \rightarrow 1 \\
00001 \rightarrow 0 & 00101 \rightarrow 1 & 01001 \rightarrow 0 & 01101 \rightarrow 1 \\
10001 \rightarrow 0 & 10101 \rightarrow 1 & 11001 \rightarrow 0 & 11101 \rightarrow 1 \\
00010 \rightarrow 1 & 00110 \rightarrow 0 & 01010 \rightarrow 1 & 01110 \rightarrow 0 \\
10010 \rightarrow 1 & 10110 \rightarrow 0 & 11010 \rightarrow 1 & 11110 \rightarrow 0 \\
00011 \rightarrow 0 & 00111 \rightarrow 1 & 01011 \rightarrow 0 & 01111 \rightarrow 1 \\
10011 \rightarrow 0 & 10111 \rightarrow 1 & 11011 \rightarrow 0 & 11111 \rightarrow 1.
\end{array}$$

Cellular automata are an important model of parallel computing and will be discussed in more detail in Section 4.5. We mention now only some basic problems concerning their global transition function.

The so called **Garden of Eden** problem is to determine, given a cellular automaton, whether its global transition function is surjective. In other words, whether there is a configuration that can not be reached in a computational process. Also problems concerning injectivity and bijectivity of the global transition function are of importance. It holds for example

Theorem 2.3.11 *The following three assertions are equivalent for one-dimensional cellular automata:*

1. *The global transition function is injective.*
2. *The global transition function is bijective.*
3. *The global transition function is reversible.*

The problem of reversibility is of special interest. Cellular automata are being considered also as a model of microscopic physics. Since microscopic physic is reversible the existence of (universal) reversible cellular automata is crucial for considering cellular automata as a model of the physical world.

2.3.2 Boolean Functions

An n-input and m-output **Boolean function** is any function from $\{0,1\}^n$ to $\{0,1\}^m$. Let $\mathcal{B}_n^m$ denote the set of all such functions.

There are three reasons why Boolean functions play an important rôle in computing in general and in foundations of computing in particular.

1. Boolean functions are exactly the functions computer circuitry implements directly. Boolean circuits, and families of Boolean circuits, discussed in Section 4.3, form the very basic model of computers.

2. A very close relation between Boolean functions and truth functions of the propositional logic, discussed later, allows to see Boolean functions—formulas—and their identities as formalizing basic rules and laws of formal reasoning.

3. Strings-to-strings functions, that so well represent functions we deal with in computing, are well modeled by Boolean functions. For example, any function $f : \{0,1\}^* \rightarrow \{0,1\}$ is sometimes called Boolean, because f can be seen as an infinite sequence $\{f_i\}_{i=1}^{\infty}$ of Boolean functions, where $f_i \in \mathcal{B}_i^1$ and $f_i(x_1,\ldots,x_i) = f(x_1 \ldots x_i)$. In this way we can identify the intuitive concept of a computational problem instance with a Boolean function from a set $\mathcal{B}_n^m$, and that of a computational problem with an infinite sequence of Boolean functions $\{f_i\}_{i=1}^{\infty}$, where $f_i \in \mathcal{B}_i^1$.

A Boolean function from $\mathcal{B}_n^m$ can be seen as a collection of m Boolean functions from $\mathcal{B}_n^1$. This is why, in discussing the basic concepts concerning Boolean functions, it is mostly sufficient to consider only Boolean functions from $\mathcal{B}_n^1$—and instead of $\mathcal{B}_n^1$ we mostly write simply $\mathcal{B}_n$.

Boolean functions look very simple. However, their space is very large. $\mathcal{B}_n$ has 2^{2^n} functions and for $n = 6$ this gives the number 18 446 744 073 709 551 616—exactly one more than the number of moves needed to solve the "original" Towers of Hanoi Problem.

The most basic way to describe a Boolean function $f \in \mathcal{B}_n$ is to enumerate all 2^n possible n-tuples of arguments and to assign to each of them the corresponding value of f. For example, the following table describes in this way the most commonly used Boolean functions of one and two variables:

		identity	negation	OR	AND	XOR	equiv.	NOR	NAND	implic.
x	y	x	$\bar{x}$	$x+y$	$x \cdot y$	$x \oplus y$	$x \equiv y$	$\overline{x+y}$	$\overline{x \cdot y}$	$\bar{x}+y$
0	0	0	1	0	0	0	1	1	1	1
0	1	0	1	1	0	1	0	0	1	1
1	0	1	0	1	0	1	0	0	1	0
1	1	1	0	1	1	0	1	0	0	1

For some of these functions several notations are used depending on the context. For example, we write also $x \vee y$ or $x\, OR\, y$ instead of $x + y$ for **conjunction**; $x \wedge y$, or $x\, AND\, y$, instead of xy for **disjunction** and $\neg x$ instead of $\bar{x}$.

A set Γ of Boolean functions is said to be a **base** if any Boolean function can be expressed as a composition of functions from Γ. From the fact that each Boolean function can be described by enumeration it follows that the set $\Gamma_0 = \{\neg, \vee, \wedge\}$ of Boolean functions forms a base.

Exercise 2.3.12 *Which of the following sets of Boolean functions forms a base? (a)* $\{OR, NOR\}$*; (b)* $\{\neg, NOR\}$*; (c)* $\{AND, NOR\}$*; (d)* $\{\bar{x} \wedge y, 0, 1\}$.

Exercise 2.3.13 *Use the NAND function to form the following functions: (a) NOT; (b) OR; (c) AND; (d) NOR.*

The so-called monotone Boolean functions play a special rôle. Let $\preceq_m$ be the so-called montone ordering on $\{0,1\}^n$ defined by $(x_1,\ldots,x_n) \preceq_m (y_1,\ldots,y_n)$ if and only if $\forall_{i=1}^n (x_i = 1 \Rightarrow y_i = 1)$. A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is called **monotone** if

$$(x_1,\ldots,x_n) \preceq (y_1,\ldots,y_n) \Rightarrow f(x_1,\ldots,x_n) \preceq f(y_1,\ldots,y_n).$$

OR and AND are examples of monotone Boolean functions, XOR is not.

Boolean expressions—formulas

Another way how to describe Boolean functions, often much more concise, is to use Boolean formulas—expressions. They can be defined over an arbitrary base. For example, Boolean expressions over the base Γ_0, described above, can be defined inductively, by assuming that an infinite pool $V = \{x_1, x_2, \ldots\}$ of Boolean variables is available, as follows:

1. $0, 1, x_1, x_2, \ldots$ are Boolean expressions.
2. If E_1, E_2 are Boolean expressions, then so are $\neg E_1, (E_1 \vee E_2), (E_1 \wedge E_2)$.

An expression of the form x_i or $\neg x_i$ (or alternatively $\bar{x}_i$), is called a **literal**.

An inductive definition of Boolean expressions can be used to define, or to prove, various properties of Boolean expressions and Boolean functions. For example, the Boolean function $f(E)$ represented by a Boolean expression E can be defined as follows:

1. $f(E) = E$ if $E \in \{0,1\} \cup V$;
2. $f(\neg E) = \overline{f(E)}; f(E_1 \vee E_2) = f(E_1) \vee f(E_2); f(E_1 \wedge E_2) = f(E_1) \wedge f(E_2)$

Two Boolean expressions, E_1 and E_2, are said to be **equivalent**, notation $E_1 \equiv E_2$, if $f(E_1) = f(E_2)$, i.e., if E_1 and E_2 are two different representations of the same Boolean function.

Exercise 2.3.14 *Show that each monotone Boolean function can be represented by a Boolean expression that uses only the functions:* 0, 1, $x \vee y$, $x \wedge y$.

Disjunctive and conjunctive normal forms

Boolean expressions are a special case of expressions in Boolean algebras and the most basic pairs of equivalent Boolean expressions can be obtained from the column 1 of Table 2.1 that contains laws of Boolean algebras. These equivalencies, especially those representing idempotence, commutativity, associativity, distributivity and de Morgan's laws, can be used to simplify Boolean expressions. For example, parentheses can be removed in multiple conjunctions and disjunctions as in $(x_3 \vee (x_2 \vee x_1)) \vee ((x_4 \vee x_5) \vee (x_6 \vee x_2))$. This allows to use the notation

$$\bigwedge_{i=1}^{k} E_i = E_1 \wedge E_2 \wedge \ldots \wedge E_k, \qquad \bigvee_{i=1}^{k} E_i = E_1 \vee E_2 \vee \ldots \vee E_k.$$

In the case E_i are literals, the expression $\bigwedge_{i=1}^{k} E_i$ is called a **minterm** and the expression $\bigvee_{i=1}^{k} E_i$ is called a **clause**. Two closely related normal forms for Boolean expressions are

$$\bigvee_{i=1}^{n}\bigwedge_{j=1}^{m_i} L_{ij} \quad \text{— } \textbf{disjunctive normal form (DNF)} \tag{2.5}$$

$$\bigwedge_{i=1}^{n}\bigvee_{j=1}^{m_i} L_{ij} \quad \text{— } \textbf{conjunctive normal form (CNF)} \tag{2.6}$$

where L_{ij} are literals. For example,

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \neg x_3),$$

$$(x_1 \wedge \neg x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3).$$

Theorem 2.3.15 *Every Boolean expression is equivalent to one in conjunctive normal form and to one in a disjunctive normal form.*

Proof. By induction on the structure of Boolean expressions E. The case $E \in \{0,1\} \cup V$ is trivial. Let now

$$E_1 = \bigwedge_{i=1}^{n_1}\bigvee_{j=1}^{r_{1,i}} L_{ij}^{(c1)}, \qquad E_1 = \bigvee_{k=1}^{m_1}\bigwedge_{l=1}^{s_{1,k}} L_{kl}^{(d1)},$$

$$E_2 = \bigwedge_{p=1}^{n_2}\bigvee_{q=1}^{r_{2,p}} L_{pq}^{(c2)}, \qquad E_2 = \bigvee_{u=1}^{m_2}\bigwedge_{v=1}^{s_{2,u}} L_{uv}^{(d2)},$$

be CNF and DNF for E_1, E_2. Using de Morgan's laws we get that

$$\neg E_1 = \bigvee_{i=1}^{n_1}\bigwedge_{j=1}^{r_{1,i}} \neg L_{ij}^{(c1)} \quad \text{and} \quad \neg E_1 = \bigwedge_{k=1}^{m_1}\bigvee_{l=1}^{s_{1,i}} \neg L_{kl}^{(d1)}$$

are DNF and CNF for $\neg E_1$, where the double negation law is used, if necessary, to make a literal out of $\neg L_x^{(y)}$. Similarly,

$$\bigwedge_{i=1}^{n_1}\bigvee_{j=1}^{r_{1,i}} L_{ij}^{(c1)} \wedge \bigwedge_{p=1}^{n_2}\bigvee_{q=1}^{r_{2,p}} L_{pq}^{(c2)} \quad \text{and} \quad \bigvee_{k=1}^{m_1}\bigwedge_{l=1}^{s_{1,k}} L_{kl}^{(d1)} \vee \bigvee_{u=1}^{m_2}\bigwedge_{v=1}^{s_{2,u}} L_{uv}^{(d2)}$$

are a CNF for $E_1 \wedge E_2$ and a DNF for $E_1 \vee E_2$. Finally,

$$\bigwedge_{i=1}^{n_1} \bigwedge_{p=1}^{n_2} (\bigvee_{j=1}^{r_{1,i}} L_{ij}^{(c1)} \vee \bigvee_{q=1}^{r_{2,k}} L_{pq}^{(c2)}) \qquad \text{and} \qquad \bigvee_{k=1}^{m_1} \bigvee_{u=1}^{m_2} (\bigwedge_{l=1}^{s_{1,k}} L_{kl}^{(d1)} \wedge \bigwedge_{u=1}^{s_{2,u}} L_{uv}^{(d2)}) \tag{2.7}$$

are a CNF for $E_1 \vee E_2$, and a DNF for $E_1 \wedge E_2$. □

The algorithm presented in Theorem 2.3.15 for the construction of a DNF or a CNF equivalent to a given Boolean expression is simple in principle. However, the size of the Boolean expression in the step 2.7 can double with respect to that for E_1 and E_2. It can therefore happen that the resulting CNF or DNF for a Boolean expression E has the size exponential with respect to the size of E.

Exercise 2.3.16 *Design disjunctive normal forms for functions:* *(a)* $x \Rightarrow y$; *(b)* $(x + \bar{y} + z)(x + \bar{y} + \bar{z})(\bar{x} + y + z)(\bar{x} + \bar{y} + z)$.

Exercise 2.3.17 *Design conjunctive normal forms for functions:* *(a)* $x \Rightarrow y$; *(b)* $\bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + x\bar{y}z + xyz$.

Satisfiability

Another important concept for Boolean expressions is that of **satisfiability** whose name is derived from a close relation between Boolean expressions and expressions of the propositional calculus.

A **(truth) assignment** T to a set S of Boolean variables is a mapping $T : S \rightarrow \{0, 1\}$. If S_E is a set of variables occurring in a Boolean expression E and $T : S_E \rightarrow \{0, 1\}$ is an initial assignment, then we say that T satisfies E, notation $T \models E$, or that T does not satisfy E, notation $T \not\models E$, if the following holds (using the inductive definition):

1. $T \models 1$, $T \not\models 0$;
2. if $x \in V$, then $T \models x$ if and only if $T(x) = 1$;
3. $T \models \neg E$ if and only if $T \not\models E$;
4. $T \models (E_1 \vee E_2)$ if and only if either $T \models E_1$ or $T \models E_2$;
5. $T \models (E_1 \wedge E_2)$ if and only if $T \models E_1$ and $T \models E_2$.

Exercise 2.3.18 *Show that two Boolean expressions* E_1 *and* E_2 *are equivalent if and only if* $T \models E_1 \Leftrightarrow T \models E_2$ *for any assignment* T *on* $S_{E_1 \vee E_2}$.

The most basic way to show the equivalence of two Boolean expressions E_1 and E_2 is to determine, using the so-called **truth table** with an enumeration of all initial assignments, step by step, the values of all subexpressions of E_1 and E_2. For example, in order to show that $x \wedge (y \vee z)$ and $x \wedge y \vee x \wedge z$ are equivalent Boolean expressions we can proceed as follows:

$x\,y\,z$	$y \vee z$	$x \wedge (y \vee z)$	$x \wedge y$	$x \wedge z$	$x \wedge y \vee x \wedge z$
0 0 0	0	0	0	0	0
0 0 1	1	0	0	0	0
0 1 0	1	0	0	0	0
0 1 1	1	0	0	0	0
1 0 0	0	0	0	0	0
1 0 1	1	1	0	1	1
1 1 0	1	1	1	0	1
1 1 1	1	1	1	1	1

A Boolean expression E is said to be **satisfiable** if there is a truth assignment T_E such that $T_E \models E$ and it is called a **tautology** (or **valid**) if $T_E \models E$, for any truth assignment to variables of E.

There is another conceptually simple algorithm to construct a DNF equivalent to a given Boolean expression E. For each truth assignment $T \models E$ one takes a minterm where the literal corresponding to a variable x is x if $T(x) = 1$ and $\bar{x}$, otherwise. For a Boolean expression with n variables this way gives a DNF of size $\mathcal{O}(n2^n)$.

The problem to find out whether a given Boolean expression E is satisfiable is called the **satisfiability problem**. In spite of its simplicity, this problem plays an important rôle in the complexity theory and we deal with it in Chapter 5. The satisfiability problem has actually been the first known **NP**-complete problem.

There is a close relation between Boolean formulas and formulas of propositional logic containing only negation, conjunction and disjunction. If 0 is interpreted as the truth value **false**, 1 as **true**, and Boolean operations of negation, OR and AND as logical negation, conjunction and disjunction, then a Boolean formula can be seen as a formula of the propositional calculus—and vice versa.

Arithmetization of Boolean functions

Of importance is also a representation of Boolean functions by multilinear polynomials.

Definition 2.3.19 *A function* $g : \mathbf{R}^n \to \mathbf{R}$ *approximates a Boolean function* $f : \{0,1\}^n \to \{0,1\}$, *if* $f(\alpha) = g(\alpha)$ *for every* $\alpha \in \{0,1\}^n$.

It is easy to see that the basic Boolean functions have the following approximation by multilinear polynomials:

true *and* **false** are approximated by 1 and 0; (2.8)

$x \wedge y$ is approximated by xy; (2.9)

$x \vee y$ is approximated by $1 - (1-x)(1-y)$; (2.10)

$\bar{y}$ is approximated by $1 - y$. (2.11)

If a given Boolean formula is first transformed into an equivalent disjunctive normal form, and then the rules (2.8), (2.9), (2.10), (2.11) are used, a multivariable polynomial approximates f can be obtained in a straightforward way. Using the identities $0^n = 0, 1^n = 1$ for any n, all powers x^n can then be replaced by x. By this way a multilinear polynomial approximating f is obtained. We have thereby shown:

Theorem 2.3.20 *Any Boolean function can be approximated by a multilinear polynomial.*

Example 2.3.21 *Boolean formula*

$$f(x,y,z) = (x \vee y \vee z) \wedge (x \vee \bar{y} \vee z)$$

can be transformed first into the polynomial

$$(x + y + z - xy - xz - yz + xyz)(1 - y + xy + yz - xyz)$$

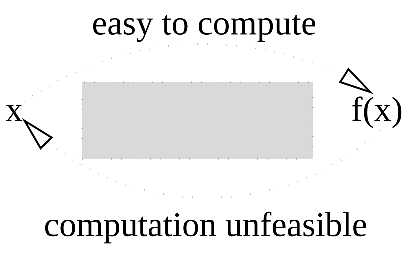
and then, after the multiplication and simplifications, the following polynomial approximation is obtained

$$x + z - xz.$$

Exercise 2.3.22 *Construct multilinear polymomial approximations for the following functions: (a) $x \Rightarrow y$; (b) $x \equiv y$; (c) $(x \wedge y \vee z)(\bar{z} \vee zx)$; (d) XOR; (e) NOR.*

2.3.3 One-way Functions

Informally, a function, say $f : \mathbf{N} \to \mathbf{N}$, is called a **one-way function** if it is easily computable—in polynomial time—but a computation of its inverse is unfeasible—it can not be done in polynomial time.



This intuitively simple concept of one way function turned out to be deeply related with some main problems of the foundations of computing and also with important applications. It plays a central rôle in Chapters 8 and 9.

It is easy to give an informal example of one-way function. For example, write a mesage on a sheet of paper, cut the paper to thousands of pieces and mix them. This is easy to make, but the resulting mesage is practically unreadable.

There are several non-equivalent definitions of one-way functions. The reason behind is that for different purposes more or less strong requirements on "one-wayness" are needed. We present now the definition of the so-called strong one-wayness. Two other definitions are discussed in Chapter 5.

Definition 2.3.23 *A function* $f : \{0,1\}^* \rightarrow \{0,1\}^*$ *is called strongly one-way if the following conditions are satisfied:*

1. f *can be computed in polynomial time.*
2. *There are* $c, \varepsilon > 0$ *such that* $|x|^\varepsilon \leq |f(x)| \leq |x|^c$*. (Otherwise, any function that shrinks the input exponentially, for example* $f(n) = \lceil \lg n \rceil$*, could be considered one-way.)*
3. *For every randomized polynomial time algorithm* $\mathcal{A}$*, and any constant* $c > 0$*, there exists an* N_c *such that for* $n > N_c$

   $$Pr(\mathcal{A}(f(x)) \in f^{-1}(f(x))) < \frac{1}{n^c}.$$

   (The probability space is that of all pairs $(x, r) \in S_n \times R_n$*, where* $S_n = \{0,1\}^n$ *and* R_n *denote all possible sequences of coin-tossings of* $\mathcal{A}$ *on inputs of length* n*.)*

Exercise 2.3.24 *Explain how it may happen that* f *and* g *are one-way functions but none of the functions* $f + g$ *and* $f \cdot g$ *is one-way.*

Observe that Definition 2.3.23 allows that a polynomial time randomized algorithm can invert f; however, only for a negligible small number of values. There are no proofs only a strong evidence that one-way functions do exist. Some candidates:

1.	Modular exponentiation	$f(x) = a^x \bmod n$, where a is a primitive root of $\mathbf{Z}_n^*$;
2.	Modular squaring	$f(x) = x^2 \bmod n$, where n is a Blum integer;
3.	Prime number multiplication	$f(p, q) = pq$, where p, q are primes of almost the same length.

All these functions are easy to compute and no one knows polynomial time deterministic algorithms to compute their inverses. As we could see in Section 1.7, in case of modular squaring there is a proof (Theorem 1.8.15) that computation of the square roots is exactly as hard as factoring of integers.

A proof that a one-way function exist (or not) would have far reaching impact on foundations of computing as we shall see later. One-way functions have also numerous applications. They will be discussed especially in Chapters 8 and 9. We present now only one of them.

Example 2.3.25 (Passwords) *Passwords are an important vehicle to protect unauthorized use of computers. After a to-be-authorized user types a password, the computer has to verify its correctness. However, it is too unsafe to keep in computer a password file with entries: (user, his/her password). There are many ways an adversary can get in and misuse the knowledge of the password. More safe is to use a one-way function* f *to compute* $f(password)$ *each time a user tries to identify herself and to use a file with entries* $(user,\ f(user\ password))$. *Since* f *is one-way even an adversary who could get this file would not be able to pretend that he is a user because from* $f(password)$ *he cannot find out the password. (Unless he tries all possibilities for passwords and for each one checks the whole password-codes table.)*

In spite of the simplicity of the informal definition of one-wayness, and technicalities of the formal definition, an additional explanation seem to be in order to make clear what makes one-way functions so special. They actually exhibit such random or chaotic relation between preimages and images, that from the knowledge of f and of $f(x)$ there is no practically realisable way to find an x' such that $f(x') = f(x)$. It also implies that f may map values that are close to each other, in some sense, to images that are far apart, and vice versa.

2.3.4 Hash Functions

Another important family of functions that perform pseudo-random mappings are the so-called **hash functions**. Informally, an ideal hash function maps elements of a large finite set into elements of a smaller set in a maximally uniform and at the same time random way—any element of the domain is equally likely to be mapped into any element of the codomain.

More formally, let h map a set U (of objects) into a set A and let $Pr(u)$ be the probability that u is picked up as the argument for h. The requirement of the so-called **simple uniform hashing** is that for any $a \in A$

$$\sum_{\{u \mid h(u)=a\}} Pr(u) = \frac{1}{|A|}.$$

The main problem with this definition is that, in general, the probability distribution Pr is not known in applications. The tendency is therefore to consider such hash functions that perform reasonably well quite independently of the pattern in which data occur.

Let us assume that $U = \{0, 1, \ldots, n\}$ for some integer n. (By a suitable coding we can usually achieve that a set of codes, or keys, of objects under consideration is such a set.) The following list contains hash functions that turned out to be useful:

$$\begin{aligned} h(u) &= u \bmod m && \text{(“division method”);} && (2.12)\\ h(u) &= \lfloor m(au - \lfloor au \rfloor)\rfloor && \text{(“multiplication method”);} && (2.13) \end{aligned}$$

where $m \in \mathbf{N}$ and $a \in \mathbf{R}$.

The choice of a suitable hash function and of its parameters, m in (2.12) or a in (2.13), depends on applications; they can be chosen also experimentally for a particular application. For example, it is usually good to choose m as a prime not close to a power of 2 in (2.12) and $a = (\sqrt{5} - 1)/2 = 0.6180339$ in (2.13), by Knuth (1969).

Exercise 2.3.26 *Explain why one should not use as m a power of 2 or 10 if the division method is used for hashing.*

Exercise 2.3.27 *Show how to implement multiplication hashing efficiently on a microcomputer.*

Exercise 2.3.28 *Show, using hash functions, that equality of two multisets with n elements can be checked in $\mathcal{O}(n)$ time with high probability.*

Hash functions play increasingly important role in foundations of computing and also in various applications. However, it has been the dictionary problem (Example 2.1.24), for which they have been invented by Luhn, in 1953, and where they first helped to get a surprisingly efficient implementation.

Example 2.3.29 (Dictionary—a hash table implementation)

Assume that any object o of a finite universe U has a unique integer key $key(o)$, and the set of these keys $K = \{0, 1 \ldots, m-1\}$ has m elements. Assume also that we have to our disposal a table $T[0 : m-1]$ of size m. Any set $S \subseteq U$ can be easily stored in T by storing every element o in the array entry $T[key(o)]$—see Figure 2.14a. This is called the "direct-access representation" of the set S. With such a set representation one clearly needs only $\mathcal{O}(1)$ time to perform any of the dictionary operations.

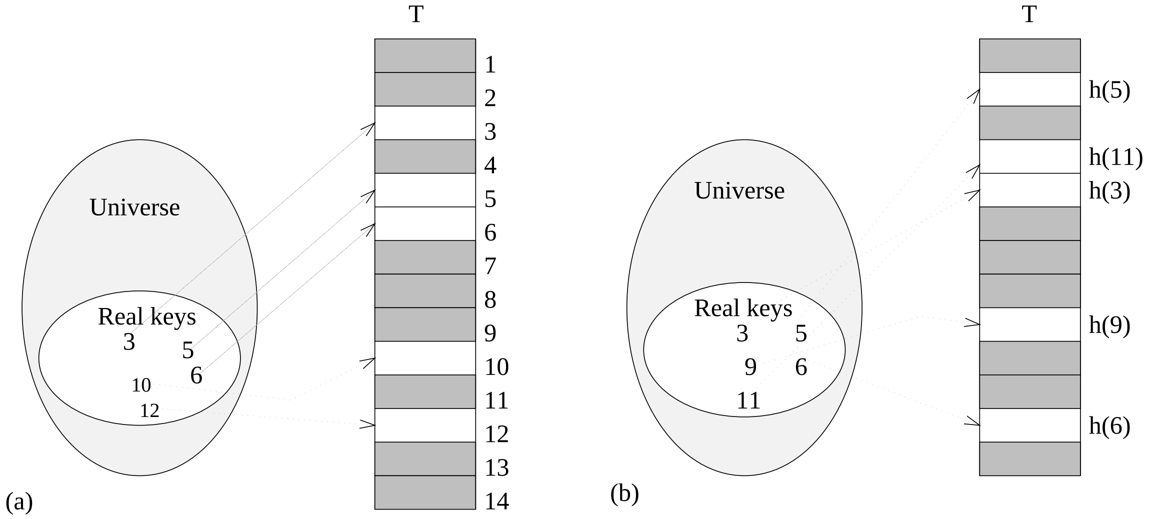


Figure 2.14: Direct-access and hash-table implementations of dictionaries-70%.

A more realistic case is when the size of U is much larger than the size m of T, but the dynamically changing set of the dictionary has never more then m elements and can therefore be stored, in principle, in the table T. Can we achieve also in such a case a constant time performance for dictionary operations? Yes, but only in the average, in the following way.

Let $h : U \rightarrow \{0, 1, \ldots m\}$ be a hash function. The basic idea is to store an object o not in the table entry $T[key(o)]$, as before, but in $T[h(key(o))]$—see Figure 2.14b. If h is a good hash function, then different elements from the dictionary set S are stored, with a large probability, in different entries of the table T—in such a case T is usually called the **hash table**. Would this happen always we could hope to get again $\mathcal{O}(1)$-time performance for all dictionary operations.

Unfortunately, if the size of U is larger than that of T, it may happen, no matter how good h is chosen, that $h(x) = h(y)$ for the keys of two elements to be stored in T. In such a case we speak about a **collision**.

Under the assumption of **uniform hashing**, that is a uniform distribution of keys, there are simple methods how to deal with the collision problem that lead to $\mathcal{O}(1)$-time performance for all dictionary operations, where the constant is really very small. For example, if the table entry $T[h(key(o))]$ is already occupied at the attempt to store o, then o is stored in the first empty entry among entries: $T[h(key(o)) + 1], \ldots, T[m-1], T[0], \ldots, T[h(key(o)) - 1]$; or we put all elements that hash to the same entry of the hash table in a list—this is the so-called **chaining method**. The worst-case performance is, however, $\mathcal{O}(n)$, where n is the size of the dictionary set. It may therefore happen that an adversary that knows h can supply such storage requests that all elements to be stored are mapped, by hashing, into the same entry of the table T.

Exercise 2.3.30 *Show how to implement dictionary operations efficiently when a hash table and chaining methods for collisions are used to store the underlying set.*

Exercise 2.3.31 *Consider a hash table of size $m = 1000$. Compute locations to which the keys 2^i, $1 \leq i \leq 9$, and $126, 127, 129, 130$ are mapped if the multiplication method is used for hashing and $a = (\sqrt{5} - 1)/2$.*

The way to avoid such a case of getting intentionally bad data, is to choose also the hash functions randomly, in the run time, from a carefully constructed family of hash functions. This approach is called the **universal hashing**. In case of dictionary implementations this approach guarantees a good average performance—provided the set of hash functions to choose from has the following property of universality:

Definition 2.3.32 *A finite family $\mathcal{H}$ of hash functions mapping the universe U into the set $\{0, 1, \ldots, m-1\}$ is called* **universal** *if*

$$\forall x, y \in U \quad |\{h \mid h \in \mathcal{H}, h(x) = h(y)\}| = \frac{|\mathcal{H}|}{m}.$$

In other words, if h is randomly chosen from $\mathcal{H}$, then a chance of a collision $h(x) = h(y)$, for $x \neq y$, is exactly $\frac{1}{m}$, that is also exactly the chance of a collision when two elements, $h(x)$

and $h(y)$, are randomly picked up from the set $\{0, \ldots m-1\}$. The following result confirms that a universal family of hash functions indeed lead to a good performance for a dictionary implementation.

Theorem 2.3.33 *If h is chosen randomly from a universal family $\mathcal{H}$ of hash functions mapping a set U to $\{0, \ldots, m-1\}$, with $|U| > m$, and h maps a set $S \subset U$, $|S| = n < m$, into $\{0, \ldots, m-1\}$, then the expected number of collisions involving an element from S is less than 1.*

Proof. For any two different elements x, y from U let X_{xy} be a random variable on $\mathcal{H}$ with value 1 if $h(x) = h(y)$, and 0 otherwise. By the definition of $\mathcal{H}$ the probability of a collision for $x \neq y$ is $\frac{1}{m}$, that is $EX_{xy} = \frac{1}{m}$. Since E is additive, see (1.72), we get for the average number of collisions involving x the estimation

$$EX_x = \sum_{y \in S} EX_{xy} = \frac{n}{m} < 1$$

□

The result is fine, but does a universal family of hash functions exist? It does, and it can be constructed, for example, as follows:

Assume that m, the size of T, is a prime and that a binary representation of any $u \in U$ can be expressed as a sequence of $r+1$ binary strings $s_0 \ldots s_r$ such that $u_i = bin(s_i) < m$ for $0 \leq i \leq r$.

To each $(r+1)$-tuple $a = (a_0, a_1, \ldots, a_r)$ of elements from $\{0, \ldots, m-1\}$ let h_a be the function from U to $\{0, \ldots, m-1\}$ defined by

$$h_a(u) = \sum_{i=0}^{r} a_i u_i \bmod m, \tag{2.14}$$

and let

$$\mathcal{H} = \{h_a \mid a \in \{0, \ldots, m-1\}^{r+1}\} \tag{2.15}$$

Clearly $|\mathcal{H}| = m^{r+1}$, and we get:

Theorem 2.3.34 *The family of functions $\mathcal{H} = \{h_a \mid a \in \{0, \ldots, m-1\}^{r+1}\}$, defined by $h_a(u) = \sum_{i=0}^{r} a_i u_i \bmod m$ is a universal family of functions.*

Proof. If $x \neq y$ are from U, then there is an i such that $x_i \neq y_i$, where x_i, y_i are the i-th substrings in the representation of x, y. Assume for simplicity that $i = 0$. (Other cases can be dealt with similarly.) For any fixed $a_1, \ldots, a_r$, there is exactly one a_0 such that

$$a_0(x_0 - y_0) \equiv \sum_{i=1}^{r} a_i(x_i - y_i) \pmod{m}$$

and therefore

$$h_a(x) = h_a(y).$$

Indeed, since m is prime and $|x_0 - y_0| < m$, then the Euclid's algorithm can be used to compute an inverse to $(x_0 - y_0) \bmod m$ and therefore a_0 does exist. Hence, each pair $x, y \in U$ collides exactly for m^r values of a because they collide exactly once for each possible r-tuple from $\{0, \ldots, m-1\}$. Since there are m^{r+1} possibilities for a, they collide with probability $\frac{m^r}{m^{r+1}} = \frac{1}{m}$ and this proves the theorem. □

In some applications the above mentioned requirement on the quality of elements in $\mathcal{H}$ can be relaxed, but the size of $\mathcal{H}$ may be an issue. (Less elements $\mathcal{H}$ has, fewer random bits are needed to choose a hash function from $\mathcal{H}$.) In other applications hash functions may need to have stronger properties. For example, that they should map elements that are 'close' to each other to elements "far apart" from each other.

Exercise 2.3.35 * *Let $\mathcal{H}$ be a family of hash functions mapping the universe of keys K into the set $[m]$. We say that $\mathcal{H}$ is k-universal if for every fixed sequence of k distinct keys $(x_1, \ldots, x_k)$ and for any h chosen randomly from $\mathcal{H}$, the sequence $(h(x_1), \ldots, h(x_k))$ is equally likely to be any of the m^k sequences of length k with elements from $[m]$. (a) Show that if $\mathcal{H}$ is 2-universal, then it is universal; (b) show that the family of functions defined by (2.14) is not 2-universal; (c) show that if the definition of the set $\mathcal{H}$, (2.15), is modified to consider functions $h_{a,b} = \sum_{i=0}^{k}(a_i u_i + b_i) \bmod m$, then $\mathcal{H}$ is 2-universal.*

Remark 2.3.36 Of large importance for applications are those hash functions that map any (binary) message up to a very large length n to a binary string of a fixed (and small) length m, which then serves as the **authenticator** (**fingerprint**) of the original message. Such hash functions should have the following properties.

1. $h(x)$ should be easy to compute, but it should be unfeasible, given an y to find an x such that $h(x) = y$ (in other words h should be a one-way hash function).
2. Given x it should be unfeasible to find a y such that $h(x) = h(y)$ and it should be also unfeasible to find a pair (x, y) such that $h(x) = h(y)$.

A very simple idea, and still quite good, even if it does not satisfies any of the above two conditions, is to partition a given binary message x into substrings $x_1, \ldots, x_k$ of length m and to compute $h(x) = \bigoplus_{i=1}^{k} x_i$.

Practical importance of such hash functions for a modern communications is so large that in 1993 National Institute of Standards and Technology (NIST) in USA developed a standard hash function (algorithm) called SHA (Secure Hash Algorithm) that maps any binary string with up to 2^{64} bits to a 160-bit binary string.

2.4 Graphs

Graph concepts are of limitless use in all areas of computing. For example, in representing computational, communication and cooperational processes, automata, networks, circuits and in describing relationships between objects. Graph theory concepts, methods, and results, and also graph algorithmic problems and algorithms, play an important rôle in the complexity theory and in the design of efficient algorithms and computing systems in general.

2.4.1 Basic Concepts

A directed (undirected) **graph** $G = \langle V, E\rangle$ consists of a set V of **vertices**, called also **nodes**, and a set $E \subseteq V \times V$ of **arcs (edges)**. A graph is **finite** if so is V; otherwise it is **infinite**. In a directed graph an arc (u,v) is depicted by an arrow from the node u to the node v (Figure 2.15a); in undirected graphs edges are depicted by simple lines connecting the corresponding vertices—Figure 2.15b, c. An edge (u,v) can be seen as consisting of two arcs (u,v) and (v,u).

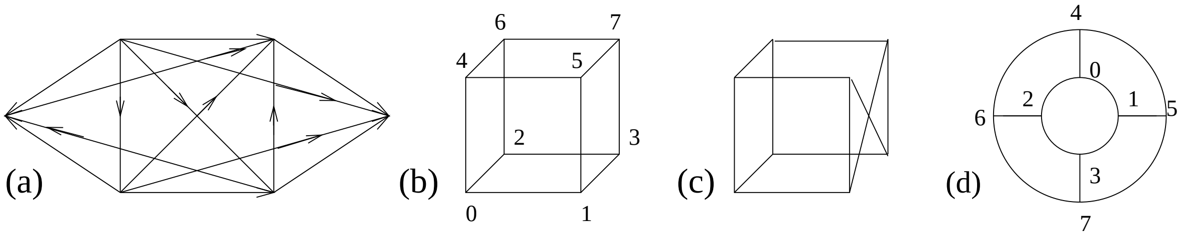


Figure 2.15: A directed graph and undirected graphs -80%

Incidences and degrees. If $e = (u,v)$ is an arc or an edge, then vertices u, v are **incident** with e and e is an arc (edge) **from** u **to** v incident with both u and v. The vertex v is called **adjacent** to u and u is also called the **neighbour** of v. Similarly, v is adjacent to u and is also its neighbour. An arc (u,v) is an **ingoing** arc of the vertex v and an **outgoing** arc of the vertex u. The **degree** of a vertex v—notation: $degree(v)$—in an undirected graph is the number of edges incident with v. In a directed graph, the **in-degree** of a vertex v—$in\text{-}degree(v)$—is the number of ingoing arcs of v; the **out-degree**—$out\text{-}degree(v)$—is the number of outgoing arcs of v. Finally, the degree of v is the sum of its in-degree and out-degree. The degree of a graph G—$degree(G)$—is the maximum of degrees of its vertices. For example, graphs in Figure 2.15a, b, c have degrees $5, 3$ and 3, respectively.

Walks, trails, paths and **cycles**. A **walk** p of length k, from a vertex u, called the origin, to a vertex v, callled the terminus, is a sequence of nodes $p = \langle u_0, u_1, \ldots, u_k\rangle$, such that $u_0 = u$, $u_k = v$, and $(u_i, u_{i+1}) \in E$, for $0 \leq i < k$. $u_0, \ldots, u_k$ are vertices on the walk p, or the vertices the walk p contains. Moreover, $(u_i, u_{i+1}), 0 \leq i < k$, are arcs (edges) on the walk p or arcs (edges) p contains. If there is a walk from a vertex u to a vertex v, then we say that v is **reachable** from u and that u and v are **connected**. The $distance(u,v)$ is the length of the shortest walk from u to v.

A walk is called a **trial** if it contains no arc (edge) twice and it is called a **path** if it contains no vertex twice.

A walk is closed if its origin and terminus coincide. A closed trial is called a **cycle** and a cycle is called a **simple cycle** if the only two identical nodes are its origin and terminus.

For example, the graph in Figure 2.15b has simple cycles only of length 4, 6 and 8. A cycle $(a, a), a \in V$ is called a **self-loop** and a simple cycle of length 3 is called a **triangle**. For example, none of graphs in Figure 2.15 has a triangle but the graph in Figure 2.16a has 16 triangles. (Find them!)

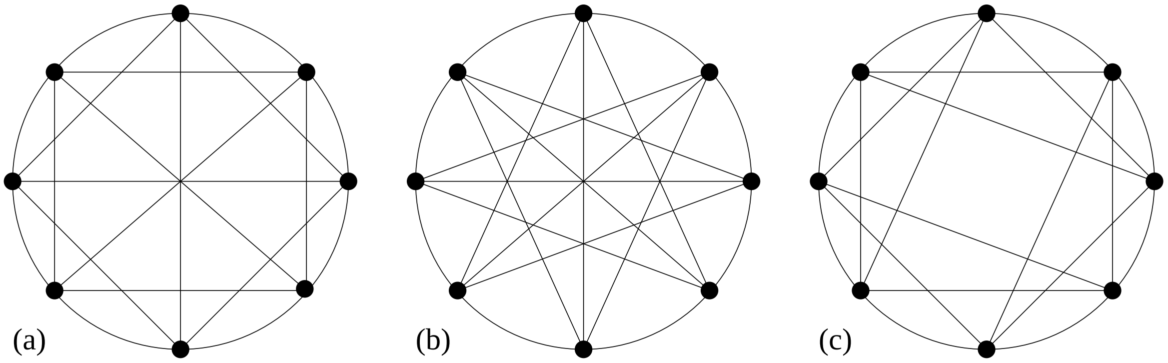


Figure 2.16: Graph isomorphism- isom-75%

A graph is called **acyclic** if it does not contain any cycle. A directed acyclic graph is also called a **dag**. For example, the graph in Figure 2.15a is acyclic.

Exercise 2.4.1 *(a) Determine the number of simple cycles of the graph in Figure 2.15c; (b) determine the number of triangles of the graph in Figures 2.16b.*

Exercise 2.4.2 * *Show that in any group of at least two people there are always two with exactly the same number of friends inside the group.*

Connectivity. In an undirected graph the relation "connected" is an equivalence relation on the set of vertices and its equivalence classes are called **connected components**. An undirected graph is called **connected** if all pairs of its vertices are connected. A directed graph is called **strongly connected** if for any two vertices u, v there is a path from u to v. The equivalence classes on the set of vertices of a directed graph, with respect to the relation "mutually conncted", are called **strongly connected components.**

In various applications it is of importance "how well connected" is a given graph. Intuitively, each successive graph in Figure 2.17 is more connected than the previous one. Indeed, the graph in Figure 2.17a can be disconnected by removing one edge, the graph in Figure 2.17b by removing one vertex. This is not the case for graphs in Figures 2.17c, d.

There are two main quantitative measures of the connectivity of graphs. **Vertex-connectivity**, v-conn(G) is the minimum number of vertices whose removal disconnects a graph G; **edge-connectivity**, e-conn(G), is the minimum number of edges whose removal disconnects G.

Exercise 2.4.3 *Show that v-conn(G) $\leq$ e-conn(G) $\leq$ degree(G) for any graph G.*

Exercise 2.4.4 ** *Show that if e-conn(G) ≥ 2 for a graph G, then any two vertices of G are connected by at least two edge-disjoint paths.*

If a graph represents a communication network, then the vertex-connectivity (edge connectivity) becomes the smallest number of communication nodes (links) whose breakdown would jeopardize communication in the network.

Isomorphism. Two graphs, $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$, are called **isomorphic** if there is a bijection (**isomorphism**) $\iota : V_1 \rightarrow V_2$ such that $(u,v) \in E_1 \Leftrightarrow (\iota(u), \iota(v)) \in E_2$. For example, graphs In Figure 2.15b, d are isomorphic and the graphs in Figure 2.15b, c are not. Any isomorphism of a graph with itself is called an **automorphism**.

To show that two graphs are isomorphic, one has to show an isomorphism between them. For example, the graphs in Figure 2.15b,d are isomorphic and the corresponding isomorphism is given by the mapping that maps a node labelled by an i in one graph into the node labelled by the same label in the second graph. Two isomorphic graphs can be seen as identical and only their representations may differ. To show that two graphs are not isomorphic is in general much harder.

Exercise 2.4.5 *Which pairs of graphs in Figure 2.16 are isomorphic and why?*

Exercise 2.4.6 *Show that if two graphs are isomorphic then there must exists a bijection between the sets of their vertices such that the corresponding nodes have the same degree and lay in the same number of cycles of any length.*

Regularity. Graphs encountred in applications and also in theory can be very complex and large. In order to manage large graphs in a transparent and effective way they have to have some degree of regularity. There are several approaches to the problem how to define regularity of graphs. Perhaps the simplest one is to consider a graph as **regular** if all its vertices have the same degree and it is k-regular, $k \in \mathbf{N}$, if all vertices have degree k. More stronger concepts, useful especially for the case of graphs modelling interconnection networks, are that of symmetry. A graph $G = \langle V, E \rangle$ is called **vertex-symmetric**, if for every pair u, v of vertices there is an automorphism α of G such that $\alpha(u) = v$. Clearly, each vertex-symmetric graph is regular. As an example, the graph in Figure 2.15b is vertex-symmetric but the graph in Figure 2.15c is not, even if it is regular. A graph G is called **arc-symmetric** if for every two arcs (u_1, v_1) and (u_2, v_2) there is an automorphism α of G such that $\alpha(u_1) = u_2$ and $\alpha(v_1) = v_2$. A graph G is called **edge-symmetric** if for every two edges (u_1, v_1) and (u_2, v_2) there is an automorphism α of G such that either $\alpha(u_1) = u_2$ and $\alpha(v_1) = v_2$ or $\alpha(u_1) = v_2$ and $\alpha(v_1) = u_2$.

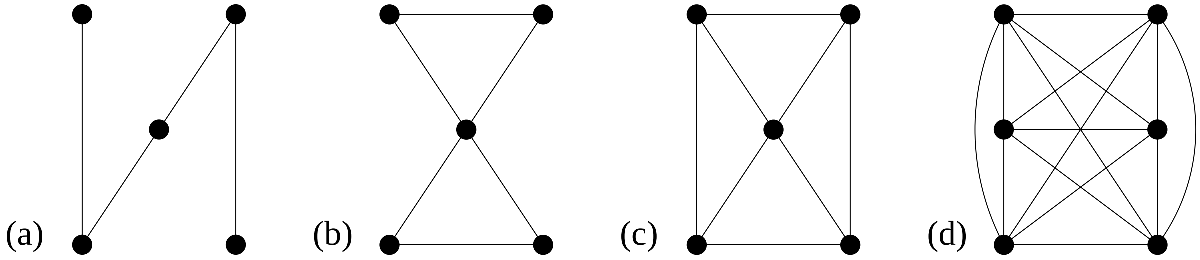


Figure 2.17: More and more connected graphs-80%

An example of a graph that is vertex-symmetric but not edge-symmetric, the so-called cube-connected cycles, is in Figure 2.35b. The importance of these concepts lies in the fact that a vertex-symmetric graph can be seen as "looking from each node (processor) the same" and an edge (arc) symmetric graph as "looking from each edge (arc) the same".

Exercise 2.4.7 *Are the graphs in Figures 2.16 and 2.18 vertex-, arc- and edge-symmetric?*

Exercise 2.4.8 * *Show that if G is a regular graph and $degree(G) \geq 3$, then $v\text{-}conn(G) = e\text{-}conn(G)$.*

Exercise 2.4.9 *Find an example of a graph that is: (a)* edge and not vertex symmetric; (b)* vertex but not edge symmetric; (c)** vertex and edge but not arc symmetric.*

Subgraphs. A graph $G_1 = \langle V', E' \rangle$ is a **subgraph** of the graph $G = \langle V, E \rangle$ if $V' \subseteq V$ and $E' \subset E$. We usually also say that a graph G' is a subgraph of a graph G if G' is isomorphic to a subgraph of G. For example, the graph in Figure 2.16c is a subgraph of the graph in Figure 2.16b—show it!

Several special types of graphs are so often used that they have special names.

Complete graphs of n nodes, notation K_n,are graphs with all nodes of degree $n-1$, see Figure 2.19 for K_5. Another name for complete graphs is a **clique**

Bipartite graphs. An undirected graph is called **bipartite** if its set of vertices can be partitioned into two subsets V_1, V_2 in such a way that each edge of G connects a vertex in V_1 and a vertex in V_2—the term **bipartition** is often used for such a partition. For example, the graphs in Figure 2.15b and Figure 2.18b, c are bipartite and the graphs in Figure 2.15c, and Figure 2.18a are not. A **complete bipartite graph** $K_{m,n}$ is a bipartite graph of $m+n$ nodes whose nodes can be partition into sets A and B with $|A| = m$, $|B| = n$, and two vertices are connected by an edge if and only if one is from A and another from B. Figure 2.18c shows $K_{4,3}$.

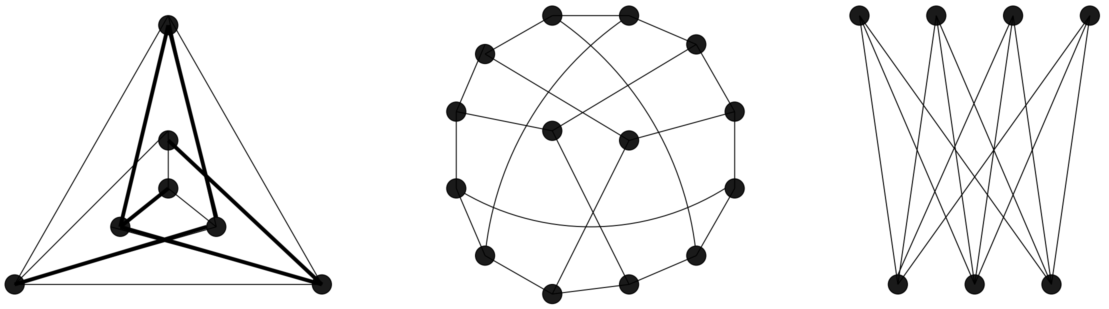
Exercise 2.4.10 *Show that graphs in Figure 2.18b and Figure 2.23d are bipartite.*

Exercise 2.4.11 * *Show that a graph is bipartite if and only if it contains no cycle of odd length.*

Bipartite graphs may seem to be too simple. However, some of the most important and also complicated graphs we shall deal with will be bipartite.

Trees. An undirected acyclic graph is called **forest**, see Figure 2.27b, and if it is connected then a **tree**—in more details we deal with trees in Section 2.4.36. A subgraph $T = \langle V, E' \rangle$ of an undirected graph $G = \langle V, E \rangle$ is called a **spanning tree** of G if T is a tree. The subgraph depicted by bold lines in Figure 2.18a is a spanning tree of the whole graph shown in Figure 2.18a. In general, if $G_1 = \langle V, E_1 \rangle$, $G_2 = \langle V, E_2 \rangle$, $E_1 \subseteq E_2$, then G_1 is called a **spanning subgraph** of G_1.

Exercise 2.4.12 *Design a spanning-tree for the graph in Figure 2.18b. How many different spanning trees does this graph have?*



(a) (b) (c)

Figure 2.18: A spanning tree and bipartite graphs-80%

Planar graphs. A graph is called **planar** if its vertices and edges can be drawn in the plane without any crossing of edges. Planarity is of importance in various applications, for example, in the design of electrical circuits. Figure 2.19a shows a graph that does not look planar but it is—see its other drawing in Figure 2.19b.

There is a simple to state condition for a graph not to be planar. To formulate this condition we define that a graph G' is a topological version of G if it is obtained from G by replacing each edge of G with arbitrarily long non-intersecting path.

Theorem 2.4.13 (Kuratowski's Theorem) *A graph is nonplanar if and only if it contains a subgraph that is a topological version either of the graph K_5 from Figure 2.19c or $K_{3,3}$ from Figure 2.19d.*

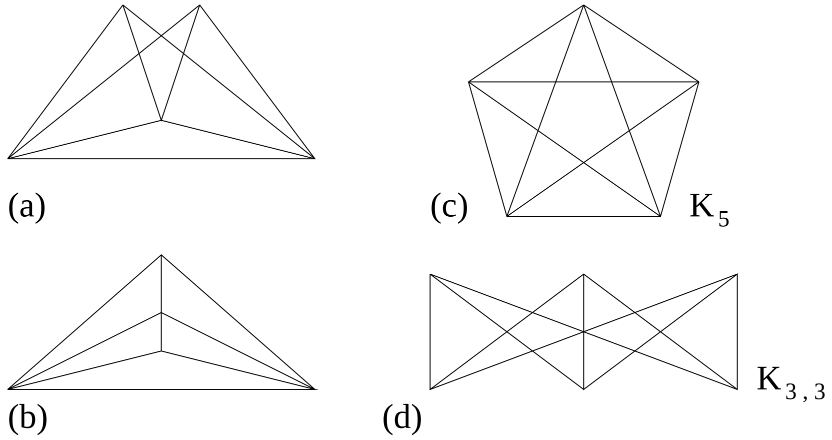


Figure 2.19: Planar and non-planar graphs-80%.

Exercise 2.4.14 *Show that each graph G is spatial, i.e. its nodes can be mapped into points of three-dimensional space in such a way that no straight-line edges connecting the corresponding nodes of G intersect neither with other edges or points representing nodes of G. (Hint: map the i-th node of G into the point (i, i^2, i^3).)*

Graph complexity measures. Number of vertices, $|V|$, and edges, $|E|$, are the main size measures of graphs $G = \langle V, E \rangle$. Clearly, $|E| \leq |V|^2$. From a variety of other graph characteristics of a special importance in computing are:

diameter — $\max\{distance(u, v) \mid u, v \in V\}$
bisection-width — the minimal number of edges one needs to remove from E to partition V into sets of $\lfloor \frac{|V|}{2} \rfloor$ and $\lceil \frac{|V|}{2} \rceil$ vertices.

For example, the graph in Figure 2.15b has diameter 3 and bisection width 4, and the graph in Figure 2.15c has diameter 2 and bisection width 4.

Exercise 2.4.15 *Determine the bisection-width of graphs in Figures 2.16 and 2.18.*

Diameter and bisection width are of importance for graphs that model communication networks; the longer the diameter of a graph, the more time two processor-nodes may need to communicate. The smaller the bisection-width, the narrower is a bottleneck through which node-processors of two parts of the graph may need to communicate.

Multigraphs and **hypergraphs**. They are several natural generalizations of the concept of an (ordinary) graph, as introduced above. A **multigraph** is like a (direted or undirected) graph, but may have multiple arcs or edges between vertices—see Figure 2.20a. Formally, a multigraph can be modelled as $G = \langle V, E, L \rangle$, where V is a set of vertices, $E \subseteq V \times V \times L$, and L is a set of labels. An arc $(u, v, l) \in E$ is an arc from the vertex u to the vertex v labeled by l. Labels are just used to distinguish different arcs between the same vertices.

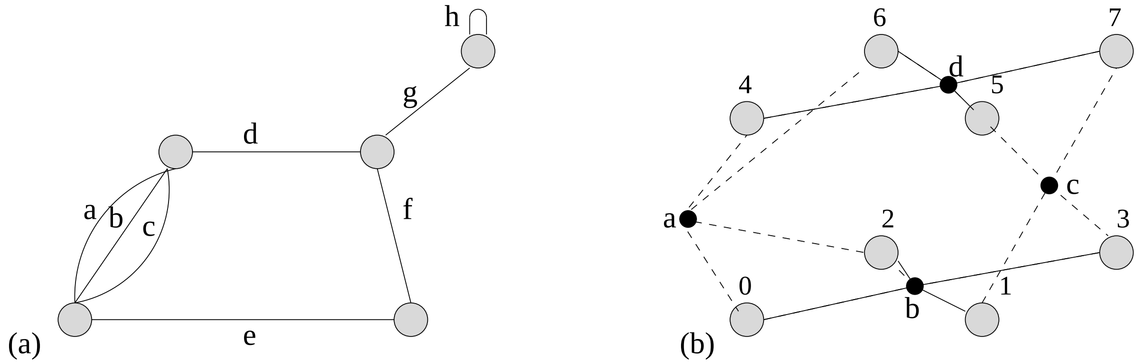


Figure 2.20: Multigraph and hypergraph-70%

A walk in a multigraph G is any sequence $w = v_0 e_1 v_1 e_2 \dots e_k v_k$ whose elements are alternatively nodes and arcs and e_i is an arc from v_{i-1} to v_i. On this base we define also trials and paths for multigraphs. Also the concept of isomorphism is defined for multigraphs similarly as for graphs.

In a similar way as graphs model binary relations, **hypergraphs** model n-ary relations for $n > 2$. An edge of a hypergraph is an n-tuple of nodes $(v_1, \dots, v_n)$. An edge can therefore connect more than two vertices. In figure 2.20b we have a hypergraph with 8 nodes and 4 hyperedges, each with four nodes: $a = (0, 2, 4, 6)$, $b = (0, 1, 2, 3)$, $c = (1, 3, 5, 7)$ and $d = (4, 5, 6, 7)$.

2.4.2 Graph Representations and Graph Algorithms

There are four basic methods to provide an explicit representation of graphs $G = \langle V, E \rangle$:

Adjacency lists. For each vertex u a list $L[u]$ of all vertices v such that $(u, v) \in E$ is given.

Adjacency matrices. A Boolean matrix of the size $|V| \times |V|$, with rows and columns labelled by vertices from V, and with 1 in the entry for a row u and a column v if and only if $(u, v) \in E$. Actually, this is a Boolean matrix represention of the relation E.

Incidence matrices. A Boolean matrix of size $|V| \times |E|$ with rows labelled by vertices and columns by arcs of G and with 1 in the entry for a row u and a column e if and only if the vertex u is incident with the arc e.

Words. $w(G)$ is a binary word $u_1 \dots u_{|V|}$, where u_i is the binary word of the i-th row of the adjacency matrix for G. In case of undirected graphs the adjacency matrix is symmetric and therefore it is sufficient to take for u_i only the last $n - i + 1$ elements of the i-th row of the adjacency matrix.

A description of a graph $G = \langle V, E\rangle$ by a list is of size $\Theta(|E|)$, by an adjacency matrix of size $\Theta(|V|^2)$, by an incidence matrix of size $\Theta(|V||E|)$ and by a word of size $\Theta(|V|^2)$. Lists are therefore mostly the most economical way to describe graphs. Matrix representation is of advantage when a direct access to its elements is needed.

Exercise 2.4.16 *Show that there are more economical representation than those mentioned above for the following graphs: (a) bipartite; (b) binary trees.*

None of the above methods is usable to describe infinite graphs, an infinite family of graphs, or very large graphs. (This is a real problem, because in some applications there is a need to work with graphs having more than 10^7 nodes.) In such cases other methods to describe graphs have to be used: specification of the set of nodes and edges by an (informal or formal) formula of logic, generation by generative, for example rewriting systems and, in applications, a variety of hierarchical descriptions is used. Implicit methods to describe families of graphs are used, for example, in Sections 2.6 and 10.1.

Computational complexity of graph problems. It is often important to decide whether a graph is connected, or planar; whether two graphs are isomorphic, or to design a spanning tree of a graph. To consider computational complexity of algorithmic problems on graphs usually one of the above graph representation techniques is used and unless explicitly specified otherwise, we assume that the adjacency matrix representation is used.

The first two of tasks mentioned above are computationally easy. Connectivity can be decided, for graphs with n nodes, in $\Theta(n)$ time on a sequential computer. A spanning tree can be constructed in $\mathcal{O}(n \lg n)$ time on a sequential computer. Surprisingly, there is also an $\mathcal{O}(n)$ time algorithm for sequential computers to decide planarity of graphs.

Graph isomorphism, on the other hand, seems to be computationally a very hard problem and, as we shall see in Section 5.5.1, it has a special position among algorithmic problems. No polynomial time algorithm is known for graph isomorphism, no proof that none exists. It is also not known whether graph isomorphism is an **NP**-complete problem—it seems that not. Interesting enough, if two graphs are isomorphic, there is a short proof (certificate) of it—just presenting an isomorphism. On the other hand, no simple proof is known in general to show that two non-isomorphic graphs are really non-isomorphic. However, as discussed in Chapter 9, there is a polynomial time interactive randomized protocol to show graph non-isomorphism.

2.4.3 Matchings and Colorings

Two simple graph concepts with numerous applications are matching and coloring.

Definition 2.4.17 *If $G = \langle V, E\rangle$ is a graph, then any subset $M \subseteq E$ is called a* **matching** *in G if no two edges of M coincide. A matching M* **saturates** *a set of nodes $V_0 \subseteq V$ if every node of V_0 is incident with an edge of M. A matching M is called* **perfect** *if it saturates V.*

Figure 2.21a shows a graph and its perfect matching and Figure 2.21b a graph that has no perfect matching.

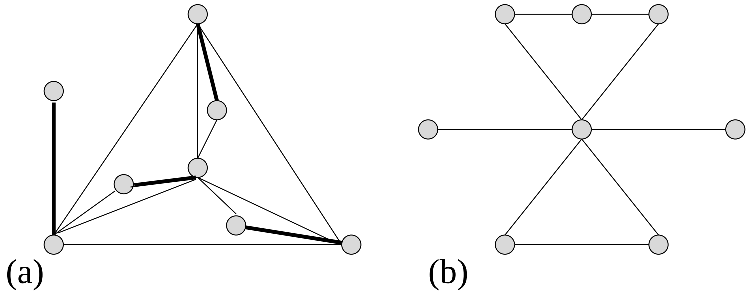


Figure 2.21: Graph matching-80%

Exercise 2.4.18 *Show that a binary tree has at most one perfect matching.*

Exercise 2.4.19 * *Two people play a game on a graph* $G = \langle V, E\rangle, |V| > 2$, *by alternatively selecting distinct nodes to form a path. The last player able to select a node wins. Show that the first player has a winning strategy if and only if* G *has no perfect matching.*

Of a special interest is matching of bipartite graphs and the following theorem contains the key result saying that if any subset of a set X of vertices of a bipartite graph "expands" by edges of the graph, then there is a matching that saturates X.

Theorem 2.4.20 (Hall's Theorem) *Let* $G = \langle V, E\rangle$ *be a bipartite graph with a bipartition* (X, Y). G *has a matching that saturates* X *if, for every* $S \subseteq X$, $|A(S)| \geq |S|$, *where* $A(S) = \{y \mid y \in Y, \exists x \in S, (x, y) \in E\}$.

As a corollary we get:

Theorem 2.4.21 *If* G *is a regular bipartite graph, then* G *has a perfect matching.*

Proof. Let G be a k-regular bipartite graph with a bipartition (X, Y). Since G is k-regular, $k|X| = |E| = k|Y|$ and therefore $|X| = |Y|$. Let now $S \subseteq X$ and let E_1 be the set of edges incident with S and E_2 the set of edges incident with $A(S)$. It follows from the definition of $A(S)$ that $E_1 \subseteq E_2$ and therefore

$$k|A(S)| = |E_2| \geq |E_1| = k|S|.$$

We therefore have $|A(S)| \geq |S|$. By Theorem 2.4.20 there is a matching M saturating X and since $|X| = |Y|$, M is a perfect matching. $\Box$

Theorem 2.4.21 is also called the Marriage Theorem, because it can be restated as follows: if every girl in a village knows exactly k boys, and every boy knows exactly k girls, then each girl can marry a boy she knows, and each boy can marry a girl he knows.

The following fundamental result is useful especially in order to prove non-existence of perfect matchings:

Theorem 2.4.22 (Talle's Theorem) *A graph has a perfect matching if and only if, for any k, if k vertices are deleted, there remains at most k connected components of odd size.*

Exercise 2.4.23 ** *Let* $G = \langle V, E\rangle$ *be a bipartite graph with the bipartition of vertices into sets* $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$. *To each edge* (a_i, b_j) *assign a variable* x_{ij}. *Let* $M_G = \{m_{ij}\}$ *be an* $n \times n$ *matrix such that* $m_{ij} = x_{ij}$ *if* $(a_i, b_j) \in E$ *and* $m_{ij} = 0$ *otherwise. Show that* G *has a perfect matching if and only if* $det(M)$ *is not identically* 0.

There are two types of graph colorings: edge and vertex coloring.

Definition 2.4.24 *An* **edge k-coloring** *of a graph* $G = \langle V, E\rangle$ *is an assignment of k elements (called colors) to edges of E in such a way that no two adjacents edges are assigned the same color. The* **chromatic index** *of G,* $\chi'(G)$*, is the minimal number of colors with which one can color edges of G. See, for example, the edge-coloring of the graph in Figure 2.22a.*

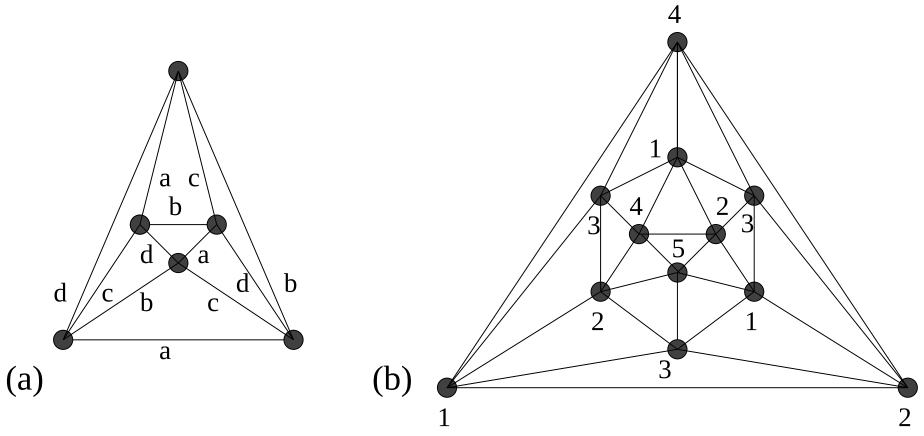


Figure 2.22: Edge and node colorings of graphs-80%

Two important results concerning the edge coloring that relate the degree of a graph and its chromatic index are now summarized:

Theorem 2.4.25 *(1) (***Vizing's Theorem***) If G has no self-loops, then either* $\chi'(G) = degree(G)$ *or* $\chi'(G) = degree(G) + 1$.[7].

(2) If G is a bipartite graph, then $\chi'(G) = degree(G)$.

---

[7] Interesting enough, to decide which of these two possibilities holds is an **NP**-complete problem even for 3-regular graphs (by Holyer (1981))

Exercise 2.4.26 *Show that* $\chi'(G) = degree(G) + 1$ *for the Petersen graph shown in Figure 2.36.*

Exercise 2.4.27 *Show how to colour a bipartite graph* $K_{m,n}$ *with* $degree(K_{m,n})$ *colours.*

Exercise 2.4.28 *Show that if* M_1 *and* M_2 *are disjoint matchings of a graph* G *with* $|M_1| > |M_2|$*, then there are disjoint matchings* M_1' *and* M_2' *such that* $|M_1'| = |M_1| - 1$*,* $|M_2'| = |M_2| + 1$ *and* $M_1 \cup M_2 = M_1' \cup M_2'$*.*

As an application of Theorem 2.4.25 we get:

Theorem 2.4.29 *If* G *is a bipartite graph and* $p \geq degree(G)$*, then there exist* p *disjoint matchings* $M_1, \ldots, M_p$ *of* G *such that*

$$E = \bigcup_{i=1}^{p} M_i$$

and, for $1 \leq i \leq p$*,*

$$\left\lfloor \frac{|E|}{p} \right\rfloor \leq |M_i| \leq \left\lceil \frac{|E|}{p} \right\rceil .$$

Proof. Let G be a bipartite graph. By Theorem 2.4.25 the edges of G can be partitioned into $k = degree(G)$ disjoint matchings $M_1', \ldots, M_k'$. Therefore, for any $p \geq k$ there exists p disjoint matchings (with $M_i' = \emptyset$ for $p > i \geq k$). Now we use the result of the Exercise 2.4.28 to get a well balanced matching. □

Finally, let us define a vertex-coloring of graphs. A **vertex k-coloring** of a graph G is an assignment of k colors to vertices of G in such a way that no incident nodes are assigned the same colour. The **chromatic number**, $\chi(G)$, of G is the minimum k for which G is vertex k-colorable. See Figure 2.22b for a vertex 5-coloring of a graph (called isocahedron).

One of the most famous problems of mathematics of this century was the so-called Four Color Problem, formulated in 1852: Is every planar graph 4-colorable?[8] The problem was solved by K. Appel and W. Haken (1977, 1977a), using ideas of B. Kempe, and their proof, made with the help of a computer, created a lot of controversy. They used a randomized approach to perform and to check a large number of reductions. The written version has more than 100 pages and at that time it was expected that one needs 300 hours of computer time for proof checking.

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[8] The problem was proposed by a student F. Guthree, who got the idea while coloring a map of counties in England. In 1879 B. Kempe published an erroneous proof that was believed for ten years to be correct.

2.4.4 Graph Traversals

Graphs are mathematical objects. In applications vertices represent processes, processors, gates, cities, plants, firms,... and arcs or edges communication links, wires, roads,... . Numerous applications and graph algorithms require to traverse graphs in some thorough and efficient way so that all vertices or edges are visited. There are several basic techniques to do that. Two of them, perhaps the most ideal ones, are Euler[9] tours and Hamilton[10] paths and cycles.

An **Euler tour** of a graph G is a closed walk that traverses each edge of G exactly once. A graph is called **eulerian** if it contains an Euler tour.

A path in a graph G that contains every node of G is called a **Hamilton path** of G; similarly, a Hamilton cycle is a simple cycle that contains every node of G. A graph is **hamiltonian** if it contains a Hamilton cycle.

For example, the graph in Figure 2.23a is eulerian but not hamiltonian; the graph in Figure 2.23b is both eulerian and hamiltonian; the graph in Figure 2.23c, called dodecahedron, is hamiltonian but not eulerian; and the graph in Figure 2.23d, called the Herschel graph, is neither hamiltonian nor eulerian.

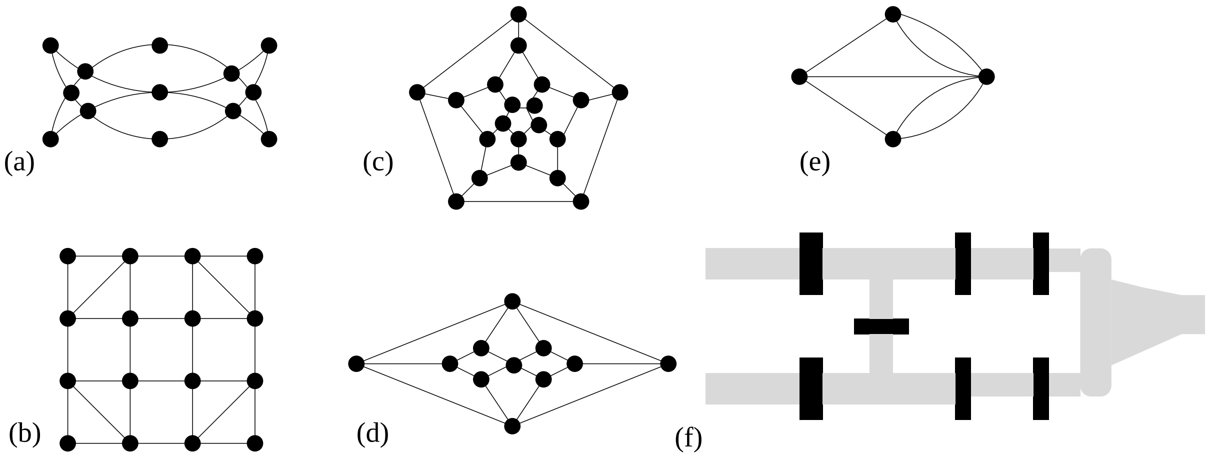


Figure 2.23: Euler tours and Hamilton cycles-65%.

Exercise 2.4.30 *Show that for every* $n \geq 1$ *there is a directed graph* G_n *with* $2n+3$ *nodes that has exactle* 2^n *Hamilton paths (and can therefore be seen as an encoding of all binary strings of length* n*).*

---

[9] Leonhard Euler (1707-1783)—a German and Russian mathematician of Swiss origin—made important contributions to many areas of mathematics and was enormously productive. He published more than 700 books and papers and left so much unpublished material that it took 49 years to publish it. His collected works to be published should have more than 95 volumes. Euler and his wife had 13 children.

[10] William Rowan Hamilton (1805-1865), an astronomer Royal of Ireland, perhaps the most famous Irish scientist of his era, made important contributions to abstract algebra, dynamics and optics.

Graph theory is rich in properties that are easy to define and hard to verify and problems easy to state and hard to solve. For example, it is easy to see whether graphs in Figure 2.23 have or do not have an Euler tour or an Hamilton cycle. The problem is whether this is easily decidable for an arbitrary graph. Euler tours cause no problem. It follows from the following theorem that one can verify in $\Theta(|E|)$ time whether a multigraph with the set E of edges is eulerian.

Theorem 2.4.31 *A connected undirected multigraph is eulerian if and only if each vertex has even degree. A connected directed multigraph is eulerian if and only if* $indegree(v) = outdegree(v)$ *for any vertex* v.

Proof. Let $G = \langle V, E, L\rangle$ be an undirected multigraph. If an Euler cycle enters a node it has to leave it unless the node is the starting node. From that the degree condition follows. Let us now assume that the degree condition is satisfied. It implies that there is a cycle in G. (Show why!) Then there is a maximal cycle that contains no edge twice. Take such a cycle C. If C contains all edges of G, we are done. If not, consider a multigraph G' with V as the set of nodes and exactly those edges of G that are not in C. Clearly also G' satisfies the even-degree condition and let C' be a maximal cycle in it with no edge twice. Since G is connected C and C' must have a common vertex. This means that from C and C' we can create a larger cycle than C having no edge twice. A contradiction to the maximality of C. The case of directed graphs is handled similarly. □

Exercise 2.4.32 *Design an algorithm to construct an Euler tour for a graph (provided it exists), and apply it to design an Euler tour for the graph in Figure 2.23a.*

Theorem 2.4.31, due to Euler (1736), is considered as initiating the graph theory. Interesting enough, the original motivation has been an intellectual curiosity whether there is such a tour for the graph shown in Figure 2.23e. This graph models paths across 7 bridges of Königsberg—Figure 2.23f—through which Euler liked to walk every day.

It may seem that the problem of Hamilton cycles is similar to that of Euler tours. For some classes of graphs it is known that they have Hamilton cycles (for example hypercubes). For others that they do not have. (For example, bipartite graphs with odd number of nodes.) There is also an easy to describe exponential time algorithm to solve the problem—check all possibilities. The problem to decide whether a graph has a Hamilton cycle or whether it has a Hamilton path is, however, **NP**-complete, see Section 5.4.

Exercise 2.4.33 *Design a Hamilton cycle for the graph in Figure 2.23c. (This is an abstraction of the original Hamilton's puzzle called "Round the World" that led to the concept of the Hamilton cycle—the puzzle was, of course, three dimensional.)*

Another way to traverse a graph so that all nodes are surely visited is to move along edges of a **spanning tree** of the graph. To construct a spanning tree for a graph G is easy. Start with S as the empty set. Check all edges of the graph, each once, and add the checked edge to S if and only if this does not make out of S a cyclic graph. (The order in which this is done does not matter.)

Two other general graph traversal methods, often useful in the design of efficient algorithms (and they actually also design spanning trees), are the **breadth-first search** and the **depth-first search**. They allow to search a graph and collect an information about the graph in linear time.

Given a graph $G = \langle V, E \rangle$ and a source node u, the **breadth-first search** first "marks" u as the node of distance 0 (from u), then visits all nodes reachable through an arc from u and markers them as nodes of distance 1. Recursively, in the i-th round, the breadth-first search visits all nodes marked by i and marks all nodes reachable from them by an arc, and not marked yet, by $i + 1$. The process ends if, in some round, no unmarked nodes are found. See Figure 2.24a for an example of a breadth-first traversal of a graph. This way the breadth-first search also computes for each node its distance from the source node u.

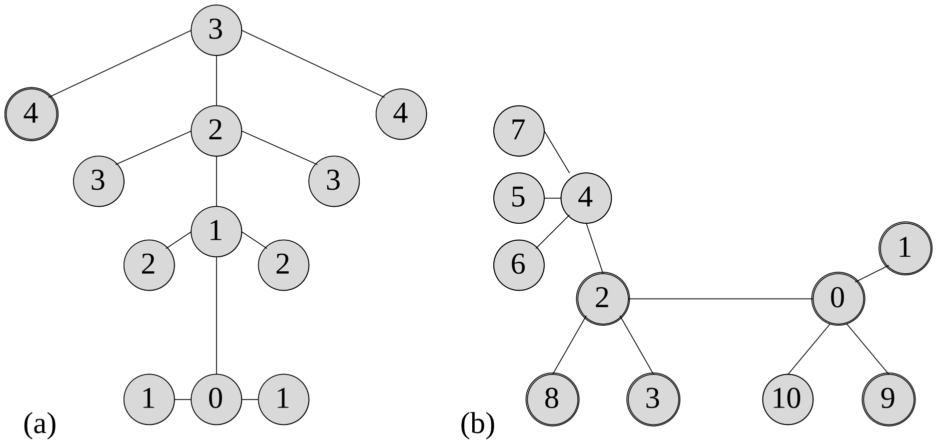


Figure 2.24: Bread-first search and Depth-first search-70%

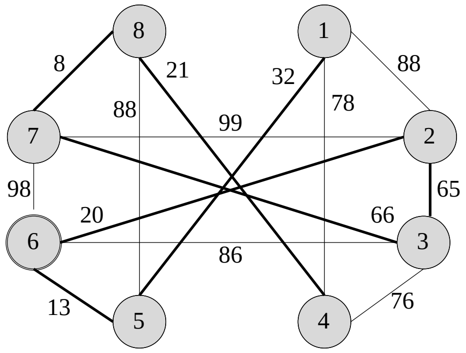
A **depth-first search** also starts traversing a graph from a source node u and marks it as "visited". Each time it gets through an edge to a node that has not been marked yet marks it as "visited" and tries to move out of that node through an edge to a node not yet marked. If there is no such edge, it backtracks to a node it came from and tries again. The process ends if there is nothing else to try—see Figure 2.24b for an example of a depth-first traversal of a graph.

The graph traversal problem gets a new dimension when to each edge a nonnegative integer—called its length—is associated, see Figure 2.25b—we then speak about a **distance graph**—and the task is then to find the most economical (shortest) traversing of the distance graph).

The first idea is to use in such a case a **minimal spanning tree**. It is a spanning tree of the distance graph with minimal sum of the length of its edges.

	1	2	3	4	5	6	7	8
1		88		78	32			
2	88		65			20	99	
3		65		76		86	66	
4	78		76					21
5	32					13		88
6		20	86		13		98	
7		99	66			98		8
8				21	88		8	

(a)



(b)

Figure 2.25: Minimal spanning tree-55%.

There are several simple algorithms to design a minimal spanning tree for a graph G. Perhaps the best known are Prim's algorithm and Kruskal's algorithm. Both start with the empty set, say T, of edges and both keep adding edges into T. **Kruskal's algorithm** takes care that edges in T always form a forest; **Prim's algorithm** that they form a tree. In a step both of them remove a shortest edge from E. Kruskal's algorithm inserts this edge to T if, after inserting the edge in, T still forms a forest. Prim's algorithm inserts the selected edge into T if, after inserting the edge in, T still forms a tree. Since dictionary operations can be implemented in $\mathcal{O}(\lg|V|)$ time, both algorithms can be easily implemented in $\mathcal{O}(|E|\lg|E|) = \mathcal{O}(|E|\lg|V|)$ time. (Prim's algorithm even in $\mathcal{O}(|E| + |V|\lg|V|)$ time—what is a better result.)

Exercise 2.4.34 *Find all distinct minimal spanning trees of the graph in Figure 2.26.*

Exercise 2.4.35 *Use Kruskal's and Prim's algorithms to design a minimal spanning tree of the graph in Figure 2.26.*

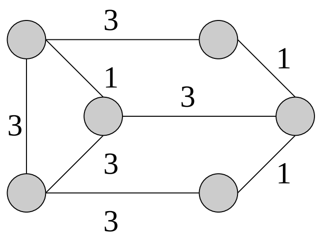


Figure 2.26: Minimal spanning tree

A closely related graph traversal problem for the distance graphs is a modification of the Hamilton cycle problem called the **travelling salesman problem**—TSP in short.

Given a complete graph $G = \langle V, V \times V \rangle$, $V = \{c_1, \dots, c_n\}$, and a distance $d(c_i, c_j)$ for each par of vertices (usually called cities in this case) c_i and c_j. Goal is to find a Hamilton path in G with the minimal sum of distances of all nodes. In other words, to find a permutation π on $\{1, \dots, n\}$ that minimizes the quantity

$$\sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)}) + d(c_{\pi(n)}, c_{\pi(1)})$$

No polynomial-time algorithm is known for this problem and also no proof is known that such an algorithm does not exist. A modification of TSP–given a graph G and an integer k, decide whether G has a travelling salesman cycle with total length smaller than k—is an **NP**-complete problem. The travelling salesman problem is perhaps the most studied **NP**-complete optimization problem, because of its importance in many applications—see Section 5.7.

2.4.5 Trees

Simple bipartite graphs very often used in computing are **trees**. As already mentioned in Section 2.4.1, a tree is an undirected, connected and acyclic graph. A set of trees is called a **forest**. See Figure 2.27 for a tree and a forest. The following theorem summarizes some of the basic properties of trees:

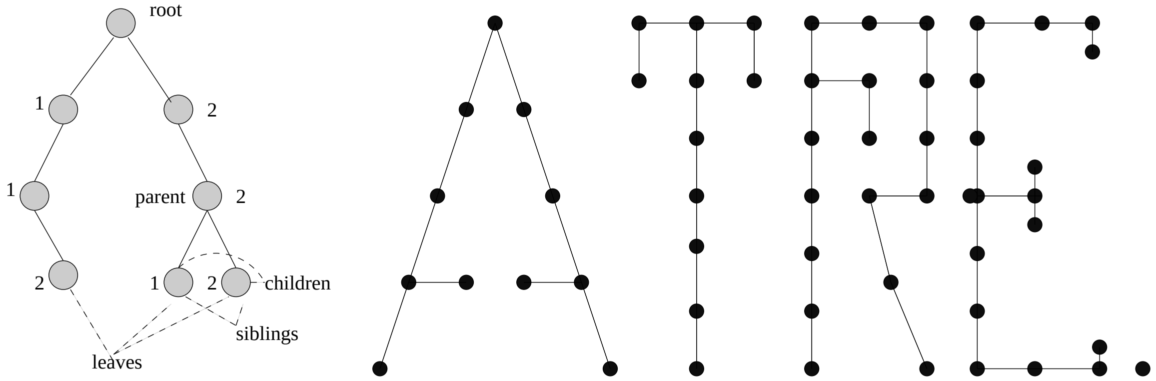
Theorem 2.4.36 *The following conditions are equivalent for a graph $G = \langle V, E \rangle$:*

1. *G is a tree.*
2. *Any two vertices in G are connected by a unique simple path.*
3. *G is connected and $|E| = |V| - 1$*
4. *G is acyclic but adding any edge to E results in a graph with a cycle.*

Exercise 2.4.37 *Prove as many equivalences in Theorem 2.4.36 as you can.*

Exercise 2.4.38 * *Determine: (a) the number of binary trees with n nodes; (b) the number of labeled trees with n nodes.*

Special terminology has been developed for trees in computing. By a tree it is usually meant a **rooted tree**—a tree where a special node is depicted as a **root**. All nodes on a path from the root to a node u, different from u, are called **ancestors** of u. If v is an ancestor of u, then u is a **descendant** of v. By the **subtree rooted in a node** x we understand the subtree containg x and all its descendants.



(a) Tree (b) Forest

Figure 2.27: A tree and a forest consisting of five trees-60%

If (y, x) is the last edge on a path from the root to a node x then y is **the parent** of x and x a **child** of y. Two nodes with the same parent are called **siblings**. A node that has no child is called a **leaf**. All other nodes are called **internal**.

The number of children of a node is its **degree** and its distance from the root is its **depth**. The **degree of a tree** is the maximal degree of its nodes and **depth of a tree** is the maximal depth of its leaves. (Observe that this meaning of degree is different from that of graphs in general.)

A tree is an **ordered tree** if to each node, except the root, a natural number is associated in such a way that siblings have always different numbers. The number associated with a node shows which child of its parent is that node. (Observe that a node can have only one, for example only the 5th child.)

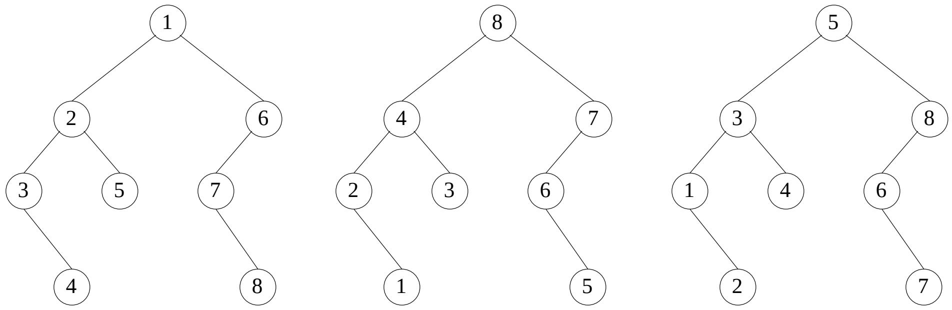
The term **binary tree** is used in two different ways. Firstly, as a tree in which any node has at most two children. Secondly, as an ordered tree in which any node has at most two children and to all nodes numbers 1 or 2 are associated. (In such a case we can talk about the first or the second child, or about the left or the right child. Observe that in such a case a node can have only the left or only the right child.) A **complete (balanced) binary tree** is a tree or an ordered tree in which each node, except leaves, has two children. More generally, a k-nary balanced tree is a tree, or an ordered tree, all nodes of which, except leaves, have k children.

Basic **tree traversal algorithms** are described and illustrated in Figure 2.28 by tree-labeling procedures, *preorder*—Figure 2.28a, *postorder*—Figure 2.28b and *inorder*—Figure 2.28b. All three procedures assume that there is a counter available that is initiated to 1 before the procedures are applied to the root and each time the procedure *Mark*(u) is used the current number of the counter is assigned to the node u and the content of the counter is increased by 1. All three procedures can be seen also as providing a labeling of tree nodes.

```
preorder($u$)

begin
Mark($u$);
preorder(leftson($u$));
preorder(rightson($u$));
end
```

```
postorder($u$)

begin
postorder(leftson($u$));
postorder(rightson($u$));
Mark($u$);
end
```

```
inorder($u$)

begin
inorder(leftson($u$);
Mark($u$);
inorder(rightson($u$);
end
```



(a) (b) (c)

Figure 2.28: Tree traversal algorithms-??%

Representation of binary trees. A binary tree with n nodes labelled by integers from 1 to n, can be represented by three arrays, say $P[1 : n]$, $L[1 : n]$, $R[1 : n]$. For each node i, the entry $P[i]$ contains the number of the parent of the node i, and entries $L[i]$ and $R[i]$ contain numbers of the left and the right child. With this tree representation any of the tree operations: (a) go to the father; (b) go to the left son; (c) go to the right son, can be implemented in $\mathcal{O}(1)$ time. (Other, more economical, representations are possible if not all three tree operations are used.)

2.5 Languages

The concept of a (formal) language is one of the key concepts in computing and becomes also one of the fundamental concepts of mathematics.

Formalization, as one of the key tools of science, leads to representation of complex objects by words and languages. Modern information processing and communication tools are also based on that. The understanding that complex objects, events, processes,... can be expressed by words and languages has developed already some time ago. Newer is the discovery that even simple languages can represent complex objects, if properly visualized.

2.5.1 Basic Concepts

An **alphabet** is an arbitrary (mostly finite) set of elements that are considered, in the given context, as having no internal structure.

Words. A **finite word (string)** w over an alphabet Σ is a finite sequence of elements from Σ; $|w|$ denotes its length. ε is the empty word of length 0.

A finite word w over Σ of length n can also be viewed as a mapping $w : \{1, \ldots, n\} \to \Sigma$, with $w(i)$ being its i-th symbol. In a similar way, an **infinite word** w over Σ (or an ω-word (ω-string)) can be seen as a mapping $w : \mathbf{N} \to \Sigma$ and a bi-infinite word w as a mapping $\mathbf{Z} \to \Sigma$. Analogically, one can consider also two dimensional rectangular words. For example, $w : \{1, \ldots, n\} \times \{1, \ldots, m\} \to \Sigma$. Two-dimensional infinite words can be defined as mappings $w : \mathbf{N} \times \mathbf{N} \to \Sigma$ or $w : \mathbf{Z} \times \mathbf{Z} \to \Sigma$.

Σ^* denotes the set of all finite words over Σ and Σ^+ the set of nonempty finite words over Σ. Σ^ω and $\Sigma^{\omega\omega}$ denote the sets of all infinite and doubly infinite words over Σ, respectively. $\Sigma^n (\Sigma^{\leq n})$ denotes the set of all strings over Σ of length n ($\leq n$).

Concatenation of a word u, of length n_1, and a word v, of lenght n_2, denoted by $u \cdot v$, or simply uv, is the word of length $n_1 + n_2$ such that $w(i) = u(i)$, for $1 \leq i \leq n_1$ and $w(i) = v(i - n_1)$, otherwise. Analogically, we define a concatenation of a word u from Σ^* and v from Σ^ω. **Powers** u^i, for $u \in \Sigma^*$ and $i \in \mathbf{N}$ are defined by $u^0 = \varepsilon, u^{i+1} = uu^i$ for $i \geq 0$.

Subwords. If $w = xyz$ for some finite words x and y and a finite or ω-word z, then x is a **prefix** of w, y is a **subword** of w and z is a **suffix** of w. If x is a prefix of w we write $x \preceq w$, and if x is a **proper prefix** of w, i.e., $x \neq w$ and $x \preceq w$, we write $x \prec w$. For a word w let $Prefix(w) = \{x \mid x \preceq w\}$.

The **reversal** of a word $w = a_1 \ldots a_n$, $a_i \in \Sigma$ is the word $w^R = a_n \ldots a_1$; a finite word w is **palindrome** if $w = w^R$.[11]

Projections. For a word $w \in \Sigma^*$ and $S \subseteq \Sigma$, w_S is the word obtained from w by deleting all symbols not in S. $\#_S w$ denotes the number of occurrences of symbols from S in w. For example, for $w = a^3b^3c^3$, $w_{\{a,c\}} = a^3c^3$ and $\#_a w = 3$.

A **morphism** ϕ from an alphabet Σ_1 to an alphabet Σ_2 is a mapping $\phi : \Sigma_1 \to \Sigma_2^*$. ϕ is often considered as being extended to map Σ_1^* into Σ_2^* as follows: for a $w = a_1 \ldots a_n$, $a_i \in \Sigma_1, \phi(w) = \phi(a_1) \ldots \phi(a_n)$. ϕ can also be naturally extended to map two-dimensional words, ω-words and $\omega\omega$-words.

Infinite words can be defined, for example, through morphisms and recurrences. Some of them have been so intensively investigated that they have their names.

Example 2.5.1 (Thue ω-word)

$$THUE = abbabaabbaababbabaababbaabbabaab\ldots$$

is defined as $\lim_{i \to \infty} x_i$, *where*

$$x_0 = a, \qquad x_i = \phi(x_{i-1}), \text{ for } i \geq 1$$

and $\phi : \{a, b\} \to \{a, b\}^*$ *a morphism with*

$$\phi(a) = ab \qquad \phi(b) = ba.$$

---

[11] Examples of palindromes in various languages: RADAR, ABLE WAS I ERE I SAW ELBA, RELIEFPFEILER, SOCORRAM ME SUBI NO ONIBUS EM MARROCOS, SAIPPUAKAUPPIAS, ANANAS OLI ILO SANANA, REVOLTING IS ERROR RESIGN IT LOVER, ESOPE RESTE ICI ET SE REPOSE, EIN LEDER GURT TRUG REDEL NIE, SATOR AREPO TENET OPERA ROTAS, NI TALAR BRA LATIN, ARARA, SOCORRAM ME SUBI NO ONIBUS EM MARROCAS, KOBYLA MA MALY BOK, JELENOVI PIVO NELEJ .

(Since $x_i \prec x_{i+1}$, for all $i \geq 0$, we can define $\lim_{i\to\infty} x_i$ as the only ω-word such that all x_i are its prefixes.

Example 2.5.2 (Fibonacci ω-word) [12]

$$FIB = abaababaabaababaababa.....$$

is defined by the recurrence $y_0 = a, y_i = \rho(y_{i-1})$, for $i \geq 1$, and the morphism $\rho(a) = ab$, $\rho(b) = a$.

Languages. If Σ is an alphabet, then any subset $L \subseteq \Sigma^*$ is called a **language** over Σ. Any subset of Σ^ω is called an ω-language. Similarly we define $\omega\omega$-languages.

Complement of a language L, notation L^c, is defined by $\Sigma^* - L$, where Σ is the smallest alphabet such that $L \subseteq \Sigma^*$.

Concatenation L_1L_2 of a language L_1 and a language, or an ω-language, L_2, is defined by

$$L_1L_2 = \{uv \,|\, u \in L_1, v \in L_2\}.$$

If L is a language, then its **powers** $L^i, i \geq 0$ are defined by $L^0 = \{\varepsilon\}$, $L^{i+1} = LL^i$ for $i \geq 0$, and its iterations L^+, L^* and L^ω are defined by

$$L^+ = \bigcup_{i=1}^{\infty} L^i; \quad L^* = \bigcup_{i=0}^{\infty} L^i,$$

and

$$L^\omega = \{u = u_1u_2u_3.... \,|\, u \text{ is an } \omega\text{-word and all } u_i \in L\}.$$

Shuffle operation on languages L_1 and L_2, notation $shuffle(L_1, L_2)$, or $L_1 \diamond L_2$ is defined by $L_1 \diamond L_2 = \{x_1y_1x_2y_2 \ldots x_ny_n \,|\, x_1 \ldots x_n \in L_1, y_1 \ldots y_n \in L_2\}$.

Morphism and inverse morphism of languages. If $L \subseteq \Sigma^*$, $L_1 \subseteq \{\Sigma_1\}^*$ and $\phi : \Sigma \to \Sigma_1^*$ is a morphism, then $\phi(L) = \{\phi(w) \,|\, w \in L\}$ and $\phi^{-1}(L_1) = \{w \,|\, w \in \Sigma^*, \phi(w) \in L_1\}$.

Exercise 2.5.3 *Which of the following identities are valid for all languages L_1, L_2, L_3 and all morphisms h: (a) $L_1(L_2 \cap L_3) = L_1L_2 \cap L_1L_3$;*

(b) $(L_1L_2)^*L_1 = L_1(L_2L_1)^*$; (c) $(L_1 \cup L_2)^* = L_2^*(L_1L_2^*)^*$; (d) $h(h(L_1)) = h(L_1)$;

(e) $h(L_1L_2) = h(L_1)h(L_2)$; (f) $h(L_1^*) = h(L_1)^*$.

---

[12] The Fibonacci word has been intensively investigated and has a variety of interesting properties, see Berstel (1985). For example, it has $n+1$ different subwords of length n for any integer n; if x is its subword, then so is x^R, there is an $x \neq \varepsilon$ such that x^3 is its subword but there is no $x \neq \varepsilon$ such that x^4 is a subword of FIB.

A **substitution**, from an alphabet Σ to an alphabet Σ_1, is a mapping $\sigma : \Sigma \rightarrow 2^{\Sigma_1}$. It can be extended to map $\Sigma^* \rightarrow 2^{\Sigma_1^*}$ as follows:

$$\sigma(a_1 \ldots a_n) = \sigma(a_1) \ldots \sigma(a_n) \text{ if } a_1 \ldots a_n \in \Sigma^*, a_i \in \Sigma,$$

and for a language L

$$\sigma(L) = \bigcup_{w \in L} \sigma(w).$$

A **left quotient** of languages A, B is defined by

$$A^{-1}B = \{y \mid \exists x \in A, xy \in B\}.$$

For example, if $L = \{a^i b^i \mid i \geq 1\}$, then $\{a^2\}^{-1}L = \{a^i b^{i+2} \mid i \geq 0\}$.

Exercise 2.5.4 *Determine* $A^{-1}B$ *for the following pairs of languages:*
(a) $A = \{a^i b^i \mid i \geq 0\}$, $B = \{a^i b^j \mid j \geq i \geq 0\}$;
(b) $A = \{ww \mid w \in \{0,1\}\}$, $B = \{ww^R \mid w \in \{0,1\}\}$.

A **prefix closure** of a language L, $Prefix(L) = \{y \mid \exists z, yz \in L\} = \bigcup_{w \in L} Prefix(w)$. A language L is said to be a **prefix-closed language** if $Prefix(L) = L$. A language L is called **prefix-free** if $x \prec y$ for no two x, y in L, i.e. no word in L is a proper prefix of another word in L.

Example 2.5.5 *If* $L = \{a^i b^i \mid i \geq j \geq 0\}$, *then* $Prefix(L) = \{a^i b^j \mid i \geq j \geq 0\}$. *The language* L *is prefix-free and the language* $Prefix(L)$ *is prefix-closed.*

Exercise 2.5.6 * *Let* $L \subset \{0,1\}^*$ *be a prefix-free language. Show that the following* **Kraft's inequality** *holds:* $\sum_{w \in L} 2^{-|w|} \leq 1$.

Families of languages. If $\mathcal{L}$ is a family of languages, then *co-*$\mathcal{L}$ is the family of languages whose complement belongs to $\mathcal{L}$, i.e.,

$$co\text{-}\mathcal{L} = \{L \mid L^c \in \mathcal{L}\} = \{L^c \mid L \in \mathcal{L}\}.$$

Example 2.5.7 *If* $\mathcal{F}$ *is the family of finite languages, then co-*$\mathcal{F}$ *is the family of "co-finite languages"—that is languages whose complements are finite.*

2.5.2 Languages, Decision Problems and Boolean Functions

As already discussed in Section 2.1 an intuitive concept of a computational problem can be formalized through a language decision problem, for an $L \subseteq \Sigma^*$, or a search problem, for an

$R \subseteq \Sigma^* \times \Sigma^*$. Both of them can be further "computerized". For languages and relations over the binary alphabet, through the concept of an infinite sequence of Boolean functions. For example, we can associate with a language $L \subseteq \{0,1\}^*$ the following sequence of Boolean functions $\{f_{L,i}\}_{i=1}^{\infty}$ defined by $f_{L,i}(x_1,\ldots,x_i) = 1$ if and only if $x_1 \ldots x_i \in L$, with each $f_{L,i}$ representing actually an instance of the given computational problem.

Example 2.5.8 *The graph isomorphism problem can be modeled by a language membership problem as follows: Let* $L_{GI} \subseteq \{0,1\}^*$ *be the set of all words* w *such that* $w = xy$, *where* $x = w(G_1)$ *and* $y = w(G_2)$ *for two isomorphic graphs. The problem to decide whether two graphs* G_1 *and* G_2 *are isomorphic is then reduced to the problem to decide whether* $w(G_1)w(G_2) \in L_{GI}$.

Example 2.5.9 *Boolean matrix multiplication problem can be modeled by the language* $L_{BMM} = \{w \mid w = w(M_1)w(M_2)w(M_3),$ *where* M_1, M_2, M_3 *are Boolean matrices and* $M_1 \cdot M_2 = M_3\}$. *The problem of multiplying two Boolean matrices* M_1 *and* M_2 *of the same degree is then reduced to the problem of searching for a matrix* M_3 *such that* $w(M_1)w(M_2)w(M_3) \in L_{BMM}$.

Example 2.5.10 *The language* $L = \{ww \mid w \in \{0,1\}^*\}$ *is represented by the following family of Boolean functions* $\mathcal{F} = \{f_L^n\}_{n=1}^{\infty}$, *where*

$$f_L^n(x_1 \ldots x_n) = \begin{cases} \bigwedge_{i=1}^{n/2}(x_i \equiv x_{\frac{n}{2}+i}), & \textit{if } n \textit{ is even;} \\ 0, & \textit{otherwise.} \end{cases}$$

In order to make the above idea to be really well defined, a fixed representation of objects of concern has first to be chosen, and also a fixed encoding of such a representation. Fortunately, as we shall see in Chapters 5 and 6, for investigation of such basic questions as "which computations are (practically) feasible", most of the usual representation of objects we work with are "equally good", and the same is with encoding methods. For example, any of the following graph representations: lists, adjacency matrices, incidence matrices, and binary words, can be easily—in low polynomial time—converted into each other.

Naturally, there are representations which are possible, and sometimes even desirable, and which do not meet the condition of an easy transferabilty to other usual representations. An important example of two "exponentially" different representations is in case of integers. All representations in positional number systems can be considered as mutually transferable one to another. The representation of integers in the unary number system is, however, essentially different because it is exponentially larger. As a result, as we shall see in Chapter 5, some algorithms that are exponential with respect to any positional representation of input integers, are becoming polynomial, with respect to the size of unary representations of input integers.

Exercise 2.5.11 *Give a Boolean function representation for the following languages; (a)* $L = \{www \mid w \in \{0,1\}^*\}$; *(b)* $L = \{w \mid w \in \{0,1\}^*, \#_1(w) = 3\}$; *(c)* $L = \{w \mid w \in \{0,1\}^*, G(w)$ *is a graph containing a triangle*$\}$.

2.5.3 Interpretations of Words and Languages

Not only computational objects have natural representations in terms of words and languages. There are also various general interpretations of words and languages that provide interesting objects.

Tree representations. Any rooted tree T, finite or infinite, of a finite degree, say k, can be represented by a language $L_T \subseteq \{1,\dots,k\}^*$, using a node labelling, as follows—see Figure 2.29, for an illustration. The root is labeled by ε and children of each node are labeled by distinct integers from the set $\{1,\dots,k\}$. Let L_T be the set of words over the alphabet $\{1,\dots,k\}$ that describe all possible paths in the tree T—for each path in the tree, starting in the root, the word consisting of labels of nodes on the path is taken into L_T. For example, for the tree in Figure 2.29 the language L_T contains the following words of length at most 2: $\varepsilon, 1, 3, 11, 12, 32, 33$.

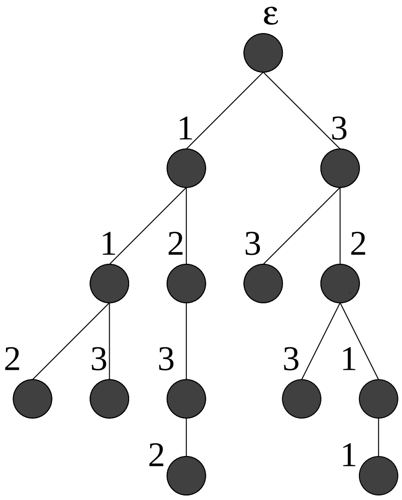


Figure 2.29: Tree languages-80%

Similarly, any language $L \subseteq \Sigma^*$ such that $Prefix(L) = L$ can be seen as a tree T_L—each word of L is a node of T_L; ε is the root and if $ua \in L, u \in L, a \in \Sigma$, then the vertex ua is a child of the vertex u. For example, the language $\{1,2,\dots,k\}^*$ describes the infinite complete[13]k-ary tree.

The fact that prefix-closed languages represent ordered trees can be used to define various families of trees using language recurrences. In the following examples, see also Figure 2.30, observe that in some cases we need to use the whole set of integers as the alphabet because there is no upper bound on the degree of trees in the family of trees that is being defined.

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[13] The term "balanced" is also used instead of "complete".

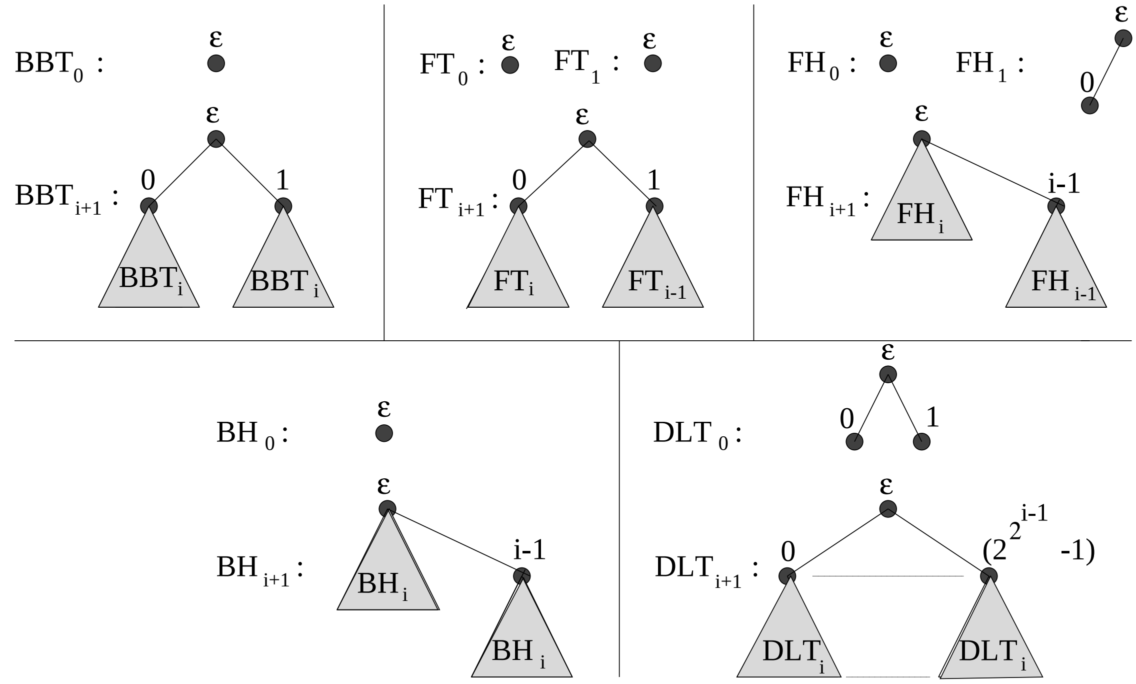


Figure 2.30: Recursively defined families of trees -70%

Balanced binary trees:	$BBT_0 = \{\varepsilon\}$, $BBT_i = \{\varepsilon\} \cup 0\, BBT_{i-1} \cup 1\, BBT_{i-1}, i \geq 1$;
Fibonacci trees:	$FT_0 = \{\varepsilon\}, FT_1 = \{\varepsilon\}$, $FT_i = \{\varepsilon\} \cup 0\, FT_{i-1} \cup 1\, FT_{i-2}, i > 1$;
Fibonacci heaps:	$FH_0 = \{\varepsilon\}, FH_1 = \{\varepsilon\} \cup \{0\}$, $FH_i = FH_{i-1} \cup (i-2)\, FH_{i-2}, i \geq 2$;
Binomial heaps:	$BH_0 = \{\varepsilon\}$, $BH_i = BH_{i-1} \cup (i-1)\, BH_{i-1}, i \geq 1$;
Doubly-logarithmic-depth trees:	$DLT_0 = \{\varepsilon, 0, 1\}$, $DLT_i = \{\varepsilon\} \cup 0\, DLT_{i-1} \cup \ldots \cup 2^{2^{i-1}-1}\, DLT_{i-1}$.

Similarly, any ω-language represents an infinite leaf-less tree.

Number representations. Any language $L \subseteq \{0, \ldots, k-1\}^*$ can be seen as representing a set of integers written in the k-nary positional system—the so-called radix notation. Similarly, any ω-language $L \subseteq \{0, \ldots, k-1\}^\omega$ can be interpreted as a set of reals in the interval $[0, 1]$.

Exercise 2.5.12 * *Show that the language* $\{0,2\}^\omega$ *can be seen as representing the Cantor set defined in the Example 2.1.2. (Since there is a trivial bijection between languages* $\{0,1\}^\omega$ *and* $\{0,2\}^\omega$*, and the first one can be seen as representing all reals—in the binary system—in* $[0,1)$*, we get that the cardinality of the Cantor set is exactly* $|\mathbf{R}|$*.)*

Image representations. Surprisingly, simple words and languages can represent complex curves and images. This is again an example how vizualisation brings a new dimension into investigations of formal objects. Basics of several such interpretations will now be discussed.

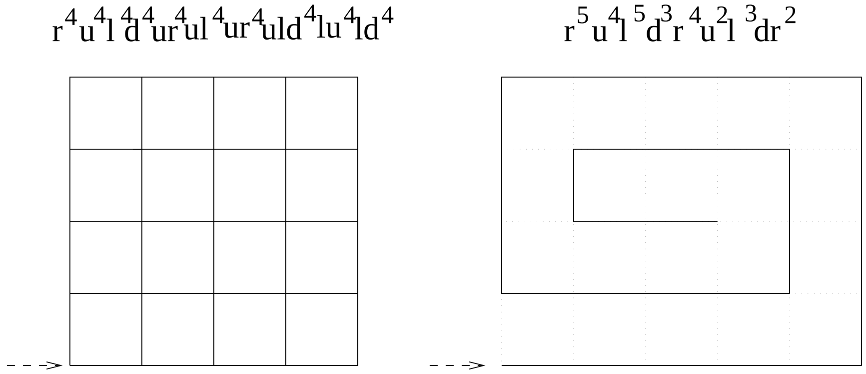


Figure 2.31: Chain code pictures and their drawing algorithms-75%

Chain code pictures. Any word over the alphabet $\{l,r,u,d\}$ can be seen as an algorithm to draw a picture along the lines of the unit grid. A starting point in the grid is assumed. Symbols l,r,u,d are interpreted as follows:

l — draw a unit-line to the left,
r — draw a unit-line to the right,
u — draw a unit-line up,
d — draw a unit-line down.

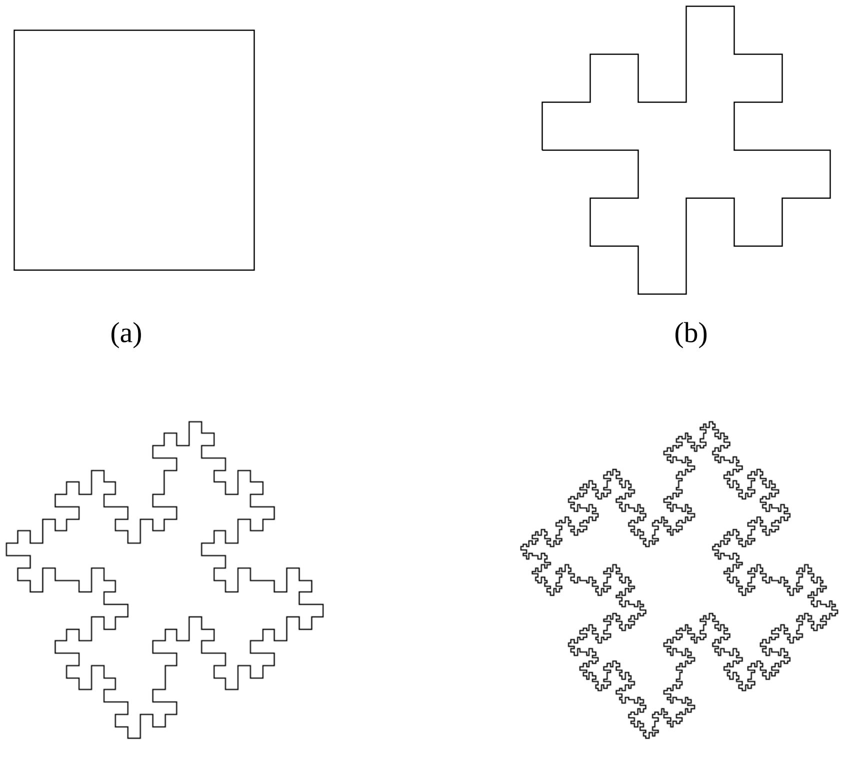
Figure 2.31 shows two chain code pictures drawing algorithms (words), and the corresponding pictures (an arrow points to the starting point). Naturally, each language $L \subseteq \{l,r,u,d\}^*$ represents a set of such pictures. For example, the language $\{r^n u^n l^n d^n \mid n \geq 1\}$ represents the infinite set of squares whose sides have integer-lengths.

Turtle interpretation of words is a natural generalization of the idea of the chain code pictures. It is allowed to draw lines of various lengths, to turn in various angles, and to interrupt drawings. This is amplified by a special choice of sets of words serving as drawing programs.

The basic idea of a turtle representation of words is the following one: At the beginning of a drawing, and after each move, the "turtle" is in a state (x, y, α), where (x, y) are coordinates of a point in the plane and α is an angle, called the **heading**—the direction the turtle is facing at that point. The picture, that the turtle "draws", is determined by the initial state (x_0, y_0, α_0) of the turtle, two parameters (d—the length, and δ—the angle), and by a **drawing algorithm—a word over the alphabet** $\{F, f, +, -\}$. Each symbol of such a word is taken as a statement with the following semantics:

F	–	draw a line of length d in the direction determined by the heading of the current state; that is move from the current state, (x, y, α) to the state $(x + d\cos\alpha, y + d\sin\alpha, \alpha)$ and draw a line between these two points;
f	–	make the same change of the state as in the case of statement F, but draw no line—this allows to interupt a drawing;
$+$	–	turn left by the angle δ;
$-$	–	turn right by the angle δ.

Figure 2.32a shows a turtle interpretation of the string "F-F-F-F" with $d = 1, \delta = 90$ and with turtle facing up initially.


(a) (b)

(c) (d)

Figure 2.32: Koch island figures for n = 0, 1, 2, 3

Interesting families of pictures can be obtained using turtle interpretations of strings obtained from some string w_0 by the iteration of a morphisms ϕ using the scheme $w_i = \phi(w_{i-1}), i \geq 1$.

Example 2.5.13 *The interpretations of words* w_1, w_2, w_3, *for* $w_0 = F - F - F - F$ *and the morphism* $\phi(F) = F - F + F + FF - F - F + F$, $\phi(-) = -$, *and* $\phi(+) = +$ *are shown in Figure 2.32b,c,d. (Resulting curves are called Koch island figures.)*

Exercise 2.5.14 *Draw a turtle interpretations of* w_2 *for* $w_0 = F + F + F + F$, $\phi(F) = F + f - FF + F + FF + Ff + FF - F - FF - Ff - FFF$, $\phi(f) = ffffff$ $\phi(+) = +, \phi(-) = -$, *and* $d = 1$, $\delta = 90$.

Turtle interpretations and various modifications of these interpretations will be discussed in more detail in Section 7.4.

Points representations. Let $A_k = \{(i,j) \,|\, 0 \leq i,j < k, i,j \in \mathbf{N}\}$ be an alphabet—where pairs (i,j) are taken as symbols. Any word

$$(x_1, y_1) \dots (x_n, y_n)$$

over the alphabet A_k can be interpreted as the point of the square $[0,1) \times [0,1)$ with the coordinates

$$(0.x_1 \dots x_n,\ 0.y_1 \dots y_n),$$

where $0.x_1 \dots x_n$ and $0.y_1 \dots y_n$ are interpreted as fractions expressed in the k-nary positional system. Any language L over A_k represents therefore a set P_L of points in the square $[0,1) \times [0,1)$.

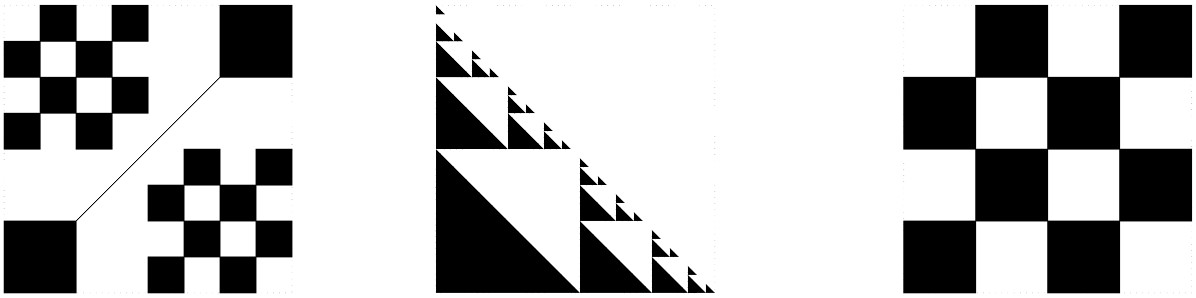
Example 2.5.15 *Points representation of languages* $\{(0,1),(1,0),(1,1)\}^*$ *and*

$$\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}^*,$$

with $k = 3$ *is shown in Figure 2.1. Figure 2.33a shows the image described by the language* $\{(0,1),(1,0)\}\{(0,0),(1,1)\}^2X^* \cup \{(0,0)(0,0),(1,1)(1,1)\}X^*$ *where* $X = \{(0,0),(0,1),(1,0),(1,1)\}$. *Figure 2.33b shows the image described by the language* $A^*(0,0)A^*(0,0)B^*$, *where* $A = \{(0,1),(1,0)\}$ *and* $B = A \cup \{(0,0),(1,1)\}$.

Let us now analyze the **image manipulation interpretations of basic language operations**. It is clear that union and intersection of languages yield union and intersection of images. More interesting are operations of concatenation, iteration and quotient.

Concatenation of two languages performs the **placement operation** on the corresponding images in the following sense: Let L_1, L_2 be languages over the alphabet A_k, for some k. Then the language $L = L_1L_2$ represents the image obtained by placing, appropriately scaled down (as described below), a copy of the image represented by the language L_2, at each point of the image represented by L_1. That is, if $(0.x_1 \dots x_n, 0.y_1 \dots y_n) \in P_{L_1}$, then a



(a) (b) (c)

Figure 2.33: Fractal pictures–75%

copy of P_{L_2}, scaled down by the factor $\frac{1}{k^n}$, is placed in the square of size $\frac{1}{k^n} \times \frac{1}{k^n}$ with its left-down corner located in the point $(0.x_1 \ldots x_n, 0.y_1 \ldots y_n)$.

For example, the language A_3^* represents the whole $[0,1) \times [0,1)$ square, and $\{(0,0),(0,2),(1,1),(2,0),(2,2)\}A_3^*$ represents the picture in Figure 2.33c.

The picture-manipulation meaning of the iteration is now clear. For example, the language $\{(0,0),(1,1)\}^*$ represents the diagonal of the unit square.

The left quotient operation represents **zooming**—an inverse of placement. The quotient $\{w\}^{-1}L$, where $w \in A_k^*, w = (x_1,y_1)\ldots(x_n,y_n)$ can be seen as picking up the subimage of P_L that is located in the square of the size $\frac{1}{k^n} \times \frac{1}{k^n}$ with its left-down corner in the point $(0.x_1 \ldots x_n, 0.y_1 \ldots y_n)$, enlarging it k^n-times and placing it in the quadrant $[0,1) \times [0,1)$. For example, $\{(0,1)\}^{-1}L$, for the language L representing the image in Figure 2.33a, describes the image shown in Figure 2.33c.

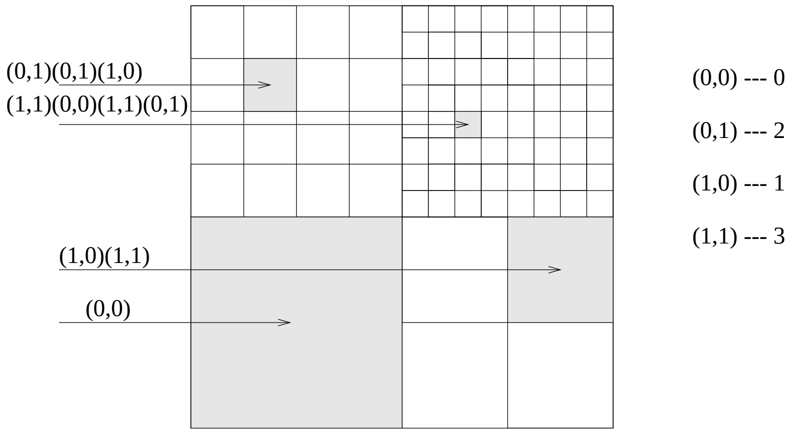


Figure 2.34: Pixel representation–55%

Exercise 2.5.16 *Draw the image obtained by the point representation of the expression* $(1,1)^*(0,0)I^*(1,0)T^*$*, where* $I = \{(0,0),(1,1)\}$ *and* $T = I \cup \{(0,1),(1,0)\}$.

Pixel representations. Words $w \in \{(0,0),(0,1),(1,0),(1,1)\}^*$ can be also seen as representing pixels of the unit square—subsquares of size $\frac{1}{2^{|w|}} \times \frac{1}{2^{|w|}}$—see Figure 2.34. More exactly a word $w = (a_1,b_1)\ldots(a_n,b_n)$ represents a pixel of size $\frac{1}{2^n} \times \frac{1}{2^n}$ with the low left corner located in the point represented by w.

To simplify the notation, symbols $0,1,2,3$ are often used instead of symbols $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, respectively. For example, $\{00,03,10,13,20,23,30,33\}$ represents the image shown in Figure 2.33c.

Exercise 2.5.17 *Describe images having the following pixel representation:*

(a) $\{0,1,2,3\}^*$*; (b)* $0\{0,1,2,3\}^*$*; (c)* $\{1,2\}^*0\{0,1,2,3\}^*$*; (d)* $0\{1,2\}^*0\{0,1,2,3\}^*$*;*

(e) $\{1,2,3\}^*0\{1,2\}^*0\{0,1,2,3\}^*$

2.5.4 Space of ω-languages*

ω-languages can be seen as formal counterparts of an intuitive concept of an (infinite) process and as such they play an important rôle in various areas of computing. For example, in semantics of computational processes and in cellular automata.

The space of ω-languages is naturally more complex than that of languages. On the other hand, in order to deal with ω-languages we can use powerful tools of mathematics. In this subsection we discuss the basic concepts of ω-languages from this point of view.

For any alphabet Σ with at least two symbols the set Σ^ω has the cardinality of the continuum and it is a (compact) metric space with respect to the metric $d : \Sigma^\omega \times \Sigma^\omega \rightarrow \mathbf{R}$ defined, for $w_1, w_2 \in \Sigma^\omega$, by

$$d(w_1,w_2) = \begin{cases} 0, & \text{if } w_1 = w_2; \\ \frac{1}{2^i}, & \text{if } i \text{ is the smallest integer such that } w_1(i) \neq w_2(i). \end{cases}$$

With this metric at our disposal we can use methods and results of topology and functional analysis to deal with ω-languages. Surprisingly, the basic concepts of this approach can be expressed in terms of (ordinary) languages over Σ. To show that, we define for a language $L \subset \Sigma^*$:

$$\overleftarrow{L} = L\Sigma^\omega; \tag{2.16}$$

$$\overrightarrow{L} = \{w \mid w \in \Sigma^\omega, Prefix(w) \cap L \text{ is infinite}\}; \tag{2.17}$$

$$\text{adherence}(L) = \overrightarrow{Prefix(L)}. \tag{2.18}$$

In the topology introduced by the metric d a set S is open and closed simultaneously if and only if $S = \overleftarrow{L}$ for a finite language $L \subset \Sigma^*$; S is open if and only if $S = \overleftarrow{L}$ for a language $L \subset \Sigma^*$; and S is closed if and only if $S = \mathit{adherence}(L)$ for a language $L \subset \Sigma^\omega$. Finally, a set is a countable intersection of open sets if and only if $S = \overrightarrow{L}$ for a language $L \subset \Sigma^*$.

Example 2.5.18 *If* $L = \{a^i b^j \mid 0 \leq i \leq j\}$, *then* $\overleftarrow{L} = \{a, b\}^\omega$, $\overrightarrow{L} = a^* b^\omega$ *and* $\mathit{adherence}(L) = a^\omega \cup a^* b^\omega$.

Informally, $\overleftarrow{L}$ is the set of all ω-words having a prefix from L; $\overrightarrow{L}$ is the set of those ω-words that have infinitely many prefixes from L. Finally, *adherence*(L) is the set of those ω-words that have infinitely many prefixes that are prefixes of some words from L.

2.6 Algebras

An algebra $\mathcal{A} = \langle \Sigma, o_1, \ldots, o_n \rangle$ is a set Σ (the carrier of $\mathcal{A}$), and operations $o_1, \ldots, o_n$ defined on Σ and satisfying some axioms. Elements of the carrier are also considered as 0-ary operations. From a variety of important algebras of a special interest for us are monoids, groups, Boolean algebras and Kleene algebras.

2.6.1 Closures

Sets extensions and closures. If $\circ$ is a binary operation and $\odot$ an unary operation on a set S, then we may **extend** these operations to apply to subsets of S in the following way:

$$\begin{aligned} A \circ B &= \{a \circ b \mid a \in A, b \in B\}, \\ \odot A &= \{\odot a | a \in A\}. \end{aligned}$$

A set $A \subseteq S$ is closed under the binary operation $\circ$ (the unary operation $\odot$) if $x, y \in A \Rightarrow x \circ y \in A$ ($x \in A \Rightarrow \odot a \in A$).

Example 2.6.1 *For the operation + of addition and the operation of square root* $\surd$, *on positive reals, we have, for the sets* $A = \{x \mid 4 \leq x \leq 36\}$ *and* $B = \{x \mid 10 \leq x \leq 40\}$, $A + B = \{x \mid 14 \leq x \leq 76\}$, $\sqrt{A} = \{x \mid 2 \leq x \leq 6\}$.

The **closure** of a set A under an operation is the least set that contains A and it is closed under that operation.

2.6.2 Semigroups and Monoids

A **semigroup** $\mathcal{S} = \langle S, \cdot \rangle$ is a set S (a carrier of $\mathcal{S}$), with an associative operation "$\cdot$" (a semigroup multiplication). A **monoid** $\mathcal{M} = \langle S, \cdot, 1 \rangle$ is a semigroup $\langle S, \cdot \rangle$, with a **unit element** 1 such that $a \cdot 1 = 1 \cdot a = a$, for each $a \in S$. A semigroup (monoid) is called abelian, or commutative, if so is its multiplication.

Example 2.6.2 *(1)* $\langle \mathbf{N},+,0\rangle, \langle \mathbf{Q},+,0\rangle, \langle \mathbf{R},+,0\rangle, \langle \mathbf{C},+,0\rangle, \langle \mathbf{N},\times,1\rangle, \langle \mathbf{Q},\times,1\rangle, \langle \mathbf{R},\times,1\rangle$ *are commutative monoids.*

(2) If Σ *is an alphabet, then* $\langle \Sigma^*,\cdot,\varepsilon\rangle$ *with "$\cdot$" as the concatenation, is a noncommutative monoid, called also the* **free monoid**.

(3) For any fixed integer n*, the set of all matrices of the degree* n*, with elements from* $\mathbf{Z}$*, is a commutative monoid with respect to matrix addition and the null matrix and also a noncommutative monoid with respect to matrix multiplication and the unit matrix.*

(4) $\langle \mathbf{N}^+,+\rangle$ *is an example of a semigroup that is not a monoid.*

If $\mathcal{M} = \langle S,\cdot,1\rangle$ is a monoid, then for any $A, B \subseteq S$ we define (multiplication) $A.B = \{a.b \,|\, a \in A, b \in B\}$. Similarly, we define $A^0, A^i, i \geq 1$, and A^*, A^+.

A **morphism** from a monoid $\mathcal{M}_1 = \langle S_1,\cdot_1,1_1\rangle$ to a monoid $\mathcal{M}_2 = \langle S_2,\cdot_2,1_2\rangle$ is a mapping $\mu : S_1 \rightarrow S_2$ such that $\mu(1_1) = 1_2$ and $\mu(u._1v) = \mu(u)._2\mu(v)$ for any $u, v \in S_1$.

A **congruence** over a monoid $\mathcal{M} = \langle S,\cdot,1\rangle$ is an equivalence relation $\equiv$ on S which is compatible with the monoid operation, i.e. which satisfies the implication

$$u_1 \equiv v_1, u_2 \equiv v_2 \Rightarrow u_1 \cdot u_2 \equiv v_1 \cdot v_2.$$

Exercise 2.6.3 *Show that if* $\equiv$ *is a congruence over a monoid* $\mathcal{M} = \langle S,\cdot,1\rangle$ *and for* $a \in S$ *we let* $[a]_\equiv$ *to denote the equivalence class in which* a *is, then the set of all such equivalence classes form a monoid, the so called* **quotient monoid** $\mathcal{M}_\equiv$*, with the multiplication defined by* $[a]_\equiv \cdot [b]_\equiv = [a \cdot b]_\equiv$*, and the unit* $[1]_\equiv$.

Exercise 2.6.4 *Let* $L \subseteq \Sigma^*$ *be a language. Show that the relation* $\equiv_L$ *on* Σ^* *defined by* $u \equiv_L v$ *if and only if* $\forall x, y \in \Sigma^*, xuy \in L \Leftrightarrow xvy \in L$ *is a congruence. (The corresponding quotient monoid is called the* **syntactical monoid** *of* L*.)*

Exercise 2.6.5 *Describe the syntactical monoids for the following languages* L *over the alphabet* $\{0,1\}$*: (a) The set of all words with no more than three consecutive* 0*'s; (b)* $L = \{0^i1^i \,|\, i \geq 0\}$.

2.6.3 Groups

A **group** $\mathcal{G} = \langle C,\cdot,^{-1},1\rangle$ is a set C (the **carrier** of $\mathcal{G}$), with a binary operation "$\cdot$" (**multiplication**), a unary operation $^{-1}$ (**inverse**) and a **unit** $1 \in C$ such that $\langle C,\cdot,1\rangle$ is a monoid, and for any $a \in C$

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

$\mathcal{G}$ is called **commutative**, or **abelian**, if so is "$\cdot$". Let $|\mathcal{G}|$ denote the cardinality of the carrier of $\mathcal{G}$.

Two elementary properties of groups are summarized in the following theorem.

Theorem 2.6.6 *If* $\mathcal{G} = \langle C,\cdot,^{-1},1\rangle$ *is a group, then*

1. *For any* $a, b \in C$ *there is a unique* x *such that* $a \cdot x = b$, *namely* $x = a^{-1} \cdot b$.

2. *For any* $a, b, c \in C$, $a \cdot c = b \cdot c \Rightarrow a = b$.

Example 2.6.7 *(1)* $\langle \mathbf{Z}, +, -, 0\rangle$ *and* $\langle \mathbf{Q} - \{0\}, \times, ^{-1}, 1\rangle$ *are commutative groups; "-" is here the unary operation of negation.*

(2) The set of all permutations of n *elements is a group, for any integer* n, *with respect to the composition of permutations, inversion of permutations and the identical permutation.*

Exercise 2.6.8 *Show that, for any integer* n, *the set* $\mathbf{Z}_n$ *of the residual classes with addition and negation (both modulo* n*) and with* 0 *is a commutative group. Similarly,* $\mathbf{Z}_n^*$ *is a commutative group with respect to multiplication and inverse (both modulo* n*) and with 1 as the unit.*

To the most basic concepts concerning groups belong that of subgroups, quotient groups, and the isomorphism of groups.

Let $\mathcal{G} = \langle C, \cdot, ^{-1}, 1\rangle$ be a group. If C_1 is a subset of C that contains 1 and is closed under multiplication and inverse, then $\langle C_1, \cdot, ^{-1}, 1\rangle$ is called a **subgroup** of $\mathcal{G}$.

Two groups $\mathcal{G}_1 = \langle C_1, \cdot_1, ^{-1_1}, 1_1\rangle$ and $\mathcal{G}_2 = \langle C_2, \cdot_2, ^{-1_2}, 1_2\rangle$ are called **isomorphic** if there is a bijection $i : C_1 \to C_2$ such that $i(1_1) = 1_2, i(a^{-1_1}) = i(a)^{-1_2}, i(a \cdot_1 b) = i(a) \cdot_2 i(b)$ for any $a, b \in C$. An isomorphism of a group $\mathcal{G}$ with itself is called **an automorphism**.

Exercise 2.6.9 * *If* $\mathcal{H} = \langle S_1, \cdot, ^{-1}, 1\rangle$ *is a subgroup of a group* $\mathcal{G} = \langle S, \cdot, ^{-1}, 1\rangle$, *then the sets* aS_1, $a \in S$, *are called* **cosets**. *Show that the family of cosets, together with the operation of multiplication,* $(aS_1) \cdot (bS_1) = (ab)S_1$, *the inverse* $(aS_1)^{-1} = a^{-1}S_1$, *and the unit element* S_1 *is a group (the so-called* **quotient group** *of* $\mathcal{G}$ *modulo* $\mathcal{H}$ *and denoted* $\mathcal{G}/\mathcal{H}$*).*

Two basic results concerning the relations between the size of a group and its subgroups are summarized in the following theorem:[14]

Theorem 2.6.10 *(1)(***Lagrange's Theorem***) If* $\mathcal{H}$ *is a subgroup of a group* $\mathcal{G}$, *then* $|\mathcal{H}|$ *is a divisor of* $|\mathcal{G}|$.

*(2) (***Cauchy's Theorem***) If a prime* p *is a divisor of* $|\mathcal{G}|$ *for a group* $\mathcal{G}$, *then* $\mathcal{G}$ *has a subgroup* $\mathcal{H}$ *with* $|\mathcal{H}| = p$.

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[14] Augustin Cauchy (1789-1857), a French mathematician and one of the developers of calculus. Cauchy wrote more than 800 papers. Joseph de Lagrange, a French mathematician (1736-1813).

Exercise 2.6.11 *Find all subgroups of the group of all permutations of: (a) 4 elements; (b) 5 elements.*

Exercise 2.6.12 * *Prove the Lagrange's Theorem.*

Exercise 2.6.13 ** *Let $\mathcal{G}$ be a finite abelian group. (a) Show that all equations $x^2 = a$ have the same number of solutions in $\mathcal{G}$; (b) extend the previous result to equations of the form $x^n = a$.*

Group theory is one of the richest mathematical theories. Only proofs concerning a complete characterization of finite groups are estimated to have about 15000 pages. A variety of groups with very different carriers is of importance. However, in a special position are groups of permutations—the so called permutation groups.

Theorem 2.6.14 (Cayley (1878)) *Any group is isomorphic to a permutation group.*

Proof Let $\mathcal{G} = \langle C, \cdot, ^{-1}, 1\rangle$ be a group. The mapping $\mu : C \to C^C$, with $\mu(g) = \pi_g$—to be the mapping defined by $\pi_g(x) = g \cdot x$—is such an isomorphism. This is easy to show. First, the mapping π_g is a permutation. Indeed, $\pi_g(x) = \pi_g(y)$ implies first $g \cdot x = g \cdot y$ and therefore, by (2) of Theorem 2.6.10, $x = y$. Moreover, μ assigns to a product of elements the product of the corresponding permutations. Indeed, $\mu(g.h) = \pi_{gh} = \pi_h \circ \pi_g$, because $\pi_h \circ \pi_g(x) = \pi_g(\pi_h(x)) = g \cdot h \cdot x = \pi_{gh}(x)$. Similarly, one can show that μ maps the inverse of an element to the inverse of the permutation assigned to that element and the unit of $\mathcal{G}$ to the identity permutation. □

Carriers of groups can be very large. It is therefore often of importance if a group can be described through a small set of its **generators**.

If $\mathcal{G} = \langle C, \cdot, ^{-1}, 1\rangle$ is a group, then a set $T \subseteq C$ is said to be a set of **generators** of $\mathcal{G}$, if any element of C can be obtained as a product of a finitely many elements of T. If $1 \notin T$ and $g \in T \Rightarrow g^{-1} \in T$, then the set T of generators is called **symmetric**.

Example 2.6.15 *For any permutation g, $T = \{g, g^{-1}\}$ is a symmetric set of generators of the group $\{g^i \,|\, i \geq 0\}$.*

It has been known since 1878 that to any symmetric set of generators of a permutation group we can associate a graph, the so-called **Cayley graph**, that is regular and has interesting properties. It has been, however, only recently realized that graphs of some of the most important communication networks for parallel computing are either Cayley graphs or closely related to them.

Definition 2.6.16 *A* **Cayley graph** *$G(\mathcal{G}, T)$, for a group $\mathcal{G} = \langle C, \cdot, ^{-1}, 1\rangle$ and its symmetric set T of generators, is defined by $G(\mathcal{G}, T) = \langle C, E\rangle$, where $E = \{(u, v) \,|\, \exists g \in T, ug = v\}$.*

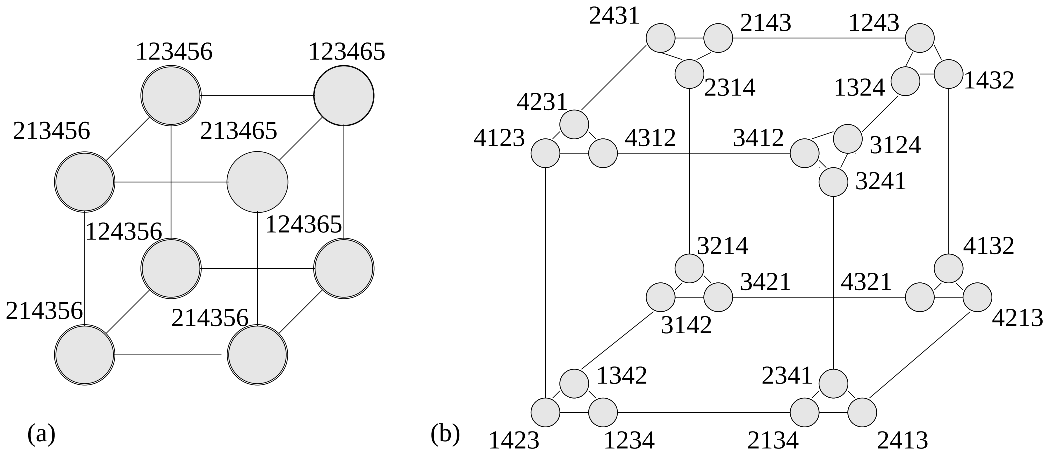

Figure 2.35: Cayley graphs-60%

Example 2.6.17 *Two Cayley graphs are shown in Figure 2.35. The first one, called the 3-dimensional hypercube, has* 8 *vertices and it is associated with a permutation group of* 8 *permutations of* 6 *elements and three transpositions* $\{[1,2],[3,4],[5,6]\}$ *as generators. The graph in Figure 2.35b, the so-called 3-dimensional cube-connected cycles, has* 24*-nodes and it is a Cayley graph associated with the set of generators* $\{[1,2],\langle 2,3,4\rangle,\langle 2,4,3\rangle\}$*. It can be shown that this is by no means a chance. Hypercubes and cube-connected cycles of any dimension, see Section 10.1, are Cayley graphs.*

An important advantage of Cayley graphs is that their graph-theoretical characterizations allow to show their various properties using purely group-theoretical means. For example:

Theorem 2.6.18 *Each Cayley graph is vertex symmetric.*

Proof. Let $G = \langle V, E\rangle$ be a Cayley graph defined by a symmetric set T of generators. Let u, v be two distinct vertices of G, that is two different elements of the group $\mathcal{G}(T)$ generated by T. The mapping $\phi(x) = vu^{-1}x$ clearly maps u into v, and, as it is easy to verify, it is also an automorphism on $\mathcal{G}(T)$ such that $(u, v) \in E$ if and only if $(\phi(u), \phi(v)) \in E$. □

Exercise 2.6.19 *Show that all three graphs in Figure 2.36 are isomorphic.*

In a Cayley graph all vertices have the same degree, equal to the cardinality of the generator set. In the Petersen graph, shown in Figure 2.36, all vertices have the same degree and in spite of that the Petersen graph is not a Cayley graph. This can be shown, using Lagrange's and Cauchy's Theorems.

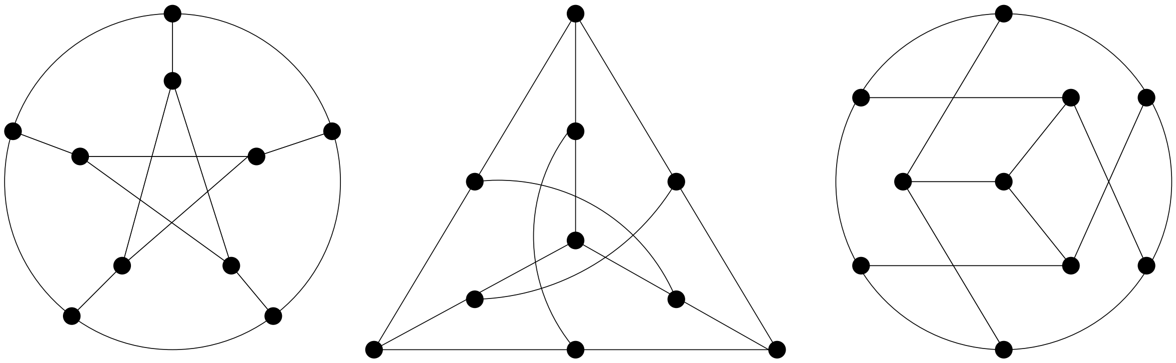

Figure 2.36: Petersen graph.

Exercise 2.6.20 * *A* **direct product of two graphs** $G_1 = \langle V_1, E_1\rangle$ *and* $G_2 = \langle V_2, E_2\rangle$ *is the graph* $G = \langle V_1 \times V_2, E\rangle$, *where* $((u_1, u_2), (v_1, v_2)) \in E$ *if and only if* $(u_1, v_1) \in E_1, (u_2, v_2) \in E$. *Show that the direct product of Cayley graphs is again a Cayley graph.*

2.6.4 Quasirings, Rings and Fields

In this section three algebras are introduced that are a natural generalization of those properties the basic number operations have. Thier importance for computing lie in the fact that many algorithmic problems, originally stated for numbers, can naturally be generalized to be algorithmic problems on these more abstract algebras and then solved using a natural generalization of number algorithms.

Definition 2.6.21 *An algebra* $\mathcal{A} = \langle S, +, \cdot, 0, 1\rangle$ *is:*

- **semiring** *if the following conditions are satisfied:*
  - $\langle S, +, 0\rangle$ *is an abelian monoid and* $\langle S, \cdot, 1\rangle$ *is a monoid;*
  - $a \cdot 0 = 0 \cdot a = 0$, *for all* $a \in S$;
  - *the following distributive laws hold for all* $a, b, c \in S$:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{and} \quad (b + c) \cdot a = (b \cdot a) + (c \cdot a).$$

- **ring** *if it is a quasiring and* $\langle S, +, 0\rangle$ *is a group for a properly defined "additive inverse"* $^{-1_a}$;
- **field** *if it is a ring,* $\langle S, \cdot, 1\rangle$ *is an abelian monoid and* $\langle S - \{0\}, \cdot, 1\rangle$ *is a group for a properly defined multiplicative inverse* $^{-1_m}$.

Example 2.6.22 $\langle \{0, 1\}, \vee, \wedge, 0, 1\rangle$ *is the so-called* **Boolean quasiring** *and* $\langle \mathbf{N}, +, \cdot, 0, 1\rangle$ *with the integer operations of addition and multiplication is the* **integer semiring**.

We can see rings as having defined an operation of subtraction (as an inverse of the operation $+$) and fields can be seen as having defined also an operation of division (as the inverse to the operation $\cdot$).

Example 2.6.23 $\langle \mathbf{Z}, +, \cdot, 0, 1\rangle$ *is a ring and, in addition, for any integer* n, $\langle \mathbf{Z}_n, +_n, \cdot_n, 0, 1\rangle$ *is also a ring if* $+_n$ *and* $\cdot_n$ *are additions and multiplications modulo* n. *The set of all polynomials of one variable with real coefficients and the operations of addition and multiplication of polynomials forms a ring, too.*

Exercise 2.6.24 *Show: (a) all matrices of a fixed degree over a quasiring form again a semiring; (b) all matrices of a fixed degree over a ring form a ring.*

Example 2.6.25 $\langle \mathbf{Q}, +, \cdot, 0, 1\rangle$ *and* $\langle \mathbf{C}, +, \cdot, 0, 1\rangle$ *are fields and, for any prime* p, $\langle \mathbf{Z}_p, +_p, \cdot_p, 0, 1\rangle$ *is a field—an example of a so-called* **finite field**.

Exercise 2.6.26 *Show that if* c *is a rational number, then the set of all numbers of the form* $a + b\sqrt{c}, a, b \in \mathbf{Q}$, *forms a field (a so-called* **quadratic field**) *with respect to the operations of addition and multiplication of numbers.*

2.6.5 Boolean and Kleene Algebras

Two other algebras of importance for computing are Boolean algebras, due to G. Boole[15], that have their origin in the attempts to formalize the laws of thought (and they found now applications in computer circuits), and Kleene algebras, that are an abstraction of several algebras playing an important rôle in computing.

A **Boolean algebra** is any algebra of the form $\mathcal{B} = \langle S, +, \cdot, -, 0, 1\rangle$ where S is a set with two distinguished elements, 0 and 1, two binary operations, $+$ (Boolean addition) and $\cdot$ (Boolean multiplication), and one unary operation $-$ (Boolean negation), satisfying axioms listed in the first column of Table 2.1.

The set of all propositions with two truth values **true** (1) and **false** (0), with disjunction $(+)$, conjunction $(\cdot)$ and negation (-), is the oldest example of a Boolean algebra. Set $\{0, 1\}$ with Boolean addition, multiplication and negation is the smallest Boolean algebra. For any n the set of all Boolean functions of n variables forms a Boolean algebra with respect to Boolean addition, multiplication and negation of Boolean functions.

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[15] George Boole (1815-1864), an English mathematician and logician. His symbolic logic is central to the study of the foundations of mathematics and also computing.

Boolean algebra	Axioms	Kleene algebra
$x + x = x$ $xx = x$	Idempotent laws	$x + x = x$
$x + 0 = x$ $x1 = x$	Identity laws	$x + 0 = x$ $x1 = x$
$x0 = 0$ $x + 1 = 1$	Dominance laws	$x0 = 0$
$x + y = y + x$ $xy = yx$	Commutative laws	$x + y = y + x$ $xy = yx$
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws	$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws	$(y + z)x = yx + zx$ $x(y + z) = xy + xz$
$\overline{(xy)} = \bar{x} + \bar{y}$ $\overline{(x + y)} = \bar{x}\bar{y}$	De Morgan's Laws	
$\bar{\bar{x}} = x$	Law of double negation	
	Iteration law	$ab^*c = \sup_{n \geq 0} ab^n c$

Table 2.1: Laws of Boolean and Kleene algebras.

Exercise 2.6.27 *There are infinitely many Boolean algebras. Show, for example: (a)* $\langle 2^A, \cup, \cap, ^c, \emptyset, A\rangle$ *is a Boolean algebra for any set* A*, where* $C^c = A - C$ *for any* $C \subseteq A$*; (This is the reason why set operations of union, intersection and negation are called Boolean operations.); (b) the set* $C = \{1,2,3,6\}$ *with binary operations gcd, lcm and* $x^{-1} = \frac{6}{x}$ *is a Boolean algebra.*

A **Kleene algebra** is any algebra of the form $\mathcal{K} = \langle S, +, \cdot, ^*, 0, 1\rangle$, where S is a set containing two distinguished elements $0, 1$, two binary operation $+$ (Kleene addition), $\cdot$ (Kleene multiplication), and one unary operation * (Kleene iteration) satisfying the axioms shown in the third column of Table 2.1.

The "iteration law" axiom requires an explanation. In a Kleene algebra we can define that $a \leq b \Leftrightarrow a + b = b$. It follows then easily from the axioms that such a relation $\leq$ is a partial order. For a set $A \subseteq S$ we define $\sup A$ to be an element y such that $x \leq y$ for all $x \in A$ (that is y is an upper bound for A) and if $x \leq y'$ for all $x \in A$ and some y', then $y \leq y'$ (that is y is the lowest upper bound). The iteration law axiom then says that $\sup\{ab^nc \,|\, n \geq 0\}$ exists and equals to ab^*c.

Exercise 2.6.28 *Show that the set* $\{0,1\}$ *with Boolean operations + and . and with* $a^* = 1$ *for any* $a \in \{0,1\}$ *forms a Kleene algebra.*

Exercise 2.6.29 * *Show that for any integer* n *the set of all Boolean matrices of degree* n *forms a Kleene algebra with respect to Boolean matrix addition, multiplication, iteration and the zero and unit matrices.*

Exercise 2.6.30 ** *Show that for any set* S *the family of all binary relations over* S *is a Kleene algebra with respect to addition, composition and iteration of relations, and with respect to the empty and identity relations.*

In all previous examples it is in principle easy to verify that all axioms are satisfied. It is more difficult to show that for the Kleene algebra in the following example.

Example 2.6.31 *For any integer* n *and Kleene algebra* $\mathcal{K}$ *the set of all matrices of degree* n *with elements from the carrier of* $\mathcal{K}$ *forms a Kleene algebra with respect to the ordinary matrix addition and multiplication, with* 0 *as the zero matrix and* 1 *as the identity matrix and with the operation* * *defined recursively by equation on page 109.*

Another example of a Kleene algebra, and actually historically the first one, due to Kleene (1956), is introduced in the following chapter.

Moral. Foundations of any mature discipline of science is based on elementary but deep and useful ideas, concepts, models and results. A good rule of thumb by dealing with foundations in computing is therefore, as in life, to remeber and to behave according to the wisdom “Wer das ABC recht kann, hat die schwerste Arbeit getan”.

2.7 Exercise

1. (a) Show that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$;

   (b) generalize previous equality for the case of the union of n sets.

2. Let A, B be sets. Do the following implications hold? (a) $A \cap B = \emptyset \Rightarrow 2^A \cap 2^B = \emptyset$; (b) $2^A = 2^B \Rightarrow A = B$.

3. Form 2^A for the following sets: (a) $A = \{1\}$; (b) $A = \{1, 2, 3, 4\}$; (c) $A = \{a, b, \{a, b\}\}$; (d) $A = \{\emptyset, a, b, \{a, b\}\}$.

4. Determine which of the following sets is the power set of a set: (a) $\emptyset$; (b) $\{\emptyset, \{a\}\}$; (c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$.

5. Show how you can simply describe the set of points of the **Menger sponge**. It is a subset of $\mathbf{R}^3$ constructed through the following infinite process. Begin with the unit cube of side 1. Divide it into 27 subcubes of identical sizes. Remove the middle one and also the middle one on each side—there remains 20 smaller cubes. Continue the process and at each step do the same with all remaining subcubes.

6. A multiset with dictionary operations forms the data type called **bag**. How can one implement efficiently bags?

7. Let $R = \{(a, b) \mid a \text{ divides } b\}$ be the relation on the set of positive integers. Find $R^{-1}, \bar{R}$.

8. List 16 different relations on the set $\{0, 1\}$ and determine which of them are: (a) reflexive; (b) transitive; (c) symmetric; (d) antisymmetric.

9. How many relations on a set of n elements are: (a) symmetric; (b) antisymmetric; (c) reflexive and symmetric?

10. Let R be a binary relation over some set A. Show that R is an equivalence if and only if the following conditions are satisfied: (i) $R = R^{-1}$; (ii) $RR \subseteq R$; (iii) $I_A \subseteq R$, where I_A is the identity relation on A.

11. Let $R = \{(1, 3), (2, 4), (3, 1), (3, 5), (5, 1), (5, 2), (5, 4), (2, 6), (5, 6), (6, 3), (6, 1)\}$. Compute R^2, R^3, R^4, R^*.

12. Determine the transitive closure of the relations: (a) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$; (b) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (d, c), (e, d)\}$; (c) $(1, 5), (2, 1), (3, 4), (4, 1), (5, 2), (5, 3)$.

13. Determine transitive closure of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

14. Which of the following relations on the set of all people or on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$ are equivalencies? (a) $\{(a,b) \mid a \text{ and } b \text{ have common parents}\}$; (b) $\{(a,b) \mid a \text{ and } b \text{ share common parents}\}$; (c) $\{(f,g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$; (d) $\{(f,g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$.

15. Modify Warshall's algorithm in such a way that it can be used to determine the shortest path between two nodes in edge labeled graphs.

16. A total ordering $\leq$ is said to be compatible with the partial ordering R, if $a \leq b$ whenever aRb. The construction of a total order compatible with a given partial order is called **topological sorting**. Design an algorithm to perform topological sorting.

17. Let $f(x) = ax + b$, $g(x) = cx + d$. Find conditions on a, b, c, d such that $f \circ g = g \circ f$.

18. Let a set A contain ten integers between 5 and 50. Show, for example using the Pigeonhole principle, that there are two disjoint subsets B, C of A, such that $\sum_{x \in A} x = \sum_{x \in B} x$.

19. Show that the mapping $f : \mathbf{N} \to \mathbf{N}^+ \times \mathbf{N}^+$ defined by $f(2^j(2k+1)) = (j,k)$ is a bijection.

20. Let $g_n : \mathbf{Z}_n^* \to \mathbf{Z}_n$ be the mapping defined by $g_n(i) = (i+1)^2 \bmod n$. Show that the mapping g_n is bijection if and only if n is a prime.

21. Let the composition of two functions $f : A \to B$ and $g : B \to C$ be surjective. Does it mean that also f is surjective?

22. Let f_A be the characteristic function of the set A. Show (a) $f_{A \cap B}(x) = f_A(x).f_B(x)$; (b)$f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x)f_B(x)$; (c) $f_{\bar{A}}(x) = 1 - f_A(x)$.

23. Let B be an n-element multiset with k distinct elements $e_1, \ldots, e_k$ and let m_i denotes the number of occurrences of the element e_i on B. Determine the number of distinct permutations of elements of B.

24. Show, using the truth table, equivalence of the following Boolean formulas: (a) $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$; (b) $(p \wedge g) \Rightarrow p$ and $p \Rightarrow (p \vee g)$.

25. Show the following implications using the truth table:

    (a) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$; (b) $[\bar{p} \wedge (p \vee q)] \Rightarrow q$.

26. Which of the following sets of Boolean functions forms a base? (a) $\{OR, NOR\}$; (b) $\{-, NOR\}$; (c) $\{AND, NOR\}$.

27. Use NAND function to form the following fuctions: (a) NOT; (b) OR; (c) AND; (d) NOR.

28. Show the following properties of the operation *NOR* ($\oplus$): (a) $x \oplus y = x\bar{y} + \bar{x}y$; (b) $x \oplus x = 0, x \oplus 0 = x, x \oplus 1 = \bar{x}$; (c) $(x \oplus y) \oplus z = xz \oplus yz$; (d) $x + y = x \oplus y \oplus xy$.

29. Show that Boolean functions *NOR* and *AND* do not form a base but Boolean functions *NOR*, *AND* and 1 do form a base.

30. A Boolean function $f(x_1, \dots, x_n)$ is said to depend essentially on an i-th variable x_i if there are $a_1, \dots, a_n \in \{0, 1\}$ such that $f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, \bar{a_i}, a_{i+1} \dots a_n)$. For $1 \leq m \leq n$ determine the number of Boolean functions of n variables that depend essentially on at most m variables.

31. **(Post's theorem) Show that a set $\mathcal{B}$ of Boolean formulas forms a Boolean base if and only if the following conditions are satisfied: (1) $\exists f \in \mathcal{B} : f(0, \dots, 0) = 1$; (2) $\exists f \in \mathcal{B} f(1, \dots, 1) = 0$; (3) $\exists f \in \mathcal{B} : f$ is not monotone; (4) $\exists f \in \mathcal{B} : \exists x_1, \dots, \exists x_n \, f(x_1, \dots, x_n) \neq \bar{f}(\bar{x_1}, \dots, \bar{x_n})$; (5) $\exists f \in \mathcal{B} : f$ cannot be displayed as $x_{i_1} \oplus x_{i_2} \oplus \dots x_{i_k} \oplus c$, where $c \in \{0, 1\}$.

32. Given any family $\mathcal{H}$ of hash functions from A to B, where $|A| > |B|$. Show that there exists $x, y \in A$ such that $|\{h \mid h(x) = h(y)\}| > \frac{|\mathcal{H}|}{|B|} - \frac{|\mathcal{H}|}{|A|}$.

33. For $a, b \in \mathbf{N}$, let $A = [a], B = [b]$ and $p \geq a$ be a prime. Let g map $\mathbf{Z}_p$ into B as closely as possible, i.e., $|\{y \in \mathbf{Z}_p \mid g(y) = i\}| \leq \lceil \frac{p}{b} \rceil$ for all $i \in B$. Let $m, n \in \mathbf{Z}_p$, $m \neq 0$. We define $h_{m,n} : A \to \mathbf{Z}_p$ by $h_{m,n}(x) = (mx + n) \bmod p$. Show that the family $\mathcal{H} = \{f_{m,n} \mid m, n \in \mathbf{Z}_p, m \neq 0, f_{m,n}(x) = g(h_{m,n}(x))\}$ is universal.

34. Let $G = \langle V, E \rangle$ be a connected directed graph. For two vertices u, v define $u \equiv v$, if u and v lie in a simple cycle, show that $\equiv$ is an equivalence relation on G. (The corresponding equivalence classes are called **biconnected components** of G.

35. A complement of a graph $G = \langle V, E \rangle$ is the graph $\bar{G} = \langle V, V \times V - E \rangle$. Show: (a) if a graph G is self-complementary, i.e. $G = \bar{G}$, then G has either $4m$ or $4m + 1$ vertices; (b) design all such graphs with at most 8 vertices.

36. For a graph $G = \langle V, E \rangle$ and $S \subset V$ let G_{-S} be the graph obtained from G by removing the set S of vertices and edges incident with them. Show that G_{-S} has less connected components than $|S|$.

37. Determine which of the pairs of graphs shown in Figure **??** are isomorphic.

38. Show that if $v\text{-}conn(G) \geq 2$ for an undirected graph G, then any two vertices (or edges) of G lie in a common cycle.

39. Show that if a graph is not 2-vertex-connected, then it is not hamiltonian.

40. Show that if a connected graph $G = \langle V, E \rangle$ has at least three vertices and each vertex has the degree at least $\frac{|V|}{2}$, then G is hamiltonian.

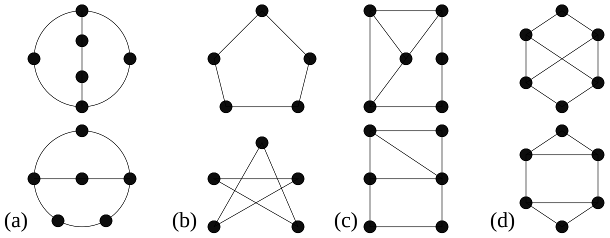


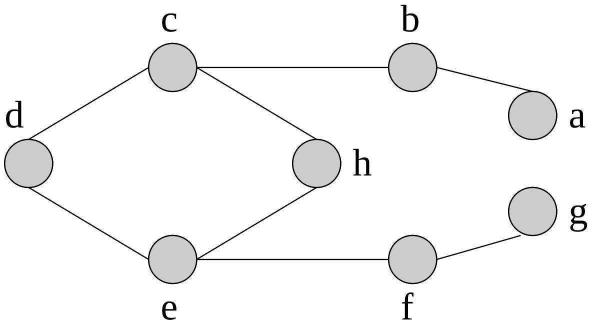
Figure 2.37: isomorphism of graphs

41. The **closure** of a graph $G = \langle V, E\rangle$ is the graph obtained from G by recursively connecting pairs of nonadjacent vertices whose sum of degrees is at least $|V|$ until no such pair remain. (a) Show that the closure of each undirected graph is uniquely defined; (b) show that a graph is hamiltonian if and only if its closure is hamiltonian.

42. Knight's tour on an $n \times m$ chessboard is a sequence of legal moves by a knight starting at some square and visiting each square exactly once. Model the chessboard by a graph with one node per square of the board and with an edge between two nodes exactly when there is a legal move by knight from one of the squares to another.

    (a) Show that a knight' tour exists if and only if there is a Hamilton path on the corresponding graph; (b) Design the Knight's tour for a 3×4 board.

43. For the following graph write down: (a) all depth-first traversals that starts in the node h; (b) all breadth-first traversals that starts in the node h.



44. If $G = \langle V, E\rangle$ is a planar graph, then each drawing of G in such a way that no two edges intersect partition the plane into the number of connected regions called **faces**. for example, the graph in Figure 2.19b partitions the plane into 6 regions. Show the **Euler's formula**: If Φ is the number of faces of a planar graph $G = \langle V, E\rangle$, then $|V| - |E| + \Phi = 2$.

45. * Show that Petersen graph, Figure 2.36 is not hamiltonian.

46. Show that for each k there is a regular graph of degree k that has no perfect matching.

47. Show that it is impossible using 1×2 dominos, to exactly cover an 8×8 square from which two opposite 1×1 corners have been removed.

48. Design an algorithm to solve the following *personnel assignment problem*: n workers are available for n jobs, with each worker being qualified for one or more of these jobs. Can all these workers be assigned, one worker per job, to jobs for which they are qualified.

49. Show that the Petersen graph is 4-edge chromatic.

50. Let $G = \langle V, E \rangle$ be a graph. A subset $S \subseteq V$ is called **independent set** of G if no two vertices of S are adjacent in G. An independent set is maximal if G has no independent set S' such that $|S| \leq |S'|$. A subset $S \subseteq V$ is called a **covering** of G if very edge of E is incident with at least one vertex of S. (a) Design a maximal independent set for graphs in Figures 2.15b, c; 2.16a, b; 2.18b; (b) show that an $S \subseteq V$ is an independent set if and only if $V - S$ is a covering of G,

51. Show that the Four Color Problem is equivalent to the problem to determine whether regions of each planar map can be colored by four colors in such a way that no two neighbouring regions have the same color.

52. Show that if u, v are words such that $uv = vu$, then there is a w such that $u = w^m$ and $v = w^n$ for some m, n.

53. Let Σ be an alphabet and $w \in \Sigma^*$. Show that $x = w^2$ is the unique solution of the equation $x^2 = wxw$.

54. Two words x and z in Σ^* are called **conjugates** if there exists $y \in \Sigma^*$ such that $xy = yz$. (a) Show that x and z are conjugates if and only if there exist $u, v \in \Sigma^*$ such that $x = uv$ and $z = vu$; (b) show that the conjugate relation is an equivalence on Σ^*; (c) show that if x is conjugate to y in Σ^*, then x is obtained from y by a circular permutation of symbols of y.

55. A word w is **primitive** if and only if it is not a nontrivial power of another word, i.e. if $w = u^n$ implies $n = 1$. (a) Show that any word is a power of a unique primitive word; (b) show that if u and v are conjugate and u is primitive, then so is v; (c) show that if $uw = wv$ and $u \neq \varepsilon$, then there are unique primitive words u', v' and integers $p \geq 1$, $k \geq 0$ such that $u = (u'v')^p$, $v = (v'u')^p$, $w = (u'v')^k u'$.

56. Show the following language identities: (a) $(A \cup B)^* = A^*(BA^*)^*$; (b) $(A \cup B)^* = (A^*B)^*B^*$; (c) $(AB)^* = \{\varepsilon\} \cup A(BA)^*B$; (d) $A^* = (\{\varepsilon\} \cup A \cup A^2 \cup \ldots \cup A^{n-1})(A^n)^*$.

57. Let $\Sigma = \{0,1\}$ and $L = \Sigma^* - \Sigma^*\{00\}\Sigma^*$. Show that the language L satisfies the identities: (a) $L = \{\varepsilon, 0\} \cup \{01, 1\}L$; (b) $L = \{\varepsilon, 0\} \cup L\{1, 10\}$; (c) $L = \bigcup_{k=0}^{\infty} L^k$.

58. Determine a language $L \subseteq \{a, b\}^*$ such that: (a) $L = \{\varepsilon\} \cup \{ab\}L$; (b) $L = \{\varepsilon\} \cup L\{ab\}$.

59. Show that there is no language $L \subseteq \{0, 1\}^*$ such that: (a) $L \cup \{01\}L = \{\varepsilon\} \cup 0L$; (b) $L = \{\varepsilon\} \cup 0L \cup L1$.

60. Determine $L_1^{-1}L_2$ and $L_2^{-1}L_1$ if: (a) $L_1 = \{a^ib^i \mid i \geq 0\}$, $L_2 = \{a^ib^j \mid 0 \leq i \leq j\}$; (b) $L_1 = \{ww \mid w \in \{0,1\}^*\}$, $L_2 = \{ww^R \mid w \in \{0,1\}^*\}$; (c) $L_1 = \{wcw \mid w \in \{0,1\}^*, c \notin \{0,1\}\}$, $L_2 = \{wcw^R \mid w \in \{0,1\}^*\}$.

61. Draw curves generated by the turtle interpretation of words w_0, w_1, w_2, w_3, where $w_0 = F-F-F-F, w_i = \phi(w_{i-1})$, $\phi(F) = F-F+F-F-F$, $\phi(-) = -$, $\phi(+) = +$, $d = 1$, $\alpha = 90$.

62. Determine $\overrightarrow{L}$, $\overleftarrow{L}$ and $adherance(L)$ for the languages: (a) $L = \{a^ib^j \mid j \geq i \geq 0\}$; (b) $L = \{ww \mid w \in \{0,1\}^*\}$; (c) $L = \{a^ib^j \mid i,j \geq 0\}$.

63. Show: (a) $shuffle(\{a\},\{b\}^*) = b^*ab^*$; (b) $shuffle(L_1, L_2 \cup L_3) = shuffle(L_1,L_2) \cup shuffle(L_1,L_3)$; (c) $shuffle(L_1,L_2) = shuffle(L_2,L_1)$.

64. Prove or disprove for any languages L, L_1 and L_2 and any homomorphism h: (a) $L\,shuffle(L_1,L_2) = shuffle(LL_1,L_2)$; (b) $h(shuffle(L_1,L_2)) = shuffle(h(L_1),h(L_2))$.

65. Solve the language equation $x = L_1 \cup L_2x$, where L_1, L_2 are languages and x is a language variable.

66. Determine the language L satisfying the identity: (a) $L \cup abL = \{\varepsilon\} \cup aL$; (b) $L \cup baL = \{\varepsilon\} \cup aL \cup bL$.

67. For a language L let $L_n = \{x \mid x \in L, |x| = n\}$. The power series $g_L(z) = \sum_{n=0}^{\infty} |L_n| z^n$ is called the generating function of the language L. Show that $g(z) = \frac{12z}{1-2z^2}$ is the generating function for the language of palindromes over the two-letter alphabet.

68. Let $\mathcal{M} = \langle S, \cdot, 1 \rangle$ be a monoid, $A, B \subseteq S$. The **left quotient** of A and B, notation $A^{-1}B$, is defined by $A^{-1}B = \{c \mid c \in S, \exists a \in A, b \in B \text{ such that } c = a \cdot b\}$. Show that $(AB)^{-1}C = B^{-1}(A^{-1}C)$.

69. Let A be a finite set. Denote $Op_2(A)$ the set of all binary operations on A. For every $f, g \in Op_2(A)$, we define $h = f \circ g \Leftrightarrow (\forall x, y \in A)(h(x,y) = f(g(x,y), g(y,x)))$. Moreover, let $I_A(x,y) = x$. Show that $\langle Op_2(A), \circ, I_A \rangle$ is a monoid for any set A.

70. The **star graph** S_n on the set of all permutations of n elements is the Cayley graph with the set $\{[1,i] \mid 1 < i < n\}$ of transpositions as generators. Draw S_2, S_3, S_4 and determine the number of edges of S_n.

71. The **pancake graph** P_n on the set of all permutations of n elements is the Cayley graph with the set of generators $\{(i, i-1, \ldots, 1, i+1, \ldots, n) \mid 1 \leq i \leq n\}$. Draw P_2, P_3, P_4 and determine the number of edges of P_n.

72. A Cayley graph is called **strongly hierarchical** if it has such a set of generators T that for each ordering $g_1, \ldots, g_k$ of its generators and each $i > 1$ the generator g_i is not expressible using generators $g_1, \ldots, g_{i-1}$. Show that a strongly hierarchical Cayley graph is edge symmetric.

73. * Show that the Petersen graph is not a Cayley graph.

74. Show that in a Boolean algebra $x = y$ if and only if $(x + \bar{y})(\bar{x} + y) = 1$.

75. Solve the following system of equations in a Boolean algebra: (a) $x + \bar{y} + x\bar{y} = x$, $xy = \bar{x} + y$; (b) $(x + \bar{y}) \cdot (\bar{x} + y) = x$; $x + y = x$

QUESTIONS

1. Which are the basic methods to specify sets and what are the advantages of particular methods?
2. In which cases is $(2^A, \subsetneq)$ a totally ordered set.
3. Is the function $f(n) = \lfloor \lg n \rfloor$ one-way?
4. How many concepts of regularity of graphs you know and what are the relations between these concepts?
5. What is the relation between coloring the nodes of planar graphs and coloring of maps?
6. Can one use depth-first search and bread-first search method to search infinite graphs?
7. How many solutions has a string equation $x^2 = wxw$ with unknown x?
8. Which languages L satisfy the equality $L \cup abL = \varepsilon \cup aL \cup bL^2$?
9. Can it happen that the same Cayley graph is generated by two different sets of generators? (If yes, give an example; if not, explain why.)
10. Which chain code pictures are generated by words of the following languages?
    (a) $\{ru^i rd^i | i \geq 1\}$; (b) $\{(ru)^i rur(dr)^i | i \geq 1\}$.

2.8 Historical and Bibliographical References

Basic mathematical concepts discussed in this chapter have been developed for centuries and presented in many textbooks of various levels of sophistication. Some of those basic books with more orientation on computing are: Rosen (1981) and Arnold and Guessarian (1996).

Georg Cantor (1845-1918), a German mathematician and Ernst F. Zermelo (1871-1956), an Italian mathematician are considered to be the main fathers of the modern set theory even though discoveries of paradoxes led to a variety of additional approaches. Sierpiński triangle, Koch curves, Mandelbrot sets and other fractal structures are treated in depth by Peitgen, Jürgens and Saupe (1992). Data structures are discussed in a variety of books, for example in Cormen, Leiserson and Rivest (1990), Gonnet (1984) and Melhorn (1984). The data type concept has been introduced by several people. In its most abstract form by ADJ group—see Guogen, Thatcher, Wagner and Wright (1977). The book by Ehrig and Mahr (1985) is currently perhaps the main reference on this topic. The binary tree implementation of dictionaries, described in Section 2.1, is due to Song (1981). Figure 2.2 is courtessy of Frank Drewers and Figure 2.6 is courtessy of Uwe Krüger and Heintz Wolf.

Two main algorithms to compute the transitive closure of a relation shown in Section 2.2 are due to Warshall (1962) and Kozen (1991).

The Garden of Eden Problem and Theorem 2.3.11 are due to Moore (1962), Myhill (1963) and Richardson (1972). For a general treatment and survey of cellular automata mappings see Garzon (1995). Boolean functions are dealt with in almost every book on discrete mathematics. There are several definitions of one-way functions—the concept that lies in the basis of modern cryptography. The one presented in Section 2.3.3 is from Goldreich (1989), where an intensive analysis of related concepts is also presented. The idea of hashing appeared first in an internal report of IBM in 1953 due to H. P. Luhn. Hashing is analysed in detail by Knuth (1973) and Gonnet (1984). The idea of universal hashing is due to Carter and Wegman (1979)—see also Cormen, Leiserson and Rivest (1990) for a presentation of hashing and universal hashing.

Graph theory initiated by Euler, is since then a very intensively developed theory with many applications and there are many books about. A careful presentation of basic concepts and results much related to computing is, for example, due to Bondy and Murty (1976), where one can also find proofs of Theorems 2.4.21 and 2.4.25. Several graphs, examples and exercises presented here are also due to them.

Salomaa's "Formal languages" (1973) is still the main references in formal language theory. Chain code languages were introduced by Maurer, Rozenberg and Welzl (1982). Turtle interpretation of words as introduced by Prusinkiewicz is in detail discussed by Prusinkiewicz and Lindenmayer (1990). Examples presented in Section 2.5.3 are due to them; the drawing programs were made by H. Fernau. Point and pixel representations of words are due to Culik and Dube (1993) and Culik and Kari (1993). Several exerciseson languages are due to Eğecioğlu (1995).

Mac Lane and Birkhoff (1967) is a standard reference on modern algebra. Theorem 1 and the concept of Cayley graph is due to Cayley (1878, 1889). Akers and Krishnamurthy (1986) started to explore properties of Cayley graphs from the interconnection networkpoint of view.

Boolean algebras are dealt with in most of the books on discrete mathematics. An abstract concept of Kleene algebra is due to Kozen (1991).

Chapter 3

AUTOMATA

INTRODUCTION

Finite state machines are the most basic model of machines, organisms and processes in technology, nature, society, universe and philosophy. A model that captures the essence of finite systems and allows to learn, demonstrate and harvest their power.

On a theoretical level, finite state machines represent the very basic model of automata to start with to design, learn, analyze and demonstrate components, principles and power of real and idealized computers and also a variety of basic computation modes.

On a practical level, finite-state machines abstract real machines, systems and processes close enough. That is why the aim of applied research and development in computing is often to reduce idealized concepts and methods to the ones realisable by finite state machines.

Finite state automata are also a good model to demonstrate how finite devices working in discrete time can be used to process infinite or continuous objects.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. basic concepts and properties of finite automata, regular expressions and regular languages;
2. finite transducers and their power and properties;
3. weighted finite transducers and their use for image generation and compression;
4. how to use discrete finite automata to process infinite and continuous objects;
5. various modifications of finite automata: nondeterministic, probabilistic, two-way, multihead and linearly bounded automata.

> The fact is that civilization requires slaves. The Greek were quite right there. Unless there are slaves to do the ugly, horrible, uninteresting work, culture and contemplation becomes almost impossible. However, slavery is wrong, insecure, and demoralizing. On mechanical slavery, on the slavery of the machine, the future of the world depends.
>
> *Oscar Wilde, 1895*

The concept of finite state devices is one of the most basic of modern science, technology and philosophy. One of those that in strikingly simple way capture the essence of the most inherent principle how machines, nature and society work. The whole process of the development of a deterministic and mechanistic view of the world, initiated by R. Descartes, revolutionary for his times, culminated in a very simple and powerful model of finite state machines, due to McCulloch and Pitts (1943), obtained by an observation of principles of neural activities.[1]

In this chapter we present, analyze and illustrate several models of automata as well as some of their, also surprising, applications. The most basic model is that of a finite state machine. It is an abstraction of a real machine (and therefore of fixed size and finite memory machines), functioning in discrete time steps.

Finite state machines are building blocks, in a variety of ways, of other models of computing, generating and recognizing devices—both sequential and parallel, deterministic and randomized. This is behind their fundamental rôle for theory and practice of computing. Because of their simplicity, efficiency and well worked-out theory, it is often a good practice to simplify sophisticated computational concepts and methods to such an extent that they can be realized by (cooperating) finite state machines.

Basic theoretical concepts and results concerning finite state machines are presented in the first part of this chapter. In the second part several applications are introduced showing a surprising power and usefulness of the basic concepts concerning finite state machines: for example, image generation, transformation and compression. Finally, various modifications of the basic model of finite state machines are considered. Some of them do not increase power of finite state machines and again show how robust their basic model is. Others turned out to be more powerful. This will result in a variety of models filling a gap between finite state machines and universal computers, discussed in the following chapter.

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[1] Automata and automatization has been already for a long time one of the most exciting ideas for mankind. Not only because it offers ways to get rid of dul work but also because it offers ways mankind can overcome its own physical and intellectual limitations. The first larger wave of such a fascination with automata came in the middle of the last century when construction of sophisticated automata, imitating functions considered inherent for living and/or intelligent creatures, flourished. The emerging automata industry—see the interesting account of Bailey (1982)—played an important rôle in the history of modern technology. The second wave, less mysterious but much more powerful, came with the advent of universal computers.

It will also be demonstrated that though finite state machines are finite and work in discrete steps, they can process infinite and continuous objects. For example, they can be seen as processing infinite words and computing (even very weird) continuous functions.

3.1 Finite State Devices November 8, 2005

The finite state machine model of a device abstracts from the technology the device is made of. Attention is payed only to a finite number of clearly distinguished **states** the device can be in, and to a finite number of clearly identified **events**, usually called **external inputs** or **signals**, that may cause the device to change its current state.

A simple finite state model of a digital watch is shown in Figure 3.1a. The model abstracts from what, how and by whom the watch is made and shows only 8 main states—depicted by boxes—the watch can be in, from the user point of view ("update hours", "display date", "display time",...), and transitions between the states caused by pushing one of four buttons a, b, c, d. Each transition is labelled by the button causing that transition. Having such a simple state transition model of a digital watch it is easy to follow the sequence of states the watch gets in if buttons are pushed in a given sequence. For example, by pushing buttons a, c, d, c, a, a, in this order, the watch gets, transition by transition, from the state "display time" back to the same state.

The finite state model of a watch from Figure 3.1a models a watch behaviour as a process that goes on and on (until the watch gets broken or the batery dies). Such processes are called **reactive**. Observe that this process has no other outputs except the states themselves—various displays. Note also that in some states, for example in "display watch", it is not specified for all buttons what happens if the button is pressed. (This can be utilized to make a more detailed model of a watch, with more states and actions; for example to manipulate the stopwatch). Note also that neither requirements nor restrictions are made how often a button is pressed and how much time a state transition takes.

There are many interesting questions one can ask/study about the model in Figure 3.1a. For example, given two states p and q, which sequence of buttons to push in order to get from the state p to the state q.

Exercise 3.1.1 *Describe five shortest sequences of buttons that make the watch from Figure 3.1a go from the state p to the state q if: (a) $p =$ "display alarm", $q =$ "display hours"; (b) $p =$ "display time", $q =$ "display alarm".*

Two other models of finite automata are depicted in Figures 3.1b, c. In both cases the states are depicted by circles, transitions by arrrows labelled by actions (external symbols or inputs) causing these transitions. These two finite-state machines are more abstract. We do not describe what states mean, only transitions between states are depicted and states are partitioned into *yes*- and *no*-states. For these two models we can also ask the question which sequences of inputs make the machine change a given state p to a given state q. Or a simpler question, which sequences of inputs makes the machine to go from the starting state to a *yes*-state. For example, in case of the model from Figure 3.1b, the sequences of letters “the”, “thee”, “their” and “then” have such a property; the sequence “tha” has not. In case of the finite state model from Figure 3.1c a sequence of inputs make the machine to go from the initial state into the single *yes*-state if and only if this sequence contains an even number of a’s.

The questions which inputs make a finite-state machines go from one state to another or to a *yes*-state turned out to be, as we shall see soon, actually the most important ones we can ask about such an abstract model of finite state machines.

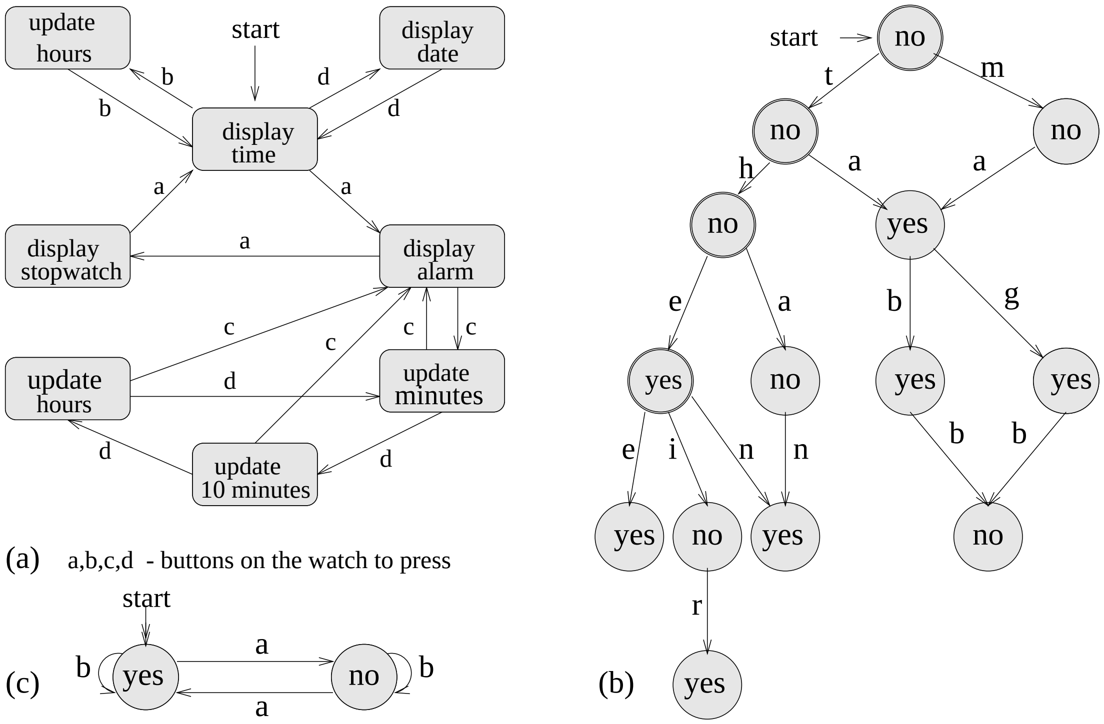


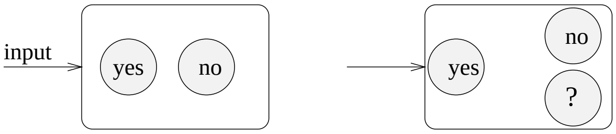
Figure 3.1: Finite state devices—65%

In our model of finite state machines we use a very general concept of a (global) state. A digital device is often composed of a large number of elementary devices, say n, such that each of them is always in one of the two binary states. Any combination of these elementary states forms the so-called “global state”. The overall number of (global) states of the device is 2^n in such a case. However, in a simple finite state model of a device very often only few of the global states are used.

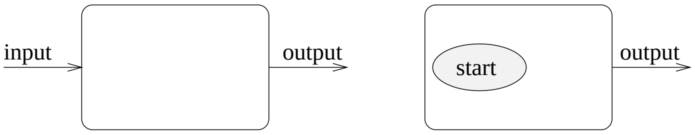
Exercise 3.1.2 *Extend the finite state model of the watch from Figure 3.1 to incorporate other functions a watch usually has.*

Exercise 3.1.3 *Express in a diagram possible states and transitions for a coffee vending machine that acts as follows: It takes 5, 10 and 25 cent coins, in any order, until the overall amount is at least 90 cents. In the moment this happens, the machine stops accepting coins, produces coffee and makes a change. (Take into consideration only money checking activity of the machine.)*

Four basic types of finite state machines are: recognizers, acceptors, transducers and generators (see Figure 3.2). A **recognizer** is a finite state machine $\mathcal{A}$ that always starts in the same initial state. Any input causes a state change (to a different or to the same state) and only a state change—no output is produced. States are partitioned into *yes*-states (**terminal** states) and *no*-states (**nonterminal** states). A sequence of inputs is said to be **recognized** (**rejected**) by $\mathcal{A}$, if and only if this sequence of inputs makes the machine to get into a **terminal** state (to a **nonterminal** state).



(a) recognizer (b) acceptor



(c) transducer (d) generator

Figure 3.2: A recognizer, an acceptor, a transducer and a generator -65%.

Example 3.1.4 *The finite state machine in Figure 3.3a recognizes an input sequence* $(a_1, b_1)\dots(a_{n-1}, b_{n-1})(a_n, b_n)$*, with* (a_1, b_1) *as the first symbol, if and only if there is a* k*,* $1 \leq k \leq n$*, such that* $a_k = b_k = 1$*. (Interesting enough, this is the case exactly if* $\binom{i+j}{i} \bmod 2 = 0$ *for the integers* $i = bin(a_n a_{n-1} \dots a_1)$ *and* $j = bin(b_n b_{n-1} \dots b_1)$*—show that!)*

An **acceptor** is also a finite-state machine that always starts in the same initial state. An input causes either a state transition or it is not accepted at all—and again no output is produced. A sequence of inputs is said to be accepted if and only if it makes the automaton

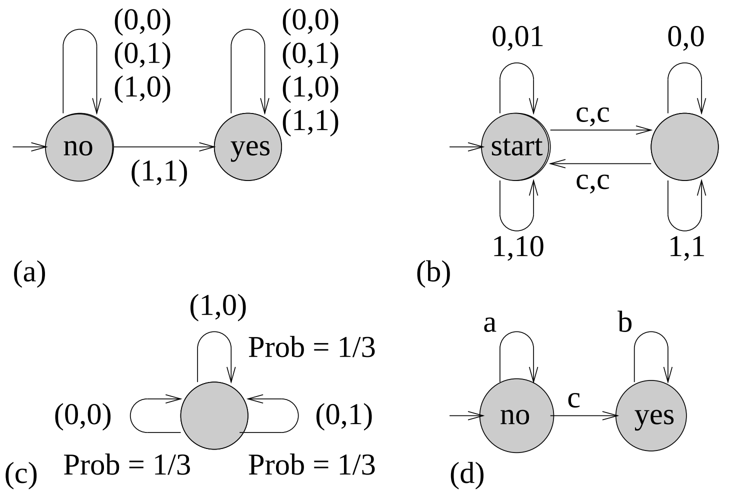


Figure 3.3: Examples of a recognizer, a transducer, a generator and an acceptor,-70%

to get into a terminal state. (The other possibilities are that a sequence of inputs makes the automaton to come to a nonterminal state or its processing is interrupted at some point, because the next transition is not defined.)

Example 3.1.5 *Figure 3.3d shows an acceptor that accepts exactly the words of the language* a^*cb^*.

A **transducer** acts as a recognizer but for each input an output is produced.

Example 3.1.6 *The transducer shown in Figure 3.3b produces for each input word* $w = w_1cw_2c\dots cw_{n-1}cw_n$, $w_i \in \{0,1\}^*$ *the output word* $w' = \phi(w_1)cw_2c\phi(w_3)c\dots cw_{n-1}\phi(w_n)$ *if* n *is odd and* $w' = \phi(w_1)cw_2c\phi(w_3)c\dots c\phi(w_{n-1})cw_n$ *if* n *is even, where* ϕ *is the morphism defined by* $\phi(c) = c$, $\phi(0) = 01$ *and* $\phi(1) = 10$. *In Figure 3.3b, in each pair "*i, o*", used as a transition label, the first component denotes the input symbol, the second one the output string.*

A **generator** has no input. It starts in a starting state, moves randomly, from a state to a state, and at each move an output is produced. For each state transition a probability is given that the transition takes place.

Example 3.1.7 *The generator depicted in Figure 3.3c has only one state and all state changes have the same probability, namely* $1/3$. *It is easy to see that if a sequence of output symbols* $(x_1, y_1)\dots(x_n, y_n)$ *is interpreted as a point of the unit square, with the coordinates* $(0.x_1\dots x_n, 0.y_1\dots y_n)$ *as in Section 2.1.2, then the generator generates the Sierpiński triangle shown in Figure 2.1.*

Is it not remarkable that a one-state generator can produce such a complex fractal structure? This is in no way an exception. As it will be seen later, finite state generators can generate very complex images, indeed.

3.2 Finite Automata

So far we have used the concepts of finite state recognizers and acceptors only intuitively. These concepts will now be formalized, generalized and analyzed. The main new idea is an introduction of nondeterminism—in some states behaviour of the automaton does not have to be determined uniquely. We show that such a generalization is fully acceptable and, in addition, sometimes very useful.

3.2.1 Basic Concepts

Definition 3.2.1 *A* **(nondeterministic) finite automaton** $\mathcal{A}$ *(in short,* **NFA** *or* **NFA***) over the (input)* **alphabet** Σ *is specified by a finite* **set of states** Q*, a distinguished* **(initial) state** q_0*, a set* $Q_F \subseteq Q$ *of* **terminal (final) states** *and a* **transition relation** $\delta \subset Q \times \Sigma \times Q$*. Formally,* $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$.

If δ *is a function, that is* $\delta : Q \times \Sigma \to Q$*, we use the notation* $\delta(q,a)$ *to specify the value of* δ *for arguments* q, a.

Informally, a computation of $\mathcal{A}$ for an input word w starts always in the initial state q_0 and continues by a sequence of **steps** (**moves** or **transitions**)—one for each input symbol. In each step the automaton moves from its current state, say p, under the next symbol of the input word, say a, into a state q such that $(p,a,q) \in \delta$—if such a q exists. If there is a unique $q \in Q$ such that $(p,a,q) \in \delta$, then the transition from the state p under the input a is uniquely determined—we usually say it is deterministic. If there are several q such that $(p,a,q) \in \delta$, then one of the possible transitions is chosen and all of them are considered as being equally likely. If, for some state p and an input a, there is no q such that $(p,a,q) \in \delta$, then we say that the input a in the state p leads to a termination of the computation. A computation ends after the last symbol of w is processed or a termination occurs.

For a formal definition of computation of a FA the concept of configuration is important. A **configuration** C of $\mathcal{A}$ is a pair $(p,w) \in Q \times \Sigma^*$. Informally, the automaton $\mathcal{A}$ is in the configuration (p,w), if it is in the state p and w is the part of the input word to be yet processed. A configuration (q_0,w) is called **initial** and any configuration $(q,\varepsilon), q \in Q_F$ is called **final**.

A **computational step** of $\mathcal{A}$ is the relation

$$\vdash_{\mathcal{A}} \subseteq (Q \times \Sigma^*) \times (Q \times \Sigma^*)$$

between configurations defined for $p,q \in Q$, $a \in \Sigma$, $w \in \Sigma^*$ by

$$(p,aw) \vdash_{\mathcal{A}} (q,w) \ \Leftrightarrow\ (p,a,q) \in \delta.$$

Informally, $(p,aw) \vdash_{\mathcal{A}} (q,w)$ means that $\mathcal{A}$ moves from the state p after the input a into the state q. A **computation** of $\mathcal{A}$ is the transitive and reflexive closure $\overset{*}{\vdash}_{\mathcal{A}}$ of the relation $\vdash_{\mathcal{A}}$ between configurations. That is, $C \overset{*}{\vdash}_{\mathcal{A}} C'$ for configurations C and C' if and only if there is a sequence of configurations $C_1, \ldots, C_n$ such that $C = C_1$, $C_i \vdash_{\mathcal{A}} C_{i+1}$, for $1 \leq i < n$, and $C_n = C'$.

Instead of $(p,w) \overset{*}{\vdash}_{\mathcal{A}} (q,\varepsilon)$, we usually use the notation $p \overset{*}{\underset{w}{\Longrightarrow}} q$. A state q is called reachable in $\mathcal{A}$ if there is an input word w such that $q_0 \overset{*}{\underset{w}{\Longrightarrow}} q$.

Exercise 3.2.2 *Let $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ be a FA. Let us define a recurrence as follows: $A_0 = \{q_0\}$, $A_i = \{q' \mid (q,a,q') \in \delta$ for some $q \in A_{i-1}, a \in \Sigma\}$, for $i \geq 1$. Show that a state q is reachable in $\mathcal{A}$ if and only if $q \in A_j$ for some $j \leq |Q|$. (This implies that it is easy to compute the set of all reachable states.)*

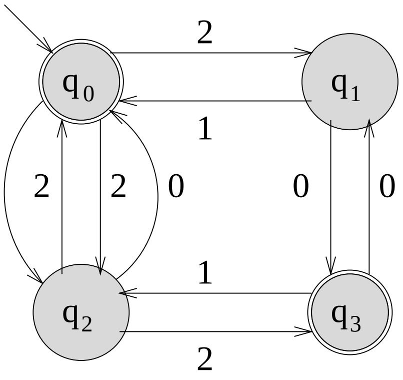
Three basic ways to represent finite automata are illustrated in Figure 3.4 on the automaton $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$, where $Q = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{0, 1\}$, and $Q_F = \{q_0, q_3\}$: an **enumeration** of transitions (Figure 3.4a), a **transition matrix** (Figure 3.4b) with rows labeled by states and columns by input symbols, and a **state graph** or a **transition diagram** (Figure 3.4c)—states are represented by circles, transitions by directed edges labeled by input symbols. The initial state is denoted by an ingoing arrow and final states by double circles. For a finite automaton $\mathcal{A}$ let $G_{\mathcal{A}}$ denote its state graph. Observe that a state q is reachable in the automaton $\mathcal{A}$ if and only if the corresponding node is reachable in the graph $G_{\mathcal{A}}$ from its starting vertex.

$$\begin{array}{lcllcl}
\delta(q_0,0) & = & \{q_2\}, & \delta(q_0,2) & = & \{q_1,q_2\}, \\
\delta(q_1,0) & = & \{q_3\}, & \delta(q_1,1) & = & \{q_0\}, \\
\delta(q_2,0) & = & \{q_0\}, & \delta(q_2,2) & = & \{q_3,q_0\}, \\
\delta(q_3,0) & = & \{q_1\}, & \delta(q_3,1) & = & \{q_2\}.
\end{array}$$

(a)

states	inputs		
	0	1	2
q_0	q_2	-	q_1, q_2
q_1	q_3	q_0	-
q_2	q_0	-	q_3, q_0
q_3	q_1	q_2	-

(b)



(c)

Figure 3.4: Finite automata representations-80%

To every finite automaton $\mathcal{A} = \langle \Sigma, Q, q_o, Q_F, \delta \rangle$ and every $q \in Q$, we associate the language $L(q)$ of those words that make $\mathcal{A}$ to move from the state q to a final state. More formally

$$L(q) = \{w \in \Sigma^* \mid q \underset{w}{\overset{*}{\Longrightarrow}} p \in Q_F\}.$$

$L(\mathcal{A}) = L(q_0)$ is then the **language recognized** by $\mathcal{A}$. A language L is called a **regular language** if there is a finite automaton $\mathcal{A}$ such that $L = L(\mathcal{A})$. The **family of languages recognizable** by finite automata, or the **family of regular languages**, is denoted by

$$\mathcal{L}(FA) = \{L(\mathcal{A}) \mid \mathcal{A} \text{ is a finite automaton}\}.$$

Exercise 3.2.3 *Let* $L_n = \{uv \mid uv \in \{0,1\}^*, |u| = |v| = n, u \neq v\}$. *Design a FA accepting the language: (a)* L_2; *(b)* L_3; *(c)* L_4.

Exercise 3.2.4 *Describe the language accepted by the FA depicted in Figure 3.5.*

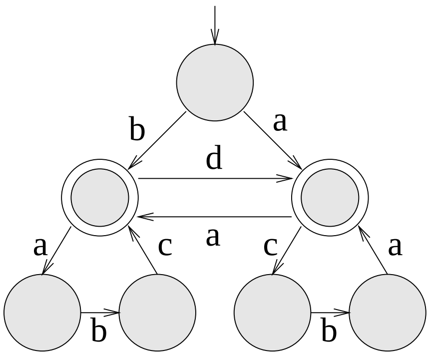


Figure 3.5: Finite automaton-80%

Another way how to define the language recognized by a finite automaton $\mathcal{A}$ is in terms of its state graph $G_{\mathcal{A}}$. A **path** in $G_{\mathcal{A}}$ is a sequence of triples $(p_1, a_1, p_2)(p_2, a_2, p_3) \ldots (p_n, a_n, p_{n+1})$ such that $(p_i, a_i, p_{i+1}) \in \delta$, for $1 \leq i \leq n$. The word $a_1 \ldots a_n$ is the label of such a path, p_1 its origin and p_{n+1} its terminus. A word $w \in \Sigma^*$ is recognizable by $\mathcal{A}$ if w is the label of a path with q_0 as its origin and a final state as its terminus. $L(\mathcal{A})$ is then the set of all words recognized by $\mathcal{A}$.

The language recognized by a finite automaton $\mathcal{A}$ can be seen as the computational process that $\mathcal{A}$ represents. That is why two finite automata $\mathcal{A}_1, \mathcal{A}_2$ are called **equivalent** if $L(\mathcal{A}_1) = L(\mathcal{A}_2)$, i.e., the corresponding languages (computational processes they represent) are equal.

Exercise 3.2.5 *A natural generalization is to consider finite automata $\mathcal{A} = \langle \Sigma, Q, Q_I, Q_F, \sigma \rangle$ with a set Q_I of initial states where computation and recognition is defined similarly. Show that to each such finite automaton $\mathcal{A}$ we can easily construct an equivalent ordinary finite automaton.*

If two FA are equivalent, i.e. if they are "the same" from the point of view that the computational processes (languages) they represent are the same, they can look very differently and can also have a different number of states. A stronger requirement for similarity is that they are **isomorphic**—they differ only in the way their states are denoted.

Definition 3.2.6 *Two FA $\mathcal{A}_i = \langle \Sigma, Q_i, q_{0,i}, Q_{F,i}, \delta_i \rangle$, $i = 1, 2$ are isomorphic if there is a bijection $\mu : Q_1 \to Q_2$ such that $\mu(q_{0,1}) = q_{0,2}$, $q \in Q_{F,1}$ if and only if $\mu(q) \in Q_{F,2}$, and for any $q, q' \in Q_1$, $a \in \Sigma$ we have $(q, a, q') \in \delta_1$ if and only if $(\mu(q), a, \mu(q')) \in \delta_2$.*

Exercise 3.2.7 *Design a finite automaton that accepts those binary words that represents integers (with the most significant bit as the first) divisible by three.*

3.2.2 Nondeterministic versus Deterministic Finite Automata

The formal definition of a FA on page 181 allows a FA to have two properties that contradict our intuition—a state transition, for a given input, does not have to be unique and it does not have to be defined. Our intuition seems to prefer that a FA is deterministic and complete in the following sense:

A finite automaton is a **deterministic finite automaton** if its transition relation is a partial function; i.e. for each state $p \in Q$, and input $a \in \Sigma$ there is at most one $q \in Q$ such that $(p, a, q) \in \delta$. A finite automaton is called **complete,** if for any $p \in Q, a \in \Sigma$ there is at least one q such that $(p, a, q) \in \delta$. In the following the notation DFA will be used for deterministic and complete FA.

The following theorem shows that our definition of finite automata, which allows "strange" nondeterminism, has not increased the recognition power of DFA.

Theorem 3.2.8 *To every finite automaton there is an equivalent deterministic and complete finite automaton.*

Proof. Given a FA $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ an equivalent DFA $\mathcal{A}'$ can be constructed, by the so-called **subset construction**, as

$$\mathcal{A}' = \langle \Sigma, 2^Q, \{q_0\}, \{B \mid B \in 2^Q, B \cap Q_F \neq \emptyset\}, \delta' \rangle,$$

where the new transition relation δ' is defined as follows:

$$(A, a, B) \in \delta' \text{ if and only if } B = \{q \mid \exists p \in A, (p, a, q) \in \delta\}.$$

The states of $\mathcal{A}'$ are therefore sets of the states of $\mathcal{A}$. There is a transition in $\mathcal{A}'$ from a state S—a set of states of $\mathcal{A}$—to another state S_1—again a set of states of $\mathcal{A}$—under an input a, if and only if to each state in S_1 there is a transition in $\mathcal{A}$ from some state in S under the input a.

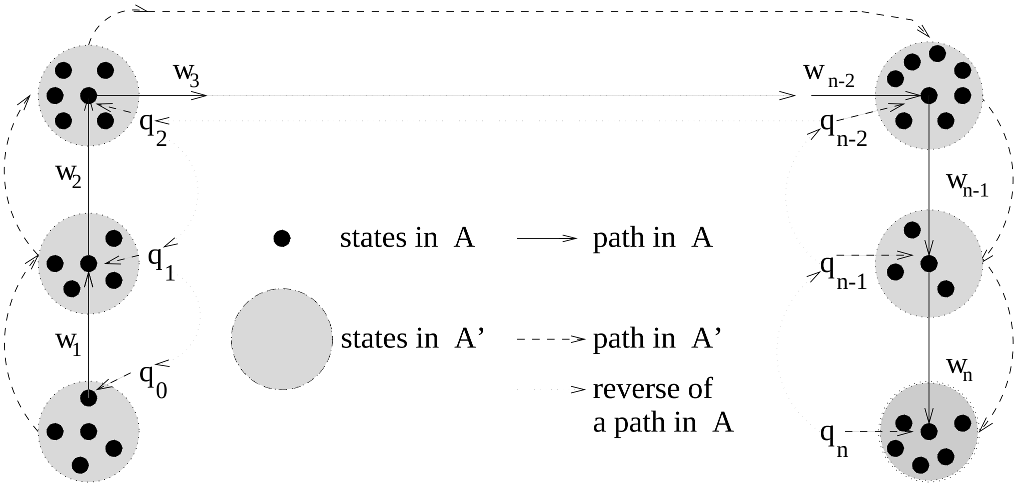


Figure 3.6: A path in a FA and in an equivalent DFA obtained by the subset construction -70%

$\mathcal{A}$ is clearly deterministic and complete. To show that $\mathcal{A}$ and $\mathcal{A}'$ are equivalent, consider the state graphs $G_{\mathcal{A}}$ and $G_{\mathcal{A}'}$. For any path in $G_{\mathcal{A}}$, from the initial state to a final state, labeled by a word $w = w_1 \dots w_s$, there is a unique path in $G_{\mathcal{A}'}$, labelled also by w, from the initial state to a final state—see Figure 3.6. The corresponding states of $\mathcal{A}'$, as the sets of states of $\mathcal{A}$, can be determined, step by step, using the transition function δ', from the initial state of $\mathcal{A}'$ and w. The state of $\mathcal{A}'$ reached by the path labeled by a prefix of w has to contain exactly the states of $\mathcal{A}$ reached, in $G_{\mathcal{A}}$, by the path labelled by the same prefix of w. Similarly, to any path in $\mathcal{A}'$, from the initial to a final state, labelled by a word w, there is a path in $G_{\mathcal{A}}$ from the initial to a final state. The states on this path can be taken from the corresponding states of the path labelled by w in $G_{\mathcal{A}'}$ in such a way that they form a path in $G_{\mathcal{A}}$—to design it, one has to start in the last state (of $\mathcal{A}'$) of the path in $G_{\mathcal{A}'}$, to pick up a state q_n (terminal in $\mathcal{A}$), from this state of $\mathcal{A}'$ and go backwards to pick up states $q_{n-1}, q_{n-2}, \dots, q_1$. This is possible, because whenever $A \underset{a}{\Longrightarrow} B$ in $\mathcal{A}'$, then for any $q' \in B$ there is a $q \in A$ such that $q \underset{a}{\Longrightarrow} q'$ in $\mathcal{A}$. □

Example 3.2.9 *Let $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ be the nondeterministic finite automaton depicted in Figure 3.7a. The finite automaton obtained by the subset construction has the following transition function δ':*

$$\begin{array}{rcl@{\qquad}rcl}
\delta'(\emptyset,0) & = & \emptyset; & \delta'(\emptyset,1) & = & \emptyset; \\
\delta'(\{p\},0) & = & \{p,q\}; & \delta'(\{p\},1) & = & \{q\}; \\
\delta'(\{q\},0) & = & \{p,r\}; & \delta'(\{q\},1) & = & \{q,r\}; \\
\delta'(\{r\},0) & = & \{p,r\}; & \delta'(\{r\},1) & = & \emptyset; \\
\delta'(\{p,q\},0) & = & \{p,q,r\}; & \delta'(\{p,q\},1) & = & \{q,r\}; \\
\delta'(\{p,r\},0) & = & \{p,q,r\}; & \delta'(\{p,r\},1) & = & \{q\}; \\
\delta'(\{q,r\},0) & = & \{p,r\}; & \delta'(\{q,r\},1) & = & \{q,r\}; \\
\delta'(\{p,q,r\},0) & = & \{p,q,r\}; & \delta'(\{p,q,r\},1) & = & \{q,r\}.
\end{array}$$

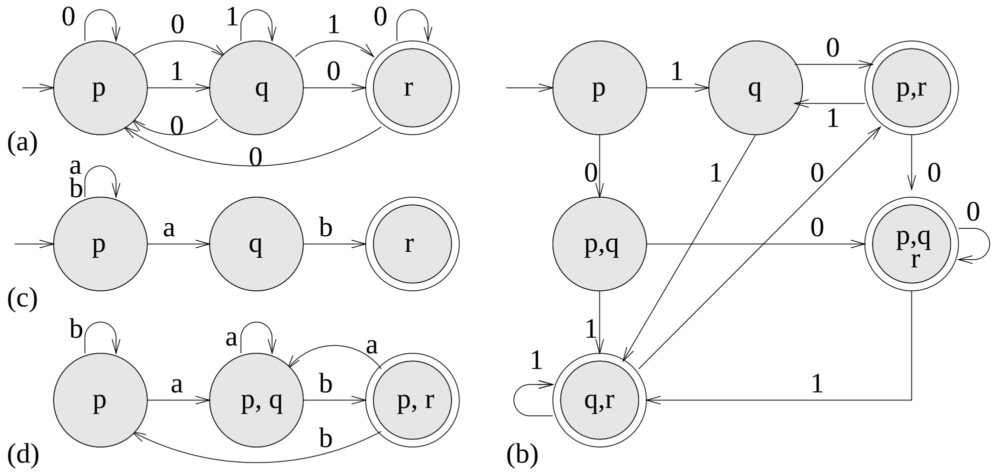


Figure 3.7: NFA and equivalent DFA obtained by the subset construction–65%

The states $\{r\}$ and $\emptyset$ are not reachable from the initial state $\{p\}$ and therefore they are not included in the state graph $G_{\mathcal{A}'}$ of $\mathcal{A}'$ in Figure 3.7b. The subset construction applied to the FA in Figure 3.7c provides the DFA shown in Figure 3.7d. Other states created by the subset construction are not reachable in this case.

Exercise 3.2.10 *Design DFA equivalent to FA in: (a) Figure 3.8a; (b) Figure 3.8b.*

Since nondeterministic and incomplete FA meet less our intuition what a finite state machine is and, moreover, they are not more powerful than DFA, it is natural to ask why to consider them at all.

There are two reasons for that. Both deal with efficiency. The first one concerns design efficiency. It is quite often easier, even significantly easier, to design a NFA accepting a language than an equivalent DFA. For example, it is straightforward to design a NFA recognizing the language $\{a,b\}^*a\{a,b\}^n$—see Figure 3.9a for the general case and Figure 3.9b for $n = 2$. On the other hand, it is much more involved to design a DFA for this language—see one in Figure 3.9c for $n = 2$. The second reason concerns size-efficiency and this one is even "provably important".

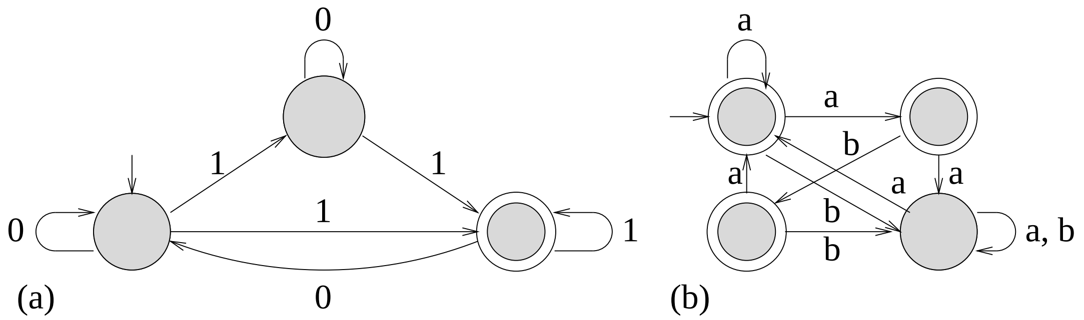


Figure 3.8: Examples of NFA –80%

The number of states of a FA $\mathcal{A}$, in short $state(\mathcal{A})$, is its **state complexity**. In case of the NFA from Figure 3.7c the subset construction does not provide a DFA with larger number of states. On the other side, the subset construction applied to the NFA from Figure 3.7a, has increased significantly the number of states. In general, the subset construction applied to an NFA with n states provides a DFA with 2^n states—this is the number of subsets of each set of n elements. This indicates that the subset construction can produce exponentially more states. However, some of these states may not be reachable as the example above shows. Moreover, it is not yet clear whether some other general method could not provide a DFA with less states and still equivalent to the given NFA.

In order to express exactly how much more economical a NFA may be, comparing with an equivalent DFA, the following economy function is introduced:

$$Economy_{NFA}^{DFA}(n) = \max\{\min\{state(\mathcal{B}) | \mathcal{B} \text{ is a DFA equivalent to } \mathcal{A}\} | \mathcal{A} \text{ is NFA}, state(\mathcal{A}) = n\}$$

The following result shows that a DFA can be, provably, exponentially larger than an equivalent NFA.

Theorem 3.2.11 *$Economy_{NFA}^{DFA}(n) = 2^n$.*

Proof idea. The inequality $\text{Economy}_{NFA}^{DFA}(n) \leq 2^n$ follows from the subset construction. In order to prove the opposite inequality, it is sufficient to show, what can be done, that the minimum DFA equivalent to the one shown in Figure 3.9d has to have 2^n states. $\square$

A simpler example, though not so perfect, of the exponential growth of states provided by the subset construction, is shown in Figure 3.9. The minimal DFA equivalent to the NFA shown in Figure 3.9a has to have 2^{n-1} states. This is easy to realize because the automaton has to remember the last $n-1$ symbols. For $n = 2$ the equivalent DFA is shown in Figure 3.9c.

Corollary 3.2.12 *Nondeterminism of FA does not increase their computational power, but can essentially (exponentially) decrease the number of states (and thereby also increase the design efficiency).*

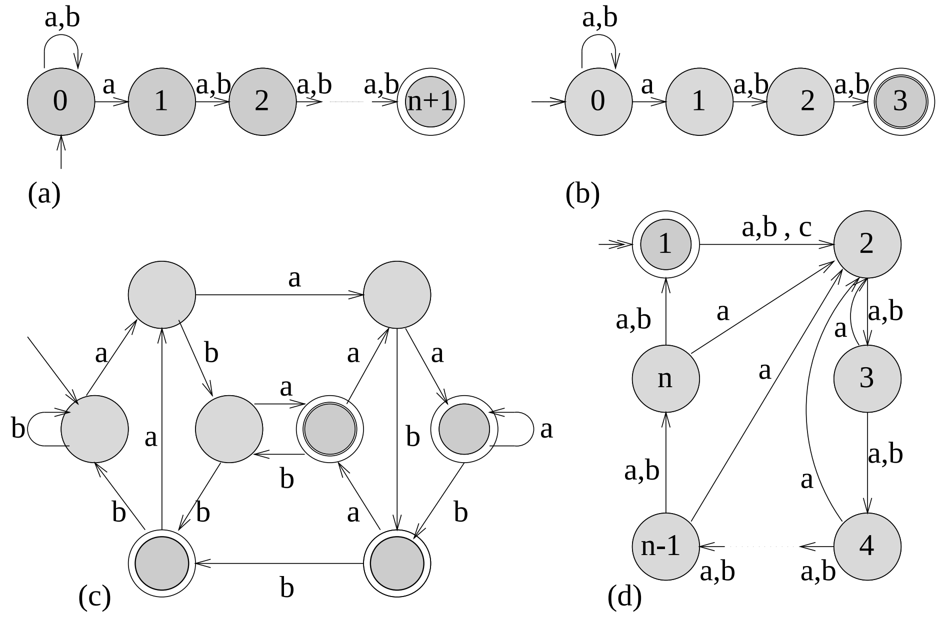


Figure 3.9: Examples showing that the subset construction can yield an exponential growth of states–70%

Exercise 3.2.13 *Design a DFA equivalent to the one in Figure 3.9d for:* *(a)* $n = 4$; *(b)* $n = 5$.

3.2.3 Minimization of Deterministic Finite Automata

Once we have a task to design a DFA recognizing a given regular language L, it is natural to try to find a "minimal" DFA , with respect to the number of states, for L. Figure 3.10 shows that two equivalent DFA may have different number of states. The following questions therefore arise naturally.

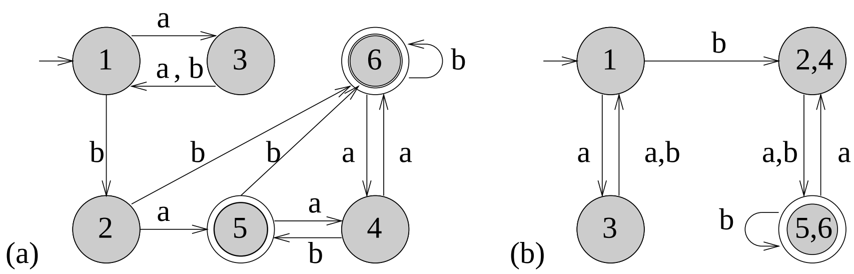


Figure 3.10: Two equivalent DFA–70%

- How many different but equivalent minimal DFA can exist for a given FA?
- How to design a minimal DFA equivalent to a given DFA?
- How fast can one construct a minimal DFA?

In order to handle these problems, new concepts have to be introduced. Two states p, q of a FA $\mathcal{A}$ are called **equivalent**, in short $p \equiv_{\mathcal{A}} q$, if $L(p) = L(q)$ in $\mathcal{A}$. A FA $\mathcal{A}$ is called **reduced** if no two different states of $\mathcal{A}$ are equivalent. A DFA $\mathcal{A}$ is called **minimal** if there is no DFA equivalent to $\mathcal{A}$ and having less states.

We show two simple methods to minimize finite automata. Both are based on the result, shown later, that if a DFA is reduced, then it is minimal.

1. **Minimization of DFA using the operations of reversal and subset construction.** The first method is based on two operations with finite automata. The operation of **reversal** assigns to a DFA $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ the finite automaton $\rho(\mathcal{A}) = \langle \Sigma, Q, Q_F, \{q_0\}, \rho(\delta) \rangle$ (that is the initial and final states are exchanged) and $q \in \rho(\delta)(q', a)$ if and only if $\delta(q, a) = q'$. The operation of **subset construction** assigns to any FA $\mathcal{A} = \langle \Sigma, Q, Q_I, Q_F, \delta \rangle$, with a set Q_I of initial states, a DFA $\pi(\mathcal{A})$ obtained from $\mathcal{A}$ by the subset construction (and containing only reachable states).

Theorem 3.2.14 *Let $\mathcal{A}$ be a finite automaton, then $\mathcal{A}' = \pi(\rho(\pi(\rho(\mathcal{A}))))$ is a reduced DFA equivalent to $\mathcal{A}$.*

Proof. Clearly $\mathcal{A}'$ is a DFA equivalent to $\mathcal{A}$. It is therefore sufficient to prove that $\pi(\rho(D))$ is reduced whenever $D = \langle \Sigma, Q', q'_0, Q'_F, \delta' \rangle$ is a FA and each of its states is reachable. Let $Q_1 \subseteq Q'$, $Q_2 \subseteq Q'$ be two equivalent states of $\pi(\rho(D))$. Since each state of D is reachable, for each $q_1 \in Q_1$ there is a $w \in \Sigma^*$ such that $q_1 = \delta'(q'_0, w)$. Thus $q'_0 \in \rho(\delta')(Q_1, w)$. As Q_1 and Q_2 are equivalent we also have $q'_0 \in \rho(\delta')(Q_2, w)$ and therefore $q_2 = \delta'(q'_0, w)$ for some $q_2 \in Q_2$. Since δ' is a mapping we get $q_1 = q_2$, and therefore $Q_1 \subseteq Q_2$. By symmetry $Q_1 = Q_2$. □

Unfortunately, there is a DFA $\mathcal{A}$ with n states such that $\pi(\rho(\mathcal{A}))$ has 2^n states—see one in Figure 3.9d. The time complexity of the above algorithm is therefore exponential in the worst case.

2. **Minimization of DFA through equivalence automata.** The second idea of designing a reduced DFA $\mathcal{A}'$ equivalent to a given DFA $\mathcal{A}$ is also quite simple and leads to a much more efficient algorithm. In the state graph $G_{\mathcal{A}}$ identify nodes corresponding to the equivalent states and then identify multiple edges with the same label between the same nodes. The resulting state graph is that of a reduced DFA. More formally,

Definition 3.2.15 *Let $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ be a DFA. For any state $q \in Q$ let $[q]$ be the equivalence class on Q with respect to the relation $\equiv_{\mathcal{A}}$. The* **equivalence automaton** *$\mathcal{A}'$ for $\mathcal{A}$ is defined by*

$\mathcal{A}' = \langle \Sigma, Q', [q_0], Q'_F, \delta' \rangle$, where $Q' = \{[q] \mid q \in Q\}$, $Q'_F = \{[q] \mid q \in Q_F\}$, and $\delta' = \{([q_1], a, [q_2]) \mid (q'_1, a, q'_2) \in \delta$ for some $q'_1 \in [q_1], q'_2 \in [q_2]\}$.

Minimization of DFA is now based on the following result:

Theorem 3.2.16 *(1) The equivalence automaton $\mathcal{A}'$ of a DFA $\mathcal{A}$ is well-defined, reduced and equivalent to $\mathcal{A}$.*

(2) $state(\mathcal{B}) \geq state(\mathcal{A}')$ for any DFA $\mathcal{B}$ equivalent to a DFA $\mathcal{A}$.

(3) Any minimal DFA $\mathcal{B}$ equivalent to a DFA $\mathcal{A}$ is isomorphic to $\mathcal{A}'$.

Proof. (1) If $q \equiv_{\mathcal{A}} q'$, then either both q, q' are in Q_F or both are not in Q_F. Final states of $\mathcal{A}'$ are therefore well defined. Moreover, if $L(q) = L(q')$ for some $q, q' \in Q$, then for any $a \in \Sigma$, $L(\delta(q,a)) = L(\delta(q',a))$, and therefore all transitions of $\mathcal{A}'$ are well defined. If $w = w_1 \ldots w_n \in \Sigma^*$ and $q_i = \delta(q_0, w_1 \ldots w_i)$, then $[q_i] = \delta'([q_0], w_1 \ldots w_i)$. This implies $L(\mathcal{A}) = L(\mathcal{A}')$. The condition of $\mathcal{A}'$ being reduced is trivially fulfilled due to the construction of $\mathcal{A}'$.

(2) It is sufficient to prove (2) assuming that all states of $\mathcal{B}$ are reachable from the initial state. Let $\mathcal{B} = \langle \Sigma, Q'', q_0'', Q_F'', \delta'' \rangle$ be a DFA equivalent to $\mathcal{A}$. Consider the mapping $g : Q'' \to Q'$ defined as follows: Since all states of $\mathcal{B}$ are reachable, for any $q'' \in Q''$ there is a $w_{q''} \in \Sigma^*$ such that $\delta''(q_0'', w_{q''}) = q''$. Define now $g(q'') = \delta'([q_0], w_{q''})$. From the minimality of $\mathcal{A}'$, and its equivalence with $\mathcal{B}$, it follows that this mapping is well defined and surjective.

(3) In case of minimality of $\mathcal{B}$ it is easy to verify that the mapping g defined in (2) is actually an isomorphism. $\square$

Corollary 3.2.17 *If a DFA is reduced, then it is minimal.*

The task to construct a minimal DFA equivalent to a given DFA $\mathcal{A}$ has therefore been reduced to that to determine which pairs of states of $\mathcal{A}$ are equivalent—or nonequivalent, what seems to be easier. This can be done as follows.

Let us call two states q, q' of $\mathcal{A}$,

1. 0-**nonequivalent**, if one of them is a final state and the other one is not,
2. i-**nonequivalent**, for $i > 0$, if they are either $(i-1)$-nonequivalent, or there is an $a \in \Sigma$ such that $\delta(q,a)$ and $\delta(q',a)$ are $(i-1)$-nonequivalent.

Let A_i be the set of pairs of i-nonequivalent states, $i \geq 0$. Clearly, $A_i \subseteq A_{i+1}$, for all $i \geq 0$ and one can show that $A_n = A_{n+k}$ for any $k \geq 0$ if $n = state(\mathcal{A})$.

Two states q, q' are not equivalent, if and only if there is a $w = w_1 \ldots w_m \in \Sigma^*$ such that $\delta(q,w) \in Q_F$ and $\delta(q',w) \not\in Q_F$. This implies that states $\delta(q, w_1 \ldots w_{m-i})$ and $\delta(q', w_1 \ldots w_{m-i})$ are i-nonequivalent. Hence, if q and q' are not equivalent, they are n-nonequivalent.

The recurrent definition of the sets A_i actually specifies an $\mathcal{O}(n^2 m)$ algorithm, $m = |\Sigma|$, to determine equivalent states and thereby the minimal DFA.

Example 3.2.18 *The construction of i-nonequivalent states for the DFA from Figure 3.10a yields $A_0 = \{(1,5),(1,6),(2,5),(2,6),(3,5),(3,6),(4,5),(4,6)\}$, $A_1 = A_0 \cup \{(1,2),(1,4),(2,3),(3,4)\}$, $A_2 = A_1 \cup \{(1,3)\}$, $A_3 = A_2$. The resulting minimal DFA is depicted in Figure 3.10b.*

It can be shown, by using a more efficient algorithm to determine the equivalence, that one can construct the minimal DFA in sequential time $\mathcal{O}(mn \lg n)$, where m is the size of the alphabet and n is the number of states of the given DFA—see references.

Exercise 3.2.19 *Design the minimal DFA accepting the language: (a) of all words over the alphabet* $\{a, b\}$ *that contain the subword "abba" and end with the subword "aaa"; (b) of all words over the alphabet* $\{0, 1\}$ *that contain at least two occurrences of the subword "111"; (c)* $L = \{w \mid \#_a w \equiv \#_b w \pmod 3\} \subseteq \{a, b\}^*$.

3.2.4 Decision Problems

To decide whether two DFA, $\mathcal{A}_1$ and $\mathcal{A}_2$, are equivalent, it suffices to construct to $\mathcal{A}_1$ the minimal equivalent DFA and to $\mathcal{A}_2$ the minimal DFA $\mathcal{A}'_2$. $\mathcal{A}_1$ and $\mathcal{A}_2$ are then equivalent if and only if $\mathcal{A}'_1$ and $\mathcal{A}'_2$ are isomorphic. If $n = \max\{state(\mathcal{A}_1), state(\mathcal{A}_2)\}$ and m is the size of the alphabet, then minimization can be done in $\mathcal{O}(mn \lg n)$ sequential time, and the isomorphism can be checked in $\mathcal{O}(nm)$ sequential time.

One way to decide the equivalence of two NFA $\mathcal{A}_1$ and $\mathcal{A}_2$ is to design equivalent DFA to $\mathcal{A}_1$ and $\mathcal{A}_2$ and then minimize these DFA. If the resulting DFA are isomorphic the original NFA are equivalent; otherwise not. However, this may take exponential time. It seems that there is no essentially better method because the equivalence problem for NFA is a **PSPACE**-complete problem—see Section 5.11.2.

Two other basic decision problems for FA $\mathcal{A}$ are the **emptiness problem**—is $L(\mathcal{A})$ empty?—and the **finitness problem**—is $L(\mathcal{A})$ finite? It follows from the following theorem that these two problems are decidable—one has only to check whether there is a $w \in L(\mathcal{A})$ such that $|w| < n$ in the first case and $n \leq |w| < 2n$ in the second case.

Theorem 3.2.20 *Let* $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ *be a DFA and* $|Q| = n$.

(1) $L(\mathcal{A}) \neq \emptyset$ *if and only if there is a* $w \in L(\mathcal{A})$ *such that* $|w| \leq n$.

(2) $L(\mathcal{A})$ *is infinite if and only if there is a* $w \in L(\mathcal{A})$ *such that* $n < |w| < 2n$.

Theorem 3.2.20 is actually a corollary of the following basic result:

Lemma 3.2.21 (Pumping Lemma for Regular Languages) *If* $\mathcal{A}$ *is a FA and there is a* $w \in L(\mathcal{A})$, $|w| > n = state(\mathcal{A})$, *then there are* $x, y, z \in \Sigma^*$ *such that* $w = xyz$, $|xz| \leq n, 0 < |y| \leq n$, *and* $xy^iz \in L(\mathcal{A})$, *for all* $i \geq 0$.

Proof. Let w be a shortest word in $L(\mathcal{A})$ with $|w| > n$ and $w = w_1 \ldots w_k, w_i \in \Sigma$. Consider the following sequence of states

$$q_i = \delta(q_0, w_1 \ldots w_i), 0 \leq i \leq k.$$

Let us now take i_1 and i_2 such that $0 \leq i_1 < i_2 \leq k$, $q_{i_1} = q_{i_2}$, and $i_2 - i_1$ is as small as possible. Such i_1, i_2 must exist (Pidgeonhole principle), and clearly $i_2 - i_1 \leq n$. Denote $x = w_1 \ldots w_{i_1}, y = w_{i_1+1} \ldots w_{i_2}, z = w_{i_2+1} \ldots w_k$. Then $\delta(q_0, xy^i) = q_{i_1} = q_{i_2}$ for all $i \geq 0$ and therefore also $xy^iz \in L(\mathcal{A})$. Because of the minimality of w we get $|xz| \leq n$. □

Exercise 3.2.22 *Show the following modification of the Pumping Lemma for Regular Languages: Let L be a regular language. There exists an $N_L \in \mathbf{N}$ such that if for some strings x_1, x_2, x_3, $x_1x_2x_3 \in L$ and $|x_2| \geq N_L$, then there exist strings u, v and w such that $x_2 = uvw$, $v \neq \varepsilon$, $|uv| \leq N_L$ and $x_1uv^iwx_3 \in L$ for all $i \geq 0$.*

Exercise 3.2.23 *Show, using one of the Pumping Lemmas for regular languages, that the language $\{wcw \mid w \in \{a,b\}^*\}$ is not regular.*

3.2.5 String Matching with Finite Automata

Finding all occurrences of a **pattern** in a **text** is a problem that arises in a large variety of applications. For example, in text editing, DNA sequences searching, and so on. This problem can be elegantly and efficiently solved using finite automata.

String matching problem: Given a string (called a **pattern**) $x \in \Sigma^*$, $|x| = m$, design an algorithm to determine, for an arbitrary $y \in \Sigma^*$, $y = y_1 \ldots y_n$, $y_j \in \Sigma$ for $1 \leq j \leq n$, all integers $1 \leq i \leq n$ such that x is a suffix of the string $y_1 \ldots y_i$.

A naive string matching algorithm,which checks in m steps, for all $m \leq i \leq n$, whether x is a sufix of $y_1 \ldots y_i$, clearly requires $\mathcal{O}(mn)$ steps.

The problem can be reduced to that of designing, for a given x, a finite automaton $\mathcal{A}_x$ capable of deciding for a given word $y \in \Sigma^*$, whether $y \in \Sigma^* x$.

If $x = x_1 \ldots x_m, x_i \in \Sigma$, then the NFA $\mathcal{A}_x$ shown for an arbitrary x in Figure 3.11a and for $x = abaaaba$ in Figure 3.11b, accepts $\Sigma^* x$. $\mathcal{A}_x$ has $m+1$ states that can be identified with the elements of the set P_x of prefixes of x, that is with the set

$$P_x = \{\varepsilon, x_1, x_1x_2, \ldots, x_1x_2 \ldots x_m\}$$

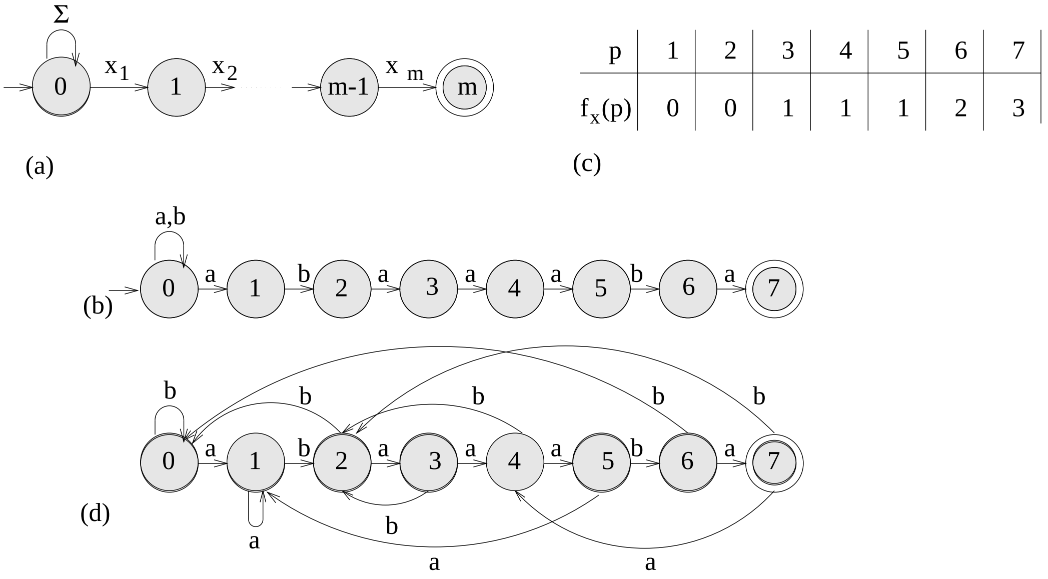
or with the integers from 0 to m; with i standing for $x_1 \ldots x_i$.

It is easy to see that the DFA $\mathcal{A}'_x$, that can be obtained from $\mathcal{A}_x$ by the subset construction, has also only $m+1$ states. Indeed, those states of $\mathcal{A}'_x$ that are reachable from the initial state by a word y form exactly the set of those elements of P_x that are suffixes of y. This set is uniquely determined by the longest of its elements, say p, since the others are those suffixes of p that are in P_x. Hence, the states of $\mathcal{A}'$ can also be identified with integers from 0 to m. (See $\mathcal{A}'_x$ for $x = abaaaba$ in Figure 3.11d.)

Let $f_x : P_x \to P_x$ be the so-called **failure function** that assignes to each $p \in P_x - \{\varepsilon\}$ the longest proper suffix of p that is in P_x. (For $x = abaaaba$ f_x is shown in Figure 3.11c—as a mapping from $\{0, \ldots, 7\}$ to $\{0, \ldots, 7\}$) Then the state of $\mathcal{A}'_x$ corresponding to the longest suffix p contains those states of $\mathcal{A}_x$ that corresponds to prefixes

$$p, f_x(p), f_x^2(p), \ldots, \varepsilon$$

To compute f_x, for an $x \in \Sigma^*$, we can use the following recursive rule: $f_x(x_1) = \varepsilon$ and for all



p	1	2	3	4	5	6	7
$f_x(p)$	0	0	1	1	1	2	3

Figure 3.11: String matching automata and a failure function–60%

$p, pa \in P_x - \{\varepsilon\}$:

$$f_x(pa) = \begin{cases} f_x(p)a, & \text{if } f_x(p)a \in P_x; \\ f_x(f_x(p)a), & \text{otherwise.} \end{cases}$$

Once f_x is known, the transition function δ_x of $\mathcal{A}'$, for $p \in P_x$ and $a \in \Sigma$, has the following form:

$$\delta_x(p, a) = \begin{cases} pa, & \text{if } pa \in P_x; \\ \delta_x(f_x(p), a), & \text{otherwise.} \end{cases}$$

This means that we actually do not need to store δ. Indeed, we can simulate $\mathcal{A}'_x$ on any input word y by the following algorithm—one of the pearls of algorithm design—with the input x, f_x, y:

Algorithm 3.2.24 (Knuth-Morris-Pratt's string matching algorithm)

```
$m \leftarrow |x|$; $n \leftarrow |y|$, $q \leftarrow 0$;
for $i \leftarrow 1$ to $n$ do while $0 < q < m$ and $x_{q+1} \neq y_i$ do $q \leftarrow f_x(q)$ od;
                  if $q < m$ and $x_{q+1} = y_i$ then $q \leftarrow q + 1$;
                  if $q = m$ then print "pattern found starting with $(i - m)$-th symbol";
                         $q \leftarrow f_x(q)$
              od
```

$\mathcal{O}(m)$ steps are needed to compute f_x and since q can get increased at most by 1 in an i-cycle, therefore the overall time of the Knuth-Morris-Pratt's algorithm is $\mathcal{O}(m+n)$. (Quite an improvement comparing with $\mathcal{O}(mn)$ for the naive algorithm.)

Exercise 3.2.25 *Compute the failure function for the patterns:*
(a) aabaaabaaaab; (b) aabbaaabbb.

Exercise 3.2.26 *Show in detail why the overall time complexity of Knuth-Morris-Pratt's algorithm is* $\mathcal{O}(m+n)$.

3.3 Regular Languages

Regular languages are one of the cornerstones of the formal language theory and they have many interesting and important properties.

3.3.1 Closure Properties

The family of regular languages is closed under all basic language operations. This can be utilized in a variety of ways, especially to simplify the design of FA recognizing given regular languages.

Theorem 3.3.1 *The family of regular languages is closed under the operations:*

1. *Union, concatenation, iteration, complement and difference.*
2. *Substitution, morphism and inverse morphism.*

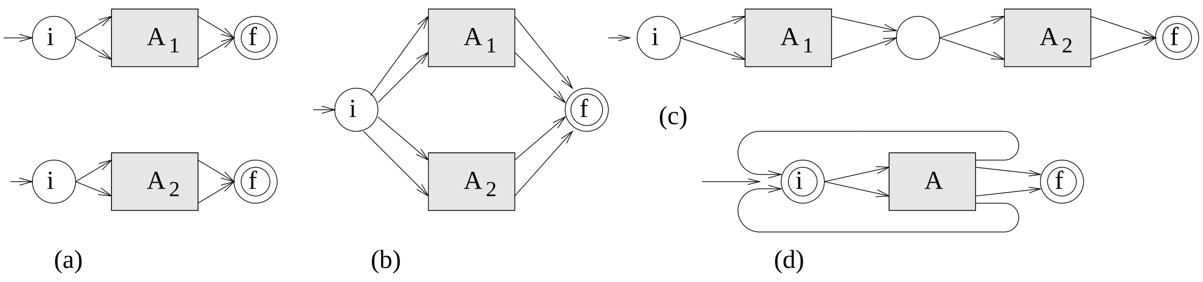


Figure 3.12: Closure of regular languages under union, concatenation and iteration–60%

Proof. To simplify the proof we assume, in some parts of the proof, that the state graphs $G_{\mathcal{A}}$ of those FA we consider are in the normal form shown in Figure 3.12a. Namely, there is no edge entering the input state i and there is a single final state f with no outgoing edge. Given a FA $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$, accepting a regular language that does not contain the empty word, it is easy to construct an equivalent FA in the above normal form. Indeed, it is enough to add two new states i—a new input state—and f—a new terminal state—and the following sets of state transitions:

- $\{(i, a, q) \mid (q_0, a, q) \in \delta\}$;
- $\{(p, a, f) \mid (p, a, q) \in \delta, q \in Q_F\}$;
- $\{(i, a, f) | (q_0, a, q) \in \delta, q \in Q_F\}$.

To simplify the proof of the theorem we assume, in addition, that languages we consider do not contain the empty word. The adjustments needed to prove our theorem in full generality are minor. For example, by taking the state i in Figure 3.12a as an additional terminal state, we add ε to the language.

Figures 3.12b,c,d show how to design a FA accepting the **union, concatenation** and **iteration** of regular languages provided that FA in the above normal form are given for these languages. (In case of the union transitions from the new initial state lead exactly to those states to which transitions from the initial states of the two automata go. In case of iteration, each transition to the final state is doubled to go also to the initial state)

Complement. If $\mathcal{A}$ is a DFA over the alphabet Σ accepting a regular language L, then by exchanging final and nonfinal states in $\mathcal{A}$ we get a DFA accepting the complement of L—the language L^c. More formally, if $L = L(\mathcal{A})$, $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$, then $L^c = L(\mathcal{A}')$, where $\mathcal{A}' = \langle \Sigma, Q, q_0, Q - Q_F, \delta \rangle$.

Intersection. Let $L_1 = L(\mathcal{A}_1)$, $L_2 = L(\mathcal{A}_2)$ where $\mathcal{A}_1 = \langle \Sigma, Q_1, q_{1,0}, Q_{1,F}, \delta_1 \rangle$ and $\mathcal{A}_2 = \langle \Sigma, Q_2, q_{2,0}, Q_{2,F}, \delta_2 \rangle$ are DFA. The intersection $L_1 \cap L_2$ is clearly the language accepted by the DFA

$$\langle \Sigma, Q_1 \times Q_2, (q_{1,0}, q_{2,0}), Q_{1,F} \times Q_{2,F}, \delta \rangle$$

where $\delta((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$ for any $p \in Q_1$, $q \in Q_2$ and $a \in \Sigma$.

Difference. Since $L_1 - L_2 = L_1 \cap L_2^c$ the closure of regular languages under difference follows from their closure under complement and intersection.

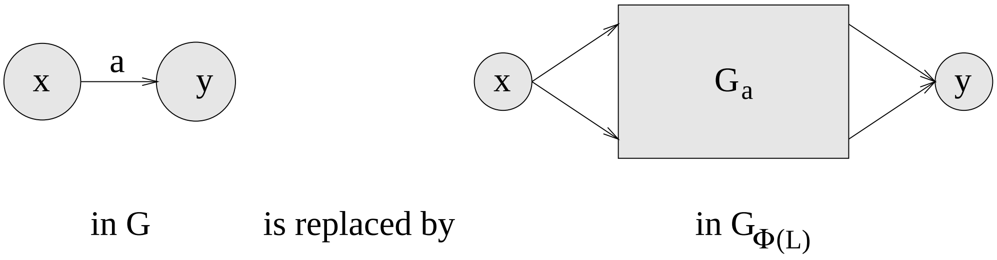


Figure 3.13: Closure of regular languages under substitution–80%

Substitution. Let $\phi : \Sigma \to 2^{\Sigma_1^*}$ be a substitution such that $\phi(a)$ is a regular language for each $a \in \Sigma$. Let L be a regular language over Σ and $L = L(\mathcal{A})$, $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$. For each $a \in \Sigma$ let G_a be the state graph in the normal form for the language $\phi(a)$. To get, the state graph for a FA accepting the language $\phi(L)$ from the state graph $G_{\mathcal{A}}$, it suffices to replace in $G_{\mathcal{A}}$ any edge labeled by an $a \in \Sigma$ by the state graph G_a in the way shown in Figure 3.13.

The closure of regular languages under **morphism** follows from the closure under substitution.

Inverse morphism. Let $\phi : \Sigma \to \Sigma_1^*$ be a morphism, $L \subset \Sigma_1^*$ a regular language, $L = L(\mathcal{A})$ for a FA $\mathcal{A}$. As defined in Section 2.5.1

$$\phi^{-1}(L) = \{w \in \Sigma^* \mid \phi(w) \in L\}.$$

Let $G_{\mathcal{A}} = \langle V, E\rangle$ be the state graph for $\mathcal{A}$. The state graph $G_{\phi^{-1}(L)}$ for a FA recognizing the language $\phi^{-1}(L)$ will have the same set of nodes (states) as $G_{\mathcal{A}}$ and the same set of final nodes (states). For any $a \in \Sigma$, $q \in V$ there will be an edge (p, a, q) in $G_{\phi^{-1}(L)}$ if and only if $p \underset{\phi(a)}{\overset{*}{\Longrightarrow}} q$ in $\mathcal{A}$. Clearly, w is a label of a path in $G_{\phi^{-1}(L)}$, from the initial to a final node, if and only if $\phi(w) \in L$. □

Using the results of Theorem 3.3.1 it is now easy to see that regular languages form a Kleene algebra. Actually, regular languages have been the original motivation for the introduction and study of Kleene algebras.

Exercise 3.3.2 *Show that if $L \subseteq \Sigma^*$ is a regular language, then so are the languages: (a) $L^R = \{w \mid w^R \in L\}$; (b) $\{u\}^{-1}L$, where $u \in \Sigma^*$.*

3.3.2 Regular Expressions

There are various formal systems that can be used to describe exactly regular languages. (That is, each language they describe is regular and they can be used to describe any regular language.) The most important one is the set of **regular expressions.** They will now be defined inductively, together with their semantics (interpretation)—a mapping that assigns a regular language to any regular expression.

Definition 3.3.3 *A regular expression E, over an alphabet Σ, is an expression formed using the following rules and it represents the language $L(E)$ defined as follows:*

1. *$\emptyset$ is a regular expression and $L(\emptyset) = \emptyset$.*
2. *a is a regular expression for any $a \in \Sigma \cup \{\varepsilon\}$ and $L(a) = \{a\}$*
3. *If E_1, E_2 are regular expressions, then so are*

$$(E_1 + E_2),\ (E_1 \cdot E_2),\ (E_1^*)$$

and

$$L((E_1 + E_2)) = L(E_1) \cup L(E_2),\ L((E_1 \cdot E_2)) = L(E_1) \cdot L(E_2),\ L((E_1^*)) = L(E_1^*),$$

respectively.

4. There are no other regular expressions over Σ.

Remark 3.3.4 Several conventions are used to simplify regular expressions. Firstly, the following priority of operators is assumed: $*, \cdot, +$. Secondly, the operators of concatenation are usually omitted. This allows to omit most of the parentheses. For example, a regular expression describing a word $w = a_1 \dots a_n$ is usually written as $a_1 a_2 \dots a_n$ and not $(\dots((a_1.a_2).a_3)\dots a_n)$. Finally, the expression $\{w_1, \dots, w_n\}$ is used to denote the finite language containing the words $w_1, \dots, w_n$.

Example 3.3.5 $\{0,1\}^*$, $\{0,1\}^*000\{0,1\}^*$ *and* $\{a,b\}^*c\{a,b\}^*c\{a,b\}^*$ *are regular expressions.*

Regular expressions and finite automata

The following theorem, identifying languages accepted by finite automata and described by regular expressions, is one of the cornerstones of formal language and automata theory as well as their applications.

Theorem 3.3.6 (Kleene's Theorem) *A language* L *is regular if and only if there is a regular expression* E *such that* $L = L(E)$.

Proof. It follows from Theorem 3.3.1 that each language described by a regular expression is regular. To finish the proof of the theorem it is therefore sufficient to show how to design, given a DFA $\mathcal{A}$, a regular expression $E_{\mathcal{A}}$ such that $L(E_{\mathcal{A}}) = L(\mathcal{A})$. This is quite a straightforward task once a proper notation is introduced.

Let $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$. Without loss of generality we assume that $Q = \{0, 1, \dots, n\}$, $q_0 = 0$ and consider, for $0 \le i, j \le n$, $-1 \le k \le n$, the set R^k_{ij} defined by:

$$R^k_{i,j} = \{w \in \Sigma^* \mid \delta(i,w) = j \text{ and } \delta(i,u) \le k \text{ for any proper prefix } u \text{ of } w\}.$$

In other words, the set $R^k_{i,j}$ contains those strings that make $\mathcal{A}$ to go from state i to state j passing only through states in the set $\{0, 1, \dots, k\}$. Clearly,

$$R^{-1}_{i,j} = \{a \mid \delta(i,a) = j\} \cup \{\varepsilon \mid \text{if } i = j\} \subseteq \Sigma \cup \{\varepsilon\},$$

and therefore there is a regular expression representing $R^{-1}_{i,j}$. Since

$$L(A) = \bigcup_{j \in Q_F} R^n_{0,j},$$

in order to prove the theorem it suffices to show the validity of the recurrence (3.1) for any $R^k_{i,j}$:

$$R^k_{i,j} = \begin{cases} \emptyset, & \text{if } k = -1, i \ne j, \delta(i,a) \ne j; \\ \{a\}, & \text{if } k = -1, i \ne j, \delta(i,a) = j; \\ \{\varepsilon\}, & \text{if } k = -1, i = j, \delta(i,a) \ne j \\ \{\varepsilon, a\}, & \text{if } k = -1, i = j, \delta(i,a) = j; \\ R^{k-1}_{i,j} \cup R^{k-1}_{i,k}(R^{k-1}_{k,k})^* R^{k-1}_{k,j}; & \text{if } k \ge 0. \end{cases} \tag{3.1}$$

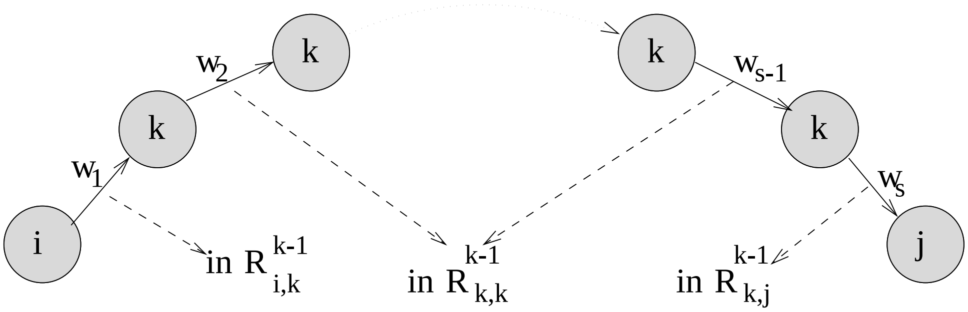


Figure 3.14: Decomposition of a computational path for a word $w \in R^k_{i,j}$, $w = w_1 w_2 \ldots w_s$-80%

Once this is done, it is straightforward to show by induction that for each language $R^k_{i,j}$ there is a regular expression representing it.

However, the validity of the recurrence 3.1 is actually easy to see. Any path in $G_{\mathcal{A}}$ from a state i to a state j that passes only through states $\{0, 1, \ldots, k\}$ can be decomposed into subpaths that may contain the state k only at the beginning or at the end of the path (if the state k occurs on the path at all)—see Figure 3.14 for the case that the state k occurs several times. These subpaths belong to one of the subsets $R^{k-1}_{i,k}$ or $R^{k-1}_{k,k}$ or $R^{k-1}_{k,j}$. ☐

Two regular expressions E_1 and E_2 are said to be equivalent if they describe the same language. Some of the most basic pairs of equivalent regular expressions are listed in the right column of Table 2.1.

Exercise 3.3.7 *Determine which of the following equalities between regular languages are valid: (a)* $(011 + (10)^*1 + 0)^* = 011(011 + (10)^*1 + 0)^*$*;*
(b) $((1 + 0)^*100(1 + 0)^*)^* = ((1 + 0)100(1 + 0)^*100)^*$.

Design of finite automata from regular expressions

One of the advantages of regular expressions is that they specify a regular language in a natural way in a linear form—by a string. Regular expressions are therefore very convenient as a specification language for regular languages, especially for processing on computers. An important practical problem is then to design, given a regular expression E, a FA (or a DFA) recognizing the language $L(E)$. An elegant way of doing it, by using the so-called **derivatives of regular expressions**, will now be described. However, in order to do that a proper notation has to be introduced.

For a regular expression E let $\rho(E)$ be a regular expression that is equal to ε if $\varepsilon \in L(E)$, and for $\emptyset$ otherwise. (That is $\rho(E)F$ equals F if the empty word is in $L(E)$, and $\emptyset$ otherwise.) To compute $\rho(E)$ for a regular expression E we can use the following inductive definition of

ρ (where $a \in \Sigma$):

$$\begin{array}{rclrclrcl} \rho(\emptyset) & = & \emptyset, & \rho(\varepsilon) & = & \varepsilon, & \rho(E+F) & = & \rho(E)+\rho(F), \\ \rho(a) & = & \emptyset, & \rho(E^*) & = & \varepsilon, & \rho(E \cdot F) & = & \rho(E)\cdot\rho(F). \end{array}$$

Definition 3.3.8 *The derivative of a regular expression E by a symbol a, notation $a^{-1}E$, is a regular expression defined recursively by $a^{-1}\emptyset = \emptyset$ and*

$$\begin{array}{rclrclrcl} a^{-1}\varepsilon & = & \emptyset & a^{-1}b & = & \emptyset, \text{ if } a \neq b, & a^{-1}(E^*) & = & (a^{-1}E)\cdot E^*, \\ a^{-1}a & = & \varepsilon & a^{-1}(E+F) & = & a^{-1}E + a^{-1}F, & a^{-1}(E \cdot F) & = & (a^{-1}E)F + \rho(E)a^{-1}F. \end{array}$$

The extension from the derivatives by symbols to derivatives by words is defined by $\varepsilon^{-1}E = E$ and $(wa)^{-1}E = a^{-1}(w^{-1}E)$.

It can been shown that with respect to any regular expression E the set Σ^* is partitioned into the equivalence classes with respect to the relation $w_1 \equiv w_2$ if $w_1^{-1}E = w_2^{-1}E$. $S_E = \{F \mid \exists w \in \Sigma^* : F = w^{-1}E\}$ has only finitely many equivalence classes—with respect to the equivalence of regular expressions.

The following method can be used to design, given a regular expression E, a DFA $\mathcal{A}_E$ recognizing the language $L(E)$:

1. The state set of $\mathcal{A}_E$ is given by the set of equivalence classes with respect to the relation $\equiv$. We write w^{-1} for $[w]$.

2. For any state $[w]$ and any symbol $a \in \Sigma$ there will be a single transition from $w^{-1}E$. Namely, that into the state $[wa]$

3. The equivalence class for the whole expression E is the initial state. A state $w^{-1}E$ is final if and only if $\varepsilon \in L(w^{-1}E)$.

To illustrate the method let us consider the regular expression $E = \{a,b\}^*a\{a,b\}\{a,b\}$. Using the notation $S = \{a,b\}$ we have $E = S^*aSS$. For derivatives we get:

$$\begin{array}{rclrcl} a^{-1}E & = & E+SS, & b^{-1}E & = & E, \\ (aa)^{-1}E & = & E+SS+S, & (ab)^{-1}E & = & E+S, \\ (aaa)^{-1}E & = & E+SS+S+\{\varepsilon\}, & (aab)^{-1}E & = & E+S+\{\varepsilon\}, \\ (aba)^{-1}E & = & E+SS+\{\varepsilon\}, & (abb)^{-1}E & = & E+\{\varepsilon\} \end{array} .$$

It is easy to verify that no two of these regular expressions are equivalent and that further derivations do not provide new regular expressions. The resulting state graph is shown in Figure 3.15. Observe that it is the same state diagram as the one in Figure 3.9c—the state diagram obtained from the one in Figure 3.9b by the subset construction. The algorithm just presented for designing a DFA accepting the language described by a regular expression always provides the minimal DFA with this property.

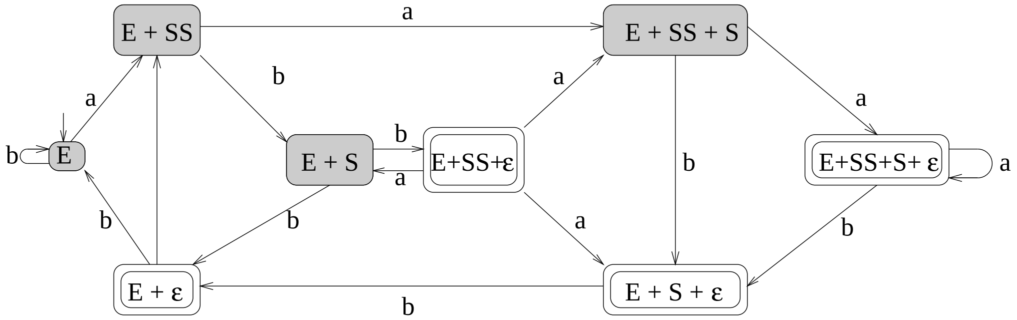


Figure 3.15: A DFA accepting the language represented by the regular expression $\{a,b\}^*a\{a,b\}\{a,b\}$ and designed using derivatives of regular expressions–60%

Exercise 3.3.9 *Use the method of derivatives to design a DFA equivalent to the following regular expressions: (a)* $\{a,b\}^*aba\{a,b\}^*$*; (b)* $\{a,b\}^*\{ab,ba\}\{a,b\}^*$.

Exercise 3.3.10 *Show that the method of derivatives creates always the minimal DFA for a given language. (Hint: show that for each string* w*,* $w^{-1}E$ *is the state of the minimal DFA that is reached from the initial state by the input* w*.)*

On the other side, one can design, using the ideas presented in the proof of Theorem 3.3.1, given a regular expression, a NFA accepting the same language in linear time.

Exercise 3.3.11 *Design a NFA describing the same language as the following regular expressions: (a)* $\{\{a,b\}^*aaa\{a,b\}^*\}^*$*; (b)* $aaa\{ab,ba\}^* + aaa\{aa,bb\}^*$.

3.3.3 Decision Problems

Finite automata and regular expressions can be seen as two different specification tools to describe regular languages. However, how different are they really? One way of understanding this difference is to compare computational complexity of some main decision problems for DFA and regular expressions.

The membership problem. This is the problem to decide, given a regular expression E, over an alphabet Σ, and a $w \in \Sigma^*$, whether $w \in L(E)$. This can be done in time $\mathcal{O}(|w||E|^2)$. Indeed, in time $\mathcal{O}(|E|)$ one can design a NFA $\mathcal{A}_E$ accepting the same language as E, and then, for each symbol of w in time $\mathcal{O}(|E|^2)$, to calculate the potential states when simulating acceptance of w on $\mathcal{A}_E$.

Exercise 3.3.12 *Design in detail an algorithm that decides, given a NFA $\mathcal{A}$ and a word w, whether $w \in L(\mathcal{A})$.*

The emptiness problem and the **finiteness problem** are, on the other side, very easy (here we assume that the symbol for the empty language is not used within the expression). They require time proportional to the length of the regular expressions. Indeed, if E contains a symbol from Σ, then the language $L(E)$ is nonempty. Similarly, if E contains a symbol from Σ in the scope of an iteration operator, then the language is infinite—and only in such a case.

Exercise 3.3.13 *Show how to decide, given a FA $\mathcal{A}$ over the alphabet Σ, whether $L(\mathcal{A}) = \Sigma^*$.*

The equivalence problem for regular expressions is, similarly as for NFA, **PSPACE**-complete.

In the rest of this section we discuss the **equivalence problem for the so-called generalized regular expressions** in order to illustrate how fantastically computational complexity of a problem can change by a seemingly unessential change of the language used to describe input data. (We shall come to this problem again in Section 5.11.

The idea to consider some generalized regular expressions is natural. We have seen in Section 3.3.1 that the family of regular languages is closed under a variety of operations, and therefore, in principle, we could enhance the language of regular expressions with all these operations. For example, with the complement and intersection—to be very modest. This would in no way increase the descriptional power of such expressions—if measured solely by the family of languages they describe.

Such generalizations of regular expressions look very natural. However, there are also very good reasons not to use them, unless there are very special reasons for doing that. The reason behind this is that **complementation and intersection have enormous descriptive power**—they can be used to describe very succinctly various "complex" regular languages. That in turn can make working with them unreasonably difficult. One can see it from the following surprising result concerning the equivalence problem for generalized regular expressions.

Theorem 3.3.14 *The following lower bound holds for the sequential time complexity $T(n)$ of any algorithm that can decide how two generalized regular expressions with the operations of union, concatenation and complementation, of length n, whether they are equivalent:*

$$T(n) = \Omega\Big(2^{2^{\cdot^{\cdot^{2^n}}}} \Big\}\log n \textit{ times}\Big)$$

An even much higher lower bound has been obtained for algorithms deciding the equivalence of regular expressions when also the iteration operation is allowed.

Why is it so? There is a simple explanation. Using the operation of complementation one can enormously shorten the description of some regular expressions. Since the time for deciding of equivalence is measured with respect to the length of the input (and regular expressions with operations of negation can be very short), the resulting time comes up very large, indeed.

Exercise 3.3.15 *Give regular expressions that describe the same language as the following generalized regular expressions: (a)* $\{a,b\}^*ab\{a,b\}^* - \{a,b\}^*ba\{a,b\}^*$*; (b)* $\{a,b\}^* - \{a^ib^3a^j \mid i,j \geq 1\}$.

3.3.4 Other Characterizations of Regular Languages

In addition to FA and regular expressions there are many other ways regular languages can be described and characterized. In this section we deal with three of them. The first one concerns the syntactical monoids—see Section 2.6.2.

Let $L \subseteq \Sigma^*$ be a language. A **context** of a string $w \in \Sigma^*$, with respect to L, is defined by

$$C_L(w) = \{(u,v) \mid uv \in \Sigma^*, uwv \in L\}.$$

The relation $\equiv_L$ on $\Sigma^* \times \Sigma^*$ defined by

$$x \equiv_L y \Longleftrightarrow C_L(x) = C_L(y),$$

is clearly an equivalence relation (the so-called **syntactical equivalence**). In addition, it is a congruence in the free monoid $(\Sigma^*, \cdot)$ because

$$x_1 \equiv_L y_1, x_2 \equiv_L y_2 \ \Rightarrow \ x_1x_2 \equiv_L y_1y_2.$$

This implies that the set of equivalence classes $[w]_L$, $w \in \Sigma^*$, with respect to the relation $\equiv_L$ and the operation $[w_1]_L.[w_2]_L = [w_1w_2]_L$ forms a monoid—the so-called **syntactical monoid** of L.

Theorem 3.3.16 (Myhill's Theorem) *A language L is regular if and only if its syntactical monoid is finite.*

Proof. Let L be a regular language and $L = L(\mathcal{A})$ for a DFA $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$. For any $p \in Q$ let $l(p) = \{w \mid \delta(q_0, w) = p\}$. Moreover, for any $w \in \Sigma^*$ let $S_w = \{(p,q) \mid \delta(p,w) = q\}$. Clearly

$$C_L(w) = \bigcup_{(p,q) \in S_w} l(p) \times L(q).$$

Since the number of different sets S_w is finite, so is the number of contexts $C_L(w)$, and therefore the syntactical monoid of L is finite.

On the other side, let us assume that the syntactical monoid $\mathcal{M}$ for a language $L \subseteq \Sigma^*$ is finite. We design a DFA recognizing L as follows: Elements of $\mathcal{M}$, that is the equivalence classes with respect to the relation $\equiv_L$, will be the states, with $[\varepsilon]_L$ being the initial state. The states of the form $[w]_L, w \in L$ will be the final states. For a state $[w]$ and a symbol $a \in \Sigma$ there will be a transition $([w]_L, a, [wa]_L)$. Clearly, the resulting DFA recognizes L. □

The second characterization is in terms of the **prefix equivalence** $\equiv_L^p$ defined for a language $L \in \Sigma^*$ by

$$w_1 \equiv_L^p w_2 \iff \forall u \in \Sigma^*(w_1u \in L \Leftrightarrow w_2u \in L).$$

Theorem 3.3.17 (Nerode's Theorem) *A language L is regular if and only if its prefix equivalence has finitely many equivalence classes. If a language L is regular, then the number of its prefix equivalence classes equals the number of states of the minimal DFA for L.*

Proof (1) If L is regular, then by Myhill's Theorem the set of syntactical equivalence classes of L is finite. Since $u \equiv_L w \Rightarrow u \equiv_L^p w$, the set of prefix-equivalence classes of L has to be finite also.

(2) Let the number of prefix equivalence classes $[w]_L^p$ be finite. These classes will be the states of a DFA $\mathcal{A}$ that will recognize L and is defined as follows: $[\varepsilon]_L^p$ is the initial state and $\{[w]_L^p \mid w \in L\}$ are final states. The transition function δ is defined by: $\delta([w]_L^p, a) = [wa]_L^p$. Since $w_1 \equiv_L^p w_2 \Rightarrow w_1a \equiv_L^p w_2a$ for all w_1, w_2 and a, δ is therefore well defined and $\mathcal{A}$ clearly recognizes L.

The resulting DFA has to be minimal because no two of its states are equivalent; this follows from the definition of the prefix equivalence classes. □

Exercise 3.3.18 *Show that syntactical monoids of the following languages are infinite (and therefore these languages are not regular):*

(a) $\{a^nb^n \mid n > 0\}$; (b) $\{a^i \mid i$ is prime$\}$.

Exercise 3.3.19 *Determine the syntax equivalence and prefix equivalence classes for the following languages: (a) $\{a,b\}^*aa\{a,b\}^*$; (b) $\{a^ib^j \mid i,j \geq 1\}$.*

Nerode's Theorem can be used also to derive lower bounds on the number of states of DFA for certain regular languages.

Example 3.3.20 *Consider the language $L_n = \{a,b\}^*a\{a,b\}^{n-1}$. Let x, y be two different strings in $\{a,b\}^n$ and let they differ in the i-th leftmost symbol. Clearly, $xb^{i-1} \in L \iff yb^{i-1} \notin L$, because one of the strings xb^{i-1} and yb^{i-1} has a and the second b in the n-th position from the right. This implies that L_n has at least 2^n prefix equivalence classes and therefore each DFA for L_n has to have at least 2^n states.*

Exercise 3.3.21 *Design an* $n+1$ *state NFA for the language* L_n *from Example 3.3.20 (and show this way that for* L_n *there is an exponential difference between the minimal number of states of NFA and DFA recognizing* L*).*

Exercise 3.3.22 *Show that the minimal DFA to accept the language* $L = \{w \mid \#_a w \bmod k = 0\} \subseteq \{a,b\}^*$ *has* k *states and that no NFA with less than* k *states can recognize* L*.*

Example 3.3.23 (Recognition of regular languages in logarithmic time.) *We show now how to use the syntactical monoid of a regular language* L *to design an infinite balanced-tree network of processors—see Figure 3.16—recognizing* L *in parallel logarithmic time.*

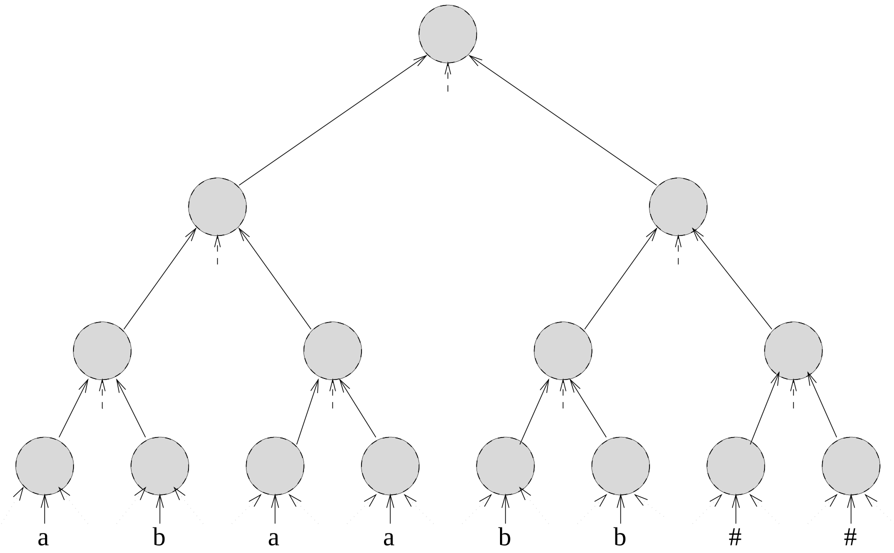


Figure 3.16: Tree automaton recognizing a regular language-60%

Since the number of the syntactical equivalence classes of a regular language is finite they can be represented by symbols of a finite alphabet. This will be used in the following design of a tree network of processors.

Each processor of the tree-network has one external input and produces, for an external input symbol $a \in \Sigma$ at its external "input pin", a symbol representing $[a]_L$ on the output. For the input symbol $\#$, a special marker, on the external input, the processor produces a symbol representing the syntactical equivalence class $[\varepsilon]_L$. Each processor has also two internal inputs. If a processor receives along these internal inputs symbols representing the syntactical equivalence classes $[w_1]_L$ and $[w_2]_L$, it produces the symbol representing $[w_1 w_2]_L$ as the output.

The tree automaton works as follows: an input word $w = a_1 \dots a_n \in \Sigma^*$ is given, one symbol per processor, to the external inputs of the leftmost processors of the topmost level of processors that has at least $|w|$ processors. The remaining processors at that level receive, at their external input, the marker #—see Figure 3.16 for $n = 6$. All processors of the input level process their inputs simultaneously and send their results to their parents. (Processors of all larger levels are "cut-off" in such a computation.) Processing in the network then goes on, synchronized, from one level to another, until the root processor is reached. All processors of these levels process only internal inputs, no external inputs are provided. An input word w is accepted if and only if at the end of this processing, the root processor produces a symbol from the set $\{[w]_L \mid w \in L\}$.

It is clear that such a network of memory-less processors accepts the language L. It is a simple and fast network—it works in logarithmic time, and therefore much faster than a DFA. However, there is a price to pay for that. It can be shown that in some cases for a regular language accepted by a NFA with n states, the corresponding syntactical monoid may have up to n^n elements. The price to be paid for recognition of regular languages in logarithmic time by a binary tree network of processors can therefore be very high—in terms of the size of the processors (they need to process a large class of inputs), and it can be shown that in some cases there is no way to avoid paying such a price.

Exercise 3.3.24 *Design a tree automaton that recognizes the language:*

(a) $\{a^{2^n} \mid n \geq 0\}$ *(Note that this language is not regular.);*

(b) $\{w \mid w \in \{a\}^*\{b\}^*, |w| = 2^k, k \geq 1\}$.

3.4 Finite Transducers

Deterministic finite automata are recognizers. However, they can also be seen as computing characteristic functions of regular languages—the output of a DFA $\mathcal{A}$ is 1 (0) for a given input w if $\mathcal{A}$ comes to a terminal (nonterminal) state on the input w. In this section several models of finite state machines computing other functions or even relations are considered.

3.4.1 Mealy and Moore Machines

Two basic models of finite transducers, as models of finite state machines computing functions, are called **Moore machine** and **Mealy machine**. They formalize an intuitive idea of an input-output mapping realized by a finite state machine in two slightly different ways.

Definition 3.4.1 *In a* **Moore machine** $\mathcal{M} = \langle \Sigma, Q, q_0, \delta, \rho, \Delta \rangle$, *the symbols* Σ, Q, q_0 *and* δ *have the same meaning as for DFA,* Δ *is an* **output alphabet** *and* $\rho : Q \to \Delta$ *an* **output function**.

For an input word $w = w_1 \dots w_n$, $w_i \in \Sigma$, $\rho(q_0)\rho(q_1) \dots \rho(q_n)$ is the corresponding output word, where $q_i = \delta(q_0, w_1 \dots w_i), 1 \leq i \leq n$. In a Moore machine the outputs are therefore "produced by states". Figure 3.17a shows a Moore machine for a serial addition of two binary numbers. (It is assumed that both numbers are represented by binary strings of the same length (leading zeroes are appended if necessary), and for numbers $x_n \dots x_1$, $y_n \dots y_1$ the input is a sequence of pairs $(x_1, y_1), \dots, (x_n, y_n)$ in this order. Observe also that the output always starts with one 0 and this is then followed by output bits of $bin^{-1}(bin(x_n, \dots, x_1) \times bin(y_n \dots y_1)$.

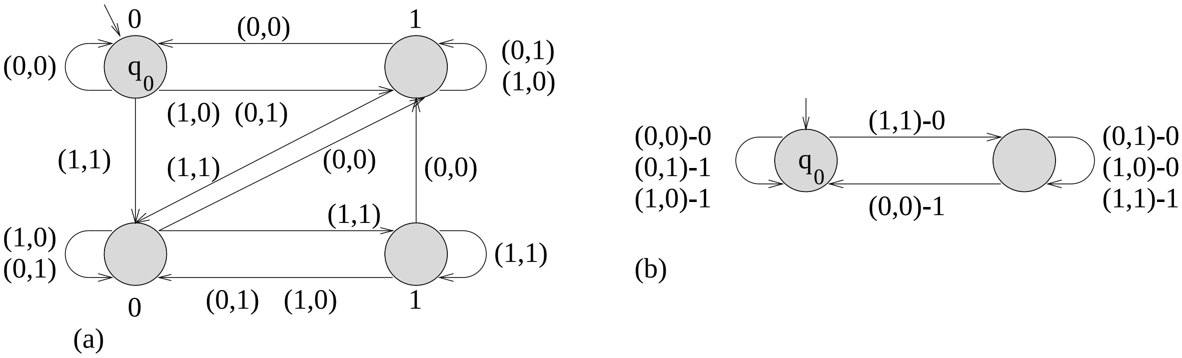


Figure 3.17: Moore and Mealy machines for serial addition–65%

Definition 3.4.2 *In a* **Mealy machine** $\mathcal{M} = \langle \Sigma, Q, q_0, \delta, \rho, \Delta \rangle$*, symbols* $\Sigma, Q, q_o, \delta, \Delta$ *have the same meaning as for a Moore machine and* $\rho : Q \times \Sigma \to \Delta$ *is an* **output function**.

For an input word $w = w_1 \dots w_n$, $w_i \in \Sigma$, $\rho(q_0, w_1) \dots \rho(q_{n-1}, w_n)$ is the corresponding output word, where $q_i = \delta(q_0, w_1 \dots w_i)$. Outputs are therefore produced by transitions. Figure 3.17b shows a Mealy machine for the serial addition of two binary numbers $x = x_1 \dots x_n$, $y = y_1 \dots y_n$ with inputs presented as above.

Let us now denote by $T_{\mathcal{M}}(w)$ the output produced by a Moore or a Mealy machine $\mathcal{M}$, for the input w. For a Moore machine $|T_{\mathcal{M}}(w)| = |w|+1$ and for a Mealy machine $|T_{\mathcal{M}}(w)| = |w|$. A Moore machine can therefore never be fully equivalent to a Mealy machine. However, it is easy to see that for any Moore machine there is a Mealy machine (and vice versa), such that they are equivalent in the following slightly weaker sense.

Theorem 3.4.3 *For every Mealy machine* $\mathcal{M}$ *over an alphabet* Σ *there is a Moore machine* $\mathcal{M}'$ *over* Σ *(and vice versa) such that* $\rho(q_0)T_{\mathcal{M}}(w) = T_{\mathcal{M}'}(w)$*, for every input* $w \in \Sigma^*$*, where* ρ *is the output function of* $\mathcal{M}'$*.*

Exercise 3.4.4 *Design (a) a Moore machine; (b) a Mealy machine, such that given an integer* x *in the binary form, the machine produces* $\lfloor \frac{x}{3} \rfloor$.

Exercise 3.4.5 * *Design (a) a Mealy machine; (b) a Moore machine that transforms a Fibonacci representation of a number into its normal form.*

Exercise 3.4.6 *Design a Mealy machine* $\mathcal{M}$ *that realizes a* 3*-step delay. (That is,* $\mathcal{M}$ *outputs in time* t *its input in time* $t-3$*.)*

3.4.2 Finite Transducers

The concept of a Mealy machine will now be generalized to that of finite state transducers. One new idea is added: nondeterminism.

Definition 3.4.7 *A* **finite (state) transducer** *(FT in short)* $\mathcal{T}$ *is described by a finite set* Q *of* **states**, *a finite* **input alphabet** Σ, *a finite* **output alphabet** Δ, *the initial state* q_0 *and a finite* **transition relation** $\rho \subseteq Q \times \Sigma^* \times \Delta^* \times Q$. *In short,* $\mathcal{T} = \langle Q, \Sigma, \Delta, q_0, \rho \rangle$.

A FT $\mathcal{T}$ can also be represented by a graph, $G_{\mathcal{T}}$, with states from Q as vertices. There is an edge in $G_{\mathcal{T}}$ from a state p to a state q, labeled by (u,v), if and only if $(p,u,v,q) \in \rho$. An interpretation of such an edge: the input u makes $\mathcal{T}$ to transfer from the state p to state q and produces v as the output.

Each finite transducer $\mathcal{T}$ defines a relation

$$\begin{aligned} R_{\mathcal{T}} &= \{(u,v) \mid \exists (q_0,u_0,v_0,q_1),(q_1,u_1,v_1,q_2),\ldots,(q_n,u_n,v_n,q_{n+1}), \\ &\qquad \text{where } (q_i,u_i,v_i,q_{i+1}) \in \rho, \text{for } 0 \le i \le n, \text{ and } u = u_0 \ldots u_n, v = v_0 \ldots v_n\}. \end{aligned}$$

The relation $R_{\mathcal{T}}$ can be seen also as a mapping from subsets of Σ^* into subsets of Δ^* such that for $L \subseteq \Sigma^*$ by $R_{\mathcal{T}}(L) = \{v \mid \exists u \in L, (u,v) \in R_{\mathcal{T}}\}$.

Perhaps the most important fact about finite transducers is that they map regular languages into regular languages.

Theorem 3.4.8 *Let* $\mathcal{T} = \langle Q, \Sigma, \Delta, q_0, \rho \rangle$ *be a finite transducer. If* $L \subseteq \Sigma^*$ *is a regular language, then so is* $R_{\mathcal{T}}(L)$.

Proof. Let $\Delta' = \Delta \cup \{\#\}$ be a new alphabet with # as a new symbol not in Δ. From the relation ρ we design first a finite subset $A_\rho \subset Q \times \Sigma^* \times \Delta'^* \times Q$ and then we take A_ρ as a new alphabet. A_ρ is designed by a decomposition of productions of ρ. We start with A_ρ being empty and for each production of ρ we add to A_ρ symbols defined according to the following rules:

1. If $(p,u,v,q) \in \rho$, $|u| \le 1$, then (p,u,v,q) is taken into A_ρ.

2. If $r = (p, u, v, q) \in \rho$, $|u| > 1$, $u = u_1 \dots u_k$, $1 \leq i \leq k$, $u_i \in \Sigma$, then new symbols $t_1^r, \dots, t_{k-1}^r$ are chosen and all quadruples
$$(p, u_1, \#, t_1^r), (t_1^r, u_2, \#, t_2^r), \dots, (t_{k-2}^r, u_{k-1}, \#, t_{k-1}^r), (t_{k-1}^r, u_k, v, q)$$
are taken into A_ρ.

Now let Q_L be the subset of A_ρ^* consisting of strings of the form
$$(q_0, u_0, v_0, q_1)(q_1, u_1, v_1, q_2) \dots (q_s, u_s, v_s, q_{s+1}) \tag{3.2}$$
such that $v_s \neq \#$ and $u_0 u_1 \dots u_s \in L$. That is, Q_L consists of strings that describe a computation of $\mathcal{T}$ for an input $u = u_0 u_1 \dots u_s \in L$. Finally, let $\tau : A_\rho \to \Delta'^*$ be the morphism defined by
$$\tau((p, u, v, q)) = \begin{cases} v, & \text{if } v \neq \#; \\ \varepsilon, & \text{otherwise.} \end{cases}$$

From the way $\mathcal{T}$ and Q_L are constructed it is readily seen that $\tau(Q_L) = R_{\mathcal{T}}(L)$.

It is also straightforward to see that if L is regular, then Q_L is regular, too. Indeed, a FA $\mathcal{A}$ recognizing Q_L can be designed as follows. A FA recognizing L is used to check whether the second components of symbols of a given word w form a word in L and, in parallel, a check is made whether w represents a computation of $\mathcal{T}$ ending with a state in Q. To verify this the automaton needs always to remember only one of the previous symbols of w and this can be done by a finite automaton.

As shown in Theorem 3.3.1, the family of regular languages is closed under morphisms. This implies that the language $R_{\mathcal{T}}(L)$ is regular. □

Mealy machines are a special case of finite transducers and so are the following generalizations of Mealy machines.

Definition 3.4.9 *In a* **generalized sequential machine** $\mathcal{M} = \langle Q, \Sigma, \Delta, q_0, \delta, \rho \rangle$, *symbols* Q, Σ, Δ *and* q_0 *have the same meaning as for finite transducers,* $\delta : Q \times \Sigma \to Q$ *is the transition mapping and* $\rho : Q \times \Sigma \to \Delta^*$ *is the output mapping.*

Computation on a generalized sequential machine is defined exactly as for Mealy machines. Let $f_{\mathcal{M}} : \Sigma^* \to \Delta^*$ be the function defined by $\mathcal{M}$. For $L \subseteq \Sigma^*$ and $L' \subseteq \Delta^*$ we therefore consider $f_{\mathcal{M}}(L)$ and define $f_{\mathcal{M}}^{-1}(L') = \{u \mid u \in \Sigma^*, f_{\mathcal{M}}(u) \in L'\}$.

It follows from Theorem 3.4.8 that if $\mathcal{M}$ is a generalized sequential machine with the input alphabet Σ and the output alphabet Δ and $L \subseteq \Sigma^*$ is a regular language, then so is $f_{\mathcal{M}}(L)$. We show now that also a reverse claim holds: if $L' \subseteq \Delta^*$ is a regular language then so is $f_{\mathcal{M}}^{-1}(L')$.

Indeed, let $\mathcal{M} = \langle Q, \Sigma, \Delta, q_0, \delta, \rho \rangle$. Consider the finite transducer $\mathcal{T} = \langle Q, \Delta, \Sigma, q_0, \delta' \rangle$ with $\delta' = \{(p, u, v, q) \mid \delta(p, u) = q, \rho(p, u) = v\} \cup \{(p, \varepsilon, \varepsilon, q)\}$. Clearly, $f_{\mathcal{M}}^{-1}(L') = R_{\mathcal{T}}(L')$ and, by Theorem 3.4.8, $f_{\mathcal{M}}^{-1}(L')$ is regular. Thus

Theorem 3.4.10 *If* $\mathcal{M}$ *is a generalized sequential machine, then both mappings* $f_{\mathcal{M}}$ *and* $f_{\mathcal{M}}^{-1}$ *preserve regular languages.*

In Section 3.3 we have seen automata independent characterizations of languages recognized by FA. There exists also a machine-independent characterization of mappings defined by generalized sequential machines.

Theorem 3.4.11 *For a mapping* $f : \Sigma^* \to \Delta^*$, *there exists a generalized sequential machine* $\mathcal{M}$ *such that* $f = f_{\mathcal{M}}$, *if and only if* f *satisfies the following conditions:*

1. f *preserves prefixes, i.e. if* u *is a prefix of* v, *then* $f(u)$ *is a prefix of* $f(v)$.
2. f *has a* bounded output, *i.e. there exists an integer* k *such that* $|f(wa)| - |f(w)| \leq k$ *for any* $w \in \Sigma^*, a \in \Sigma$.
3. $f(\varepsilon) = \varepsilon$.
4. $f^{-1}(L)$ *is a regular language if* L *is regular.*

Exercise 3.4.12 *Let* f *be the function defined by* $f(a) = b$, $f(b) = a$ *and* $f(x) = x$ *for* $x \in \{a,b\}^* - \{a,b\}$ *(a) Does* f *preserve regular languages? (b) Can* f *be realized by a generalized sequential machine?*

Exercise 3.4.13 * *Show how to design, given a regular language* R, *a finite transducer* $\mathcal{T}_R$ *such that* $\mathcal{T}_R(L) = L \Diamond R$ *(where* $\Diamond$ *denotes the shuffle operation introduced in Section 2.5.1).*

3.5 Weighted Finite Automata and Transducers

A seemingly minor modification of the concepts of finite automata and transducers—an assingment of weights to transitions and states—results in finite state devices with unexpected computational power and importance for image processing, as we shall see soon. In addition, the weighted finite automata and also transducers introduced in this section illustrate a well known experience that one often gets powerful practical tools by slightly modifying and "twisting" abstract theoretical concepts.

3.5.1 Basic Concepts

The concept of a weighted finite automaton is very simple and tricky at the same time. Let us therefore start with its informal interpretation for the case it is used to generate images. Each state p determines a function that assigns a grayness value to each pixel, represented by an input word w, and therefore it represents an image. This image is computed as follows: to each path starting in p and labeled by w a value (of grayness) is computed by multiplying the weights of all transitions along the path and, in addition, by the so-called terminal weight of the final state of the path. These values are then added for all paths from p labeled by w.

The initial weights of all nodes are then used to form a linear combination of these functions to get the final image generating function. More formally:

Definition 3.5.1 *A* **weighted finite automaton** *(in short WFA)* $\mathcal{A}$ *is described by a finite set* Σ *of* **input symbols**, *a finite set* Q *of* **states**, *an* **initial distribution** $i : Q \to \mathbf{R}$,*and a* **terminal distribution** $t : Q \to \mathbf{R}$ *of states, as well as a* **weigted transition function** $w : Q \times \Sigma \times Q \to \mathbf{R}$. *In short,* $\mathcal{A} = \langle \Sigma, Q, i, t, w \rangle$.

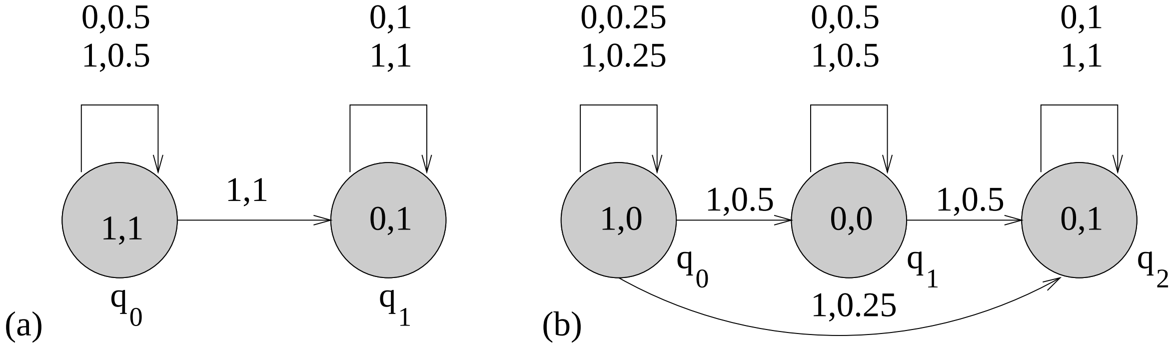


Figure 3.18: Two WFA computing functions on rationals and reals—80%

To each WFA $\mathcal{A}$ we first associate the following distribution function $\delta_{\mathcal{A}} : Q \times \Sigma^* \to \mathbf{R}$:

$$\delta_{\mathcal{A}}(p, \varepsilon) = t(p); \tag{3.3}$$

$$\delta_{\mathcal{A}}(p, au) = \sum_{q \in Q} w(p, a, q)\delta_{\mathcal{A}}(q, u) \text{ for each } p \in Q, a \in \Sigma, u \in \Sigma^*. \tag{3.4}$$

A WFA $\mathcal{T}$ can be represented by a graph $G_{\mathcal{T}}$, see Figures 3.18a,b, with states as vertices and transitions as edges. A vertex representing a state q is labeled by the pair $(i(q), t(q))$. If $w(p, a, q) = r$ is nonzero then there is, in $G_{\mathcal{T}}$, a directed edge from p to q labeled by the pair (a, r).

A WFA $\mathcal{A}$ can now be seen as computing a function $f_{\mathcal{A}} : \Sigma^* \to \mathbf{R}$, defined by

$$f_{\mathcal{A}}(u) = \sum_{p \in Q} i(p)\delta_{\mathcal{A}}(p, u).$$

Informally, $\delta_{\mathcal{A}}(p, u)$ is the sum of all "final weights" of all paths starting in p and labeled by u. The final weight of each such path is obtained by multiplying weights of all transitions on the path and also the final weight of the last node of the path. $f_{\mathcal{A}}(u)$ is then obtained by taking a linear combination of all $\delta_{\mathcal{A}}(p, u)$, defined by the initial distribution i.

Example 3.5.2 *For the WFA* $\mathcal{A}_1$ *from Figure 3.18a we get:*

$\delta_{\mathcal{A}_1}(q_0, 011) = 0.5 \cdot 0.5 \cdot 0.5 \cdot 1 + 0.5 \cdot 0.5 \cdot 1 \cdot 1 + 0.5 \cdot 1 \cdot 1 \cdot 1 = 0.875; \delta_{\mathcal{A}_1}(q_1, 011) = 1 \cdot 1 \cdot 1 \cdot 1 = 1$

and therefore $f_{\mathcal{A}_1}(011) = 1 \cdot 0.875 + 0 \cdot 1 = 0.875$. *Similarly,* $\delta_{\mathcal{A}_1}(q_0, 0101) = 0.625$ *and* $f_{\mathcal{A}_1}(0101) = 0.625$. *For the WFA* $\mathcal{A}_2$ *from Figure 3.18b we get, for example:*

$$\delta_{\mathcal{A}_2}(q_0, 0101) = 0.25 \cdot 0.5 \cdot 0.5 \cdot 1 \cdot 1 + 0.25 \cdot 0.25 \cdot 1 \cdot 1 \cdot 1 = 0.125$$

and therefore also $f_{\mathcal{A}_2}(0101) = 0.125$.

Exercise 3.5.3 *Determine, for the WFA* $\mathcal{A}_1$ *from Figure 3.18a and for* $\mathcal{A}_2$ *from Figure 3.18b: (a)* $\delta_{\mathcal{A}_1}(q_0, 10101)$, $f_{\mathcal{A}_1}(10101)$*; (b)* $\delta_{\mathcal{A}_2}(q_0, 10101)$, $f_{\mathcal{A}_1}(10101)$.

Exercise 3.5.4 *Determine* $f_{\mathcal{A}_3}(x)$ *and* $f_{\mathcal{A}_4}(x)$ *for the WFA* $\mathcal{A}_3$ *and* $\mathcal{A}_4$ *obtained from* $\mathcal{A}_1$ *from Figure 3.18a by changing the initial and terminal distributions as follows:*

(a) $i(q_0) = 1$, $i(q_1) = 0$, $t(q_0) = 0$, *and* $t(q_1) = 1$*;*

(b) $i(q_0) = i(q_1) = 1$, *and* $t(q_0) = t(q_1) = 1$.

Exercise 3.5.5 *(a) Show that* $f_{\mathcal{T}_1}(x) = 2bre(x) + 2^{-|x|}$ *for the WFT* $\mathcal{T}_1$ *depicted in Figure 3.18; (b) Determine functions computed by WFAs obtained from the one in Figure 3.18a by considering several other initial and terminal distributions.*

If $\Sigma = \{0,1\}$ is the input alphabet of a WFA $\mathcal{A}$, then we can extend $f_{\mathcal{A}} : \Sigma^* \to \mathbf{R}$ to a (partial) **real function** $f^{\omega}_{\mathcal{A}} : [0,1] \to \mathbf{R}$ defined as follows: for $x \in [0,1]$ let $bre^{-1}(x) \in \Sigma^{\omega}$ be the unique binary representation of x, see Page 93. Then

$$f^{\omega}_{\mathcal{A}}(x) = \lim_{n\to\infty} f_{\mathcal{A}}(Prefix_n(bre^{-1}(x))),$$

provided the limit exists; otherwise $f^{\omega}_{\mathcal{A}}(x)$ is undefined.

For the rest of this section, to simplify the presentation, a binary string $x_1 \ldots x_n$, $x_i \in \{0,1\}$ and an ω-string $y = y_1 y_2 \ldots$, over the alphabet $\{0,1\}$ will be interpreted, depending on the context, either as strings $x_1 \ldots x_n$ and $y_1 y_2 \ldots$ or as reals $0.x_1 \ldots x_n$ and $0.y_1 y_2 \ldots$. Instead of $bin(x)$ and $bre(y)$ we shall often write simply x or y and take them as strings or numbers.

Exercise 3.5.6 *Show, for the WFA* $\mathcal{A}_1$ *from Figure 3.18a: (a) if* $x \in \Sigma^*$*, then* $f_{\mathcal{A}_1}(x0^n) = 2bre(x) + 2^{-(n+|x|)}$*; (b)* $f_{\mathcal{A}_1}(x) = 2bre(x)$.

Exercise 3.5.7 *Show that* $f_{\mathcal{A}}(x)^{\omega}(x) = x^2$ *for the WFA* $\mathcal{A}_2$ *from Figure 3.18b.*

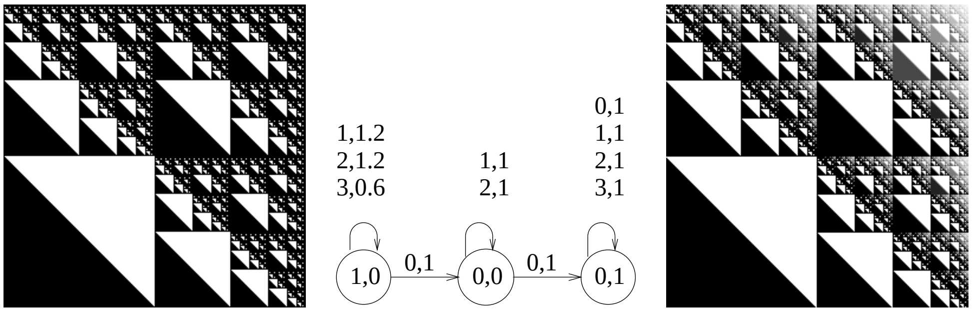
Exercise 3.5.8 *Determine* $f^{\omega}_{\mathcal{A}_3}(x)$ *for the WFA obtained from WFA* $\mathcal{A}_2$ *by taking other combinations of values for the initial and final distributions.*

Of a special importance are WFA over the alphabet $P = \{0,1,2,3\}$. As shown in Section 2.5.3 a word over P can be seen as a pixel in the square $[0,1] \times [0,1]$. A function $f_{\mathcal{A}} : P^* \to \mathbf{R}$ is then considered as a multiresolution image with $f_{\mathcal{A}}(u)$ being the grayness of the pixel specified by u. In order to have compatibility of different resolutions, it is usually required that $f_{\mathcal{A}}$ is **average preserving**. That is, it holds:

$$f_{\mathcal{A}}(u) = \frac{1}{4}[f_{\mathcal{A}}(u0) + f_{\mathcal{A}}(u1) + f_{\mathcal{A}}(u2) + f_{\mathcal{A}}(u3)].$$

In other words, the grayness of a pixel is the average of the graynesses of its four main subpixels. (One can also say that images in different resolutions look similar if $f_{\mathcal{A}}$ is average preserving—multiresolution images contain only more details.)

It is easy to see that at the pixel representation of words over the alphabet P the language $L = \{1,2,3\}^*0\{1,2\}^*0\{0,1,2,3\}^*$ represents the image shown in Figure 3.19a (see also Exercise 2.5.17). At the same time L is the set of words w such that $f_{\mathcal{A}}(w) = 1$ for the WFA obtained from the one in Figure 3.19b by replacing all weights by 1. Now it is easy to see that the average preserving WFA shown in Figure 3.19b generates the gray-scale image from Figure 3.19c.



(a) (b) (c)

Figure 3.19: Generation of a fractal image.

The concept of a WFA will now be generalized to a **weighted finite transducer** (in short WFT).

Definition 3.5.9 *In a WFT* $\mathcal{T} = \langle \Sigma_1, \Sigma_2, Q, i, t, w \rangle$, Σ_1 *and* Σ_2 *are input alphabets,* Q, i *and* t *have the same meaning as for a WFA and* $w : Q \times (\Sigma_1 \cup \{\varepsilon\}) \times (\Sigma_2 \cup \{\varepsilon\}) \times Q \to \mathbf{R}$ *is a* **weighted transition function**.

We can associate also to a WFT $\mathcal{T}$ the state graph $G_{\mathcal{T}}$ with Q being the set of nodes and with an edge from a node p to a node q with the label $(a_1, a_2 : r)$ if $w(p, a_1, a_2, q) = r$.

A WFT $\mathcal{T}$ specifies a **weighted relation** $R_{\mathcal{T}} : \Sigma_1^* \times \Sigma_2^* \to \mathbf{R}$ defined as follows: For $p, q \in Q$, $u \in \Sigma_1^*$ and $v \in \Sigma_2^*$, let $A_{p,q}(u,v)$ be the sum of weights of all paths $(p_1, a_1, b_1, p_2)(p_1, a_2, b_2, p_2) \ldots (p_n, a_n, b_n, p_{n+1})$ from the state $p = p_1$ to the state $p_{n+1} = q$ that are labeled by $u = a_1 \ldots a_n$ and $v = b_1 \ldots b_n$. Moreover, we define:

$$R_{\mathcal{T}}(u,v) = \sum_{p,q \in Q} i(p) A_{p,q}(u,v) t(q).$$

That is, only the paths from an initial to a final state are taken into account. In this way $R_{\mathcal{T}}$ relates some pairs (u,v), namely those that $R_{\mathcal{T}}(u,v) \neq 0$, and assigns some weight to the relational pair (u,v).

Observe that $A_{p,q}(u,v)$ does not have to be defined. Indeed, for some p, q, u and v, it can happen that $A_{p,q}(u,v)$ is infinite. This is due to the fact that if a transition is labeled by $(a_1, a_2 : r)$, then it may happen that either $a = \varepsilon$ or $a_2 = \varepsilon$ or $a_1 = a_2 = \varepsilon$, and therefore there may be infinitely many paths between p and q labeled by u and v. To overcome this problem, we restrict ourselves to those WFT that have the property that if the product of weights of a cycle is nonzero then either not all first labels or not all second labels on the edges of the path are ε.

The concept of a weighted relation may seem artificial. However, its application to functions turned out to be a powerful tool. In image processing applications, weighted relations represent an elegant and powerful way to transform images.

Definition 3.5.10 *Let $\rho : \Sigma_1^* \times \Sigma_2^* \rightarrow \mathbf{R}$ be a weighted relation and $f : \Sigma_1^* \rightarrow \mathbf{R}$ a function. An application of ρ on f, in short $g = \rho \circ f = \rho(f) : \Sigma_2^* \rightarrow \mathbf{R}$ is defined by*

$$g(v) = \sum_{u \in \Sigma_1^*} \rho(u,v) f(u),$$

for $v \in \Sigma_2^*$, if the sum, that can be infinite, converges, otherwise $g(u)$ is undefined. (The order of summation is given by a strict ordering on Σ_1^*.)

Informally, an application of ρ on f produces a new function g. The value of this function for an argument v is obtained by taking f-values of all $u \in \Sigma^*$ and each $f(u)$ is multiplied by the weight of the paths that stand for the pair (u,v). This simply defined concept is very powerful. The concept itself, as well as its power, can be in the best way illustrated by examples.

Exercise 3.5.11 *Describe the image transformation defined by the WFT shown in Figure 3.20a that produces, for example, the image shown in Figure 3.20c from the image depicted in Figure 3.20b.*

Example 3.5.12 (Derivation) *The WFT $\mathcal{T}_3$ in Figure 3.21a defines a weighted relation $R_{\mathcal{T}_3}$ such that for any function $f : \{0,1\}^* \rightarrow \mathbf{R}$, interpreted as a function on fractions, we get*

$$R_{\mathcal{T}_3} \circ f(x) = \frac{df(x)}{dx}$$

(and therefore $\mathcal{T}_3$ acts as a functional), in the following sense: for any fixed n and any function $f : \Sigma_1^n \rightarrow \mathbf{R}$, $R_{\mathcal{T}_3} \circ f(x) = \frac{f(x+h)-f(x)}{h}$, where $h = \frac{1}{2^n}$. (This means that if x is chosen to have n bits then also the least significant 0, in a binary representation of x, matter.) Indeed, $R_{\mathcal{T}_3}(x,y) \neq 0$, for $x, y \in \{0,1\}^*$ if and only either $x = y$ and then $R_{\mathcal{T}_3}(x,y) = -2^{|x|}$ or $x = x_1 10^k$, $y = x_1 01^k$, for some k, and in such a case $R_{\mathcal{T}_3}(x,y) = 2^{|x|}$. Hence $R_{\mathcal{T}_3} \circ f(x) = R_{\mathcal{T}_3}(x,x)f(x) + R_{\mathcal{T}_3}(x + \frac{1}{2^{|x|}}, x) f(x + \frac{1}{2^{|x|}}) = -2^{|x|} f(x) + 2^{|x|} f(x + \frac{1}{2^{|x|}})$. Take now $n = |x|$, $h = \frac{1}{2^n}$.

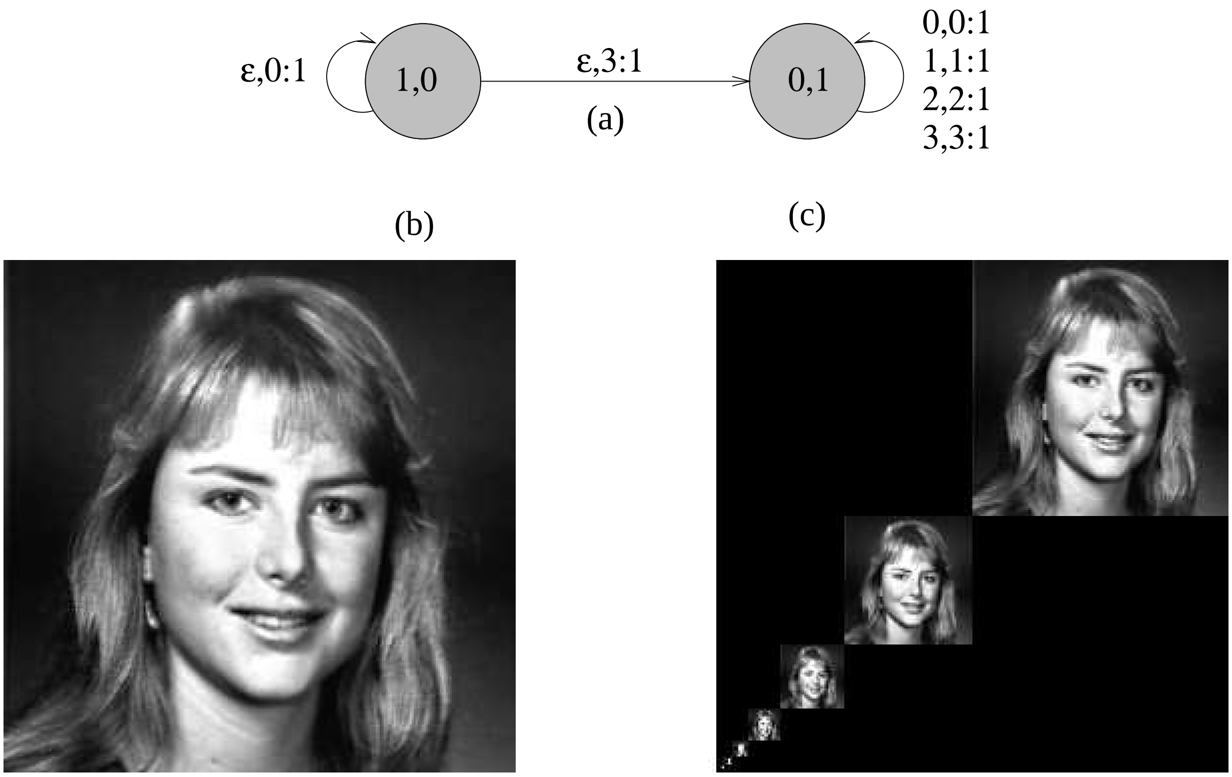


Figure 3.20: Image transformation –80%

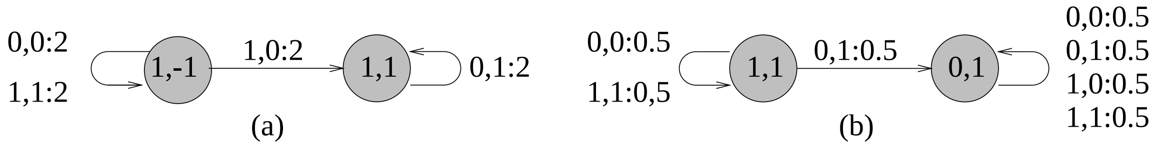


Figure 3.21: WFT for derivation and integration–70%

Example 3.5.13 (Integration) *The WFT $\mathcal{T}_4$ in Figure 3.21b determines a weighted relation $R_{\mathcal{T}_4}$ such that for any function $f : \Sigma^* \to \mathbf{R}$*

$$R_{\mathcal{T}_4} \circ f(x) = \int_0^x f(t)\, dt,$$

in the following sense: $R_{\mathcal{T}_4} \circ f$ computes $h(f(0) + f(h) + f(2h) + \ldots + f(x))$ (for any fixed resolution $h = \frac{1}{2^k}$, for some k, and all $x \in \{0,1\}^k$).

Exercise 3.5.14 *Explain in details how the WFT in Figure 3.21b determines a functional for integration.*

Exercise 3.5.15 * *Design a WFT for partial derivation of functions of two variables with respect: (a) to the first variable; (b) to the second variable.*

The following theorem shows that the family of functions computed by WFA is closed under the weighted relations realized by WFT.

Theorem 3.5.16 *Let* $\mathcal{A}_1 = \langle \Sigma_1, Q_1, i_1, t_1, w_1 \rangle$ *be a WFA and* $\mathcal{A}_2 = \langle \Sigma_2, Q_2, i_2, t_2, w_2 \rangle$ *be an* ε*-loop free WFT. Then there exists a WFA* $\mathcal{A}$ *such that* $f_{\mathcal{A}} = R_{\mathcal{A}_2} \circ f_{\mathcal{A}_1}$.

This result actually means that to any WFA $\mathcal{A}$ over the alphabet $\{0,1\}$ other WFAs $\mathcal{A}'$ and $\mathcal{A}''$ can be designed such that for any $x \in \Sigma^*$, $f_{\mathcal{A}'}(x) = \frac{df_{\mathcal{A}}(x)}{dx}$ and $f_{\mathcal{A}''}(x) = \int_0^x f_{\mathcal{A}}(x)dx$.

Exercise 3.5.17 *Construct a WFT to perform: (a)** *rotation by 45 degree clockwise; (b) circular left shift by one pixel in two-dimension.*

Exercise 3.5.18 *Describe the image transformations realized by WFT in: (a) Figure 3.22a; (b) in Figure 3.22b.*

Exercise 3.5.19 * *Prove Theorem 3.5.16.*

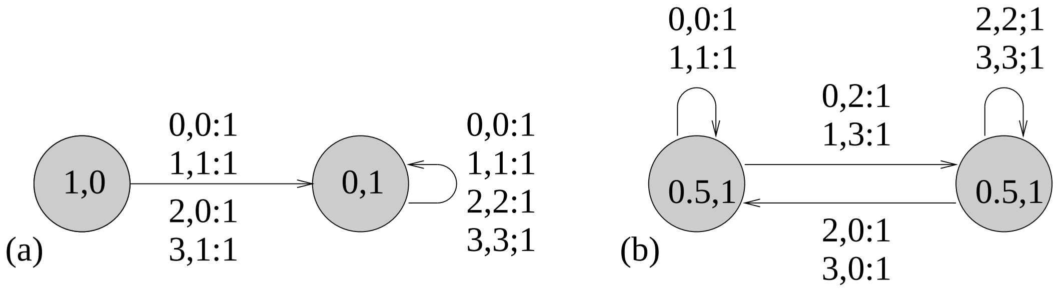


Figure 3.22: Two WFT–80%

3.5.2 Functions Computed by WFA

For a WFA $\mathcal{A}$ on the alphabet $\{0,1\}$, the real function $f^{\omega}_{\mathcal{A}} : [0,1] \to \mathbf{R}$ does not have to be total. However, it is always total for a special type of WFT introduced in the Definition 3.5.20. As it will be seen later even such simple WFT have unexpected power.

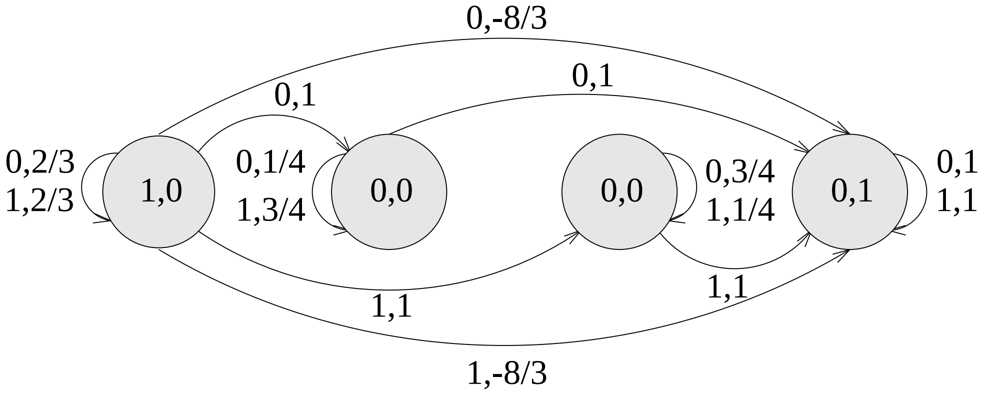


Figure 3.23: A LWFT computing an everywhere continuous function having nowhere a derivative—80%

Definition 3.5.20 *A WFA $\mathcal{A} = \langle \Sigma, Q, i, t, w \rangle$ is called* **level weighted finite automaton** *(in short LWFA) if*

1. *all weights are between 0 and 1;*
2. *the only cycles are self-loops;*
3. *if the weight of a self-loop is 1, then it must be a self-loop of a node that has no other outgoing edges than self-loops.*

For example, the WFA in Figure 3.18b is a LWFA; the one in Figure 3.18a is not. LWFA have unexpected properties summarized in the following theorem

Theorem 3.5.21 *LWFA have the following properties*

1. *It is decidable, given a LWFA, whether the real function it computes is continuous. It is also decidable, given two LWFA, whether the real functions they compute are identical.*
2. *Any polynomial of one variable with rational coefficients is computable by a LWFA. In addition, for any integer n there is a fixed, up to the initial distribution, LWFA $\mathcal{A}_n$ that can compute any polynomial of one variable and degree at most n. (To compute different polynomials only different initial distributions are needed).*
3. *If arbitrary negative weights are allowed, then there exists a simple LWFA—see Figure 3.23—computing a real function that is everywhere continuous and has no derivatives at any point of the interval $[0, 1]$.*

Exercise 3.5.22 * *Design a LWFA computing all polynomials of one variable of degree 3 and show how to fix the initial and terminal distributions to compute a particular polynomial of the degree 3.*

3.5.3 Image Generation and Transformation by WFA and WFT

As already mentioned an average preserving mapping $f : P^* \to \mathbf{R}$ can be considered as a multiresolution image. There is a simple way to ensure that a WFA on P defines an average preserving mapping and thereby a multiresolution image.

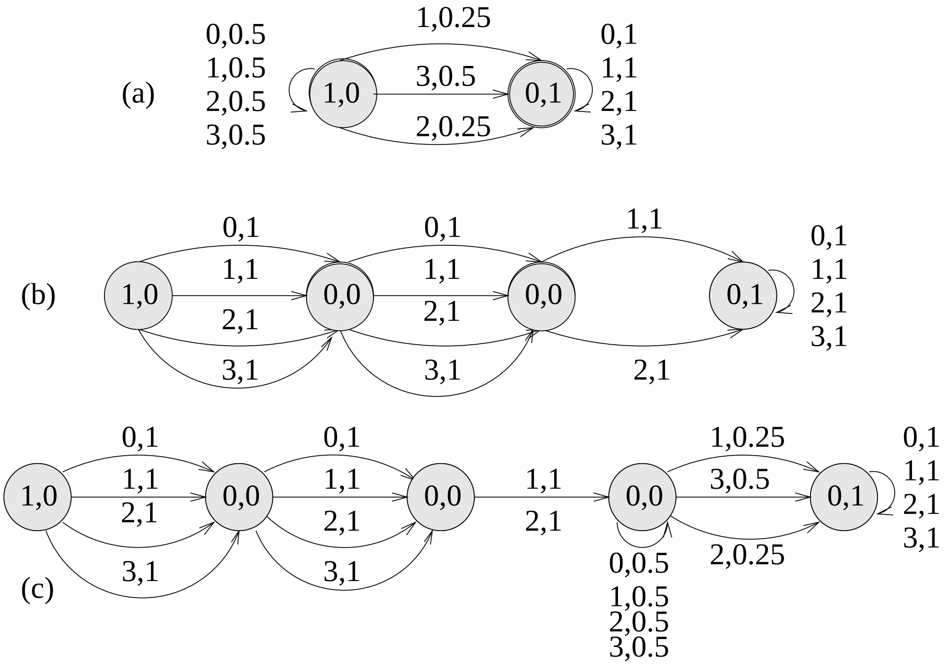


Figure 3.24: WFA generating two images and their concatenation–70%

Definition 3.5.23 *A WFA $\mathcal{A} = \langle P, Q, i, t, w\rangle$ is average preserving if for all $p \in Q$*

$$\sum_{a\in\Sigma, q\in Q} w(p,a,q)t(q) = 4t(p).$$

Indeed, we have

Theorem 3.5.24 *Let $\mathcal{A}$ be a WFA on P. If $\mathcal{A}$ is average preserving, then so is $f_{\mathcal{A}}$.*

Proof. Let $u \in P^*, a \in P$. Since

$$\begin{aligned} f_{\mathcal{A}}(ua) &= \sum_{q\in Q} \delta(q, ua)t(q) && (3.5)\\ &= \sum_{p,q\in Q} \delta(p,u)w(p,a,q)t(q) && (3.6) \end{aligned}$$

we have

$$\begin{aligned} \sum_{a\in P} f_{\mathcal{A}}(ua) &= \sum_{p\in Q} \delta(p,u) \sum_{a\in P, q\in Q} w(p,a,q)t(q) && (3.7)\\ &= \sum_{p\in Q} \delta(p,u)4t(p) = 4f_{\mathcal{A}}(u). && (3.8) \end{aligned}$$

□

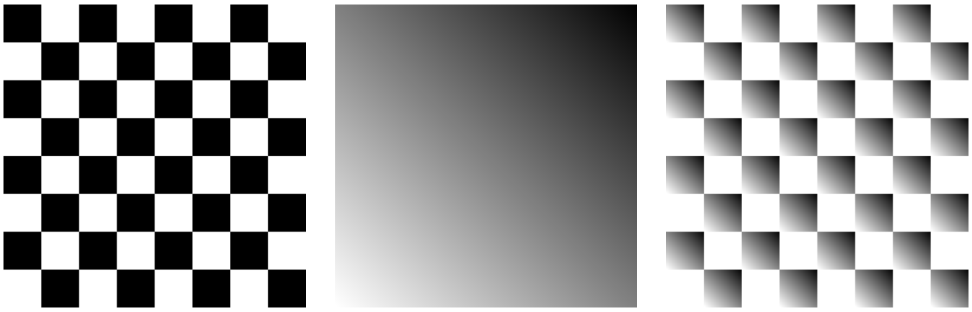

(a) (b) (c)

Figure 3.25: Concatenation of two images generated by WFA

The family of multiresolution images generated by WFA is closed under various operations as addition, multiplication by constants, cartesian product, concatenation, iteration, various affine transformations, zooming, rotating, derivation, integration, filtering and so on. Concatenation of WFA, see also Section 2.5.3, is defined as follows:

Definition 3.5.25 *Let $\mathcal{A}_1, \mathcal{A}_2$ be WFA over P and $f_{\mathcal{A}_1}, f_{\mathcal{A}_2}$ multiresolution images defined by $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Their concatenation $\mathcal{A}_1\mathcal{A}_2$ is defined as*

$$f_{\mathcal{A}_1\mathcal{A}_2}(u) = \sum_{u_1u_2=u} f_{\mathcal{A}_1}(u_1)f_{\mathcal{A}_2}(u_2).$$

Exercise 3.5.26 *(a) Show that the WFA from Figure 3.24b generates the chess board shown in Figure 3.25a; (b) show that the WFA in Figure 3.24a generates a linear slope shown in Figure 3.25b; (c) show that concatenation of the two images from Figures 3.24a,b, see the result in Figure 3.25c, generates the WFA from Figure 3.24c.*

Observe that several WFA we have considered, for example the one in Figure 3.24b, are nondeterministic in the sense that if the weights are discarded a nondeterministic FA is obtained. It can be shown that nondeterministic WFA generate more images than deterministic ones. For example, there is no deterministic WFA generating the same linear slope as does the WFA from Figure 3.24a.

3.5.4 Image Compression

We have seen several examples of WFT generating images. From the application point of view of special importance is the inverse problem: given an image, design a WFT generating this image. Indeed, to store a multiresolution image directly a lot of memory is needed. A

WFT generating the same image requires usually much less memory. There is a simple to formulate algorithm to do image compression.

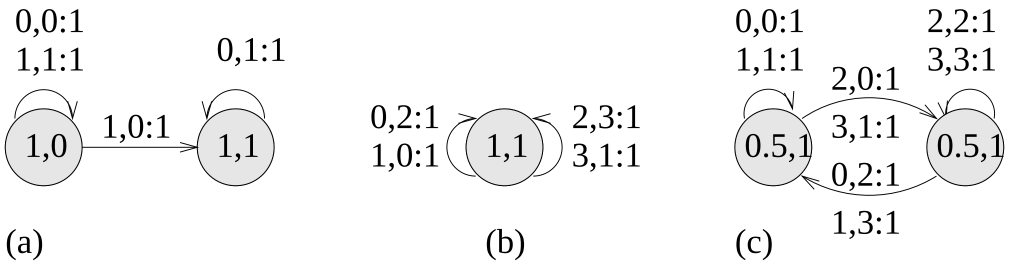


Figure 3.26: Image transformations defined by WFT: (a) circular shift left, (b) rotation, (c) vertical squeezing

Algorithm 3.5.27 (Image compression) *Assume as input an image I given by a function $\phi : P^* \to \mathbf{R}$. (It can be also a digitalized photo.)*

1. *Assign the initial state q_0 to the image represented by the empty word, i.e. to the whole image I, and define $i(q_0) = 1, t(q_0) = \phi(\varepsilon)$—the average grayness of the image I.*

2. *Recursively, for a state q assigned to a square specified by a string u, consider four subsquares specified by strings $u0, u1, u2, u3$. Denote the image in the square ua by I_{ua}. If this image is everywhere 0, then there will be no transition from the state q with the label a. If the image I_{ua} can be expressed as a linear combination of the images I_{v_i} corresponding to the states $p_1, \dots, p_k$, i.e. $I_{ua} = \sum_{i=1}^{k} c_i I_{v_i}$, then add a new edge from q to each p_i with label a and with the weight $w(q,a,p_i) = c_i (i = 1, \dots, k)$. Otherwise, assign a new state r to the pixel ua and define $w(q,a,r) = 1, t(r) = \phi(I_{ua})$—the average grayness of the image in the pixel ua.*

3. *Repeat step 3 for each new state and stop if no new state is created.*

Since any real image has a finite resolution the algorithm has to stop in practice. If this algorithm is applied to the picture shown in Figure 3.19a we get a WFA as the one shown in Figure 3.19b but with all weights equal 1. Using the above "theoretical algorithm" a 5-10 times compression can be obtained. However, when a more elaborate "recursive algorithm" is used than a larger compression—50-60 times for gray-scale images and 100-150 times for color images (and still providing pictures of good quality), has been obtained.

Of practical importance are also WFT. They can perform most of the basic image transformations as changing contrasts, shifts, shrinking, rotation, vertical squeezing, zooming, filters, mixing images, creating regular patterns of images,. . . .

Exercise 3.5.28 *Show that the WFT from Figure 3.26a performs circular shift left.*

Exercise 3.5.29 *Show that the WFT from Figure 3.26b performs rotation by 90 degrees counterclockwise.*

Exercise 3.5.30 *Show that the WFT from Figure 3.26c performs vertical squeezing defined as the sum of two affine transformations:* $x_1 = \frac{x}{2}, y_1 = y$ *and* $x_2 = \frac{x+1}{2}, y_2 = y$ *— makes two copies of the original image and puts them next to each other in the unit square.*

3.6 Finite Automata on Infinite Words

A natural generalization of the concept of finite automata recognizing/accepting finite words and languages of finite words is that of finite automata recognizing ω-words and ω-languages. These concepts also have numerous applications in many areas of computing. Many processes modeled by finite-state devices (for instance, the watch in Section 3.1) are potentially infinite and therefore it is the most appropriate to see their inputs as ω-words. Two types of FA play the basic rôle here.

3.6.1 Büchi and Muller automata

Definition 3.6.1 *A* **Büchi automaton** $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ *is formally defined exactly as a FA, it is only used to process ω-words and the acceptance is defined in a special way. An ω-word* $w = w_0 w_1 w_2 \ldots \in \Sigma^\omega, w_i \in \Sigma$, *is accepted by $\mathcal{A}$ if there is an infinite sequence of states $q_0, q_1, q_2, \ldots$ such that $(q_i, w_i, q_{i+1}) \in \delta$, for all $i \geq 0$ and a state in Q_F occurs infinitely often in this sequence. Let $L^\omega(\mathcal{A})$ denote the set of all ω-words accepted by $\mathcal{A}$.*

An ω-language L is called regular if there is a Büchi automaton accepting L.

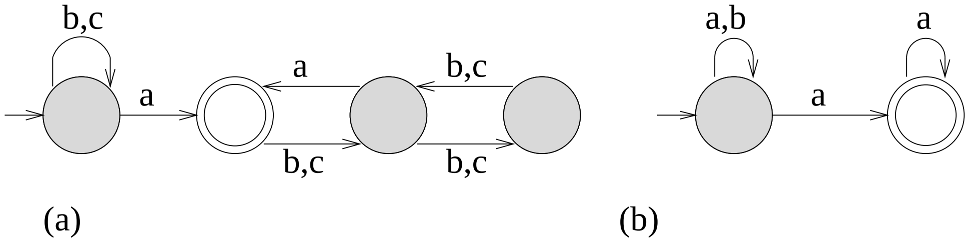


Figure 3.27: Büchi automata–80%

Example 3.6.2 *Figure 3.27a shows a Büchi automaton accepting the ω-language over the alphabet $\{a, b, c\}$ consisting of ω-words that contain infinitely many a's and between any two* *occurrences of a there is an odd number of occurrences of b and c. Figure 3.27b shows a Büchi automaton recognizing the language* $\{a,b\}^*a^\omega$.

Exercise 3.6.3 *Construct a Büchi automaton accepting the language* $L \subseteq \{a,b,c\}^*$ *defined as follows: (a)* $w \in L$ *iff after any occurence of the symbol a there is some occurence of the symbol b in w; (b)* $w \in L$ *iff between any two occurences of the symbol a there is a multiple of* 4 *occurences of b's or c's.*

The following theorem summarizes those properties of ω-regular languages and Büchi automata that are similar to that of regular languages and FA . Except for the closure under the complement, they are easy to show.

Theorem 3.6.4 *(1) The family of regular* ω*-languages is closed under the operations of union, intersection and complement.*

(2) An ω*-language L is regular if and only if there are regular languages* $A_1,\dots,A_n$ *and* $B_1,\dots,B_n$ *such that* $L = A_1B_1^\omega \cup \ldots \cup A_nB_n^\omega$.

(3) The emptiness and the equivalence problems are decidable for Büchi automata.

Exercise 3.6.5 *Show: (a) if L is a regular language, then* L^ω *is a regular* ω*-language; (b) if* L_1 *and* L_2 *are regular* ω*-languages, then so are* $L_1 \cup L_2$ *and* $L_1 \cap L_2$*; (c)*** *the emptiness problem is decidable for Büchi automata.*

The result stated in point (2) of Theorem 3.6.4 shows how to define regular ω-expressions in such a way that they define exactly regular ω-languages.

One of the properties of FA not shared by Büchi automata concerns the power of nondeterminism. Nondeterministic Büchi automata are more powerful than deterministic. This actually easily follows from the fact that languages accepted by deterministic Büchi automata can be nicely characterized using regular languages. To show that is the task of the next exercise.

Exercise 3.6.6 *Show that an* ω*-language* $L \subseteq \Sigma^\omega$ *is accepted by a deterministic Büchi automaton if and only if*

$$L = \overrightarrow{W} = \{w \in \Sigma^\omega \mid Prefix_n(w) \in W, \ for\ infinitely\ many n\},$$

for some regular language W.

Exercise 3.6.7 * *Show that the language* $\{a,b\}^\omega - \overrightarrow{(b^*a)^*}$ *is accepted by a Büchi automaton but not by a deterministic Büchi automaton.*

There is, however, a modification of deterministic Büchi automata, with a different acceptance mode, the so-called **Muller automata**, that are deterministic and recognize all regular ω-languages.

Definition 3.6.8 *In a Muller automaton* $\mathcal{A} = \langle \Sigma, Q, q_0, \mathcal{F}, \delta \rangle$, *where* Σ, Q, q_0 *and* δ *have the same meaning as for DFA, but* $\mathcal{F} \subseteq 2^Q$ *is a family of sets of final states.* $\mathcal{A}$ *recognizes an* ω*-word* $w = w_0 w_1 w_2 \ldots$ *if and only if the set of states that occur infinitely often in the sequence of states* $\{q_i\}_{i=0}^{\infty}$, $q_i = \delta(q_0, w_0 w_1 w_2 \ldots w_i)$, *is an element of* $\mathcal{F}$. *(That is the set of those states the automaton* $\mathcal{A}$ *takes infinitely often when processing* w, *is an element of* $\mathcal{F}$.*)*

Exercise 3.6.9 * *Show the so-called McNaughton's theorem: Muller automata accept exactly regular* ω*-languages.*

Exercise 3.6.10 *Show, for the regular* ω*-language* $L = \{0,1\}^*\{0\}^\omega$, *(that is not a deterministic regular* ω*-language), that there are three nonisomorphic minimal (with respect to the number of states) Muller automata for* L. *(This indicates that the minimization problem for Muller automata has different features than for DFA.)*

3.6.2 Finite State Control of Reactive Systems*

In many areas of computing, for example in operating systems, communication protocols, control systems, robotics and so on, the appropriate view of computation is that of a non-stop interaction between two agents or processes. They will be called **controller** and **disturber** (or **plant**). Each of them is supposed to be able to perform at each moment one of finitely many actions. Programs or automata representing such agents are called **reactive**; their actions are modeled by symbols from finite alphabets and their continuous interactions are modeled by ω-words.

In this section we only illustrate, as a case study, that (regular) ω-languages and ω-words are a proper framework to state precisely and to solve satisfactorily basic problems concerning such reactive systems. A detailed treatment of the subject and methods currently being worked out, is outside the scope of the book.

A **desirable interaction** of such agents can be specified through an ω-language $L \subseteq (\Sigma\Delta)^\omega$, where Σ and Δ are disjoint alphabets. An ω-word w from L has therefore the form $W = c_1 d_1 c_2 d_2 \ldots$, where $c_i \in \Sigma$ ($d_i \in \Delta$). The symbol c_i (d_i) denotes the i-th action controller (disturber) performs. The idea is that the controller tries to respond to the actions taken by the disturber in such a way that these actions make the disturber "behave accordingly".

Three basic problems arise in case such a desirable behaviour is specified through an ω-language. The **verification problem** is to decide, given a controller, whether it is able

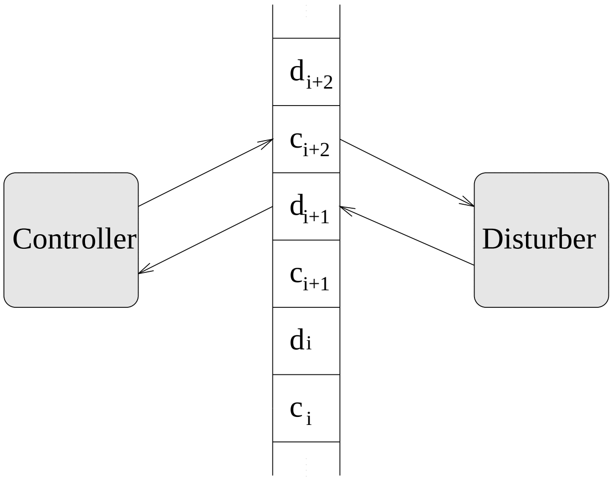


Figure 3.28: Controller and disturber–65%

to interact with the disturber in such a way that the resulting ω-word is in the given ω-language. The **solvability problem** is to decide, given an ω-language description, whether there exists a controller, of a certain type, capable of achieving an interaction with the disturber resulting always in an ω-word from the given ω-language. Finally, the **synthesis problem** is to design a controller, from a given specification of an ω-languge for the desired interaction with the disturber. Interesting enough, all these problems are solvable in the case that the ω-language specifying the desirable behaviour of the controller-disturber interactions is a regular ω-language.

Problems of verification and synthesis for such reactive automata can be nicely formulated, as many problems in computing, in the framework of games. This time in the framework of the so-called **Gale-Steward games** of two players. They are called again **controller** (C) and **disturber** (D). Their actions are modeled by symbols from alphabets Σ_C and Σ_D, respectively. Let $\Sigma = \Sigma_C \cup \Sigma_D$.

A **Gale-Steward game** is specified by an ω-language $L \subseteq (\Sigma_C\Sigma_D)^\omega$. A **play** of the game is an ω-word $p \in \Sigma_C(\Sigma_D\Sigma_C)^\omega$. (An interpretation is that C starts an interaction by choosing a symbol from Σ_C and then D and C keep choosing, in turn and indefinitely, symbols from their alphabets (depending, of course, on the interactions to that moment).) Player C **wins** the play p if $p \in L$, otherwise D wins. A **strategy** for C is a mapping $s_C : \Sigma_D^* \to \Sigma_C$ specifying a choice of a symbol from Σ_C (a move of C), for any finite sequence of choices of symbols by D — moves of D to that moment. Any such strategy determines a mapping $\bar{s}_C : \Sigma_D^\omega \to \Sigma_C^\omega$, defined by

$$\bar{s}_C(d_0d_1d_2\ldots) = c_0c_1c_2\ldots, \text{ where } c_i = s_C(d_0d_1\ldots d_{i-1}).$$

If D chooses an infinite sequence $\mu = d_0d_1\ldots$ of events (symbols) to act and C has a strategy s_C, then C chooses the infinite sequence $\gamma = \bar{s}_C(\mu)$ to create, together with D, the play $p_{\mu,\bar{s}_C} = c_0d_0c_1d_1\ldots$

The main problem, the so-called **uniform synthesis problem**, can now be described as follows. Given a specification language for a class $\mathcal{L}$ of ω-languages, design an algorithm, if it exists, such that given any specification of an ω-language $L \in \mathcal{L}$, the algorithm designs a (winning) strategy s_C for C, such that no matter what strategy D chooses, that is, no matter which sequence μ disturber D chooses, the play $p_{\mu,\bar{s}_C}$ will be in L.

In general it holds:

Theorem 3.6.11 (Büchi-Landweber's Theorem) *Let Σ_C, Σ_D be finite alphabets. To any ω-regular language $L \subseteq \Sigma_C(\Sigma_D\Sigma_C)^\omega$ and any Muller automaton recognizing L, a Moore machine $\mathcal{A}_L$ with Σ_D as the input alphabet and Σ_C as the output alphabet can be constructed such that $\mathcal{A}_L$ provides the winning strategy for the controller with respect to the language L.*

The proof is quite involved. Moreover, this result has the drawback that the resulting Moore machine may have superexponentially more states than the Muller automaton defining the game. The problem of designing a winning strategy for various types of behaviours, is being intensively investigated.

3.7 Limitations of Finite State Machines

Once a machine model is designed and its advantages are demonstrated, an additional important task is to find out its limitations.

In general it is not easy to show that a problem is not within the limits of a machine model. However, for finite state machines, especially finite automata, there are several simple and quite powerful methods to show that a language is not regular. We illustrate some of them.

Example 3.7.1 (Proof of non-regularity of languages using Nerode's theorem) *For the language $L_1 = \{a^ib^i \mid i \geq 0\}$ it clearly holds that $a^i \not\equiv_{L_1} a^j$ if $i \neq j$. This implies that the syntactical monoid for the language L_1 is infinite. L_1 is therefore not recognizable by a FA.*

Example 3.7.2 (Proof of non-regularity of languages using the Pumping Lemma) *Let us assume that the language $L_2 = \{a^p \mid p \text{ prime}\}$ is regular. By the Pumping Lemma for regular languages, there exist integers $x, y, z, x+z \neq 0, y \neq 0$, such that all words $a^{x+iy+z}, i \geq 0$, are in L_2. However, this is impossible because, for example, $x+iy+z$ is not prime for $i = x+z.0$.*

Example 3.7.3 [Proof of non-regularity of languages using a descriptional finiteness argument] *Let us asume that the language $L_3 = \{a^icb^i \mid i \geq 1\}$ is regular and $\mathcal{A}$ be a DFA recognizing L_3. Clearly, for any state q of $\mathcal{A}$, there is* **at most** *one $i \in \mathbf{N}$ such that $b^i \in L(q)$. If such an i exists we say that q specifies that i. Since $a^icb^i \in L_3$ for each i, for any integer j there must exist a state q_j (the one reachable after the input a^jc) that specifies j. A contradiction exists, because there are only finitely many states in $\mathcal{A}$.*

Exercise 3.7.4 *Show that the following languages are not regular:*
(a) $\{a^i b^{2i} \mid i \geq 0\}$*; (b)* $\{a^i \mid i$ *is composite*$\}$*; (c)* $\{a^i \mid i$ *is a Fibonacci number*$\}$*;*
(d) $\{w \in \{0,1\}^* \mid w = w^R\}$.

Example 3.7.5 *We show now that neither Moore nor a Mealy machine can multiply two arbitrary integers given the corresponding pairs of bits as the input as in the case of binary adders in Figure 3.17. (To be consistent with the model in Figure 3.17, we assume that if the largest number has* n *bits then the most significant pair of bits is followed by additional* n *pairs* $(0,0)$ *on the input.)*

If the numbers x *and* y *to be multiplied are both equal to* 2^{2m}*, the* $2m+1$*-th input symbol will be* $(1,1)$ *and all others are* $(0,0)$*. After reading the* $(1,1)$ *symbol, the machine still has to perform* $2m$ *steps before producing a* 1 *on the output. However, this is impossible, because during these* $2m$ *steps* $\mathcal{M}$ *has to get into a cycle. (It has only* m *states and all inputs after the input symbol* $(1,1)$ *are the same—*$(0,0)$*.) This means that either* $\mathcal{M}$ *produces a* 1 *before the* $(4m+1)$*-th step or* $\mathcal{M}$ *never produces a* 1*. This is a contradiction to the assumption that such a machine exists.*

Exercise 3.7.6 *Show that there is no finite state machine to compute the function: (a)* $f_1(n) =$ *the* n*-th Fibonacci number; (b)* $f(0^n 1^m) = 1^{n \bmod m}$.

Example 3.7.7 *It follows from Theorem 3.4.11 that no generalized sequential machine can compute the function* $f : \{0,1\}^* \rightarrow \{0,1\}^*$ *defined by* $f(w) = w^R$*. Indeed, the prefix condition from that theorem is not fulfilled.*

Example 3.7.8 *Let* $L \subset \{0,1\}^\omega$ *be a language of such* ω*-words* w *for which there is an integer* $k > 1$ *such that* w *has a symbol* 1 *exactly in the positions* k^n *for all integers* n*. We claim that* L *is not regular* ω*-language. Indeed, since the distances between two consecutive* 1*'s are getting bigger and bigger, a finite automaton can not check whether they are correct.*

Concerning weighted finite transducers it has been shown that they can compute neither exponential functions nor trigonometric functions.

3.8 From Finite Automata to Universal Computers

Several natural ideas to enhance the power of finite automata will now be explored. Surprisingly, some of these ideas do not increase the computational power of finite automata at all. Some of them, also surprisingly, increase this power very much. All these models have one in common. The only memory they need to process an input is the memory needed to store the input. One of these models illustrates a new important mode of computation—probabilistic finite automata. The importance of others is mainly in the fact that they can be used to represent in an isolated form various techniques for Turing machines discussed in the next chapter.

3.8.1 Transition Systems

A **transition system** $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ is defined similarly as a finite automaton, only the finite transition relation δ is a subset of $Q \times \Sigma^* \times Q$ and not of $Q \times \Sigma \times Q$ as for finite automata. In other words, in a transition system , that cause a single transition, may be arbitrary words. Computation and acceptance are defined for transition systems in the same way as for finite automata. Namely, an input word w is accepted if there is a path from the initial state to a final state labeled by w.

Each finite automaton is a transition system. On the other hand, to each transition system $\mathcal{A}$ it is easy to design an equivalent FA—accepting the same language. To show that we sketch a way to modify the state graph $G_{\mathcal{A}}$ of a transition system $\mathcal{A}$ in order to get a state graph of an equivalent FA.

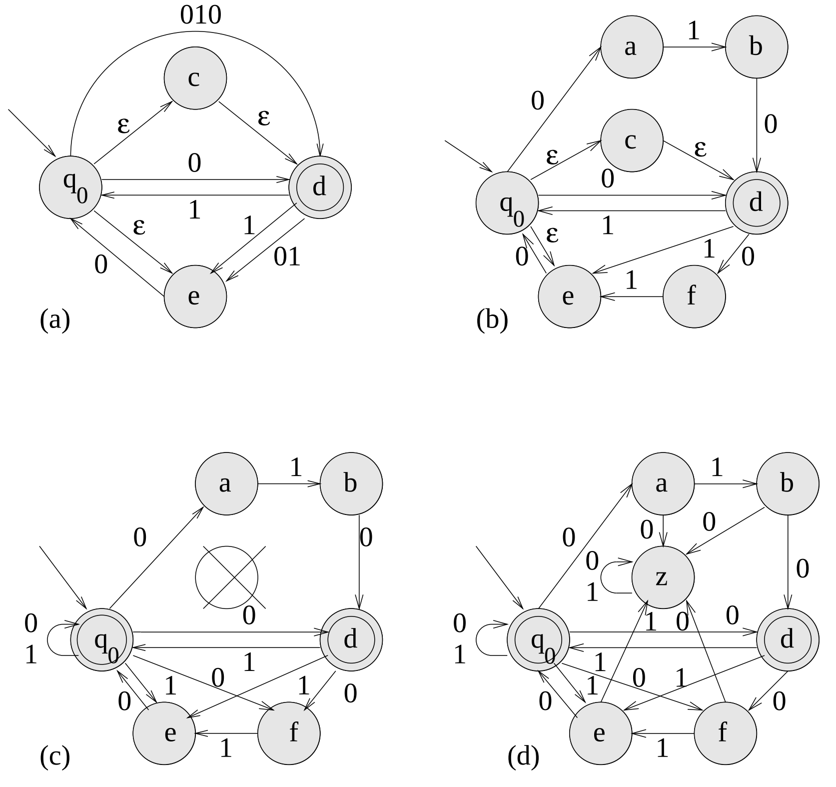


Figure 3.29: Derivation of a complete FA from a transition system—65%

1. Replace each transition (edge) $p \underset{w}{\Longrightarrow} q, w = w_1 w_2 \ldots w_k$, $w_i \in \Sigma$, $k > 1$ by k transitions $p \underset{w_1}{\Longrightarrow} p_1 \underset{w_2}{\Longrightarrow} p_2 \ldots p_{k-2} \underset{w_{k-1}}{\Longrightarrow} p_{k-1} \underset{w_k}{\Longrightarrow} q$, where $p_1, \ldots, p_{k-1}$ are newly created states. (See

the step from Figure 3.29a to Figure 3.29b.)

2. Remove ε-transitions. This is slightly more involved task. One needs first to compute the transitive closure of the relation $\underset{\varepsilon}{\Longrightarrow}$ between states. Then for any triple of states p, q, q' and each $a \in \Sigma$ such that $p \overset{*}{\underset{\varepsilon}{\Longrightarrow}} q \underset{a}{\Longrightarrow} q'$, the transition $p \underset{a}{\Longrightarrow} q'$ is added. If, in addition, after such modifications, $q' \overset{*}{\underset{\varepsilon}{\Longrightarrow}} q$ for some $q' \in Q$ and $q \in Q_F$, then add also q' to the set of final states and remove all ε-transitions and unreachable state. (See the step from Figure 3.29b to Figure 3.29c).

3. Finally, if we need that the resulting automaton is complete, a new "sink-state" is added and all missing transition are added and directed to this sink-state (see the step from Figure 3.29c to Figure 3.29d). By this construction we have shown:

Theorem 3.8.1 *The family of languages accepted by transition systems is exactly the family of regular languages.*

The main advantage of transition systems is that they may have much shorter descriptions and smaller numbers of states than any equivalent FA. Indeed, for any integer n a FA accepting the one-word language $\{a^n\}$ has to have $n-1$ states but there is a two state transition system to do it.

Exercise 3.8.2 *Design a transition system with as few states as possible that accepts those words over the alphabet $\{a, b, c\}$ that either begin or end with the string "baac", or contain the substring "abca" and use then the above method to design an equivalent FA*

Exercise 3.8.3 *Design a minimal, with respect to the number of states, transition system accepting the language $L = (a^4b^3)^* \cup (a^4b^6)^*$ and then transform its state graph to get a state graph for a FA accepting the same language.*

3.8.2 Probabilistic Finite Automata

We have mentioned already several times the power of randomization. It is therefore natural to explore how much randomization can increase the power of finite automata.

Definition 3.8.4 *A* **probabilistic finite automaton** $\mathcal{P} = \langle \Sigma, Q, q_0, Q_F, \phi \rangle$ *has an* **input alphabet** Σ, *a* **set of states** Q, *the* **initial state** q_0, *a set of* **final states** Q_F *and a* **probability distribution mapping** ϕ *that assigns to each* $a \in \Sigma$ *a* $|Q| \times |Q|$ *matrix* M_a *of nonnegative reals with rows and columns of each* M_a *labeled by states and such that* $\sum_{q \in Q} M_a(p, q) = 1$ *for any* $a \in \Sigma$, *and* $p \in Q$. *Informally,* $M_a(p, q)$ *determines the probability that the automaton* $\mathcal{P}$ *goes, under the input* a, *from the state* p *into the state* q; $M_a(p, q) = 0$ *means that there is no transition from* p *to* q *under the input* a.

If $w = w_1 \dots w_n$, $w_i \in \Sigma$, *then the entry* $M_w(p,q)$ *of the matrix* $M_w = M_{w_1} M_{w_2} \dots M_{w_n}$ *is exactly the probability that* $\mathcal{P}$ *goes, under the input word* w, *from the state* p *to the state* q. *Finally, for a* $w \in \Sigma^*$, *we define:*

$$Pr_{\mathcal{P}}(w) = \sum_{q \in Q_F} M_w(q_0, q).$$

$Pr_{\mathcal{P}}(w)$ *is the probability with which* $\mathcal{P}$ *recognizes* w.

There are several ways how to define acceptance by a probabilistic finite automaton. The most basic one is very natural. It is called **an acceptance with respect to a cut-point**. For a real number $0 \leq c < 1$ we define the language:

$$L_c(\mathcal{P}) = \{u \mid Pr_{\mathcal{P}}(u) > c\}.$$

The language $L_c(\mathcal{P})$ is said to be the language recognized by $\mathcal{P}$ with respect to the **cut-point** c. (Informally, $L_c(\mathcal{P})$ is the set of input strings that can be accepted with a probability larger than c.)

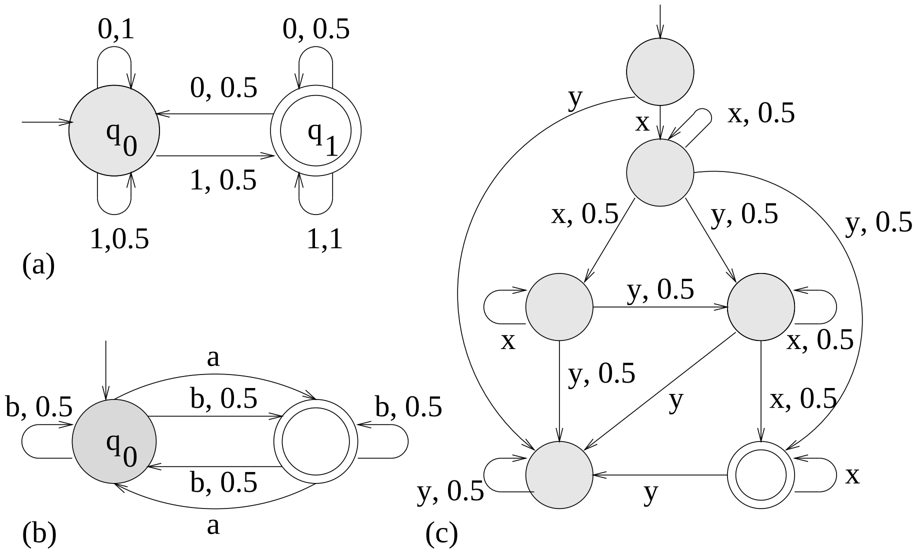


Figure 3.30: Probabilistic finite automata—missing probabilities are 1–70%

Example 3.8.5 *Let* $\Sigma = \{0, 1\}$, $Q = \{q_0, q_1\}$, $Q_F = \{q_1\}$,

$$M_0 = \begin{vmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}, \qquad M_1 = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix}.$$

Figure 3.30a shows the corresponding probabilistic finite automaton $\mathcal{P}_0$. Each edge is labeled by an input symbol and by the probability that the corresponding transition takes place. By induction it can be easily shown that for any $w = w_1 \dots w_n \in \Sigma^n$, the matrix $M_w =$

$M_{w_1}M_{w_2}\dots M_{w_n}$ has in the right upper corner the number $0.w_n\dots w_1$ in binary notation. (Show that!)

Exercise 3.8.6 *Determine, for all possible c, the language accepted by the probabilistic automaton from Figure 3.30b, with respect to the cut-point c.*

Exercise 3.8.7 *Determine the language accepted by the probabilistic automaton in Figure 3.30c with respect to the cut-point* 0.5. *(Don't get surprised if you get a nonregular language.)*

At first we show that with this general concept of acceptance, with respect to a cut-point, the probabilistic finite automata are more powerful than ordinary FA.

Theorem 3.8.8 *For the probabilistic finite automaton $\mathcal{P}_0$ from Example 3.8.5 there exists a real $0 \le c < 1$ such that the language $L_c(\mathcal{P}_0)$ is not regular.*

Proof. If $w = w_1\dots w_n$, then—as already mentioned above—$Pr_{\mathcal{P}_0}(w) = 0.w_n\dots w_1$ (because q_1 is the single final state). This implies that if $0 \le c_1 < c_2 < 1$ are arbitrary reals, then $L_{c_1}(\mathcal{P}_0) \subsetneq L_{c_2}(\mathcal{P}_0)$. The family of languages that $\mathcal{P}_0$ recognizes, with different cut-points, is therefore not countable. On the other side, the set of regular expressions over Σ is countable and therefore so is the set of regular languages over Σ. Hence there exists an $0 \le c < 1$ such that $L_c(\mathcal{P}_0)$ is not a regular language. □

The situation is, however, different for acceptance with respect to the so-called **isolated cut-points**. A real $0 \le c < 1$ is an isolated cut-point with respect to a probabilistic FA $\mathcal{P}$, if there is a $\delta > 0$ such that for all $w \in \Sigma^*$:

$$|Pr_{\mathcal{P}}(w) - c| > \delta. \tag{3.9}$$

Theorem 3.8.9 *If $\mathcal{P} = \langle \Sigma, Q, q_0, Q_F, \phi\rangle$ is a probabilistic FA with c as an isolated cut-point, then the language $L_c(\mathcal{P})$ is regular.*

To prove the theorem we shall use the following combinatorial lemma:

Lemma 3.8.10 *Let P_n be the set of all n-dimensional random vectors, i.e., $P_n = \{x = (x_1,\dots,x_n),\ x_i \ge 0,\ 1 \le i \le n$ and $\sum_{i=1}^n x_i = 1\}$ and let, for an $\varepsilon > 0$, U_ε be such a subset of P_n that for any $x, y \in U_\varepsilon$, $x \ne y$ implies $\sum_{i=1}^n |x_i - y_i| \ge \varepsilon$. Then the set U_ε contains at most $(1 + \frac{2}{\varepsilon})^{n-1}$ vectors.*

Proof of the theorem. Assume that $Q = \{q_0, q_1, \dots, q_{n-1}\}$ and, for simplicity, and without loss of generality, that $Q_F = \{q_{n-1}\}$. In such a case the probability that $\mathcal{P}$ accepts some w is $Pr_{\mathcal{P}}(w) = M_w(q_0, q_{n-1})$, where M_w is an $n \times n$ matrix defined as on page 228.

Consider now the language $L = L_c(\mathcal{P})$ and assume that we have a set of k words $v_1, \dots, v_k$ such that no two of them are in the same prefix equivalence class with respect to the relation

$\equiv_L^p$. This implies, due to the definition of the prefix equivalence, that for each pair $i \neq j, 1 \leq i, j \leq k$ there exists a word y_{ij} such that $v_i y_{ij} \in L$ and $v_j y_{ij} \notin L$—or vice versa.

Let now $(s_1^i, \ldots, s_n^i)$, $1 \leq i \leq k$, be the first row of the matrix M_{v_i}, and let $(r_1^{ij}, \ldots, r_n^{ij})$ be the last column of the matrix $M_{y_{ij}}$ Since $M_{v_i y_{ij}} = M_{v_i} M_{y_{ij}}$ and q_{n-1} is the only accepting state, we get $Pr_{\mathcal{P}}(v_i y_{ij}) = s_1^i r_1^{ij} + \ldots + s_n^i r_n^{ij}$ and $Pr_{\mathcal{P}}(v_j y^{ij}) = s_1^j r_1^{ij} + \ldots + s_n^j r_n^{ij}$ and therefore

$$c < s_1^i r_1^{ij} + \ldots + s_n^i r_n^{i,j} \quad \text{and} \quad s_1^j r_1^{ij} + \ldots + s_n^j r_n^{ij} \leq c.$$

If we now use the inequality (3.9), we get

$$\sum_{l=1}^{n} (s_l^i - s_l^j) r_l^{ij} \geq 2\delta. \tag{3.10}$$

In addition, it holds

$$\begin{array}{rcl}
\sum_{l=1}^n (s_l^i - s_l^j) r_l^{ij} & \leq & (\sum_{l=1}^n (s_l^i - s_l^j))^+ \max\{r_l^{ij} \mid 1 \leq l \leq k\} \\
& & +(\sum_{l=1}^n (s_l^i - s_l^j))^- \min\{r_l^{ij} \mid 1 \leq l \leq k\} \\
& = & (\sum_{l=1}^n (s_l^i - s_l^j))^+ (\max\{r_l^{ij} \mid 1 \leq l \leq k\} - \min\{r_l^{ij} \mid 1 \leq l \leq k\}) \\
& \leq & (\sum_{l=1}^n (s_l^i - s_l^j))^+ = \frac{1}{2} \sum_{l=1}^n |s_l^i - s_l^j|,
\end{array}$$

where $(\ldots)^+$ denotes that only the positive numbers from the expression inside the parentheses are taken and, similarly, $(\ldots)^-$ stands for taking only the negative numbers. In deriving these inequalities we have used essentially the fact that $|r_l^{ij}| \leq 1$ for all l, i, j.

A combination of the last inequality with the inequality 3.10 yields $\sum_{l=1}^n |s_l^i - s_l^j| \geq 4\delta$. An application of the Lemma 3.8.10 then gives $k \leq \left(1 + \frac{1}{2\delta}\right)^{n-1}$. Now we can use Myhill's theorem to get that the language $L_c(\mathcal{P})$ has to be regular.

□

It has been shown that from the point of view of randomized computations the acceptance with respect to an isolated cut-point is very natural. Theorem 3.8.9 is therefore often seen as the main theorem showing the power of the probabilistic finite automata. Unfortunately, it is still an open problem as to whether it is decidable, given a probabilistic finite automaton $\mathcal{P}$ with rational probabilities of transitions and a rational λ, whether λ is an isolated cut-point of $\mathcal{P}$.

Exercise 3.8.11 * *A cutpoint* λ *is weakly isolated for a probabilistic finite automaton* $\mathcal{P}$ *if* $|Pr_{\mathcal{P}}(w) - \lambda| \geq \varepsilon$ *or* $Pr_{\mathcal{P}}(w) = \lambda$ *for all* $w \in \Sigma^*$ *and some fixed* ε. *Prove that if* λ *is a weakly isolated cut-point for* $\mathcal{P}$, *then the language* $L_\lambda(\mathcal{P})$ *is regular.*

Exercise 3.8.12 ** *Two probabilistic finite automata* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are called mutually isolated if* $|Pr_{\mathcal{P}_1}(w) - Pr_{\mathcal{P}_2}(w)| \geq \varepsilon$ *for all* $w \in \Sigma^*$. *Prove that if* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are mutually isolated, then the language* $L = \{w \mid Pr_{\mathcal{P}_1}(w) > Pr_{\mathcal{P}_2}(w)\}$ *is regular.*

The concept of a probabilistic finite automaton is usually generalized and instead of a fixed initial state an initial distribution of states is considered, i.e. each state is an initial state of a given probability. In order to count the overall probability that a word is accepted, the probability of each path has to be multiplied with the probability that its starting state is initial. Languages accepted by such probabilistic finite automata with respect to a cut-point c are called c-stochastic. A language is called (finite state) **stochastic** if there is a probabilistic finite automaton $\mathcal{A}$ and a cut-point c such that $L = L_c(\mathcal{A})$.

Exercise 3.8.13 *Show that any regular language is c-stochastic for any cut-point* $0 \leq c \leq 1$.

Exercise 3.8.14 *Show that every* 0*-stochastic language is regular.*

Of a special interest are probabilistic finite automata with uniform probability distribution of transitions—for each state q and each input symbol a all transitions from q under a have the same probability. Such probabilistic automata are formally defined exactly as nondeterministic automata and therefore it is natural to ask what is the difference between them. Actually, it is a very big one. Nondeterministic automata are very convenient to deal with but completely unrealistic models of computations. On the other hand probabilistic finite automata are a very realistic model of computation.

Another way how to see probabilistic finite automata, and often very useful way for applications, is as defining a probability distribution on the set of inputs.

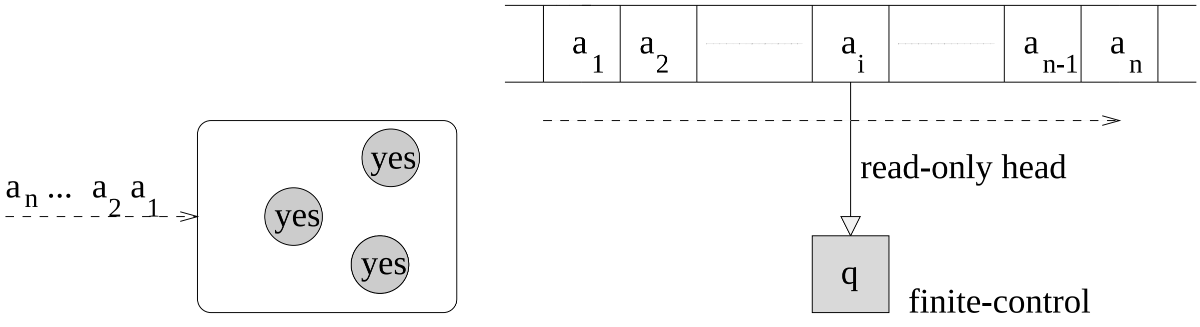
3.8.3 Two-way Finite Automata

In addition to the usual view of a FA as a finite state device, see Figure 3.31a, that processes an external input string symbol by symbol, there is another view of FA—see Figure 3.31b—that is a base of several natural generalizations of FA.

As illustrated in Figure 3.31b, a (one-way) deterministic finite automaton $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ can be seen as consisting of a **finite control** (determined by δ) that is always in one of the states in the set Q, a **bi-infinite tape**, each cell of which may contain a symbol from Σ or a special blank symbol $\sqcup$ not in Σ, and a **read-only head** that always stays on a cell of the tape and can move only to the right—one cell per move.

At the beginning of a computation the input word w is written, one symbol per cell, in $|w|$ consecutive cells and the head is positioned on the cell with the first symbol of w. In each computational step $\mathcal{A}$ reads the symbol from the cell the head is on at that moment, and depending on the state of the finite control, $\mathcal{A}$ goes, according to the transition function of the finite control δ, to a new state q and also moves its head to the next cell. In case when $\mathcal{A}$ reaches a final state in the moment when the head moves over the right end of the input word, then the input word is considered as accepted. In a similar way nondeterministic finite automata can be defined.

With such a model of a finite automaton the question naturally arises whether a more



(a) (b)

Figure 3.31: Two ways to see a FA–80%

powerful device could be obtained if the head is allowed to move also to the left. Let us explore this idea.

Definition 3.8.15 *A* **two-way finite automaton** $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ *is defined similarly as an ordinary FA, only the transition function has the form*

$$\delta : Q \times \Sigma \to Q \times \{\leftarrow, \downarrow, \to\}$$

and $\delta(p, a) = (q, d)$*, where* $d \in \{\leftarrow, \downarrow, \to\}$*, means that the automaton* $\mathcal{A}$ *in the state* p *moves, under the input* a*, to the state* q *and the head moves one cell in the direction* d*; i.e. to the right if* $d = \to$*, to the left if* $d = \leftarrow$*, and does not move at all if* $d = \downarrow$*. The language* $L(\mathcal{A})$*, accepted by* $\mathcal{A}$*, is then defined as follows:*

$L(\mathcal{A}) = \{w \in \Sigma^* |$ $\mathcal{A}$ *starting with the head on the first symbol of* w *and in the state* q_0 *moves, after a finite number of steps, over the right end of* w *exactly when* $\mathcal{A}$ *comes to a final state*$\}$.

Nondeterministic two-way finite automata can be defined similarly.

Example 3.8.16 *The two-way finite automaton* $\mathcal{A}_0$ *in Figure 3.32 recognizes the language*

$$L = x_0(\Sigma - \{x_0\})^* \cap \Sigma^* x_1 \Sigma^* \cap \Sigma^* x_2 \Sigma^* \cap \Sigma^* x_3 \Sigma^*$$

of all words over the alphabet $\Sigma = \{x_0, x_1, x_2, x_3\}$ *that begin with* x_0*, do not contain another occurrence of* x_0 *and contain all remaining symbols from* Σ *at least once. Indeed,* $\mathcal{A}_0$ *first verifies whether the first symbol is really* x_0*. If so* $\mathcal{A}_0$ *starts a move to the right to search for an* x_1*. If found,* $\mathcal{A}_0$ *moves back to the left end and starts a search for an* x_2*. If found* $\mathcal{A}_0$ *again moves to the leftmost end and starts a search for an* x_3*. If* x_0 *is found twice, or the first symbolis not* x_0*, then* $\mathcal{A}_0$ *moves to the "sink" state* $q_{\#}$*.*

Two-way finite automata seem to be more powerful than FA. This is, however, misleading.

Theorem 3.8.17 *Nondeterministic two-way finite automata accept exactly the regular languages.*

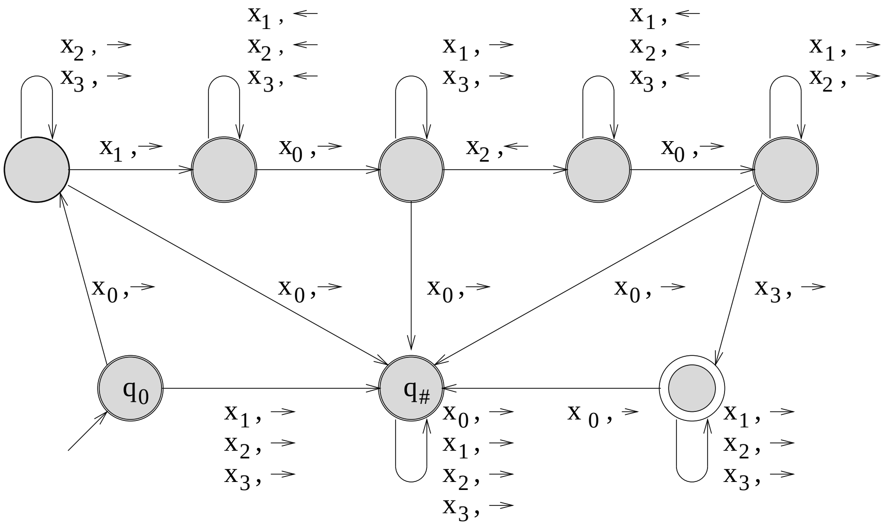


Figure 3.32: An example of a two-way finite automaton–65%

The proof of this result is quite involved. This is quite natural because the result is contra-intuitive.

Informally seen, the theorem actually says that **multiple readings do not help if writing is not allowed and the machine has only finite memory.**

On the other hand, a two-way finite automaton can be much smaller than an equivalent FA.

Example 3.8.18 *Let* $\Sigma = \{a_0, a_1, \ldots, a_n\}$ *and let* L_0 *be the set of all words over* Σ *that start with the symbol* a_0*, do not contain any other occurrence of* a_0 *and contain each of the remaining symbols from* Σ *at least once. Similarly, as in Example 3.8.16, a two-way finite automaton with* $2n+2$ *states can be constructed to accept* L_0*. On the other side, it can be shown that any finite automaton recognizing* L_0 *has to have at least* 2^n *states.*

One can show that even larger savings of the description length can be achieved by using two-way finite automata compared with ordinary FA. It holds, for example,

$$Economy_{2DFA}^{DFA}(n) = \Omega((n/5)^{n/5}).$$

where economy function for replacing a DFA with an equivalent 2DFA is defined similarly as on page 187.

Exercise 3.8.19 *Show, given an integer* n*, how to design a two-way finite automaton accepting the language*

$$L_n = \{10^{i_1}10^{i_2}1\ldots10^{i_n}2^k0^{i_k} \mid 1 \le k \le n, 1 \le i_j \le n, 1 \le j \le n\}.$$

3.8.4 Multihead Finite Automata

Another natural idea to enhance the power of finite automata is to admit a sort of parallelism by allowing to use several read-only heads—see Figure 3.33a. Two types of multihead finite automata are natural: one-way—heads move only in one direction—and two-way—heads can move in both directions.

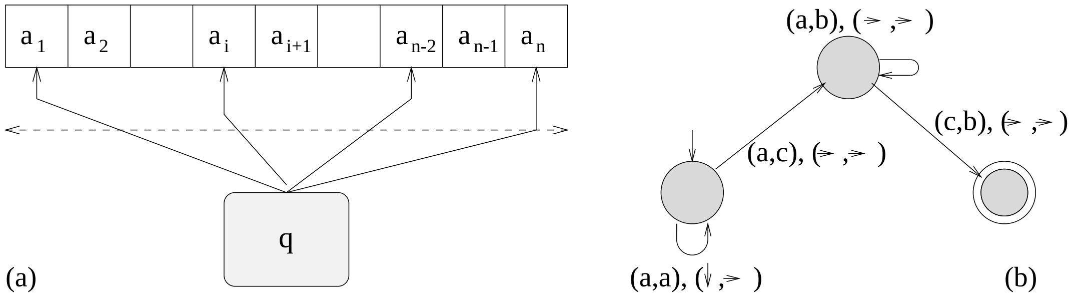


Figure 3.33: Multihead finite automata–65%

Informally, a k-head two-way finite automaton (in short k-2FA) has k heads and at the beginning of any computation all heads stay on the cell with the first symbol of the input word. Each computation step is uniquely determined by the current state of the automaton and by the symbols the heads read. A step consists of a state change and moves of heads as specified by the transition function. More formally:

In a k-head two-way finite automaton $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \delta \rangle$ the symbols Σ, Q, q_0, Q_F have the usual meaning and

$$\delta : Q \times \underbrace{\Sigma \times \ldots \times \Sigma}_{k} \to Q \times \underbrace{\{\leftarrow, \downarrow, \rightarrow\} \times \ldots \times \{\leftarrow, \downarrow, \rightarrow\}}_{k},$$

where $\delta(p, b_1, \ldots, b_k) = (q, d_1, \ldots, d_k)$ means that if $\mathcal{A}$ is in the state p and the i-th head reads b_i, for $1 \leq i \leq k$, then $\mathcal{A}$ goes to the state q and the j-th head moves in the direction determined by d_i. Nondeterministic k-head two-way finite automata are defined similarly and so are one-way multihead FA.

The language $L(\mathcal{A})$ is defined as the set of words w such that if $\mathcal{A}$ starts with all heads on the first symbol of w, then after some number of steps $\mathcal{A}$ moves to a final state exactly when one of the heads leaves the cells that are occupied by w, at the right end.

Example 3.8.20 *It is easy to see that the 2-head one-way finite automaton from Figure 3.33b recognizes the language* $\{a^i c b^i \mid i \geq 1\}$ *that is not regular.*

Exercise 3.8.21 *(a) Design a 2-head FA accepting the language* $\{a^i b^i c^i \mid n \geq 1\}$*; (b) design a 3-head FA accepting the language* $\{a^i b^{2i} c^{3i} \mid i \geq 1\}$.

Once we know that even one-way 2-head finite automata are more powerful than one-head finite automata, it is natural to ask whether any increase of the number of heads provides more power. To formulate the result let us denote by $\mathcal{L}(k\text{-2DFA})$ ($\mathcal{L}(k\text{-2NFA})$) the family of languages accepted by deterministic (nondeterministic) two-way finite automata with k heads. In an analogical way we use notation $\mathcal{L}(k\text{-1DFA})$ and $\mathcal{L}(k\text{-1NFA})$.

Theorem 3.8.22 *For each* $k \geq 1$ *and* $i = 1, 2$, $\mathcal{L}(k\!-\!iDFA) \subsetneq \mathcal{L}((k+1)\!-\!iDFA)$ *and* $\mathcal{L}(k\!-\!iNFA) \subsetneq \mathcal{L}((k+1)\!-\!iNFA)$.

The proofs are quite involved and represent solutions of long standing open problems. It follows from Theorem 3.8.22 that k-head finite automata, for $k = 1, 2, \ldots$ form an infinite hierarchy of more and more powerful machines!

How is all that possible? It seems that also k-head finite automata have only finitely many states and use only finite memory. This impression, however, is wrong. The actual state of such a machine is determined not only by the state of its finite control but also by the positions of the heads. If an input word w has length n, then the overall number of global states (configurations) a k-head FA $\mathcal{A}$ can be in is $|Q|n^k$ where Q is the set of internal states of $\mathcal{A}$. The total number of global states of a k-head FA therefore grows polynomially with respect to the length of the input.

The following two closely related families of languages

$$\bigcup_{k=1}^{\infty} \mathcal{L}(k\text{-2DFA}) \quad \text{and} \quad \bigcup_{k=1}^{\infty} \mathcal{L}(k\text{-2NFA})$$

play an important rôle in the complexity theory and are equal to two families of languages defined with respect to the space complexity **L** and **NL** intruduced in Section 5.2.

3.8.5 Linearly Bounded Automata

Another natural generalization of finite automata are linearly bounded automata (LBA). The head of a LBA is not only allowed to move in both directions but **the head may also write**—see Figure 3.34a. This is an essentially new and very powerful step in the generalization of the finite automata concept.

Since the head of a LBA can move in both directions and can also write, two markers \$ and # are used to delimit the beginning and the end of the tape section on which the input word is written. A LBA is neither allowed to move left from \$ nor right from #, and it is also not allowed to write these markers on the tape or to erase them.

Formally, an LBA $\mathcal{A}$ is specified as $\mathcal{A} = \langle \Sigma, \Delta, Q, q_0, Q_F, \$, \#, \delta \rangle$, where Σ, Q, q_0, Q_F have the same meaning as for FA and

- $\Delta \supseteq \Sigma$ is the **tape alphabet**;
- $\$, \# \in \Delta - \Sigma$ are special **markers**;
- $\delta \subseteq Q \times \Delta \times Q \times \Delta \times \{\leftarrow, \downarrow, \rightarrow\}$ is a **transition relation** satisfying the above mentioned conditions (i.e., if $(p, a, q, b, d) \in \delta$, then $b \notin \{\$, \#\}, a = \$ \Rightarrow d \neq \leftarrow, a = \# \Rightarrow d \neq \rightarrow$ $), a = \$ \Rightarrow b = \$, a = \# \Rightarrow b = \#$.

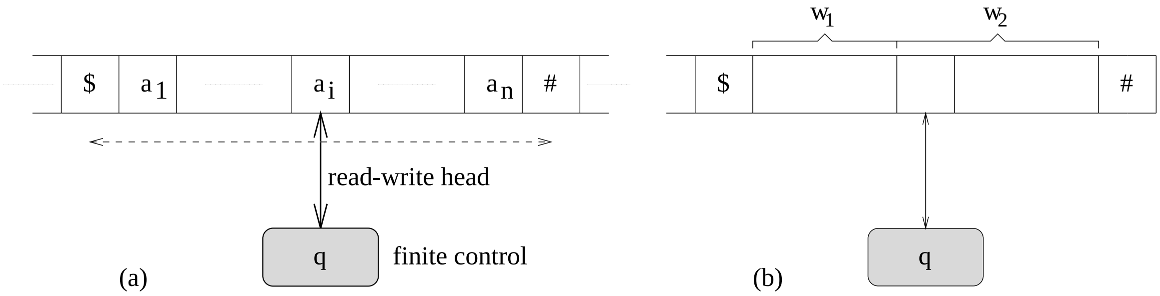


Figure 3.34: A linearly bounded automaton–60%-

A LBA $\mathcal{A}$ may perform a transition $(p, a, b, q, d) \in \delta$ when $\mathcal{A}$ is in the state p and the head reads a. In such a case the finite control of $\mathcal{A}$ goes to state q, the head rewrites a by b and moves in the direction d. Of course, in general there may be more quintuples in δ than one that start with the same p and a and therefore a move of the head may be nondeterministic. If δ is a function of its first two arguments then we have a **deterministic LBA (DLBA)**.

To describe a **computation** on a LBA $\mathcal{A}$, the concept of **configuration** is again useful. It is a word of the form $w_1qw_2 \in \Delta^*Q\Delta^*$. A LBA $\mathcal{A}$ is in the configuration w_1qw_2 if q is its current state, w_1w_2 is the contents of the tape and the head is positioned on the first symbol of w_2, see Figure 3.34b. A configuration is **initial** if it has the form q_0w, $w \in \Sigma$ (the input alphabet), and it is **final** if its state is final.

The concept of a configuration is very helpful to define a computation on a LBA formally. In order to do that we first introduce the concept of a computation step. A configuration C' is a **direct successor** of a configuration C, in short $C \vdash C'$, if C' is configuration that can be obtained from C by performing a transition—a computation step. A configuration is called **terminating** if there is no configuration that would be its direct successor. (In a terminating computation the LBA "halts".) If $C \vdash^* C'$, then C' is called a **successor configuration** of C or a configuration that can be **reached** from C. A **computation** of a LBA is a finite or infinite sequence of configurations that starts with the initial configuration and, for any integer $i > 1$, the i-th configuration is a direct successor of the $(i-1)$-th configuration. A **terminating computation** is a finite computation that ends with a terminating configuration.

The language accepted by a LBA $\mathcal{A}$ is defined as follows:

$$L(\mathcal{A}) = \{w \in \Sigma^* \mid \exists \text{ computation starting in } q_0w \text{ and ending in a final configuration}\}.$$

To describe a LBA formally, its transition relation has to be specified. To do that in details may be a tedious but basically straightforward task when a high-level algorithm describing its behaviour is given as in the following example.

Example 3.8.23 *Describe the behavior of a LBA recognizing the language* $\{a^ib^i \mid i \geq 1\}$.

```
begin Check if the input word has the form $a^ib^j$—if not, then reject;
    while there are at least one $a$ and one $b$ on the tape
        do erase one $a$ and one $b$;
```

```
    if there is still a symbol $a$ or $b$ on the tape then reject else accept
end
```

Exercise 3.8.24 *Describe a LBA accepting the language* $\{a^i b^i c^i \mid i \geq 1\}$.

The above examples show that DLBA can accept languages that are not regular and therefore DLBA are more powerful than finite automata. On the other hand it is not known whether nondeterminism brings new power in case of LBA.

Open problem 3.8.25 (LBA Problem) *Are LBA more powerful as DLBA?*

This is one of the longest standing open problems in foundations of computing.

The next natural question is how powerful are LBA compared with multihead FA (because multihead FA have also been shown to be more powerful than finite automata). It is in a sense a question what provides more power: a possibility to write (and thereby to store immediate results and to make use of memory of the size proportional to the size of the input) or a possibility to use more heads (and thereby parallelism).

Let us denote by $\mathcal{L}(LBA)$ the family of languages accepted by LBA and by $\mathcal{L}(DLBA)$ the family of languages accepted by DLBA. For the reason that will be made clear in Chapter 7, languages from $\mathcal{L}(LBA)$ are called **context-sensitive** and those from $\mathcal{L}(DLBA)$ are called **deterministic context-sensitive**.

Theorem 3.8.26 *The following relations hold between the families of languages accepted by multihead finite automata and LBA.*

$$\bigcup_{k=1}^{\infty} \mathcal{L}(k\text{-2DFA}) \subsetneq \mathcal{L}(\text{DLBA}) \tag{3.11}$$

$$\bigcup_{k=1}^{\infty} \mathcal{L}(k\text{-2NFA}) \subsetneq \mathcal{L}(\text{NLBA}) \tag{3.12}$$

We show here only that each multihead 2DFA can be simulated by a DLBA. Simulation of a multihead 2NFA by a NLBA can be done similarly. The proof that there is a language accepted by a DLBA and not accepted by a multihead 2DFA, and similarly for the nondeterministic case, is beyond the scope of the book.

In order to simulate a k-head 2DFA $\mathcal{A}$ by a DLBA $\mathcal{B}$ we need:
(a) to represent a configuration of $\mathcal{A}$ by a configuration of $\mathcal{B}$;
(b) to simulate one transition of $\mathcal{A}$ by a computation on $\mathcal{B}$.

(a) **Representation of configurations.** A configuration of $\mathcal{A}$ is given by a state q, a tape content $w = w_1 \ldots w_n$ and the positions of k heads. In order to represent this information in a configuration of $\mathcal{B}$, the j-th symbol of w, that is w_j, is represented at any moment of computation by a $(k+2)$-tuple $(q, w_j, s_1, \ldots, s_k)$, where $s_i = 1$ if the i-th head of $\mathcal{A}$ stays, in the given configuration of $\mathcal{A}$, on the i-th cell, and $s_i = 0$, otherwise. Moreover, in order to create the representation of the initial configuration of $\mathcal{A}$, $\mathcal{B}$ replaces the symbol w_1 in the given input word w by $(q_0, w_1, 1, \ldots, 1)$ and all other $w_i, 1 < i \le |w|$ by $(q_0, w_i, 0, \ldots, 0)$.

(b) **Simulation of one step of $\mathcal{A}$.** $\mathcal{B}$ reads the whole tape content and remembers in its finite state control the state of $\mathcal{A}$, and the symbols read by heads in the corresponding configuration of $\mathcal{A}$. This information is enough for $\mathcal{B}$ to simulate a transition of $\mathcal{A}$. $\mathcal{B}$ needs only to make an additional pass through the tape in order to replace the old state of $\mathcal{A}$ by the new one and update the positions of all heads of $\mathcal{A}$. □

It can happen that a LBA gets into an infinite computation. Indeed, the head can get into a cycle, for example one step right and one step left, without rewriting the tape. However, in spite of that it holds:

Theorem 3.8.27 *The membership problem for LBA is decidable.*

Proof. First an observation. The number of configurations of a LBA $\mathcal{A} = \langle \Sigma, \Delta, Q, q_0, Q_F, \$, \#, \delta \rangle$ that can be reached from an initial configuration $q_0 w$ is bounded by $c_w = |Q||\Delta|^{|w|}(|w| + 2)$. ($\Delta^{|w|}$ is the number of possible contents of the tape of length $|w|$, $|w| + 2$ is the number of cells the head can stand on and $|Q|$ is the number of possible states.) This implies that if $\mathcal{A}$ is a DLBA, then it is sufficient to simulate c_w steps of $\mathcal{A}$ in order to find out whether there is a terminal configuration reachable from the initial configuration $q_0 w$—that is, whether w is accepted by $\mathcal{A}$. Indeed, if $\mathcal{A}$ does not terminate in c_w steps, then it has to be in an infinite loop. If $\mathcal{A}$ is not deterministic, then configurations reachable from the initial configuration $q_0 w$ form a configuration tree (see Figure 3.35) and in order to find out whether $w \in L(\mathcal{A})$ it is enough to check all configurations of this tree up to the depth c_w. □

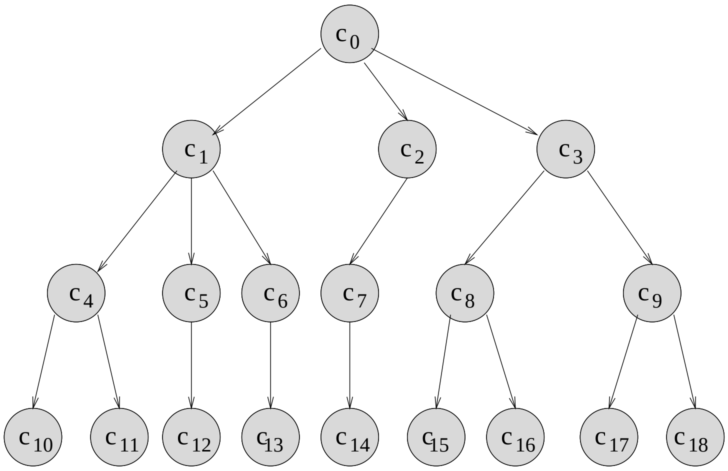


Figure 3.35: Configuration tree–60%

The fact that a LBA may not halt is not pleasant. This makes it hard to design more complex LBA from simpler ones, for example using sequential composition of LBA. The following result is therefore of importance.

Theorem 3.8.28 *For each LBA there is an equivalent LBA that always terminates.*

To prove this theorem we apply a new and often useful technique of dividing the tape into more tracks—see Figure 3.36—in this case into two. Informally, each cell of the tape is divided into an upper and a lower subcell. Each of these subcells can contain a symbol and the head can work on the tape in such a way as it would read and write only to a subcell of one of the tracks. Formally, this is nothing else than to use pairs $\frac{x}{y}$ of symbols as symbols of the tape alphabet and at each writing to change either none or only one of them or both of them.

(a) one tape with two tracks (b) one tape with one track

Figure 3.36: A tape with two or one tracks–80%

Proof. Given an LBA $\mathcal{A}$ with the input alphabet Σ we design from $\mathcal{A}$ another LBA $\mathcal{B}$ the tape of which consists of two tracks. At the beginning of a computation the input word w is seen as being written in the upper track. $\mathcal{B}$ first computes the number $c_w = |Q||\Delta|^{|w|}(|w|+2)$—the maximum number of possible configurations and stores this number in the second track. (Such a computation is not a problem with LBA power.) Space is another issue. There is, however, enough space to write c_w on the second track because $|Q||\Delta|^{|w|}(|w|+2) \leq (2|Q||\Delta|)^{|w|}$, and therefore it is enough to use a number system with sufficiently large base, for example $(2|Q||\Delta|)$, the size of which does not depend on the input word w. $\mathcal{B}$ then simulates the computation of $\mathcal{A}$ step by step. Whenever the simulation of a step of $\mathcal{A}$ is finished, $\mathcal{B}$ decreases the number on the second track by 1. If $\mathcal{A}$ accepts before the number on the second track is zero, $\mathcal{B}$ accepts as well. If $\mathcal{B}$ decreases the number on the second track to zero, then $\mathcal{B}$ moves to a terminating but not final state. Clearly, $\mathcal{B}$ accepts an input word w if and only if $\mathcal{A}$ does. $\square$

The family of context-sensitive languages contains practically all formal languages one has to deal with in practice. It is a rich family and one of its basic properties is stated in the following theorem.

Theorem 3.8.29 *Both families $\mathcal{L}(LBA)$ and $\mathcal{L}(DLBA)$ are closed under Boolean operations (union, intersection and complement).*

Proof. Given two LBA (or DLBA) $\mathcal{A}_1, \mathcal{A}_2$ that always terminate, it is easy to design a LBA (or DLBA) that for a given input w simulates first the computation of $\mathcal{A}_1$ on w and then the computation of $\mathcal{A}_2$ on w and accepts w if and only if both $\mathcal{A}_1$ and $\mathcal{A}_2$ accept w (in case of intersection) or if at least one of them accepts (in case of union). This implies closure under union and intersection. To show the closure under complement is fairly easy for a DLBA $\mathcal{A} = \langle \Sigma, \Delta, Q, q_0, Q_F, \delta, \$, \#, \delta \rangle$, which always terminates. It is enough to take $Q - Q_F$ instead of Q_F as the set of the final states. The proof that also the family $\mathcal{L}(LBA)$ is closed under complement is much more involved. $\square$

Another natural idea to enhance the power of finite automata is to allow the head to move everywhere on the tape and do writing and reading everywhere, not only on cells occupied by the input word. This will be explored in the following chapter and, as we shall see, it leads already to the most powerful concept of machines we can have.

All automata we have dealt with in this chapter can be seen as more or less restricted variants of the Turing machines discussed in the next chapter and all techniques used to design automata of this chapter can be used also as techniques "to program " Turing machines. This is also one of the reasons why we discussed such models as LBA in detail.

Moral. Automata, like people, can look so similar and be so different and can look so different and be so similar. A good rule of thumb by dealing with automata is, as in real life, to think twice and expolre carefully before making a final judgement.

3.9 Exercises

1. Let $\mathcal{A}$ be the FA over the alphabet $\{a, b\}$ with the initial state 1, the final state 3 and the transition relation $\delta = \{(1, a, 1), (1, b, 1), (1, a, 2), (2, b, 3)\}$. Design an equivalent deterministic and complete FA.

2. Design state graphs for FA accepting the following languages: (a) $L = \{w \mid w \in \{a, b\}^*, aaa$ is not a subword of $w\}$; (b) $L = \{w \mid w \in \{a, b\}^*, w = xbv, |v| = 2\}$; (c) $L = \{w \mid w \in \{a, b\}^*, aaa$ is not a subword of of w and $w = xby, |y| = 2\}$.

3. Design a finite automaton to decide whether a given number n is divided by 3 for the case: (a) n is given in binary, the most significant digit first; (b) n is given in binary, the least significant digit first; (c) n is given in decimal; (d)* n is given in Fibonacci number representation.

4. Show that if a language L_1 can be recognized by a DFA with n states and L_2 by a DFA with m states, then there is a DFA with $n2^m$ states recognizing the language L_1L_2 (and in some cases no smaller DFA for L_1L_2 exists).

5. Show that for any n state DFA $\mathcal{A}$ there exists a DFA $\mathcal{A}'$ having at most $2^{n-1} + 2^{n-2}$ states and such that $L(\mathcal{A}') = (L(\mathcal{A}))^*$.

6. Show that a language $L \subseteq \{a\}^*$ over an one-symbol alphabet is regular if and only if there are two finite sets $M_1, M_2 \subset \{a\}^*$ and a $w \in \{a\}^*$ such that $L = M_1 \cup M_2\{w\}^*$.

7. Show that if R is a regular language, then so is the language $R_{half} = \{x \mid \exists y |x| = |y|, xy \in R\}$.

8. Show that the following languages are not regular: (a) $\{ww \mid w \in \{a, b\}^*\}$;

   (b) $\{a^ib^jc^j \mid i, j \geq 1\} \cup b^*c^*$; (c) $L = \{w \mid w \in \{a, b\}^*, w$ contains more a's than b's$\}$.

9. Which of the following languages is regular: (a) UNEQUAL=$\{a^nb^m | n, m \in \mathbf{N}, n \neq m\}$; (b) $\{a\}^*$UNEQUAL; (c) $\{b\}^*$UNEQUAL.

10. Show that the following languages are not regular (a) $\{a^{2^n} | n \geq 1\}$; (b) $\{a^{n!} \,|\, n \geq 1\}$.

11. Let w be a string. How many states has the minimal DFA recognizing the set of all substrings of w.

12. Let $L_n = \{x_1\#x_2\#\ldots x_m\#\#x \,|\, x_i \in \{a,b\}^n, x = x_j \text{ for some } 1 \leq j \leq m\}$. Show that each DFA accepting L_n has to have 2^{2^n} states.

13. Let R_1 and R_2 be regular languages recognized by DFA $\mathcal{A}_1$ and $\mathcal{A}_2$ with r and s states, respectively. Show that the languages $R_1 \cup R_2$ and $R_1 \cap R_2$ can be recognized by DFA with rs states. (Hint: take the cartesian product of the states of $\mathcal{A}_1$ and $\mathcal{A}_2$ as the new set of states.)

14. Find two languages L_1, L_2 such that none of them is regular but their union and also their intersection are.

15. For two languages $K, L \subset \Sigma^*$ define the right-quotient $K/L = \{u \in \Sigma^* \,|\, \exists v \in L, uv \in K\}$. Show that the family of regular languages is closed under the operation of right-quotient.

16. * Let us assume that there exists a morphism ϕ from Σ^* to a finite monoid $\mathcal{M}$ and let for some $L \subseteq \Sigma^*, \phi^{-1}(\phi(L)) = L$. Prove that L is a regular language.

17. * Show that the regular language $\{10101\}^*$ can be expressed without operation of iteration using only the operations of union, concatenation, complement and intersection.

18. Show that the family of regular languages is closed under the shuffle operation.

19. Show that if a tree automaton has the property that for the acceptance of an input word it does not matter which level of processors is used to start the computation with an input word w, provided the level has at least $|w|$ processors and the input is given to the leftmost processor, one symbol per processor, then such a tree automaton accepts always a regular language.

20. Let $R \subset \Sigma^*$ be a regularlanguage. Is also the language $twist(R)$ defined by the following mapping regular?: $TWIST : \Sigma^* \rightarrow \Sigma^*; TWIST(x) = x$ if $x \in \Sigma \cup \{\varepsilon\}$, $TWIST(awb) = abTWIST(w)$ if $a, b \in \Sigma$.

21. Design a Moore or Mealy machine with three states which for the input 00001000100010 produces as output 01010000101001.

22. Show that the family of relations defined by finite transducers is closed under composition.

23. * Let f be a bijection between Σ^* and Δ^* that preserves prefixes, length and regular sets. Show that f is realized by a generalized sequential machine.

24. Which image transformations realize the WFT in Figure 3.37:

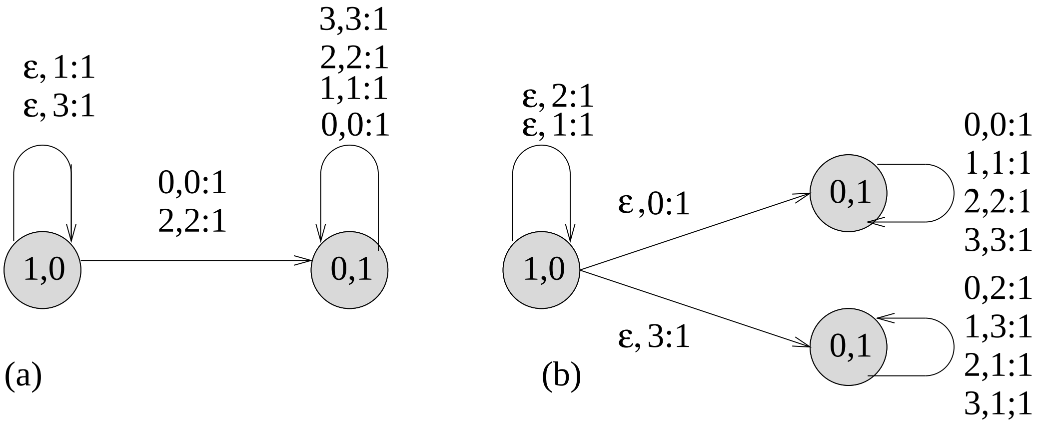


Figure 3.37: WFA

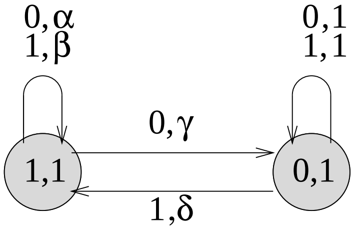


Figure 3.38: LWFT

25. Construct a WFT to perform: (a)** a rotation by 90 degrees counterclockwise with linear slope; (b) * stretching defined by the mapping $(x, y) \to (x, \frac{3}{2}y)$.

26. ** Show that the function computed by the LWFT depicted on the Figure 3.38 is continuous if and only if the following two conditions are satisfied: (1) $\alpha + \beta = 1$, (2) $\delta(1-\alpha) = \gamma(1-\delta)$, $0 \leq \alpha, \beta \leq 1$, $0 \leq \gamma, \delta$.

27. * Show that the polynomial x^n can be computed by a WFA with $n+1$ states $(q_n, \ldots, q_0)$ with the initial distribution $(1, 0, \ldots, 0)$, final distribution $(1, \ldots, 1)$ and the following transitions: (1) $q_i \xrightarrow{j,2^{-i}} q_i$ for $j = 0, 1, i = 0, 1, \ldots, n$; (2) $q_i \xrightarrow{1,2^{-i}\binom{i}{t}} q_{i-t}$, for $i = 1, \ldots, n$ and $t = 1, \ldots, i$.

28. Prove the second assertion of Theorem 3.6.4.

29. Show that the ω-language $\{a^n b^n c^\omega \mid n \geq 0\}$ is not regular.

30. Design Büchi automata recognizing the following ω-languages: (a) An ω-language consisting of ω-words over the alphabet $\{a, b, c\}$ with infinitely many a's and b's and such that there is an odd number of c's between any two symbols from $\{a, b\}$; (b) an ω-language consisting of all ω-words over $\{a, b, c\}$ with infinitely many a's and also b's and with never more than three c's in a row.

31. Determine whether the following ω-languages are regular: (a) $\{a^ib^i \mid i \geq 0\}^\omega$; (b) $\{a^ib^j \mid i,j \geq 0\}^\omega$; (c) $\{a^ib^j \mid 1 \leq i \leq j\}^\omega$.

32. Show that there is no finite state machine to compute the following functions $f : \mathbf{N} :\rightarrow \mathbf{N}$: (a) $f(n) = n^2$; (b) $f(n) = \lfloor\sqrt{n}\rfloor$.

33. Design a transition system with as few states as possible to recognize the languages: (a) $\{a^{3i}b^{4j} \mid i,j \geq 1\}$; (b) $^*L_n = \{a^i \mid 1 \leq i \leq n\}$.

34. Let $\mathcal{A} = \langle\Sigma, Q, Q_I, Q_F, \delta\rangle$ be a transition system with the alphabet $\Sigma = \{a,b,c\}$, states $Q = \{1,2,\ldots,7\}$, the initial states $Q_I = \{1,2\}$, the final states $Q_F = \{4,5\}$ and the transitions $\{(1,abc,5),(2,\varepsilon,4),\ (3,b,4),\ (4,a,6),\ (4,c,7),\ (6,c,5)\}$. Transform $\mathcal{A}$, step by step, into an equivalent transition system with the following properties. (a) with only one initial state; (b) with transitions only on symbols from $\Sigma \cup \{\varepsilon\}$; (c) with transitions on all symbols from all states; (d) with all states reachable from the initial state; (e) complete and deterministic.

35. Show that every stochastic language is c-stochastic for any $0 \leq c \leq 1$.

36. * Give an example of a probabilistic finite automaton, which accepts a non-regular language with the cut-point $\frac{1}{3}$.

37. Design a multihead FA recognizing the languages: (a) $\{a^ib^jc^id^j \mid i,j \geq 1\}$; (b) $\{ww^R \mid w \in \{0,1\}^*\}$.

38. Design LBA recognizing languages: (a) $\{a^i \mid i \text{ is a prime }\}$; (b) $\{ww^R \mid, w \in \{0,1\}^*\}$.

39. Which of the following strings-to-strings functions over the alphabet $\{0,1\}$ can be realized by a finite transducer? (a) $w \rightarrow w^R$; (b) $w_1 \ldots w_n \rightarrow w_1w_1w_2w_2 \ldots w_nw_n$; (c) $w_1 \ldots w_n \rightarrow w_1 \ldots w_nw_1 \ldots w_n$.

QUESTIONS

1. When does the subset construction yield the empty set of states as a new reacheable state?

2. Are minimal nondeterministic finite automata always unique?

3. Is the set of regular languages closed under the shuffle operation?

4. Is the mapping $1^i \rightarrow 1^{F_i}$ realizable by a finite transducer?

5. What is the rôle of initial and terminal distributions for WFA?

6. How can one define WFA generating three-dimensional images?

7. Weighted finite automata and probabilistic finite automata are defined very similarly. Where are the differences?

8. Does the power of two-way finite automata change if we assume that input is put between two endmarkers?

9. Are LBA with several heads on the tape more powerful than ordinary LBA?

10. Which are natural ways to define finite automata on $\omega\omega$-words and how to define in a natural way the concept of regular $\omega\omega$-languages?

3.10 Historical and Bibliographical References

It is surprising that such a basic and elementary concept as that of finite state machine was discovered only in the middle of this century—this concept is usually credited to McCulloch and Pitts (1943). Its modern formalization is due to Moore (1956) and Scott (1959). (Dana Scott received the Turing award in 1976)

Finite automata are the subject of numerous books. For example, Salomaa (1969), Hopcroft and Ullman (1969), Brauer (1984) and Floyd and Beigel (1994). (John E. Hopcroft received the Turing award in 1986 for his contribution in data structures; Robert Floyd in 1978 for his contribution to program correctness.) A very comprehensive but also very special treatment of the subject is due to Eilenberg (1974). See also the survey by Perrin (1990).

Bar-Hillel and his collaborators, see Bar-Hillel (1964), were the first to deal with finite automata in more details. The concept of NFA and Theorem 3.2.8 are due to Rabin and Scott (1959). The proof that there is a NFA with n states such that each equivalent DFA has 2^n states can be found in Trakhtenbrot and Barzdin (1973) or in Lupanov (1963). Minimization of finite automata and Theorem 3.2.16 are due to Huffman (1954) and Moore (1956). The first minimization algorithm, based on two operations, is from Brauer (1988) and credited to van de Snepscheut (1985). Asymptotically the fastest known minimization algorithm, in time $\mathcal{O}(mn \lg n)$, is due to Hopcroft (1971). The pumping lemma for regular language has emerged in the course of time and for two variants and detailed discussion see Floyd and Beigel (1994). For string-matching algorithms see Knuth, Morris and Pratt (1977).

The concepts of regular language and regular expression and Theorem 3.3.6 are due to Kleene (1956). The concept of derivatives of regular languages is due to Brzozowski (1964). Very high lower bounds for the inequivalence problem for generalized regular expressions are due to Stockmeyer and Meyer (1973). The characterization of regular languages in terms of syntactical congruences, Theorems 3.3.16 and 3.3.17, are due to Nerode (1958) and Myhill (1957). The recognition of regular languages in logarithmic time using syntactical monoids is due to Culik, Salomaa, and Wood (1984) and the existence of regular languages for which each processor of the recognizing tree-network of processors has to be huge is due to Gruska, Napoli and Parente (1994).

For two main models of finite state machines see Mealy (1955) and Moore (1956) and a detailed analysis of the two models is in Brauer (1984). The results concerning finite transducers and generalized sequential machines, theorems 3.4.8 till 3.4.11 are due to Ginsburg and Rose (1963, 1966)—see also Ginsburg (1966).

The concept of weighted finite automaton and weighted finite transducer are due to Culik II and his collaborators (Culik and Kari (1993), (1994) and (1995), Culik and Friš (1995), Culik and Rajčáni (1996)). See also Culik and Kari (1995) and Rajčáni (1995) for a survey. Section 3.4.2 and examples, exercises and images are derived from these and related papers. For a more practical "recursive image compression algorithm" see Culik and Kari (1994). The idea to use finite automata to compute continuous functions is due to Culik and Karhumäki (1994); the existence of a function everywhere continuous, having nowhere derivatives and still computable by WFA, is due to Derencourt, Karhumäki, Latteux and Terlutte (1994). An interesting and powerful generalization of WFT, the so-called iterative WFT, have been introduced by Culik and Rajčáni (1995).

The idea to consider finite automata on infinite words is due to Büchi (1960) and McNaughton (1966) and together with the concept of finite automata on infinite trees, due to Rabin (1969), this created the foundations for those areas of computing where one deals with nonterminating processes. For Muller automata see Muller (1963). A detailed overview of computations on infinite objects is due to Thomas (1990). For a presentation of problems and results concerning Gale-Stewards (1953) games see Thomas (1995).

The concept of transition system and Theorem 3.8.1 are due to Myhill (1957). Probabilistic finite automata were introduced by Rabin (1963), Carlyle (1964) and Bucharaev (1964). Theorems 3.8.8 and 3.8.9 are due to Rabin (1963) and the proof of the second theorem presented here is due to Paz (1971). See also Salomaa (1969), Starke (1969) and Bucharaev (1995) for a detailed presentation of probabilistic finite automata.

Two-way finite automata were introduced already by Rabin and Scott (1959) and they made also a sketch of tghe proof of Theorem 3.8.17. A simpler proof is due to Shepherdson (1959)—see also Hopcroft and Ullman (1969). Example 3.8.16 is due to Barnes (1971) and Brauer (1984). For results concerning economy of description of regular languages with two-way FA see Meyer and Fischer (1971). Multihead finite automata were introduced by Rosenberg (1966) and the existence of infinite hierarchies was shown by Yao and Rivest (1978) for the one-way case and Monien (1980) for two-way k-head finite automata.

Deterministic linearly bounded automata were introduced by Myhill (1960); nondeterministic ones by Kuroda (1964). The closure of DLBA under intersection and complement was shown by Landweber (1963), and the closure of NLBA under the complement independently by Szelepcsényi (1987) and Immerman (1988).

Chapter 4

COMPUTERS

INTRODUCTION

The discovery that there are universal computers, and in principle very simple, is the base of modern computing theory and practice. The aim of this chapter is to present and demonstrate main models of universal computers, their properties, mutual relations and various deep conclusions one can draw from their existence and properties. Computer models help us not only to get insights what computers can do and how but also to discover tasks they can not do. The following computer models are considered: several variants of Turing machines, several variants of random access machines, including parallel random access machines, families of Boolean circuits and cellular automata.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. several basic models of universal computers, their properties and basic programming techniques for them,
2. methods how to simulate main models of universal computer on each other,
3. two classes of universal computers that correspond to inherently sequential and inherently parallel computers,
4. main theses of computing: Church's Thesis, Sequential Computation Thesis and Parallel Computation Thesis.

> "There is no use in trying" she said: "one can't believe impossible things."
> "I daresay you haven't had much practice" said the Queen. "When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."
>
> *Lewis Carrol: The Walrus and the Carpenter*

The discovery of universal computers belong to the most important successes of 20'th century science. This discovery can be seen as a natural culmination of a centuries long process of searching for principles and limitations of mind and machines. Amplified by enormous information processing power of matter and advances of modern technology, the discovery of the very simple universal computers resulted very soon in the most powerful tool of mind and mankind.

Several basic models of universal computers are introduced, demonstrated and analyzed in this chapter. Each of them plays an important rôle in some theoretical and methodological considerations in computing. On one side, a large variety of these models demonstrates convincingly the robustness of the concept of universality in computing. On the other side, different models allow to deal in a transparent way with different modes and aspects of computing. Mutual simulations of these models, on which we also concentrate in this chapter, show, on one side, a variety of methods how to transform programs for one universal computer to programs for another universal computers. On the other side, they show that there are actually two main classes of computer models: inherently sequential and inherently parallel.

Each model of a universal computer is essentially a universal programming language. However, these programming languages have too simple control and data structures to be practically useful. On the other hand, their simplicity and elegance makethem excellent tools to discover the laws and limitations of computing and allow to use the exact methods to demonstrate correctness of our findings.

Models of sequential computers seem to be already quite satisfactorily developed. Some of them fully correspond to theory needs, others sufficiently well model real computers and their theoretical analysis provides both theoretically and practically deep insights and useful forecasts. This does not seem to be yet so in the area of parallel computing.

A clear tendency in computer development is to build larger and larger finite machines for larger and larger tasks. Though the detailed structure of larger machines is usually different from smaller ones, there is some uniformity among computers of different size. Computer models therefore consist either of an infinite family of uniformly designed finite computers or this uniformity has to be pursued to the limit and computer models infinite in size (of memory) have to be considered.

The concept of a universal computer demonstrates how little is sufficient to do everything one can do with algorithmic methods. It has turned out that the most important/fruitful way to study the power of various computer models and computational problems is to investigate the amount of computational resources needed to solve problems and to simulate one computer model on another. The main resources are time, storage, processors, program, communication and randomness.

Time is the most natural resource that is potentially unbounded for computers. It is therefore natural to consider as reasonable the cases when the amount of time needed to solve a problem grows with the size of the problem. Storage and processors, for the case of parallel computing, seem, however, to be qualitatively different resources, because their amount is clearly bounded for any real computer. In spite of that it has turned out very fruitful to consider also for these resources that their amount grows with the size of the problem.

With time, storage and processors as resources we deal in this chapter. With (the size of) programs in Chapter 6 and with communication in Chapter 11.

4.1 Turing Machines November 8, 2005

Historically, the very first (infinite) model of computer, and since then of the key importance for theory, invented in 1936 by A. M. Turing[1], one of the fathers of modern computer science and technology, is called, in his honour, **(one-tape) Turing machine**—in short TM—see Figure 4.1. This model serves also as a basis on which several other basic computer and computational models and modes are based, on which complexity theory is developed, and some of the key concepts of modern science are built. The main reason for the enormous importance of this model is its simplicity, elegance and flexibility and the fact that the basic step of Turing machines is indeed elementary, both from the operational and communication point of view.

4.1.1 Basic Concepts

Informally, a one-tape TM is similar to a linearly bounded automaton discussed in Section 3.8, but without any restriction concerning the moves of the head. The head can move, write, and read also outside the cells occupied by the input. This immediately implies that for Turing machines we can apply the basic concepts and also programming techniques introduced for various generalization of finite automata in Section 3.8.

Formally, see Figure 4.1, a (one-tape) TM $\mathcal{M}$ consists of a **bi-infinite tape** divided into an infinite number of cells in both directions and with one distinguished starting cell—or 0-th cell. Cells of the tape can contain any symbol from a finite **tape alphabet** Γ, or a symbol $\sqcup$ (that may also be in Γ), representing an empty cell; a **read-**

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[1] Alan M. Turing (1912-54) an English mathematician. He wrote fundamental papers on computability and artificial intelligence. During the Second World War Turing participated in the cryptographical project ULTRA in Bletchey Park and in the design of Colossus, the first powerful electronic computer. After the war Turing supervised the design and building of ACE, a large electronic digital computer at the National Physical Laboratory. His last and longest papers laid foundation to mathematical biology.

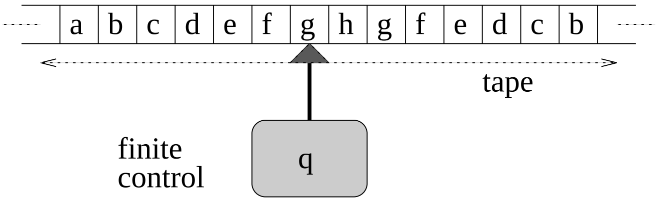


Figure 4.1: One-tape Turing machine

write head positioned at any moment of the discrete time on a cell; a **finite-control unit** being always in one of the states: either from a finite **set** Q **of nonterminating states** (containing the initial state, say q_0) or in one of the **terminal states** from the set $H = \{HALT, ACCEPT, REJECT\}$,—indexACCEPT—textbf and implementing a (partial) **transition function**

$$\delta : Q \times \Gamma \rightarrow (Q \cup H) \times \Gamma \times \{\leftarrow, \downarrow, \rightarrow\}.$$

Interpretation: $\delta(q, x) = (q', x', d)$ means that if $\mathcal{M}$ is in the state q, the head reads x, then $\mathcal{M}$ enters the state q', stores x' in the cell the head is currently on, and the head moves in the direction d, i.e., right if $d = \rightarrow$, left if $d = \leftarrow$, and does not move at all if $d = \downarrow$.[2] Formally, $\mathcal{M} = \langle \Gamma, Q, q_0, \delta \rangle$, but sometimes, if there is a need to consider explicitly a subset $\Sigma \subseteq \Gamma - \{\sqcup\}$ as the input alphabet, we consider a TM $\mathcal{M}$ in the form $\mathcal{M} = \langle \Sigma, \Gamma, Q, q_0, \delta \rangle$.

A **computation** of a TM $\mathcal{M}$ can be defined formally using the concept of **configuration** of the form (q, w, w'), where $q \in Q \cup H$, and $w, w' \in \Gamma^*$. Each configuration contains a complete description of the current **"global" state** of the computation: the state q the machine is in, the content ww' of the tape, and the position of the head—on the first symbol of w' (if $w' \neq \varepsilon$, or on the first cell after last symbol of w). (We assume that only such tapes are used all but finitely many symbols of which are blanks $\sqcup$ and in writing down a configuration infinitely many leftmost and rightmost $\sqcup$'s on the tape are discarded.) If $\mathcal{M}$ moves in one step from a configuration C to a configuration C', we write $C \vdash_{\mathcal{M}} C'$. By writing $C \overset{m}{\underset{\mathcal{M}}{\vdash}} C'$, $(C \overset{*}{\underset{\mathcal{M}}{\vdash}} C')$ we denote that the configuration C yields in m steps (in some finite number of steps) the configuration C'. Each configuration (q_0, ε, w), $w \in (\Gamma - \{\sqcup\})^*$ is called **initial** and each configuration (q, w, w'), $q \in H$ is called **terminating**. A finite sequence of configurations $C_1, C_2, \ldots, C_m$ is called a **terminating computation** of $\mathcal{M}$, if C_1 is an initial configuration, C_m a terminating configuration and $C_i \vdash_{\mathcal{M}} C_{i+1}$ for $i \geq 1$.T(here are two ways how to interpret a terminating computation of a TM $\mathcal{M}$. The first one is that $\mathcal{M}$ stops, and there is no next configuration. This will be called halting. The second one is that $\mathcal{M}$ keeps staying in the same configuration. This will be called an idling termination.) An infinite sequence of configurations $C_1, C_2, \ldots$ such that $C_i \vdash_{\mathcal{M}} C_{i+1}$ for all $i \geq 1$ is called an **infinite computation**.

There are four types of computations of a TM $\mathcal{M}$ when $\mathcal{M}$ starts in an initial configuration (q_0, ε, x)—with the first symbol of x in the cell of the tape the head is on. If $\mathcal{M}$ yields a

---

[2] If conciseness is very important, we use notation YES and NO to denote states $ACCEPT$ and $REJECT$, respectively.

terminating configuration with the state $ACCEPT$ ($REJECT$), then $\mathcal{M}$ is said to **accept** (**reject**) x. If the terminating configuration is (q, w, w'), then $\mathcal{M}$ is said to terminate with the output ww', i.e. $\mathcal{M}(x) = ww'$. Finally, if a computation of $\mathcal{M}$ does not terminate, then we say that $\mathcal{M}$ diverges on the input x, in short $\mathcal{M}(x) = \nearrow$. If $\mathcal{M}$ does not diverge, we say that $\mathcal{M}$ converges, in short $\mathcal{M}(x) = \searrow$.

4.1.2 Acceptance of Languages and Computation of Functions

Turing machines are a natural tool to study the language acceptance and decision problems as well as computation of strings-to-strings functions. This can be easily extended, as we shall see soon, to deal with the computation of integers-to-integers functions. Since finite objects can be encoded by strings this allows to deal with a variety of decision and computational problems.

Definition 4.1.1 *(1) Let $\mathcal{M} = \langle \Sigma, \Gamma, Q, q_0, \delta \rangle$ be a TM with the input alphabet Σ. Then*

$$L(\mathcal{M}) = \{w \mid w \in \Sigma^*, \mathcal{M}(w) = ACCEPT\}$$

is the language, over Σ, **accepted** *by $\mathcal{M}$. In addition, if $\mathcal{M}(x) \in \{ACCEPT, REJECT\}$ for any $x \in \Sigma^*$, then $L(\mathcal{M})$ is said to be the language* **decided (recognized)** *by $\mathcal{M}$. (2) A language $L \subseteq \Sigma^*$ is said to be* **recursively enumerable**, *if there is a TM $\mathcal{M}$ that accepts $L = L(\mathcal{M})$, and it is called* **recursive**, *if there is a TM that decides (recognizes) L.*

The distinction between the concepts of recursivity and recursive enumerability of languages is, as we shall see, important and essential. For any recursive language $L \subseteq \Sigma^*$ there is a TM $\mathcal{M}$ that terminates for any input $x \in \Sigma^*$ and always says whether $x \in L$ or not—one only has "to wait patiently". For a recursively enumerable language $L \subseteq \Sigma^*$, it is only guaranteed that there is a TM $\mathcal{M}$ such that $\mathcal{M}$ certainly stops and accepts for any $x \in L$. However, $\mathcal{M}$ may or may not stop for $x \notin L$—and one has no idea how long to wait in order to learn that $\mathcal{M}$ halts or that it does not halt.

Definition 4.1.2 *(1) A (partial) strings-to-strings function $f : \Sigma^* \to \Sigma^*$ is said to be (partially) computable by a TM $\mathcal{M} = \langle \Sigma, \Gamma, Q, q_0, \delta \rangle$, $\Sigma \subseteq \Gamma$, if $\mathcal{M}(x) = f(x)$ for any $x \in \Sigma^*$ from the domain of f and $\mathcal{M}(x) = \nearrow$, otherwise.*

(2) If there is a TM $\mathcal{M}$ that (partially) computes a function $f : \Sigma^* \to \Sigma^*$, then f is called **(partially) recursive**.

(3) A function $f : \mathbf{N}^t \to \mathbf{N}^s$ is called **(partially) recursive**, *if there is a TM $\mathcal{M}$ such that $f(x_1, \ldots, x_t) = (y_1, \ldots, y_s)$, if and only if*

$$\mathcal{M}(1^{x_1+1}01^{x_2+1}0 \ldots 01^{x_t+1}) = 1^{y_1+1}0 \ldots 01^{y_s+1}.$$

Exercise 4.1.3 *A TM, as defined above, can perform in one step three actions; a state change, a writing and a head move. Show that to each TM $\mathcal{M}$ we can design a TM $\mathcal{M}'$ which performs in each step at most two of these three elementary actions and: (a) accepts the same language as $\mathcal{M}$; (b) computes the same function as $\mathcal{M}$.*

Exercise 4.1.4 *Explore the possibility that to each TM $\mathcal{M}$ we can construct another TM $\mathcal{M}$" that behaves "essentially as $\mathcal{M}$" and in each move performs only one of the three elementary actions.*

In the following examples we illustrate three basic ways how to specify a TM. They are similar to those used to describe finite automata: transition tables, enumeration of transition tuples and state-graphs.

Example 4.1.5 *The TM $\mathcal{M}_1$ described by the transition table in Figure 4.2a decides whether an input $x \in \{0,1\}^*$ is a* **palindrome**. *Informally, starting in the initial state q_0, $\mathcal{M}_1$ reads the first symbol of the word on the tape, erases this symbol, enters one of the states r_0 or r_1, depending on the symbol read, and moves one cell to the right. If now $\mathcal{M}_1$ reads $\sqcup$, then $\mathcal{M}_1$ accepts, otherwise $\mathcal{M}_1$ goes from the stae r_0 (r_1) to the state r_0' (r_1') and moves to the right end of the input string. When coming to the first cell with $\sqcup$, $\mathcal{M}_1$ moves one symbol to the left and goes from the state r_0 (r_1) to the state s_0 (s_1). If $\mathcal{M}_1$ reads 0 (reads 1) in the state s_0 (in s_1), then $\mathcal{M}_1$ replaces the symbol being read by $\sqcup$, goes to the state l, and, being in the state l, $\mathcal{M}_1$ keeps moving left until a $\sqcup$ is reached. $\mathcal{M}_1$ then moves the head one cell to the right, goes to the state q_0 and repeats the procedure. If $\mathcal{M}_1$ reads 1(0) in the state s_0 (s_1), then $\mathcal{M}_1$ rejects. If $\mathcal{M}_1$ reads $\sqcup$ in the state q_0, then $\mathcal{M}_1$ accepts.*

state	$\sqcup$	0	1
q_0	$YES, \sqcup, \downarrow$	$r_0, \sqcup, \rightarrow$	$r_1, \sqcup, \rightarrow$
r_0	$YES, \sqcup, \downarrow$	$r_0', 0, \rightarrow$	$r_0', 1, \rightarrow$
r_1	$YES, \sqcup, \downarrow$	$r_1', 0, \rightarrow$	$r_1', 1, \rightarrow$
r_0'	$s_0, \sqcup, \leftarrow$	$r_0', 0, \rightarrow$	$r_0', 1, \rightarrow$
r_1'	$s_1, \sqcup, \leftarrow$	$r_1', 0, \rightarrow$	$r_1', 1, \rightarrow$
s_0	-	$l, \sqcup, \leftarrow$	$NO, \sqcup, \downarrow$
s_1	-	$NO, \sqcup, \downarrow$	$l, \sqcup, \leftarrow$
l	$q_0, \sqcup, \rightarrow$	$l, 0, \leftarrow$	$l, 1, \leftarrow$

(a)

q_0	1	q'	$\sqcup$	$\rightarrow$
q'	0	HALT	$\sqcup$	$\downarrow$
q'	1	q''	$\sqcup$	$\rightarrow$
q''	0	HALT	1	$\downarrow$
q''	1	q''	1	$\rightarrow$

(b)

Figure 4.2: Turing machines recognizing palindromes and computing $x + y$

Exercise 4.1.6 *The head movement of the TM from Figure 4.2a is depicted in Figure 4.3a. Design another (faster) Turing machine with the head movement as shown in Figure 4.3b.*

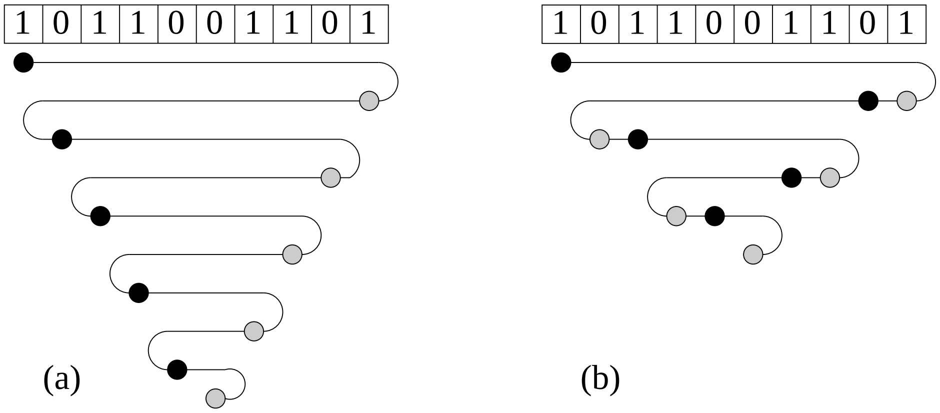


Figure 4.3: Movement of heads when recognizing palindromes-80%

Example 4.1.7 *TM $\mathcal{M}_2$, described by an enumeration of transitions in Figure 4.2b, computes the function $f(x, y) = x + y$. Indeed, $\mathcal{M}_2$ erases on the input $1^{x+1}01^{y+1}$ the first 1 and also the next 0, if $x = 0$; otherwise, $\mathcal{M}$ erases also the second 1 in 1^{x+1} and moves to find the first 0. $\mathcal{M}$ replaces this 0 by 1 and halts.*

Example 4.1.8 *TM $\mathcal{M}_3$ depicted by the state diagram in Figure 4.4a computes, given an integer n, the Fibonacci number F_n using the recurrence 1.6, i.e. $\mathcal{M}_3$ produces for an input 0^{n+1} the output $0^{F_n‘+1}$, where $F'_0 = 0$, $F'_1 = 1$ and $F'_{n+1} = F'_n + F'_{n-1}$, for $n > 1$.*

Example 4.1.9 *TM $\mathcal{M}_4$ in Figure 4.4b generates the Fibonacci ω-word, see Section 2.4.1, defined by $F(1) = ab, F(i) = h(F(i-1))$ for $i \geq 2$, where $h(a) = ab, h(b) = a$. (The verb "generates" means here that $\mathcal{M}_4$, starting on the empty tape, keeps writing on its tape larger and larger words and their prefixes from the alphabet $\{a, b\}$ form larger and larger prefixes of the Fibonacci ω-word.)*

Exercise 4.1.10 *Design a TM that multiplies two integers, i.e. from the input $1^{x+1}01^{y+1}$ produces the output 1^{xy+1}.*

Exercise 4.1.11 *Design a TM that generate the Thue ω-word.*

Exercise 4.1.12 *Design a TM that recognizes whether a given integer is a prime.*

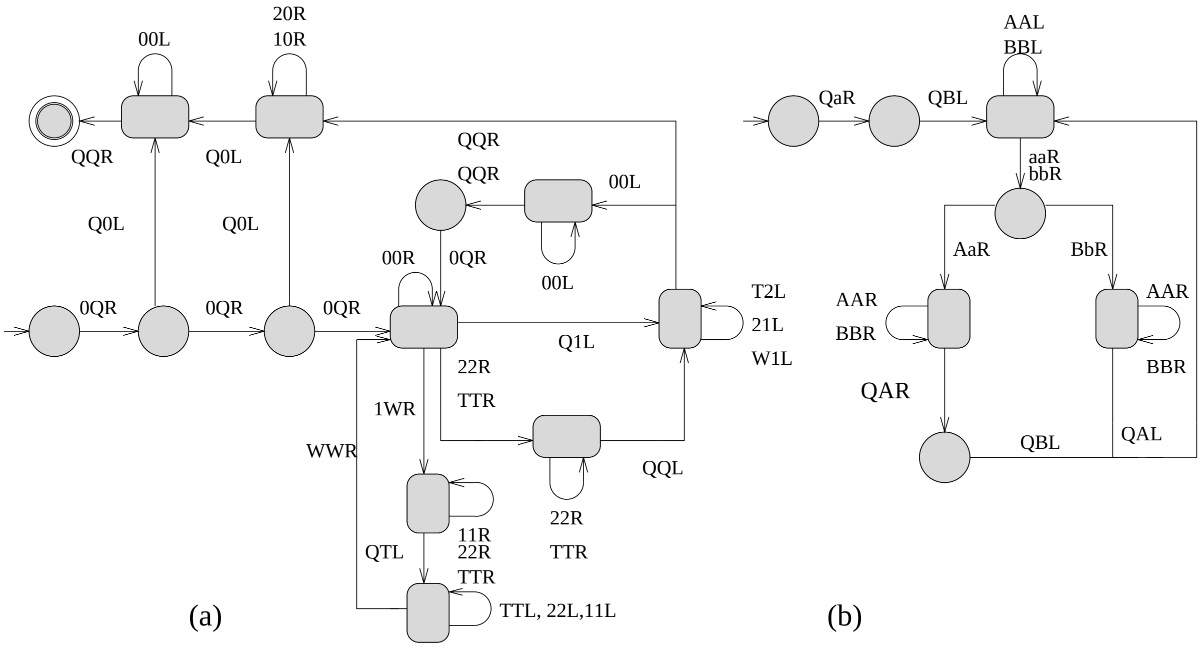


Figure 4.4: Turing machines computing Fibonacci numbers and generating Fibonacci ω-words

4.1.3 Programming Techniques, Simulations and Normal Forms

There are several basic techniques that are useful for the design of Turing machines. Dividing a tape into tracks and carrying symbols in the states are perhaps two most basic ones. (We have actually already encountered them in Section 3.8).

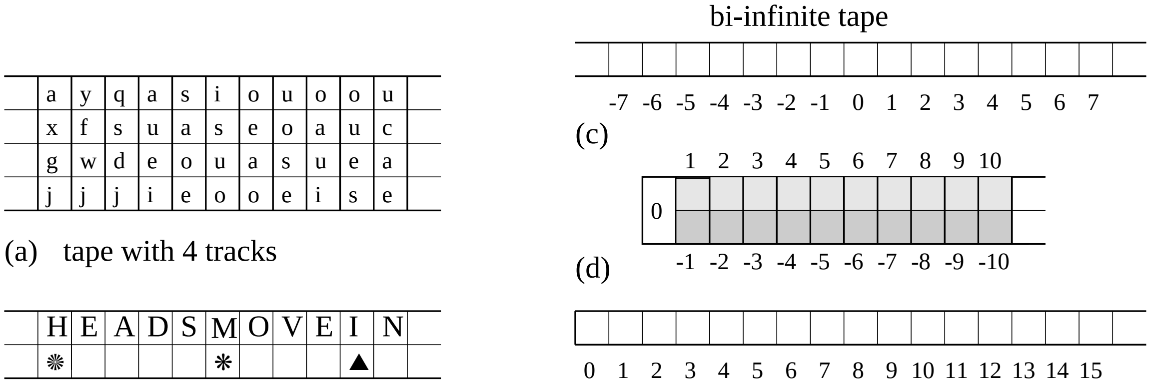
Dividing a tape into tracks. If Σ is a tape alphabet, then a division of a tape into k tracks, see Figure 4.5a for $k = 4$, actually corresponds to using the alphabet Σ^k instead of Σ. Informally, we can speak in such a case about "changing a symbol of the i-th track" but this actually means to replace a symbol from Σ^k by another one which differs only in the i-th component. One use of this techniques is to leave markers on cells as demonstrated in Figure 4.5b.

Carrying symbols in states. States can be used to "carry symbols" or information about symbols from one part of the tape to another. Formally, this is done by creating "compound states" (q, a), where q is an "elementary" state and a is a symbol to be carried out. (This technique has actually been used in the Example 4.1.5.)

Example 4.1.13 *A TM with the states* $\{q, q', (q, 0), (q, 1)\}$ *and transitions*

$$\begin{array}{ll} q, 0, (q, 0), \sqcup, \rightarrow, & q', j, q', i, \leftarrow, \\ q, 1, (q, 1), \sqcup, \rightarrow, & q', \sqcup, q, \sqcup, \rightarrow, \\ (q, i), j, (q, i), j, \rightarrow, & q, c, HALT, \sqcup, \downarrow, \\ (q, i), \sqcup, i, q', \leftarrow, & \end{array}$$

where i *stands for any element from the set* $\{0, 1\}$ *and* j *stands for any element from the set* $\{0, 1, c\}$*, transforms any input of the form* ucw*, where* $uw \in \{0, 1\}^*$*, into the form* wu*.*



(a) tape with 4 tracks

(b) putting marks

(c)

(d)

(e) one-way infinite tape

Figure 4.5: Tapes with tracks and a folding of a bi-infinite tape-70%

Simulation. One of the most important concept for dealing with various models of computers is that of simulation of one machine $\mathcal{M}$ on another machine $\mathcal{M}'$. In the strong sense, this means that $\mathcal{M}'$ simulates steps of $\mathcal{M}$ in such a way that from a sequence of configurations representing a computation of $\mathcal{M}'$ one can easily obtain a sequence of configurations representing a computation of $\mathcal{M}$ for the same input. A very formal and fully general definition of this intuitive concept would be cumbersome and therefore we shall use this concept quite informally. In any particular case it will be clear what a simulation means. More restricted and more precise versions of simulation will be introduced later.

Turing machines with one-way infinite tapes. The most often considered normal form of TM is that with one-way infinite tape—see Figure 4.5e. Such a TM can be seen as an ordinary Turing machine the head of which never moves left from its starting position. It is easy to show that each TM $\mathcal{M}$ can be transformed to another TM $\mathcal{M}'$, with one way infinite tape only, that simulates $\mathcal{M}$ in the same number of steps. The basic idea of such a construction is illustrated in Figure 4.5d: to fold the tape and then to consider a folded tape as a two-track tape.

Exercise 4.1.14 *Show in detail how to transform any TM working on a bi-infinite tape to a TM working on a one-way infinite, two-track tape.*

Turing machines with end-markers. To each TM $\mathcal{M} = \langle \Sigma, \Gamma, Q, q_0, \delta \rangle$ we can easily design a TM $\mathcal{M}' = \langle \Sigma, \Gamma \cup \{\#, \$\}, q_0, Q', \delta' \rangle$ which for an input $\#w\$$ with two endmarkers $\#$ and $\$$ behaves in a similar way as $\mathcal{M}$ does on w. However, each time $\mathcal{M}'$ reaches an endmarker and $\mathcal{M}$ at that moment would write or move outside the area between markers $\#$ and $\$$, $\mathcal{M}'$ first moves the marker accordingly and then performs the step $\mathcal{M}$ would do.

Exercise 4.1.15 *Show that to each TM $\mathcal{M}$ we can design another TM $\mathcal{M}'$ that simulates $\mathcal{M}$ and in no state it has a possibility to move both righ and left (i.e. in no state it has transitions of the type $qaq'b \rightarrow$ and also $qcq''d \leftarrow$).*

4.1.4 Church's Thesis

Alan Turing defined his machine model in order to formalize the intuitive concept of the **effective procedure** or, as we call it today, **algorithm**.[3]

Intuitively, an effective procedure is a *set of rules which tell us, from step to step, precisely what to do*. For example, an effective procedure to compute a function f is a set of rules which tell us, precisely what to do, from step to step, in order to obtain, from the input arguments of f, the resulting value of f after a finite number of steps.

The problem to have an exact definition of an effective procedure becomes of crucial importance when one needs to show that there is no effective procedure to solve a given problem. It has actually never been a large dispute whether a specific description is really an effective procedure. This is the main reason why an attempt to formalize such a very basic concept of science got urgent only at the beginning of this century.

A need to have an exact definition of an effective procedure actually arose already in connection with the three famous problems of antiquity: **the squaring of the circle, the duplication of the cube** and **the trisection of the angle**—using a straightedge and a compass only. However, these problems were very special and there was actually only the intellectual curiosity behind them. The problem to formulate an intuitive concept of effective procedure started to be of prime importance only at the beginning of this century, in connection with the Hilbert's famous project to formalize all our knowledge and his *Entscheidungsproblem*: given a reasonably powerful formal system, can we decide whether a given statement has a proof?

It is evident that a TM specifies an effective procedure and that any function computable by a Turing machine is effectively computable in an intuitive sense. The concept of a TM is therefore clearly a formalization of the intuitive concept of an effective procedure. However, since this is a very simple concept, it is natural to bother whether this formalization is not too narrow. In other words, are there algorithms or effective procedures that can not be carried out by a Turing machine? Are there functions that are effectively computable but not by a Turing machine?

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[3] The name for this intuitive concept of algorithm is considered as being derived from the name of Arab astronomer and mathematician Mohammed ibn Musa al-Khovârizmi (9th century), considered also as the father of algebra—this name being derived from the titel *Al-jabr wa'l mugābalah* of one of his two books that played an important rôle in the history of mathematics and computing. In this book he provides algorithms to solve various linear and quadratic equations. His description of algorithms is entiretily rhetorical and deals always with particular equations. For example, the description of the procedure to solve the equation $x^2 + 10x = 39$ goes as follows: "Take half the number of roots, that is, in this case five, then multiply this by itself and the result is five and twenty. Add this to nine and thirty, which gives sixty four; take the square root, or eight, and subtract from it half the number of roots, namely five, and there remains three. This is the root."

One of the cornerstones not only of computing but actually of all modern science is the thesis which is usually called the Church's Thesis (or Church-Turing's Thesis).[4]

> **Church's Thesis.** Any process which could naturally be called an effective procedure can be realized by a Turing machine.

This thesis identifies a formal and an intuitive concept and therefore can not be proven formally. There is, however, such an overwhelming evidence supporting this thesis that one can think about it as being a sort of natural law. The evidences supporting the thesis are of three types.

1. There have been developed many other formalizations of the concept of effective procedure, based on very different ideas, and all of them have turned out to be provably equivalent: μ-recursive functions of A. C. Kleene (1936), λ-definable functions of A. Church (1941), rewriting systems of E. L. Post (1936), normal algorithms of A. A. Markov (1954) and several others.
2. The class of functions computed and the class of languages accepted or decided by Turing machines are very robust. Numerous modifications of the concept of Turing machines had no effect on these classes. Moreover, they are closed under all operations they should intuitively be closed.
3. No effective procedure is known that would not have its formal counterpart in terms of Turing machines. There are no indications that there could exist an effective procedure that could not be programmed as a Turing machine.

Church's Thesis can be used, and it is often used, as an essential tool in proofs and we will do that also in this book. Once something can be done by an effective procedure we can assume that there is a Turing machine capable of doing it (after a proper encoding of the problem). This can be always replaced by a formal proof, though it can be a very tedious task. A formal construction of a TM resembles designing a program in an assembly code. Arguments based on Church's Thesis resemble programming using a very high-level programming language.

Church's Thesis is also an expression of limitations of the concept of an effective procedure. It is therefore of importance to look at this thesis also from a point of view of modern physics and to ask the question whether there may be something *beyond Turing machine.*

There is an important physical principle underlying the Church's Thesis that can be formulated as follows: *Every finitely realizable physical system can be perfectly simulated by a model of a computing machine operating by finite means.*

Since classical physics is continuous and Turing machines are discrete the above principle is not obeyed by them. On the other hand it is possible to develop a quantum generalization of Turing machines such that this machine and quantum physics are compatible with the principle.

One reason for assuming that there may be something beyond Turing machines are problems encountered when trying to understand the mind. There are serious views, though not

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[4] Alonzo Church (1903-1994), a US logician.

shared by everybody, that the mind is more than a Turing machine and also that the current physical theories are insufficient to comprehend the mind—see Penrose (1990).

4.1.5 Universal Turing Machines

As an illustration of the power of Church's thesis we establish the existence of a universal Turing machine—a specific Turing machine $\mathcal{M}_u$ that can simulate any other Turing machine, if properly encoded. The existence of universal computers is one of the cornerstones on which the power of modern computers and computing is based. An effective encoding and enumeration of Turing machines plays by that an important rôle.

There are many ways how a TM $\mathcal{M} = \langle \Gamma, Q, q_1, \delta \rangle$, where $\Gamma = \{a_1, a_2, \ldots, a_n\}$, $Q = \{q_1, \ldots, q_m\}$, $H = \{q_{m+1}, q_{m+2}, q_{m+3}\}$, can be encoded by a binary string. For example,

$$\text{a transition } \delta(q_i, a_j) = (q_k, a_l, d) \text{ can be encoded as } 0^i10^j10^k10^l10^{d'},$$

where $d' = 1$ if $d = \leftarrow$, $d' = 2$ if $d = \downarrow$, and $d' = 3$ if $d = \rightarrow$. The whole TM $\mathcal{M}$ can then be encoded by the string

$$\langle \mathcal{M} \rangle = 10^n110^m11\underbrace{0^{i_1}10^{j_1}10^{k_1}10^{l_1}10^{d'_1}}_{\text{first transition}}11\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{\text{other transitions}}111.$$

and a $w \in \Gamma^*$, $w = a_{i_1} \ldots a_{i_k}$ can be encoded by $\langle w \rangle = 0^{i_1}10^{i_2}1 \ldots 10^{i_k}$. These encodings will be often used.

Remark 4.1.16 The above encoding can be generalized to all Turing machine by assuming that tape symbols are from an infinite alphabet $\Gamma = \{a_1, a_2, a_3, \ldots\}$ and states are also from an infinite set $Q = \{q_1, q_2, q_3, \ldots\}$ with the first three states being three halting states.

Notice also that the string "11" is used as a marker between two components and "111" as the suffix endmarker. Observe also that such an encoding of TM has the so-called **self-delimiting property**: no prefix of a code of a TM is again a code of another TM.

With a fixed encoding of Turing machines, such as one above, we can use a strict ordering of encodings of all Turing machines and enumerate all Turing machines as $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$ For the rest of the book the actual details of such encodings will not be mostly of importance. The only requirement, clearly fulfilled by the above encoding, will be that there is an effective procedure for determining the i-th Turing machine TM_i given i, and for determining i such that $TM_i = \mathcal{M}$, given $\mathcal{M}$. Any such enumeration of TM will be called a **Gödel numbering** of Turing machines, and the number i such that $TM_i = \mathcal{M}$ will be called the **index** or **Gödel number** of $\mathcal{M}$, with respect to a choosen Gödel numbering. Sometimes it will be required that encodings have the self-delimiting property.

Let us now fix an integer k and denote, for any integer i, by f_i the partial recursive function of k integer variables computed by $\mathcal{M}_i$, as defined in Definition 4.1.2. The number i is then referred to as an index of f_i. Clearly, any partial recursive function has infinitely many indices.

We show now that for any k there is a single Turing machine $\mathcal{M}_u^k$ that is universal for computing of all partial recursive functions of k variables in the following sense: If $\mathcal{M}_u^k$ gets on the input an index i and the values $x_1, \ldots, x_k$, then $\mathcal{M}_u^k$ computes $f_i(x_1, \ldots, x_k)$.

Theorem 4.1.17 (The existence of a k-universal Turing machine) *For every $k \geq 1$ there is an integer u_k such that for any $i \geq 1$, and all $x_1, \ldots, x_k$, $f_i(x_1, \ldots, x_k) = f_{u_k}(i, x_1, \ldots, x_k)$.*

Proof. Consider the following informal algorithm to compute a function of $k+1$ variables $i, x_1, \ldots, x_k$. Construct $\mathcal{M}_i$ and use it to compute with the arguments $x_1, \ldots, x_k$ as the inputs. If the computation halts, output the final result of computation. By Church's Thesis, this algorithm can be carried out by a Turing machine $\mathcal{M}_u$, and this u is the index of the universal partial recursive function of $k+1$ variables to compute any partial recursive function of k variables. □

In Section 4.1.7 we discuss another variant of the above theorem and show in more detail how to design a universal Turing machine capable to simulate efficiently any other Turing machine. Completely detailed constructions of universal Turing machines can be found in literature—for example in Minsky (1967). It is interesting to see such a construction, though one does not learn from that much more than from the above proof based on the Church's Thesis.

Because of the enormous power of universal Turing machines one is inclined to expect that they have to be quite complicated. Actually, just the opposite is true and the search for **minimal universal Turing machines** has demonstrated that.

Intellectual curiosity is main but not the only reason why the problem of finding minimal universal Turing machine is of interest. Extreme micro-applications, search for principles and power of genetic information processing, as well as the tendency to minimize size and maximize performance of computers are additional reasons behind.

A nontrivial problem is what to choose as a complexity measure of Turing machines with respect to which one should try to find a minimal universal TM. Number of states? Number of tape symbols? The following theorem indicates that this is not the way to go.

Theorem 4.1.18 *There is a universal Turing machine that has only two nonterminating states, and there is another universal Turing machine that uses only two tape symbols.*

Exercise 4.1.19 * *Show that to any TM $\mathcal{M}$ there is another TM $\mathcal{M}'$ that uses only two states and computes the same integers-to-integers functions as $\mathcal{M}$.*

Exercise 4.1.20 *Show that no one-state Turing machine can be universal.*

A better reflection of the intuitive concept of the **size** of Turing machines is their **product complexity**:

$$\text{number of states} \times \text{number of tape symbols},$$

or the total number of transitions. Concerning the product complexity the problem of finding a minimal universal Turing machine is still open and currently the best upper and lower bounds are summarized in the following theorem:

Theorem 4.1.21 *There is a universal Turing machine with product-complexity* 24 (4 *states and* 6 *symbols and* 22 *transitions) and there is no universal Turing machine with product-complexity smaller than* 7.

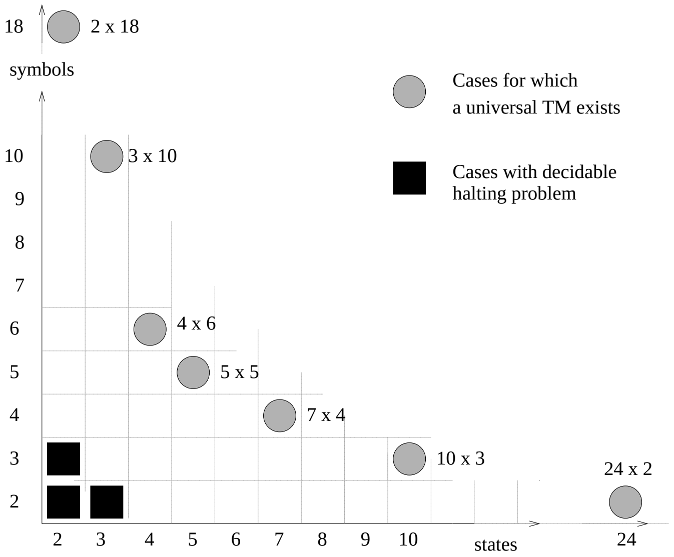

Figure 4.6: Minimal Turing machines-45%

Figure 4.6 shows, for different number of states (tape symbols), the current minimal number of tape symbols (states) needed to design a universal Turing machine. Figure 4.7 contains the transition tables of three universal Turing machines: one with 2 states and 18 symbols, one with 4 states and 6 symbols, and one with 24 states and two symbols. (To describe the movements of the heads, the symbols R instead of $\rightarrow$ and L instead of $\leftarrow$ are used.) The way Turing machines and inputs are encoded is the key point here in which much of the complexity is hidden. (All these machines achieve universality by simulating a tag-system—see Section 7.1.)[5]

Remark 4.1.22 The existence of a **universal quantum computer** (universal quantum Turing machine) Q_u has also been shown. Such a universal quantum computer has the property that for each physical process $\mathcal{P}$ there is a program that makes Q_u to perform that process. In particular, the universal quantum computer can, in principle, perform any physical experiments.

4.1.6 Undecidable and Unsolvable Problems

A decision problem is called **undecidable** if there is no algorithm (Turing machine) to decide it. A search problem is called **unsolvable** if there is no algorithm (Turing machine) to solve it.

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[5] In order to see the merit of these results it is worth to remember that the first really powerful electronic computer ENIAC had 18000 lamps, 70000 capacitors, 60 tons weight, and 30m length.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
q_0	$q_0 17L$	$q_0 4R$	$q_0 17L$	$q_0 0R$	$q_0 3L$	$q_0 7R$	$q_0 9R$	$q_0 5L$	$q_0 5R$	$q_0 8L$	$q_1 11L$	$q_1 8L$	$q_1 1L$	$q_0 14L$	$q_0 15L$	$q_1 16R$	—	$q_0 2R$
q_1	$q_1 2R$	$q_1 2R$	$q_1 1L$	$q_1 4R$	$q_1 0L$	$q_0 10R$	$q_1 7R$	$q_1 6L$	$q_1 9R$	$q_1 6L$	$q_0 5R$	$q_1 9R$	$q_1 14R$	$q_1 14R$	$q_1 13L$	$q_1 17R$	$q_0 17L$	$q_1 12L$

UTM(2,18), Rogozhin, 1995

UTM(4,6), Rogozhin, 1982

UTM(24,2), Rogozhin, 1982

	q_0	q_1	q_2	q_3
0	$q_0 3L$	$q_1 4R$	$q_2 0R$	$q_3 4R$
1	$q_0 2R$	$q_2 2L$	$q_3 3R$	$q_1 5L$
2	$q_0 1L$	$q_1 3R$	$q_2 1R$	$q_3 3R$
3	$q_0 4R$	$q_1 2L$	—	—
4	$q_0 3L$	$q_1 0L$	$q_0 5R$	$q_1 5L$
5	$q_3 4R$	$q_1 1R$	$q_0 0R$	$q_3 1R$

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}	q_{15}	q_{16}	q_{17}	q_{18}	q_{19}	q_{20}	q_{21}	q_{22}	q_{23}
0	$q_4 0R$	$q_0 1R$	$q_3 0L$	$q_{11} 1L$	$q_0 1R$	$q_6 0L$	$q_7 0L$	$q_6 0L$	$q_{18} 0R$	$q_3 1L$	$q_3 0L$	$q_{18} 0R$	$q_9 0R$	$q_{14} 0L$	$q_{15} 0R$	$q_{14} 0R$	$q_{15} 0R$	$q_{18} 0R$	$q_2 1L$	$q_{17} 1R$	$q_{21} 0R$	$q_9 1L$	$q_{20} 1R$	$q_{12} 0R$
1	$q_1 1R$	$q_2 1L$	$q_1 0L$	$q_8 0L$	$q_5 0L$	$q_6 1L$	$q_5 0L$	$q_1 1R$	$q_3 1L$	$q_{12} 0R$	—	$q_{13} 1L$	$q_{23} 1R$	$q_{10} 1L$	$q_{16} 1R$	$q_9 1R$	$q_{20} 1R$	$q_{19} 1R$	$q_{17} 1R$	$q_{17} 0R$	$q_{22} 1R$	$q_{20} 1R$	$q_{20} 0R$	$q_2 0L$

Figure 4.7: Three minimal universal TM

We show first undecidability of two basic problems concerning Turing machines. By doing that we assume that a Gödel numbering of Turing machines is fixed.

- The **self-applicability problem** is to decide, given an integer i, whether the i-th Turing machine halts on the input i, i.e., whether $TM_i(i) = \searrow$.
- The **halting problem** is to decide, given a Turing machine $\mathcal{M}$ and an input w whether $\mathcal{M}$ halts on w.

Theorem 4.1.23 *The self-applicability problem and the halting problem are undecidable for Turing machines.*

Proof. Let us define the function

$$f(n) = \begin{cases} \mathcal{M}_n(n) + 1, & \text{if } \mathcal{M}_n \text{ converges for the input } 1^n; \\ 0, & \text{otherwise.} \end{cases}$$

If either self-applicability or halting problem are decidable, then f is computable and, by the Church's thesis, there is an m such that $f(n) = \mathcal{M}_m(n)$. In such a case $f(m) = \mathcal{M}_m(m) = \mathcal{M}_m(m) + 1$—a contradiction that implies that neither self-applicability nor halting problem are decidable. $\Box$

Remark 4.1.24 The proof of Theorem 4.1.23 is based on the so-called **diagonalization method**. First, an infinite matrix M is defined the rows and columns of which are labeled by integers and $M(i,x)$ is the value of the i-th Turing machine for the input 1^{x+1}. The diagonal of the matrix $M(i,j)$ is then considered and a function f is constructed such that $f(i) \neq M(i,i)$, for all i.

The unsolvability case—the existence of a well defined but not computable function will now be demonstrated by the so-called **Busy Beaver Function** BB. $BB(n)$ is the maximal number of 1's a Turing machine with n states and two symbol tape alphabet $\{\sqcup, 1\}$ can write on the tape when starting with the empty tape and terminating after a certain number of steps. Since the number of such TM is finite, $BB(n)$ is well defined and $BB(n) \leq BB(n+1)$ for all n.

Theorem 4.1.25 *For any total recursive function f and any sufficiently large x the inequality $f(x) < BB(x)$ holds. (As a consequence, the Busy Beaver Function is not recursive.)*

Proof. Given any recursive function $f(x)$ let us consider the function

$$g(x) = max\{f(2x+2), f(2x+3)\}. \tag{4.1}$$

Clearly, the function g is total and therefore by Church's Thesis and Theorem 4.1.18 there is a Turing machine $\mathcal{M}_g$ with the tape alphabet $\{\sqcup, 1\}$ computing g. Let $\mathcal{M}_g$ have m states.

For each integer x we can easily construct a Turing machine $\mathcal{M}_x$ such that when $\mathcal{M}_x$ starts on the empty tape, it first writes 1^{x+1} on the tape, then moves to the leftmost 1 and starts to work as $\mathcal{M}_g$. Clearly, $\mathcal{M}_x$ with this property can be so designed that it has $n = m + x + 2$ states and uses $\{\sqcup, 1\}$ as the tape alphabet. When started on the blank tape $\mathcal{M}_x$ halts with precisely $g(x)$ symbols 1 on the tape. Thus,

$$g(x) < BB(m + x + 2)$$

and for $x = k \geq m$, we get $g(k) < BB(2k+2)$ and therefore

$$f(2k+2) < BB(2k+2), \qquad f(2k+3) < BB(2k+3).$$

Thus $f(x) < BB(x)$, for $x \geq m$. □

It is known that $BB(1) = 1, BB(2) = 4, BB(3) = 6, BB(4) = 13$ and Turing machines that achieve these maximal values are shown in Figure 4.8a,b,c,d, where 0 is written instead of blank symbol. For larger n only the following lower bounds are currently known: $BB(5) \geq 4098, BB(6) \geq 136612, BB(8) \geq 10^{44}, BB(12) \geq 6x^4_{164}$, where $x_0 = 4096$ and $x_i = x^{4096}_{i-1}$ for $i \geq 1$. TM in Figure 4.8b, c, d and e achieve writing the indicated number of 1's in 4, 11, 96 and 47176870 steps, respectively.

	A
0	1H
1	

(a)

	A	B
0	1BL	1AR
1	1BR	H

(b)

	A	B	C
0	1BR	1CR	1AL
1	1CL	1H	0BL

(c) Lin, Rado, 1963

	A	B	C	D
0	1BR	1AL	1H	1DL
1	OCR	1AR	1DR	OBL

(d) Weimann, Casper, Fenzl 1973

	A	B	C	D	E
0	1BR	1CR	1DR	1AL	1H
1	1CL	1BR	0EL	1DL	0AL

(e) Marxen, Buntrock 1990

Figure 4.8: Turing machines computing the busy beaver function for $n = 1, 2, 3, 4, 5$ ("H" stand for the halting state HALT

Exercise 4.1.26 ** *Verify that Marxen-Buntrock's TM really needs 47176870 steps to write 4098* 1*'s.*

Exercise 4.1.27 * *Get, by designing a TM, an as good lower bound for* $BB(6)$ *as you can.*

The existence of undecidable and unsolvable problems belong to the main discoveries of science of this century with various applications concerning the limitations of our knowledge. This will be discussed in more details in Chapter 6, where also more insights into the structure of decidable and solvable problems as well as several examples of undecidable and unsolvable problems are discussed.

4.1.7 Multitape Turing Machines

There are many generalizations of one-tape Turing machines. Two main schemas of such generalizations are shown in Figure 4.9: **off-line Turing machines**, Figure 4.9a, and **on-line Turing machines**, Figure 4.9b. In both cases the Turing machine has an **input tape**, an **output tape** with a write-only head moving from the left to the right only, a **control unit** connected by heads with the input tape, the output tape and the "memory" (or storage). The memory S has a potentially infinite number of cells. Each of them can contain a symbol of a finite alphabet. Cells of S are interconnected by some regular interconnection network (graph). A **configuration** of such a machine is determined by its state, contents of the memory cells and the positions of heads. A **step** is determined by the current state and by the symbols the heads read. A step results in a state change, a replacement of symbols in cells of the memory on which the heads are at that moment, and the moves of the heads to the neighbouring cells—along the interconnection structure of S.

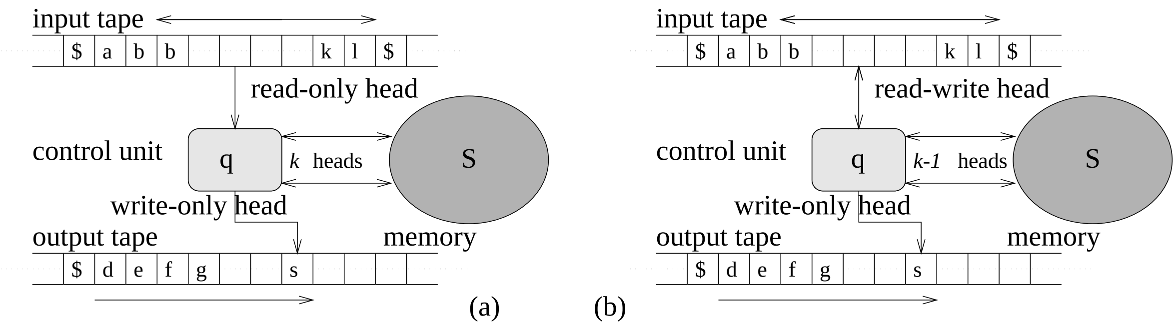

Figure 4.9: Off-line and on-line Turing machines-65%

Four interconnection schemes for memory are illustrated in Figure 4.10: multitape Turing machines with one head on each tape (Figure 4.10a), a one-tape multihead Turing machine (Figure 4.10b), a multihead Turing machine with a two-dimensional tape (Figure 4.10c), and a Turing machine with a tree-structured memory (Figure 4.10d).

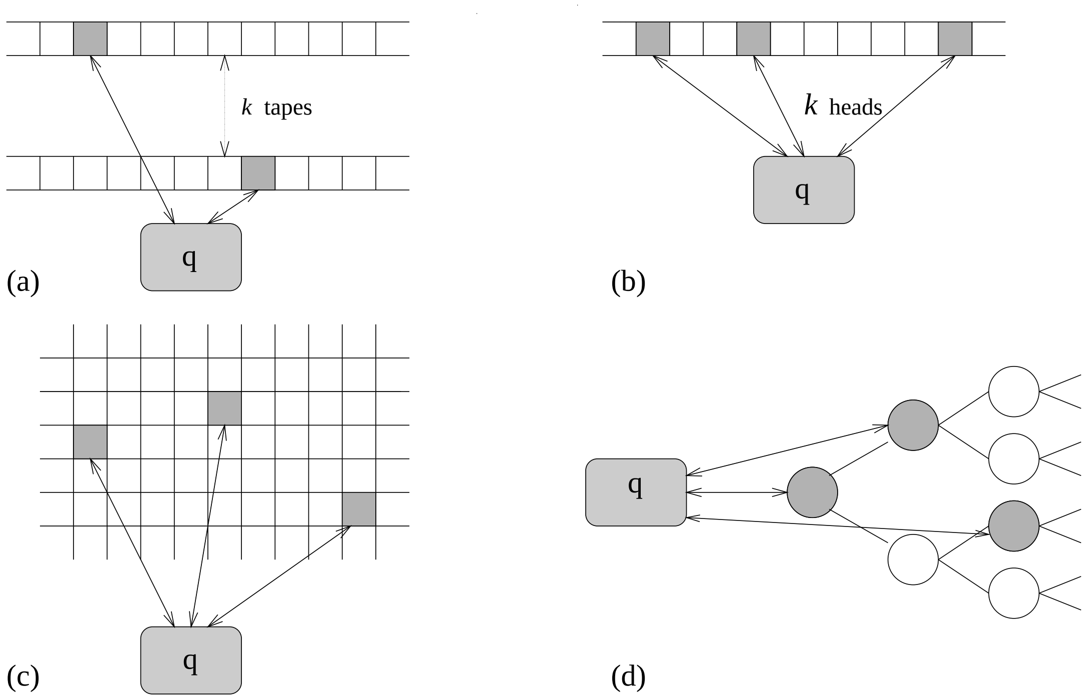

Figure 4.10: Four-types of Turing machines–70%

On-line and off-line versions differ only in the way the input tape is processed. In off-line Turing machines the input tape head is a read-only head that can move in both directions.[6] In on-line models, the input tape has a read-write head that can move in both directions. The

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[6] Sometimes it is assumed that the head on the input tape moves only to the right.

main advantage of the off-line models of TM is that both input and output are completely separated from the memory.

Off-line Turing machines are of interest mostly when considering the space complexity as discussed later. We shall use on-line multitape Turing machines—MTM in short—as our basic model of Turing machines, unless it is mentioned explicitly that the off-line model is used. For that reason we define basic concepts for (on-line) MTM only. The extension to off-line MTM is straightforward.

Formally, a k-tape MTM $\mathcal{M} = \langle \Gamma, Q, q_0, \delta \rangle$ is specified by the **tape alphabet** Γ, the set of **states** Q, the **initial state** q_0, and the **transition function**

$$\delta : Q \times \Gamma^k \to (Q \cup H) \times \Gamma^k \times D^k,$$

where $D = \{\leftarrow, \downarrow, \rightarrow\}$ are the directions the heads can move.

The concepts of a **configuration**, a computation **step**, **yield** relations $\vdash_{\mathcal{M}}, \vdash^n_{\mathcal{M}}, \vdash^*_{\mathcal{M}}$ and a **computation** are defined similarly as for one-tape TM. For example, a configuration is a $(2k+1)$-tuple of the form $(q, w_1, w'_1, w_2, w'_2, \ldots, w_k, w'_k)$, where q is the current state and the i-th tape contains the word $w_i w'_i$ with the head on the first symbol of w'_i. The **initial configuration** with an input word w has the form $(q_0, \varepsilon, w, \varepsilon, \ldots, \varepsilon)$. The contents of the output tape at termination is the overall output of a MTM.

Time and space bounds and complexity classes

It is straightforward to introduce basic concepts concerning time resources for computations on MTM. If a MTM $\mathcal{M}$ starts with a string w on its input tape and with all other tapes empty and yields in m steps a terminating configuration, then m is the time of the computation of $\mathcal{M}$ on w. Denote $Time_{\mathcal{M}}(n)$ the maximal number of steps of $\mathcal{M}$ for inputs of length n. $\mathcal{M}$ is said **to operate within the time bound** $f(n)$, for a function $f : \mathbf{N} \to \mathbf{N}$, or to be $f(n)$-**time bounded**, if $\mathcal{M}$ terminates within $f(|w|)$ steps, for any input $w \in \Sigma^*$. If a language $L \subseteq \Sigma^*$ is decided by a $f(n)$-time bounded MTM, then we write $L \in Time(f(n))$. Thus, $Time(f(n)$ is a family of those languages that can be decided by a $f(n)$-time bounded MTM—a **time complexity class**. Observe also that concerning the time requirements there is no essential difference between on-line and off-line MTM. Sometimes we need to be more precise and therefore we use the notation $Time_k(f(n))$ to denote the family of languages accepted by k-tape MTM within the time bound $f(n)$.

Theorem 4.1.28 *For any on-line $f(n)$-time bounded k-tape MTM $\mathcal{M}$, $f(n) \geq n$, there is an off-line $\mathcal{O}(f(n))$-time bounded $(k+2)$-tape MTM $\mathcal{M}'$ that accepts the same language.*

Proof. $\mathcal{M}'$ first copies the input onto the second tape, moves the head on the second tape to the first symbol of the input word, and then simulates $\mathcal{M}$ on $k+1$ tapes numbered $2, \ldots, k+2$. Finally, $\mathcal{M}'$ writes the output on its output tape. □

Before we define space bounds for MTM let us consider two examples of MTM that recognize palindromes.

Example 4.1.29 *The MTM $\mathcal{M}$ from Figure 4.11 first copies the input w from the first tape to the second tape, then moves the head on the first tape to the leftmost symbol, and, finally, moves both heads in the opposite directions and while doing that keeps comparing, symbol by symbol, the corresponding symbols of w and w^R until it either encounters a different pair of symbols or gets safely through. Time bound is clearly $\mathcal{O}(|w|)$ and $\mathcal{M}$ uses $|w|$ cells of the second tape.*

Example 4.1.30 *We sketch the behaviour of a 3-tape TM $\mathcal{M}$ that requires only $\mathcal{O}(\lg n)$ space on its non-input tapes to recognize whether an input string w of length n is a palindrome. The third tape will be used to store an integer $i \leq \frac{n}{2}$. This requires $\mathcal{O}(\lg n)$ space. To start with, $\mathcal{M}$ writes $\frac{n}{2}$ on the third tape and 1 on the second tape. For each i on the third tape $\mathcal{M}$ uses the counter on the second tape to find the i-th symbols from the left and the right in the input word. (To keep the counter requires $\mathcal{O}(\lg n)$ space.) $\mathcal{M}$ compares the two symbols found and if they do not agree, then $\mathcal{M}$ rejects, otherwise $\mathcal{M}$ decreases i by 1 and the process continues until either $\mathcal{M}$ rejects, or i on the third tape reaches 1.*

$q_0, 0, \sqcup, q_0, 0, 0, \rightarrow, \rightarrow$	{Copying the content of the first
$q_0, 1, \sqcup, q_0, 1, 1, \rightarrow, \rightarrow$	tape onto the second tape.}
$q_0, \sqcup, \sqcup, q_l, \sqcup, \sqcup, \leftarrow, \leftarrow$	{Reaching the right end.}
$q_l, 0, x, q_l, 0, x, \leftarrow, \downarrow$	{Moving the first head to the left end.}
$q_l, 1, x, q_l, 1, x, \leftarrow, \downarrow$	{Moving the first head to the left end.}
$q_l, \sqcup, x, q_c, \sqcup, x, \rightarrow, \downarrow$	{Left end reached, start to move right.}
$q_c, 0, 0, q_c, \sqcup, \sqcup, \rightarrow, \leftarrow$	{Comparison of two symbols on two tapes.}
$q_c, 1, 1, q_c, \sqcup, \sqcup, \rightarrow, \leftarrow$	{Comparison of two symbols on two tapes.}
$q_c, 0, 1, REJECT, 0, 1, \downarrow, \downarrow$	{Corresponding symbols do not agree.}
$q_c, 1, 0, REJECT, 1, 0, \downarrow, \downarrow$	{Corresponding symbols do not agree.}
$q_c, \sqcup, \sqcup, ACCEPT, \sqcup, \sqcup, \downarrow, \downarrow$	{Hurra, palindrome.}

Figure 4.11: A multitape Turing machine for palindrome recognition (x stands here for any symbol from $\{0, 1\}$)

Exercise 4.1.31 *Design an $\mathcal{O}(nk^2)$-time bounded 3-tape Turing machine that lexicographically orders given strings $x_i \in \{a, b\}^k$, $1 \leq i \leq n$, given as an input string $x_1\#x_2\#\ldots\#x_n$.*

There are three basic ways how to count space for MTM. The first one is to take the maximum over all configurations of a computation, of the sum of the lengths of all strings on all tapes. The second way is to take again the maximum over all configurations, but counting for each configuration only the longest string on a tape. For a k-tape MTM these two ways of counting the space may differ only by a constant multiplicative factor (at most k). Therefore, we use the second one only. The third way is used only for off-line MTM. It is actually similar to the second one but the contents of the input and output tape are not counted.

With the first two ways of counting the space used during a computation for an input w is always at least $|w|$. The last approach allows to have sublinear space complexity of a computation. This is actually the case of the MTM from Example 4.1.30.

A MTM (or an off-line MTM) $\mathcal{M}$ is said to be $s(n)$-space bounded, where $s : \mathbf{N} \rightarrow \mathbf{N}$ is a function, if $\mathcal{M}$ uses at most $s(|w|)$ cells for any input w.

Suppose now that a language $L \subseteq \Sigma^*$ is decided by a MTM, or an off-line MTM within the space bound $s(n)$. In such a case we say $L \in \mathit{Space}(s(n))$. $\mathit{Space}(s(n))$ is therefore a family of languages—a **space complexity class**.

Mutual simulations of Turing machines

The Examples 4.1.29 and 4.1.30 indicate that by using more tapes we may speed-up computations and sometimes also decrease space needed. In general, it is of interest and importance to find out how powerful different machine models are with respect to the time and space requirements. In order to deal with this problem a general but quite weak concept of simulation of one machine on another is introduced

Definition 4.1.32 *A machine $\mathcal{M}$ simulates a machine $\mathcal{M}'$ for inputs from Σ^*, if $\mathcal{M}'(x) = \mathcal{M}(x)$, for all $x \in \Sigma^*$.*

The following theorem shows that not so much can be gained by using Turing machines with more heads, more tapes or more dimensional tapes.

Theorem 4.1.33 *To any Turing machine $\mathcal{M}$ with several tapes, or several heads or with a two-dimensional tape that operates within the time bound $t(n) \geq n$ and space bound $s(n)$, one can effectively construct a one-tape TM $\mathcal{M}'$ that simulates $\mathcal{M}$ and operates within the time bound $\mathcal{O}(t^2(n))$, and the space bound $\Theta(s(n))$.*

Proof. We carry out the proof only for MTM and in order to simplify the proof we assume that the input is written always between two endmarkers. The other cases are left to the reader—see Exercises 4.1.34 and 4.1.35. Let $\mathcal{M}$ be a k-tape MTM. We describe a one-tape TM $\mathcal{M}'$ that simulates $\mathcal{M}$. To each state q of $\mathcal{M}$ a state q' of $\mathcal{M}'$ will be associated in such a way that if $\mathcal{M}$ moves, in one step, from a state q_1 to q_2, then, in a number of steps proportional to $t(n)$, $\mathcal{M}'$ moves from the state q'_1 to q'_2.

In order to simulate k tapes of $\mathcal{M}$, the only tape of $\mathcal{M}'$ is divided into k tracks, and the i-th track is used to store the contents of the i-th tape of $\mathcal{M}$. Each configuration C of $\mathcal{M}$, see Figure 4.12a, is simulated by a configuration C' of $\mathcal{M}'$, see Figure 4.12b, where all symbols simultaneously read by the heads of $\mathcal{M}$ in C are in one cell of the tape of $\mathcal{M}'$—this is the key point of the whole construction. Thus, $\mathcal{M}'$ can read in one step all symbols k heads of $\mathcal{M}$ read, and therefore $\mathcal{M}'$ knows, by reading one cell, how to change the contents of its tape and its state in such a way that it corresponds to the next configuration of $\mathcal{M}$. The main difficulty in doing that lies in the fact that some heads of $\mathcal{M}$ can move in one direction, others in the opposite direction, and some may not move at all. In order to implement all

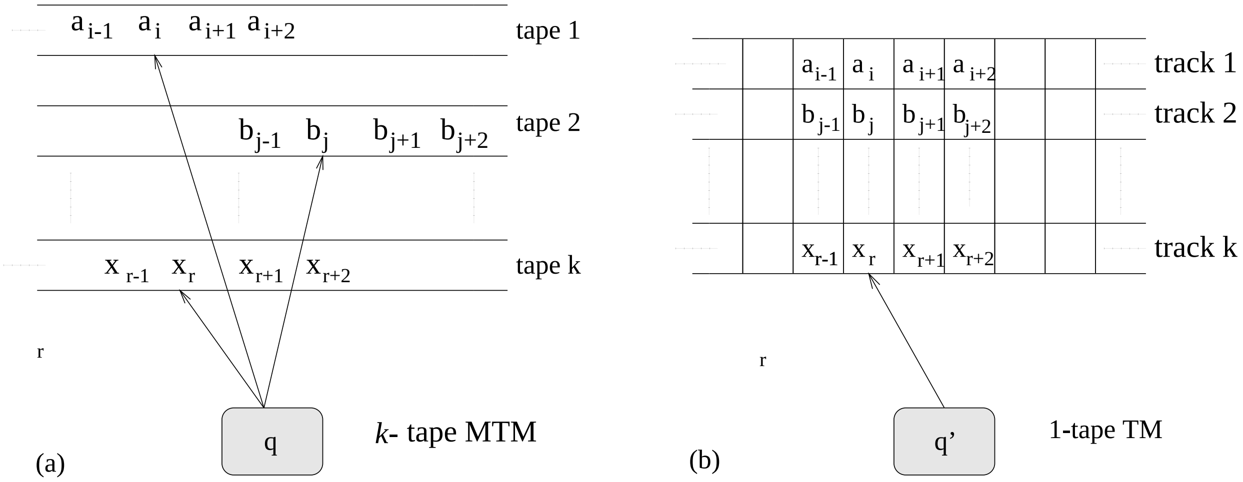


Figure 4.12: Simulation of a multitape TM by a one-tape TM–70%

these changes, and still to have all heads of $\mathcal{M}$ on one cell of $\mathcal{M}'$, $\mathcal{M}'$ has to move some tracks to the left and some to the right. This is, however, no problem, because $\mathcal{M}'$ can store information which track to shift and in which direction in its state. $\mathcal{M}'$ then moves to the right end of the occupied portion of the tape and, in one scan from the right to the left, $\mathcal{M}'$ can make all needed adjustments—to shift some tracks left, some right. After that the head of $\mathcal{M}'$ moves to the cell that contains the contents of all cells the heads of $\mathcal{M}$ will be on in the next configuration of $\mathcal{M}$, and $\mathcal{M}'$ also moves to the new state. It is clear that in this way the space requirement of $\mathcal{M}'$ may be at most twice of that for $\mathcal{M}'$. Concerning the time requirements, the fact that $\mathcal{M}$ makes at most $t(n)$ moves implies that two ends of the occupied portion of the tape are never more than $n + t(n)$ cells apart. In order to simulate one step of $\mathcal{M}$, $\mathcal{M}'$ has therefore to make at most $\mathcal{O}(t(n))$ moves: moving first to the one end of the occupied tape, then to the other end, and, finally, to the cell the heads of $\mathcal{M}$ are on. The overall time complexity is therefore $\mathcal{O}(t^2(n))$. Space bound is clearly $\Theta(s(n))$ □

Exercise 4.1.34 *Show how to simulate, as fast as possible, a TM with a two-dimensional tape by a one-tape TM.*

Exercise 4.1.35 *Show how to simulate, as fast as possible, a multihead one-tape TM by a one-head one-tape TM.*

Universal multitape Turing machines

We show again the existence of a universal TM. This time for the class of all k-tape MTM for a fixed k and the proof will be constructive this time. The main new aim is to show that a universal Turing machine can simulate any Turing machine efficiently.

Theorem 4.1.36 *Let an integer k be fixed and also an alphabet $\Gamma \supseteq \{0,1\}$. Then there exists a universal k-tape MTM U_k with the following properties:*

1. *If $\langle \mathcal{M} \rangle$ is a self-delimiting Gödel encoding of a k-tape MTM $\mathcal{M}$ with the tape alphabet Γ, then on the input $\langle \mathcal{M} \rangle w$, $w \in \{0,1\}^*$, U_k simulates $\mathcal{M}$ on the input w.*

2. *The maximal number of steps U_k needs to simulate one step of $\mathcal{M}$ is bounded by $c_{U_k}|\langle \mathcal{M} \rangle|^\alpha$, where $\alpha = 2$ if $k = 1$, $\alpha = 1$ if $k \geq 2$ and c_{U_k} is a constant.*

Proof. Let $\Gamma = \{a_1, \ldots, a_m\}$. Given an input $\langle M \rangle w$, U_k makes first a copy of $\langle \mathcal{M} \rangle$ on the third track of its first tape—see Figure 4.13. During the simulation this string will always be positioned in such a way that at the beginning of a simulation of a step of $\mathcal{M}$ the head of U_k reads always the leftmost symbol of $\langle \mathcal{M} \rangle$. The current state q_j of $\mathcal{M}$ will be stored as 0^j on the second track of the first tape of U_k, and again in such a way that U_k reads its leftmost symbol whenever starting to simulate a step of $\mathcal{M}$. Strings on the tapes of $\mathcal{M}$ are stored in the first tracks of k-tapes of U_k. Each a_i is encoded by the string $10^i1^{|\Gamma|-i}0$. Thus, the encoding of any symbol of Γ takes exactly $|\Gamma| + 2$ bits. Whenever U_k starts to simulate a step of $\mathcal{M}$, the heads of U_k read the first symbols of encodings of the corresponding symbols on the tapes of $\mathcal{M}$ at the beginning of that step of $\mathcal{M}$.

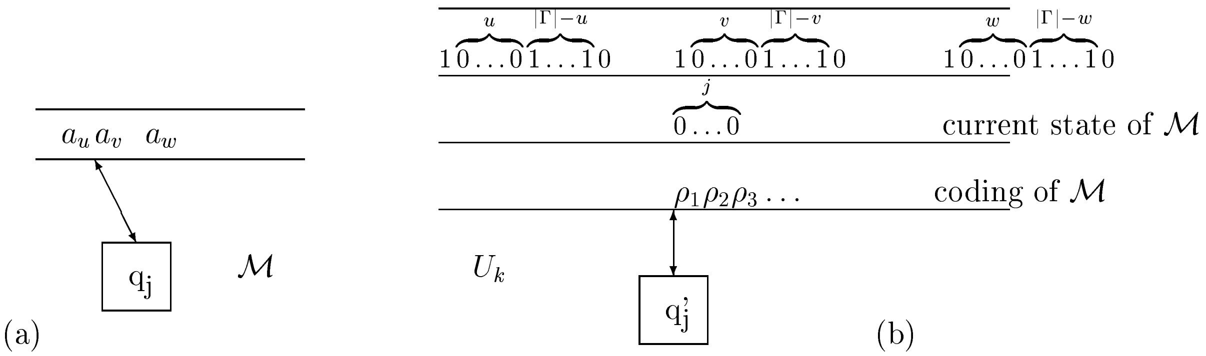


Figure 4.13: Universal Turing machine

In order to simulate one step of $\mathcal{M}$, U_k reads and writes down, say on a special track of a tape, the current state of $\mathcal{M}$ and all the symbols the heads of $\mathcal{M}$ read in the corresponding configuration of $\mathcal{M}$. This pattern is then used to search through $\langle \mathcal{M} \rangle$ on the third track of its first tape, for the corresponding transition. This requires $\mathcal{O}(\langle \mathcal{M} \rangle)$ time. Once the transition is found, U_k replaces the old state of $\mathcal{M}$ by the new one, replaces also those symbols the heads of $\mathcal{M}$ would replace, and starts to realize all moves of heads of $\mathcal{M}$. Finally, U_k has to shift $\langle \mathcal{M} \rangle$ by at most $|\Gamma| + 2$ cells, depending on the move of the head of $\mathcal{M}$, on the third track of its first tape.

It is readily seen that in order to simulate one step of $\mathcal{M}$, U_k has to make the number of steps proportional to $|\langle \mathcal{M} \rangle|$. In order to shift $\langle \mathcal{M} \rangle$, U_k needs time $c(|\Gamma|+2+|\langle \mathcal{M} \rangle|) \leq 2c|\langle \mathcal{M} \rangle|$, for a constant c, if $k > 1$ and therefore another tape is available for shifting $\langle \mathcal{M} \rangle$. The time is $c(|\Gamma| + 2)|\langle \mathcal{M} \rangle| \leq |\langle \mathcal{M} \rangle|^2$, if $k = 1$. □

Exercise 4.1.37 *Show that one can design a single universal Turing machine* $\mathcal{M}_u$ *that can simulate any other MTM* $\mathcal{M}$ *(no matter how many tapes* $\mathcal{M}$ *has).*

Exercise 4.1.38 *Show how to simulate a one-tape* k*-head Turing machine by a* k*-tape Turing machine. Analyze the time complexity of the simulation.* (*It has been shown that a simulation of a* k*-head* $t(n)$*-time bounded Turing machine can be done in time* $\mathcal{O}(t(n))$.*)*

The existence of universal computers that can efficiently simulate any other computer of the same type is the key result behind an enormous successes of computing based on classical physics. The existence of such an efficient universality is far from obvious. For example, the existence of a universal quantum Turing machine was shown already in 1985 but the fact that there is a universal quantum Turing machine that can simulate any other quantum Turing machine efficiently (i.e. in polynomial time) was shown only in 1993.

4.1.8 Time Speed-up and Space Compression

In Chapter 1 we argued why multiplicative constants in the asymptotic time bounds for computational complexity of algorithms are clearly of large practical importance but not of a deeper theoretical interest. One reason for that is that the hardware advances have been so rapid until now that algorithm designers could compete only by improving the rate of growth of algorithms. Two results of this section confirm that once MTM are taken as a computer model, then multiplicative constants are not of importance at all neither for the time nor for the space complexity. In other words, their improvements can be compensated by the so-called "hardware improvements", e.g. by enlarging the tape alphabet.

Lemma 4.1.39 *If* $L \in Time_k(f(n))$, *then for any* $\varepsilon > 0$, $L \in Time_{k+1}(n + \varepsilon(n + f(n)) + 5)$.

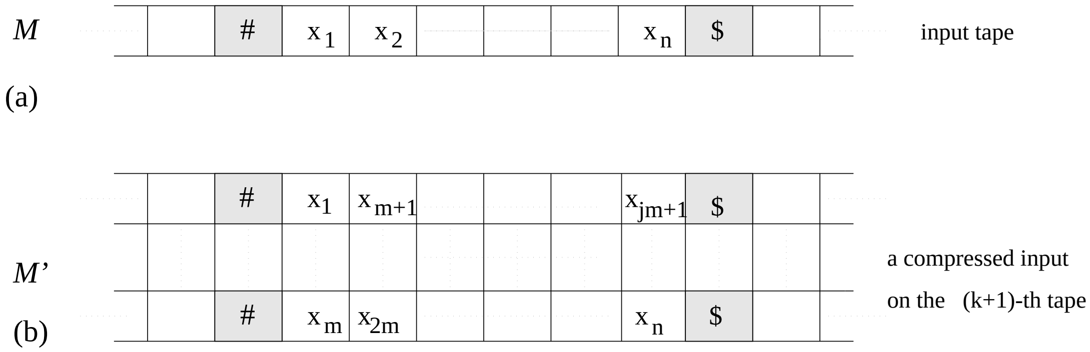


Figure 4.14: Linear speed-up of Turing machines-70%

Proof. Let $\mathcal{M}$ be an off-line k-tape MTM with time bound $f(n)$ and m be an integer (we show later how to chose m—the choice of m will depend on $\mathcal{M}$ and ε). We design a $(k+1)$-MTM $\mathcal{M}'_m$ that will simulate $\mathcal{M}$ as follows. $\mathcal{M}'_m$ starts its simulation by reading the input of $\mathcal{M}$ and, using the technique of storing symbols in the state, compresses each block of m input symbols into a single symbol (into an m-tuple of input symbols), and writes this symbol on its $(k+1)$-th tape—see Figure 4.14. This compression corresponds to using a tape with m tracks instead of a single-track input tape. (In case the length of the input is not a multiple of m some $\sqcup$'s are added.) This process takes n steps, where n is the length of the input. $\mathcal{M}'_m$ then moves its head in $\lceil \frac{n}{m} \rceil$ steps to the leftmost symbol on the $(k+1)$-th tape and the simulation of $\mathcal{M}$ starts. During the simulation $\mathcal{M}'_m$ works with such m-tuples on all its tapes.

$\mathcal{M}'_m$ simulates $\mathcal{M}$ in such a way that m steps of $\mathcal{M}$ are simulated by 4 steps of $\mathcal{M}'_m$. At the beginning of each simulation of a sequence of m steps of $\mathcal{M}$, the machine $\mathcal{M}'_m$ reads an m-tuple of symbols on each tape. This includes not only information about symbols in the corresponding cells of $\mathcal{M}$, but also information on which of these symbols are the heads of $\mathcal{M}$, and in which state $\mathcal{M}$ is. Observe that in the next m moves $\mathcal{M}$ can visit only cells of that block and of only one of the neighbouring blocks of m symbols. By reading these two blocks on each tape, during two steps, $\mathcal{M}'_m$ gathers all information needed to simulate m steps of $\mathcal{M}$. $\mathcal{M}'_m$ can then make the resulting changes in these two blocks in only two additional steps.

Time estimations: Choose $m = \lceil \frac{4}{\varepsilon} \rceil$. The number of steps $\mathcal{M}'_m$ has to perform is:

$$\begin{aligned} n + \left\lceil \frac{n}{m} \right\rceil + 4\left\lceil \frac{f(n)}{m} \right\rceil &\leq n + \lceil \varepsilon n \rceil + 4\left\lceil \frac{\varepsilon f(n)}{4} \right\rceil \\ &\leq n + \varepsilon n + 1 + 4\left(\frac{\varepsilon f(n)}{4} + 1 \right) \\ &\leq n + \varepsilon(n + f(n)) + 5. \end{aligned}$$

□

In case of an on-line k-tape MTM, $k \geq 2$, any tape can be used to write down the compressed input and therefore the $(k+1)$-th tape is superfluous. Observe that the trick which we have used, namely a compression of m-tuples of symbols into one symbol of a bigger alphabet, corresponds actually to the "increasing of the word length of the computer". If $f(n) = cn$, then it follows from Lemma 4.1.39 that c can be compressed to be arbitrarily close to 1. In case $f(n) \succ n$, Lemma 4.1.39 says that the constant factor at the leading term can be arbitrarily small. To summarize:

Theorem 4.1.40 (Speed-up theorem) *For any integer* $k \geq 2$ *and a real* $\varepsilon > 0$, $Time_k(f(n)) \subseteq Time_k(f_\varepsilon(n))$, *where* $f_\varepsilon(n) \leq \varepsilon f(n)$ *for sufficiently large* n, *if* $f(n) \succ n$, *and* $f_\varepsilon(n) \leq n + \varepsilon f(n)$ *for sufficiently large* n, *otherwise.*

Theorem 4.1.40 justifies the use of asymptotic notation to express time complexity of MTM. In particular, if a language L is decided by some MTM in polynomial time, then

$L \subseteq Time(n^k)$ for some k. From that it follows that the time complexity class

$$\mathbf{P} = \bigcup_{k=0}^{\infty} Time(n^k)$$

contains all languages that can be decided by MTM in polynomial time.

Exercise 4.1.41 *Show the following modification of the speed up theorem: For every TM $\mathcal{M}$ and $\varepsilon > 0$, there is a TM $\mathcal{M}'$ over the same alphabet which recognizes the same language and for which $Time_{\mathcal{M}'}(n) \leq \varepsilon Time_{\mathcal{M}}(n) + n$. (Hint: instead of a compression requiring an enlargement of the alphabet use more tapes.)*

Using the same compression technique as in the proof of Theorem 4.1.40, we can prove an analogous result for the space compression.

Theorem 4.1.42 (Linear space compression theorem) *For any function $s(n) > n$ and any real $\varepsilon > 0$ we have $Space(s(n)) = Space(\varepsilon s(n))$.*

Theorem 4.1.42 allows us to define

$$\mathbf{PSPACE} = \bigcup_{k=0}^{\infty} Space(n^k)$$

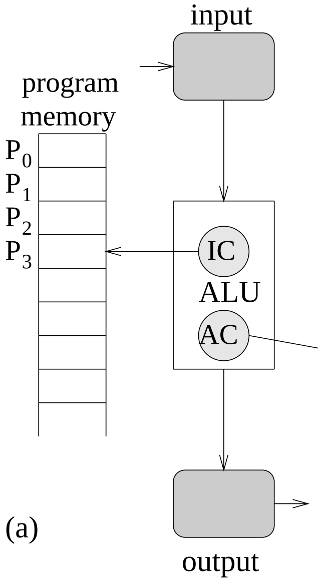
as the class of all languages that can be decided by MTM with a polynomial space bound.

4.2 Random Access Machines

Turing machines are an excellent computer model to study fundamental problems of computing. On the other side, the architecture of Turing machines has little in common with that of modern computers and programming of Turing machines has little in common with programming of modern computers. The most essential clumsiness distinguishing Turing machines from real sequential computer is that its memory is not accessible immediately. In order to read a memory far away all intermediate cells have to be read also. This is bridged over by the **Random access machine** model (RAM in short), introduced and analyzed in this section, that has turned out to be a simple but adequate abstraction of sequential computers of von Neumann type. Algorithm design methodologies for RAM and sequential computers are basically the same. Complexity analysis of algorithms and algorithmic problems for RAM well reflect and predict complexity analysis of programs to solve these problems on typical sequential computers. At the same time, surprisingly, if time and space requirements for RAM are properly measured, there are mutually very efficient simulations between RAM and Turing machines.

4.2.1 Basic Model

The **memory** of a RAM—see Figure 4.15a—consists of a data memory and a program memory. The **data memory** is an infinite **random access array of registers** $R_0, R_1, R_2, \ldots$ each of which can store an arbitrary integer. The register R_0 is called **accumulator** and plays a special rôle. The **program memory** is also a random access array of registers $P_0, P_1, P_2, \ldots$ each of them capable to store an instruction from the instruction set shown in Figure 4.15b. A **control unit** (also called ALU for "arithmetical logical unit"), contains two special registers, an **address counter** AC, and an **instruction counter** IC. In addition, there are **input** and **output** units.



(a)

READ	operand	$\{$input $\longrightarrow R_{op}\}$
WRITE	operand	$\{R_{op} \longrightarrow$ output$\}$
LOAD	operand	$\{R_{op} \longrightarrow R_0\}$
STORE	operand	$\{R_0 \longrightarrow R_{op}\}$
ADD	operand	$\{R_0 + R_{op} \rightarrow R_0\}$
SUB	operand	$\{R_0 - R_{op} \rightarrow R_0\}$
MULT	operand	$\{R_0 * R_{op} \rightarrow R_0\}$
DIV	operand	$\{R_0 / R_{op} \rightarrow R_0\}$
JUMP	label	$\{$go to label$\}$
JZERO	label	$\{$if $R_0 = 0,$ then go to label$\}$
JGZERO	label	$\{$if $R_0 > 0,$ then go to label$\}$
HALT		
ACCEPT		
REJECT		

(b)

Figure 4.15: Random Access Machine-70%

At the beginning of a computation all data memory and control unit registers are set to 0 and a program is stored in the program memory. A **configuration** of a RAM is described by a tuple $(i, i_1, n_{i_1}, \ldots, i_m, n_{i_m})$, where i is the content of IC, $i_1, \ldots, i_m$ are the addresses of the registers used up to that moment during the computation, and n_{i_k} is the current content of the register R_{i_k}.

The operand of an instructions is of one of the following three types:

$=i$ — a constant i;
i — an address, referring to the register R_i,
$*i$ — an indirect address; referring to the register$R_{c(R_i)}$,

where $c(R_i)$ denotes the contents of the register R_i. (In Figure 4.15 R_{op} means i, if the operand has the form $= i$; R_{op} means R_i, if the operand is of the form i and, finally, R_{op} stands for $R_{c(R_i)}$, if the operand has the form $*i$.)A **computation** of a RAM is a sequence of computation steps. Each step leads from one configuration to another. In each computational step a RAM executes the instruction contained currently in the program register $P_{c(IC)}$. In order to perform a non-jump instruction its operand is stored in AC and through AC the data memory is accessed, if necessary. The READ instruction reads the next input number; the WRITE instruction writes the next output number. The memory management instructions (LOAD and STORE), arithmetical instructions and conditional jump instructions use the accumulator R_0 as one of the registers. The second register, if needed, is specified by the contents of AC. After a non-jump instruction has been performed, the content of IC is increased by 1 and the same happens if the test in a jump instruction fails. Otherwise, the label of a jump instruction explicitly defines the new contents of IC.

A computation of a function is naturally defined for a RAM. The arguments have to be provided at the input and a convention has to be taken to determine their number. Either their number is a constant, or the first input integer determines the total number of inputs, or there is some special number denoting the last input.[7] Language recognition requires, in addition, an encoding of symbols by integers.

Figure 4.16 depicts RAM programs to compute two functions: (a) $f(n) = 2^{2^n}$ for $n > 0$; (b) F_n—the n-th Fibonacci number. In both cases n is given as the only input. Fixed symbolic addresses, as N, i, F_{i-1}, F_i, *aux* and *temp*, are used in Figure 4.16 to make programs more readable. Comments in curly brackets serve the same purpose.

```
        READ     $N$      $\{N \leftarrow n\}$
        LOAD     $= 2$
while:  STORE    temp     $\{temp \leftarrow 2^{2^{n-N}}\}$
        LOAD     $N$
        JGZERO   body     {while $N > 0$ do}
        WRITE    temp
        HALT
body:   SUB      $= 1$
        STORE    $N$      $\{N \leftarrow N - 1\}$
        LOAD     temp
        MULT     0        $\{R_0 \leftarrow temp^2\}$
        JUMP     while
```

(a)

```
        READ        $N$   $\{N \leftarrow n\}$
        LOAD      $= 1$
        STORE       $i$   $\{i \leftarrow 1\}$
        STORE   $F_{i-1}$
        STORE     $F_i$
while:  SUB         $N$
        JZERO     print   {while $i < N$ do}
        LOAD      $F_i$
        STORE       aux
        ADD     $F_{i-1}$
        STORE     $F_i$   $\{F_i^{new} \leftarrow F_i + F_{i-1}\}$
        LOAD        aux
        STORE   $F_{i-1}$ $\{F_{i-1}^{new} \leftarrow F_i\}$
        LOAD        $i$
        ADD          =1
        STORE       $i$
        JUMP      while
print:  WRITE     $F_i$
        HALT
```

(b)

Figure 4.16: RAM programs to compute (a) $f(n) = 2^{2^n}$; (b) F_n is n-th Fibonacci number.

The instruction set of a RAM, as presented in Figure 4.15, is a typical one but not the only one possible. Actually, any “usual” microcomputer operation could be added. However,

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[7] For example, the number 3 can denote the end of a binary vector.

in order to get relevant complexity results in the analysis of RAM-programs, sometimes only a subset of the instructions listed in Figure 4.15 is allowed—namely, those without multiplication and division. (It will soon become clear why.) Such a model is usually called RAM$^+$. To this new model the instruction SHIFT with the semantics $R_0 \leftarrow \lfloor R_0/2 \rfloor$ is sometimes added.

Figure 4.17 shows how a RAM$^+$ with the SHIFT operation can be used to multiply two positive integers x and y to get $z = x \cdot y$ using the ordinary school method. In comments in Figure 4.17 k stands for the number of cycles performed to that point. At the beginning $k = 0$. The basic idea of the algorithm is simple: if the k-th rightmost bit of y is 1, then $x2^k$ is added to the resulting sum. The SHIFT operation is used to determine, using the instructions numbered 4 to 9, the k-th bit.

```
 1:  READ     0   {$R_0 \leftarrow x$}
 2:  STORE   $x1$  {$x1 \leftarrow x$}
 3:  READ     0   {$R_0 \leftarrow y$}
 4:  STORE   $y1$  {$y1 \leftarrow \lfloor y/2^k \rfloor$}
 5:  SHIFT
 6:  STORE   $y2$  {$y_2 \leftarrow \lfloor y/2^{k+1} \rfloor$}
 7:  ADD     $y2$  {$R_0 \leftarrow 2\lfloor y/2^{k+1} \rfloor$}
 8:  SUB     $y1$  {$R_0 \leftarrow 2\lfloor y/2^{k+1} \rfloor - \lfloor y/2^k \rfloor$}
 9:  JZERO   13   {if the $k$-th bit of $y$ is 0}
10:  LOAD    $z$   { zero at the start}
11:  ADD     $x1$
12:  STORE    z   {z $\leftarrow x \cdot (y \bmod 2^k)$}
13:  LOAD    $x1$
14:  ADD     $x1$
15:  STORE   $x1$
16:  LOAD    $y2$
17:  JZERO   19   {if $\lfloor y/2^k = 0 \rfloor$}
18:  JUMP     4
19:  WRITE   $z$
20:  HALT
```

Figure 4.17: Integer multiplication on RAM$^+$

If we use similar complexity measures as for Turing machines, that is one instruction as one time step, and one used register as one space unit—the so-called **uniform complexity measures**, then the complexity analysis of the program from Figure 4.16, which computes $f(n) = 2^{2^n}$, yields the estimations $T_u(n) = \mathcal{O}(n) = \mathcal{O}(2^{\lg n})$ for time and $S_u(n) = \mathcal{O}(1)$ for space. Both estimations are clearly unrealistic because just to store these numbers one needs time proportional to their length $\mathcal{O}(2^n)$. One way out is to consider only the RAM$^+$ model (with or without the shift instruction). In a RAM$^+$ an instruction can increase the length of the binary representations of the numbers involved at most by one (multiplication can double it), and therefore the uniform time complexity measure is realistic. The second, more general, way out is to consider the so-called **logarithmic complexity measures**. The time to perform an instruction is considered to be equal to the sum of lengths of binary representations of all numbers involved in the instruction (i.e. all operands as well as all addresses). The space needed for a register is then the maximum length of binary representations of numbers stored in that register during the program execution. The logarithmic space complexity of a computation is then the sum of the logarithmic space complexities of all registers involved. With respect to these logarithmic complexity measures, the program from Figure 4.16a, for the function $f(n) = 2^{2^n}$, has the time complexity $T_l(n) = \Theta(2^n)$ and the space complexity $S_l(n) = \Theta(2^n)$, what corresponds to practice and our intuition. Similarly, for the complexity of the program from Figure 4.17, to multiply two n-bit integers x and y we get: $T_u(n) = \Theta(n)$, $S_u(n) = \Theta(1)$, $T_l(n) = \Theta(n^2)$, $S_l(n) = \Theta(n)$, where, the subscript u refers to the uniform and the subscript l to the logarithmic measures. In the last example, uniform and logarithmic measures differ only by a polynomial factor, with respect to length of the input. In the first example the differences are exponential.

4.2.2 Mutual Simulations of Random Access and Turing Machines

In spite of the fact that random access machines and Turing machines seem to be very different computer models, they can simulate each other efficiently.

Theorem 4.2.1 *A one-tape Turing machine $\mathcal{M}$ of time complexity $t(n)$ and space complexity $s(n)$ can be simulated by a RAM$^+$ of the uniform time complexity $\mathcal{O}(t(n))$ and space complexity $\mathcal{O}(s(n))$, and with the logarithmic time complexity $\mathcal{O}(t(n)\lg t(n))$ and the space complexity $\mathcal{O}(s(n))$.*

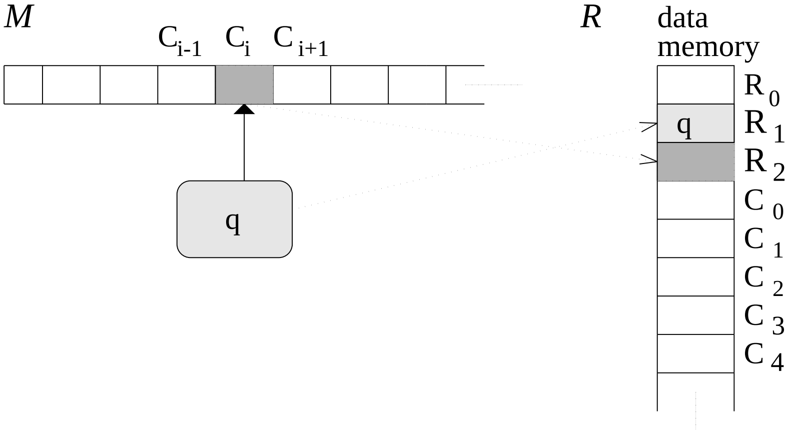


Figure 4.18: Simulation of a TM on a RAM$^+$-80%

Proof. As mentioned in Section 4.1.3 we can assume without loss of generality that $\mathcal{M}$ has a one-way infinite tape. Data memory of a RAM$^+$ $\mathcal{R}$ simulating $\mathcal{M}$ is depicted in Figure 4.18. It uses the register R_1 to store the current state of $\mathcal{M}$ and the register R_2 to store the current position of the head of $\mathcal{M}$. Moreover, the contents of the j-th cell of the tape of $\mathcal{M}$ will be stored in the register R_{j+2}, if $j \geq 0$.

$\mathcal{R}$ will have a special subprogram for each instruction of $\mathcal{M}$. This subprogram will simulate the instruction using the registers $R_0 - R_2$. During the simulation the instruction LOAD $*2$, with the indirect addressing, is used to read the same symbol as the head of $\mathcal{M}$. After the simulation of an instruction of $\mathcal{M}$ is finished, the main program is entered that uses registers R_1 and R_2 to determine which instruction of $\mathcal{M}$ is to be simulated as the next one. The number of operations $\mathcal{R}$ needs to simulate one instruction of $\mathcal{M}$ is clearly constant, and the number of registers used is only by 2 larger than the number of cells used by $\mathcal{M}$. This gives the uniform complexity time and space estimations. The size of numbers stored in registers (except in R_2), is bounded by a constant because the alphabet of $\mathcal{M}$ is finite. This yields the $\mathcal{O}(s(n))$ bound for the logarithmic space complexity. The logarithmic factor, $\lg t(n)$, for the logarithmic time complexity comes from the fact that the number representing the head position in the register R_2 may be as large as $t(n)$. ▯

It is easy to see that the same result holds also for a simulation of MTM on RAM$^+$, only slightly more complicated mapping of k tapes into a sequence of memory registers of a RAM has to be used.

Exercise 4.2.2 *Show that the same complexity estimations as in Theorem 4.2.1 can be obtained for the simulation of k-tape MTM on RAM$^+$.*

The fact that also RAM can be efficiently simulated by Turing machines is more surprising.

Theorem 4.2.3 *A RAM$^+$ of uniform time complexity $t(n)$ and logarithmic space complexity $s(n) \leq t(n)$ can be simulated by a MTM in time $\mathcal{O}(t^4(n))$ and space $\mathcal{O}(s(n))$. A RAM of logarithmic time complexity $t(n)$ and logarithmic space complexity $s(n)$ can be simulated by a MTM in time $\mathcal{O}(t^3(n))$ and space $\mathcal{O}(s(n))$.*

Proof. If a RAM$^+$ has uniform time complexity $t(n)$ and logarithmic space complexity $s(n) \leq t(n)$, then its logarithmic time complexity is $\mathcal{O}(t(n)s(n))$ or $\mathcal{O}(t^2(n))$, because each RAM$^+$ instruction can increase the length of integers stored in the memory at most by one and the time needed by a Turing machine to perform a RAM$^+$ instruction is proportional to the length of operands.

We show now how a RAM$^+$ $\mathcal{R}$ with the logarithmic time complexity $t(n)$ and the logarithmic space complexity $s(n)$ can be simulated by a 7-tape MTM $\mathcal{M}$ in time $\mathcal{O}(t^2(n))$. From that the first statement of the theorem follows.

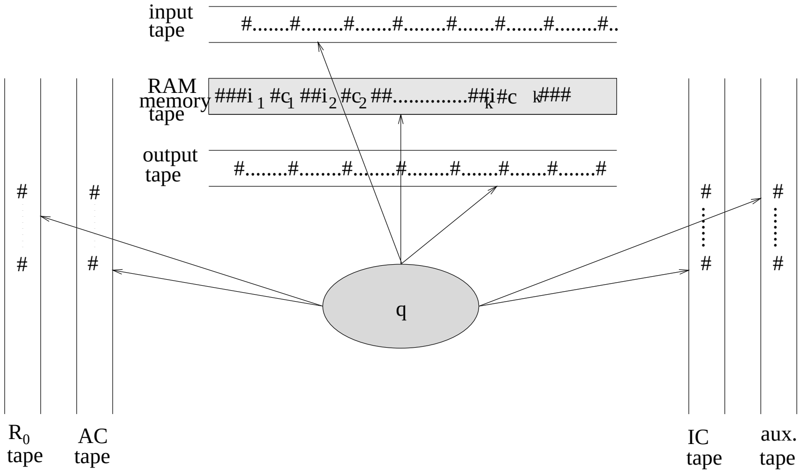


Figure 4.19: Simulation of a RAM on a TM-50%

$\mathcal{M}$ will have a general program to preprocess and postprocess all RAM instructions and a special group of instructions for each RAM-instruction. The first, read-only input tape contains the inputs of $\mathcal{R}$, separated one from another by the marker #. Each time a RAM instruction is to be simulated, the second tape contains the addresses and the contents of all registers of $\mathcal{R}$, used by $\mathcal{R}$ up to that moment, in the form:

$$\#\#\#i_1\#c_1\#\#i_2\#c_2\#\#i_3\#\ldots\#\#i_{k-1}\#c_{k-1}\#\#i_k\#c_k\#\#\#$$

where $\#$ is a marker, $i_1, i_2, \ldots, i_k$ are addresses of registers used until then, stored in the binary form, and c_j is the current contents of the register R_{i_j}—again in the binary form. The accumulator tape contains the current contents of the register R_0. The AC tape contains the current contents of AC, and the IC tape the current value of IC. The output tape is used to write the output of $\mathcal{R}$, and the last tape is an auxiliary working tape—see Figure 4.19.

The simulation of a RAM instruction begins with the updating of AC and IC. A special subprogram of $\mathcal{M}$ is used to search on the second tape for the register $\mathcal{R}$ has to work with. If the operand of the instruction has the form "$= j$", then the register is the accumulator. If the operand has the form "j", then j is the current contents of AC and one scan through the second tape, together with the comparison of integers i_k with the number j written on the AC tape, are enough either to locate j and c_j on the second tape, or to find out that the register R_j has not been used yet. In the second case, the string $\#\#j\#0$ is added at the end of the second tape just before the string $\#\#\#$. In case of an indirect addressing "$*j$", two scans through the second tape are needed. In the first one, the register address j is found and the contents of the corresponding register, c_j is written on the auxiliary tape. In the second scan the register address c_j is found to get c_{c_j}. (In case j or c_j is not found as a register address, we insert on the second tape a new register with 0 as its contents.)

In case of instructions that use only the contents of the register stored on the second tape, i.e. WRITE and LOAD, or an arithmetic instruction, those contents are copied either on the output tape, or on the accumulator tape, or on the auxiliary tape.

A little bit more complicated is the simulation of a RAM instruction that should change the contents of a register R_i found on the second tape. In such a case $\mathcal{M}$ first copies the contents of the second tape after the string $\#\#i\#c_i\#$ to the auxiliary tape, then replaces $c_i\#$ with contents of the AC tape, appends $\#$, and copies the content of the auxiliary tape back to the second memory tape. In case of arithmetical instructions the accumulator tape (with the content of R_0) and the auxiliary tape, with the second operand, are used to perform the operation. The result is then used to replace the old contents of the accumulator tape.

The key factor for the complexity analysis is that the contents of none of the tapes can ever be larger than $\mathcal{O}(s(n))$. This immediately implies the space complexity bound. In addition, this implies that the scanning of the second tape can be done in time $\mathcal{O}(t(n))$. Simulations of an addition and a subtraction also require only time proportional to the length of arguments. This provides a time bound $\mathcal{O}(t^2(n))$. In case of multiplication, a similar algorithm as that described in Figure 4.17 can be used to implement a multiplication in $\mathcal{O}(t^2(n))$ time. (Actually the SHIFT instruction has been used only to locate the next bit of one of the arguments in the constant time.) This is easily implementable on a TM. A similar time bound holds for division. This yield in total $\mathcal{O}(t^3(n))$ time estimation for the simulation of a RAM with the logarithmic time complexity $t(n)$. $\square$

Exercise 4.2.4 *Could we perform the simulation shown in the proof of Theorem 4.2.3 without having a special tape for IC?*

4.2.3 Sequential Computation Thesis

Church's thesis concerns basic, idealized limitations of computing. In this chapter we present also two quantitative variations of Church's Thesis: the **Sequential Computation Thesis** (called also the **Invariance Thesis**) and the **Parallel Computation Thesis**. Both of them deal with the robustness of certain quantitative aspects of computing, namely with the mutual simulations of computer models.

Turing machines and RAM are examples of computer models—or computer architectures (in a modest sense). For a deeper understanding of merits, potentials and applicability of various computer models the following concept of **time** (and **space**) **simulation** is the key one.

Definition 4.2.5 *We say that a computer model* $\mathcal{CM}'$ *simulates a computer model* $\mathcal{CM}$ *with time (space) overhead* $f(n)$*, notation*

$$\mathcal{CM}' \leq \mathcal{CM}\ (\textit{time } f(n)) \qquad \textit{or} \qquad \mathcal{CM}' \leq \mathcal{CM}\ (\textit{space } f(n))$$

if for every machine $\mathcal{M}_i \in \mathcal{CM}$ *there exists a machine* $\mathcal{M}_{s(i)} \in \mathcal{CM}'$ *such that* $\mathcal{M}_{s(i)}$ *simulates* $\mathcal{M}_i$*; that is, for an encoding* $c(x)$ *of an input* x *of* $\mathcal{M}_i$, $\mathcal{M}_{s(i)}(c(x)) = \mathcal{M}_i(x)$ *and, moreover, if* $t(|x|)$ *is the time (space) needed by* $\mathcal{M}_i$ *to process* x*, then the time (space) needed by* $\mathcal{M}_{s(i)}$ *on the input* $c(x)$ *is bounded by* $f(t(|x|))$*. If, in addition, the function* $s(i)$ *is computable in polynomial time, then the simulation is called effective. (Another way to consider a simulation is to admit also an encoding of outputs.)*

As a corollary of Theorems 4.2.1 and 4.2.3 we get:

Theorem 4.2.6 *One-tape Turing machines and* RAM^+ *with the uniform time complexity and the logarithmic space complexity (or RAM with the logarithmic time and space complexity) can simulate each other with a polynomial overhead in time and a linear overhead in space.*

We have introduced the RAM as a model of von Neumann type of (sequential) computers. However, is it really so? Perhaps the most important contribution of von Neumann was the idea that programs and data are stored in the same memory and programs can modify themselves. (What RAM programs can not do.)

A computer model closer to the original von Neumann idea is called RASP (**Random Access Stored Program**). A RASP is as a RAM but RASP programs can modify themselves. The instruction set for RASP (RASP^+) is the same as for RAM (RAM^+), only the indirect addressing is not allowed. On the other side, a RASP program is stored in data registers, one instruction per two registers. The first of these two registers contains the operation, encoded numerically, for example, as in Table 4.1; the second register contains either

Instruction	Encoding	Instruction	Encoding	Instruction	Encoding
LOAD i	1	SUB $=i$	7	WRITE i	13
LOAD $=i$	2	MULT i	8	WRITE$=i$	14
STORE i	3	MULT $= i$	9	JUMP i	15
ADD i	4	DIV i	10	JGZERO i	16
ADD $=i$	5	DIV $=i$	11	JZERO i	17
SUB i	6	READ i	12	HALT	18

Table 4.1: Encoding of RASP instructions

the operand or the label—in case of a jump instruction.

Exercise 4.2.7 * *Show that RAM and RASP and also RAM*$^+$ *and RASP*$^+$ *can simulate each other with only linear time and space overheads, no matter whether uniform or logarithmic complexity measures are used.*

Since RAM and RASP can simulate each other with linear time and space overhead, for the complexity investigations it is of no importance which of these two models is used. However, since RAM programs can not modify themselves, they are usually more transparent and this is the reason why RAM are nowadays almost exclusively used for the study of basic problems of the design and analysis of algorithms for sequential computers. The results concerning mutual simulations of Turing machines, RAM and RASP machines are the base of the following thesis on which the modern ideas of **feasible computing**, **complexity theory** as well as of program design theory are based.

> **Sequential Computation Thesis.** There exists a standard class of computer models, which includes among others all variants of Turing machines, many variants of RAM and RASP with logarithmic time and space measures, and also the RAM$^+$ and RASP$^+$ with the uniform time measure and logarithmic space measure, provided only the standard arithmetical instructions of additive type are used. Machine models in this class can simulate each other with polynomial overhead in time, and linear overhead in space.

Computer models satisfying the Sequential Computation Thesis are said to form the **First Machine Class**. In other words, a computer model belongs to the First Machine Class if and only if this model and one-tape Turing machines can simulate each other within polynomial overhead in time and simultaneously with linear overhead in space. The Sequential Computation Thesis therefore becomes a guiding rule for determination of inherently sequential computer models that are equivalent with other such models in a reasonable sense.

The First Machine Class is very robust. In spite of that it may be far from easy to see whether a computer model is in this class. For example, the RAM with the uniform time and space complexity measures is not in the First Machine Class. Such RAM cannot be simulated by MTM with a polynomial overhead in time. Even more powerful is the RAM augmented with the operation of integer division as we shall see later. Also the following exercise demonstrates the large power of such machines.

Exercise 4.2.8 * *Show that RAM with the integer-division can compute* $n!$ *in* $\mathcal{O}(\lg^2 n)$ *steps (or even in* $\mathcal{O}(\lg n)$ *steps). (Hint: use the recurrence* $n! = n(n-1)!$ *if* n *is odd and* $n! = \binom{n}{n/2}((n/2)!)^2$ *if* n *is even and the identity* $(2^l + 1)^{2k} = \sum_{j=0}^{2k} \binom{2k}{j} 2^{lj}$*, for sufficiently large* l*.)*

Example 4.2.9 *Another simple computer model that is a modification of RAM*$^+$ *but is not in the First Machine Class are the so-called* **register machines**. *Only nonnegative integers can be stored in the registers of a register machine. A program for a register machine is a finite sequence of the labeled instructions of one of the following types:*

l: *PUSH* α	$\{c(\alpha) \leftarrow c(\alpha) + 1\}$;
l: *POP* α	$\{c(\alpha) \leftarrow \max\{0, c(\alpha) - 1\}$;
l: *TEST* $\alpha : l_1$	**if** $c(\alpha) = 0$ **then go to** l_1;
l: *HALT,*	

where $c(\alpha)$ *denotes the current content of the register* α *and each time a non-jumping instruction is performed or the test in a jump instruction fails, the following instruction is performed as the next one (if there is any).*

Exercise 4.2.10 *Show that each one-tape Turing machine can be simulated with only linear time overhead by a Turing machine that has two so-called pushdown tapes. On a pushdown tape the machine can only read and remove only the leftmost symbol and can write at the leftmost end of these tapes (pushing all other symbols into the tape).*[8]

Exercise 4.2.11 *Show that each pushdown tape can be simulated by a register machine with two registers. (Hint: if* $\Gamma = \{Z_1, \ldots, Z_{k-1}\}$ *is the pushdown tape alphabet, then each word* $Z_{i_1} Z_{i_2} \ldots Z_{i_m}$ *on the pushdown tape can be represented in one register of the register machine by the integer:* $i_1 + ki_2 + k^2 i_3 + \ldots + k^{m-1} i_m$*. In order to simulate a pushdown tape operation the contents of one register is transferred, symbol by symbol—or 1 by 1—to another register and during that process the needed arithmetical operation is performed.)*

Exercise 4.2.12 *Show that each one-tape TM can be simulated by a register machine with two registers. (Hint: according to the previous exercise it is enough to show how to simulate a 4-register machine by a two register machine. The basic idea is to represent contents* i, j, k, l *of four registers by one number* $2^i 3^j 5^k 7^l$*.)*

Register machines are not powerful enough to simulate TM in polynomial time, but they can simulate any TM—see the exercises above.

4.2.4 Straight-line Programs

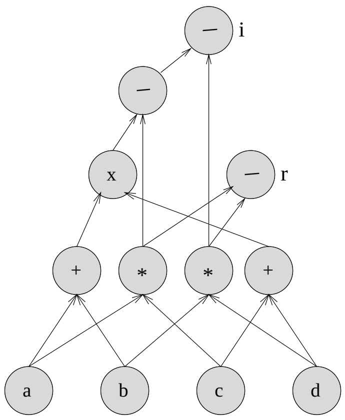
Of a particular interest and importance are special RAM programs, the so-called **straight-line programs**. Formally, they can be defined as finite sequences of simple assignment statements

$$\begin{array}{lcl} X_1 & \leftarrow & Y_1 \odot_1 Z_1, \\ X_2 & \leftarrow & Y_2 \odot_2 Z_2, \\ \vdots & \vdots & \vdots \\ X_n & \leftarrow & Y_n \odot_n Z_n, \end{array}$$

where each X_i is a variable, Y_i and Z_i are either constants, input variables or some X_j with a $j < i$. $\odot$ is one of the operations $+, -, \times, /$. (A variable that occurs on the right hand side of a statement and does not occur on the left-hand side of a previous statement is called an input variable.) Figure 4.20a shows a straight-line program with 4 input variables.

$$\begin{array}{lcl} x & \leftarrow & a+b \\ y & \leftarrow & c+d \\ z & \leftarrow & x \times y \\ u & \leftarrow & a \times c \\ v & \leftarrow & b \times d \\ r & \leftarrow & u - v \\ i & \leftarrow & z - u \\ i & \leftarrow & i - v \end{array}$$

(a)



(b)

Figure 4.20: Complex number multiplication-70%

A straight-line program can be seen as a RAM program without jump instructions and can be depicted as a circuit, the leaves of which are labeled by the input variables and internal nodes by the arithmetical operations—a so-called **arithmetical circuit** (see Figure 4.20b). The number of instructions of a straight-line program, or equivalently, the number of internal nodes of the corresponding arithmetical circuit, is its **size**.

---

[8] A pushdown tape is as one-way infinite tape but its head keeps staying on the leftmost symbol of the tape. Moreover, the machine can read only the leftmost symbol or replace it by a string. If this string has more than one symbol all symbols on the tape are pushed to right to make space just for a new string. If the string is empty, then all tape symbols are pushed one cell left.

Straight-line programs look very simple but actually they are the most proper framework to formulate some of the most basic and most difficult computational problems. For example, given two matrices $A = \{a_{ij}\}, B = \{b_{ij}\}$ of fixed degree n, what is the minimum number of arithmetical operations needed to compute: (a) the **product** $C = A{\cdot}B$; (b) the **determinant** $\det(A)$ of A; (c) the **permanent** $perm(A)$, of A, where

$$det(A) = \sum_{\sigma \in Perm_n} (-1)^{i(\sigma)} a_{1\sigma(1)} \dots a_{n\sigma(n)}, \qquad perm(A) = \sum_{\sigma \in Perm_n} a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

and where $Perm_n$ is the set of all permutations of the set $\{1, \dots, n\}$ and $i(\sigma)$ is the number of inversions in σ.

Exercise 4.2.13 *Show that the problem of computing permanent of a* $0-1$-*matrix is equivalent to the problem of counting the number of perfect matchings of a bipartite graph.*

Let us now discuss in more detail the matrix multiplication problem. In case $n = 2^k, k \in \mathbf{N}$, in order to multiply two matrices A and B of degree n, we can use the following divide-and-conquer method: decompose matrices A and B into four matrices of degree $n/2$ as shown in Figure 4.21a, then multiply recursively A and B, as matrices of the degree 2 the elements of which are matrices of the degree $n/2$. The classical "school algorithm" to multiply two matrices $\{a_{ij}\}$ and $\{b_{ij}\}$ of degree 2, using formulas $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j}$, $1 \le i, j \le 2$, requires 8 multiplications and 4 additions—see Figure 4.21b. However, there exists, surprisingly, another algorithm to multiply two matrices of degree 2, due to Strassen, see Figure 4.21c, that uses only 7 multiplications (and 18 additions/subtractions).[9] At the first sight it seems that nothing was gained, just the opposite, because the new algorithm requires to perform more operations than the old one. However, if this new algorithm to multiply matrices of the degree 2 is used recursively with the above divide-and-conquer method, then the following recurrence holds for the number of arithmetic operations of the resulting algorithm

$$T(n) = 7T(\frac{n}{2}) + \Theta(n^2)$$

because $\Theta(n^2)$ operations are needed to add/subtract two matrices of the degree $n/2$. By Theorem 1.6.3, $T(n) = \mathcal{O}(n^{\lg_2 7}) = \mathcal{O}(n^{2.81})$, what is asymptotically less than the $\Theta(n^3)$ steps required by the "classical algorithm", see Example 1.1.11.

Strassen's algorithm to multiply matrices of the degree 2, Figure 4.21b, and the resulting asymptotically faster algorithm to multiply two matrices, were at the time of their discovery very surprising results. They demonstrated that century long and never challenged beliefs concerning the best possible algorithms for basic algorithmic problems can be wrong. In addition, these results initiated a rapid development of modern algorithm design and analysis

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[9] It has been shown that there is no algorithm to multiply matrices of degree 2 with 6 multiplications only and that any straight-line program that can multiply matrices of degree 2 with 7 multiplications only has to use at least 15 operations of addition/subtraction.

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

(a)

$$\begin{array}{rcl}
z_1 & \leftarrow & a_{11} \times b_{11} \\
z_2 & \leftarrow & a_{12} \times b_{21} \\
c_{11} & \leftarrow & z_1 + z_2 \\
z_1 & \leftarrow & a_{11} \times b_{12} \\
z_2 & \leftarrow & a_{12} \times b_{22} \\
c_{12} & \leftarrow & z_1 + z_2 \\
z_1 & \leftarrow & a_{21} \times b_{11} \\
z_2 & \leftarrow & a_{22} \times b_{21} \\
c_{21} & \leftarrow & z_1 + z_2 \\
z_1 & \leftarrow & a_{21} \times b_{12} \\
z_2 & \leftarrow & a_{22} \times b_{22} \\
c_{22} & \leftarrow & z_1 + z_2
\end{array}$$

(b)

$$\begin{array}{rcl}
i & \leftarrow & a_{12} - a_{22} \\
j & \leftarrow & b_{21} + b_{22} \\
x & \leftarrow & i \times j \\
i & \leftarrow & a_{11} + a_{22} \\
j & \leftarrow & b_{11} + b_{22} \\
y & \leftarrow & i \times j \\
i & \leftarrow & a_{11} - a_{12} \\
j & \leftarrow & b_{11} + b_{12} \\
z & \leftarrow & i \times j \\
i & \leftarrow & a_{11} + a_{12} \\
u & \leftarrow & i \times b_{22} \\
i & \leftarrow & b_{12} - b_{22} \\
v & \leftarrow & a_{11} \times i
\end{array}
\qquad
\begin{array}{rcl}
i & \leftarrow & b_{12} - b_{11} \\
w & \leftarrow & a_{22} \times i \\
i & \leftarrow & a_{21} + a_{22} \\
t & \leftarrow & i \times b_{11} \\
i & \leftarrow & x + y \\
j & \leftarrow & w - u \\
c_{11} & \leftarrow & i + j \\
c_{12} & \leftarrow & u + v \\
c_{21} & \leftarrow & w + t \\
i & \leftarrow & y - z \\
j & \leftarrow & v - t \\
c_{22} & \leftarrow & i + j
\end{array}$$

(c)

Figure 4.21: Matrix multiplication

theory, with enormous impacts on efficiency of computation that matches the hardware successes.[10]

Exercise 4.2.14 *A natural way to multiply two matrices of the degree* $n = m^k$ *is to use the divide-and-conquer method in the way that the given matrices are decomposed into* m^2 *submatrices of the degree* n/m *and this method is then used recursively. In case we find a way to multiply* $m \times m$ *matrices with less than* m^3 *operations this leads to an algorithm that is asymptotically faster than the classical one. The best current algorithm to multiply matrices of degree* 3 *uses* 23 *multiplications. Is it enough to beat Strassen's algorithm? If not, how much do we need to improve the above mentioned algorithm for multiplication of* 3×3 *matrices in order to beat Strassen's algorithm? (In 1978 Pan showed how to multiply* 48×48 *matrices using only* 47216 *multiplications and that lead to the general matrix algorithm with* $\mathcal{O}(n^{2.78014})$ *arithmetical operations.)*

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[10] These results also initiated an exciting competition to find the best possible algorithm (i.e. with as small number of operations as possible), to multiply two matrices. Consequent improvements: $\mathcal{O}(n^{2.78})$—Pan in 1978; $\mathcal{O}(n^{2.779})$—Bini in 1979; $\mathcal{O}(n^{2.542})$—Schönhage in 1980; $\mathcal{O}(n^{2.522})$—Pan (1980); $\mathcal{O}(n^{2.495})$—Winograd (1981); $\mathcal{O}(n^{2.376})$—Coppersmith and Winograd (1987). (Since the best known lower bound is currently $2n^2$, there is still quite a space for improvements.)

Matrix multiplication is one of the key computational problems in scientific and engineering applications. In addition, many other practically important algorithmic problems are computationally equivalent to matrix multiplication. For many other algorithmic problems $\mathcal{P}$ an asymptotically faster algorithm for matrix multiplication immediately yields an asymptotically faster algorithm for $\mathcal{P}$. For example, matrix inversion is equivalent with respect to the computational complexity measured by the size of the straight-line programs to matrix multiplication. Moreover, each asymptotic upper bound for the number of operations for matrix multiplication is at the same time the asymptotic upper bound for the number of operations needed to compute determinants. The determinant of a matrix of degree n, can therefore be computed within $\mathcal{O}(n^{2.376})$ arithmetical operations and also in time $\mathcal{O}(n^{2.376})$. On the other hand, to compute the permanent of a matrix—what looks like a problem very similar to that of computing the determinant—seems to be essentially more difficult. No polynomial time algorithm is known to compute permanents (even if the computation of permanent can be done on a RAM with the operation of integer division in $\mathcal{O}(n^2)$ steps). A special way how to compute the permanent will be discussed in Chapter 9.

Observe that Strassen's algorithms works properly of any ring of matrix elements. On the other hand, Strassen's algorithm cannot be used directly to multiply Boolean matrices since Boolean quasiring is not a ring. However, with a simple trick, Boolean matrix multplication can be reduced to a matrix multiplication over a ring.

Exercise 4.2.15 *Show that if $m(n)$ denotes the number of arithmetical operations needed to multiply two marices of degree n, then two Boolean matrices of degree n can be multiplied using $\mathcal{O}(m(n))$ arithmetical operations. (Hint: interpret* 0-1 *elements of Boolean matrices as integers, multiply these integer matrices,...*

Exercise 4.2.16 *Show that using Boolean matrix multiplication one can decide efficiently whether a given undirected graph has a triangle.*

Another simple example, showing again how misleading is sometimes our common sense concerning computational complexity of algorithmic problems, deals with the complex number multiplication:

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i.$$

How many multiplications of reals are needed to multiply two complex numbers? Four, as the common sense says? No, Figure 4.20a shows that this can be done with three multiplications only. (In case when a, b, c, d are matrices and we have actually multiplication of two complex matrices, this new algorithm represents a significant improvment.)

Exercise 4.2.17 *Some “innocent” arithmetical operations have surprising power. Show, in order to demonstrate it, that multiplication of two matrices of degree* n *with nonnegative integers can be done by a straight-line program with* $\mathcal{O}(n^2)$ *operations if integer-division is allowed. (Hint: in order to multiply two matrices* $A = \{a_{ij}\}$, $B = \{b_{ij}\}$ *to get* $C = A \cdot B$, $C = \{c_{ij}\}$ *compute*

$$z = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\right)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\right) + 1$$

and $A_i = \sum_{j=1}^{n} a_{ij} z^{(j-1)}$, $B_j = \sum_{i=1}^{n} b_{ij} z^{n-i+1} + 1$ *for* $i, j = 1, 2, \ldots, n$ *and use the operation of integer division and of modulus (that can be reduced to integer division) to compute all* c_{ij} *in* $\mathcal{O}(n^2)$ *operations.)*

4.2.5 RRAM—Random Access Machines over Reals

RRAM are defined similarly as RAM, the only difference is that any real number can be an input, a constant in a register or an operand of an instruction. Moreover, any RRAM operation is supposed to take one time unit, no matter how large operands are and no matter whether they are rational or irrational, and arithmetical operations are always peformed exactly.

The motivation behind the RRAM model is to have a computer model that allows to deal well with problems of numerical mathematics, scientific computing, and calculus and to bring computational complexity paradigms into these areas of continuous mathematics.

At first sight it may seem that RRAM are not too much different from RAM. This is, however, misleading. We present some examples illustrating the enormous power of RRAM, as well as machines with slightly adjusted instruction sets. As we shall see RRAM seem to go even beyond Turing machines, and violate Church's thesis, because they can “compute” also functions not computable by Turing machines. This, however, only means that RRAM, in spite of being quite natural, are not in their full generality appropriate models to study the fundamental questions of computability even though they seem to be quite an appropriate model to study many important questions related to the efficiency and the complexity of algorithmic problems of calculus and numerical mathematics.

In the following examples RRAM programs are written in an informal high-level language, in a way that makes their translation into proper RRAM programs straightforward.

Example 4.2.18 (Complement of the Cantor set.) *The following RRAM program accepts those* $x \in [0, 1)$ *that are not in the Cantor set (see Example 2.1.16).*

```
        if $x < 0$ or $x > 1$ then REJECT fi;
cycle:  $x \leftarrow 3x$;
        if $0 \leq x \leq 1$ then JUMP cycle fi;
        if $2 \leq x \leq 3$ then $x \leftarrow x - 2$; JUMP cycle fi;
```

```
        $ACCEPT$
```

Example 4.2.19 (Computation of $\lceil x \rceil$) *The following RRAM program computes $\lceil x \rceil$, for $x > 0$.*

```
if $0 \leq x \leq 1$ then WRITE 1;
$k \leftarrow 1$;
while $x \geq k$ do $k \leftarrow k \times 2$;        {a search for an upper bound};
$l \leftarrow \frac{k}{2}; r \leftarrow k$;
while $r > l + 1$ do if $l \leq x \leq \frac{r+l}{2}$ then $r \leftarrow \frac{r+l}{2}$ else $l \leftarrow \frac{r+l}{2}$ od      { binary search}

if $l = x$ then WRITE $l$ else WRITE $r$
```

Clearly, each cycle is performed $\mathcal{O}(\lg x)$ times and therefore $\mathcal{O}(\lg x)$ is the total number of steps to compute $\lceil x \rceil$. Interesting enough, it is an open problem whether one can compute $\lceil x \rceil$ on an RRAM in $\mathcal{O}(1)$ steps. This may seem to be a problem of minor interest but actually the opposite is true. Indeed, if it would be possible to compute $\lceil x \rceil$ in $\mathcal{O}(1)$ time, then we could extend the RRAM instruction repertoir by the instruction $R_0 \leftarrow \lceil R_0 \rceil$, what does not seem to be a big deal. However, in such a case we could factor integers and test satisfiability of Boolean expressions in a polynomial number of steps on RRAM.

Our last example shows how large the computing power of RRAM is.

Example 4.2.20 (Decidability of arbitrary sets of natural numbers) *Let $S \subset \mathbf{N}$ be any set of integers. Let us define*

$$s_S = 0.s_1 s_2 s_3, \ldots$$

to be a real number where each $s_i \in \{0, 1\}$ and $s_i = 1$, if and only if $i \in S$. The following RRAM program with the build-in constant s_S and the ceiling operation can decide, given an $n \in \mathbf{N}^+$, whether $n \in S$.

```
if $\lfloor 2^n s_S \rfloor - 2\lfloor 2^{n-1} s_S \rfloor \neq 0$ then ACCEPT else REJECT
```

More realistic RRAM models are obtained if it is required that all inputs and constants are rational numbers. An additional useful step seems to be to restrict arithmetical operations to addition and subtraction and/or to consider a logarithmic complexity measure for rational numbers r—to be the minimum of $\lceil \lg p \rceil + \lceil \lg q \rceil$ where $r = \frac{p}{q}$ and p, q are integers.

Remark 4.2.21 Since a RAM is a single machine (A RAM program is its input), the problem of universality for RAM can not be stated exactly in the same way as for Turing machines. However, the property of self-simulation discussed in Exercise 21 comes close to it.

4.3 Boolean Circuits Families

At the lowest level of computation a typical computer processes bits. All numbers and characters are represented by bits and all basic operations are composed of bit operations. The real bit computers are well modeled by Boolean circuits.

Uniformly designed families of Boolean circuits are another very basic computer model, very different from Turing machines and RAM. Since the structure and work of Boolean circuits is the most transparent and tractable, they play an important rôle in theoretical studies.

Both TM and RAM are examples of **uniform** and **infinite computer models** in the sense that each particular computer can process inputs of an arbitrary size. For example, a single TM (or a RAM) program can be used to multiply matrices of an arbitrary degree. On the other side, a single Boolean circuit computes only a single Boolean function. It can process binary strings only of a fixed size—interpreted as an assignment of Boolean values to variables. In order to make a (universal) computer model of the same power as Turing machines out of Boolean circuits, uniformly designed **families** of Boolean circuits have to be considered. For each integer n there has to be a circuit in such a family with n inputs and all circuits of the family have to be designed in a uniform way, as described later.

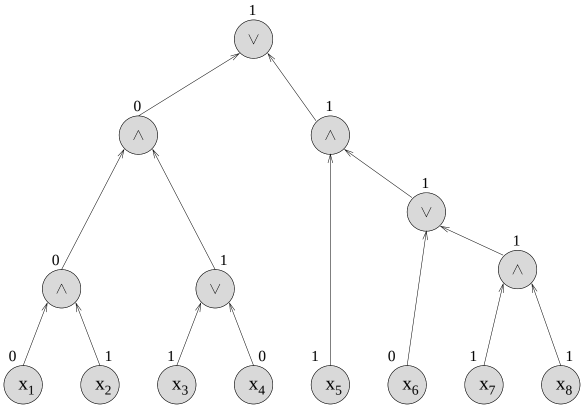


Figure 4.22: A Boolean circuit-40%

4.3.1 Boolean Circuits

A Boolean circuit over a Boolean base $\mathcal{B}$ is a finite labeled directed acyclic graph—see Figure 4.22—whose nodes of in-degree 0—the **input nodes** or **leaves**—are labeled by different Boolean variables and all other nodes—the **gates**—are labeled by Boolean functions (operators) from $\mathcal{B}$, always of the same arity as is the in-degree of the node. The nodes of out-degree 0 are called **output nodes**. We shall consider mostly the base $\mathcal{B} = \{\text{NOT, OR, AND}\}$ ($\mathcal{B} = \{\neg, \vee, \wedge\}$) unless explicitly stated otherwise.

Each Boolean circuit C with n input nodes (labeled by variables $x_1, \ldots, x_n$) and m output nodes (labeled by variables $y_1, y_2, \ldots, y_m$), represents a Boolean function $f_C : \mathbf{B}_n \to \mathbf{B}_m$. The value of f_C for a truth assignment $T : \{x_1, \ldots, x_n\} \to \{0, 1\}$ is the vector of values produced by the output nodes (gates) of C. At this computation process each input node produces the value of its variable at the given truth assignment T, and each gate produces the value of the Boolean function (operator) assigned to that node—for arguments obtained along the input edges from its predecessors. One such computation is shown in Figure 4.22.

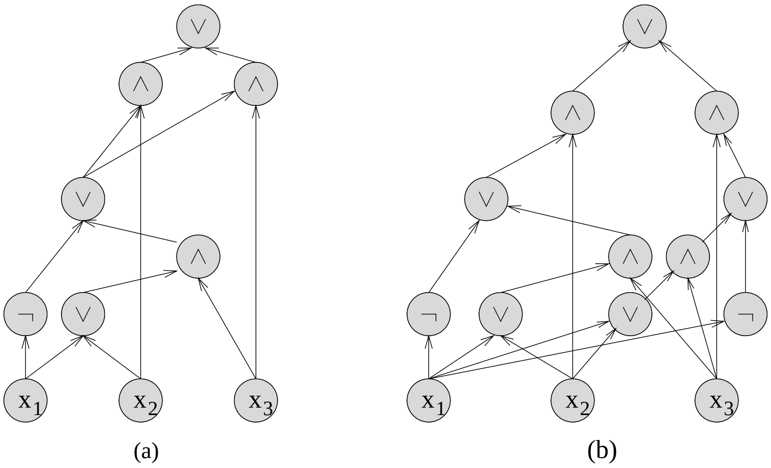


Figure 4.23: Boolean circuits-60%

To each Boolean expression corresponds in a natural way a Boolean circuit. Each variable corresponds to an input node and each occurrence of an operator to a gate. See Figure 4.23b for the circuit corresponding to the Boolean expression

$$((x_1 \vee x_2) \wedge x_3 \vee \neg x_1) \wedge x_2 \vee ((x_1 \vee x_2) \wedge x_3 \vee \neg x_1) \wedge x_3. \tag{4.2}$$

Boolean circuits often represent a more economical way to describe Boolean functions than Boolean expressions. This is due to the fact that several identical subexpressions of a Boolean expression can be represented by a single subcircuit—see the two Boolean circuits in Figure 4.23 computing the same Boolean function, namely the one represented by the Boolean expression (4.2).

Exercise 4.3.1 *Design a Boolean circuit over the base* $\{\vee, \wedge, \neg\}$ *to compute the Boolean function* $f \in \mathcal{B}_3^2$ *such that* $f(0, x_1, x_2) = (x_1, x_2)$ *and* $f(1, x_1, x_2) = (x_2, x_1)$.

Exercise 4.3.2 * *Define in a natural way the concept of a Boolean circuit that is universal for the set of all circuits with the same number of inputs.*

Exercise 4.3.3 * *Design a Boolean circuit over the base* $\{\vee, \wedge, \neg\}$ *that is universal for the base* $\{\vee, \wedge, \neg\}$.

Boolean circuits are a natural abstraction of sequential combinational circuits used to design electronic digital devices. Gates in such circuits are electronic elements each of which can have on its inputs and on the output values 0 and 1—usually represented by two different voltage levels. For the most common gates the standard notation is used—see Figure 4.24. AND and OR may have more inputs (they are easily replaced by subcircuits consisting only of the gates with two inputs). To draw graphical representations for sequential circuits another convention is usually used concerning interconnections—wires: they are not connected unless, at a point of intersection, a dot is placed. Figure 4.25 shows a sequential circuit for a one-bit adder. It has two bit inputs x and y, a carry-input c, and two outputs z and c_0, where

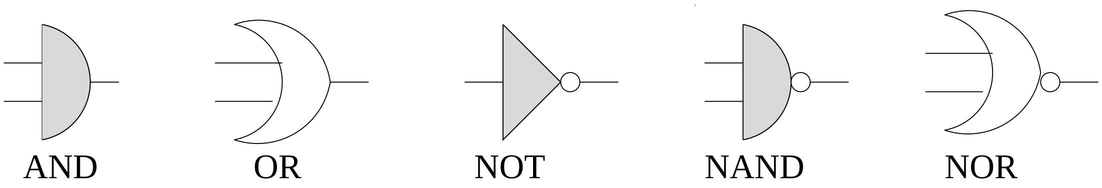


Figure 4.24: Basic gates-80%

$$z = \bar{x}\bar{y}c \vee \bar{x}y\bar{c} \vee x\bar{y}\bar{c} \vee xyc \qquad c_0 = \bar{x}yc \vee x\bar{y}c \vee xy\bar{c} \vee xyc,$$

i.e., z is the resulting sum-bit and c_0 is the new carry bit.

Exercise 4.3.4 *Let a fixture have three switches and flipping any of the switches turns the light on (off) when it is off (on). Design a sequential circuit that accomplishes it.*

Exercise 4.3.5 *Construct a sequential circuit that computes the product of two 3-bit integers.*

Clocked circuits versus finite-state machines

The most obvious element of computation missing from Boolean circuits is repetition: timing of the work of computing elements and storage of the results between consecutive computation

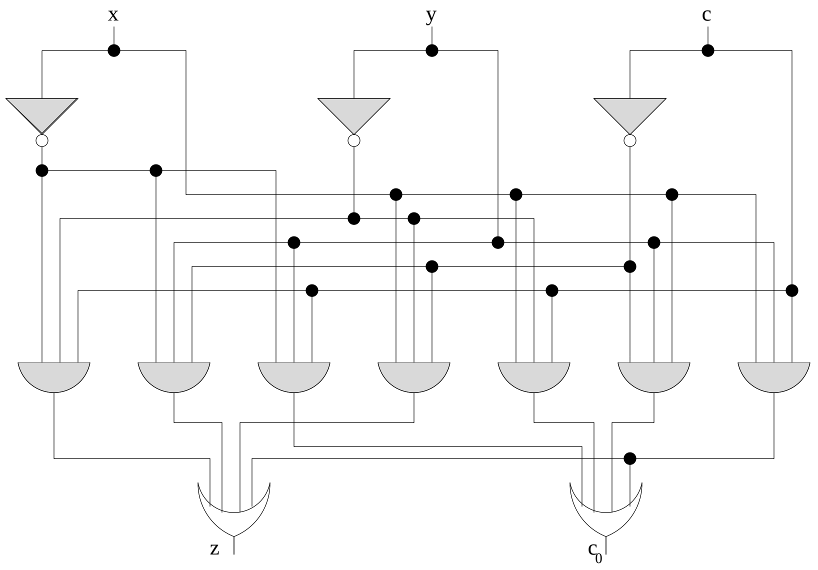
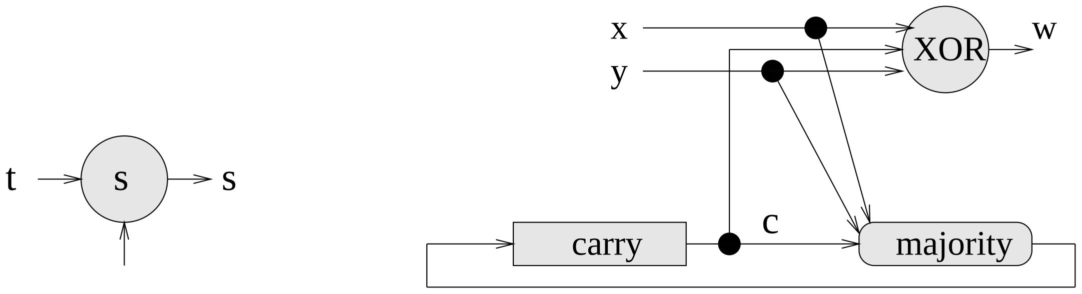


Figure 4.25: One bit adder-50%

steps. These two functions are performed in computer circuitry using **shift registers** (called also flip-flop registers) controlled by a central clock (usually missing from diagrams). A shift register (Figure 4.26a) has two inputs (one, usually invisible, is from the clock) and at each clock pulse the bit value t on its ingoing edge becomes the new value of the register and its old value s "jumps" on the outgoing edge and becomes the value of the register.



(a) shift register

(b) binary adder

Figure 4.26: A shift register and a binary adder% 90

A **clocked circuit** is a directed graph with vertices being either input nodes, Boolean gates, shift registers or output nodes and with no cycle going through Boolean gates only. Computation on a clock circuit goes as follows: At the beginning initial values are written into the shift registers and on input edges and all computations through Boolean gates propagate within one clock cycle. Then a clock pulse is sent to all shift registers and new input values are submitted to inputs. After that this new input assignment is processed and the process goes on. The response of the clock circuit to inputs depends only on its inputs and the current contents of its shift registers. If a clock circuit has n inputs, k shift registers and m outputs, and if we denote the inputs at the clock cycle t by $x^t = (x_i^t, \dots, x_n^t)$, the states of the shift register by $q^t = (q_1^t, \dots, q_k^t)$ and the output by $y^t = (y_1^t, \dots, y_m^t)$, then to each clock circuit two functions are associated.

$$\lambda : \{0,1\}^{k+n} \to \{0,1\}^m, \tag{4.3}$$

$$\delta : \{0,1\}^{k+n} \to \{0,1\}^k, \tag{4.4}$$

such that

$$y^t = \lambda(q_t, x_t), \tag{4.5}$$

$$q^{t+1} = \delta(q_t, x_t). \tag{4.6}$$

From a user point of view a clock circuit is therefore a Mealy machine.

Example 4.3.6 *The clock circuit shown in Figure 4.26b is the binary adder whose behaviour is described by equations:* $w^t = x^t \oplus y^t \oplus c^t$ *and* $c^{t+1} = majority(x^t, y^t, c^t)$.

The relation of clock circuits and Mealy machines goes also in the opposite direction in the following way. Both inputs and states of any Mealy machine can be encoded in binary. Once this is done each Mealy machine can be seen as being specified by mappings (4.3) and (4.4) and equations (4.5) and (4.6). Now it holds:

Theorem 4.3.7 *Let* $\lambda : \{0,1\}^{k+n} \to \{0,1\}^m$ *and* $\delta : \{0,1\}^{k+n} \to \{0,1\}^k$ *be any two functions. Then there is a clock circuit with the input* $x^t = (x_1^t, \dots, x_n^t)$*, states of clock registers* $q^t = (q_1^t, \dots, q_k^t)$ *and output* $y^t = (y_1^t, \dots, y_m^t)$*, at time* t*, whose behaviour is described by equations (4.5) and (4.6).*

Proof. The clock circuit will have k shift registers, which will contain at time t the string q^t, n input nodes with x^t as inputs at time t, and m output nodes with outputs y^t at time t. It contains two Boolean circuits the inputs of which are the outputs of the shift registers and inputs of the whole circuit. One circuit computes the λ function and its m outputs are the overall outputs of the circuit. The second Boolean circuit computes the function δ and its outputs are the inputs of the shift registers. □

Exercise 4.3.8 *Design a clocked circuit for a memory cell (a flip-flop element).*

Any real computer is on its basic logical level a clock circuit. For example, by combining flip-flop elements with the decoder (Exercise 27), we can build a random access memory.

4.3.2 Circuit Complexity of Boolean Functions

Boolean circuits are an appropriate model to deal in a transparent way with the three most basic computational resources: **sequential time, parallel time** and **space**. The three basic and corresponding complexity measures for Boolean circuits are defined as follows:

Size(C) — the **size complexity** of C, i.e. the number of gates of C.
Depth(C) — the **depth complexity** of C, i.e. the maximal distance of a gate of C from an input node.
Width(C) — the **width complexity** of C is defined by Width(C) $= \max_{i \leq Depth(C)} Width(C,i)$, where Width($C,i$)[11] is the number of gates of the maximal distance i from an input node

Complexity measures for circuits induce in a natural way complexity measures for Boolean functions f, relative to a chosen Boolean base $\mathcal{B}$:

$$c_{\mathcal{B}}(f) = \min\{c(C) \mid C \text{ is a Boolean circuit for } f \text{ over the base } \mathcal{B}\}$$

where c is any of the measures Size, Depth and Width.

Between the size and depth complexity of Boolean functions the following relations hold—the first one is easy to show:

$$Depth_{\mathcal{B}}(f) \leq Size_{\mathcal{B}}(f), \qquad Depth_{\mathcal{B}}(f) = \mathcal{O}\left(\frac{Size_{\mathcal{B}}(f)}{\lg Size_{\mathcal{B}}}\right).$$

Exercise 4.3.9 *Show that the choice of the base is not essential. Namely show that if c is any of the above complexity measures and $\mathcal{B}_1, \mathcal{B}_2$ are arbitrary bases, then $c_{\mathcal{B}_1}(f) = \mathcal{O}(c_{\mathcal{B}_2}(f))$.*

Exercise 4.3.10 *A Boolean circuit with only $\vee$- and $\wedge$-gates is called monotone. Show that to each Boolean circuit C over the base $\{\vee, \wedge, \neg\}$ with variables $x_1, \ldots, x_n$ one can construct a monotone Boolean circuit C' with inputs $x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}$ such that $Size(C') = \mathcal{O}(Size(C))$ and C' computes the same function as C.*

Exercise 4.3.11 *Derive the following upper bounds for the size of Boolean circuits over the base of all Boolean functions of two variables: (a) $\mathcal{O}(n2^n)$; (b) $\mathcal{O}(2^n)$ (Hint: $f(x_1, \ldots, x_n) = (x_1 \wedge f(1, x_2, \ldots, x_n)) \vee (\bar{x_1} \wedge f(0, x_2, \ldots, x_n))$); (c)* $\mathcal{O}(\frac{2^n}{n})$; *(d)*** $(1 + o(1))\frac{2^n}{n}$.

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[11] $Width(C,i)$ is sometimes defined as the number of such gates of C that have depth at most i and outgoing edge into a node of depth larger than i.

Boolean circuits can also be seen as another representation of Boolean straight-line program (straight-line programs that do not use arithmetical but logical operations). Size complexity of Boolean circuits is then the same as time complexity of the corresponding Boolean straight-line program.

Circuit complexity of Boolean functions is a much investigated area aimed to get a very fundamental understanding of the complexity of computation. We present now perhaps the most basic result concerning the size complexity of Boolean functions, and by doing that we arrive at the perhaps most puzzling problem in foundations of computing.

In the following lemma and theorem we consider for simplicity the base $\mathcal{B}_0 = \{\text{AND, OR, NAND, NOR}\}$ consisting of Boolean functions of two arguments. This simplifies technical details of the proofs but has no essential effect on the main result. The problem we deal with is the size complexity of Boolean functions of n arguments.

Lemma 4.3.12 *At most* $S(b,n) = \frac{(b+n-1)^{2b}4^b b}{b!}$ *Boolean functions from* $\mathbf{B}_n$ *can be computed by Boolean circuits, over the base* $\mathcal{B}_0$*, of size* b*.*

Proof. Let us estimate the number of Boolean circuits of size b. For each node there are 4 Boolean functions to choose from (AND, OR, NAND, NOR), and there are $b-1+n$ possibilities for the starting node of each of the two ingoing edges of the node ($b-1$ other (Boolean) gates and n input nodes). Each such circuit computes at most b different Boolean functions. (Because, there are at most b possibilities to choose an output node). Finally, one has to take into account that each circuit has been counted $b!$ times—for $b!$ different numberings of nodes. Altogether we get the estimation claimed in the lemma. □

Let now $b = \max\{\text{Size}_{\mathcal{B}_0}(f) \,|\, f \in \mathbf{B}_n\}$. If a Boolean function can be computed by a Boolean circuit of size k, then it can be computed by a circuit of the size $k+1$ and therefore, by Lemma 4.3.12, $S(b,n) \geq |\mathbf{B}_n|$, and we shall use this inequality to get an estimation for b. In doing that we use the inequality $b! \geq cb^{b+0.5}e^{-b}$, for some constant c, that follows from the Stirling's approximation of $b!$—see page 33. Therefore,

$$\lg S(b,n) \geq \lg |\mathbf{B}_n| \Rightarrow 2b\lg(b+n-1) + 2b + \lg b - (b+\frac{1}{2})\lg b + b\lg e - \lg c \geq 2^n. \tag{4.7}$$

Since $b \geq n-1$ for sufficiently large n, the inequality 4.7 implies that

$$b\lg b + (4+\lg e)b + \frac{1}{2}\lg b - \lg c \geq 2^n. \tag{4.8}$$

Let us now assume that $b \leq 2^n n^{-1}$. In such a case we get from the above inequality a new one

$$2^n n^{-1}(n - \lg n + 4 + \lg e) + \frac{1}{2}(n - \lg n) - \lg c \geq 2^n \tag{4.9}$$

and the last inequality clearly does not hold for large n. Therefore,

$$b \geq 2^n n^{-1}$$

has to hold for sufficiently large n.

Observe that the inequalities 4.7 and 4.8 hold and 4.9 does not hold for $b \leq 2^n n^{-1}$ if on the right side of all these inequalities 2^n is replaced by $2^n - 2^n n^{-1} \lg\lg n$. This implies that if we take instead of the class $\mathbf{B}_n$ only a subclass, $\mathbf{B}'_n \subset \mathbf{B}_n$, such that $\lg|\mathbf{B}'_n| \geq 2^n - 2^n n^{-1} \lg\lg n$, then also in such a case we get the inequality $b \geq 2^n n^{-1}$, for $b = \max\{Size_{\mathcal{B}_0}(f) \mid f \in \mathbf{B}'_n\}$. Note also that for such an estimation it does not really matter which functions are in $\mathbf{B}'_n$, only their number is of importance. We can therefore take as $\mathbf{B}'_n$ those $2^{(2^n - 2^n n^{-1} \lg\lg n)}$ Boolean functions from $\mathbf{B}_n$, that have the **smallest Boolean circuit size complexity**. By the same considerations as above we then get that all remaining Boolean functions in $\mathbf{B}_n$ have to have the circuit size complexity at least $2^n n^{-1}$. Therefore $2^{2^n}(1 - 2^{-2^n n^{-1} \lg\lg n})$ Boolean functions have to have the circuit size complexity at least $2^n n^{-1}$. Since $\lim_{n\to\infty} 2^{-2^n n^{-1} \lg\lg n} = 0$ we have:

Theorem 4.3.13 (Shannon's effect) *For sufficiently large n, at least $|\mathbf{B}_n|(1 - 2^{-2^n n^{-1} \lg\lg n})$ out of $|\mathbf{B}_n| = 2^{2^n}$ Boolean functions of n-variables have the circuit size complexity at least $2^n n^{-1}$. (In other words, almost all Boolean functions in $\mathbf{B}_n$ have the circuit size complexity at least $2^n n^{-1}$).*

Now wehave a puzzling situation. In spite of the fact that almost all Boolean functions of n variables have, for large n, exponential complexity, nobody was so far able to find a specific family of Boolean functions $\{f_n\}_{n=1}^{\infty}$, $f_n \in \mathbf{B}_n$, for which we would be able to prove more than linear asymptotic lower bound for the circuit size complexity for $f(n)$, dispate the large effort of scientific community. This has even led to the suggestion to start to consider as an axiom that no "explicit Boolean function" has non-polynomial Boolean circuit size complexity. Interesting enough, such an approach provides so far results corresponding well to our intuition and therefore can be considered to be plausible.

An important task is to design Boolean circuits, as best as possible, for the key computing problems. Size and depth are by that the most important criteria. For example, if the school algorithm to multiply two n-bit integers can be turned into a Boolean circuit of size $\mathcal{O}(n^2)$. A better solution is due to Schönhage and Strassen—a Boolean circuit of size $\mathcal{O}(n \lg n \lg\lg n)$.

4.3.3 Mutual Simulations of Turing Machines and Families of Circuits*

A single Boolean circuit computes only one Boolean function. In order to be able to compare Turing machines and Boolean circuits as models of computers we have to consider infinite **families of Boolean circuits** in $\mathcal{C} = \{C_1, C_2, \ldots\}$, where C_i is a Boolean circuit with i input nodes. We say that such a family of circuits computes a (Boolean) function $f : \{0,1\}^* \to \{0,1\}$ if $f_{C_i} = f_{\mathbf{B}_i}$, i.e., the circuit C_i computes the reduction of f to the domain $\mathbf{B}_i = \{0,1\}^i$. For example, we could have a family of circuits $\mathcal{C}^{(m)} = \{C_1^{(m)}, C_2^{(m)}, \ldots\}$ such that $C_{2i^2}^{(m)}$ computes the product of two Boolean matrices of the degree i.

For a family $\mathcal{C} = \{C_i\}_{i=1}^{\infty}$ of Boolean circuits size and depth complexity bounds are defined as follows: Let $t : \mathbf{N} \to \mathbf{N}$ be a function. We say that the size complexity (depth complexity) of $\mathcal{C}$ is bounded by $t(n)$, if for all n

$$\mathrm{Size}(C_n) \leq t(n) \qquad (\mathrm{Depth}(C_n) \leq t(n)).$$

The concept of a Boolean circuit family, as introduced above, allows to "compute" non-recursive functions. Indeed, let $f : \mathbf{N} \to \mathbf{B}$ be a non-recursive function. Then so is the function $h : \mathbf{B}^* \to \mathbf{B}$ defined by

$$h(w) = \begin{cases} 0, & \text{if } f(|w|) = 0; \\ 1, & \text{if } f(|w|) = 1, \end{cases}$$

and since $h(w)$ depends only on $|w|$ it is easy to see that h is computable by an infinite family of very simple circuits. In order to exclude such "computations" only **uniformly created families of circuits** will be considered.

There are several definitions of uniformity. The following one is guided by the intuition that a circuit constructor should have no more computational power than the objects it constructs. A family of circuits $\mathcal{C} = \{C_i\}_{i=1}^{\infty}$ is called **uniform** if there is an off-line MTM $M_{\mathcal{C}}$ which for any input 1^n constructs in $\mathcal{O}(Size(C_n) \lg(C_n))$ time and $\mathcal{O}(\lg Size(n))$ space a description $\overline{C_n}$ of C_n in the form $\overline{C_n} = (\bar{v}_1, \ldots, \bar{v}_k)$, where $\bar{v} = (v, l(v), p(v))$ is a complete description of the node v and its neighbourhood in C_n: $l(v)$ is the variable or the Boolean operator associated with v and $p(v)$ is the list of predecessors of v. Moreover, it is assumed that nodes are "topologically sorted" in the sense that v_i precedes v_j then $i < j$. For the length $l_{\mathcal{C}}(n)$ of $\overline{C_n}$ we clearly have $l_{\mathcal{C}}(n) = \mathcal{O}(\mathrm{Size}(C_n) \lg \mathrm{Size}(C_n))$. (Observe also that since $Size(C_n) = \mathcal{O}(2^{Depth(C_n)})$, the uniformity requirement actually asks that circuits are constructed in $\mathcal{O}(Depth(C_n))$ space.)

Our requirement of uniformity for a family of Boolean circuits therefore asks that all circuits of the family are constructed "in the same way"—using a single TM and reasonably easy: the time needed for their construction is proportional to the length of the description and space needed is logarithmic.[12]

In the rest of this section we present several simulations of uniform families of Boolean circuits by Turing machines and vice versa. In order to compare the computational power of these two computer models we consider the computation of functions $f : \mathbf{B}^* \to \mathbf{B}$. In addition, for $s, d : \mathbf{N} \to \mathbf{N}$ let

- Size($s(n)$) denote the family of Boolean functions $f : \mathbf{B}^* \to \mathbf{B}$, for which there is a uniform family $\mathcal{C} = \{C_n\}$ of circuits such that C_n computes $f_{\mathbf{B}_n}$ and Size(C_n)$\leq s(n)$.
- Depth($d(n)$) denote the family of functions $f : \mathbf{B}^* \to \mathbf{B}$, for which there is a uniform family $\mathcal{C} = \{C_n\}$ of circuits such that C_n computes $f_{\mathbf{B}_n}$ and Depth(C_n)$\leq d(n)$.

Before going into the details of simulations let us emphasize again that there is a one-to-one correspondence between (Boolean) functions $f : \mathbf{B}^* \to \mathbf{B}$ and languages over the alphabet $\{0,1\}$. To each such function f a language $L_f = \{w \,|\, w \in \mathbf{B}^*, f(w) = 1\}$ is associated and to each language $L \subseteq \{0,1\}^*$ a function $f_L : \mathbf{B}^* \to \mathbf{B}$ is associated with $f_L(w) = 1$ if and only if $w \in L$. The notation Time($t(n)$) and Space($s(n)$) used to denote the families of

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[12] Notice that we assume that off-line Turing machines are used to design circuits. This implies that the space needed to write down the description of the circuit that is being constructed does not count to the overall space complexity—the description is written on the output tape. Because of that we take only $\mathcal{O}(\lg Size(n))$ bound on the space complexity of $M_{\mathcal{C}}$.

languages (over the alphabet $\{0,1\}$) can therefore also be used to denote families of functions $f:\mathbf{B}^* \rightarrow \mathbf{B}$ accepted by MTM within the given time or space bounds.

We start with a series of lemmas that show polynomial relations between Turing machine and Boolean circuits complexity classes.

Lemma 4.3.14 *If $s(n) \geq n$, then Size$(s(n)) \subseteq$ Time$(s^2(n)\lg s(n))$.*

Proof. Let $f \in \text{Size}(s(n))$. We describe a MTM $\mathcal{M}_f$ which, given an input w of the length n, at first generates a circuit C_n that computes the function $f_{\mathbf{B}_n}$ and the size of which is bounded by $s(n)$. Then $\mathcal{M}_f$ determines for all nodes v of C_n the value computed by the gate in the node v, when w is processed by C_n.

$\mathcal{M}_f$ starts by constructing, given an input w with $|w| = n$, in time $\mathcal{O}(s(n)\lg s(n))$ a description $\overline{C_n}$ of the circuit that computes $f_{\mathbf{B}_n}$ where in $\overline{C_n} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ the nodes $v_1, v_2, \ldots, v_k$ are topologically ordered. $\mathcal{M}_f$ then computes in succession $v_1^*, v_2^*, \ldots, v_k^*$, where $v_i^* = C_n(v_i, w)$, and $C_n(v_i, w)$ is the value the gate v_i outputs when the input w is processes by C_n. Since each node has at most two predecessors, $\mathcal{M}_f$ needs at most $\mathcal{O}(s(n)\lg s(n))$ time to search through $\overline{C_n}$ to find the values produced by the gates of the predecessors of the node v_i, the value of which is just being computed. Since C_n has at most $s(n)$ nodes, the overall time needed to compute the output value for the input w is $\mathcal{O}(s^2(n)\lg s(n))$. □

Exercise 4.3.15 * *If a more sophisticated divide-and-conquer algorithm is used in the previous proof to evaluate nodes of C_n, then the overall time needed can be reduced to $\mathcal{O}(s(n)\lg^2 s(n))$. Show that.*

Lemma 4.3.16 *If $d(n) \geq \lg n$, then Depth$(d(n)) \subseteq$ Space$(d(n))$.*

Proof. Let $f \in \text{Depth}(d(n))$. First we show how to design an $\mathcal{O}(d^2(n))$-space bounded MTM $\mathcal{M}_f$ to recognize L_f.

Since $f \in \text{Depth}(d(n))$, there exists a $\mathcal{O}(d(n))$-space bounded off-line MTM $\mathcal{M}'_f$ that constructs, for an input w of length n, a description of the circuit C_n of depth at most $d(n)$, such that C_n computes f restricted to $\mathbf{B}_n$. During the simulation $\mathcal{M}'_f$ will be repeatedly activated. However, and this is essential, each time $\mathcal{M}'_f$ will be used only a part of the description of C_n will be stored, namely that corresponding to the description of a single node of C_n.

$\mathcal{M}_f$ starts by activating $\mathcal{M}'_f$ and storing only the description of the output node of C_n. $\mathcal{M}_f$ then uses a depth-first search traversal through C_n to compute, gate by gate, the values produced by the gates of C_n for a given input w. Each time a new node of C_n is visited during this depth-first search, $\mathcal{M}'_f$ is activated and only the description of the node searched for is stored. Since $\text{Size}(C_n) = \mathcal{O}(2^{d(n)})$, $\mathcal{O}(d(n))$ space is sufficient to store the description of a single node. In order to perform the whole depth-first search evaluation of the circuit C_n, the descriptions of at most $\mathcal{O}(d(n))$ nodes have to be stored simultaneously. This yields the overall space estimation $\mathcal{O}(d^2(n))$. It can be reduced to $\mathcal{O}(d(n))$ by using the following trick: to store the information which part of the tree has not been processed yet it is not necessary to store the full descriptions of the nodes on a path but for each node only one or two of the numbers 1 and 2, specifying the successors of the node to be yet processed. This requires $\mathcal{O}(1)$ space per node. This way the overall space requirement can be reduced to $\mathcal{O}(d(n))$. □

Let us now turn to a more complicated task—depth- and size-efficient simulations of Turing machines by families of Boolean circuits.

In order to formulate these results new technical terms are needed. They will be used also in the following chapter.

Definition 4.3.17 *A function* $f : \mathbf{N} \rightarrow \mathbf{N}$ *is* $t(n)$**-time-constructible** *and* $s(n)$**-space-constructible** *if the function* $f' : \{1\}^* \rightarrow \{0,1\}^*$*, defined by* $f'(1^n) = bin^{-1}(f(n))$*, is computable by a* $t(n)$*-time bounded and* $s(n)$*-space bounded 2-tape TM.* f *is called* **time-constructible (space-constructible)** *if* f *is* f*-time-constructible (*f*-space-constructible).* f *is called* **linearly time- (space-) approximable** *if there is a function* f' *such that* $f(n) \leq f'(n) = \mathcal{O}(f(n))$ *and* f' *is time- (space-) constructible.*[13]

Practically all functions that one encounters during the analysis of computations are both time- and space-constructible.

Exercise 4.3.18 * *Show that if functions* f *and* g *are time- (space-) constructible then so are functions: (a)* $f+g$*; (b)* $f \cdot g$*; (c)* 2^g*.*

Exercise 4.3.19 ** *Show that the following functions are time (space) constructible: (a)* n^2*; (b)* 2^n*; (c)* $n!$*; (d)* $n\lceil \lg n \rceil$*.*

Lemma 4.3.20 *Space*$(s(n)) \subseteq$ *Depth*$(s^2(n))$*, if* $s(n) \geq \lg n$ *and* $s(n)$ *is a* $\mathcal{O}(\lg n)$*-space-constructible.*

Proof. Let $f \in \text{Space}(s(n))$ and $\mathcal{M} = \langle \Gamma, Q, q_0, \delta \rangle$ be an $s(n)$-space bounded one-tape off-line MTM computing f. $\mathcal{M}$ has at most

$$t = n|Q||\Gamma|^{s(n)}s(n) = 2^{\mathcal{O}(s(n))}$$

"partial configurations" for inputs of length n—in which also the position of the head on the input tape is taken into account. Let us assume that exactly one of them, c_t, is the terminating

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[13] The concept of time- and space-constructible functions is often used in the following slightly stronger sense; f is time-constructible if there is a TM $\mathcal{M}$ computing f and such that $\mathcal{M}$ halts after exactly $f(n)$ steps for input 1^n. s is space-constructible if there is an $s(n)$ space bounded TM that writes $1^{s(n)}$ for an input 1^n (i.e. $\mathcal{M}$ marks exactly $s(n)$ cells of the tape).

partial configuration and that c_1 is the initial partial configuration. The behaviour of $\mathcal{M}$ on the input w can be described by the transition matrix $A_w = \{a_{ij}\}_{i,j=1}^{t}$ with

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \text{ or } c_i \vdash_{\mathcal{M}} c_j \text{ for the input } w; \\ 0, & \text{otherwise.} \end{cases}$$

For $k = 1, \ldots, \lceil \lg t \rceil$, we define the matrices

$$A^{(1)} = A_w \odot A_w \qquad \text{and} \qquad A^{(k)} = A^{(k-1)} \odot A^{(k-1)}, \text{ if } k > 1,$$

where $\odot$ stands for the Boolean product of two matrices. Clearly, $a_{1j}^{(l)} = 1$, if and only if there is, for the input w, a sequence of configurations $c_1 \vdash_{\mathcal{M}} c_{i_1} \vdash_{\mathcal{M}} \ldots \vdash_{\mathcal{M}} c_j$ of the length at most 2^l, and therefore

$$\mathcal{M} \text{ accepts } w \text{ if and only if } a_{1t}^{(\lceil \lg t \rceil)} = 1.$$

A Boolean product of two matrices of the degree t can be computed by a simple Boolean circuit of the size $t^2(2t-1)$ and depth $\lceil \lg 2t \rceil$. In order to compute $A^{(\lceil \lg t \rceil)}$ one only needs to connect sequentially $\lceil \lg t \rceil$ of such circuits. The resulting circuit has depth $\mathcal{O}(\lg(2t) \lg t) = \mathcal{O}(s^2(n))$.

It is easy to compute elements of the matrix $A^{(i)}$, and since $s(n)$ is $\lg n$-space-constructible, it is possible to determine t and the resulting circuit using only $\mathcal{O}(\lg n)$ space. Hence the lemma. $\square$

An efficient simulation of a $t(n)$-time bounded MTM $\mathcal{M}$ by a circuit family is a more difficult problem. The source of difficulties lies in the fact that $\mathcal{M}$ may behave differently for different inputs of the same length. In order to show that a size-efficient simulation of a Turing machine by a uniform circuit family exists, it is useful to use a technical result (see references), showing that each MTM can be simulated efficiently by the so-called oblivious MTM.

Definition 4.3.21 *A MTM $\mathcal{M}$ is called* **oblivious** *if the sequence of moves of the heads of $\mathcal{M}$ is the same for all inputs of the same length.*

Theorem 4.3.22 *Let $t(n)$ be $\mathcal{O}(t(n))$-time-constructible and $s(n)$ be $\mathcal{O}(s(n))$-space-constructible. Then each $t(n)$-time and $s(n)$-space bounded MTM can be simulated by an $\mathcal{O}(t(n) \lg s(n))$-time bounded and $\mathcal{O}(s(n))$-space bounded oblivious MTM.*

Equipped with the result of Theorem 4.3.22 we can handle the problem of simulation of time bounded Turing machines by uniform families of Boolean circuits.

Lemma 4.3.23 *If $t(n)$ is $\mathcal{O}(t(n))$-time- and space-constructible, then*

$$\mathit{Time}(t(n)) \subseteq \mathit{Size}(\mathcal{O}(t(n) \lg t(n))).$$

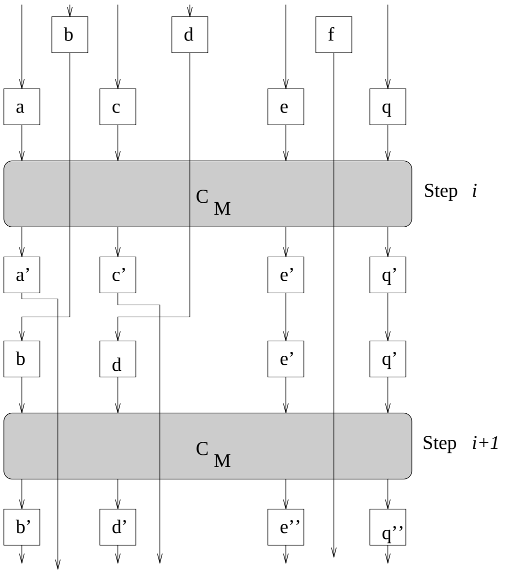

Figure 4.27: A Boolean circuit simulating an oblivious MTM

Proof. If a Boolean function $f : \mathbf{B}^* \to \mathbf{B}$ is in Time$(t(n))$, then, by Theorem 4.3.22, there is an $\mathcal{O}(t(n)\lg t(n))$-time bounded oblivious MTM $\mathcal{M}'$ computing f.

We can encode states of $\mathcal{M}'$, and also its tape symbols, by binary words of a fixed and equal length. Then we can design a fixed size circuit $C_{\mathcal{M}'}$ that implements that part of the transition function of $\mathcal{M}'$ that transfers a state and the symbols read by heads into a new state and new symbols on the tape. Since $\mathcal{M}'$ is oblivious we can now design, given a fixed n, using $t(n)\lg t(n)$ copies of the circuit $C_{\mathcal{M}'}$, a Boolean circuit of the size $\mathcal{O}(t(n)\lg t(n))$ that computes $f_{\mathbf{B}_n}$—see Figure 4.27. (For each copy of $C'_{\mathcal{M}}$, its inputs are either the overall inputs or are connected with the outputs of some of the previous copies of $C_{\mathcal{M}'}$. The same holds for outputs. They either enter inputs of some other copies of $C_{\mathcal{M}}$ or provide total outputs.) To simulate the i-th step of $\mathcal{M}'$ the inputs of $C_{\mathcal{M}'}$ are connected with the last outputs of gates representing the cells the heads of $\mathcal{M}'$ are on at the end of the $(i-1)$-th step. □

As a corollary of the above series of lemmas we get the following main result concerning the mutual simulation of Turing machines and uniform families of circuits.

Theorem 4.3.24 *The following mutual simulations of Turing machines and Boolean circuits are possible for all "nice" bounds $t(n)$ and $s(n)$.*

$$\begin{array}{rclrcl} \mathit{Size}(s(n)) & \subseteq & \mathit{Time}(s^2(n)\lg s(n)), & \mathit{Depth}(s(n)) & \subseteq & \mathit{Space}(s(n)), \\ \mathit{Time}(t(n)) & \subseteq & \mathit{Size}(t(n)\lg t(n)), & \mathit{Space}(s(n)) & \subseteq & \mathit{Depth}(s^2(n)). \end{array}$$

Notice that for the uniform families of Boolean circuits, as a computer model, the parallel time is polynomially related to the the space on Turing machines.

4.4 PRAM—Parallel RAM

Several random access machines sharing a common (**shared** or **global**) memory—the so-called **PRAM**—are from the parallel algorithm design and analysis point of view perhaps the most attractive model of parallel computing. The amount of the RAM processors used to solve an algorithmic problem depends usually on the size of input.

There are two basic variants of PRAM. In a MIMD (Multiple Instruction Multiple Data) PRAM, see Figure 4.28a, each RAM-processor may perform a different program. In a SIMD (Single Instructions Multiple Data) PRAM, see Figure 4.28b, all RAM perform the same program and they can be seen as being controlled by a "super-RAM".

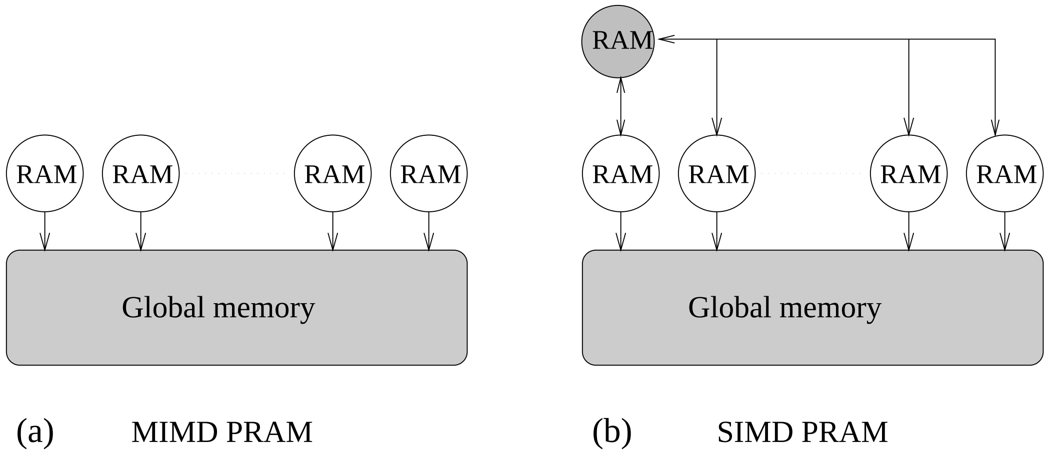

Figure 4.28: MIMD and SIMD PRAM-80%

There are several reasons why such models of parallel computing are of importance in spite of the fact that their direct implementation can hardly be efficient with the current technology.

1. PRAM programming allows to abstract much from that levels of parallel program design that deal with concurrency, synchronization, memory organization and contention, network topology and routing. In designing PRAM algorithms one does not have to be concerned how communication is accomplished, but only what is to be communicated and where.
2. Parallel program design techniques developed for PRAM turned out to be of importance also for other models of parallel computing.
3. Theoretical results indicate that an efficient simulation of PRAM, or closely related models, on (bounded-degree) multiprocessor networks is feasible—see Section **??**.

In order to be able to develop PRAM programs in a clear way and to formulate precisely complexity results we use the following formal model of PRAM.

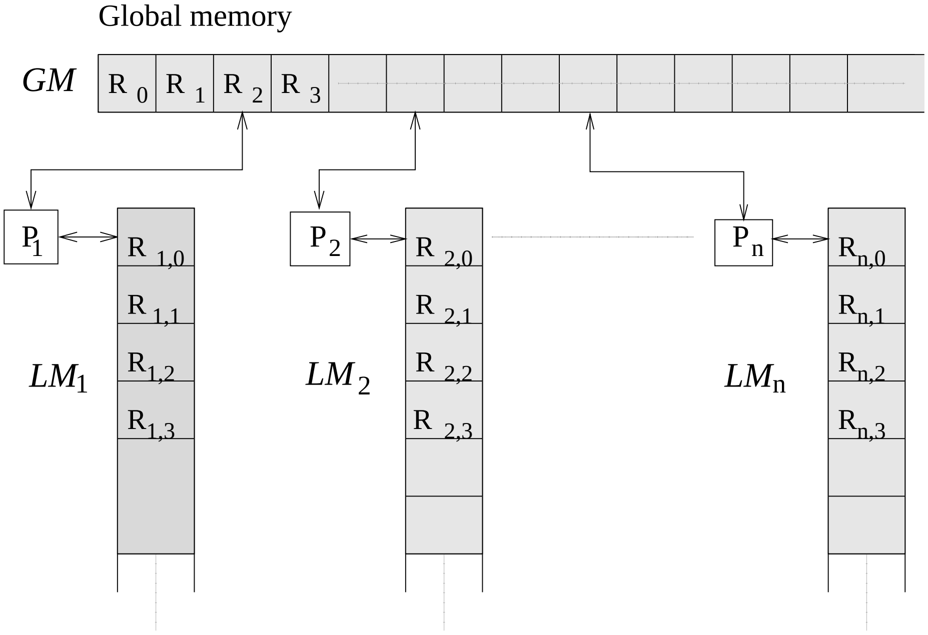

Figure 4.29: Shared (global) memory and local memories of a PRAM-80%

4.4.1 Basic Model

A PRAM consists of an infinite sequence of processors $P_1, P_2, \ldots$ (Figure 4.29). The index i of P_i serves as the **processor identifier** and is denoted by PID. All processors are RAM. A processor P_i has its local (potentially infinite) memory LM_i consisting of a sequence of registers $R_{i,0}, R_{i,1}, R_{i,2}, \ldots$ and P_i is the only processor with access to these registers. In addition, there is a global memory GM consisting of a (potentially infinite) sequence of registers $R_0, R_1, R_2, \ldots$ Each of the registers of the global, and also local memories, can contain an arbitrary integer. Instead of R_i we also write $GM[i]$ or GM_i.

A processor P_i accesses the shared memory through the instructions READ and WRITE. The instruction "READ j", performed by P_i causes the transfer $R_{i,0} \leftarrow GM_j$ and the instruction "WRITE j" causes the transfer $GM_j \leftarrow R_{i,0}$. All other instructions that P_i performs deal with its local memory. In addition to the usual RAM instructions, see Figure 4.15b, there is one special instruction LOAD PID that causes loading of the processor identifier into its accumulator, namely $R_{i,0} \leftarrow$ PID. Concerning the inputs and outputs, the following convention will be used: In the case of n inputs $x_1, x_2, \ldots, x_n$, n is in GM_0 and x_i in GM_i, $1 \le i \le n$. In case of m outputs $y_1, \ldots, y_m$, m is in GM_0 and y_i in GM_i, for $1 \le i \le m$.(Other input-output conventions are, naturally, possible.)

All processors of a PRAM perform the same program, are synchronized, and each of them performs one instruction per time step. A **computation** starts by all processors performing the first instruction of the program and ends when the processor P_1 stops. The contents of the shared memory registers at that point is the output of the computation.

The fact that all processors perform the same program does not mean that at any time moment all of them perform the same instruction and with the same data. This is due to the fact that each processor can use its own PID as a data. This can influence the outcomes of conditional jumps, and therefore different processors may eventually perform at the same time different instructions of the program. For example, it can happen that a processor P_i can read a register GM_j only if $j < n_0$, for some fixed n_0. As a consequence, in most of naturally designed programs, at any moment only finitely many processors can perform some meaningful computation (i.e. computations having an impact on outputs). In the examples we consider only such programs.[14]

A PRAM with potentially infinite number of processors and shared memory cells is a proper model to investigate the most basic question concerning efficiency of parallel computing. Of a theoretical and also practical importance is also a model of PRAM with a fixed number of processors and shared memory registers. In such a case a larger input is usually assumed to be proportionally distributed through all processors. Limitations on amount of shared memory corresponds to restricting the amount of information that can be communicated between processors.

4.4.2 Memory Conflicts

The main problem of PRAM, theoretically and also practically, are conflicts, read and especially write conflicts, when more than one processor try, concurrently, to read from the same register or to write to the same register.

Concurrent reads of a shared memory register cause actually no theoretical and negligible practical problems. In a PRAM reading, computation and writing are considered as being separated. One concurrent step of PRAM is considered as consisting of a reading phase (where all readings are performed), a computational phase and, finally, a writing phase. However, a concurrent writing does creates conflicts and causes major problems. Several methods are used to deal with write conflicts and according to these methods several variants of PRAM are considered. The three most basic models are:

EREW PRAM (**E**xclusive **R**ead, **E**xclusive **W**rite)

— Neither concurrent reads nor concurrent writes are allowed.

CREW PRAM (**C**oncurrent **R**ead, **E**xclusive **W**rite)

— Concurrent reads are allowed, but not concurrent writes.

CRCW PRAM (**C**oncurrent **R**ead, **C**oncurrent **W**rite)

— Both concurrent reads and concurrent writes are allowed, but a fixed strategy is chosen to deal with concurrent writes. Such a strategy specifies what are the contents of the register after a concurrent write to the register has been executed. Different writing strategies lead to different CRCW PRAM models. Three of the most commonly investigated models of CRCW PRAM are the following ones:

---

[14] In case of MIMD PRAM it is usually assumed that the processor P_1 has a special register and its contents specify the maximal index of the processor currently active. In this way a PRAM$^+$ can at most double in each step the amount of processors being active.

CRCW^{com} PRAM (Common PRAM

— Concurrent writes are allowed if and only if all processors that try to write to the same register try to write the same; if this is not the case the computation is aborted and its results are considered as undefined.

CRCW^{arb} PRAM (Arbitrary PRAM)

— If several processors try to write to the same register, then one of them is arbitrarily chosen to write.

CRCW^{pri} PRAM (Priority PRAM)

— From those processors trying to write to the same register the one with the smallest (largest) index succeeds.

Other models of CRCW PRAM are discussed in the exercises.

4.4.3 PRAM Programming

In order to describe PRAM programs in a transparent form the following abstractions will be used:

1. Programs will be described using a high-level programming language constructs that can be translated in a straightforward way into RAM instructions.
2. Direct data transports between registers of the shared memory, or between the shared and a local memory will be used. Going to the basic RAM level, these transports have to be realised through transfers via accumulators. For example, a transfer $A \leftarrow B$ in the global memory is realised through instructions: "READ B, WRITE A".
3. To deal with global memory we use freely various higher level data structures. For example, arrays $A[1:n]$, $B[1:n]$. They may be implemented, for example, as segments $GM[1:n]$, $GM[n+1:2n]$ of the shared memory.
4. Various indexings of processors will be used that will meet the structure of the underlying algorithm. For example, processors $P_{ij}, 1 \leq i,j \leq n$ can be considered. This stands for using processors $P_k, 1 \leq k \leq n^2$ and considering decomposition of each k as $k = (i-1)n + j, 1 \leq i,j \leq n$. To get i and j from k a fixed length sequence of RAM instructions can be used.

Our first example shows an extraordinary power of concurrent writing.

Example 4.4.1 (Maximum finding on a CRCW^{com} PRAM)

Input: integer n stored in GM_0; integers $x_1, \ldots, x_n$ stored in $GM[1:n]$.
Output: $\max\{x_1, \ldots, x_n\}$ stored in GM_0.
Processors: $P_{ij}, 1 \leq i,j \leq n$.
Auxiliary array: $Y[1:n]$.

Algorithm 4.4.2 *For the processor P_{ij}*

```
$Y[i] \leftarrow 0$;
if $GM[i] < GM[j]$ then $Y[i] \leftarrow 1$;        $\{Y[i] = 0$ iff $GM[i] = \max\{GM[1], \ldots, GM[n]\}$
if $Y[i] = 0$ then $GM[0] \leftarrow GM[i]$.
```

If the statement "$Y[i] \leftarrow 0$" is replaced by "**if** $j = 1$ **then** $Y[i] \leftarrow 0$" then the concurrent write in the first statement can be avoided. By inserting the statement "**if** $i < j \wedge Y[i] = 0$, **then** $Y[j] \leftarrow 1$" between the second and the third statement, we can achieve that after this added statement $Y[i] = 0$ if and only if i is the smallest index such that $GM[i] = \max\{GM[i], \ldots, GM[n]\}$. Finally, by replacing the last statement with the statement "**if** $j = 1 \wedge Y[i] = 0$ **then** $GM[0] \leftarrow GM[i]$", we could avoid concurrent writing also in the last statement. Note, that in all concurrent writings all processors try to write the same value. The same program would therefore work correctly also on models CRCW^{arb} PRAM and CRCW^{pri} PRAM.

In our second example we use another method, more global at the first sight, how to describe PRAM programs. Those parts of the program that can be executed with different data on different processors will be described by the concurrent constructs

$$\mathbf{par}[a \le i \le b] P_i : S_i \qquad \text{or} \qquad \mathbf{par}[a \le i \le b] : S_i$$

with the semantics that the statement S_i is executed (by the processor P_i), concurrently, for all i between a and b. In addition, from now on we use in the descriptions of PRAM programs freely various arrays or sequences without referring formally to the PRAM shared memory.

Example 4.4.3 (Maximum finding on an EREW PRAM)

Input: d, $n = 2^d$, numbers $a_n, \ldots, a_{2n-1}$.
Output: $a_1 = \max\{a_n, \ldots, a_{2n-1}\}$.

Algorithm 4.4.4 *For a EREW PRAM.*

```
for $d \leftarrow d - 1$ downto 0 do
par$[2^d \le j \le 2^{d+1} - 1] : a_j \leftarrow \max\{a_{2j}, a_{2j+1}\}$.
```

The computation process described by Algorithm 4.4.4 is depicted in Figure 4.30 for $n = 8$. Note that no indication is made in the program which processors perform statements $a_j \leftarrow \max\{a_{2j}, a_{2j+1}\}$. At least two implementations are natural. In the first one the processor P_j performs such a statement. In this case we need n processors. In the second implementation, only processors $P_1, \ldots, P_{n/2}$ are needed—see Figure 4.30. However, in this case it takes more time for each processor to compute the addresses of the data the processors should work with.

Exercise 4.4.5 *Design an EREW PRAM program to add n numbers in $\mathcal{O}(\lg n)$ time.*

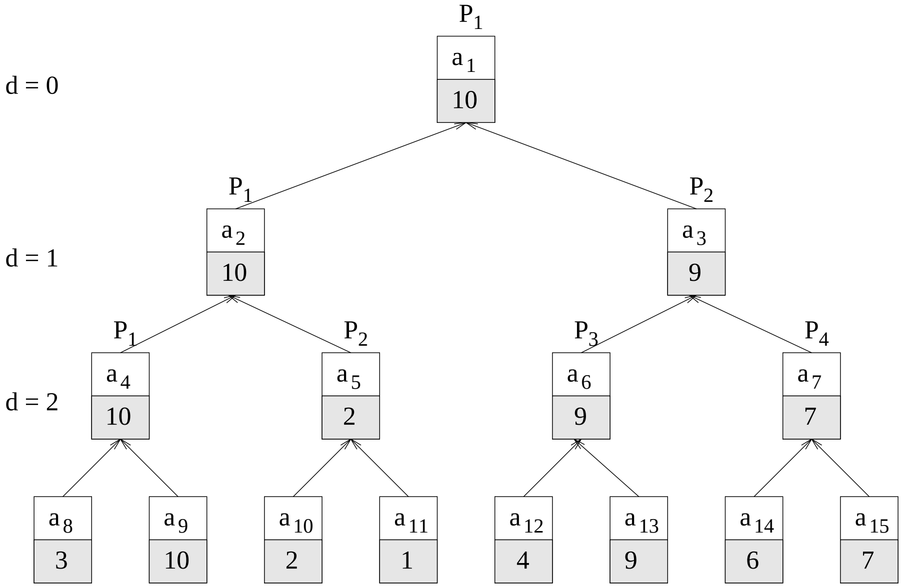

Figure 4.30: Maximum finding on a EREW PRAM

Example 4.4.6 (Matrix multiplication on an EREW PRAM)

Input: An integer d, matrices $A[i,j]$, $B[i,j]$, $1 \leq i,j \leq n = 2^d$.
Output matrix $C[i,j]$, $1 \leq i,j \leq n$, $C = A \cdot B$
Processors: P_{ijl}, $1 \leq i,j,l \leq n$.
Auxiliary array: $D[i,j,l]$, $1 \leq i,j,l \leq n$.

Algorithm 4.4.7 *For processor* P_{ijl} :

```
$D[i,j,l] \leftarrow A[i,l]B[l,j]$;
for $k \leftarrow 1$ to $d$ do
    if $l \leq \frac{n}{2^k}$ then $D[i,j,l] \leftarrow D[i,j,2l-1] + D[i,j,2l]$;
if $l = 1$ then $C[i,j] \leftarrow D[i,j,1]$
```

The first statement performs, in parallel, all multiplications and the next **for**-cycle all additions. The last statement forms the output. In case of CRCW^{com} PRAM, the last statement could be simplified to $C[i,j] \leftarrow D[i,j,1]$.

Example 4.4.8 (Boolean matrix multiplication on CRCW^{com} PRAM$^+$) *Two Boolean matrices* $A[i,j]_{i,j=1}^n$, $B[i,j]_{i,j=1}^n$ *can be multiplied to get* $C = A.B$ *on a CRCW^{com} PRAM$^+$ using* n^3 *processors* $\{P_{ijk}\}_{i,j,k=1}^n$ *in* $\mathcal{O}(1)$ *steps. In the first step all* c_{ij} *can be initiated to* 0. *Then the processor* P_{ijk} *reads* a_{ik} *and* b_{kj} *and writes* 1 *in* c_{ij} *if and only if* $a_{ik} \wedge b_{kj} = 1$.

4.4.4 Efficiency of Parallelization

Intuitively, it seems to be clear that parallelization of computing—that is the use of parallel computers—is the way to go in order to increase the performance of computers. On the other hand, theoretical considerations show and practical experiences demonstrate that to harvest the power of parallelism is not easy at all.

In this section we take a closer look on the problems of efficiency of parallelization. First for PRAM models, then on a more general level.

PRAM complexity measures

Three computational resources are of importance for PRAM computing: **parallel time** consumed, **number of processors** employed, and the overall **work** done by processors. For a PRAM $\mathcal{R}$ and an input x they are defined as follows:

- $\mathrm{Time}_{\mathcal{R}}(x) =$ the number of concurrent steps of $\mathcal{R}$ until $\mathcal{R}$ stops.
- $\mathrm{Proc}_{\mathcal{R}}(x) = \max\{i | i = 1$ or P_i performs a WRITE statement before $\mathcal{R}$ stops$\}$.
- $\mathrm{Work}_{\mathcal{R}}(x) = \mathrm{Proc}_{\mathcal{R}}(x) \times \mathrm{Time}_{\mathcal{R}}(x)$.

The definition of the processor complexity requires some explanations. Our intuition says that we should count only those processors the work of which really influences the outputs. The above definition tries to capture this intuition in a simple way. The definition is clearly not perfect, because it may happen that a processor makes computations having impact on outputs but its predecessor does not. In spite of that this definition has turned out to be good enough to deal with not too degenerated cases.

The work complexity is also a very natural and useful measure of efficiency of PRAM programs. It can also be used to set up benchmarks one should try to reach when designing PRAM programs. We say that a PRAM $\mathcal{R}$ is **optimal** for an algorithmic problem $\mathcal{P}$ if $\mathrm{Work}_{\mathcal{R}}(n) = \Theta(t(n))$, where $t(n)$ is asymptotically best known sequential time complexity of $\mathcal{P}$ on RAM.

The ultimate goal in designing PRAM programs is to achieve work-optimality. However, this is not the end of the optimization effort. As soon as the work optimality is achieved, the next aim is to keep this optimality and to minimize time—with a philosophy/belief behind that processors will be ultimately cheaper than time.

Trade-offs between time and number of processors is an important issue in parallel computing in general, and in PRAM computing n particular. For example, with a single RAM one needs $\Omega(n \lg n)$ time to sort n numbers, with $\mathcal{O}(n^2)$ processors sorting can be done in $\mathcal{O}(\lg n)$ time While for a RAM an algorithm is considered to be really fast if its time complexity is almost linear, i.e. $\mathcal{O}(n \lg n)$ or less, a PRAM algorithm is considered as really fast if its time complexity is $\mathcal{O}(\lg n)$ or less. (For example, $\mathcal{O}(\lg \lg n)$, or $\mathcal{O}(\lg^* n), \ldots$)

Example 4.4.9 *Let us analyze the computational complexity of our first three PRAM algorithms Algorithm 4.4.2 has the time complexity* $\mathcal{O}(1)$ *and uses* $\mathcal{O}(n^2)$ *processors. Its work complexity is* $\mathcal{O}(n^2)$*—much more than* $\Theta(n)$ *for the best possible sequential algorithm. Algorithm 4.4.4 needs* $\Theta(\lg n)$ *time and* $\Theta(n)$ *processors and therefore its work complexity*

$\Theta(n \lg n)$ is still more than the optimal one. The matrix multiplication algorithm needs $\Theta(\lg n)$ time and $\Theta(n^3)$ processors. The product is at least by a $\Omega(n^{0.5} \lg n)$ factor larger than optimality suggest.

Remark 4.4.10 It is important to realize why a practically unrealistic assumption, namely that of having the amount of processors available to grow with the size of input, does not make PRAM computation theory neither wrong nor irrelevant. It only means that the application of theory is indirect. It is fruitful to view the processor requirement for PRAM, or other model of parallel computation as growing function of the size of the problem not because the machine are (or are expected to be) so flexible, but because such a requirement allows to focus on important intrinsic issues, like parallel decomposition of the problem, and translate insights obtained into answers to various pragmatic questions—(like "how big problems can be solved using a certain amount of time and certain number of processors"). Similar arguments apply to storage as a resource.

Parallelization

One of the basic and general questions concerning the advantages parallelism can bring is the following one: Given an algorithmic problem $\mathcal{A}$ and an integer k, how much can we speed up solution of $\mathcal{A}$ using k processors?

Two measures are used to express how much parallelization helps:

$$\mathrm{Speedup}(\mathcal{A}) : p \to \frac{T_1(\mathcal{A})}{T_p(\mathcal{A})} \qquad \mathrm{Efficiency}(\mathcal{A}) : p \to \frac{\mathrm{Speedup}(\mathcal{A})}{p},$$

where $T_i(\mathcal{A})$ is the best time known to solve $\mathcal{A}$ using i processors.

Unfortunately, there are algorithmic problems $\mathcal{A}$ such that $T_1(\mathcal{A}) = T_p(\mathcal{A})$ for any p and therefore $\mathrm{Efficiency}(\mathcal{A})(p) = \frac{1}{p}$. In other words, parallelization does not help a bit. Exponentiation x^n is one such problem. No parallel algorithm can do asymptotically better than the sequential one presented on the page 8—with respect to the number of arithmetical operations.

The goal of parallel computation is to develop algorithms that use only a reasonable amount of processors and are very fast. The goal is fundamentally limited by the so-called speed-up inequality.

$$\text{(best known sequential time)/(number of processors)} \leq \text{ parallel time}$$

This inequality implies that in order to achieve a subpolynomial time parallel algorithm, a polynomial number of processors has to be used.

There is a simple and easy to see principle that can sometimes be used to improve performance of an almost optimal parallel algorithm by a logarithmic factor and to get a fully optimal one.

Remark 4.4.11 No general and really good method for parallelization of sequential algorithms is likely to exists. The best parallel algorithms are often designed on strikingly different ideas than the best sequential algorithms for the same problem.

Brent‘s Scheduling Principle. If a parallel computation can be performed on $n = \max\{x_i \,|\, 1 \leq i \leq t\}$ processors in time t, where x_i is the number of operations in the i-th parallel step, then on $p < n$ processors the same computation can be performed in time $t + \lceil \frac{x}{p} \rceil$, where $x = \sum_{i=1}^{t} x_i$.

Exercise 4.4.12 *Show the correctness of Brent's Scheduling Principle in detail.*

By taking $x = \mathcal{O}(n), t = \mathcal{O}(\lg n)$, and $p = \mathcal{O}(\frac{n}{\lg n})$ we get the following corollary from the Brent's Scheduling Principle:

Lemma 4.4.13 *If a computation that needs sequential time $\mathcal{O}(n)$ can be performed on $\mathcal{O}(n)$ processors in $\mathcal{O}(\lg n)$ time, then the same computation can be performed on $\mathcal{O}(\frac{n}{\lg n})$ processors in time $\mathcal{O}(\lg n)$. (Such a computation is optimal because $\mathcal{O}(\lg n)\mathcal{O}(\frac{n}{\lg n}) = \mathcal{O}(n)$.)*

If Brent's Scheduling Principle and its corollary are applied to the Algorithm 4.4.4, we get a work-optimal algorithms.

4.4.5 PRAM Programming—Continuation

We present several other PRAM algorithms to demonstrate the power of PRAM and some parallel algorithm design methodologies.

Example 4.4.14 (Prefix sum) *Given a sequence of numbers $x_1, \ldots, x_n$, compute the sequence $y_1, \ldots, y_n$ of all its prefix sums, that is*

$$y_j = \sum_{i=1}^{j} x_i.$$

This problem seems to be inherently sequential. Indeed, it seems that in order to compute y_i one needs to compute first y_{i-1}. However, the problem has an $\mathcal{O}(\lg n)$ parallel solution.

The prefix sum problem is so important for parallel computing that algorithms to solve it were wired into some parallel computers. Its importance is due to the fact that fast parallel algorithms for many problems use a fast prefix sum algorithm as their key subalgorithm.

In order to make our first prefix sum algorithm transparent we assume that inputs $x_1, \ldots, x_n$ are represented by $a_n, \ldots, a_{2n-1}$ and we need to compute $y_n, \ldots, y_{2n-1}$, represented by $b_n, \ldots, b_{2n-1}$.

Input: $n \in N$, $n = 2^d$, $x_1, \ldots, x_n \in \mathbf{R}$.
Output: $y_1, \ldots, y_n$, with $y_j = \sum_{i=1}^{j} x_i$.

Algorithm 4.4.15 (Prefix sum for EREW PRAM)

```
for $l \leftarrow d-1$ downto 0 do par$[2^l \le j < 2^{l+1}] : a_j \leftarrow a_{2j} + a_{2j+1}$;
$b_1 \leftarrow a_1$;
for $l \leftarrow 1$ to $d-1$ do
    par$[2^l \le j < 2^{l+1}] : b_j \leftarrow$ if $j$ is odd then $b_{(j-1)/2}$ else $b_{j/2} - a_{j+1}$
```

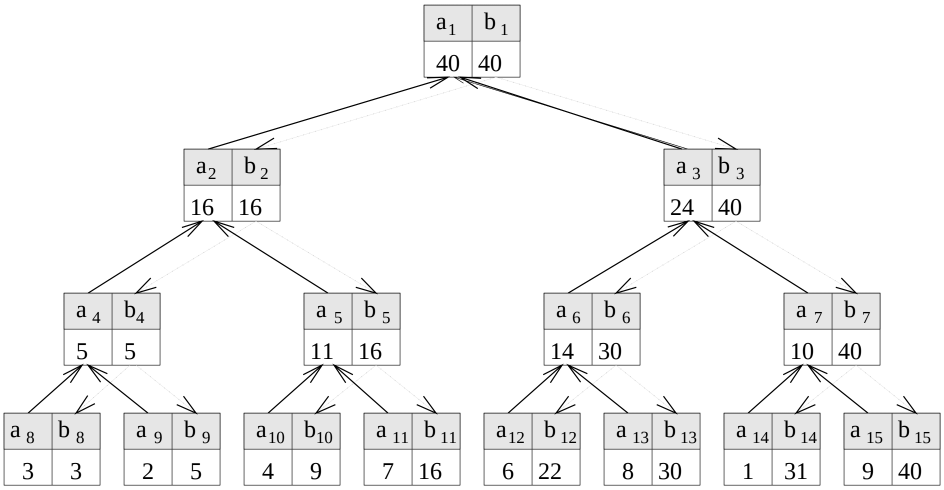

Figure 4.31: A prefix sum algorithm—50%

The computation of the algorithm is depicted, for $n = 8$, in Figure 4.31. In the first phase, the leaves-to-root phase—described by the first **for**-statement—all $a_i, 1 \le i \le n-1$, are computed. Each a_i is the sum of all a's in the leaves of the subtree with a_i in the root. In the second phase, the root-to-leaves phase—described by the second **for**-statement—b's are computed, with each b_i being the sum of all a's from a_1 to the rightmost one in the subtree with b_i in the root. (That is why all b's along any rightmost path of a subtree have the same value.) To compute new b's each parent sends its b-value to its children. The right child keeps this value, but the left child subtracts from it the current a-value of its sibling.

Exercise 4.4.16 *Observe that the memory accesses are data independent in both above PRAM algorithms for the prefix sum. Show that these algorithms and their generalizations with any associative operation $\odot$ instead of $+$, can be represented by a circuit of depth $\mathcal{O}(\lg n)$, size $\mathcal{O}(n)$ and $\odot$-gates.*

Exercise 4.4.17 *Using the prefix sum algorithm design an $\mathcal{O}(\lg n)$ an EREW PRAM algorithm for the following variant of the knapsack problem: Given n objects $o_1, \dots, o_n$, their weights $w_1, \dots, w_n$, their values $v_1, \dots, v_n$ and an integer k, find a vector $(x_1, \dots, x_n)$, $0 \le x_i \le 1$ such that $\sum_{i=1}^n x_i v_i$ is maximized and $\sum_{i=1}^n x_i w_i \le k$.*

Example 4.4.18 (List ranking problem) *Given a linked list of* n *elements* $a_1, \ldots, a_n$*, the task is to compute* $b_1, \ldots, b_n$*, where* $b_i = \sum_{j=i}^{n} a_j$*. (The requirement that* $a_1, \ldots, a_n$ *form a list means that an access to* a_{i+1} *is only through a pointer from* a_i*—see Figure 4.32a.)*

Assume that the linked list is specified by two arrays

$$\textit{contents-array}: \ C[1:n] \qquad \textit{and} \qquad \textit{successor-array}: \ S[1:n]$$

where $S[i]$ *is a pointer to the successor of* $a[i]$*. (That is, if* $C[i] = a_j$*, then* $C[S[i]] = a_{j+1}$*. In the following algorithm we assume that the last element is in* $C[n]$ *and* $C[n] = 0$*,* $S[n] = n$*.*

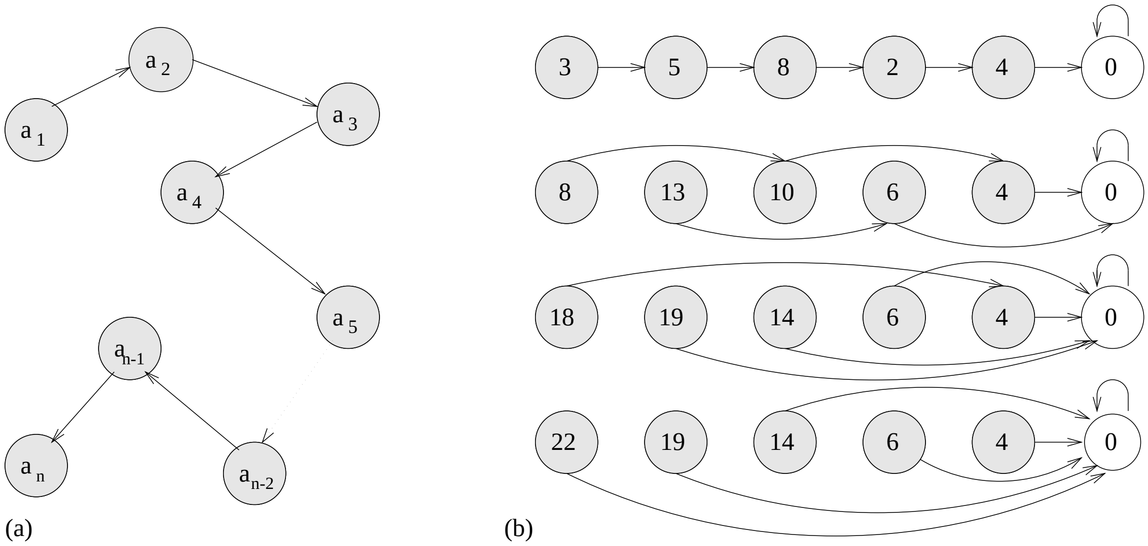

Figure 4.32: List ranking algorithm execution–70%

Program 4.4.19 (Link ranking algorithm for an EREW PRAM)

```
**do** $\lceil \lg n \rceil$ times
   **par**$[1 \le i \le n]$: **begin** $C[i] \leftarrow C[i] + C[S[i]]; S[i] \leftarrow S[S[i]]$
**par**$[1 \le i \le n]$: WRITE $C[i]$
```

This simple program is based on the so-called **pointer jumping method** and its execution is illustrated for $n = 6$ in Figure 4.32b.

The execution time is $\Theta(\lg n)$ and $\Theta(n)$ processors are needed. A work-optimal algorithm can be obtained using a more sophisticated technique.

Example 4.4.20 (Maximum finding on doubly-logarithmic-depth trees.) *Our first CRCW PRAM algorithm for the maximum finding has the work complexity* $\Theta(n^2)$*, the second algorithm, for EREW PRAM, the work complexity* $\Theta(n \lg n)$*. Is there a better solution? Yes, it is. Consider a doubly-logarithmic tree of processors with* $n = 2^{2^k}$ *leaves, where each internal node-processor performs the constant time algorithm to find a maximum. The*

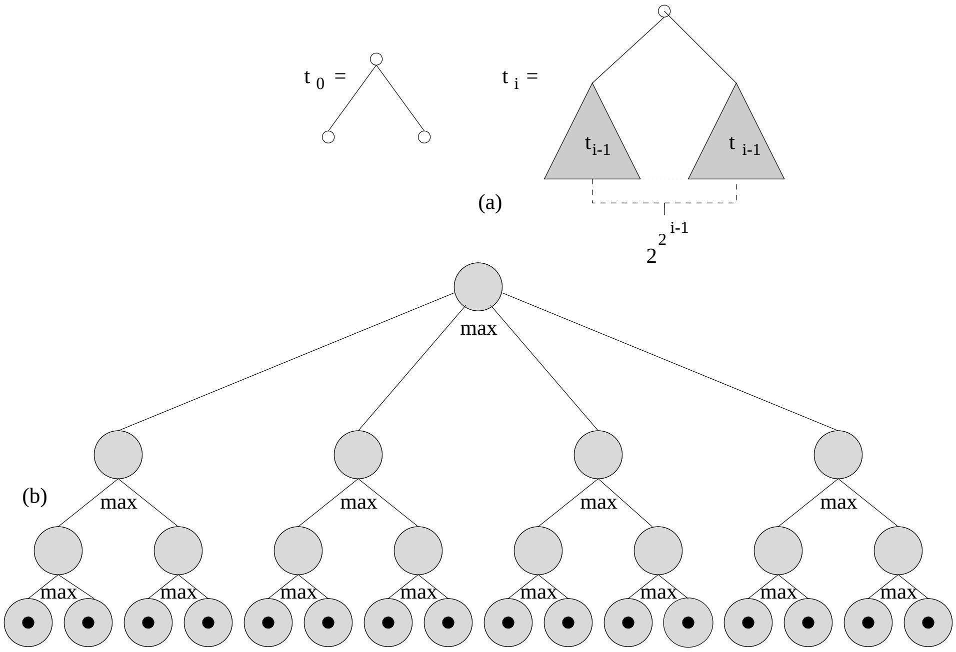

Figure 4.33: Maximum finding on a doubly-logarithmic-depth tree-50%

inductive definition of the doubly-logarithmic-depth trees t_i *is given in Figure 4.33a—see also Section 2.5.3. The case* $i = 2$ *is depicted in Figure 4.33b. By induction we can show that the depth of* t_k *is* $\lg\lg n + 1$*, the* i*-th level of* t_k *has* $2^{2^k - 2^{k-i}}$ *nodes, and each of them has* $2^{2^{k-i-1}}$ *children. The total number of operations needed to be performed on the* i*-th level is:*

$$\underbrace{\Theta((2^{2^{k-i-1}})^2)}_{\#\textit{ of operations}} \times \underbrace{\Theta(2^{2^k - 2^{k-i}})}_{\#\textit{ of processors}} = \Theta(2^{2^k}) = \Theta(n).$$

The overall work complexity is therefore $\Theta(n \lg\lg n)$*.*

The examples we have shown, and the way they have been presented, illustrate the main advantage of the PRAM model: a possibility to develop the basic ideas of the program on a "machine-independent level" and not too much a need to consider communication and synchronization problems.

Exercise 4.4.21 *Design an EREW PRAM program to sort* n *distinct numbers on* $\mathcal{O}(\lg n)$ *time.*

Exercise 4.4.22 (Shortest path in a graph.) *Design an* $\mathcal{O}(\lg^2 n)$ *algorithm to determine a shortest path between any two nodes of an* n *node complete graph whose edges are labeled by integers representing their length.*

4.4.6 Parallel Computation Thesis

Basic concepts concerning (feasible) parallel computing are very robust and so are its main complexity classes. In addition, these classes are closely related to the complexity classes of sequential computing.

We start with a general concept of a PRAM complexity class. If Φ is a class of PRAM and $t, p : \mathbf{N} \to \mathbf{N}$ are functions (complexity bounds), then

$$\Phi\text{-TimeProc}(t(n), p(n))$$

is the family of languages accepted by a $t(n)$-time bounded and $p(n)$-processor bounded PRAM of the family Φ. Of a special importance is the class

$$\text{PRAM-Time}(t(n)) = \text{CRCW}^{+}\text{-TimeProc}(t(n), 2^{t(n))})$$

where CRCW^{+} stands for PRAM CRCW^{pri} all RAM of which are RAM^{+}, i.e. without multiplication and division. The main advantage of CRCW^{+} is that for the inputs of size $\mathcal{O}(1)$, for example, for language recognition problems, the maximal length of a register, after $t(n)$ computational steps, is $\mathcal{O}(t(n))$. This allows to compare CRCW^{+} complexity classes with those of Turing machines computations.

A relation between time complexity and processor complexity of PRAM is presented in the following lemma. This lemma also shows the "price of sequentialization". Φ stands there again for any of the PRAM models considered so far.

Lemma 4.4.23 *Let $t, p, k : \mathbf{N} \to \mathbf{N}$, and let $k(n)$ be computable by a RAM^{+} in time $t(n)$. Then*

$$\Phi\text{-}\mathit{TimeProc}(t(n), p(n)) \subseteq \Phi\text{-}\mathit{TimeProc}(\mathcal{O}(k(n)t(n)), \left\lceil \frac{p(n)}{k(n)} \right\rceil)$$

Proof. Given a PRAM $\mathcal{R} \in \Phi$ of processor complexity $p(n)$, we show how to construct another PRAM $\mathcal{R}' \in \Phi$ that uses only $\lceil \frac{p(n)}{k(n)} \rceil$ processors and simulates $\mathcal{R}$ in $\mathcal{O}(k(n)t(n))$ time. For an input of length n, $\mathcal{R}'$ first computes $k(n)$ and then simulates $\mathcal{R}$ in such a way that each processor P'_i of $\mathcal{R}'$ simulates the processors $P_{(i-1)k(n)+1}, \ldots, P_{ik(n)}$ of $\mathcal{R}$. Each concurrent step of $\mathcal{R}$ is therefore simulated by $\mathcal{O}(k(n))$ steps of $\mathcal{R}'$. To do this, the local memory of P'_i is so structured that it simulates local memories of all simulated processors.

Care has to be taken that a concurrent step of $\mathcal{R}$ is simulated properly in the sense that it does not happen that a value is overwritten before needed by some other processor in the same parallel step. Another problem is the priority handling by CRCW^{pri} PRAM. All that can be easily taken care for if each register R of $\mathcal{R}$ is simulated by three registers of $\mathcal{R}'$: one with the old contents of R, one with new contents and one that keeps the smallest PID of those processors of $\mathcal{R}$, that try to write into R. To read from a register the old contents are used. To write into a register, in case of the CRCW^{pri} PRAM model, the priority stored in one of the three corresponding registers has to be checked. This way $\mathcal{R}'$ needs $\mathcal{O}(k(n))$ steps to simulate one step of $\mathcal{R}$. □

As a consequence we have:

Theorem 4.4.24 $\bigcup_{k=1}^{\infty} \mathit{CRCW}^{+}\text{-}\mathit{TimeProc}(n^k, n^k) = \mathbf{P}$ *(polynomial time).*

Proof. For $k(n) = p(n) = t(n) = n^k$ we get from the previous lemma:

$$\begin{aligned} \text{CRCW}^{+}\text{-TimeProc}(n^k, n^k) &\subseteq \text{CRCW}^{+}\text{-TimeProc}(\mathcal{O}(n^{2k}), 1) \\ &= \text{RAM}^{+}\text{-Time}(\mathcal{O}(n^{2k})) \subseteq \mathbf{P} \end{aligned}$$

and the opposite inclusion is trivial. □

We are now in the position to look more clearly on problem of feasibility in the framework of parallel computation.

It has turned out that the following concepts are to a very large degree independent of parallel computer models.

- A problem is **feasible** if it can be solved by a parallel algorithm with polynomial worst case time and processor complexity.
- A problem is **highly parallel** if it can be solved by an algorithm with worst case polylog time complexity ($\lg^{\mathcal{O}(1)} n$) and polynomial processor complexity.
- A problem is **inherently sequential** if it is feasible but not highly parallel.

Observe that Theorem 4.4.24 implies that the class of inherently sequential problems is identical with the class of **P**-complete problems.

One of the main results that justifies an introduction of the term of highly parallel computational problem is now presented:

Theorem 4.4.25 *A function $f : \{0,1\}^* \to \{0,1\}^*$ can be computed by a uniform family of Boolean circuits $\{C_i\}_{i=1}^{\infty}$ with Depth$(C_n) = \lg^{\mathcal{O}(1)} n$, if and only if f can be computed by a CREW$^+$ PRAM in time $t(n) = (\lg n)^{\mathcal{O}(1)}$ and Proc$(n) = n^{\mathcal{O}(1)}$ for inputs of length n.*

The main result of this section concerns the relation between the space for TM-computations and time for PRAM-computations.

Lemma 4.4.26 *Space$(s(n))$ $\subseteq$PRAM-Time$(\mathcal{O}(s(n)))$, if $s(n) \geq \lg n$ is a time-constructible function.*

Proof. Let $\mathcal{M}$ be a $s(n)$ space bounded MTM. The basic idea of the proof is the same as in the proof of Lemma 4.3.20. The space bound $s(n)$ allows to bound the number of possible configurations of $\mathcal{M}$ by $t(n) = 2^{\mathcal{O}(s(n))}$. For each input x of length n let us consider a $t(n) \times t(n)$ Boolean transition matrix $T_{\mathcal{M}}(x) = \{a_{ij}\}_{i,j=1}^{t}$ with $t = t(n)$ and

$$a_{ij} = 1 \Longleftrightarrow i = j \text{ or } c_i \vdash_{\mathcal{M}} c_j \qquad \text{on the input } x,$$

where c_i, c_j are configurations of $\mathcal{M}$, describing the potential behaviour of $\mathcal{M}$.

A PRAM with t^2 processors can compute all a_{ij} in one step. In order to decide whether $\mathcal{M}$ accepts x, it is enough to compute $T_{\mathcal{M}}^{t}(x)$ (and from the resulting matrix to read whether x is accepted). This can be done using $\lceil \lg t \rceil$ Boolean matrix multiplications. By Example 4.4.8, Boolean matrix multiplication can be done by a CRCW$^+$ PRAM in the constant time. Since $\lceil \lg t \rceil = \mathcal{O}(s(n))$, this implies the Lemma. □

Lemma 4.4.27 *If* $t(n) \geq \lg n$ *is time-constructible, then* $\mathit{PRAM\text{-}Time}(t(n)) \subseteq \mathit{Space}(t^2(n))$.

Sketch of the proof. The basic idea is very simple, details are technical and cumbersome. A $t(n)$-time bounded PRAM$^+$ $\mathcal{R}$ has after $t(n)$ steps all registers of the size $\mathcal{O}(t(n))$ and at most $2^{t(n)}$ registers have been used. One needs therefore $\mathcal{O}(t(n))$ bits to store the number of any of the registers that have been used.

$\mathcal{R}$ can be simulated in $\mathcal{O}(t^2(n))$ space as follows. For any of the possible registers of $\mathcal{R}$ and any of its possible contents, it is verified whether it is the correct contents at the end of the computation. To perform such a verification a backtracking has to be done to guess previous steps of $\mathcal{R}$ and, in the worst case, all possibilities are to be checked. This requires enormous time. However, since the depth of the recursion is $t(n)$ and at each recursion level only $\mathcal{O}(t(n))$ bits have to be stored to remember the numbers of the registers that have been used and their contents. The overall space needed is therefore $\mathcal{O}(t^2(n))$. □

As a corollary of the last two lemmas we have:

Theorem 4.4.28 *Turing machines with respect to the space complexity and CRCW$^+$ PRAM with the respect to the time complexity are polynomially related.*

As another corollary we get

Theorem 4.4.29 $\bigcup_{k=1}^{\infty} \mathit{PRAM\text{-}Time}(n^k) = \mathbf{PSPACE}$.

Observe that in principle the result stated in the last theorem is analogous to the one in Theorem 4.3.24—namely that the space on MTM and the parallel time on uniform families of Boolean circuits are polynomially related. The same results have been shown also for other natural models of parallel computers and this has lead to the following thesis:

> **Parallel Computation Thesis.** There exists a standard class of (inherently parallel) computer models, which includes among others several variants of PRAM machines and uniform families of Boolean circuits, for which the polynomial time is as powerful as the polynomial space for machines from the First Machine Class.

Computer models that satisfy the parallel computation thesis form the so-called **Second Machine Class**.

It seems intuitively clear that **PSPACE** is a much richer class than **P**. However, no proof of that is known and the problem

$$\mathbf{P}{=}\mathbf{PSPACE}$$

is another important open problems in foundations of computing.

In order to see how subtle this big problem is, notice that RAM$^+$ with the uniform time and space complexity measures are in the First Machine Class, but RAM with division and the uniform time and space complexity measures are in the Second Machine Class! (So powerful are multiplication and division!) This result also clearly demonstrates often ignored fact that not only the overall architecture of a computer model and its instruction repertoir but also complexity measures form an inherent part of the model.

Exercise 4.4.30 *Is it true that the First and Second Machine Class coincide if and only if* $\mathbf{P} = \mathbf{PSPACE}$*?*

4.4.7 Relations between CRCW PRAM Models

There is a natural ordering between basic PRAM computer models:

$$\text{EREW PRAM} \prec \text{CREW PRAM} \prec \text{CRCW}^{com} \text{ PRAM} \prec \text{CRCW}^{arb} \text{ PRAM} \prec \text{CRCW}^{pri} \text{ PRAM}$$

in the sense that each program for one of these models can be run on all "higher" models with respect to the ordering $\prec$. This is evidently true. Less easy is to see that each higher model is strictly more powerful than the previous one. In spite of that, and this seems to be even less obvious, differences between the power of these models are actually not too big.

Theorem 4.4.31 *A step of a CRCW*^{pri} *PRAM* $\mathcal{R}$ *with* p *processors and* m *memory registers can be simulated by an EREW PRAM* $\mathcal{R}'$ *in* $\mathcal{O}(\lg p)$ *steps with* p *processors and* mp *registers.*

Proof. The i-th processor P_i of $\mathcal{R}$, $1 \leq i \leq p$, will be simulated by the i-th processor $\mathcal{P}_i'$ of $\mathcal{R}'$. To each register $GM_{\mathcal{R}}[j]$, $1 \leq j \leq m$, there will correspond p registers of $\mathcal{R}'$. One of them will simulate the corresponding register of $\mathcal{R}$; $p-1$ other registers will be used to resolve the access conflicts for $GM_{\mathcal{R}}[j]$.

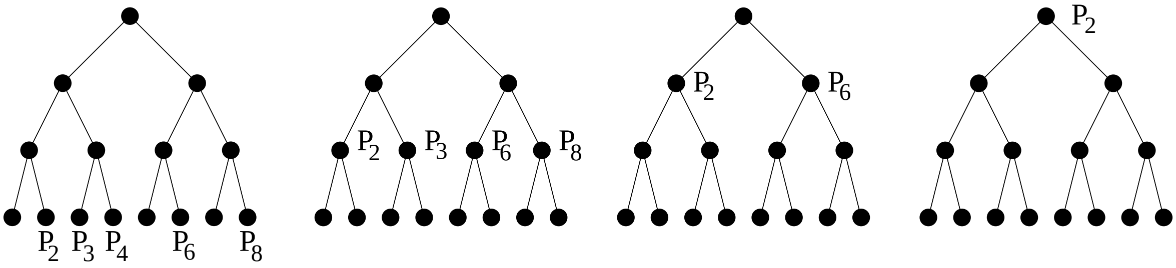

Figure 4.34: Determination of the left most processor70%

In order to perform a concurrent write to a register $GM_{\mathcal{R}}[j]$, each processor of $\mathcal{R}$ has first to find out whether it is the processor with the smallest PID, wishing to write to $GM_{\mathcal{R}}[j]$. This can be done, on $\mathcal{R}'$, in parallel for $1 \leq j \leq m$, as follows. Let us imagine for the moment that if the i-th processor wants to write to $GM_{\mathcal{R}}[j]$, then it is located in the i-th leaf of a binary tree T_j with p leaves—see Figure 4.34a for the case processors P_2, P_3, P_4, P_6, P_8 wants to write to $GM_{\mathcal{R}}[j]$. The process of determining the leftmost of these processors can be done, simultaneously for all j, by having processors "to climb T_j trees" according to the following rules: A processor in a left-child-node can always go up to the parent node; a processor in a right-child-node can go up only if there is no processor in its sibling node. This way the leftmost processor gets to the root and can then perform writing. Concurrent reads are performed in a similar way. The leftmost processor moves to the root, reads and then starts to go down the tree to its original position. On this way down the tree, the processor distributes data obtained to processors "waiting for it" and they, in turn, do the same on their way to their original positions.

This "tree climbing" can be simulated on $\mathcal{R}'$ using $p-1$ registers available for each register of $\mathcal{R}$ in time $\mathcal{O}(\lg p)$, proportional to the depth of trees. $\Box$

A CRCW^{pri} PRAM can be simulated by a CRCW^{com} PRAM without any time loss, but using a larger number of processors.

Theorem 4.4.32 *Any CRCW^{pri} PRAM with $p(n)$ processors and computation time $t(n)$ can be simulated by a CRCW^{com} PRAM with $\binom{p(n)}{2} + p(n)$ processors in time $\mathcal{O}(t(n))$.*

Proof. Let $P_1, \ldots, P_{p(n)}$ be the processors used by a CRCW^{pri} PRAM $\mathcal{R}$. A CRCW^{com} PRAM $\mathcal{R}'$ will use additional processors P_{ij}, $1 \leq i < j \leq p(n), i \neq j$, and new shared memory registers $R_1, \ldots, R_{p(n)}$ initialized to 0. In order to simulate one step of $\mathcal{R}$, P_{ij} writes 1 to R_j if $i < j$ and both P_i and P_j try to write to the same memory register. A processor P_i can then determine whether it is the processor with the lowest PID wishing to write to a register by checking whether $R_i = 0$. $\Box$

Finally, we present some results showing that there is a strict hierarchy between the power of main models of PRAM, with respect to the computational power when allowing the same computational time.

Theorem 4.4.33 *(1) CREW PRAM are strictly more powerful than EREW PRAM. (2) To compute the function $OR(x_1, \ldots, x_n)$ a CREW PRAM needs $\Omega(\lg n)$ steps.*

Theorem 4.4.34 *The CRCW^{com} PRAMs are strictly more powerful than CREW PRAM.*

Proof. By Theorem 4.4.33 it is sufficient to show that a CRCW^{com} can compute $\mathrm{OR}(x_1, \ldots, x_n)$ in a constant number of steps. Indeed, GM_0 is first initialized to 0 and then, for $1, \ldots, n$, the j-th processor reads x_j and writes 1 into GM_0, if and only if $x_j = 1$. $\Box$

Exercise 4.4.35 *Show that a CRCW^{com} PRAM with a polynomial number of processors can add two n bit integers in constant time.*

Exercise 4.4.36 *Show that with $n2^n$ processors a CRCW^{com} PRAM can compute any Boolean function of n variables in constant time.*

Theorem 4.4.37 *The CRCW^{pri} PRAM is a strictly more powerful model than CRCW^{arb} PRAM and this model is in turn strictly more powerful than CRCW^{com} PRAM.*

4.5 Cellular Automata

Around 1953, in an attempt to find an answer to one of the most intriguing scientific questions of that period:

Can machines reproduce themselves?

John von Neumann[15] known today as one the fathers of modern (sequential) computers of "von Neumann type", developed a cellular automata model of the biological world. This allowed him to get an affirmative answer to the question of concern. Since then cellular automata have been developed to be one of the basic models of information processing not only for the biological world but also for the world of microscopic physics (to study dynamical, complex, and chaotic systems and fractal phenomena), and also a fundamental model of massively parallel computing, often called, paradoxically, as a "non von Neumann type" computing.

Useful models of massive parallel computers have to be constructed in hardware and should be scalable. As such they can be based on local interactions only. This is why since the early days of modern computing a large effort has been spent to understand what is meaningfully achievable, in principle and in practice, by computational devices consisting of large numbers of simple and interacting processing units lacking a central executor that could provide global coordination and control. Cellular automata are a framework to deal with these problems.

4.5.1 Basic Concepts

Informally, a d-dimensional cellular automaton $\mathcal{A}$ with a finite set of **states** Q, a finite **neighbourhood** $N \subseteq \mathbf{Z}^d$, and a **local transition function** $\delta : Q^N \to Q$, is a d-dimensional array network of identical finite automata with Q as their set of states. For each node $n \in \mathbf{Z}^d$, the neighbourhood N determines the set $\{n\} + N$ of "neighbours" of the node n. Formally, $\mathcal{A} = \langle d, Q, N, \delta \rangle$ and elements (nodes) of $\mathbf{Z}^d$ are ragarded as representing those finite automata which $\mathcal{A}$ consists of.

Figure 4.35a depicts three neighbourhoods of the dotted nodes for one-dimensional cellular automata: $N_1 = \{0,1\}$, $N_2 = \{-1,0,1\}$, $N_3 = \{-2,-1,0,1,2,\}$. Figure 4.35b illustrates two often used neighbourhoods (again for the dotted nodes) for two-dimensional cellular automata: **von Neumann neighbourhood** $N_4 = \{(-1,0),(0,-1),(0,0),(0,1),(1,0)\}$, and **Moore neighbourhood** $N_5 = \{(-1,-1),(-1,0),(-1,1),(0,-1),(0,0),(0,1),(1,-1),(1,0),(1,1)\}$.

All finite automata of a cellular automaton work concurrently, synchronized and in discrete time steps. At each time moment the new state of each finite automaton is defined to be the value of the local transition function applied to its current state and to the states of all its neighbours.

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[15] John von Neumann (1903-1957), a USA mathematician and physicist of Hungarian origin, one of the leading scientists of his period. He made fundamental contributions to almost all areas of modern mathematics and its applications as well as to theoretical physics: axiomatization of set theory, functional analysis and quantum physics, mathematical logic, automata theory, probability theory, game theory, numerical mathematics, mathematical methods in economics. Von Neumann was one of the leading persons in the development of the first very powerful electronic computer and also of the first atomic bomb.

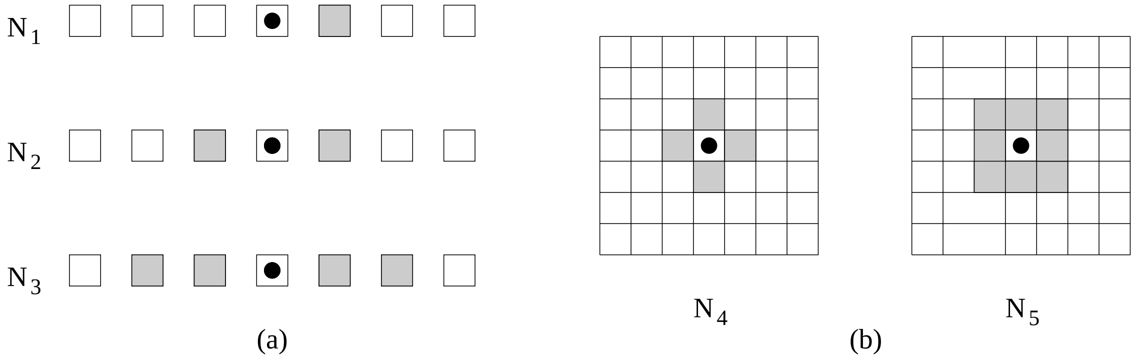


Figure 4.35: Neighbourhoods for one and two-dimensional cellular automata-65%

In order to describe more formally the overall behaviour of a cellular automaton $\mathcal{A} = \langle d, Q, N, \delta \rangle$ the concept of configuration is again the basic one.

A **configuration** of $\mathcal{A}$ is a mapping $c \in Q^{\mathbf{Z}^d}$ assigning to each finite automaton of $\mathcal{A}$ (that is to each node of $\mathbf{Z}^d$) a state. The **global transition function**

$$G_\delta : Q^{\mathbf{Z}^d} \rightarrow Q^{\mathbf{Z}^d},$$

$$G_\delta(c)(w) = \delta(c(\{w\} + N)),$$

is used to define a **computation (behaviour)** of $\mathcal{A}$ to be a sequence of configurations $c_0, c_1, c_2, \ldots$ such that $c_i = G_\delta(c_{i-1})$, for $i \geq 1$.

For example, if $d = 1$, $N = \{-1, 0, 1\}$, then

$$G_\delta(c)(i) = \delta(c(i-1), c(i), c(i+1));$$

if $N = \{-k, -k+1, \ldots, -1, 0, 1, \ldots, k-1, k\}$, then

$$G_\delta(c(i)) = \delta(c(i-k), c(i-k+1), \ldots, c(i+k-1), c(i+k));$$

if $d = 2$ and the von Neumann neighbourhood is considered, then

$$G_\delta(c(i,j)) = \delta(c(i-1,j), c(i,j-1), c(i,j), c(i,j+1), c(i+1,j)).$$

It is often assumed that the set of states Q has one special state—a sleeping (or quiescent) state—such that if a finite automaton A of $\mathcal{A}$ and all its neighbours are in the sleeping state, then the next state of A is the sleeping state, too.

One of the first problems investigated in cellular automata was the question whether there is a configuration, called Garden of Eden, that cannot be obtained from another configuration by a cellular automaton step. This problem is related, as already discussed in Section 2.3.1, with the injectivity-bijectivity-reversibility problem for global transition functions.

The basic decision problem for cellular automata is the so-called **reachability problem**: given a cellular automaton $\mathcal{A}$ with the global transition function G_δ and two configurations c and c', is there an i such that $G^i_\delta(c) = c'$?

Remark 4.5.1 John von Neumann designed a cellular automaton consisting of 29-state finite automata and showed that there is a configuration of about 200 of such finite automata that is universal (it can simulate a universal Turing machine) and a larger configuration, estimated to consist of several hundred thousands of finite automata that is also self-reproducible.

Exercise 4.5.2 * *Self-reproducibility in a trivial sense—without having a configuration that simulates a universal Turing machine—is easy to achieve. Consider a two states* ($\{0,1\}$) *two-dimensional cellular automaton with the neighbourhood* $N = \{(-1,0),(1,0),(0,1),(0,-1)\}$ *and the transition function* $f(x,y,z,u) = (x+y+z+u) \bmod 2$ *and show that for each initial distribution of the live states (states* 1*), with only finitely many* 1 *states, there is an integer* n *such that the initial configuration reproduces itself* 4 *times after* n *steps.*

Exercise 4.5.3 *Determine the number of two-state two-dimensional cellular automata with: (a) von Neumann neighbourhood; (b) Moore neighbourhood.*

4.5.2 Case Studies

With several examples we illustrate the power of cellular automata.

The behaviour of two-state one-dimensional cellular automata with the neigbourhood $N = \{-1,0,1\}$

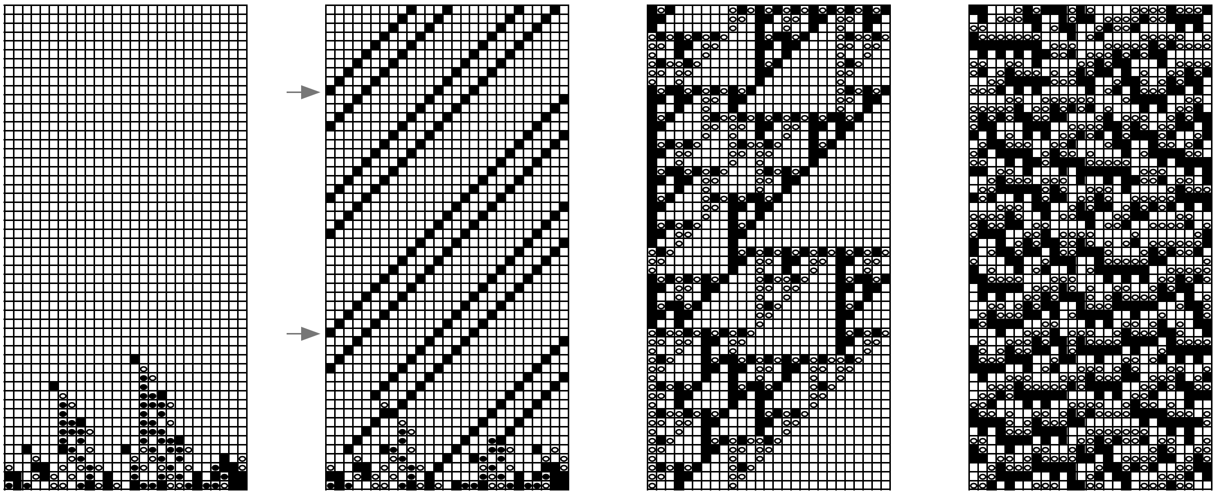
Even apparently very simple one-dimensional cellular automata, with only two-states and a nearest neigbours neighbourhood, can exhibit a very complex/chaotic behaviour.

Each such cellular automaton is uniquely defined by a mapping $\{0,1\}^N \rightarrow \{0,1\}$. There are therefore 2^8 of such cellular automata and each of them is uniquely specified by the following (transition) rule

$$\begin{array}{lllllllllll}
000 & \rightarrow & a_0 & \quad 010 & \rightarrow & a_2 & \quad 100 & \rightarrow & a_4 & \quad 110 & \rightarrow & a_6 \\
001 & \rightarrow & a_1 & \quad 011 & \rightarrow & a_3 & \quad 101 & \rightarrow & a_5 & \quad 111 & \rightarrow & a_7
\end{array}$$

and therefore by an integer i with the binary representation $bin_8^{-1}(i) = a_7a_6a_5a_4a_3a_2a_1a_0$. Such an automaton can therefore be specified by an integer $0 \leq i < 256$ and we can speak about the cellular automaton with the rule i. This terminology is often used in the literature about cellular automata.

Some of these 256 cellular automata are not very interesting, but some really are. For example those with the numbers 18, 20, 30, 52, 54, 60, 80, 90, 102, 120, 129, 150, 184, 190, 216. The behaviour of some of them is so complex that attempts to get deeper and as full as possible understanding are clearly a never ending story. Some behaviours of these one-dimensional cellular automata are shown in Figure 4.36, in which some of the so-called



Cellular automata with convergent behaviour

(a)

Cellular automata with periodic behaviour

(b)

Cellular automata with fractal behaviour

(c)

Cellular automata with chaotic behaviour

(d)

Figure 4.36: Behaviour of one-dimensional cellular automata

spatio-temporal evolution patterns are depicted. The i-th line shows the states of the corresponding finite automaton at the time i (counting is bottom-up). Figure 4.36a depicts a convergent behaviour, Figure 4.36b a periodic behaviour, Figure 4.36c a fractal behaviour and Figure 4.36d a chaotic behaviour.

Exercise 4.5.4 * *One-dimensional cellular automata can also be considered, in several ways, as language recognizers. For example, we say that a cellular automaton* $\mathcal{A} = \langle 1, Q, \{-1, 0, 1\}, \delta \rangle$, *recognizes a language* $L \subseteq \Sigma^* \subset Q^*$, *with respect to the set* $Q_F \subset Q - \Sigma$ *of final states, if* $w \in L$ *if and only if the automaton* $\mathcal{A}$ *starting with the configuration* w *(and all other automata in the quiescent state* $\sqcup \in Q - \Sigma$*) comes to a state from* Q_F *in the finite automaton with the first symbol of* w *at the beginning. Design a cellular automaton recognizing the language* $\{a^n b^n c^n \,|\, n \geq 1\}$.

LIFE Game

In an attempt to develop a model of "artificial life" in the late 1960's J. H. Conway from Cambridge, England, designed a two-state two-dimensional cellular automaton with the Moore neighbourhood that has become perhaps the most intensively investigated cellular automaton so far. It is called the LIFE game because one of the first tasks was to find initial configurations that would exhibit behaviour of "living creatures". The transition rule is simple:

> The new state of each finite automaton is 1 if and only if one of the following conditions is satisfied:
>
> 1. its old state was 1 and at least two and at most three of its neighbours were in the state 1;
> 2. its old state was 0 and exactly three of its neighbours were in the state 1.

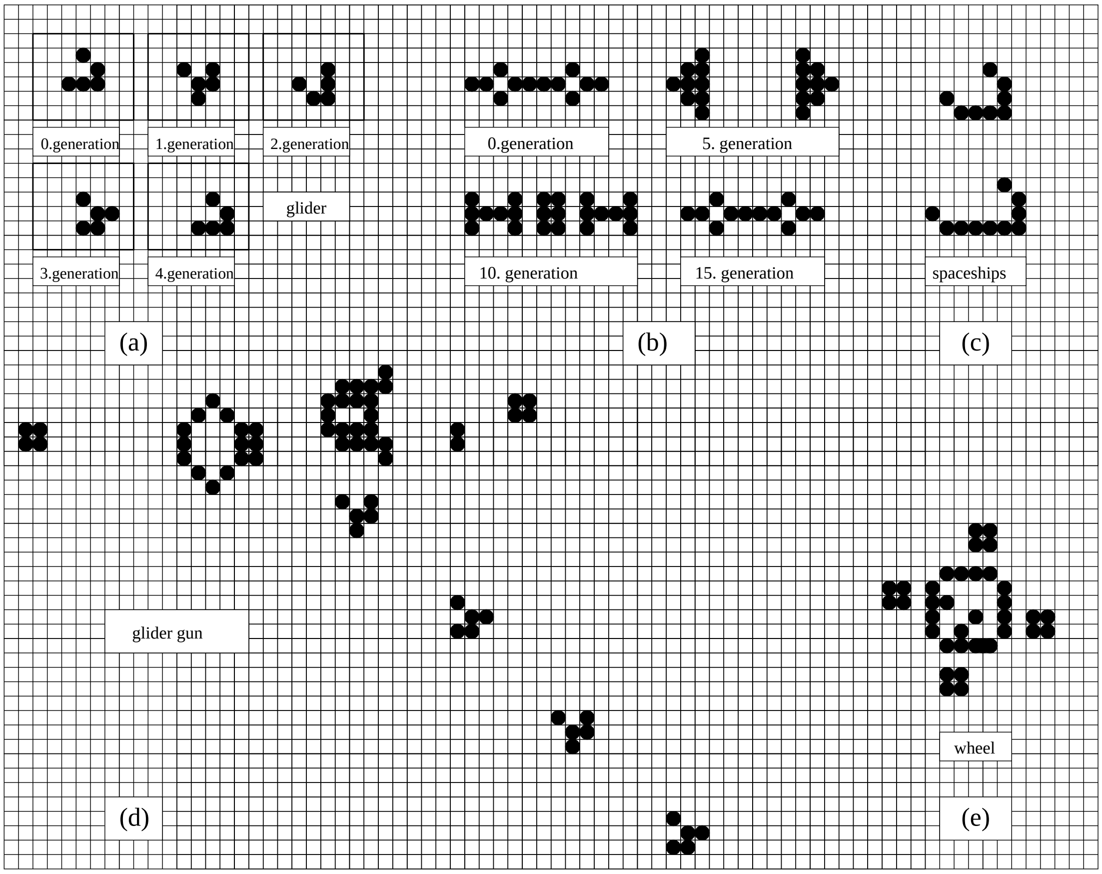


Figure 4.37: LIFE game configurations-60%

Figure 4.37a shows an initial configuration, called **glider**, and its four next configurations. (Black squares are used to denote the state 1; the empty square denotes the state 0.) Figure 4.37b shows a configuration that repeats itself in site after 15 steps. The configuration shown in Figure 4.37d is called Gosper's **glider gun**, because periodically, with the period of 30 steps, fires again and again a glider that starts to move "south east". In addition, if the initial configuration from Figure 4.37b is properly positioned into the way gliders move, it starts “to eat” them. The glider gun itself can be created after 174 steps from 13 gliders properly positioned. One can also position two glider guns in such a way that the intersecting streams of gliders “build a factory that assembles and fires a spaceship every 300 moves,...”. (The term “spaceship” is used to denote a configuration that keeps moving in the plane “throuwing away some smoke”. Two of them are depicted in Figure 4.37c.) Possibilities to create configurations with interesting behaviour for LIFE seem to be limitless—see for example a “wheel” in Figure 4.37e that keeps “rotating in site” in the sense that three internal 1’s rotate.

On a more theoretical side, the power of gliders to act as “pulses” was used to show that there is an initial configuration that can simulate a universal Turing machine! It has also been shown that LIFE has a Garden of Eden configuration.

Exercise 4.5.5 *Design a program to simulate LIFE and use it: (a)* to simulate several steps of the behaviour of spaceships, wheel and the glider gun depicted in Figure 4.37; (b)** to determine how to position configuration from Figure 4.37b into the way of the gliders fired from the glider gun in order to let the configuration to destroy the gliders.*

(c)** Let us call mosaic a configuration that contains 1 *in the automaton position in the node* (i,j) *if and only if* $1 \leq i \leq 17$, $i \bmod 3 = 0$, $1 \leq j \leq 17$, $j \bmod 3 = 0$. *The mosaic is clearly a stable configuration. We say that such a mosaic has a “virus” if one of the automata in middle originally in state* 0 *is in state* 1. *Find the way to position virus into mosaic in such a way that: (a) the mosaic destroys the virus; (b) the virus destroys the mosaic.*

Firing Squad Synchronization Problem

Cellular automata are by definition infinite networks of finite automata. This full generality of the concept is needed when simulation of physical phenomena is considered. In order to deal with the computational problems either only potential infiniteness is needed—each initial configuration has only finitely many cells with non-sleeping states—or even only finite networks of finite automata are considered. Perhaps the most famous problem for the later case is the so-called Firing Squad Synchronization Problem (FSSP), due to J. Myhill (1957), solutions of which shows a surprising synchronization power of finite automata networks.

The task is to design, if it exists, a single finite automaton, called popularly a **soldier**, with two inputs and two outputs, each new state of which depends on its old state and the two old states received at its two inputs from two neighbouring automata; the new state is then sent through its outputs to two neighbouring automata. The soldier should have among his states a **sleeping state**, a state **fire**, and a **fire when ready** state and it should have the following properties.

1. If a soldier is in a sleeping state and so are its neighbours, then it remains in the sleeping state.

2. If n of soldiers are connected into a row, n arbitrary, all in the sleeping state at the beginning, and the left one (called usually general), receives the input "fire when ready", then after a finite number of steps all soldiers simultaneously, and each of them for the first time, come into the state **fire** (and can therefore fire simultaneously as a squad should do)—see Figure 4.38. Missing inputs of the first and the last soldier are the sleeping states.

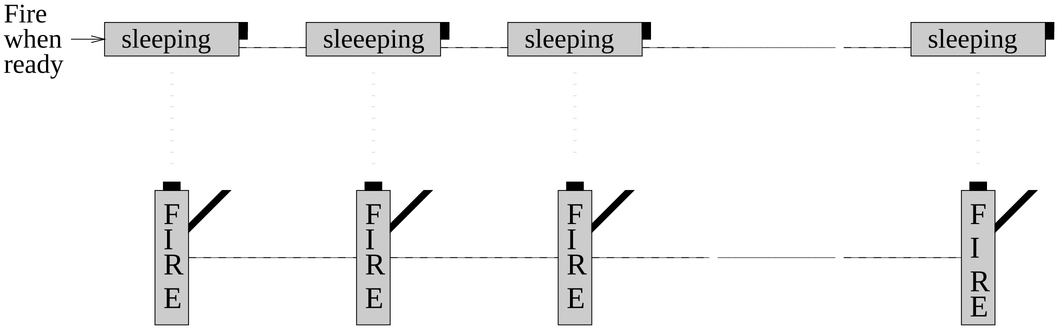


Figure 4.38: Firing squad - 70%

The main point is that the soldier has to be designed as a finite automaton in such a way that the firing squad works properly no matter how many soldiers are in the squad—that is, no matter how big n is.

Four questions are of immediate interest. Has this problem a solution at all? If yes, what is the minimal time the squad needs to get synchronized? How small (simple) the soldier can be (with respect to the number of states)? Can we achieve minimality simultaneously for the time the squad needs to fire, and for the size of the soldier?

It is easy to see that the minimal time needed for synchronization is $2n-2$. Indeed, this time is required that the leftmost soldier get a feedback that the general's order has reached all soldiers.[16]

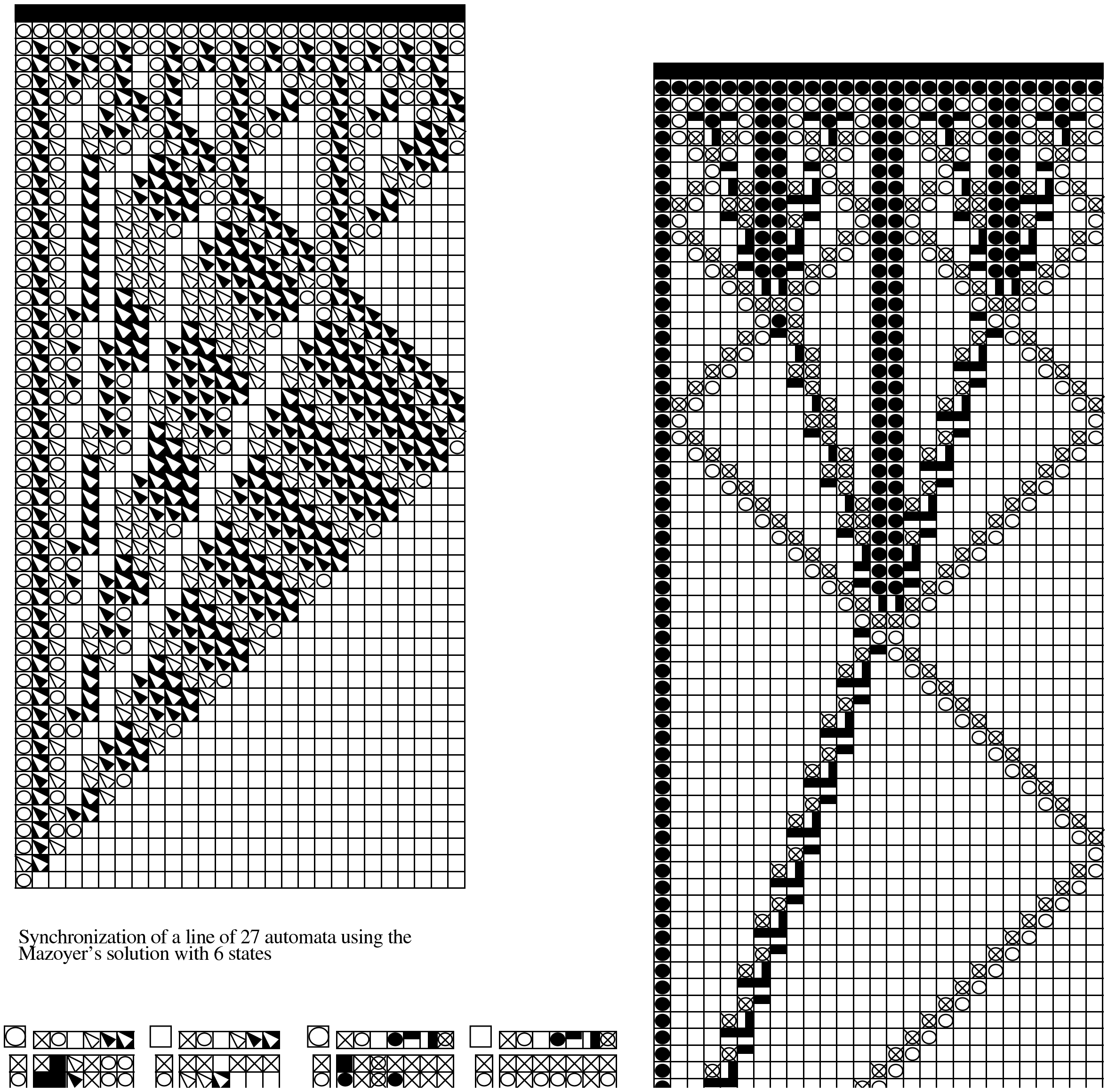
There is a simple, divide-and-conquer, solution, due to Minsky, though neither time nor size optimal, for $n = 2^k$. To describe the basic idea, we start with the situation that there is exactly one general and he is at the left end of the array—see Figure 4.40. (Warning: in this figure not all configurations are shown and time flows from top to bottom.)

1. Each general ($\clubsuit$) turns itself into a lieutenant ($\triangle$) and initializes on its both sides (or only on his right-hand side if he stands at the left end), a process to create a new general in the middle between him and the next lieutenant, or at the right end of the squad, whatever comes first.

2. A lieutenant goes into the firing state $\dagger$ if and only if both of his neighbours are lieutenants.

---

[16] The first minimal time solution, due to E. Goto in 1962 was estimated at that time as needing a soldier with about a milion of states. Currently the state-minimal solution, due to J. Mazoyer (1987) uses a 6-state soldier described and demonstrated in Figure 4.39. It has been shown that no 4-state soldier can yield the minimal solution and the case n=5 is open.



Synchronization of a line of 27 automata using the Mazoyer's solution with 6 states

Figure 4.39: Firing Squad Synchronization Problem Solutions

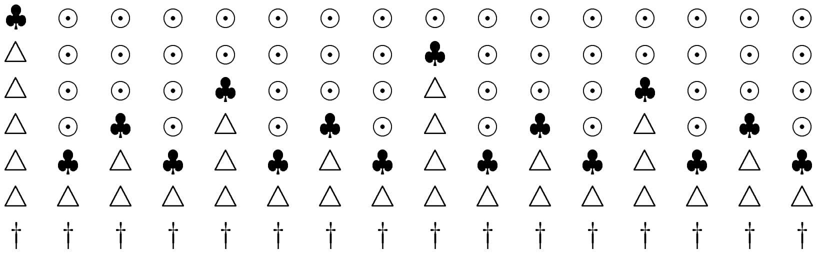


Figure 4.40: Divide and conquer solution of the FSSP-

It is easy to see that if the step 1 is synchronized, what is not a problem for $n = 2^k$, if the method indicated in Exercise 4.5.6 is used, then all lieutenants simultaneously enter the firing state after $3n - 1$ steps.

Exercise 4.5.6 ** *Design a one-dimensional cellular automaton with neighbourhood* $\{-1, 0, 1\}$ *and with states (*`left, right, sleeping, middle`} *in its set of states that is able to determine which automaton is in the middle of the row of an odd number of finite automata bounded by one in the state* `left` *and one in the state* `right`*. In other words a cellular automaton that when starting in the initial configuration* `left, sleeping`$^{2k-1}$`,right`*, and with all other finite automata in the* `sleeping` *state, then after a certain number of steps the automaton in the middle, between the ones in the initial states* `left` *and* `right` *comes (and for the first time) into the state* `middle` *and none of the other automata came to that state before. (Hint: The leftmost automaton sends two signals to the right, each with different speed, in such a way they meet in the middle.)*

The first minimal time solution for FSSP, with a small, 16-state, soldier, was due to Waksman (1966). A similar 7-state soldier solution, due to J.-B. Yunes, is also described and demonstrated in Figure 4.39. Surprisingly, the Firing Squad Synchronization Problem has a solution also in the case that soldiers form a balanced tree or an arbitrary graph all nodes of which have the same degree and also in case that some soldiers malfunction in the sense that they only let signals to go through them.

The Firing Squad Synchronization Problem solutions are the base of solutions of various synchronization tasks for cellular automata. They also show a surprising global synchronization power of local communications.

Exercise 4.5.7 *Sketch (design) a solution of the FSSP for the more general case of having one general in the squad in an arbitrary poistion.*

Exercise 4.5.8 ** *Show how to solve the FSSP for an arbitrary two-dimensional array of FA.*

4.5.3 A Normal Form

In the case of the cellular automaton LIFE a new state of the cell depends on its old state and on the number of 1-states of the neighbours—and not at all on the actual distribution of states 1 among neighbours. Many cellular automata, especially those used in physical applications, have a similar property. They are totalistic in the following sense:

Definition 4.5.9 *A cellular automaton* $\mathcal{A} = \langle 1, Q, \{-1,0,1\}, \delta\rangle$, *is called* **totalistic** *if* $Q \subset \mathbf{N}$ *and there is a function* $\Phi : \mathbf{N} \rightarrow \mathbf{N}$ *such that*

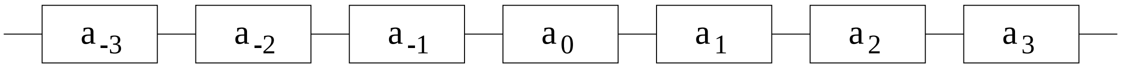
$$\delta(x,y,z) = \Phi(x+y+z).$$

As the following theorem shows a restriction to the totalistic cellular automata is not essential.

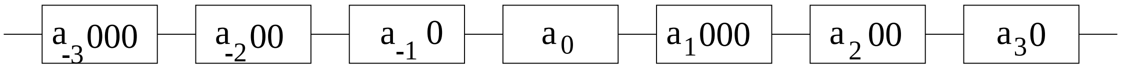
Theorem 4.5.10 *For each cellular automaton* $\mathcal{A} = \langle 1, Q, \{-1,0,1\}, \delta\rangle$, $Q \subset \mathbf{N}$, *there exists a totalistic cellular automaton* $\mathcal{A}' = \langle 1, Q', \{-1,0,1\}, \delta' \rangle$ *that has 4-times more states than* $\mathcal{A}$ *and simulates* $\mathcal{A}$ *without any time overhead.*

Proof. Assume that $Q = \{1,\ldots,n\}$ and denote $b = n+1$. We will use b as the basis to represent all integers and to perform all additions.

Take $Q' = \{q \cdot m | q \in Q, m \in \{1, 10, 100, 1000\}\}$ (i.e., all states of Q' have either the form q or $q0$ or $q00$ or $q000$, where $q \in Q$ and all these numbers are in the b-nary number system). A configuration of $\mathcal{A}$



will be simulated in $\mathcal{A}'$ by the configuration



In other words,

$$\textbf{if } n \bmod 4 = \begin{cases} 0, & \textbf{then } a_n \text{ is simulated by } a_n; \\ 1, & \textbf{then } a_n \text{ is simulated by } a_n000; \\ 2, & \textbf{then } a_n \text{ is simulated by } a_n00; \\ 3, & \textbf{then } a_n \text{ is simulated by } a_n0. \end{cases}$$

If Φ' is now defined, for $1 \leq x, y, z \leq b$, by

$$\begin{array}{rclcrcl}\Phi'(xyz) & = & \delta(x,y,z)0 & \qquad & \Phi'(xy0z) & = & \delta(z,x,y)000 \\ \Phi'(x0yz) & = & \delta(y,z,x) & \qquad & \Phi'(xyz0) & = & \delta(x,y,z)00,\end{array}$$

where $xyz = xb^2 + yb + z$ and similar number representations are used for other arguments), and $\delta'(x,y,z) = \Phi'(x+y+z)$, then $\mathcal{A}'$ is totalistic and simulates $\mathcal{A}$, step by step, in real time. □

Definition 4.5.9 and Theorem 4.5.10 can be easily extended to deal in a straightforward way with larger neighbourhoods and multi-dimensional cellular automata.

Exercise 4.5.11 *One-dimensional cellular automata with neighbourhood* $\{-1, 0\}$*, the so-called* **one-way cellular automata***, can also be seen as representing a normal form for one-dimensional cellular automata.Show that one-way one-dimensional cellular automata can simulate in a natural way any other one-dimensional cellular automaton.*

Exercise 4.5.12 *Show that to every one-dimensional CA* $\mathcal{A}$ *with the neighbourhood* $\{-1, 0, 1\}$ *and* k *states there is a one-dimensional cellular automaton* $\mathcal{A}'$ *with the neighbourhood* $\{-1, 0\}$ *that simulates* $\mathcal{A}$ *twice as slow and* $\mathcal{A}'$ *needs at most* k^2 *states.*

4.5.4 Mutual Simulations of Turing Machines and Cellular Automata

Time as the computational complexity measure is defined for cellular automata in a natural way—as the number of steps during a computation. Space as the computational complexity measure is defined only for the case that the initial configuration has only finitely many finite automata in a non-sleeping state. In such a case, for one-dimensional cellular automata, the space complexity of a computation is the maximal distance during the computation between two finite automata that have everr been (during a computation), in a non-sleeping state. For multi-dimensional cellular automata the space complexity is defined analogously. On this base we can in a natural way define $t(n)$-time and $s(n)$-space bounded computations on a cellular automaton.

Mutual simulations between Turing machines and cellular automata are in principle very easy and they show, surprisingly, that cellular automata, as the main model of massive parallelism, are in the First Machine Class.

Theorem 4.5.13 *(1) Each* $t(n)$*-time and* $s(n)$*-space bounded one-tape Turing machine can be simulated by a one-dimensional cellular automaton with neighbourhood* $\{-1, 0, 1\}$ *in time* $t(n)$ *and space* $s(n)$*.*

(2) Each $t(n)$*-time and* $s(n)$*-space bounded one-dimensional cellular automaton with the neighbourhood* $\{-k, \ldots, 0, \ldots, k\}$ *and* $t(n) > n$ *can be simulated by a* $\mathcal{O}(kt^2(n))$*-time and* $\mathcal{O}(s(n))$*-space bounded one-tape Turing machine.*

Proof. (1) Let $\mathcal{M} = \langle \Gamma, Q, q_0, \delta \rangle$ be a $t(n)$-time bounded one-tape Turing machine with a set of states Q, a tape alphabet Γ, and a transition function δ. We show how to simulate $\mathcal{M}$ in time $t(n)$ on a one-dimensional cellular automaton $\mathcal{A}$ with the neighbourhood $\{-1, 0, 1\}$ and the set of states $Q' = \Gamma \cup Q \times \Gamma$. The overall simulation is based on the representation of a configuration $a_1 \dots a_n q a_{n+1} a_{n+2} \dots a_m$ of $\mathcal{M}$, by the following sequence of states of the finite automata of $\mathcal{A}$: $a_1, \dots, a_n, (q, a_{n+1}), a_{n+2}, \dots, a_m$. In order to simulate one transition of $\mathcal{M}$ at most two finite automata of $\mathcal{A}$ change their states. The transition function δ' of $\mathcal{A}$ is defined as follows: if $x, y, z \in \Gamma$, then

$$\delta'(x, y, z) = y;$$

if $\delta(q, x) = (q', x', \rightarrow)$, then for $y, z \in \Gamma$:

$$\delta'(y, z, (q, x)) = z \qquad \delta'(y, (q, x), z) = x' \qquad \delta'((q, x), y, z) = (q', y).$$

Similarly we can define the values of the transition function δ' for other cases.

(2) The simulation of a cellular automaton $\mathcal{A}$ with the neighbourhood $\{-k, \dots, k\}$ by a one-tape Turing machine $\mathcal{M}$ is straightforward once we assume that an input of $\mathcal{A}$ is written on a tape of $\mathcal{M}$ with end-markers.

The simulation of one step of $\mathcal{A}$ is done by $\mathcal{M}$ in one right-to-left or a left-to-right sweep. In a right-to-left sweep $\mathcal{M}$ first erases the end-marker and next $2k+1$ symbols and remembers them in its finite control. Then $\mathcal{M}$ writes the end-marker and, cell by cell, new states of finite automata of $\mathcal{A}$. After reaching the left end-marker $\mathcal{M}$ keeps writing $2k+1$ new states for automata of $\mathcal{A}$ and then the new left end-marker. Once this is done a new left-to-right sweep can start. Since $\mathcal{A}$ can extend the number of non-sleeping finite automata in $t(n)$ steps maximally to $kt(n) + n$, $\mathcal{M}$ needs $\mathcal{O}(kt(n))$ steps to simulate one step of $\mathcal{A}$. Hence the theorem. □

Exercise 4.5.14 *Show that one can simulate Turing machines with several tapes and several heads per tape in real time on cellular automata.*

4.5.5 Reversible Cellular Automata

Let us recall the basic definition from Section 2.3.11.

Definition 4.5.15 *A cellular automaton $\mathcal{A} = \langle d, Q, N, \delta \rangle$ is reversible, if there is another cellular automaton $\mathcal{A}' = \langle d, Q, N', \delta' \rangle$ such that for each configuration c of $\mathcal{A}$ it holds*

$$G_\delta(c) = c_1 \qquad \textit{if and only if} \qquad G_{\delta'}(c_1) = c.$$

In other words, a cellular automaton $\mathcal{A}$ is reversible if there is another cellular automaton $\mathcal{A}'$ such that for any sequence of configurations $c_1, c_2, \dots, c_{n-1}, c_n$ of $\mathcal{A}$, where $c_i \vdash c_{i+1}$, for $1 \le i < n$, $\mathcal{A}'$ can reverse this computation to get the following sequence of configurations

$c_n, c_{n-1}, \ldots, c_2, c_1$. (Observe that the reverse cellular automaton $\mathcal{A}'$ may use a much smaller or larger neighbourhood than $\mathcal{A}$ does.)

There are two main reasons why the concept of reversibility of cellular automata is of importance.

1. The main physical reason why a computation needs energy is the loss of information that mostly occurs during a computation and each loss of information leads to energy dissipation. (For example, a computation starting with the input x and performing the statement $x \leftarrow x \times x$ causes a loss of information.) On the other hand, if a computation is reversible, then there is no loss of information and therefore in principle such a computation can be carried out without a loss of energy.

2. Cellular automata are used to model phenomena of microscopic physics, especially those in gas and fluid dynamics. Since microscopic physics is in principle reversible, then so have to be also those cellular automata that model well these microscopic processes. For this reason the problem to decide whether a given cellular automaton is reversible is of importance for cellular automata models of microscopic physics.

The very basic problem is whether there are at all reversible cellular automata. They do exist and the following example shows one of them. It is a cellular automaton with two states, the neighbourhood $N = \{-1, 0, 1, 2\}$, and the following transition function:

$$\begin{array}{lllllllllll}
0\underline{0}00 & \rightarrow & 0 & \quad 0\underline{1}00 & \rightarrow & 1 & \quad 1\underline{0}00 & \rightarrow & 0 & \quad 1\underline{1}00 & \rightarrow & 1 \\
0\underline{0}01 & \rightarrow & 0 & \quad 0\underline{1}01 & \rightarrow & 1 & \quad 1\underline{0}01 & \rightarrow & 0 & \quad 1\underline{1}01 & \rightarrow & 1 \\
0\underline{0}10 & \rightarrow & 1 & \quad 0\underline{1}10 & \rightarrow & 0 & \quad 1\underline{0}10 & \rightarrow & 0 & \quad 1\underline{1}10 & \rightarrow & 1 \\
0\underline{0}11 & \rightarrow & 0 & \quad 0\underline{1}11 & \rightarrow & 1 & \quad 1\underline{0}11 & \rightarrow & 0 & \quad 1\underline{1}11 & \rightarrow & 1
\end{array}$$

where the underlined digits indicate states to be changed by the transition. It is quite easy to verify that this cellular automaton is reversible. There are only two transitions that change the state: both have environment $(0, 10)$. It is now sufficient to observe that this environment can not be changed by any transition.

There do not seem to be many reversible cellular automata. For two-state automata with a neighbourhood N with $|N| = 2$ or $|N| = 3$ there are none. For the neighbourhood $N = \{-1, 0, 1, 2\}$ there are 65536 cellular two-state automata but only 8 of them are reversible and all of them are a insignificant modification of the one presented above. The following theorem, of importance for cellular automata applications, is therefore quite a surprise.

Theorem 4.5.16 *(1) Any k-dimensional CA can be simulated in real time by a $(k+1)$-dimensional reversible CA. (2) There is a universal cellular automaton that is reversible. (3) It is decidable whether a one-dimensional cellular automaton is reversible but it is undecidable whether a two-dimensional cellular automaton is reversible.*

Example 4.5.17 *A simple four state cellular automaton with the neighbourhood $\{0, 1\}$ is depicted in Figure 4.41a and its reversible counterpart, with the neighbourhood $\{-1, 0\}$ in Figure 4.41b.*

$*$	0	1	2	3
0	0	1	1	0
1	2	3	3	2
2	0	1	1	0
3	2	3	3	2

(a)

$*$	0	1	2	3
0	0	0	3	3
1	2	2	1	1
2	0	0	3	3
3	2	2	1	1

(b)

Figure 4.41: A cellular automaton and its reversible counterpart-??%

Exercise 4.5.18 *Show that the one-dimensional cellular automaton with the neighbourhood* $N = \{0,1\}$ *and states* $\{0,1,\ldots,9\}$ *and the transition function* $\delta(x,y) = (5x + \lceil \frac{5y}{10} \rceil) \bmod 10$ *is reversible.*

Remark 4.5.19 The concept of reversibility applies also to other models of computers, for example to Turing machines. It is surprising that any one-tape TM can be simulated by a one-tape two-symbols reversible TM.

Moral. There is a surprising variety of forms in which universality of computing can exhibit itself. A good rule of thumb by computing is therefore as in life, to solve problems with tools that fit best and to apply tools to problems that fit them best.

4.6 Exercises

1. Design a Turing machine to compute the following strings-to-strings functions over the alphabet $\{0,1\}$, where w_i are symbols and w strings: (a) $w \rightarrow w^R$; (b) $w \rightarrow ww$; (c) $w_1 w_2 \ldots w_n \rightarrow w_1 w_1 w_2 w_2 \ldots w_n w_n$.
2. Design a Turing machine that performs unary-to-binary conversion.
3. Design a Turing machine that generates binary representations of all positive integers separated by the marker $\#$.
4. Design a Turing machine that for an input string x makes exactly $2^{|x|}$ steps.
5. Design a TM that generates all well-parenthetized sequences over the alphabet $\{(,)\}$, and each only once, i.e. it generates an infinite string like $()\$()()\$(())\$()()()\$()(())\$\ldots$.
6. * Show that for any Turing machine there is an equivalent two-symbols Turing machine (with symbols $\sqcup$ and 1, which can always replace the blank by 1 but never rewrite 1 by the blank).
7. ** Show that any computation that can be done by a Turing machine can be simulated by a Turing machine which has two one-way infinite tapes and which can neither write nor read on these tapes but can only sense when the head comes to the end of the tape.

8. Show that any TM can be simulated by a TM whose tape is always entirely emtpy with exception of at most three 1's.

9. Design a TM which, when started with an empty tape, writes down its own description and halts.

10. Show that a k-tape $t(n)$-time bounded TM can be simulated by a 2-tape TM in $\mathcal{O}(t(n) \lg t(n))$ time. (Hint: move tapes not simulated heads.)

11. Show that for any function $f \in \omega(n)$ the complexity class Time$(f(n))$ is closed: (a) under union; (b) under intersection; (c) under complement.

12. Define formally the concept of Turing machines with a binary-tree like memory.

13. Show how a TM with a tree-like memory can be simulated by a two-tape ordinary TM and estimate the efficiency of the simulation.

14. ** Find a problem that can be solved significantly more efficiently on a TM with a tree-like memory than on any TM with a finite-dimensional tape.

15. Design a RAM that computes a product of two polynomials if the coefficients of these polynomials are given.

16. Design a RAM that for a given integer n computes: (a) $\lfloor \lg n \rfloor$; (b) a binary representation of n; (c) Fibonacci representation of n.

17. Design a RASP program to compute in $\Theta(n)$ steps g_n defined by $g_0 = -1$, $g_1 = 0$, $g_2 = 1$, $g_3 = 0$, $g_n = 5g_{n-1} - 4g_{n-4}$, for $n \geq 4$.

18. Consider a version of RAM with successor and predecessor operations as the only arithmetical operations. Design programs for such RAM for addition and subtraction.

19. *Show that the permanent of an $n \times n$ integer matrix $A = \{a_{ij}\}$ can be computed in $\mathcal{O}(n^2)$ arithmetical operations if the integer division is allowed. (Hint: Use numbers $z = (\prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij})^n$, $\alpha_i = z^{n^i}, i = 1, \ldots, n$, $B = \sum_{i=1}^{n} n^i$, $T = \prod_{i=1}^{n}(\sum_{j=1}^{n} \alpha_j a_{ij})$ as well as integer division (and modulus) operations.

20. Show how to encode by an integer $\langle p, x \rangle$ a RAM program p and a RAM input x.

21. For a RAM program p and an input x let $R(p,x)$ be the corresponding output. Show that there is a RAM program u such that $R(u, \langle p, x \rangle) = R(p,x)$, for any p and x

22. Design a Boolean circuit over the base $\{\vee, \wedge, \neg\}$ to compute the function $f(x,y,z) =$ **if** x **then** y **else** z.

23. Design a Boolean circuit over the base $\{NOR\}$ to compute: (a) $x \Rightarrow y$; (b) $x \equiv y$.

24. Design a Boolean circuit to recognize palindromes among binary strings of length: (a) 8; (b) 2^n.

25. Design a Boolean circuit, over the base $\{\vee, \wedge, \neg\}$, of depth $\mathcal{O}(\lg n)$ for the Boolean function: $f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = 1 \Leftrightarrow \forall i \in \{1, \ldots, n\} : x_i \neq y_i$

26. Design a Boolean circuit, over the base $\{\vee, \wedge, \neg\}$, of depth $\mathcal{O}(\lg n)$ for the Boolean function:
$$g_n(x_{n-1}, \ldots, x_0, y_{n-1}, \ldots, y_0) = 1$$
if and only if $bin(x_{n-1} \ldots x_0) > bin(y_{n-1} \ldots y_0)$.

27. (Decoder) Show how to design a Boolean circuit over the base $\{\vee, \wedge, \neg\}$, called decoder, with n inputs and 2^n outputs such that for an input $x_1, \ldots, x_n$ there is 1 on exactly the $bin(x_1 \ldots x_n)$-th output. (This is a way a random access memory is addressed.)

28. k-threshold function $t_k : \{0,1\}^n \to \{0,1\}$ is the Boolean function of n arguments that has value 1 if and only if at least k of its arguments has value 1. Show: (a) how to design t_k for $1 \leq k \leq n$; (2) design one Boolean circuit that computes all $t_1, \ldots, t_n$ and has size as small as possible.

29. Design a Boolean circuit that determines whether 3 or more of four people have a common vote on a committee that votes yes on an issue and any committee member has a switch to vote.

30. Let $\mathcal{B}$ be a base of Boolean functions that contains the identity function and let k be the maximal arity of functions in $\mathcal{B}$. Show that if a Boolean function f can be computed by a Boolean circuit C over the base $\mathcal{B}$, then f can be computed by a Boolean circuit C' over the base $\mathcal{B}$ such that each gate of C' has the outdegree at most 2 and $Size(C') \leq (k+1)Size(C)$.

31. Show that each Boolean function $f \in \mathbf{B}_n$ can be computed by a Boolean circuit of the size $3 \cdot 2^n$. (Hint: Use the disjunctive normal form for f.)

32. Show that every time-constructible function is space constructible.

33. * Show that if s is a space-construct-ble function, then $2^{s(n)}$ is time-constructable.

34. (Universal Boolean circuit.) Show that for each integer n there is a circuit UC_n of the size $\mathcal{O}(2^n)$ with $2^n + n$ inputs such that for all binary strings p of length 2^n and any string x of length n the output of the circuit UC_n with the input px is the value of the Boolean function determined by the string p for the input x.

35. Design an EREW PRAM program to compute in $\lceil \lg n \rceil + 1$ time the Boolean function $x_1 \vee x_2 \ldots \vee x_n$.

36. Show that the following program, for an EREW PRAM, computes the function $x_1 \vee x_2 \vee \ldots \vee x_n$, where x_i is stored in the shared memory location $GM[i]$ and its computation time is strictly less than $\lceil \lg n \rceil$ steps. (F_i stands here for the i-th Fibonacci number):

```
begin $t \leftarrow 0; Y[i] \leftarrow 0;$
      until $F_{2t-1} \le n$
            do if $i + F_{2t} \le n$ then $Y[i] \leftarrow (Y[i] \vee GM[i + F_{2t}]);$
               if $(Y[i] = 1) \vee (i > F_{2t+1})$ then $GM[i - F_{2t+1}] \leftarrow 1;$
               $t \leftarrow t + 1$
            od
end
```

37. **(Preemptive scheduling)** Let m machines $M_1, \ldots, M_m$ and n jobs J_j, $1 \le j \le n$, with the processing times $p_j, 1 \le j \le n$ be given. Design an EREW PRAM algorithm to construct a feasible and optimal preemptive scheduling of n jobs on m machines in time $\mathcal{O}(\lg n)$. (A preemtive schedule assigns to each job J_j a set of triples (M_i, s, t), where $1 \le i \le m$ and $0 \le s \le t$, to denote that J_j is to be processed by M_i from time s to time t. A preemptive schedule is *feasible* if the processing intervals for different jobs on the same machine are nonoverlapping, and processing intervals of each job J_j on different machine are also nonoverlapping and have the total length p_j for j-th job. A preemptive schedule is optimal if the maximum completion time is minimal.)

38. * Show that one step of a CRCW^{pri} PRAM with p processors and m registers of shared memory can be simulated by a CRCW^{com} PRAM with p processors and m registers of shared memory in $\mathcal{O}(\lg p)$ steps.

39. ** Show that one step of a CRCW^{pri} PRAM with p processors and m registers of shared memory can be simulated by a CRCW^{com} in $\mathcal{O}(\lg\lg p)$ steps using p processors and $m(p-1)$ registers of shared memory.

40. EROW (**E**xclusive **R**ead **O**wner **W**rite) PRAM is a PRAM model in which each processor has a single register of shared memory assigned (it "owns this register") and it can write only to that register. Show that any Boolean circuit of depth d and size s can be simulated by an EROW PRAM in $\mathcal{O}(d)$ steps using s processors and s shared memory 1-bit registers.

41. Show: (a) any problem in DLOGSPACE can be solved by a EROW PRAM in $\mathcal{O}(\lg n)$ steps using $n^{\mathcal{O}(1)}$ processors; (b) any problem in NLOGSPACE can be solved by a CRCW^{com} PRAM in $\mathcal{O}(\lg n)$ steps using $n^{\mathcal{O}(1)}$ processors.

42. ** An **abstract** PRAM is the one in which no restriction is made on the instruction set. Show that any Boolean function of n variables can be computed by an abstract EROW PRAM in $\mathcal{O}(\lg n)$ steps using $\frac{n}{\lg n}$ processors on $\frac{n}{2\lg n}$ shared memory registers, provided n input values are in n different registers of the shared memory.

43. * Also very simple "finite cellular automata" can exhibit chaotic behaviour. Demonstrate that by designing and running a program to simulate the following $n \times n$ array of two state—0 and 1—finite automata with the following transition function:

$$c_{i,j}(t) = (c_{i,j}(t-1) \wedge c_{i-1,1}(t-1)) \oplus c_{i,j-1}(t-1) \oplus c_{i,j+1}(t-1)$$

that exhibit chaotic behaviour almost for any intitial configuration (provided the automata on the border of the rectangle keep getting 0 along their disconnected inputs (to the environment).

44. Sketch (design) a solution of the Firing Squad Synchronization Problem for the case the squad has two generals, one at each end of the squad, and they simultaneously send the order “fire when ready”.

45. Sketch (design) a solution of the Firing Squad Synchronization Problem for the case the soldiers of the squad are interconnected to form a balanced binary tree all leaves of which have the same distance from the root—the general.

46. Show that one-dimensional cellular automata can recognize the language $\{a^{2^n} \mid n \geq 0\}$ in real-time (i.e. in time equal to the length of input).

47. $*$ Show that any one-tape Turing machine with m symbols and n states can be simulated by a one-dimensional cellular automaton with $m + 2n$ states.

48. * Consider a two-dimensional cellular automaton with the Moore neighbourhood and 4-state finite automata with the states $\{0, 1, 2, 3, 4\}$ and the local transition function that maps the state 0 to 0, 1 to 2, 2 to 3 and 3 either to one, if at least one and at most two neighbours are in the state 1, and to 3, otherwise. The initial configuration shown in Figure 4.42b is called “electron” and the one in Figure 4.42c is called a“wire with an electron“ because if this configuration develops the electron “moves along the wire”. The initial configuration shown in Figure 4.42d is called “a diode” because if the electron is attached to its input, indicated by the in-comming arrow, then it moves through the diode. However, if the electron is attached to the output then it does not get through. Show that one can design initial configurations that behave as the following computational elements: (a) an OR-gate; (b) an inverter; (c) and AND-gate (without using inverters); (d) an XOR-gate; (e) a one-bit memory; (f) a crossing of two wires which “are not connected”.

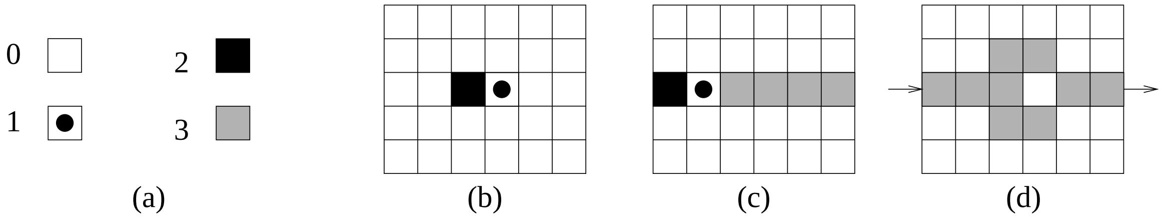


Figure 4.42: A cellular automaton to simulate computational elements–65%

49. (Universal cellular automaton.) A natural way how to see a universal two-dimensional cellular automaton U (for two-dimensional cellular automata with the same neighbourhood) is that for any other two-dimensional cellular automaton A with the same neighbourhood the plane of U is divided into rectangular blocks B_{ij} (of the size that depends on A). With appropriate starting conditions and a fixed k, if U is run k steps,

then each block B_{ij} performs a step of simulation of one cell c_{ij} of A. Show how to construct such a universal cellular automaton. (Hint: design a cellular automaton that can simulate any Boolean circuit over the base $\{NOR\}$ in such a way that cells behave either like NOR gates or as horizontal or vertical wires or as crossings or turns of wires. The transition function of any given CA is then expressed by a Boolean circuit and ...)

50. (**Prime recognition by one-dimensional CA**) Design a one-dimensional CA that has in a fixed cell the state 1 in the i-th step if and only if i is a prime). Due to I. Korec this can be done with a 14-state CA and neighbourhood $\{-1, 0, 1\}$.

51. (**Limit sets**) Let $G : Q^{\mathbf{Z}} \rightarrow Q^{\mathbf{Z}}$ be the global function computed by a one-way one-dimensional cellular automaton $\mathcal{A}$ with the sleeping state. Let us define a sequence of sets of configurations $\Omega_0 = Q^{\mathbf{Z}}, \Omega_i = G(\Omega_{i-1})$, for $i > 0$. The **limit set** Ω of $\mathcal{A}$ is defined by $\Omega = \bigcap_{i=0}^{\infty} \Omega_i$.

    Show; (a) Ω is a non-empty set; (b) Ω is included in its preimages, that is $\forall c \in \Omega, \exists d \in \Omega$, $G(d) = c$; (c) find an example of a cellular automaton such that $\Omega_i = \Omega$ for some i; (d) find an example of a cellular automaton such that $\Omega \neq \Omega_i$ for all i.

QUESTIONS

1. What is the evidence that Church's Thesis holds?
2. Can there be a universal Turing machine with one state only?
3. What are the differences, if any, between laws of physics and such theses as Church's?
4. What a modification of straigt-line programs is suitable to study computational complexity of such problems as sorting, merging and maximum or median finding?
5. What are the relations between Church's Thesis, Sequential Computation Thesis and Parallel Computation Thesis?
6. Why Strassen's algorithm does not work over the Boolean quasiring?
7. How many single Boolean functions of two variables form a complete basis?
8. What is the relation between the length of a Boolean expression and the size of the corresponding Boolean circuit?
9. How can one naturally generalize von Neumann and Moore neighborhoods for three-dimensional cellular automata?
10. Does the FSSP have a solution for reversible CA?

4.7 Historical and Bibliographical References

History of Turing machines goes back to the seminal paper of Turing (1937) and basic results concerning Turing machines presented in Section 4.1 can be found in any of the numerous books on computability and computational complexity. Concerning Turing himself, an interesting bibliography has been written by Hedges (1983). A survey of Turing machines and their rôle in computing and in science in general is due to Hopcroft (1984). For a broader and also futuristic impacts of Turing machines see the book edited by R. Herken (1988).

The fundamental original papers of Turing, Kleene, Church, Post and others are collected in Davis (1965). For an analysis of Church's thesis see Kleene (1952) and also the illuminating discussions in Rozenberg and Salomaa (1994). Basic relations between quantum theory and Church's thesis and also the idea of a universal quantum Turing machine are analyzed by Deutsch (1985).

The existence of a universal Turing machine with two nonterminating and one terminating state was shown by Shannon (1956). Minsky (1967) with his 4-symbol and 7-state universal Turing machine represents an end of one period of searching for minimal universal Turing machines—see Minsky (1962) for an older history of this competition. For newer results concerning minimal universal Turing machines see Rogozhin (1996). Various approaches to the concept of universality are analyzed by Priese (1979).

The Busy Beaver Problem is due to Rado (1962) and for a presentation of various results on this problem see Dewdney (1984). The concept of the off-line Turing machine and the basic results on resource bounded Turing machines are due to Hartmanis and Stearns (1965) and Hartmanis, Lewis and Stearns (1965).

The model of RASP machine was introduced and investigated by Shepherdson and Sturgis (1963), Elgot and Robinson (1964) and Hartmanis (1971). Cook and Reckhow (1973) introduced the RAM model as a simplification of RASP and showed basic simulations between RAM, RASP and Turing machines. A detailed analysis of the computational power of various types of RAM models is due to Schönhage (1979). Exercises 19 and 4.2.17 are due to Vyskoč (1983). Another natural modification of RAM, the so-called array-processing machines (APM), due to van Leeuwen and Wiedermann (1987), represents an extension of RAM with vecorized versions of the usual RAM-instructions. APM are also in the Second Machine Class.

A detailed presentation and analysis of various computer models and their simulations is due to van Emde Boas (1990) and Vollmar and Worsch (1995). Van Emde Boas also introduced the concepts of the First and Second Machine Class. Our formulation of Sequential Computational Thesis is from van Emde Boas (1990).

The concept of register machines, also called the successor RAM, or Minsky machines, is due to Minsky (1967) who also showed that each Turing machine can be simulated by a two-register machine and even by a one-register machine if multiplication and division are allowed.

For the history and references of the search for the fastest algorithms for matrix multiplication and related problems see Winograd (1980) and Pan (1984).

Blum, Shub and Smale (1989) initiated an investigation of RRAM and our presentation is derived from their results.

The investigation of Boolean circuit complexity goes back to Shannon (1949) and so does Theorem 4.3.13. Our proof of this theorem follows Wegener (1987). His book contains also a systematic presentation of the "older results" on the Boolean circuit complexity. See also the book of Savage (1986). Basic results concerning mutual simulations of Turing machines and uniform families of Boolean circuits are due to Schnorr (1976), Borodin (1977) and Pippenger and Fischer (1979). The first concept of uniformity for families of Boolean circuits was introduced by Borodin (1977). For references to other concepts of uniformity see, for example, Greenlaw, Hoover and Ruzzo (1995). Our presentation is derived from a systematic treatment of this subject by Reischuk (1990) where one can also find a proof of the existence of oblivious TM. The assumption that no "explicit" Boolean function has non-polynomial size Boolean circuit complexity was formulated and investigated by Lipton (1994).

PRAM models of parallel computing were introduced by Fortune and Willie (1978)—CREW PRAM—Goldschlager (1982)—CRCW PRAM—and Shiloach and Vishkin (1981). The basic result on the relation between parallel time on the machines of the Second Machine Class and space on the machines of the First Machine Class is due to Goldschlager (1977). Parallel Computation Thesis seems to appear first in Chandra and Stockmeyer (1976) and got well known especially through thesis of Goldschlager (1977). Systematic presentations of various parallel complexity classes and simulations between models of parallel and sequential computing are due to Parberry (1987)and Reischuk (1990). Basic hierarchy results between various models of PRAM were established by Cook, Dwork and Reischuk (1986). For a detail overview of relations between various models of PRAM see Fich (1993). For fast circuits for parallel prefix sum problem and their application to fast parallel computation of mappings computable by finite state transducers see Ladner and Fischer (1986). $\mathcal{O}(\sqrt{t(n)})$-time simulation of $t(n)$-time bounded TM on CREW PRAM is due to Dymond and Tompa (1985). For the design of parallel algorithms see, for example, Karp and Ramachandran (1990) and Ja'Ja' (1992). A work-optimal algorithm for the list ranking problem is due to Cole and Vishkin (1986). Algorithm 4.4.1 is due to Kučera (1982) and the idea to use the doubly-logarithmic-depth trees is due to van Emde Boas (1975). John von Neumann's decision, inspired by S. L. Ulam, to consider cellular automata as a model of biological world to investigate within this model the problem of self-reproducibility—see von Neumann (1966)—started a research in the area of parallelism. Since then cellular automata have been investigated from several other points of view: as a model of physical world, chaotic systems and dynamical systems and also as a model of massive parallelism.

The original von Neumann's solution of the self-reproducibility problem with 29-state FA has been improved first by E. F. Codd (1968) who found an elegant solution with an 8-state FA. (By the way, Codd received Turing award in 199? for introducing an elegant, minimal and powerful model of relational data bases.). von Neumann solution was further improved by E. R. Banks (1971) who found a solution with 4-state FA.)

For an analysis of the behaviour of one-dimensional cellular automata see Wolfram (1983, 1984, 1986) and Garzon (1995). An exciting account of the history and achievements in Conway's LIFE game is due to Gardner (1970,1971,1983). Universality of LIFE game was shown by Berlekamp, Conway and Guy (1983) and generalizations of LIFE game to three-dimensional cellular automata are suggested by Bays (1987). The first solution to the FSSP was due to Minsky and Mc Carthy in time $3n$using a divide-and-conquer method. (Minsky has received Turing award in 197? and Mc Carthy in 19??—both for their contributions to artificial intelligence.) A survey of results on Firing Squad Synchronization Problems is due to Mazoyer (1976). The existence of the totalistic normal form for cellular automata was shown by Culik and Karhumäki (1987).

The history of reversible computation goes back to the "Garden of Eden" problem of Moore and got an explicit formulation in papers by Amoroso and Patt (1972), Richardson (1972) and Bennett (1973). The first claim of Theorem 4.5.16 is due to Toffoli (1977). The existence of universal reversible cellular automata was shown by Toffoli (1977) for two- and multi-dimensional cellular automata and by Morita and Harao (1989) for one-dimensional cellular automata. The reversible cellular automaton in Figure 4.41 and in the Exercise 4.5.18 are due to Korec (1996).The fact that any one-tape TM can be simulated by a one-tape, two symbol reversible Turing machine was shown by Morita, Shirasaki and Gono (1989). The decidability of reversibility for one-dimensional automata is due to Amoroso and Patt (1972) and the undecidability for two-dimensional cellular automata is due to Kari (1990). Surveys of results on reversibility and related problems of energy-less computations are due to Bennett (1988) and Toffoli and Margolus (1990).

Chapter 5

COMPLEXITY

INTRODUCTION

Computational complexity is about quantitative laws and limitations that govern computing. It explores the space of algorithmic problems, its structure and develops techniques how to reduce the search for efficient methods for the whole class of algorithmic problems to the search for effecient methods for few key algorithmic problems. Computational complexity discovers inherent quantitative limitations to develop effecient algorithms and designs/explores methods how to cope with these limitations, using randomness, approximations and heuristics. Finally, computational complexity tries to understand what is feasible and what is efficient in sequential and parallel computing and by doing that to capture practical limitations not only of computing but also of scientific theories and rational reasoning.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. main computational complexity classes for deterministic, nondeterministic, randomized and parallel computing; their structure and relations between them,
2. basic resource-bounded reductions and the concept of complete problems,
3. a variety of complete problems for such classes as **NP, P** and **PSPACE** and methods to show their completeness,
4. several algorithmic problems that play a special rôle in complexity investigations: graph isomorphism problem, prime recognition and travelling salesman problem,
5. methods to overcome limitations **NP**-completeness imposes (using the average case and randomized computations. approximations and heuristics) and their limitations.

> To find specific candidate problems on which pure science can be expected to have the greatest impact, we have to look among the most difficult ones where no solutions are known, rather than the easier ones where several alternatives already exist. Physics has had greater influence on space travel than on violin making.
>
> *Leslie G Valiant, 1989*

Complexity theory is about quantitative laws and limitations that govern computations. Discovery that computational problems have an intrinsic nature that obeys strong quantitative laws and that the understanding of these laws yields deeper theoretical insights and pays practical dividends in computing, is one of the main outcomes of computational theory and practice. Moreover, since the computing paradigm is universal and widespread, the quantitative laws of computational complexity apply to all information processing from numerical simulations and computations to automatic theorem proving and formal reasoning and from hardware to physical and biological computations.

Classification of computational problems into computational complexity classes with respect to the amount of computational resources needed to solve them has turned out to be very fruitful. Computational complexity classes have deep structure and their understanding allows to develop powerful tools for algorithm design and analysis. Especially the concepts of resource-bounded reducibility and completeness, presented in this chapter, belong to the most useful algorithm design methodologies.

The central task of complexity theory is the search for borderlines between what is and is not feasibly computable. With this task complexity theory aims and impacts go far beyond computing, because the search for the limits of what is feasibly computable is the search for limits of scientific methods, rational reasoning and knowable.

The development of new paradigms for computing that allow to solve satisfactory previously practically unsolvable problems, is one of another main aims and outcomes of complexity theory.

Complexity theory has been able to discover several key algorithmic problems, both from theoretical and practical point of view, and to concentrate on their in-depth study. The rôle of these problems can be compared with that some differential equations play in calculus and in our capability to create mathematical models of behaviour of nature.

The key problem of complexity theory, the $\mathbf{P} = \mathbf{NP}$ problem, is simultaneously one of the basic problems of current science.

Interesting enough, as it is often the case with the basic science, negative results of complexity theory—showing that this and that is impossible or infeasible—have also strong positive impacts—for instance for cryptography, secure communication, or random number generators, see Chapters 8 and 9—and belong to the practically most useful outcomes of complexity theory.

5.1 Nondeterministic Turing Machines—November 8, 2005

We have seen in Chapter 3 that nondeterministic finite automata are of large importance for our capability to harvest the concept of finite state machines in spite of the fact that they do not represent a realistic model of computers. This is even more true on the level of universal computers. Nondeterministic Turing machines have a hardly replaceable rôle in developing and exploring the key concepts concerning computational complexity.

A one-tape **nondeterministic Turing machine**, in short NTM, $\mathcal{M} = \langle \Gamma, Q, q_0, \delta \rangle$ is defined formally in a similar way as a one-tape deterministic Turing machine (DTM or TM for short), only instead of a transition function we have a transition relation:

$$\delta \subset Q \times \Gamma \times (Q \cup H) \times \Gamma \times D, \tag{5.1}$$

where $H = \{HALT, ACCEPT, REJECT\}$ and $D = \{\leftarrow, \downarrow, \rightarrow\}$. As a consequence a configuration c of NTM $\mathcal{M}$ can have several potential next configurations and $\mathcal{M}$ can go nondeterministically from c to one of them. We can therefore see the overall computational process of a NTM not as a sequence of configurations but as a tree of configurations—see Figure 5.1. If we use, for each state and each tape symbol, a fixed numbering $1, 2, \ldots$ of possible transitions, then we can use this numbering to label edges of the configuration tree as shown in Figure 5.1. In a similar way nondeterministic multitape TM are defined and in the following the notation NTM will be used to denote such TM.

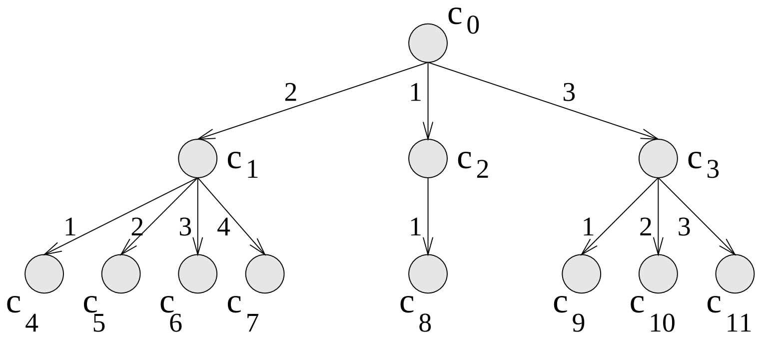


Figure 5.1: Tree of configurations-80%

We say that a NTM $\mathcal{M}$ accepts an input w (in time $t(|w|)$ and space $s(|w|)$), if there is at least one path in the configuration tree with q_0w being the configuration at the root, that ends in the accepting state (and it has the length at most $t(|w|)$, and none of the configurations on this path has larger length than $s(|w|)$). This can be used to define in a natural way when a NTM computes a relation or a function within certain time and space bounds. For a NTM $\mathcal{M}$ let $L(\mathcal{M})$ be the language accepted by $\mathcal{M}$.

Exercise 5.1.1 *Show that for each NTM $\mathcal{M}$ we can design a NTM $\mathcal{M}'$ that can make in each nonterminating configuration at least two moves, and accepts the same language as $\mathcal{M}$. Moreover, $\mathcal{M}$ accepts an input w in t steps iff so does $\mathcal{M}'$.*

Exercise 5.1.2 *Show that for each NTM $\mathcal{M}$ we can design a NTM $\mathcal{M}'$ that can make in each nonterminating configuration exactly two moves, accepts the same language as $\mathcal{M}$ does and there is an integer k such that $\mathcal{M}$ accepts an input w in t steps iff $\mathcal{M}'$ accepts w in kt steps.*

Complexity classes for NTM: Denote by NTime($t(n)$) (NSpace($s(n)$)) the family of languages accepted by $t(n)$-time bounded ($s(n)$-space bounded) NTM and denote

$$\mathbf{NP} = \bigcup_{k=0}^{\infty} \mathrm{NTime}(n^k), \qquad \mathbf{NPSPACE} = \bigcup_{k=0}^{\infty} \mathrm{NSpace}(n^k).$$

A computation of a NTM can be seen as a sequence of transition steps. Some of them are deterministic—there is only one possibility for the next configuration—others should be seen as the result of a choice or of a guess (how to proceed to accept the input or to solve the problem). And it is exactly this guessing potential of NTM that makes them a useful tool.

Example 5.1.3 *It is easy to design a NTM that decides in $\mathcal{O}(n^2)$ time whether a Boolean formula of length n is satisfiable. Indeed, the machine first goes through the formula and replaces all variables by 0 or 1 in a consistent way—each variable is replaced by the same value all over the formula. Each time a new variable is encountered a nondeterministic choice is made whether to assign 0 or 1 to this variable. This way the machine chooses an assignment. In the next step the formula is evaluated. Both stages can be done in $\mathcal{O}(n^2)$ time. If the formula is satisfiable, then there is a sequence of correct guesses for assignments of values to the variables and therefore there is an accepting path of length $\mathcal{O}(n^2)$ in the configuration tree.*

Exercise 5.1.4 *Describe behaviour of a NTM that accepts in polynomial time encodings of graphs with a Hamilton cycle.*

We now come to one of the main points why to deal with NTMs. For many important algorithmic problems not known to be in **P**, it can be easily shown that they are in **NP**. Typically, they are problems for which the only known deterministic algorithms are those making an exhaustive search of all possibilities, but for each such possibility it is easy to verify whether it is correct or not. An NTM just guesses one of these possibilities and verifies the correctness of the guess. In addition, and this is another key point, no one has shown yet that $\mathbf{P} \neq \mathbf{NP}$. It is therefore possible, though unlikely, that $\mathbf{P} = \mathbf{NP}$.

As we shall see, similarly as in the case of finite automata, nondeterminism does not increase essentially power of Turing machines. It seems, however, that an NTM can be "much faster".

Theorem 5.1.5 *If a language L is accepted by a $t(n)$-time bounded NTM, then it is accepted by a $2^{\mathcal{O}(t(n))}$-time bounded DTM.*

Proof. We show how to simulate a $t(n)$-time bounded NTM $\mathcal{M}_{non} = \langle \Gamma, Q, q_0, \delta \rangle$ by a $2^{\mathcal{O}(t(n))}$-time DTM $\mathcal{M}_{det} = \langle \Gamma', Q', q_0', \delta' \rangle$. Let

$$k = \max_{q \in Q, x \in \Gamma} \{\text{number of transitions for } q \text{ and } x\},$$

and denote by T_w the configuration tree of $\mathcal{M}_{non}$ for the computation with the initial configuration $q_0 w$ and assume that the edges of this tree are labeled by symbols from $\{1, \dots, k\}$ to specify the transition used (as shown in Figure 5.1).

$\mathcal{M}_{det}$ will simply try to go through all possible computations of $\mathcal{M}_{non}$—in other words through all possible paths in the configuration tree T_w. Some of these paths can be infinite and therefore $\mathcal{M}_{det}$ can not use the depth-first search to traverse the configuration tree. However, the breadth-first search will work fine. This leads to a very simple simulation method. A strict ordering of all words from $\{1, \dots, k\}^*$ is considered and consequently, word by word, for each $u \in \{1, \dots, k\}^*$, the computation along the path labeled by u is simulated (if there is such a path in the configuration tree at all). If such a computation leads to the accepting state, then $\mathcal{M}_{det}$ accepts, otherwise $\mathcal{M}_{det}$ goes into the simulation of the computation corresponding to the next word, in the strict ordering of words in $\{1, \dots, k\}^*$. This way $\mathcal{M}_{det}$ has to check at most $k^{t(n)}$ paths. Simulation of a single path takes at most $\mathcal{O}(t(n))$ time. Altogether, $\mathcal{M}_{det}$ needs $k^{t(n)}\mathcal{O}(t(n)) = 2^{\mathcal{O}(t(n))}$ time.

Let us now be more technical. The tape of M_{det} will be divided into three tracks. The first one will contain only the input word; the second one always a word from $\{1, \dots, k\}^*$ representing the path to be just simulated. The third track will be used to do all simulations according to the following simulation algorithm:

1. $\mathcal{M}_{det}$ starts a simulation by generating the word "1" on the second track (as the first nonempty word of the strict ordering of $\{1, \dots, k\}^*$).

2. $\mathcal{M}_{det}$ simulates on the third track the sequence of configurations specified by the word on the second tape (in case this word really describes a sequence of computational steps). If $\mathcal{M}_{non}$ reaches the accepting state during this simulation, M_{det} accepts; if not, $\mathcal{M}_{det}$ goes into step 3.

3. $\mathcal{M}_{det}$ changes the word on the second track to the next one in the strict ordering of the words of the set $\{1, \dots, k\}^*$ and goes to the step 2.

□

In a similar way nondeterministic versions of other models of Turing machines can be simulated by deterministic ones with at most exponential increase in time. Moreover, in a similar way we can prove:

Exercise 5.1.6 *Show that* $NTime(f(n)) \subseteq Space(f(n))$ *for any time-constructible function* f.

So far nobody has been able to come with a polynomial time simulation of NTM by DTM and therefore from the time complexity point of view nondeterminism seems to have huge advantage for Turing machines. Interesting enough, this is not so with the space.

Theorem 5.1.7 (Savitch's Theorem) $NSpace(s(n)) \subseteq Space(s^2(n))$ *for any space-constructible function* $s(n) \geq \lg n$.

Proof. Let $\mathcal{M}$ be a $s(n)$-space bounded NTM and $L = L(\mathcal{M})$. We describe an algorithm that accepts L and can be easily transformed into a $s^2(n)$-space bounded Turing machine accepting L.

Similarly, as in the proof of Lemma 4.3.20, we can show that there is a constant k (that depends only on the size of Q and Γ), such that for any input w of the size n, $\mathcal{M}$ can be in at most $k^{s(n)}$ configurations and each of them can have length at most $s(n)$. This immediately implies that if $w \in L$, then there is an accepting computation with at most $k^{s(n)} = 2^{s(n)\lg k}$ steps.

The following algorithm, one of the pearls of algorithm design, which uses the divide-and-conquer procedure *test* presented below, recognizes whether $w \in L$. The procedure *test*, with argument c, c' and i, just checks whether there is a way to get from a configuration c to c' in 2^i steps.

Algorithm 5.1.8

```
compute $s(n)$;
for all accepting configurations $c_\tau$ such that $|c_\tau| \leq s(n)$
   do if $test(q_0w, c_\tau, s(n)\lg k)$ then accept

procedure $test(c, c', i)$
if $i = 0 \wedge [(c = c') \vee (c \vdash c')]$ then return true
  else for all configurations $c''$ with $|c''| \leq s(n)$
          do if $test(c, c'', i-1) \wedge test(c'', c', i-1)$ then return true;
return false
```

Space complexity analysis: Each call of the procedure *test* requires $\mathcal{O}(s(n))$ space. The depth of the recursion is $\lg\lceil 2^{s(n)\lg k}\rceil = \mathcal{O}(s(n))$. Total space bound is therefore $\mathcal{O}(s^2(n))$. Moreover, $s(n)$ can be computed in $\mathcal{O}(s(n))$ space because $s(n)$ is space-constructible. □

Corollary 5.1.9 PSPACE = NPSPACE.

The proof of Theorem 5.1.7 uses the so-called **reachability method** to simulate space-bounded computations. With a slight modification of this method we can show:

Exercise 5.1.10 *$NSpace(f(n)) \subseteq Time(\mathcal{O}(1)^{\mathcal{O}(f(n))})$ for any time-constructible function $f(n) \geq \lg n$.*

Many results shown in Section 4.1 also hold for nondeterministic Turing machines. For example, the speed-up theorem and the compression theorem.

Nondeterministic TM form also a basis for the definition of a variety of other models of computations. For example, for randomized computations. In doing that the following normal form of NTM is often of importance

Lemma 5.1.11 *Let a NTM $\mathcal{M}$ accept a language L within the time bound $f(n)$, where f is a time-constructible function. Then there is a NTM $\mathcal{M}'$ that accepts L in time $\mathcal{O}(f(n))$ and all its computations for inputs of size n have the same length. Moreover, we can assume that $\mathcal{M}'$ has exactly two choices to make in each nonterminating configuration.*

Proof. In order to transform $\mathcal{M}$ into a NTM $\mathcal{M}'$ that accepts the same language as $\mathcal{M}$ and for all inputs of size n has computations of the same length $\mathcal{O}(f(n))$, we proceed as follows. $\mathcal{M}'$ uses first, on an input w of size n, a TM $\mathcal{M}_f$ that in time $f(n)$ produces a "yardstick" of length exactly $f(|w|)$ (we are making here a use of the time-constructibility of f). After $\mathcal{M}_f$ finishes its job, $\mathcal{M}'$ starts to simulate $\mathcal{M}$, using the "yardstick" to design an alarm clock. $\mathcal{M}'$ advances its pointer in the "yardstick" each time $\mathcal{M}'$ ends simulation of a step of $\mathcal{M}$, and halts if and only if $\mathcal{M}'$ comes to the end of the "yardstick", i.e., exactly after $f(x)$ steps. Should $\mathcal{M}$ finish sooner $\mathcal{M}'$ keeps going on, making dummy moves, until $\mathcal{M}'$ comes to the end of the yardstick, and then accepts or rejects as $\mathcal{M}$ did. For the rest of proof we make use of the results of Exercises 5.1.1 and 5.1.2.

Exercise 5.1.12 (**Parallel one tape Turing machines**) *Formally, they are specified as nondeterministic Turing machines, i.e. for each state and symbol read by the head a finite number of possible transitions is given. A parallel Turing machine starts to process a given input word with the only one ordinary Turing machine active that has its head on the first symbol of the input. During the computation of the parallel Turing machine several ordinary Turing machines can work in parallel on the same tape. In each step each of the currently active ordinary Turing machine reads its tape symbol and performs simultaneously all possible transitions creating for each transition a new (ordinary) Turing machine hat has the state and the head position determined by that transition. In this way several (ordinary) Turing machine can have simultaneously their heads over the same cell. If several of them tries to write different symbols into the same cell and at the same time, the computation is abrupted.*

(a) Show that the number of distinct ordinary Turing machines that can be active on the tape of a parallel Turing machine grows only polynomially with the number of steps.

(b) Design a parallel Turing machine that can recognize palindromes in linear time.

5.2 Complexity Classes, Hierarchies and Trade-offs

The amount of computational resources needed to solve a problem is clearly of importance in general. This is especially evident for time in such real-time applications as spacecraft and plane control, surgery-support systems and banking systems. It is therefore of prime practical and also theoretical interest to classify computational problems with respect to computational resources needed to solve them. By limiting the overall resources, the range of solvable problems gets narrower. This way we arrive to various complexity classes.

In addition to the complexity classes that have been already introduced:

$$\mathbf{P} = \bigcup_{k=0}^{\infty} \mathrm{Time}(n^k), \qquad \mathbf{NP} = \bigcup_{k=0}^{\infty} \mathrm{NTime}(n^k),$$
$$\mathbf{PSPACE} = \bigcup_{k=0}^{\infty} \mathrm{Space}(n^k), \qquad \mathbf{NPSPACE} = \bigcup_{k=0}^{\infty} \mathrm{NSpace}(n^k),$$

there are four other complexity classes that play a major rôle in complexity theory. The first two of them deal with the logarithmic space complexity:

$$\mathbf{L} = \mathbf{LOGSPACE} = \bigcup_{k=1}^{\infty} \mathrm{DSpace}(k \lg n), \qquad \mathbf{NL} = \mathbf{NLOGSPACE} = \bigcup_{k=1}^{\infty} \mathrm{NSpace}(k \lg n).$$

$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P}$ are basic inclusions between these new classes and the class **P**. The first of these inclusions is trivially true, the second follows from Exercise 5.1.10. These inclusions imply that in order to show that a problem is solvable in polynomial time, it is enough to show that the problem can be solved using only logarithmic space. Sometimes this is easier.

The last two main complexity classes deal the with the exponential time bounds:

$$\mathbf{EXP} = \bigcup_{k=0}^{\infty} \mathrm{Time}(2^{n^k}), \qquad \mathbf{NEXP} = \bigcup_{k=0}^{\infty} \mathrm{NTime}(2^{n^k}).$$

As we shall see all these classes represent certain limits what can be considered as feasible in computation.

Some of the complexity classes are closed under complement. For example, **P, PSPACE** and **EXP**. However, this does not seem true for the classes **NP** and **NEXP**. Of importance are therefore also the classes

co-NP and **co-NEXP**

that contain complements of languages in **NP** and in **NEXP**, respectively.

With the space complexity classes the situatiuon is different due to the following result:

Theorem 5.2.1 (Immerman-Szelepcsényi's Theorem) *If* $f(n) \geq \lg n$, *then* $\mathit{NSpace}(f(n)) = \mathit{co\text{-}NSpace}(f(n))$.

Later we shall deal with some other complexity classes that are so important that they got also names. However, only some of the complexity classes have broadly accepted names and special notation. As the following deep and very technical result shows, there are infinitely many different complexity classes. In addition, the following theorem shows that even a very small increase in bounds on time and space resources provides an enlargement of complexity classes.

Theorem 5.2.2 *(1) If f_1 and f_2 are time-constructible functions, then*

$$\liminf_{n\to\infty} \tfrac{f_1(n)\lg f_1(n)}{f_2(n)} = 0 \implies \mathit{Time}(f_1(n)) \subsetneq \mathit{Time}(f_2(n));$$

$$\liminf_{n\to\infty} \tfrac{f_1(n)}{f_2(n)} = 0 \implies \mathit{NTime}(f_1(n)) \subsetneq \mathit{NTime}(f_2(n)).$$

(2) If $f_2(n) \geq f_1(n) \geq \lg n$ are space-constructible functions, then

$$\liminf_{n\to\infty} \frac{f_1(n)}{f_2(n)} = 0 \implies \mathit{Space}(f_1(n)) \subsetneq \mathit{Space}(f_2(n)).^{1}$$

The following relations among the main complexity classes follow from the results stated in Exercises 5.1.6 and 5.1.10 and an obvious fact that $\mathrm{Time}(f(n)) \subseteq \mathrm{NTime}(f(n))$ and $\mathrm{Space}(S(n)) \subseteq \mathrm{Nspace}(s(n))$ for any f:

$$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} = \mathbf{NPSPACE} \subseteq \mathbf{EXP} \subseteq \mathbf{NEXP}. \tag{5.2}$$

It follows from Theorem 5.2.2 that $\mathbf{L} \subsetneq \mathbf{PSPACE}$, $\mathbf{P} \subsetneq \mathbf{EXP}$, $\mathbf{NP} \subsetneq \mathbf{NEXP}$. We therefore know for sure that some of the inclusions in 5.2 are proper. Probably all of them. However,

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[1] On the other hand the class $\mathrm{Space}(\mathcal{O}(1))$ is exactly the class of regular languages and the class $\mathrm{Space}(s(n))$ with $s(n) = o(\lg n)$ contains only regular languages.

no one knows which of these inclusions are proper. One of the main problems of foundations of computing is to solve this puzzle.

If f is a time-constructible function and $f(n)/lgf(n) = \Omega(n)$ then the obvious relation $\mathrm{Time}(f(n)) \subseteq \mathrm{Space}(f(n))$ can be strengthened to show that space is strictly more powerful than time. Indeed, it holds:

$$\mathrm{Time}(f(n)) \subseteq \mathrm{Space}\left(\frac{f(n)}{\lg f(n)}\right).$$

It is also interesting to observe that the requirement that f is time-constructible is important in Theorem 5.2.2. Without this restriction we have the result:

Theorem 5.2.3 (Gap Theorem) *To every recursive function $\phi(n) > n$, there is a recursive function $f(n)$ such that $Time(\phi(f(n))) = Time(f(n))$.*

For example, there is a recursive function $f(n)$ such that $\mathrm{Time}(2^{2^{f(n))}})$ $=\mathrm{Time}(f(n))$.

Finally, we present a result indicating that our naive believe in the existence of best programs is wrong. Indeed, if a language $L \in \mathrm{Time}(t_1(n)) - \mathrm{Time}(t_2(n))$, then we say that $t_1(n)$ is the upper bound and $t_2(n)$ is a lower bound on time complexity of L on MTM. At this point it may seem that we can define the time complexity of a language (algorithmic problem) L as the time complexity of the asymptotically optimal MTM (algorithm) recognizing L. Surprisingly enough, this is not the way to go, because there are languages (algorithmic problems) with no best MTM (algorithm). This fact is more precisely formulated in the following weak version of the speed-up theorem

Theorem 5.2.4 (Blum's Speed-up Theorem) *There exists a recursive language L such that for any MTM $\mathcal{M}$ accepting L there exists another MTM $\mathcal{M}'$ for L such that $Time_{\mathcal{M}'}(n) \leq \lg(Time_{\mathcal{M}}(n))$ for almost all n.*

5.3 Reductions and Complete Problems

One of the main tasks and contributions of theory in general is to localize the key problems to which other problems can be reduced "easily" and then to investigate in depth these key problems and reduction methods. This approach has turned out to be extraordinary successful in the area of algorithmic problems.

The study of the so-called complete problems for main complexity classes and algorithmic resource bounded reductions has brought deep insights into the nature of computing, revealed surprising neat structure in the space of algorithmic problems and unexpected relations between algorithmic problems having seemingly nothing in common. In this way the study of computational complexity discovered new unifying principles for different areas of science, technology, ... On the other hand, the complexity theory results on algorithmic reductions represent a powerful methodology to design algorithms for new algorithmic problems by making use of algorithms for already solved problems.

The basic tools are time or space bounded reductions of one algorithmic problem to another one. On the most abstract level this idea is formalized through the concept of oracle Turing machines.

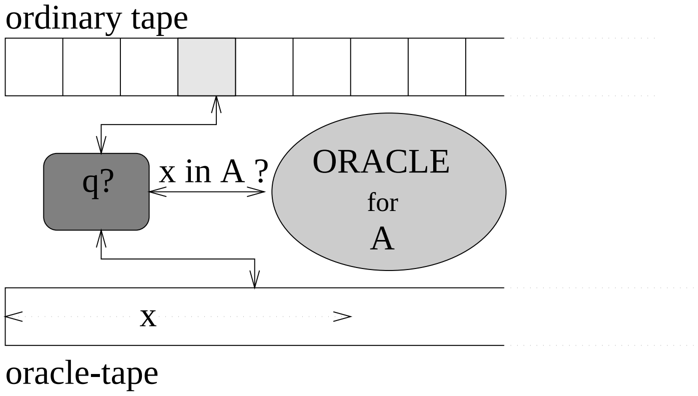

Figure 5.2: An oracle Turing machine–80 %

A (one-tape) **oracle Turing machine** $\mathcal{M}$, with the oracle-tape alphabet Δ and a language $A \subseteq \Delta^*$ as an **oracle**, is a Turing machine with two tapes, an ordinary read-write tape and a special read-write **oracle-tape**. In addition, $\mathcal{M}$ has three special states, say $q?$, q^+ and q^-, such that whenever $\mathcal{M}$ comes to the state $q?$, then the next state is either q^+ or q^-, depending on whether the contents x of the oracle-tape at that moment is, or is not, in A. In other words, when $\mathcal{M}$ gets into the "query" state $q?$, this can be seen as $\mathcal{M}$ asking the oracle the question about the membership of the word x, written on the oracle-tape, in A. In additon, and this is crucial, it is assumed that the oracle's answer is "free" and immediate (because the oracle is supposed to be all powerful—as oracles should be). In other words, a transition from the state $q?$ to one of the states q^+ or q^- takes one step, as for all other transitions.

Denote by $\mathcal{M}^A$ an oracle Turing machine $\mathcal{M}$ with the oracle A—the same TM can be connected with different oracles—and let $L(\mathcal{M}^A)$ denote the language accepted by such an oracle machine with the oracle A.

To see how such a concept can be utilized let us assume that there is an oracle Turing machine $\mathcal{M}^A$ that can solve in polynomial time an algorithmic problem $\mathcal{P}$. If the oracle A is then replaced by a polynomial time algorithm to decide membership for A, we get a polynomial time algorithm to accept $L(\mathcal{M}^A)$.

Example 5.3.1 *We describe behaviour of an oracle TM* $\mathcal{M}^A$ *with the oracle* $A = \{a^nb^nc^n \mid n \geq 1\}$ *that recognizes the language* $L = \{0^i1^i0^i1^j0^j1^j0 \mid i, j \geq 1\}$. $\mathcal{M}^A$ *starts by reading the first group of* 0*'s in the input word and for each* 0 *writes one symbol* a *on the oracle tape, then writes the symbol* b *for each* 1 *in the first group of* 1*'s and a symbol* c *for each* 0 *in the second group of* 0*'s. After encountering the first one in the second group of* 1*'s the machine asks the oracle whether the string written on the oracle tape is in* A*. If not, the machine rejects the input. If yes, the machine empties the oracle tape and then proceeds by writting on the oracle tape one* a *for each* 1 *in the second group of* 1*'s, then one* b *for each* 0 *in the third group of* 0*'s and, finally, one* c *for each* 1 *in the third group of* 1*'s. After encountering the next* 0 *the machine asks again the oracle whether the content of its oracle tape is in* A*. If not, the machine rejects. If yes,* $\mathcal{M}^A$ *checks whether there are any additional* *input symbols and if not, then accepts.*

Exercise 5.3.2 *Design in detail a simple oracle Turing machine with the oracle* $A = \{1^i \,|\, i \text{ is prime}\}$ *to accept the language*
$L = \{a^i b^j c^k \,|\, i + j \text{and } i + k \text{ are primes }\}$.

The concept of the oracle Turing machine is the base of our most general definition of time-bounded reducibilities.

Definition 5.3.3 *A language* L_1 *is* **polynomial time Turing reducible** *to the language* L_2*, in short* $L_1 \leq_p^T L_2$*, if there is a polynomial-time bounded oracle Turing machine* $\mathcal{M}^{L_2}$ *that accepts* L_1.

Very often we do not need the whole power of the Turing reducibility, which makes no restriction on how often the oracle is consulted during a computation. The following weaker concept of reducibility actually corresponds to the case when the oracle is consulted only once—at the very end of a reduction.

Definition 5.3.4 *A language* $L_1 \subseteq \Sigma_1^*$ *is* **polynomial time (many-to-one) reducible** *to the language* $L_2 \subseteq \Sigma_2^*$*, in short* $L_1 \leq_p^m L_2$*, if there is a polynomial time computable function* $f : \Sigma_1^* \to \Sigma_2^*$ *such that, if* $x \in \Sigma_1^*$*, then*

$$x \in L_1 \Longleftrightarrow f(x) \in L_2. \tag{5.3}$$

A language L_1 *is* **logspace reducible** *to* L_2*, in short* $L_1 \leq_{\text{lg}} L_2$*, if there is a logarithmic space computable (on an off-line MTM) function* f *such that* (5.3) *holds.*

Exercise 5.3.5 *The concept of the polynomial reduction was defined only for languages but it is often used in a more general sense. Show, for example, that the problem to compute the transitive closure of a Boolean matrix is both polynomial time and logarithmic space reducible in a natural way to the problem of multiplication of two Boolean matrices.*

The following result is crucial for a broader usefulness of all three concepts of reducibility.[2]

Lemma 5.3.6 *Reducibilities* $\leq_p^T$, $\leq_p^m$ *and* $\leq_{lg}$ *are transitive.*

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[2] It can be shown that all three above concepts of reducibilities are different in the sense that for any pair of them there is a pair of languages L_1 and L_2 such that L_1 is reducible to L_2 with respect to one of these reducibilities but not with respect to the second one.

Proof. Only the case of logspace reducibility is nontrivial. Let us first point out where the difficulties lie. If $x \in L_1 \iff f_1(x) \in L_2$ and $y \in L_2 \iff f_2(y) \in L_3$, where f_1 and f_2 are computable in logarithmic space on off-line MTM, then clearly $x \in L_1 \iff f_2(f_1(x)) \in L_3$. However, is $f_2(f_1(x))$ computable in $(\lg n)$-space on an off-line MTM with $n = |x|$? It may happen that to store $f_1(x)$ we need more than $(\lg n)$-space. (Notice, that in order to compute $f_1(x)$ from x we are using in the first reduction an off-line MTM the output of which is not counted into the overall space estimation.) This implies that we can not simply connect sequentially two $(\lg n)$-space bounded off-line MTM, one to compute $f_1(x)$ from x, and the second one to compute $f_2(f_1(x))$ from $f_1(x)$, and to expect automatically to get a $(\lg n)$-space bounded MTM. The trick to use is the following one.

Let $\mathcal{M}_1$ be a $(\lg n)$-space bounded off-line Turing machine to compute f_1 and $\mathcal{M}_2$ a $(\lg n)$-space bounded off-line Turing machine to compute f_2. We design a $(\lg n)$-space bounded off-line Turing machine $\mathcal{M}$ to compute $f_2(f_1(x))$ that works as follows. $\mathcal{M}$ has a counter C that is used to store information which bit of $f_1(x)$ the machine $\mathcal{M}_2$ needs as the next one. $\mathcal{M}$ starts by simulating $\mathcal{M}_2$ on x. Each time $\mathcal{M}_2$ needs a bit of $f_1(x)$, say the i-th bit, then $\mathcal{M}$ writes the integer i into C and then simulates $\mathcal{M}_1$ on x. However, it does not produce the whole output of $\mathcal{M}_1$, only a single bit, namely the i-th bit. This way $\mathcal{M}_2$ needs at most $(\lg |f_1(x)|)$-space to write i. Since f_1 is $(\lg n)$-space computable, $|f_1(x)| \leq |x|^k$ must hold for some k. Therefore, $\lg |f_1(n)| = \mathcal{O}(\lg n)$. □

We are now ready to introduce two additional concepts concerning resource bounded reducibilities that play a central rôle in complexity theory.

Definition 5.3.7 *Let $\mathcal{L}$ be a family of languages. A language L_0 is called* **hard** *for $\mathcal{L}$, with respect to a reducibility $\mathcal{R}$, if each language $L \in \mathcal{L}$ is $\mathcal{R}$-reducible to L_0. L_0 is* **complete** *for $\mathcal{L}$, with respect to a reducibility $\mathcal{R}$, if: (1) L_0 is in $\mathcal{L}$; (2) L_0 is hard for $\mathcal{L}$, with respect to $\mathcal{R}$.*

For the main complexity classes the following types of reducibilities and completeness are usually considered and for them the following notation is used:

NLOGSPACE-completeness — completeness for **NLOGSPACE** w.r.t. logspace reduc.
P-completeness — completeness for **P** w.r.t. logspace reduc.
NP-completeness — completeness for **NP** w.r.t. polynomial time reduc.
PSPACE-completeness — completeness for **PSPACE** w.r.t. polynom. time reduc.
NEXP-completeness — completeness for **NEXP** w.r.t. polynom. time reduc.

Directly from Definition 5.3.7 we get

Lemma 5.3.8 *If $L_1 \leq_p^m L_2$, and $L_3 \leq_{\lg} L_4$, then it holds:*[3]

$$\begin{aligned}
L_4 \in \mathbf{NLOGSPACE} &\implies L_3 \in \mathbf{NLOGSPACE},\\
L_2 \in \mathbf{P} &\implies L_1 \in \mathbf{P},\\
L_2 \in \mathbf{NP} &\implies L_1 \in \mathbf{NP},\\
L_2 \in \mathbf{PSPACE} &\implies L_1 \in \mathbf{PSPACE},\\
L_2 \in \mathbf{EXP} &\implies L_1 \in \mathbf{EXP}.
\end{aligned}$$

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[3] Two other concepts of **P**-completeness are dealt with in Section 5.10

As a corollary of Lemma 5.4 we have

Theorem 5.3.9 **P** = **NP** *if and only if there is an* **NP***-complete problem that is in* **P**. *Similarly,* **EXP** = **NEXP** *if and only if there is an* **NEXP***-complete problem that is in* **EXP**.

5.4 NP Complete Problems

The concept of **NP**-completeness is of a special theoretical and practical importance . Actually it is one of the most important concepts in theory and practice of computing and therefore we deal with it in detail.

A proof that a problem is **NP**-complete has two consequences. At first it implies that currently no polynomial time algorithm is known to solve such a problem—only algorithms that essentially perform a complete search of exponentially many possibilities. This means that an **NP**-complete problem can not be solved in practice for inputs of larger size. This is a bad news because many practically important algorithmic problems are **NP**-complete. Secondly, and this is a better news, **NP**-completeness of a problem implies that there is still a chance, though very very small, that for such a problem a polynomial time algorithm will be constructed. Thirdly, and this is the best news, if we discover a polynomial time algorithm for one **NP**-complete problem, we could easily design polynomial time algorithms for all **NP**-complete problems.

There are two basic methods to prove **NP**-completeness of a language L:

Direct method. One shows first that L is in **NP**, what is mostly easy, and then shows that each language in **NP** can be reduced to L in polynomial time. This means to show that for any polynomial time bounded NTM $\mathcal{M}$, and any input x for $\mathcal{M}$ we can, in polynomial time, construct string $y_{\mathcal{M},x}$ such that $x \in L(\mathcal{M})$, if and only if $y_{\mathcal{M},x} \in L$.

Reduction method. One shows $L \in$ **NP** and $L' \leq_p^m L$ for a **NP**-complete language L'. **NP**-completeness of L then follows from the transitivity of the polynomial-time reducibility.

5.4.1 Direct Proofs of NP-completeness

Direct proofs of **NP**-completeness for three problems will now be described. All these problems have the property that several of their modifications have been used to show the existence of complete problems also for other complexity classes and, in addition, these problems serve also well to show **NP**-completeness of other problems using the reduction method.

Bounded halting problem

In our first **NP**-completeness result we assume, see Section 4.1, that a fixed Gödel numbering $\langle\mathcal{M}\rangle$ of Turing machines $\mathcal{M}$, and $\langle w\rangle$ of their inputs w is given.

Theorem 5.4.1 (Time-bounded version of the halting problem) *The language*

$$L^*_{halt} = \{\langle\mathcal{M}\rangle\langle w\rangle\#^t \mid \mathcal{M} \textit{ is a one-tape one-head NTM that accepts } w \textit{ in at most } t \textit{ steps}\}$$

is **NP**-*complete (# represents a marker).*

Proof. We design a NTM $\mathcal{M}_0$ that, for an input w', first checks whether $w' = \langle\mathcal{M}\rangle\langle w\rangle\#^t$ for some NTM $\mathcal{M}$, and an input w of $\mathcal{M}$. If not, $\mathcal{M}_0$ rejects w'. If yes, $\mathcal{M}_0$ generates, nondeterministically, the string

$$\langle\mathcal{M}\rangle\langle w\rangle\#^t c_0\$c_1\$\ldots\$c_{t'},$$

where $t' \leq t$. $\mathcal{M}'$ then checks whether c_0 is the code of the initial configuration of $\mathcal{M}$ for the input w, $c_{t'}$ is the code of a terminating configuration of $\mathcal{M}$, and whether, for all $1 \leq i \leq t'$, c_i is the code of a configuration of $\mathcal{M}$ that can be obtained from the configuration c_{i-1} by one step of $\mathcal{M}$. If all these conditions are satisfied, $\mathcal{M}'$ accepts; otherwise it rejects the input.

If we denote $n = |\langle\mathcal{M}\rangle\langle w\rangle\#^t\,|$, then $\mathcal{M}_0$ accepts L^*_{halt} in $\mathcal{O}(n^3)$ time steps, because $t \leq n$, and $\mathcal{M}_0$ needs at most $\mathcal{O}(n^2)$ time to generate a new c_i. Thus, $L^*_{halt} \in$ **NP**.

Let now $L \in$ **NP**, i.e. there is a NTM $\mathcal{M}_L$ and a polynomial p_L such that $\mathcal{M}_L$ accepts L in time $p_L(n)$. Let $L \subset \Gamma^*$ and τ_L be a function defined on Γ^* and such that $\tau_L(w) = \langle\mathcal{M}_L\rangle\langle w\rangle\#^{p_L(|w|)}$. Since p_L is a polynomial, $\tau_L(w)$ can be computed in polynomial time. Moreover, $w \in L$ if and only if $\tau_L(w) \in L^*_{halt}$. Thus, $L \leq^m_p L^*_{halt}$. □

Bounded tiling problem

Tiling of the plane or other geometrical objects by various types of tiles is one of the exciting, deep and also applied problems discussed in more detail in Section 6.4. Now we deal with a special bounded version of the tiling problem.

We consider a tiling of an $n \times n$ quadratic grid G, with the outer sides of the boundary cells (1×1-squares—or unit squares) colored by colors from a set C (see Figure 5.3a), and a finite set T of 1×1-tiles, called also Wang tiles (see Figure 5.3b), the sides of which are also colored by colors from C. The problem is to decide whether there is such a tiling of G, with tiles from T, that satisfies the following two conditions:

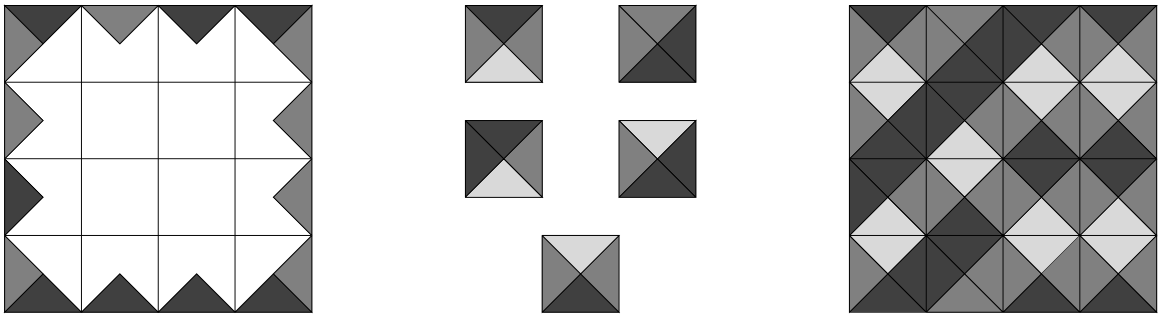
(a) (b) (c)

Figure 5.3: Bounded tiling–80%

Boundary consistency condition: Colors of outer sides of boundary tiles match colors of boundary cells of G.

Adjacency condition: Colors of adjacent sides of any two neighbouring tiles are the same.

An important restriction is that neither rotation nor reflection of tiles is allowed. On the other side, an infinite number of copies of tiles from T are available.

We show now how to associate a bounded tiling problem with a Turing machine computation. In doing so we assume that a NTM $\mathcal{M}$ has the following properties:

1. $\mathcal{M}$ uses only one-way infinite tape (see Section 4.1).
2. $\mathcal{M}$ has a unique accepting (rejecting) configuration with the empty tape and the state "ACCEPT" ("REJECT").
3. In no state $\mathcal{M}$ can move both to left and to right. (In other words, there is no state q with transitions $qaq'b \leftarrow$ and $qcq''d \rightarrow$.)

It is easy to show that each NTM $\mathcal{M}$ can be transformed into a NTM $\mathcal{M}'$ that satisfies these three conditions, accepts the same language as $\mathcal{M}$, and requires only a linear time overhead compared to $\mathcal{M}$.

Let $\mathcal{M} = \langle \Gamma, Q, q_0, \delta \rangle$ be a NTM satisfying the three conditions defined above. We describe a set of tiles $T_{\mathcal{M}}$ with the following properties.

1. Each configuration c of $\mathcal{M}$ can be encoded by a row of tiles from $T_{\mathcal{M}}$ that satisfy the adjacency condition. (Such a sequence of tiles can be seen as a tiling of a horizontal strip of heigth 1.
2. Each extension of such a row-tiling of a configuration c to a consistent tiling of the row below encodes a configuration obtained from c by one step of $\mathcal{M}$.

Construction. Colors: $C = \Gamma \cup Q \cup (Q \times \Gamma) \cup \{\sqcup\}$. Tiles of T_M are designed as follows: For each $x \in \Gamma, q \in Q$, there are tiles shown in Figure 5.4a, where filling stands for the symbol $\sqcup$.

The first of these tiles encodes a tape cell that contains the symbol x for the case that the head is not on that cell. The second tile encodes a cell with x in, the head on that cell and with the finite state control in the state q. Moreover, for each $x \in \Gamma$ and $q \in Q$ there is exactly one of the tiles shown in Figure 5.4b, depending on whether $\mathcal{M}$ moves, in the state q, only to the left, or only to the right. Finally, to each of the transitions

$$(q, x, q', x', \leftarrow) \qquad (q, x, q', x', \rightarrow) \qquad (q, x, q', x', \downarrow)$$

corresponds one of the tiles shown in Figure 5.4c.

For example, the encoding of the configuration $aqbcdefgh$ and of the three next configurations obtained by the transitions

$$(q, b, q', b', \rightarrow), \qquad (q', c, q'', c', \rightarrow), \qquad (q'', d, q''', d', \downarrow),$$

has the form shown in Figure 5.5.

Theorem 5.4.2 *The bounded tiling problem (BTP) is* **NP***-complete.*

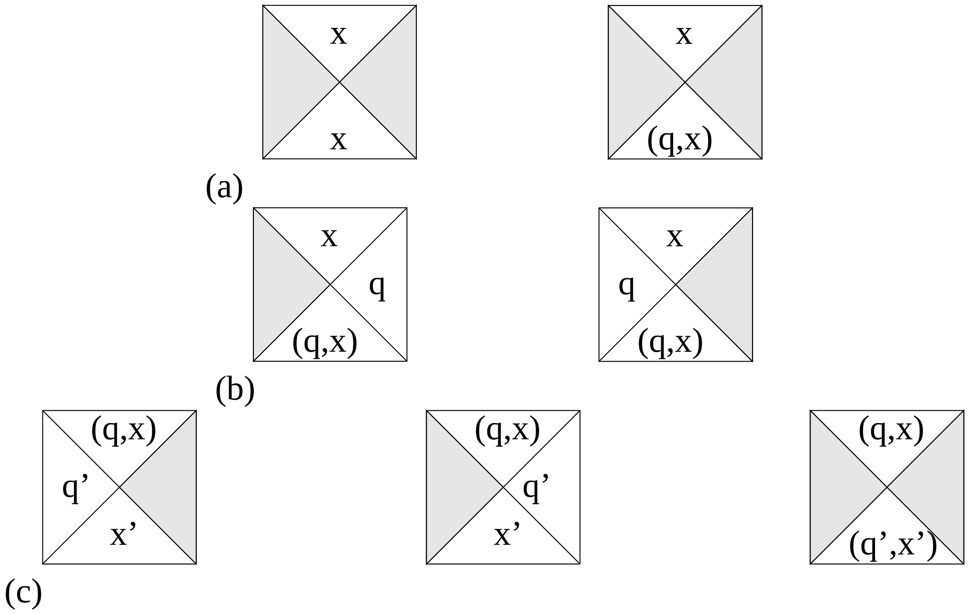

Figure 5.4: Tiles design to simulate a TM

Proof. BTP is clearly in **NP**—one just guesses a tiling and verifies whether boundary consistency and adjacency conditions are satisfied.

Let now $L \in \mathbf{NP}$ and $\mathcal{M}$ be a NTM in the normal form defined above that accepts L in time cn^k for some constant c. Finally, let w be an input word of $\mathcal{M}$. We show how to design, in polynomial time, an instance of BTP, denoted $\mathrm{BTP}_{\mathcal{M},w}$ such that $\mathcal{M}$ accepts w in time $c|w|^k$, if and only if $\mathrm{BTP}_{\mathcal{M},w}$ has a solution. Once this is shown the theorem is proven. Let $T_{\mathcal{M}}$ be the set of tiles corresponding to $\mathcal{M}$ and $G_{\mathcal{M},w}$ be the $N \times N$ grid, where $N = c|w|^k$ and sides of which are colored as follows: In the top row there are colors determined by the word $q_0 w \sqcup^{N-|w|}$, i.e., by the initial configuration extended by blanks. The colors of the sides

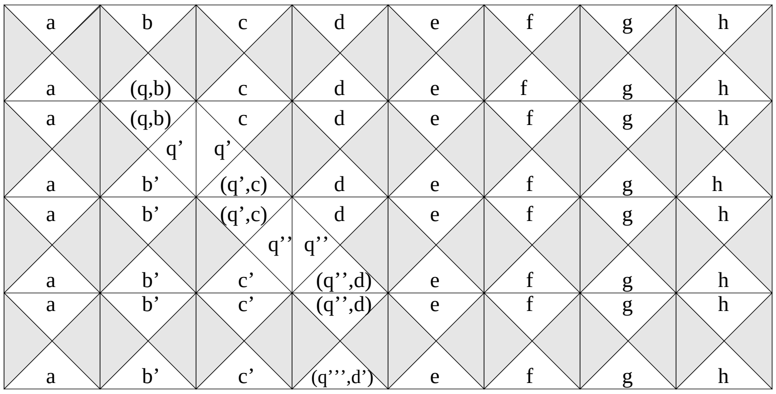

Figure 5.5: Tiling simulating a computation -60%

of the last row are determined by the encoding of the only accepting configuration. Left and right sides of $G_{\mathcal{M},w}$ are colored by blanks (depicted in the figures by filled areas). Clearly, $\mathcal{M}$ accepts w in time $c|w|^k$, if and only if there is a tiling of $G_{\mathcal{M},w}$ with tiles from $T_{\mathcal{M}}$. ☐

Satisfiability of Boolean formulas

Historically, the first known **NP**-complete problem was the satisfiability problem for Boolean formulas, due to S. A. Cook (1971). Since then various variants of the satisfiability problem have turned out to play an important rôle in complexity theory.

Each Boolean formula $\mathcal{F}$ can be encoded in a natural way by a binary word. If $\mathcal{F}$ has length n and m variables one needs for encoding $\mathcal{O}(n\lceil\lg m\rceil)$ bits. The set of all encodings of all satisfiable Boolean formulas into binary words will be denoted SAT. Formally, we show that SAT is an **NP**-complete language. Less formally this result is stated as follows:

Theorem 5.4.3 (Cook's Theorem) *The satisfiability problem for Boolean formulas is* **NP***-complete.*

Proof. It is easy to see that SAT is in **NP**. One checks whether a given input string is an encoding of a Boolean formula, then guesses a satisfying assignment and evaluates the formula. This can be done in a polynomial time (and actually in linear time on a 2-tape TM, because the set of Boolean formulas is a deterministic context-free language—see Section 7.3).

Let now $L \in \mathbf{NP}$ and $\mathcal{M}$ be a one-tape NTM that accepts L in a time bounded by a polynomial p. We show how to design, given an input w for $\mathcal{M}$, in polynomial time a Boolean formula $\mathcal{F}_{\mathcal{M},w}$ that is satisfiable if and only if $\mathcal{M}$ accepts w.

Let $\mathcal{M} = \langle \Gamma, Q, q_0, \delta \rangle$, $Q = \{q_0, \dots, q_s\}$, where $q_{s-2} = HALT$, $q_{s-1} = ACCEPT$ and $q_s = REJECT$ and $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ with $\gamma_1 = \sqcup$. If $\mathcal{M}$ accepts w, then $\mathcal{M}$ makes on the input w, with $|w| = n$, at most $p(n)$ steps and the length of any configuration is at most $p(n)$. (We are assuming that $\mathcal{M}$ has one-way infinite tape and an idling termination.)

In order to describe $\mathcal{F}_{M,w}$, in short F', we use the following variables (that are supposed to have value one if the attached condition is satisfied):

$C(i,j,t)$ — i-th cell contains the symbol γ_j at time t,
$S(r,t)$ — $\mathcal{M}$ is in state q_r at time t,
$H(i,t)$ — the head of $\mathcal{M}$ stays on the i-th cell at time t,

for $1 \le i \le p(n)$, $1 \le j \le m$, $0 \le t \le p(n)$, $0 \le r \le s$. The total number of variables is: $\mathcal{O}(p^2(n))$. The formula F' will have the form

$$F' = A \wedge B \wedge C \wedge D \wedge E \wedge F \wedge G,$$

where $A, \dots, G$ are Boolean formulas that describe $\mathcal{M}$ and its computation on the input w. In order to shorten the descriptions of these formulas we use the predicate:

$$\bigoplus(a_1, \dots, a_{\mathcal{T}}) = (a_1 \vee a_2 \vee \dots \vee a_{\mathcal{T}}) \bigwedge_{i \neq j} (\neg a_i \vee \neg a_j)$$

that has the value 1 if and only if $a_i = 1$ for exactly one i. The length of this formula is $\mathcal{O}(\mathcal{T}^2)$. Description of subformulas:

1. $A = A_1 \wedge A_2 \wedge \cdots \wedge A_{p(n)}$,
   $A_t = \bigoplus(H(1,t), H(2,t), \ldots, H(p(n),t)), 0 \le t \le p(n)$,
   i.e. $A_t = 1$, if and only if, at time t, the head of $\mathcal{M}$ stays on exactly one of the cells.

2. $B = \bigwedge\limits_{0\le t\le p(n), 0\le i\le p(n)} B(i,t) \quad \text{with} \quad B(i,t) = \bigoplus(C(i,1,t), \ldots, C(i,m,t))$,
   i.e. $B(i,t) = 1$ if and only if, the i-th cell contains exactly one symbol at the time t.

3. $C = \bigwedge\limits_{0\le t\le p(n)} C_t \quad \text{with} \quad C_t = \bigoplus(S(0,t), \ldots, S(s,t))$,
   i.e. $C = 1$, if and only if, M is, in exactly one state at each moment,.

4. $D = \bigwedge\limits_{0\le i,t\le p(n), 1\le j\le m} [(C(i,j,t) \equiv C(i,j,t+1)) \vee H(i,t)]$,

   i.e., $D = 1$, if and only if, at each moment, at most one cell changes its state.

5. $E = \bigwedge\limits_{0\le i,t\le p(n), 0\le j\le m, 0\le k\le s} E_{ijkt} \quad \text{and} \quad E_{ijkt} = \neg C(i,j,t) \vee \neg S(k,t) \vee \neg H(i,t) \vee$

   $$\bigvee\nolimits_l [C(i,j_l,t+1) \wedge S(k_l,t+1) \wedge H(i+r_l,t+1)],$$

   i.e., $E = 1$ if and only if each next configuration is obtained from the previous one by a valid transition of $\mathcal{M}$. In the formula $E_{i,j,k,t}$, l runs through all possible transitions of $\mathcal{M}$, for the case that $\mathcal{M}$ is in the state q_k and the head reads x_j. It is assumed here that an l-th transition has the form $(q_k, x_j, q_{k_l}, x_{j_l}, r_l)$, where $r_l \in \{-1,0,1\}$ depending on the direction the head moves.

6. $F = S(0,0) \wedge H(1,0) \wedge \bigwedge\limits_{0\le i\le n} C(i,j_i,0) \wedge \bigwedge\limits_{n<i\le p(n)} C(i,1,0)$,

   where $w = x_{j_1}, \ldots, x_{j_n}$, and $F = 1$ if and only if at time $t = 0$, $\mathcal{M}$ is in the state q_0 and the tape contains w.

7. $G = S(s-1, p(n))$

   i.e. $G = 1$, if and only if M comes finally into the accepting state.

The overall length of the Boolean formula F' is therefore $\mathcal{O}(p^3(n))$. It is now easy to verify that F' is satisfiable if and only if $\mathcal{M}$ accepts w. $\square$

5.4.2 Reduction Method to Prove NP-completeness

The **NP**-completeness of many problems can been shown by a chain of reductions starting with the satisfiability problem.

Theorem 5.4.4 *The language* CNFF $= \{w \mid w$ *is a satisfiable Boolean formula in CNF*$\}$ *is* **NP**-*complete.*

Proof. Deciding if a formula is in conjuctive normal form requires only polynomial time. Since SAT $\in$ **NP**, we have immediately that CNFF $\in$ **NP**. To show **NP**-completeness of CNFF, it is now enough to show SAT $\leq_p^m$ CNFF, i.e. that there is a polynomial time algorithm to reduce a Boolean formula F to a Boolean formula F' in CNF such that F is satisfiable if and only if F' is. (Observe that we do not require that formulas F and F' are equivalent.)

To design an equivalent Boolean formula in CNF we cannot use the method described in Section 2.3.2, because this method can increase the size of a formula exponentially. However, we can use the following polynomial time transformation.

Using de Morgan laws we can shift all negations directly to the variables and this way we get a formula with literals and conjuctions and disjunctions as the only operations. Shifting all negations can be done easily in linear time by one pass through the formula. One has only to keep track of the depth of subformulas with respect to negation. The resulting formula will be at most twice as long as the original one.

In order to transform the resulting Boolean formula into a conjunction of clauses the following trick can be used. If $F = F_1 \vee F_2$ and both F_i are already in CNF, then we choose a new variable y that does not occur in the formulas F_1 and F_2, and replace F with the formula $F' = (F_1 \vee y) \wedge (F_2 \vee \overline{y})$. Clearly, F is satisfiable if and only if F' is. If now

$$F_i = G_1 \wedge \cdots \wedge G_k$$

with clauses G_j, $1 \leq j \leq k$, then

$$F_i \vee y \equiv (G_1 \vee y) \wedge \cdots \wedge (G_k \vee y),$$

and $G_i \vee y$ are again clauses. If we continue this way, and the original formula is of the size n, then the above construction increases its size at most by n, and the whole reduction can be performed in $\mathcal{O}(n^2)$ steps. $\square$

Theorem 5.4.5 *The language* 3-*CNFF* $= \{w \mid w$ *is a Boolean formula in conjunctive normal form all clauses of which have exactly 3 literals (in the so-called* $3-CNF$ *form)*$\}$ *is* **NP**-*complete.*

Proof. CNFF $\in$ **NP** implies 3-CNFF $\in$ **NP**. To show CNFF $\leq_p$ 3-CNFF, we describe a polynomial time algorithm that transforms any Boolean formula in CNF into a CNF in which each clause has exactly three literals. We show how to transform a clause $C = x_1 \vee \ldots \vee x_k$ into such a form. This is done by introducing new variables. Let us distinguish three cases:

$$\begin{array}{ll} k = 1, & C \equiv (x_1 \vee y_1 \vee y_2) \wedge (x_1 \vee \neg y_1 \vee y_2) \wedge (x_1 \vee y_1 \vee \neg y_2) \wedge (x_1 \vee \neg y_1 \vee \neg y_2); \\ k = 2, & C \equiv (x_1 \vee x_2 \vee y_1) \wedge (x_1 \vee x_2 \vee \neg y_1); \\ k > 3, & C \equiv (x_1 \vee x_2 \vee y_1) \wedge (\neg y_1 \vee x_3 \vee y_2) \wedge (\neg y_2 \vee x_4 \vee y_3) \\ & \wedge \cdots \wedge (\neg y_{k-3} \vee x_{k-1} \vee x_k). \end{array}$$

It is clear that the resulting Boolean formula is satisfiable if and only if the original one is satisfiable. □

Exercise 5.4.6 *Let CNFF-k be the language consisting of those satisfiable Boolean formulas in CNF in which each variable occurs in at most k clauses. Show that the language CNFF-*3 *is* **NP**-*complete.*

Exercise 5.4.7 * *Show, for example by a reduction from the* 3-*CNFF problem, that the CLIQUE problem is* **NP**-*complete. (It is the problem to decide, given a graph G and an integer k, whether G has a clique of size k.)*

Exercise 5.4.8 *Show, for example by a reduction from the* 3-*CNFF problem, that NODE-COLORABILITY problem is* **NP**-*complete. (It is the problem to decide, given a graph G and an iteger k, whether G is node-colorable with k colors.)*

The **knapsack problem**, **NP**-completeness of which we show next and that has played an important rôle in the modern cryptography—see Chapter 8—is defined as follows: Given integers $w_1, \ldots, w_n$ and c, does there exist a Boolean vector $(x_1, \ldots, x_n)$ such that

$$\sum_{j=1}^{n} x_i w_i = c\,?$$

(The story behind: Given n objects, $o_1, \ldots, o_n$, with weights $w_1, \ldots, w_n$, and a knapsack with the capacity c. Is there a subset of objects with the total weight exactly c? (To be loaded into the knapsack.)) In the following we assume that each such a knapsack problem instance, given by weights and a capacity, is encoded by a binary string and let us denote by KNAPSACK the corresponding language of instances of solvable knapsack problems[4].

Theorem 5.4.9 *KNAPSACK is an* **NP**-*complete problem.*

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[4] There are several variants of the knapsack problem. The one discussed in this section is often called SUBSET-SUM problem. Other version is discussed in Section 5.7.2.

Proof. The problem is clearly in **NP**—one just chooses a subset of weights and checks whether their sum equals the given capacity. **NP**-completeness of KNAPSACK will be proven by showing that 3-CNFF $\leq_p^m$ KNAPSACK. To do that we design an algorithm that transforms, in polynomial time, a Boolean formula F, in 3-CNFF, into an instance of the knapsack problem that is solvable if and only if the Boolean formula F is satisfiable.

Algorithm: Let $F = F_1 \wedge \ldots \wedge F_k$, where each F_i is a clause with exactly three literals from the set $\{x_1, \ldots, x_m, \neg x_1, \ldots, \neg x_m\}$. We show now how to construct the knapsack capacity c and the weights $w_1, \ldots, w_{2k+2m}$ (all in decimal—this choice of number system will be of no significance because a transformation into the binary form can be done in polynomial time). Let c be an integer the decimal representation of which has $(m+k)$ digits, all equal to 3:

$$c = \underbrace{3 \ldots 3}_{m+k}$$

and $w_1, \ldots, w_{2k+2m}$ be $(m+k)$-digit numbers denoted by

$$\overline{x}_1, \ldots, \overline{x}_m \qquad \neg\overline{x}_1, \ldots, \neg\overline{x}_m \qquad y_1, \ldots, y_k \qquad y'_1, \ldots, y'_k \tag{5.4}$$

and defined as follows: (The notation used in (5.4) is to indicate the relation of these numbers to literals $x_1, \ldots, x_m, \neg x_1, \ldots, \neg x_m$ or to the clauses $F_1, \ldots, F_k$.)

(1) For $i = 1, \ldots, m$, the i-th rightmost digit in $\overline{x}_i$ and also in $\neg\overline{x}_i$, is 3. For $j = 1, \ldots, k$, the $(m+j)$-th rightmost digit in $\overline{x}_i$ (in $\neg\overline{x}_i$) is 1 if F_j contains x_i (if F_j contains $\neg x_i$). All other digits in $\overline{x}_i$ and $\neg\overline{x}_i$ are 0.

Observe that in this way $\overline{x}_i$ contains, in its decimal repesentation, uniquely and easily decodable information to which of the variables it corresponds (namely x_i) and in which of the clauses x_i occurs. The same holds for $\neg\overline{x}_i$. Therefore the numbers $\overline{x}_1, \ldots, \overline{x}_m, \neg\overline{x}_1, \ldots \neg\overline{x}_m$ represent a full description of the formula F.

(2) For $j = 1, \ldots, k$, the $(m+j)$-th rightmost digit in y_j is 1, all other digits are 0. Moreover, $y_i = y'_i$ for all $1 \leq i \leq k$.

Example 5.4.10 *If* $F = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$, *then* $k = 2, m = 3$ *and*

$$\begin{array}{llll} c = 33333 & \overline{x_1} = 01003 & \neg\overline{x_1} = 10003 & y_1 = 01000 = y'_1 \\ & \overline{x_2} = 01030 & \neg\overline{x_2} = 10030 & y_2 = 10000 = y'_2 \ . \\ & \overline{x_3} = 10300 & \neg\overline{x_3} = 01300 & \end{array}$$

Clearly: $c = \overline{x_1} + \overline{x_2} + \overline{x_3} + y_1 + y'_1 + y_2$

We show now that the knapsack problem K_F, given by the weights $w_1, \ldots, w_{2k+2m}$ and the capacity c, as described above, has a solution if and only if the formula F is satisfiable.

1. Let F be satisfiable. Fix a satisfying assignment α for F. For each $i = 1, \ldots, m$, we take $\overline{x}_i$ or $\neg\overline{x}_i$ into the solution of K_F, depending on whether x_i has the value 1 or 0 in the assignment α. The sum of all these numbers has the last m digits equal 3, and each of the first k digits of this sum is either 1, 2 or 3. By adding a proper number of (y)'s and (y')'s, we get exactly c.

2. Let the knapsack problem K_F have a solution. In other words, assume that there is a subset of numbers from (5.4) that is a solution of K_F. This is possible only if, for each $i = 1, \ldots, m$, exactly one of the numbers $\overline{x}_i$ or $\neg\overline{x}_i$ is in a sum forming c. By taking exactly that one from each pair $(\overline{x_i}, \neg\overline{x_i})$ we get a satisfying assignment for F. □

The last problem, whose **NP**-completeness we now show, again by a reduction from 3-CNFF, is the Hamilton cycle problem for directed graphs—one of the basic problems concerning traversals of directed graphs. In doing so we denote by HAMILTOND the set of encodings of directed graphs that have a Hamilton cycle.

Theorem 5.4.11 *HAMILTOND is an* **NP***-complete problem.*

Proof. It is trivial to see that HAMILTOND $\in$ **NP** and so we concentrate on showing a polynomial time reducibility of 3-CNFF to HAMILTOND.

Let a Boolean formula in 3-CNF have the form $F = F_1 \wedge \ldots \wedge F_k$, where each $F_i = y_{i1} \vee y_{i2} \vee y_{i3}$ is a clause with three literals. We show how to design in polynomial time a graph G_F such that F is satisfiable if and only if G_F has a Hamilton cycle.

Let $x_1, \ldots, x_m$ be all variables that occur in F, and for each $1 \leq i \leq m$, let m_i be the total number of occurrences of literals x_i and $\neg x_i$ in F. First, for each variable x_i we design a subgraph H_i, depicted in Figure 5.6a, with $2m_i + 4$ nodes, $4m_i + 6$ edges and $m_i + 3$ levels, and also we design, for each clause F_j, a subgraph G_j, with 6 nodes, 9 internal edges, 3 ingoing and 3 outgoing edges—see Figure 5.6c. Moreover, let us connect all H_i graphs into a cycle as indicated in Figure 5.6b to get the graph H.

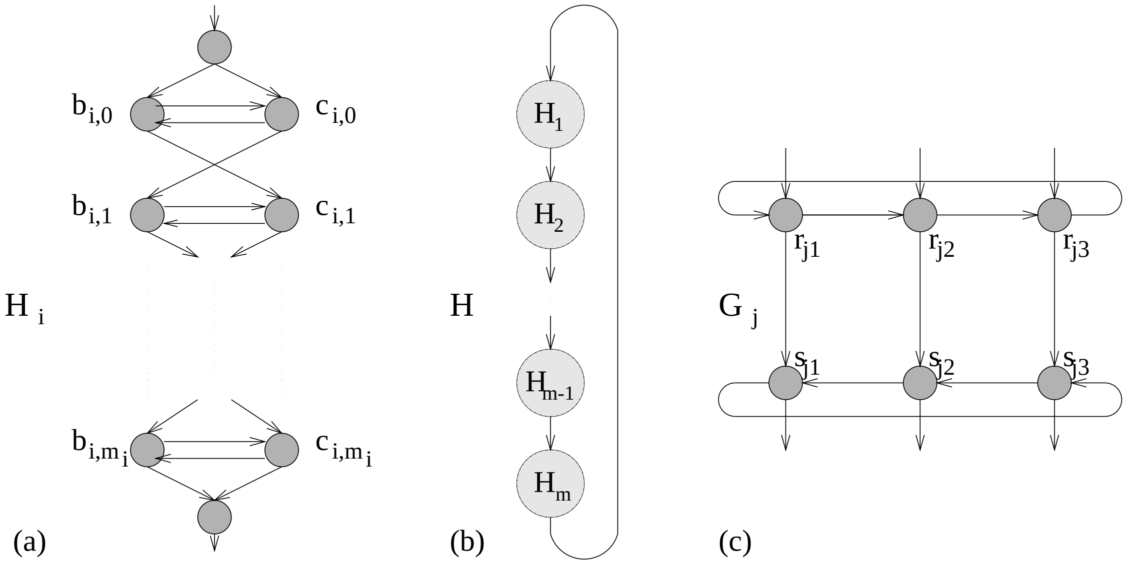

Figure 5.6: Design of a hamiltonian graph for a satisfiable Boolean formula-70%

Observe that each H_i has exactly two edge-disjoint Hamilton paths. Each of them is uniquely determined by the edge chosen at the starting node—either to $b_{i,0}$ or to $c_{i,0}$. Observe also that in one of these two paths all edges from one level to the next level are always from a b-node to a c-node and in the second path this is exactly in the opposite direction. The key point for our reduction now is that one of the Hamilton paths through the graph H_i, say the one through $b_{i,0}$, will correspond to the assignment of 1 to the variable x_i and the second to the assignment of 0 to x_i.

As the next step we connect all G_j graphs with H to get a graph that encodes the whole formula F. For each clauseF_j, and each of its literals y_{jk}, we do the following: if $y_{jk} = x_i$, then in the subgraph H_i we introduce an edge from a c-node to the node r_{jk}, and an edge from the s_{jk}-node to the corresponding b-node. If $y_{jk} = \neg x_i$, we make a similar construction, but this time a detour from a b-node to a c-node of the next level is introduced, and the corresponding nodes are connected with nodes r_{jk} and s_{jk}. It is not important which of the not-yet-used levels of H_i is taken to make these interconnections between H_i and G_j graphs. However, since each H_i has exactly $m_i + 1$ levels with two nodes, the whole construction can take place. We perform this construction for all clauses and all literals in them. As the result, all input and output edges of all G_j-graphs are connected with some nodes of H. The resulting graph G_F uniquely encodes F.

It remains to show that G_F has a Hamilton cycle if and only if Boolean formula F is satisfiable. Let us first assume that F is satisfiable.

Observe that if one enters a G_j graph through a node r_{jk} in a Hamilton cycle through the graph G_F, then there is only one way to visit all nodes of G_j exactly once and to get out—through the node s_{jk}. If one leaves through another node then there is a node of G_j that has not been visited yet, and there is no other way to visit this node on this to-be-Hamilton cycle.

If now F is satisfied, then we pick up in each clause F_j one literal that takes the value 1 in the chosen assignment. If this literal corresponds to a variable x_i, this determines the way H_i should be traversed. In addition, for this single literal we make a detour from H_i to visit G_j and to take a Hamilton path through G_j. This way we get a Hamilton cycle for G_F.

The rest of the proof—to show that the existence of a Hamilton cycle for G_F implies the existence of a satisfying assignment is now not difficult using the above arguments, and is left to the reader. □

Exercise 5.4.12 *Show that the problem to decide for a given undirected graph G whether G is hamiltonian, the so-called HAMILTON problem, is* **NP***-complete.*

Exercise 5.4.13 *Show that the problem to decide whether a given graph has a Hamilton path is* **NP***-complete.*

5.4.3 Analysis of NP-completeness

The concept of **NP**-completeness is one of the most important but also one of the most complex in computing. Let us analyze various aspects of it.

NP-completeness of the optimization problems

The so-called decision versions of many important optimization problems—actually almost all one needs to solve in practice—are **NP**-complete in the sense discussed later. On the theoretical side this is, for example, the problem to determine the maximal number of satisfying assignments of a Boolean formula—the so-called MAXSAT problem. Perhaps the most important and most widely investigated optimization problem is the **Travelling Salesman Problem**, in short TSP, defined as follows.

We are given n cities, denoted by $c_1, \ldots, c_n$, and a nonnegative integer-distance d_{ij} between any two cities c_i and c_j (assuming $d_{ij} = d_{ji}$, for all i, j). The task is to find the shortest tour of the cities—that is a permutation π on $\{1, \ldots, n\}$ such that $\sum_{i=1}^{n}, d_{\pi(i),\pi(i+1)}$ is as small as possible (where $\pi(n+1)$ is defined to be $\pi(1)$).

Actually, none of these two problems is a decision problem, and therefore the concept of **NP**-completeness does not apply formally to them. In spite of that it is common to speak about **NP**-completeness of such problems. By that we usually understand **NP**-completeness of the decision versions of the problems, where a bound b is given, as a part of the input data, and the question is to decide whether there is a solution smaller (or larger) than this bound. (For example, in the case of the MAXSAT problem, the question to decide is whether the number of satisfying assignments is larger than b? Or is there a tour through the graph shorter than b in case of TSP?)

Theorem 5.4.14 *The decision version of the travelling salesman problem is* **NP***-complete and the traveling salesman problem is* **NP***-hard.*

Proof. We show how to reduce, in polynomial time, the HAMILTON problem to a bounded decision version of TSP. Let G be a graph with n nodes $c_1, \ldots, c_n$. We design a distance matrix $\{d_{ij}\}_{ij}^{n}$, and a cost limit b, such that there is a Hamilton cycle in G, if and only if there is a tour in G of length at most b. The distance d_{ij} is defined to be 1, if there is an edge between the nodes c_i and c_j in G; otherwise $d_{ij} = 2$, and b is defined to be n. The rest of the proof is now straightforward. $\square$

Decision versions of hundreds, or even thousands, of various optimization problems have turned out to be **NP**-complete. The rational behind this view of optimization problems is that once we know a polynomial time algorithm for such a decision-version of an optimization problem, then it is easy to find the optimal solution using the binary search approach.

Example 5.4.15 *Assume we have a polynomial time algorithm $\mathcal{A}$ for the bounded version of TSP. Given a graph G and a distance matrix $\{d_{ij}\}_{ij}^{n}$, we can use $\mathcal{A}$ to compute the shortest tour of G as follows. First we take a trivial upper bound on the length of such tour, say b. Then we use $\mathcal{A}$ to decide whether G has a tour shorter than $\frac{b}{2}$. If yes, we call $\mathcal{A}$ with $\frac{b}{4}$, if not with $\frac{3b}{4}$. Continuing this binary search we can get, with logarithmic many attempts, the shortest tour.*

Exercise 5.4.16 *Show that in case of TSP the task to find the shortest tour can be reduced in polynomial time to the problem to compute the cost of the minimal tour and to the corresponding decision problem.*

Fragility of NP-completeness

There is a simple and quite good "rule of thumb" for finding out whether a decision problem is **NP**-complete: if checking exponentially many possibilities seems to be the only way to solve the problem and, at the same time, it is easy to verify correctness of any to-be-solution. However, as Table 5.1 shows, this "rule of thumb" has its limits. There are pairs of algorithmic problems that look very similar but one of them is in **P** and the other one is **NP**-complete.

POLYNOMIAL TIME	**NP**-COMPLETE
EDGE-COVER **Input**: Graph $G = (V, E)$, integer k. **Problem**: Is there a subset $E' \subset E$ with $\lvert E'\rvert \leq k$ such that every vertex is incident with an edge in E'?	VERTEX-COVER **Input**: Graph $G = (V, E)$, integer k. **Problem**: Is there a subset $V' \subset V$ with $\lvert V'\rvert \leq k$ such that every edge is incident with a node in V'?
EULER-CYCLE **Input** Graph $G = (V, E)$. **Problem**: Does G have an Euler tour?	HAMILTON-CYCLE **Input** Graph $G = (V, E)$. **Problem**: Does G have a Hamilton cycle?
LINEAR-DIOPHANTINE-EQUATIONS **Input**: Positive integers a, b and c. **Problem**: Are there positive integers x and y such that $ax + by = c$?	QUADRATIC-DIOPHANTINE-EQUATIONS **Input** Positive integers a, b and c. **Problem**: Are there positive integers x and y such that $ax^2 + by = c$?
LINEAR-PROGRAMMING **Input**: $k \times l$ matrix A over $\mathbf{Z}$, $b \in \mathbf{Z}^k$. **Problem**: Does there exist $y \in \mathbf{Q}^l$ such that $Ay \geq b$?	INTEGER-LINEAR-PROGRAMMING **Input**: $k \times l$ matrix A over $\mathbf{Z}$, $b \in \mathbf{Z}^k$. **Problem**: Does there exist $y \in \mathbf{Z}^l$ such that $Ay \geq b$?

Table 5.1: Problems in **P** and **NP**-complete problems

Isomorphism of NP-complete problems

NP-complete problems occur in all areas of computing, mathematics, science, engineering, ... and often one sees at the first sight no relation among them. In spite of that, and this is a quite surprising and important discovery, there are no two known **NP**-complete problems that would not be isomorphic in the following sense: there is a bijection, computable in polynomial time, that maps instances of one **NP**-complete problem into instances of the second one, and also the inverse mapping is computable in polynomial time. This seems to indicate that any two **NP**-complete problems are actually very similar! However, there is no proof yet that this is really true. Therefore we have only a conjecture:

> **Hypothesis** [Berman-Hartmanis' hypothesis] All **NP**-complete problems are mutually polynomial time isomorphic.

Currently there seems to be more doubt than belief that the hypothesis is true. The following fact also contributes to that: if the existence of good approximation algorithms for **NP**-complete problems is considered, see Section 5.7, then we can prove that there are, from the approximation point of view, essential differences among **NP**-complete problems.

NP-completeness

dependence on the form of inputs. By saying that a decision problem is **NP**-complete, it is implicitly assumed that all numerical input data are given in binary (or in decimal), and that the total number of bits needed to represent all input data is taken as the size of the problem. This implicit assumption is sometimes very essential because an **NP**-complete problem can be in **P**, if input data are given using the unary representation, or even outside of **NP**, if input data are given in some very compressed form!

Example 5.4.17 (Knapsack problem with input data in unary form) *Let us recall that*

$$KNAPSACK = \{a_1, \ldots, a_n, c \mid \exists(x_1, \ldots, x_n) \in \{0,1\}^n, \sum_{i=1}^{n} a_i x_i = c\},$$

and let us consider a vector $B[0, 1, \ldots, c]$, defined, for $0 \leq s \leq c$, by

$$B[s] = \begin{cases} 1, & \textbf{if } \exists(x_1, \ldots, x_n) \in \{0,1\}^n : s = \sum_{i=1}^{n} a_i x_i; \\ 0, & \textit{otherwise.} \end{cases}$$

We can compute the vector B in $\mathcal{O}(nc)$ time using the following algorithm:

```
$B[0] \leftarrow 1$;
for $s \leftarrow 1$ to $c$ do $B[s] \leftarrow 0$
for $i \leftarrow 1$ to $n$ do for $s \leftarrow c$ downto $a_i$ do
                    if $B[s - a_i] = 1$ then $B[s] \leftarrow 1$
```

In case the input data for the KNAPSACK problem are given in unary form, their representation has length at least $n + c$ bits (c bits are needed to represent c and at least one is needed for each a_i). Time complexity of the above algorithm is therefore quadratic, with respect to the size of the input.

However, there are **NP**-complete problems that remain **NP**-complete even in case their input data are given in the unary form. Such problems are called **strongly NP-complete**.

Exercise 5.4.18 *Show that the following* **NP***-complete problems are strongly* **NP***-complete: (a) CLIQUE problem (see Exercise 5.4.7; TSP problem (decision version).*

Example 5.4.19 *The so-called* 3*-PARTITION problem is also an example of a problem that is* **NP***-complete in the strong sense. It is the following problem: given numbers* $a_i, 1 \leq i \leq 3m$*, partition these numbers into* m *sets in such a way that the sum of elements in each set is the same. (If* $\sum_{i=1}^{3m} a_i = mb$*, and it is required that, in addition, each* $a_i \in (\frac{b}{4}, \frac{b}{2})$*, then each partition has to have three elements—from this comes the name of the problem.)*

The existence of a strong dependence of the running time of some algorithms on the way inputs are presented leads to the following concept:

Definition 5.4.20 *A* **pseudopolynomial time algorithm** *is one whose running time would be polynomial if all inputs were expressed in unary notation.*

On the other side, there are also **NP**-complete problems that are no longer in **NP** if inputs are given in a very compressed form. This has been shown for various graph problems on regular graphs with n nodes that can be represented using $\mathcal{O}(\log n)$ bits.

Average-case NP-completeness

Most of the complexity theory deals with the worst-case complexity. The main reason is that the worst-case complexity is much more easy to deal with than the average-case complexity and still provides deep insights in the nature of computing and also pays large practical dividends.

In some cases, however, the average-case complexity of an algorithm or an algorithmic problem provides more adequate estimations of real computational complexity. (One area, where it is insufficient for a problem to have large worst-case complexity and we actually need high average-case complexity, is cryptography discussed in Chapter 8.) Moreover, the average-case complexity provides a different hierarchy of algorithmic problems. Indeed, there are pairs of algorithmic problems with the same worst-case complexity but much different average-case complexity. In addition, there are **NP**-complete problems with polynomial time average-case complexity.

In order to extend the concept of **NP**-completeness also to the average-case complexity, first of all we have to take a more general view what a decision problem is. By that we will mean, for the rest of this section, a pair (L, μ), a so-called **randomized decision problem**, where $L \subseteq \Sigma^*$ is a language, and μ is a probability distribution over Σ^*, such that $L = \{w \in \Sigma^* \mid \mu(w) \neq 0\}$, and μ is "computable in polynomial time"; i.e. the so-called cumulative distribution

$$M(x) = \sum_{y \leq x} \mu(y)$$

is computable in polynomial time. (Observe that this is a strong requirement, because the sum is taken over the exponentially many substrings smaller than x with respect to the strict ordering.)

We say that a decision problem (L, μ) can be solved in the average polynomial time, if there is a TM $\mathcal{M}$, and an integer $k > 0$, such that

$$\sum_{x \in \Sigma^*} \mu(x) \frac{(T_{\mathcal{M}}(x))^{\frac{1}{k}}}{|x|} < \infty,$$

where $T_{\mathcal{M}}(x)$ is the number of steps of $\mathcal{M}$ for the input x. (Note that this definition is well motivated and robust. Indeed, since the outcome is not affected by raising $T_{\mathcal{M}}(x)$ or $|x|$ to a constant power, the definition is to a sufficient degree machine-model independent and it is also not affected by polynomial reductions).

A polynomial reduction from a problem (L, μ), $L \subseteq \Sigma^*$, to a problem (L', μ'), is a polynomial reduction f of L to L' such that the following property holds: there exists an integer l such that, for all $x \in \Sigma^*$,

$$\mu'(x) \geq \frac{1}{|x|^l} \sum_{y \in f^{-1}(x)} \mu(y).$$

(In other words, the distribution μ' should be nowhere more than polynomially smaller than the distribution induced by μ).

Exercise 5.4.21 *Show that a reduction from one randomized algorithmic problem (L, μ) to another (L', μ'), defined as above, is transitive.*

Exercise 5.4.22 *Show that if there is a polynomial time reduction from (L, μ) to (L', μ'), and (L', μ') can be solved in the average polynomial time, then (L, μ) can also be solved in the average polynomial time.*

We can now define that a randomized problem(L, μ) is the average-case **NP**-complete, if L is in **NP**, μ is computable in polynomial time and all problems (L', μ') with $L' \in$ **NP** and μ' polynomially computable, can be reduced in polynomial time to (L, μ). The proofs of the average-case **NP**-completeness are in general more technical than for **NP**-completeness, because probability distributions are involved. A handful of randomized versions of **NP**-complete problems have been shown to be the average-case **NP**-complete. For example, the following randomized version of the bounded halting problem (RBHP) for a NTM $\mathcal{M}$:

Input: A string $w01^n$ with $n > |w|$.

Question: Is there a halting computation of $\mathcal{M}$ on $|w|$ with at most n steps.

Probability: Proportional to $n^{-3}2^{-|w|}$.

(The above probability distribution for RBHP corresponds to the following experiment: randomly choose n, then $k < n$ and, finally, a string w of length k.)

There is also a randomized version of the bounded tiling problem that is the average-case **NP**-complete.

The class of problems that can be solved in the average polynomial time is usually denoted by **RNP**.

5.5 Graph Isomorphism and Prime Recognition

Two important algorithmic problems seem to have a special position in **NP**: graph isomorphism and prime recognition. All efforts to show that they are either in **P** or **NP**-complete have failed.

5.5.1 Graph Isomorphism and Non-isomorphism

As we shall see in Section 5.11.1 a proof that the graph isomorphism problem is **NP**-complete would have consequences that do not agree with our current intuition.

On the other hand, it is interesting to notice that a seemingly small modification of the graph isomorphism problem, the **subgraph isomorphism problem** is **NP**-complete. This is the problem to decide, given two graphs G_1 and G_2, whether G_1 is isomorphic with a subgraph of G_2.

Exercise 5.5.1 *Explain how it can happen that we can prove that the subgraph isomorphism problem is* **NP***-complete but we can have big difficulties to prove the same for the graph isomorphism problem?*

In addition, the graph isomorphism problem is in **P** for various important classes of graphs, for example for planar graphs.

Exercise 5.5.2 *Show that the following graph isomorphism problems are decidable in polynomial time: (a) for trees; (b) for planar graphs.*

A complementary problem, the so-called **graph non-isomorphism problem**, is even not known to be in **NP**. It is the problem to decide, given two graphs, whether they are non-isomorphic.

It is worth to point out why there is such a difference between graph isomorphism and graph non-isomorphism problems. In order to show that two graph are isomorphic, it is enough to provide and check an isomorphism. To show that two graph are non-isomorphic, one has to prove that no isomorphism exists. This seems to be much more difficult.

We deal with graph isomorphism and non-isomorphism problems also in Chapter 9.

5.5.2 Prime Recognition

This is an algorithmic problem par excellence—a king of algorithmic problems. For more than two thousands years some of the best mathematicians have worked on it and the problem is still far from being solved. Moreover, a large body of knowledge in mathematics has its origin in the study of this problem.

There are several easy to state criteria for an integer to be a prime. For example:

Wilson's test: n is prime iff $(n-1)! \equiv -1 \pmod{n}$.

Lucas' test: n is prime iff $\exists g \in \mathbf{Z}_n^*$ such that $g^{n-1} \equiv 1 \pmod{n}$ but $g^{(n-1)/p} \not\equiv 1 \pmod{n}$ for all prime factors p of $n-1$.

None of the known criteria for primality seems to lead to a polynomial time algorithm for primality testing. The fastest known deterministic algorithm to test primality of a number n has complexity $\mathcal{O}((\lg n)^{c \lg \lg \lg n})$. However, it is also far from clear that no deterministic polynomial time primality testing algorithm exists. For example, it has been shown that primality testing of an integer n can be done in the deterministic polynomial time $\mathcal{O}(\lg^5 n)$ if the Generalized Rieman Hypothesis holds.[5]

On the other hand, the following reformulation of the Lucas' test provides a nondeterministic polynomial time algorithm to recognize primes and therefore prime recognition is in **NP**.

Algorithm 5.5.3 (Nondeterministic prime recognition)

```
if $n = 2$ then accept;
if $n = 1$ or $n > 2$ is even, then reject;
if $n > 2$ is odd then choose an $1 < x < n$;
                      verify whether $x^{n-1} \equiv 1 \pmod{n}$;
                      guess a prime factorization $p_1, \ldots, p_k$ of $n-1$;
                      verify that $p_1 \ldots p_k = n-1$;
                      for $1 \leq i \leq k$, check that $p_i$ is prime and $x^{(n-1)/p_i} \not\equiv 1 \bmod n$;
                      accept, if none of the checks fails
```

If $\lg n = m$, then the computation of x^{n-1} and $x^{(n-1)/p_i}$ takes time $\mathcal{O}(m^4)$. Since $p_k \leq n/2$, one can in a quite straightforward way derive a recurrence for time complexity of the above algorithm and show that its computational complexity is $\mathcal{O}(\lg^5 n)$.

Exercise 5.5.4 * *Show in detail how one can derive* $\mathcal{O}(\lg^5 n)$ *time upper bound for the complexity of the above nondeterministic implementation of the Lucas' test.*

---

[5] Rieman Hypothesis says that all complex roots of the equation $\sum_{n=1}^{\infty} \frac{1}{n^z} = 0$, with the real part between 0 and 1, have the real part exactly 1/2. This is one of the major hypotheses of number theory and has been computationally verified for $1.5 \cdot 10^9$ roots. Generalized Rieman hypothesis makes the same claim about the roots of the equation $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^y} = 0$, where $\chi(a) = \chi_n(a \bmod n)$ if $gcd(a,n) = 1$ and 0, otherwise and χ_n is a homomorphism of the multiplicative group Z_n^* into the multiplicative group of all complex numbers. Also the Generalized Rieman Hypothesis has been verified for a very large number of roots.

Recognition of the composite numbers is clearly in **NP**—one just guesses and verifies factorization—and therefore prime recognition is in **NP** $\cap$ **co-NP**.

5.6 NP versus P

In spite of the fact that the complexity class **NP** seems to be, at the first sight, of purely theoretical interest, because the underlying machine model is unrealistic, it actually plays a very important rôle in practical computing. This will be discussed in this section. In addition, we analyse the structure and basic properties of the complexity classes **NP** and **P**, as well as their mutual relation.

5.6.1 Rôle of NP in Computing

There is another characterization of **NP** that allows to see better the overall rôle of **NP** in computing. In order to show this characterization two new concepts are needed.

Definition 5.6.1 *A binary relation $R \subset \Sigma^* \times \Sigma^*$ is called* **polynomially decidable** *if there is a deterministic Turing machine that decides the language $\{x\#y \mid (x,y) \in R\}$, with $\#$ being a marker not in Σ, in polynomial time. Moreover, a binary relation R is called* **polynomially balanced** *if there is an integer k such that $(x,y) \in R$ implies $|y| \leq |x|^k$. (In other words the size of y is polynomially bounded by the size of x and therefore, if $(x,y) \in R$, then y can be only polynomially larger than x).*

Theorem 5.6.2 *A language L is in* **NP** *if and only if there is a polynomially decidable and polynomially balanced relation R such that $L = \{x \mid (x,y) \in R \text{ for some } y\}$.*

Proof. (1) If $L \in$ **NP**, then there is a NTM $\mathcal{M}$ that decides L in polynomial time n^k for some k. Let R now be the relation defined as follows: $(x,y) \in R$ if and only if y is an accepting computation for $\mathcal{M}$ and its input x. R is clearly polynomially balanced and decidable. Moreover, $L = \{x \mid (x,y) \in R\}$.

(2) Let a polynomially balanced and decidable relation R exist for a language L, and let k be such that $(x,y) \in R$ implies $|y| \leq |x|^k$. We show how to design a polynomial time bounded NTM $\mathcal{M}$ that decides L. For an input x, $\mathcal{M}$ first guesses an y such that $|y| \leq |x|^k$, and then decides whether $(x,y) \in R$. Both tasks can be performed in polynomial time because R is polynomially balanced and decidable. ▯

Observe that our new characterization of **NP**, from Theorem 5.6.2 does not use the concept of nondeterminism at all.

Theorem 5.6.2 throws a light on the substance of the class **NP**. If a problem (language) L is in **NP**, then each "yes" instance x of the problem ($x \in L$) has a short—of polynomial size—**certificate** (**witness—proof of membership**) y that x is in L, and one can easily, in polynomial time, verify the correctness of a chosen certificate for x.

In case of the satisfiability problem a satisfying assignment is such a certificate. In case of the Hamilton cycle problem it is a particular Hamilton cycle that is a certificate. In case of the graph isomorphism problem it is a particular isomorphism that plays the rôle of a certificate. Prime recognition is an example of a problem that has a certificate but showing it is far from trivial.

Many important properties in **NP** are such that also their negation is in **NP** and one can demonstrate it using witnesses. For example, it is easy to give a certificate that a graph is connected and also for the case it is not connected. However, in many cases deep mathematical results have to be used to show the existence of a witness. For example, it is easy to give a witness that a graph is planar even this is far from trivial (coordinates of nodes could have too many digits) and it is based on the result from graph theory saying that if a graph is planar, then it can be drawn in the plane in such a way that each edge is a straight line segment and the coordinates of every vertex are integers whose number of digits is polynomial in the size of the graph. The existence of a certificate for nonplanarity of a graph follows from Kuratowski's Theorem. To give a witness that a graph has a perfect matching is easy; one just presents such a matching. On the other hand, the existence of a witness for non-existence of a perfect matching follows from the Tutte's Theorem.

NP is certainly the most important complexity class containing most, though not all, of the natural problems one encounters in computing and Theorem 5.6.2 can serve as a basis to explain why. Indeed, many computational problems require to design objects with certain properties. Sometimes an optimal solution is required, sometimes any solution is good. These mathematical objects are often an abstraction of real objects and it is therefore only natural that they can not be of enormous size, and that it should be easily verifiable whether they have the required properties. Such objects are actually certificates Theorem 5.6.2 talks about.

5.6.2 Structure of NP

The study of the complexity of computational problems concerns mostly **NP**, and the main task is to sort out which problems have a polynomial time solution and which do not have. The concept of **NP**-completeness plays the key rôle here. If a problem is shown to be **NP**-complete it is certainly among those least likely to be in **P**.

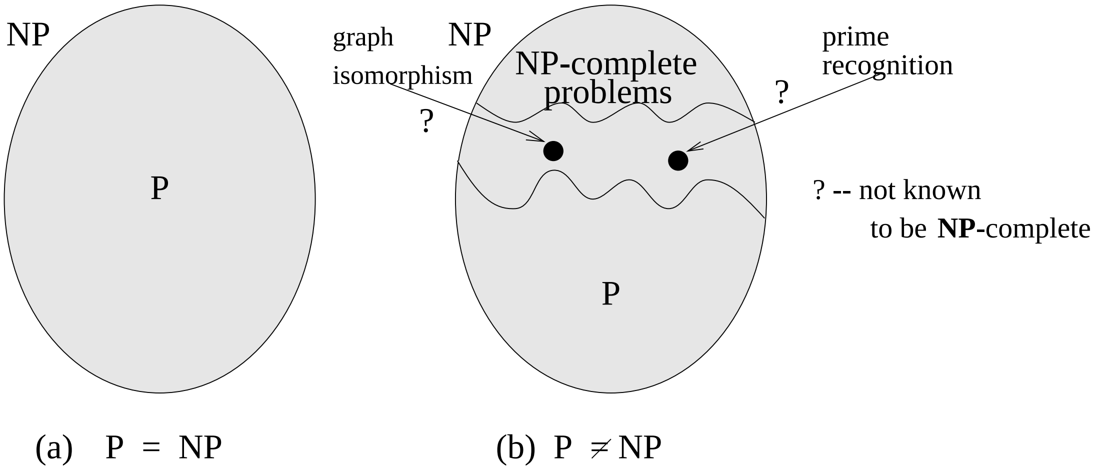

Figure 5.7: Structure of NP–80%

What structure does actually **NP** have? If $\mathbf{P} = \mathbf{NP}$, we get a simple picture shown in Figure 5.7a, and the whole theory of **NP**-completeness loses its merit. This is, however, very unlikely, with respect to our current knowledge. If $\mathbf{P} \neq \mathbf{NP}$, we know at least two important subclasses of **NP**. They are **P** and the class of **NP**-complete problems. Is that all? Two problems discussed in the previous section indicate that it is not. Indeed, the following theorem shows that the structure of **NP** in such a case is as depicted in Figure 5.7b.

Theorem 5.6.3 *If* $\mathbf{P} \neq \mathbf{NP}$, *then (a) there is a language* $L \in \mathbf{NP} - \mathbf{P}$ *that is not* **NP**-*complete; (b) there is an infinite hierarchy of more and more complexity classes between* **P** *and* **NP**.

5.6.3 P = NP Problem

A proof that $\mathbf{P} = \mathbf{NP}$ could have clearly a large impact on practical computing, because it would imply the existence of polynomial time algorithms for many algorithmic problems we badly need to solve efficiently.

Exercise 5.6.4 *Why it is not absolutely sure that a proof that* $\mathbf{P} = \mathbf{NP}$ *would significantly affect practical computing. What could happen?*

A proof that $\mathbf{P} \neq \mathbf{NP}$ could have also large impact on our understanding of the nature of computing. This is mainly due to the fact that nowadays there are many results known of the following type

$$\mathbf{P} = \mathbf{NP} \text{ iff } \ldots\ldots\ldots\ldots\ldots\ldots \text{ holds.}$$

For example,

$$\mathbf{P} = \mathbf{NP} \text{ iff } \begin{array}{l}\text{every honest partial function computable in polynomial}\\ \text{time has an inverse computable in polynomial time.}\end{array}$$

A function f is honest if and only if for every value y in the range of f there is an x in the domain of f such that $f(x) = y$ and $|x| \leq p(|y|)$ for some fixed polynomial p. The last result is sometimes interpreted as $\mathbf{P} \neq \mathbf{NP}$ if and only if there is a one-way function. However, the requirement for one-way function formulated in Section 2.3.3 is stronger than in the above result.

Exercise 5.6.5 *Show that if a function has an iverse computable in polynomal time, then it is honest.*

Theorem 5.6.2 suggests also another way to see the $\mathbf{P} = \mathbf{NP}$ problem. A certificate y for a "yes" instance x of an **NP**-complete problem can be seen as a proof for x. By Theorem 5.6.2, **NP**-complete problems have proofs that are of a reasonable size and easily verifiable. However, they may not be easy to find. On the other side, **P** can be seen as the class of problems for which one can find the proof in polynomial time. The problem $\mathbf{P} = \mathbf{NP}$ can therefore be seen as the problem to resolve the following puzzle: what is more difficult, to find a proof, or to check a proof? The answer seems to be intuitively clear—however, only until one tries to prove it.

It is natural that a lot of current research in foundations of computing concentrates on areas that could lead to the solution of $\mathbf{P} = \mathbf{NP}$ problem. One way that seems to be promising is through circuit complexity. By Theorem 4.3.24, for each problem in **P** there is a uniform family of polynomial size circuits. Thus, in order to show that an **NP**-complete problem is outside of **P** it would suffice to show that its circuit complexity is superpolynomial. Since almost all Boolean functions have the worst possible circuit complexity this may not seem to be a big deal. However, currently the best lower bound for circuit complexity of an **NP**-complete problem is $3n - o(n)$.

It can also happen that the $\mathbf{P} = \mathbf{NP}$ problem is not solvable within the framework the current logic, mathematics and computing theory are developed. It is therefore natural to explore also this possibility. One attempt of doing that is through the so-called **relativized complexity theory**, where relativization is understood with respect to different oracles.

5.6.4 Relativization of the P = NP Problem*

The concept of the oracle, introduced in Section 5.3, is a formalization of an intuitive idea of computing with an advice or with a help (from an oracle), or more plainly, with subprograms.

If we fix a language A as an oracle, and consider complexity classes relativized to this oracle, we speak about relativized complexity classes. If $\mathcal{L}$ is a complexity class, then $\mathcal{L}^A$ denotes the relativized complexity class, namely the class of languages, accepted by oracle Turing machines $\mathcal{M}^A$ where $\mathcal{M}$ is a Turing machine accepting a language from $\mathcal{L}$. If $\mathcal{A}$ is a family of languages, then $\mathcal{L}^{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} \mathcal{L}^A$.

We show now that the solution of a relativized version of the $\mathbf{P} = \mathbf{NP}$ problem, namely the $\mathbf{P}^A = \mathbf{NP}^A$ problem, depends on the choice of the oracle.

Theorem 5.6.6 *There are oracles A and B such that $\mathbf{P}^A = \mathbf{NP}^A$ and $\mathbf{P}^B \neq \mathbf{NP}^B$.*

Proof. Let A be a **PSPACE**-complete language. According to Savitch's theorem we have

$$\mathbf{P}^A = \mathbf{P}^{\mathbf{PSPACE}} = \mathbf{PSPACE},$$

$$\mathbf{NP}^A = \mathbf{NP}^{\mathbf{PSPACE}} = \mathbf{NPSPACE} = \mathbf{PSPACE},$$

and therefore $\mathbf{P}^A = \mathbf{NP}^A$.

The construction of the language B is more involved and the diagonalization method will be used to do that.

For each language B let us define the language

$$L_B = \{0^n \mid \exists w \in B, |w| = n\}.$$

Clearly, $L_B \in \mathbf{NP}^B$. In order to prove the second statement of the theorem it is sufficient to show that there is a B such that $L_B \notin \mathbf{P}^B$. This will now be done.

Let $\mathcal{M}_1, \mathcal{M}_2, \ldots$ be an enumeration of polynomial time bounded deterministic oracle Turing machines, and let $p_1, p_2, \ldots$ be their corresponding polynomial time bounds. Such an enumeration can be obtained from an enumeration of all TM and polynomials, by putting a "polynomial clock" to each of them, similarly as in Lemma 5.1.11.

We design a set $B \subset \{0,1\}^*$ in phases using the following method:

```
Phase 0. Set $B \leftarrow \emptyset$, and $n \leftarrow 0$.
Phase $k > 0$.
(1) Take $m$ to be the smallest integer such that $2^m > p_k(m)$ and $m$ is larger than the length of the longest string that any of the machines $\mathcal{M}_1, \ldots, \mathcal{M}_k$ asks its oracle for inputs of length at most $n$. (Observe that $m$ is well defined.)
(2) Set $n \leftarrow m$.
(3) if $0^n \in L(\mathcal{M}_k^B)$
      then go to the $(k+1)$-th phase
      else let $w$ be a string such that $|w| = n$ and $\mathcal{M}_k$ for the input $0^n$ never asks
          the oracle whether it contains $w$. (Since $2^m > p_k(m)$ such a string does exist.)
          Set $B \leftarrow B \cup \{w\}$ and go to the $(k+1)$-th phase.
```

We show now that the assumption $L_B \in \mathbf{P}^B$ leads to a contradiction. Let k be such that $L_B = L(\mathcal{M}_k^B)$. (Since $\mathcal{M}_1, \mathcal{M}_2, \ldots$ is an enumeration of polynomial time bounded oracle TM, such a k must exist.) Moreover, let n_k be the integer value n receives in the k-th phase. If $0^{n_k} \in L(\mathcal{M}_k^B)$, then no string of the length n_k is added to B in the k-th phase (and therefore $0^{n_k} \notin L_B$).

If $0^{n_k} \notin L(\mathcal{M}_k^B)$, then in the k-th phase a string of the length n_k is added to B. Observe also that two different phases do not mix in the sense that they deal with different sets of strings. Thus,

$$0^{n_k} \in L_B \Leftrightarrow 0^{n_k} \notin L(\mathcal{M}_k^B) = L_B$$

and this is a contradiction. □

Exercise 5.6.7 *Show: (a) there are oracles A, B such that: (a)* $\mathbf{NP}^B \neq \mathbf{PSPACE}^B$*; (b)* $\mathbf{P}^C \neq \mathbf{co}\text{-}\mathbf{NP}^C$*.*

Remark 5.6.8 There are many other results showing identity or differences of various complexity classes (not known to be neither identical nor different) with respect to some oracles. For example, there are oracles A, B, C and D such that: (1) $\mathbf{NP}^A \neq \mathbf{co} - \mathbf{NP}^A$; (2) $\mathbf{NP}^B \neq \mathbf{PSPACE}^B$ and $\mathbf{co} - \mathbf{NP}^B \neq \mathbf{PSPACE}^B$; (3) $\mathbf{P}^C \neq \mathbf{NP}^C$ and $\mathbf{NP}^C = \mathbf{co} - \mathbf{NP}^C$; (4) $\mathbf{P}^D \neq \mathbf{NP}^D$ and $\mathbf{NP}^D = \mathbf{PSPACE}^D$. Clearly technically these are interesting results. However, what they actually implies? This is often discussed in literature. The main outcome seems to be an understanding that some techniques can hardly be used to separate some complexity classes (i.e. to show they are different. For example, if a technique "relativizes" in the sence that a proof $\mathbf{P} \neq \mathbf{NP}$ by this technique would imply $\mathbf{P^A} \neq \mathbf{NP^A}$, for any oracle A, then this technique can not be used to show that $\mathbf{P} \neq \mathbf{NP}$.

5.6.5 P-completeness

The original motivation for an introduction of the concept of **P**-completeness, by S. Cook in 1972, with respect to the logspace reducibility, was to deal with (still open) problem whether everything computable in polynomial time is computable in polylogarithmic space. Other concepts of **P**-completeness will be discussed in section 5.10.

Many problems have been shown to be **P**-complete. Some of them are a natural modification of the known **NP**-complete problems.

Exercise 5.6.9 *Show, for example by a modification of the proof of* **NP***-completeness of the bounded halting problem, that the following deterministic version of the bounded halting problem is* **P***-complete:*

$$L^D_{halt} = \{\langle\mathcal{M}\rangle\langle w\rangle\#^t \,|\, \mathcal{M} \textit{ is a deterministic TM that accepts } w \textit{ in } t \textit{ steps}\}.$$

Some **P**-complete problems look surprisingly simple. For example, the **circuit value problem**, an analog of the satisfiability problem for Boolean formulas: Given a Boolean circuit C and an assignment α to its Boolean variables, decide whether C has the value 1 for the assignment α. If we take self-delimiting encodings $\langle C\rangle$ of circuits C, then we have:

$$\text{CIRCUIT-VALUE} = \{\langle C\rangle\alpha \,|\, C \text{ has the value 1 for the assignment } \alpha\}.$$

Theorem 5.6.10 *CIRCUIT-VALUE problem is* **P***-complete.*

Proof. An evaluation of a circuit can be clearly done in polynomial time and therefore the problem is in **P**. It has been shown in Chapter 4, Lemma 4.3.23, that to any deterministic polynomial time bounded Turing machine that accepts a language $L \subset \Sigma^*$, and any $x \in L \subseteq \Sigma^*$, we can design in polynomial time a circuit $C_{L,x}$ such that $x \in L$ if and only if $C_{L,x}$ has the value 1 for the assignment determined by x. It is not difficult to see that this construction can actually be carried out in $O(\lg|x|)$ space. This shows **P**-completeness. □

In order to demonstrate a subtle difference between **NP**-completeness and **P**-completeness, let us mention two of the very important and closely related optimization problems: rational linear programming (RLP), and integer linear programming (ILP). See Table 5.1.

A widely used method to solve the rational linear programming problem is the **simplex method**. For many practically important inputs the method runs very fast but its worst-case complexity is exponential. The discovery that there is a polynomial time algorithm to solve this problem (due to Khachyian (1983)), was an important step in the development of efficient algorithms. On the other hand, ILP seems to be an essentially more difficult problem, in spite of the fact that the set of potential solutions is smaller than that for RLP. Interesting enough, these two problems have their firm place in computational complexity classes:

Theorem 5.6.11 *The rational linear programming problem is* **P***-complete, integer linear programming problem is* **NP***-complete.*

Exercise 5.6.12 *Show that 3-CNFF problem can be reduced in polynomial time to the* **integer linear programming problem**. *(To show* **NP***-completeness of ILP is a much harder task.)*

5.6.6 Structure of P

As mentioned in Section 5.2, we have the inclusions **LOGSPACE** $\subseteq$ **NLOGSPACE** $\subseteq$ **P**. It is not known which of these inclusions is proper, if any. The problem

$$\mathbf{LOGSPACE} = \mathbf{NLOGSPACE}$$

or, in other notation, $\mathbf{L} = \mathbf{NL}$, is another important open problem of the complexity theory.

Also for the class **NLOGSPACE** various natural complete problems are known. One of them is the 2-CNFF problem: to decide whether a Boolean formula in conjunctive normal form with two literals in each clause is satisfiable. Another **NLOGSPACE**-complete problem is the graph-accessability problem (GAP): Given a directed graph G and two of its nodes: s (source) and t (sink), decide whether there is a path in G from s to t.

5.6.7 Functional Version of the P = NP Problem

Most of the complexity theory deals with the decision problems—how to recognize strings from a language. However, most of the computing practice deals with the function problems—how to compute functions—or with the search problems.

There are two reasons for such heavy concentration of the complexity theory on decision problems: (1) simplicity, elegance and power of such theory; (2) a transfer to computational problems does not seem to bring much more insights and, moreover, it is often quite easy.

There are two natural connections between decision and computational problems. To decide whether $x \in L$ for a language L is equivalent to computing $f_L(x)$ for the characteristic function of L. Another important relation between decision and function problems can be formulated for languages from **NP** as follows:

Definition 5.6.13 *(1) Let* $L \in$**NP** *and* R_L *be a polynomial time decidable and polynomially balanced relation for* L*—see the proof of Theorem 5.6.2. The search problem associated with* L *and denoted by* FL *is the following problem:*

Given x*, find a string* y *such that* $R(x,y)$ *holds, if such a string exists and return "no", if no such* y *exists.*

Denote **FNP** *(***FP***) the class of search problems associated with languages in* **NP** *(in* **P***).*

We can therefore talk about such search problems as FSAT and FKNAPSACK.

Intuitively, there should be a very close relation between **P** and **FP**, between **NP** and **FNP** and between the problems **P** = **NP** and **FP** = **FNP**. Indeed, this is true. However, in order to reveal such a close relation fully, the following subtle, and tricky at the first sight, definition of the polynomial time reducibility in **FNP** is needed:

Definition 5.6.14 *A function problem $F_1 : \Sigma_1^* \to \Sigma_1^*$ is polynomial time reducible to a function problem $F_2 : \Sigma_2^* \to \Sigma_2^*$, if there are polynomial time computable functions $f_1 : \Sigma_1^* \to \Sigma_2^*$ and $f_2 : \Sigma_2^* \to \Sigma_1^*$ such that the following condition hold:*

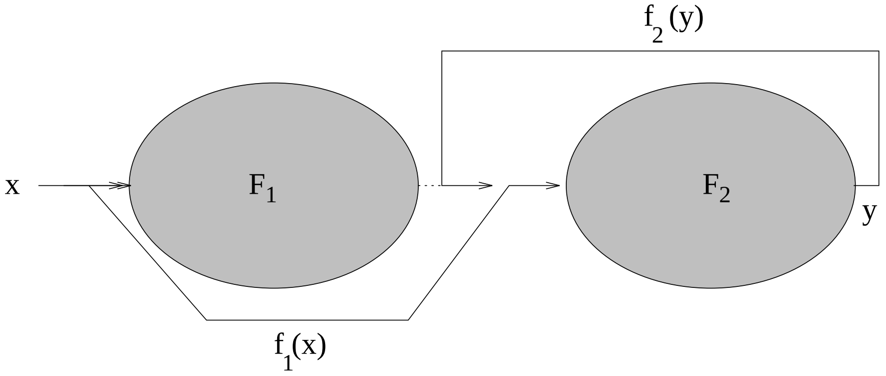

Figure 5.8: Reducibility in **FNP**

1. *If F_1 is defined for an x, then F_2 is defined for $f_1(x)$.*
2. *If y is an output of F_2, for the input $f_1(x)$, then $f_2(y)$ is the correct output of F_1 for the input x.*

Observe a subtlety of this definition: from the output of F_2, for the input $f_1(x)$, we can construct, in polynomial time, the correct output of F_1.

A function problem F is **FNP***-complete, if $F \in$* **FNP***, and each problem in* **FNP** *can be reduced to F in polynomial time.*

It is easy to see that the SAT problem is decidable in polynomial time if and only if FSAT is computable in polynomial time. Indeed, the only nontrivial task is to show that if there is a polynomial algorithm to decide SAT, then we can solve FSAT in polynomial time.

Let us assume that there is a polynomial time algorithm $\mathcal{A}$ to decide the SAT problem. Let F be a Boolean function of n variables $x_1, \ldots, x_n$. We can use $\mathcal{A}$ to decide whether F has a satisfying assignment. If not, we return "no". If yes, we design formulas F_0 and F_1 by fixing, in F, $x_1 = 0$ and $x_1 = 1$. We then use $\mathcal{A}$ to decide which of those two formulas has a satisfying assignment. One of them has to have . Assume that it is F_0. This implies that there is a satisfying assignment for F with $x_1 = 0$. We keep doing this restriction of F and, step by step , find values for all variables in a satisfying assignment for F.

Remark 5.6.15 In case of sequential computations, function problems can often be reduced with a small time overhead to decision problems. For example, the problem to compute a function $f : \mathbf{N} \to \mathbf{N}$ can be reduced to the problem to decide, given an n and k, whether $f(n) < k$, in case that a reasonable upper bound on $f(n)$ is easy to establish. (What is often the case.) Using the binary search in the interval $[0, b]$ one can then determine $f(n)$ using $\lceil \lg b \rceil$ times an algorithm the for the corresponding decision problem for f and k.

Using the ideas from the proof of Theorem 5.4.3 we can easily show that FSAT is **FNP**-complete problem. Therefore, we have the relation we expected:

Theorem 5.6.16 **P** = **NP** *if and only if* **FP** = **FNP**.

Important candidates for being in **FNP** − **FP** are one-way functions in the following sense:

Definition 5.6.17 *Let* $f : \Sigma^* \to \Sigma^*$. *We say that* f *is a (weakly) one-way function if the following holds:*

1. f *is injective and for all* $x \in \Sigma^*$, $|x|^{\frac{1}{k}} \leq |f(x)| \leq |x|^k$ *for some* $k > 0$ *(i.e.* $f(x)$ *is at most polynomially larger or smaller than* x*).*
2. f *is in* **FP** *but* f^{-1} *is not in* **FP**. *(In other words, there is no polynomial time algorithm which, given a* y*, either computes* x *such that* $f(x) = y$ *or returns "no", if there is no such* x*.)*

Exercise 5.6.18 *Show that if* f *is a one-way function, then* f^{-1} *is in* **FNP**.

In order to determine more exactly the rôle of one-way functions in complexity theory let us denote by **UP** the class of languages accepted by the so-called **unambiguous polynomial-time bounded NTM**. They are polynomial time bounded NTMs such that for any input there is at most one accepting computation.

Exercise 5.6.19 *Assume there is a one-way function* $f : \Sigma^* \to \Sigma^*$. *Define the language* $L_f = \{(x, y) \mid$ *there is a* z *such that* $f(z) = y$ *and* $z \preceq x$ *(in strict ordering of strings)*$\}$.

(a) Show that $L_f \in$ **UP**;

(b) Show that $L_f \notin$ **P** *(for example, by showing, using the binary search, that if* $L_f \in$ **P**, *then* $f^{-1} \in$ **FP**);

(c) Show that **P** $\subseteq$ **UP** $\subseteq$ **NP**.

Theorem 5.6.20 **P** $\neq$ **UP** *if and only if there are one-way functions.*

Proof. It follows from Exercice 5.6.19 that if there is a one-way function, then **P** $\neq$ **UP**. Let us now assume that there is a language $L \in$ **UP**−**P** and let $\mathcal{M}$ be an unambiguous polynomial time bounded NTM accepting L. Denote by $f_{\mathcal{M}}$ the function defined as follows:

$$f_{\mathcal{M}}(x) = \begin{cases} 1y, & \text{if } x \text{ is an accepting computation of } \mathcal{M} \text{ for } y \text{ as an input;} \\ 0y, & \text{if } x \text{ is not an accepting computation of } \mathcal{M} \text{ for } y \text{ as an input.} \end{cases}$$

Clearly, $f_{\mathcal{M}}$ is well defined, one-to-one (because of unambiguity of $\mathcal{M}$), and computable in polynomial time. Moreover, the lengths of inputs and outputs are polynomially related. Finally, were $f_{\mathcal{M}}$ invertible in polynomial time we would be able to recognize $L \notin$ **P** in polynomial time. ▯

Exercise 5.6.21 *Define the class* **FUP***and show that* **P** = **UP** *if and only if* **FP**=**FUP***.*

5.6.8 Counting problems—Class #P

In a decision problem we ask whether there is a solution; in a search problem we ask to find a solution; in a **counting problem** we ask how many solutions exist.

Counting problems are clearly of importance, have their specificity and may be computationally hard even for problems in **P**.

It is common to use notation **#P** for a counting version of a decision problem P. For example, #SAT is the problem to determine how many satisfying assignments has a given Boolean formula. #HAMILTON-PATH is the problem to determine how many Hamilton paths has a given graph.

A "counting analog" of the class **NP** is the class **#P**, prononciation "sharp P" (or "number P" or "pond P"), defined as follows.

Definition 5.6.22 *Let Q be a polynomially balanced and polynomially decidable binary relation. The counting problem associated with Q is the problem to determine, for a given x, the number of y such that $(x,y) \in Q$. The output is required to be in binary.* **#P** *is the class of all counting problems associated with polynomially balanced and decidable relations.*

The number of solutions of a counting problem can be exponentially large and this is the reason why in the definition of **#P** it is required that the output is in binary.

There is another definition of the class **#P**, actually the original one, considered in the following exercise.

Exercise 5.6.23 *Show that* **#P** *is the class of functions f for which there is a NTM $\mathcal{M}_f$ such that $f(x)$ is the number of accepting computations of $\mathcal{M}_f$ for the input x.*

#P-completeness is defined with respect to the same reduction as **FP**-completeness. Actually, a more restricted form of reduction is often sufficient to prove **#P**-completeness. A reduction f of instances x of one counting problem $\mathcal{P}_1$ into instances of another counting problem $\mathcal{P}_2$ is called **parsimonious** if x and $f(x)$ have always the same number of solutions.

Many reductions used to prove **#P**-completeness are either parsimonious or can be easily modified to be so. This is true for example for the proof of the Cook's Theorem. This way **#P**-completeness has been shown for counting versions of a variety of **NP**-complete problems,for example for #SAT and #HAMILTON-PATH.

Of a special interest are those **#P**-complete problems for which the corresponding search problem can be solved in polynomial time. Of such a type is PERMANENT problem for Boolean matrices (that is equivalent to the problem of counting perfect matchings in bipartite graphs).

5.7 Approximability of NP Complete Problems

From the theory point of view **NP**-completeness results are beautiful and powerful. The hardest problems ("trouble-makers") have been localized and close relations between them discovered. Could we eliminate one of them, all would be eliminated.

From the practical point of view **NP**-completeness is a disastrous phenomenon, and "practically unacceptable". Too many too important problems are **NP**-complete. Computing practice can hardly accept such a limitation. One has to look for all feasible ways to get around them—if this is possible.

There are several approaches to overcome the limitations **NP-completeness** imposes: approximation algorithms, randomized algorithms, average-case fast algorithms (with respect to "main" probability distributions of inputs, heuristics, even exponential time algorithms that are fast for most of the inputs we encounter in applications could do, if there is nothing better.

In this section we deal with the approximation algorithms for **NP**-complete optimization problems. Perhaps the most surprising discovery in this connection is that in spite of the isomorphism between **NP**-complete problems they can be surprisingly different from the point of view of the design of good approximation algorithms. In addition, the existence of good approximation algorithms for some **NP**-complete problems has turned out to be deeply connected with some fundamental questions of computing.

As we shall see the search for good approximation algorithms for **NP**-complete problems brought good news but also bad news. For some **NP**-complete problems there are approximation algorithms as good as we need. For some **NP**-complete problems quite good approximations can be obtained. A bad news is that for some **NP**-complete problems there are good reasons to believe that no good approximation algorithms exist.

5.7.1 Performance of Approximation Algorithms

The issue is to understand how good approximation algorithms can exist for particular **NP**-complete problems. In order to deal with this task we need some quantitative measures for "goodness" of approximations. We start therefore with such criteria.

For each instance x of an optimization problem $\mathcal{P}$ let $F_{\mathcal{P}}(x)$ be the set of feasible solutions of $\mathcal{P}$, and for each $s \in F_{\mathcal{P}}(x)$ let a $c(s) > 0$—a cost of the solution s—be given. The optimal solution of $\mathcal{P}$ is then defined by

$$OPT(x) = \min_{s \in F_{\mathcal{P}}(x)} c(s) \quad \text{or by} \quad OPT(x) = \max_{s \in F_{\mathcal{P}}(x)} c(s)$$

depending on whether the minimal or maximal solution is required. (For example, for TSP the cost is the length of a tour.)

We say that an approximation algorithm $\mathcal{A}$, mapping each instance x of an optimization problem $\mathcal{P}$ to one of its solutions in $F_{\mathcal{P}}(x)$, has the **ratio bound** $\rho(n)$ and the **relative error bound** $\varepsilon(n)$ if

$$\max_{|x|=n}\left\{\frac{c(\mathcal{A}(x))}{c(OPT(x))},\frac{c(OPT(x))}{c(\mathcal{A}(x))}\right\}\leq\rho(n)\qquad\max_{|x|=n}\frac{|c(\mathcal{A}(x))-c(OPT(x))|}{\max\{c(OPT(x)),c(\mathcal{A}(x))\}}\leq\varepsilon(n).$$

Both definitions are chosen to correspond to our intuition and to apply simultaneously to minimization and maximization problems. Both of these bounds compare an approximation solution with the optimal one, but in two different ways.

Exercise 5.7.1 *Show how one can simplify definitions of relative error bound and/or ratio bound if only maximalization (minimalization) problems are considered.*

The ratio bound is never less than 1. An optimal algorithm has the ratio bound 1. The larger is the best possible ratio bound of an approximation algorithm, the worser is the algorithm. The relative error bound is never more than 1 and always between 0 and 1.

Exercise 5.7.2 *Show: (a) An algorithm $\mathcal{A}$ of a maximization problem with the relative error bound ε never produces a solution more than $(1-\varepsilon)$-times smaller than the maximum; (b) an algorithm $\mathcal{A}$ for a minimization problem with the relative error bound ε, never produces a solution that would be more than $(\frac{1}{1-\varepsilon})$-times larger than the minimum.*

We shall now concentrate on two problems concerning approximation of **NP**-complete problems.

- The **constant relative error bound problem**: given an **NP**-complete problem $\mathcal{P}$ with a cost of solutions and an $\varepsilon > 0$, does there exist an approximation polynomial time algorithm for $\mathcal{P}$ with the relative error bound ε?
- **The approximation scheme problem:** does there exist for a given **NP**-complete problem $\mathcal{P}$ a polynomial time algorithm to design, given an ε and an input instance x, an approximation for $\mathcal{P}$ with the relative error bound ε?

Let us first deal with the constant relative error bounds. We say that an algorithm $\mathcal{A}$ is an ε-approximation algorithm for an optimization problem $\mathcal{P}$, if ε is its relative error bound. The **approximation threshold** for $\mathcal{P}$ is the greatest lower bound of all $\varepsilon > 0$ such that there is a polynomial time ε-approximation algorithm for $\mathcal{P}$.

5.7.2 NP-complete Problems with a Constant Approximation Threshold

We show now that **NP**-complete optimization problems can differ very much with respect to their approximation thresholds. Notice, that if an optimization problem $\mathcal{P}$ has the approximation threshold 0, then this means that an approximation arbitrary close to the optimum is possible, and the approximation threshold 1 means that essentially no approximation is possible.

As the first example let us consider the following optimization version of the knapsack problem: Given n items with weights $w_1, \dots, w_n$ and values $v_1, \dots, v_n$, and a knapsack limit c, the task is to find a bit vector $(x_1, \dots, x_n)$ such that $\sum_{i=1}^{n} x_i w_i \leq c$ and $\sum_{i=1}^{n} x_i v_i$ is as large as possible.

Exercise 5.7.3 *We get a decision version of the above version of the knapsack problem by fixing a goal K and asking whether there is such a solution vector that $\sum_{i=1}^{n} x_i v_i \geq K$. Show that also this new version of the knapsack problem is* **NP***-complete.*

Theorem 5.7.4 *The approximation threshold for the optimization version of the KNAPSACK problem is* 0.

Proof. The basic idea of the proof is very simple. We take a modification of the algorithm from Example 5.4.17 and make out of it a polynomial time algorithm by applying it to an instance with truncated input data. The larger truncation we make, the better approximation we get. Details follow:

Let a knapsack instance $(w_1, \dots, w_n, c, v_1, \dots, v_n)$ be given, and let $V = \max\{v_1, \dots, v_n\}$. For $1 \leq i \leq n, 1 \leq v \leq nV$ we define:

$$W(i,v) = \min_{x \in \{0,1\}^i} \{\sum_{j=1}^{i} x_j w_j \mid \sum_{j=1}^{i} x_j v_j = v\}.$$

Clearly, $W(i,v)$ can be computed using the recurrencies: $W(0,v) = \infty$, and, for all $i > 0$, and for $1 \leq v \leq nV$,

$$W(i+1,v) = \min\{W(i,v), W(i, v - v_{i+1}) + w_{i+1}\}.$$

Finally, we take the largest v such that $W(n,v) \leq c$.

The time complexity of this algorithm is $\mathcal{O}(n^2 V)$. The algorithm is therefore not polynomial, with respect to the size of the input. In order to make out of it a polynomial time approximation algorithm we use the following "truncation trick":

Instead of the knapsack instance $(w_1, \dots, w_n, c, v_1, \dots, v_n)$ we take a b-approximate instance $(w_1, \dots, w_n, c, v_1', \dots, v_n')$, where $v_i' = 2^b \lfloor \frac{v_i}{2^b} \rfloor$, i.e. v_i' is obtained from v_i by replacing the least significant b bits by zeros. (We show later how to choose b.)

If we now apply the above algorithm to this b-truncated instance, we get its solution in time $\mathcal{O}(\frac{n^2 V}{2^b})$,because we can ignore the last b zeros in v_i's.

The vector $x^{(b)}$ which we obtain as the solution for this b-truncated instance may be quite different from the vector $x^{(0)}$ that provides the optimal solution. However, and this is essential, as the following inequalities show, the values these two vectors produce can not be too different. Indeed, it holds

$$\sum_{i=1}^{n} x_i^{(0)} v_i \geq \sum_{i=1}^{n} x_i^{(b)} v_i \geq \sum_{i=1}^{n} x_i^{(b)} v_i' \geq \sum_{i=1}^{n} x_i^{(b)} (v_i - 2^b) \geq \sum_{i=1}^{n} x_i^{(b)} v_i - n2^b.$$

The first inequality holds because $x^{(0)}$ provides the optimal solution for the original instance; the second inequality holds because $v_i \geq v_i'$; the third inequality holds because $x^{(b)}$ is the optimal solution for the b-truncated instance.

We can assume without loss of generality that $w_i \leq c$ for all i. In such a case V is the lower bound on the value of the optimal solution.The relative error bound for the algorithm is therefore $\varepsilon = \frac{n2^b}{V}$.

Given an $0 < \varepsilon \leq 1$, we can take $b = \left\lceil \lg \frac{\varepsilon V}{n} \right\rceil$, in order to obtain an ε-approximation algorithm for the optimization version of the knapsack problem. Time complexity of the algorithm is therefore: $\mathcal{O}(\frac{n^2 V}{2^b}) = \mathcal{O}(\frac{n^3}{\varepsilon})$—and therefore we have a polynomial time algorithm. □

The second problem approximability of which we now discuss is the VERTEX-COVER problem: given a graph $G = (V, E)$, we seek the smallest set of nodes C such that each edge of G coincides with at least one node from C. Let us consider the following approximation algorithm for VERTEX-COVER;

Algorithm 5.7.5 (VERTEX-COVER approximation algorithm)

```
$C \leftarrow \emptyset; E' \leftarrow E;$
while $E' \neq \emptyset$ do take any edge $(u, v)$ from $E'$;
                   $C \leftarrow C \cup \{u, v\};$
                   $E' \leftarrow E' - \{$ all edges in $E'$ that are incident with one of the nodes $u, v\}$
              od.
```

Let C_G be the vertex cover this algorithm provides for a graph $G = (V, E)$. C_G can be seen as representing $\frac{|C_G|}{2}$ edges of G, no two of which have a common vertex. This means that if $OPT(G)$ is an optimal node covering of G, then it has to have at least $\left|\frac{C_G}{2}\right|$ nodes. Thus, $|OPT(G)| \geq \frac{|C_G|}{2}$, and therefore

$$\frac{|C_G| - |OPT(G)|}{|C_G|} \leq \frac{1}{2}.$$

We have actually proved:

Theorem 5.7.6 *The approximation threshold for VERTEX-COVER problem is* $\leq \frac{1}{2}$.

Surprisingly, the above very simple approximation algorithm is the best known approximation algorithm for the VERTEX-COVER problem.

Exercise 5.7.7 *Give an example of the graph, for which Algorithm 5.7.5 never produces an optimal solution.*

Exercise 5.7.8 *Design a good $\mathcal{O}(1)$ approximation algorithm for MAX-3-CNFF problem to determine the maximal number of satisfying assignments for a Boolean formula in 3-CNF. (Hint: determine how many clauses satisfies the average assignment.)*

The Theorem 5.7.12 in the next subsection shows that, very unfortunately, there are **NP**-complete problems that do not seem to have approximation algorithms with a relative error bound smaller than 1. Not only that. Practically such an important optimization problem as the TSP problem is, has this property.

Exercise 5.7.9 *BIN-PACKING problem is given by n integers $a_1, \dots, a_n$ (items), a capacity c and a number b of bins. The task is to determine whether there is an assignment of integers a_i into n bins such that in no bin is the total sum of integers larger than c. This problem is* **NP***-complete. Consider now the minimization version of the problem that requires to determine, given $a_1, \dots, a_n$ and c, the minimum number of bins such that bin-packing is possible. Show that the approximation threshold for BIN-PACKING problem is at least $\frac{1}{3}$.*

Exercise 5.7.10 * *The* **asymptotic approximation threshold** *for an optimization problem $\mathcal{P}$ is the smallest integer ε such that there is a δ and an approximation algorithm $\mathcal{A}$ for $\mathcal{P}$ such that for all instances x*

$$|c(\mathcal{A}(x)) - OPT(x)| \leq \varepsilon \max\{OPT(x), c(\mathcal{A}(x))\} + \delta.$$

Show that the asymptotic approximation threshold for BIN-PACKING problem is at most $\frac{1}{2}$.

Exercise 5.7.11 ***SUBSET-SUM problem. Given a finite set Q of rational numbers and a bound c. Compute $\max_{X \subset Q}\{\sum_{x \in X} x \mid \sum_{x \in X} \leq c\}$. Determine for the SUBSET-SUM problem: (a) approximation threshold; (b) asymptotic approximation threshold.*

5.7.3 Travelling Salesman Problem

This is one of the most studied **NP**-complete optimization problems. Our first result concerning its approximation is far from encouraging:

Theorem 5.7.12 *Unless* $\mathbf{P} = \mathbf{NP}$, *the approximation threshold for TSP is* 1. [6]

Proof. In order to prove the theorem it is sufficient to show that the existence of a polynomial time approximation algorithm$\mathcal{A}$ for TSP with the relative error bound $\varepsilon < 1$ implies the existence of a polynomial time algorithm to determine whether a given graph has a Hamilton cycle (what is an **NP**-complete problem). This we show now.

Given a graph $G = (V, E)$, first we construct the following instance of TSP: Let V be the set of cities and the distance between two cities c_i and c_j be 1 if $(i, j) \in E$, and $\frac{|V|}{1-\varepsilon}$ otherwise. To this instance of TSP we now apply the algorithm $\mathcal{A}$. If $\mathcal{A}$ returns a tour of the cost $|V|$, then we know that G has a Hamilton cycle. If $\mathcal{A}$ yields a tour of larger length, then the total length of such a tour has to be larger than $\frac{|V|}{1-\varepsilon}$. Since $\mathcal{A}$ is a polynomial ε-approximation algorithm, by the assumption, we know that the optimum is never less than $(1-\varepsilon)$-times the solution found by $\mathcal{A}$, and therefore larger than $|V|$. Thus, G has no Hamiltonian cycle.[7] $\square$

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[6] Similar results have been shown for several other optimization **NP**-complete problems. For example, for the CLIQUE problem: Given a graph G determine the maximal k such that G has a complete subgraph with k nodes.

[7] Very pesimistic results of Theorems 5.4.14 and 5.7.12 do not represent the whole reality concerning our capabilities to solve TSP. Actually TSP is an important example of a success story in the area of combinatorial optimization methods. A large effort has been made to solve as large TSP instances as possible and as good as possible. Advances in technology, theory, algorithms and data structures are behind the following progress in the area of getting optimal solution. The largest nontrivial TSP instances solved: in 1980—318 cities; in 1989—2392 cities and in 1994—7397 cities (taking 3-4 years). In all cases state-of-the-art computers and methods were used.

A success story the of approximations of TSP is also remarkable: with a variety of approximation algorithms tested, including those based on the ideas of simulating annealing from physics, genetic algorithms and neural networks.

One successful and elegant way to attack TSP is through the so-called Held-Karp's lower bound. This approach is based on an observation that if we use variables x_{ij} to represent the edge between cities c_i and c_j, $1 \leq i < j \leq n$, taking $x_{ij} = 1$ to mean that the edge (c_i, c_j) is in the optimal tour, then the TSP is equivalent to the following integer linear programming problem: Minimize

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ij} x_{ij}$$

subject to conditions

$$\sum_{i=k \text{ or } j=k} x_{ij} = 2, \; 1 \leq k \leq n,$$

$$\sum_{|S \cap \{i,j\}|} x_{ij} \geq 2 \text{ for all } S \subseteq \{1, \ldots, n\}$$

and $x_{ij} \in \{0, 1\}$, for $1 \leq i < j \leq n$. If we relax the last condition and replace it by the condition $0 \leq x_{ij} \leq 1$, we get a (rational) linear programming problem that is in $\mathbf{P}$ and the solution of which provides a lower bound, the so-called Held-Karp's lower bound (HK-lower bound), for the original instance of TSP. Experiences show that for a large variety of randomly generated instances of TSP the optimal tour length averages less than 0.8% over the HK-bound and for real-world instances this is almost always less than 2%. However, computation

It has also been shown that for several important special cases of TSP there are quite good approximation algorithms.

Let us first consider the case in which all weights are 1 or 2. Observe that in such a case each approximation algorithm is a $\frac{1}{2}$-approximation algorithm. In such a case it is even possible to prove the existence of a $\frac{1}{7}$-approximation algorithm.

A natural generalization of the above case are TSP instances that satisfy the so-called **triangle inequality**: $d(i,j) \leq d(i,k) + d(k,j)$ for any nodes i,j,k and the distance $d(x,y)$ betwen any two nodes x,y. It has been shown that in such a case there is a $\frac{1}{3}$-approximation algorithm.

Exercise 5.7.13 * *Show that the approximation threshold for TSP is at most $\frac{1}{2}$ for graphs satisfying the triangle inequality. (Hint: evaluate "goodness" of the following approximation algorithm to design a traveling salesman tour:*

1. *Choose a node v and design minimum spanning tree with the root in the node v.*
2. *Label vertices of the tree in a preorder way and tour the graph accordingly.)*

Exercise 5.7.14 * *Show that the decision version of the TSP is* **NP***-complete also when restricted to labelled graphs satisfying the triangle inequality.*

5.7.4 Nonapproximability

The ideal case concerning approximability of a problem $\mathcal{P}$ is when there is a polynomial time algorithm $\mathcal{A}$ such that given a problem instance x and an $\varepsilon > 0$, $\mathcal{A}$ returns a solution with the relative error bound ε. This motivation is behind the following concepts:

Definition 5.7.15 *A* **polynomial time approximation scheme** *for an optimization problem $\mathcal{P}$ is an algorithm $\mathcal{A}$ which, given an $\varepsilon > 0$ and an instance x of $\mathcal{P}$, provides a solution of $\mathcal{P}$ for x with the relative error bound ε in time bounded by a polynomial in $|x|$ (that may depend also on ε). If, in addition, this polynomial depends polynomially on $\frac{1}{\varepsilon}$ as well, then the approximation scheme is said to be a* **fully polynomial approximation scheme***.*

Observe that our approximation algorithm for the optimization version of the knapsack problem provides a fully polynomial approximation scheme.

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of the HK-bound is also computationally a very demanding task. In 1994, a top result was to compute the HK-bound exactly for TSP instances with up to 30 000 cities and to get a good approximation of this bound for TSP instances with up to a milion cities.

For a detailed presentation of the current methods and results, including Held-Karp's bound computations, to solve TSP see Johnson and McGeoch (1996) and Johnson, McGeoh and Rothberg (1996).

Another interesting optimization problem is MAXSAT: we are given a set of clauses and we search for an assignment that satisfies the largest number of them. A decision version of this problem has a bound k, and the task is to decide whether there is a satisfying assignment that satisfies at least k of the clauses.

MAXSAT problem is **NP**-complete even in case when we consider only clauses with two literals. MAXSAT is a surprisingly hard problem with respect to approximation. It holds:

Theorem 5.7.16 *If there is a polynomial time approximation scheme for MAXSAT problem. then* $\mathbf{P} = \mathbf{NP}$.

Similar results hold also for some other **NP**-complete problems. For example, for MAX2SAT, VERTEX-COVER and so on.

5.8 Randomized Complexity Classes

Randomization seems to be a powerful methodology to overcome limitations deterministic computation imposes and to speed-up computations. In this section we explore the power of this methodology.

5.9 Randomized algorithms

Before introducing main acceptance modes and complexity classes for randomized computing let us discuss some examples of randomized algorithms to see their advantages and methods of their analysis.

In some cases randomization is the only way to deal with the problem.

Example 5.9.1 (Symmetry breaking) *n identical processors connected into a ring have to choose one of the processors to be a "leader"; provided each of processors knows n.*

Algorithm 5.9.2 (Election of a leader)

1. *Each processor sets its local variable m to n and gets "active".*
2. *Each active processor chooses, randomly and independently, an integer between 1 and m.*
3. *Those processors that choose 1 (if any) send one-bit message around the ring.*
4. *After $n-1$ steps each processor knows the number l of processors that choosed 1. If $l = 1$, the election ends; if $l = 0$, the election continues by repeating Step 2; if $l > 1$, then only those processors remain active that choosed 1. They set their local variables m to l and the election continues by Step 2.*

The correctnes of the algorithm is obvious and one can also show that its time efficiency is good and number of bits needed to be exchanged is small.

Randomized algorithms are often very simple and their efficiency is either comparable or better than that of deterministic ones for the same problem.

Example 5.9.3 (Randomized QUICKSORT) *To sort a sequence* $S = \langle a_1, \dots, a_n \rangle$ *of* n *different elements we can use the following algorithm:*

Algorithm 5.9.4 (RQUICKSORT)

1. *Choose randomly* $1 \leq j \leq n$.
2. *Partition* S *into sequences* S_1 *and* S_2 *of elements* $a_i \leq a_j$ *and* $a_i > a_j$, *respectively.*
3. *Sort recursively* S_1 *and* S_2.

It is again clear that RQUICKSORT correctly sorts any sequence and if all input sequences are equally probable, then it can be shown that RQUICKSORT requires in average $\Theta(n \lg n)$ *time.*

Exercise 5.9.5 *Design a simple randomized algorithm to determine, given a sequence of* n *elements and* $1 \leq k \leq n$, *the* k*-th smallest element.*

Example 5.9.6 (Zero polynomial testing) *In order to decide whether a polynomial* p *with integer coefficients and* n *variables, each of the degree at most* k, *is identically zero, we can use* N *times the following simple randomized algorithm: compute* $p(\xi_1, \dots, \xi_n)$, *with integer values* $\xi_1, \dots, \xi_n$ *chosen randomly and independently according to the uniform distribution in the interval* $[0, 2kn]$. *If once a value different from* 0 *is obtained, then* p *is not identically* 0. *If all* N *times the value is zero, then we can consider* p *to be identically zero. By Schwartz' Lemma 1.9.4, the probability of error is at most* 2^{-N}. *(The algorithm can be used to text equality of two polynomials.)*

Such a randomized algorithm is of importance if p is given implicitly. For instance as the value of the determinant of a matrix of polynomials.

Example 5.9.7 (Cuts in multigraphs) *The task is to determine the smallest set of edges of a given multigraph* G *whose removal disconnects* G. *We show that there is a simple randomized algorithm doing the job correctly, with large probability.*

The basic idea of the algorithm presented below is that an edge contraction of a multigraph does not reduce the size of its minimal cut. (By an edge contraction of a multigraph G, *with respect to an edge* (u, v) *we understand a merging of vertices* u *and* v *into one, redirecting edges to* u *and* v *to the newly created vertex and removing self-looops of the new vertex.)*

Algorithm 5.9.8 (Minimal cut in a multigraph)

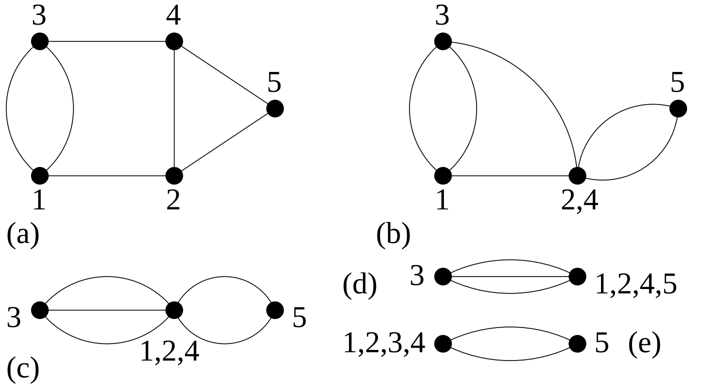

Figure 5.9: Contraction in a multigraph-70%

while *there are more than two vertices in the multigraph*
do *an edge contraction with respect to a randomly choosen edge* **od**;
Output the size of the maximal cut of the resulting two-vertex graph.

As shown in Figure 5.9a,b,c,d,e, an application of the Algorithm 5.9.8 may produce the correct result (Figure 5.9e), but may not (Figure 5.9d). Let us therefore investigate how probable is that the above algorithm makes an error.

Let G *be a multigraph of* n *nodes,* k *the size of its minimal cut and* C *its particular minimal cut.* G *has to have at least* $\frac{kn}{2}$ *edges—otherwise* G *would have a vertex of degree less than* k *and its incidental edges would form a minimal cut of size smaller than* k.

We derive a lower bound on the probability that no edge of C *is ever contracted during an execution of the algorithm—and therefore the edges surviving till the end are exactly those of* C.

Denote by E_i *the event of* not choosing *an edge of* C *at the* i*-th step of algorithm, for* $1 \leq i \leq n-2$. *The probability that the edge randomly choosen in the first step is in* C *is at most* $k/(nk/2) = \frac{2}{n}$, *and therefore* $Pr(E_1) \geq 1 - \frac{2}{n}$. *If* E_1 *occurs, then at the second contraction step there are at least* $\frac{k(n-1)}{2}$ *edges and therefore the probability of picking an edge in* C *is at most* $\frac{2}{n-1}$. *Hence,* $Pr(E_2|E_1) \geq 1 - \frac{2}{n-1}$. *Similarly, at the* i*-th step,* $i \geq 2$, *the number of remaining vertices is* $n-i+1$, *the size of the minimal cut is still at least* k, *and therefore the multigraph has at least* $\frac{k(n-i+1)}{2}$ *edges remaining at this step. Thus* $Pr(E_i|\bigcap_{j=1}^{i-1} E_i) \geq 1 - \frac{2}{n-i+1}$. *The probability that no edge of* C *is ever picked during the execution of the algorithm is therefore, by a natural generalization of (1.70),*

$$Pr[\bigcap_{i=1}^{n-2} E_i] \geq \prod_{i=1}^{n-2} \left(1 - \frac{2}{n-i+1}\right) = \frac{2}{n(n-1)}.$$

The probability of discovering a particular minimal cut is therefore larger than $\frac{2}{n^2}$.

A big deal? Not so. However, and this is the point, if we repeat the above algorithm $\frac{n^2}{2}$ *times, making independent random decisions each time, then the probability that a minimal* *cut is not found in any of the* $\frac{n^2}{2}$ *runs is at most*

$$\left(1-\frac{2}{n^2}\right)^{\frac{n^2}{2}} < \frac{1}{e}.$$

(Observe, that by the repetition of the algorithm we have reduced the probability of failure from $1-\frac{2}{n^2}$ *to a more respectable* $\frac{1}{e}$*. Further execution of the algorithm can make the failure probability arbitrarily small (of course, by increasing computation time).*

Algorithm 5.9.8 is remarkable simple comparing with the known deterministic ones for the same problem.

It is nowadays clear that randomization can speed-up many computations. However, and this is one of the most important questions, can randomization help also to cope with **NP**-completeness?

Satisfiability of Boolean formulas, as one of the basic **NP-complete** problems, is certainly one of the most proper choices to explore what we can achieve with radomization in case of **NP**-complete problems. Surprisingly, also for satisfiability we have a very simple randomized algorithm:

Algorithm 5.9.9 (Satisfiability of Boolean formulas)

```
Choose randomly a truth assignment $T$.
while there is a truth assignment $T'$ that differs from $T$ in exactly
      one variable and satisfies more clauses than $T$
      do choose $T'$ that satisfies the most clauses and set $T \leftarrow T'$ od;
return $T$
```

We show now that if a Boolean formula F in 3-CNF, with n variables, is satisfiable, then Algorithm 5.9.9 discovers, with very high probability, a satisfying assignment T' for F.

If Algorithm 5.9.9 starts with a truth assignment that agrees with T' in values of very few variables, then it is likely that a local maximum that is not a global optimum is found. Let us therefore call a truth assignment *bad* if it agrees with T' in fewer than $(\frac{1}{2}-\theta)n$ variables, for some $\theta > 0$ and let us call all other assignments *good*. Since Algorithm 5.9.9 chooses the starting truth assignment randomly, the probability that this assignment is bad is, by Chernoff's bound from Exercise 1.9.15, with $p=\frac{1}{2}$, $\varepsilon = 2\theta$, at most $e^{-\theta^2 n}$. This means that our algorithm usually starts with a good truth assignment.

Consider now a good truth assignment T that agrees with a satisfying assignment T' in exactly $k \geq (\frac{1}{2}-\theta)n$ variables and let V be a fixed variable on which T and T' disagree. Suppose we change the value of V. How many clausees we gain and how many we loose? Let us denote by C_+ the set of all clauses, of three literals, on n variables, that are satisfied by T' (no matter whether they are in F or not), and that change from false to truth if the value of V is flipped. Moreover, denote by C_- the set of clauses that change from truth to false.

Our next aim is to calculate the size of C_+ and C_-. In order to do that we can assume, without loss of generality, that T' is a constantly true assignment (otherwise we can flip literals in F, if needed). In such a case C_+ is the set of 3-clausees that contain the variable V and two other variables that occur positively if and only if they are false in T. Hence $|C_+| = \binom{n-1}{2}$. Similarly, C_- is the set of clauses that contain the variable V negatively (that is $\bar{V}$) and two other variables that occur positively if and only if they are false in T'; excluding the case that all three variables occur negatively and two of them are among those k in which T and T' agree, because for such triples of variables T' is not satisfied. Hence, $|C_-| = \binom{n-1}{2} - \binom{k}{2}$.

Observe that by changing the value of V we do not gain anything if and only if F has at least as many clauses in C_+ as in C_-. Our next aim is to determine an upper bound of the probability that this happens. In order to do that denote $B(p,n)$ a binomial random variable the value of which gives the number of successes in n independent trials. In such a case the probability that we do not gain anything by changing the value of V is $Pr(B(p,|C_-|) \geq B'(p,|C_+|))$, where B and B' are independent binomial random variables. This probability is clearly at most

$$Pr(B(p,|C_-|) \geq m) + Pr(B'(p,|C_+|) \leq m),$$

for every m. We can now apply Chernoff's bounds from Lemma 1.9.13 and Exercise 1.9.15 to get that this probability is at most

$$e^{-((m-p|C_-|)^2)/(3p|C_-|)} + e^{-((m-p|C_+|)^2)/(2p|C_+|)} \leq e^{-((m-p|C_-|)^2)/(3p|C_-|)} + e^{-((m-p|C_+|)^2)/(3p|C_+|)}. \tag{5.5}$$

If we take $m = p\sqrt{|C_-|\,|C_+|}$, then, after elementary adjucements of the right side of formula in (5.5) we get that the probability that by flipping V we do not gain anything is at most $2e^{-cpn^2}$, where

$$c \doteq (1 - \sqrt{1 - (0.5 - \varepsilon)^2})^2.$$

This means that the probability that our algorithm, starting from a good truth asignment, will ever be misled by flipping a variable is atmost $n2^n e^{-cpn^2}$, because there are at most $n2^{n-1}$ such possible flippings. This yields, for $p = \frac{1}{2}$, the following theorem:

Theorem 5.9.10 *If* $0 < \varepsilon < \frac{1}{2}$, *then there exists a constant* $c \doteq (1 - \sqrt{(1 - (1/2 - \varepsilon)^2)})^2/6$ *such that for all but a fraction of at most* $n2^n e^{-cn^2/2}$ *of satisfiable* 3*-CNF Boolean formulas with* n *variables, the probability that the greedy algorithm succeeds in discovering a truth assignment in each independent trial from a random start is at least* $1 - e^{-\varepsilon^2 n}$.

Randomization is of importance also for design of approximation algorithms.

Example 5.9.11 (MAX-CNF problem) *The task is, given a Boolean formula* F, *to determine a truth assignment that satisfies maximum of clauses of* F. *This problem is* **NP**-*hard, but can be solved by just randomly choosing an assignment.*

Indeed, suppose that F *has* m *clauses and each its variable is set randomly to* 1 *or* 0. *For* $1 \leq i \leq m$, *let* Z_i *be a random variable the value of which is* 1 *if the* i*-th clause is satisfied and* 0, *otherwise. The probability that a clause with* k *literals is not satisfied by a random assignment is* 2^{-k} *and the probability that it is satisfied is* $1 - 2^{-k} \geq \frac{1}{2}$. *Hence* $EZ_i \geq \frac{1}{2}$, *for all i and therefore the expected number of clauses satisfied by a random assignment is* $\sum_{i=1}^{m} EZ_i \geq \frac{m}{2}$. *A random choice of an assignment yields therefore a very simple randomized* $\frac{1}{2}$*-approximation algorithm.*

Remark 5.9.12 An interestibg problem with good but not arbitrarilily good approximation algorithms is the problem to detrmine the maximal number of clauses that can be satisfied by a given Boolean formula in 3-CNF. Indeed, given a random assignment an arbitrary clause has at least $\frac{7}{8}$ chance of being satisfied. This implies that the average assignment satisfies at least $\frac{7}{8}$-th of all clauses. Since the maximal assignment has to satisfy more clauses than the average number we get a very good approximation by saying always, in $\mathcal{O}(1)$-time, that $\frac{9}{10}$-th of clauses are satisfied. More sophisticated methods yield even a better approximations. However, there is a constant $c < 1$, very close to one (about 0.99999) such that one can get c-approximation only if $\mathbf{P} = \mathbf{NP}$.

In general, we talk about **randomized algorithms** in case algorithms use "coin-tossing" or various RANDOM procedures to make random decisions. The corresponding formal machine model is that of **probabilistic Turing machine**—a Turing machine that can generate a random bit in one step or, what is only another point of view, has a special "random tape" filled up with new random bits each time a computation starts.

Exercise 5.9.13 *Let* $t : \mathbf{N} \to \mathbf{N}$ *be a space-constructible function. Show that each probabilistic* $t(n)$*-time bounded MTM can be simulated by a deterministic MTM in space* $\mathcal{O}(t(n))$ *and in time* $\mathcal{O}(2^{t(n)})$.

Randomized elements of random algorithms or probabilistic Turing machines are not so easy to deal with such an elegance and exactness as we have been used studying deterministic computations and their models. However, and fortunately, in order to model random steps formally and to study the power of randomization, we do not need actually this coin-tossing or random number generators at all. It is enough to consider nondeterministic Turing machines that have in each configuration exactly two choices to make and all computations of which have the same length, and then just to consider different acceptance modes. By Lemma 5.1.11, this is no essential restriction on the power of NTM and each NTM with time bounded by a polynomial can be transformed in polynomial time into one satisfying the above properties. This means, that we can consider only NTM $\mathcal{M}$ such that if, for some input x, the length of a computation is t, then the configuration tree of $\mathcal{M}$ is a complete binary tree of depth t.

5.9.1 Models and Complexity Classes of Randomized Computing

In order to get a deeper insight into the power of randomization we need to have simple and realistic models of randomized computation in polynomial time. The following four complexity classes of randomized computations, namely **ZPP, RP, PP,** and **BPP**, are closely related with four natural modes of randomized computing. (In the following definition only those NTM are considered that have exactly two choices to make in each nonterminating configuration and all computations of which have the same length for inputs of the same size.)

RP A language L is in **RP** (**R**andom **P**olynomial time), if there is a polynomial time bounded NTM $\mathcal{M}$ such that if $x \in L$, then **at least half** of all computations of $\mathcal{M}$ on x terminate in the accepting state, and if $x \notin L$, then **all** computations of $\mathcal{M}$ on x terminate in the rejecting state. (The so-called *Monte Carlo acceptance* or *one-sided Monte Carlo acceptance.*)

PP A language L is in **PP** (**P**robabilistic **P**olynomial time), if there is a polynomial time bounded NTM $\mathcal{M}$ such that $x \in L$, if and only if **more than half** of computations of $\mathcal{M}$ on x terminate in the accepting state. (The so-called *acceptance by majority.*)

BPP A language L is in **BPP** (**B**ounded error (away from $\frac{1}{2}$) **P**robabilistic **P**olynomial time), if there is a polynomial time bounded NTM $\mathcal{M}$ such that:

1. If $x \in L$, then at least $\frac{3}{4}$ of the computations of $\mathcal{M}$ on x terminate in the accepting state.
2. If $x \notin L$, then at least $\frac{3}{4}$ of the computations of $\mathcal{M}$ on x terminate in the rejecting state.

(The so-called *acceptance by clear majority* or *two-sided Monte Carlo acceptance.*)

ZPP a language L is in **ZPP** (**Z**ero error **P**robabilistic **P**olynomial time), if $L \in$ **RP**$\cap$**co-RP** (The so-called Las Vegas acceptance).

Exercise 5.9.14 *Show that any constant* $0 < c < 1$ *could be substitute for* $\frac{1}{2}$ *in the definition of the class: (a)* **RP**; *(b)*** **PP**.

Exercise 5.9.15 *Show that any constant* $\frac{1}{2} < c < 1$ *could be substituted for* $\frac{3}{4}$ *in the definition of the class* **BPP**.

Las Vegas algorithms, as the one for sorting in Example 5.9.4, are of course ideal ones, because an incorrect solution may be a disastre in some applications. The following exercise shows a way how to derive a Las Vegas algorithm from a Monte Carlo one, provided we can verify efficiently the correctness of a to-be-solution of the problem.

Exercise 5.9.16 *Let $\mathcal{A}$ be a $t(n)$ average time bounded Monte Carlo algorithm for a problem $\mathcal{P}$ that produces a correct solution with probability $\gamma(n)$. Moreover, let correctness of a to-be-solution to $\mathcal{P}$ can be verified in time $v(n)$. Show how to obtain a Las Vegas algorithm for $\mathcal{P}$ that runs in expected time at most $(t(n)+v(n))/\gamma(n)$. (Hint, make a use of the mean of a geometric random variable—see Exercise 1.9.7.)*

Less formally, classes **RP, PP** and **BPP** can be defined as classes of problems for which there is a randomized algorithm A with the following properties:

RP	$x \in L \Rightarrow Pr(A(x) \text{ accepts}) \geq \frac{1}{2}$	$x \notin L \Rightarrow Pr(A(x) \text{ accepts}) = 0$
PP	$x \in L \Rightarrow Pr(A(x) \text{ accepts}) > \frac{1}{2}$	$x \notin L \Rightarrow Pr(A(x) \text{ accepts}) \leq \frac{1}{2}$
BPP	$x \in L \Rightarrow Pr(A(x) \text{ accepts}) \geq \frac{3}{4}$	$x \notin L \Rightarrow Pr(A(x) \text{ accepts}) \leq \frac{1}{4}$

The classes **ZPP, RP** and **PP** fit nicely in our basic hierarchy of complexity classes.

Theorem 5.9.17 P $\subseteq$ ZPP $\subseteq$ RP $\subseteq$ NP $\subseteq$ PP $\subseteq$ PSPACE.

Proof. Inclusions **P** $\subseteq$ **ZPP** $\subseteq$ **RP** are trivial. If $L \in$ **RP**, then there is a NTM $\mathcal{M}$ accepting L with Monte Carlo acceptance. Hence $x \in L$ if and only if $\mathcal{M}$ has at least one accepting computation for x. Thus, $L \in$ **NP**, and this proves the inclusion **RP** $\subseteq$ **NP**.

To show **NP** $\subseteq$ **PP** we proceed as follows. Let $L \in$ **NP** and let $\mathcal{M}$ be a polynomial time bounded NTM accepting L. Design a NTM $\mathcal{M}'$ such that $\mathcal{M}'$, for an input w, chooses nondeterministically and performs one of the steps:

1. $\mathcal{M}'$ accepts.
2. $\mathcal{M}'$ simulates $\mathcal{M}$ on input w.

Using the ideas presented in the proof of Lemma 5.1.11, $\mathcal{M}'$ can be transformed into an equivalent NTM $\mathcal{M}''$ that has exactly two choices in each nonterminating configuration, all computations of which have the same length bounded by a polynomial, and which accepts the same language as $\mathcal{M}'$. We show that $\mathcal{M}''$ accepts L by majority and therefore $L \in$ **PP**.

If $w \notin L$, then exactly half of the computations accepts w—those that start with the step 1. This is, however, not enough, and therefore $\mathcal{M}''$, as a probabilistic TM, does not accept w.

If $w \in L$, then there is at least one computation of $\mathcal{M}$ that accepts w. This means that more than one half of all computations of $\mathcal{M}''$ accept w—all those computations that take the step 1, as the first one, and at least one of those going through the step 2. Hence $\mathcal{M}''$ accepts w by majority.

The last inequality **PP** $\subseteq$ **PSPACE** is again easy to show. Let $L \in$ **PP** and $\mathcal{M}$ be a NTM accepting L by majority and with time bounded by a polynomial p. In such a case no configuration of $\mathcal{M}$ is longer than $p(|w|)$ for an input w. Using the method to simulate a NTM by a DTM as shown in the proof of Theorem 5.1.5, we get easily that $\mathcal{M}$ can be simulated in polynomial space. □

Since there is a polynomial time Las Vegas algorithm to recognize primes (see reference), prime recognition is in **ZPP** (and it may be in $\mathbf{RP}-\mathbf{P}$).

Exercise 5.9.18 *Denote MAJSAT the problem to decide for a given Boolean formula F whether more than half of all possible assignments to variables in F satisfy F. Show: (a) MAJSAT is in* **PP**; *(b) MAJSAT is* **PP***-complete (with respect to polynomial time reducibility).*

Exercise 5.9.19 ** *Let $0 \leq \varepsilon \leq 1$ be a rational number. Let $\mathbf{PP}_\varepsilon$ be the class of languages L for which there is a NTM $\mathcal{M}$ such that $x \in L$ if and only if at least an ε fraction of all computations are acceptances. Show that $\mathbf{PP}_\varepsilon = \mathbf{PP}$.*

Main complexity classes of randomized computing have also been shown to be separated by oracles. For example, there are oracles A, B, C, D, E, F and G such that: (a) $BPP^A \subsetneq NP^A$; (b) $NP^B \subsetneq BPP^B$; (c) $P^C \neq BPP^C$; (d) $P^D \neq RP^D$; (e) $P^E \subsetneq ZPP^E$; (f) $RP^F \neq ZPP^F$; (g) $RP^G \neq BPP^G$.

5.9.2 The Complexity Class BPP

Acceptance by clear majority seems to be the most important concept of randomized computation. In addition, the class **BPP** is often considered as a plausible formalization of the concept of **feasible computation** and therefore deserves more analysis.

First of all, the number $\frac{3}{4}$, used in the definition of the class **BPP**, should not be taken as a magic number. Any number strictly larger than $\frac{1}{2}$ can do and results in the same class. Indeed, let us assume that we have a machine $\mathcal{M}$ that decides a language by a strict majority of $\frac{1}{2}+\varepsilon$. We can use this machine $2k+1$ times and accept as the outcome the majority of outcomes. By Chernoff's Bound, Lemma 1.9.13, the probability of the false answer is at most $e^{-2\varepsilon^2 k}$. By taking sufficiently large k this probability can be reduced as much as needed. For $k = \lceil \frac{\ln 2}{\varepsilon^2} \rceil$ we get a probability of error at most $\frac{1}{4}$ as desired.

Exercise 5.9.20 * *Show that in the definition of the class* **BPP** *ε does not have to be a constant. It can be $|x|^{-c}$ for any $c > 0$. Show also that the bound $\frac{3}{4}$ can be replaced by $1 - \frac{1}{2^{|x|}}$.*

The concept of decision by clear majority seems to be therefore a robust one. Few words are also in order concerning a relation between the classes BPP and PP. BPP-algorithms allow to diminish, by a repeated use, the probability of error as much as needed. This is not true for PP-algorithms.

Let us now turn to another argument which shows that the class **BPP** has properties indicating that it is a reasonable extension of the class **P**.

In order to formulate the next theorem we need to define when a language $L \subseteq \{0,1\}^*$ has polynomial size circuits. This is in case that there is a family of Boolean circuits $C_L = \{C_i\}_{i=1}^{\infty}$ and a polynomial p such that the size of C_n is bounded by $p(n)$, C_n has n inputs and for all $x \in \{0,1\}^*$, $x \in L$ if and only if the output of $C_{|x|}$ is 1 if its i-th input is the i-th symbol of x.

Theorem 5.9.21 *All languages in* **BPP** *have polynomial size Boolean circuits.*

Proof. Let $L \in \mathbf{BPP}$ and $\mathcal{M}$ be a polynomial time bounded NTM that decides L by clear majority. We show that there is a family of circuits $\mathcal{C} = \{C_n\}_{n=1}^{\infty}$, the size of which is bounded by a polynomial, and such that C_n accepts the language L restricted to $\{0,1\}^n$. The proof is elegant, but not constructive, and the resulting family of Boolean circuits is not uniform. (If certain uniformity conditions are satisfied this would imply $\mathbf{P} = \mathbf{BPP}$.)

Let $\mathcal{M}$ be time bounded by a polynomial p. For each $n \in \mathbf{N}$ a circuit C_n will be designed using a set of strings $A_n = \{a_1, \ldots, a_m\}$, where $m = 12(n+1)$ and $a_i \in \{0,1\}^{p(n)}$. The idea behind is that each string a_i represents a possible sequence of random choices of $\mathcal{M}$ during a computation and therefore completely specifies a computation of $\mathcal{M}$ for inputs of length n.

Informally, on an input w with $|w| = n$, C_n simulates $\mathcal{M}$ with each sequence of choices from $A_{|w|}$, and then, as the outcome, takes the majority of $12(|w|+1)$ outcomes. From the proof of Lemma 4.3.23 we know how to design a circuit simulating a polynomial time computation on TM and using those ideas we can construct C_n with the above property and of the polynomial size, with respect to n.

The point now is to show that there exists a set A_n such that C_n works correctly. To see that, we need the following lemma:

Lemma 5.9.22 *For all $n > 0$ there is a set A_n of $12(n+1)$ binary (Booean) strings of the length $p(n)$ such that for all inputs x of length n fewer than half of the choices in A_n lead $\mathcal{M}$ to a wrong decision (either to accept $x \notin L$ or to reject $x \in L$).*

Assume now, for the moment, that Lemma 5.9.22 holds and the set A_n has the required property. With the ideas from the proof of Lemma 4.3.23 we can design a circuit C_n with polynomially many gates that simulates $\mathcal{M}$ with each of the sequences from A_n and then takes the majority of outcomes. It follows from the property of A_n, stated in Lemma 5.9.22, that C_n outputs 1 if and only if the input w is in $L \cap \{0,1\}^n$. Thus, L has a polynomial size circuit. □

Proof of Lemma 5.9.22. Let A_n be a set of $m = 12(n+1)$ Boolean strings, taken randomly and independently, of length $p(n)$. We show now that the probability (which refers to the choice of A_n) is at least $\frac{1}{2}$, that for each $x \in \{0,1\}^n$ more than half of the choices in A_n make $\mathcal{M}$ to perform a correct computation.

Since $\mathcal{M}$ decides L by a clear majority, for each $x \in \{0,1\}^n$ at most one quarter of the computations are bad (in the sense that they either accept an $x \notin L$ or reject an $x \in L$). Since Boolean strings in A_n have been taken randomly and independently, the expected number of bad computations with vectors from A_n is at most $\frac{m}{4}$. By the Chernoff's Bound, Lemma 1.9.13, the probability that the number of bad Boolean string choices is $\frac{m}{2}$ or more is at most $e^{-\frac{m}{12}} < \frac{1}{2^{n+1}}$. This is therefore the probability that x is wrongly accepted by $\mathcal{M}$ when simulating computations specified by A_n.

The last inequality holds for each $x \in \{0,1\}^n$, and therefore the probability that there is an x that is not correctly accepted at the given choice of A_n is at most $2^n \frac{1}{2^{n+1}} = \frac{1}{2}$. This means that most of the choices for A_n provide the correct acceptance for all $x \in \{0,1\}^n$.

This implies that there is always a choice of A_n with the required property, in spite of the fact we have no idea how to find it. □

What does this result imply? It follows from Theorem 4.3.24 that a language L is in **P** if and only if there is a uniform family of polynomial size Boolean circuits for L. On the other side, for no **NP**-complete problem a family of polynomial size Boolean circuits is known! **BPP** seems to be therefore a very small/reasonable extension of **P** (if any). Since **BPP** does not seem to contain an **NP**-complete problem the acceptance by clear majority does not seem to help us to cope with **NP**-completeness.

Exercise 5.9.23 *Show the inclusions:* **RP** $\subseteq$ **BPP** $\subseteq$ **PP**.

Exercise 5.9.24 *Show that a language* $L \subseteq \Sigma^*$ *is in* **BPP** *if and only if there is a polynomially decidable and polynomially balanced (by p) relation* $R \subseteq \Sigma^* \times \Sigma^*$ *such that* $x \in L$ ($x \notin L$) *if and only if* $(x,y) \in R$ ($(x,y) \notin R$) *for more than* $\frac{3}{4}$ *of words* y *with* $|y| \leq p(|x|)$.

Another open problem is the relation between **NP** and **BPP**. Currently these two classes seem to be incomparable, but no proof of this fact is known. It is also not clear whether there are complete problems for classes **RP, BPP, ZPP**. (The class **PP** has complete problems.)

As already mentioned there is some evidence, but no proofs, that polynomial time randomized computing is more powerful than polynomial time deterministic computing. However, this statement is true only for computing on computers working on principles of classical physics. It has been proven by Simon (1994) that polynomial time (randomized) computing on (potential) quantum computers is more powerful than polynomial time randomized computing on classical computers. This again allows to extend our concept of feasibility.

5.10 Parallel Complexity Classes — November 8, 2005

The most important problem concerning complexity of parallel computing seems to be to find out what can be computed in polylogarithmic time with polynomially many processors and the most interesting complexity class for parallel computing seems to be the class **NC** - this stands for "Nick's class", and refers to Nicholas Pippenger, who was the first to define and explore this class.

There are several equivalent definitions of **NC**. From the point of view of algorithm design and analysis a useful and natural one is in terms of PRAM computations:

$$\mathbf{NC} = \text{PRAM-TimeProc}(\lg^{O(1)}(n), n^{O(1)}).$$

From the theoretical point of view, the following definition, in terms of uniform families of Boolean circuits, bounded by polylogarithmic depth and polynomial size, seems to be more easy to work with:

$$\mathbf{NC} = \text{UCIRCUIT-DepthSize}(\lg^{O(1)}(n), n^{O(1)}).$$

To get a more detailed insight into the structure of the class **NC** an appropriate question to ask is the following one: What can be computed using different amounts of parallel computation time? This leads to the following refinements of **NC**:

$$\mathbf{NC^i} = \text{UCIRCUIT-DepthSize}(\lg^{i}(n), n^{O(1)}), \qquad i \geq 0.$$

In this way a possibly infinite family of complexity classes has been introduced that is related with the sequential ones in the following way:

$$\mathbf{NC^1} \subseteq \mathbf{DLOGSPACE} \subseteq \mathbf{NLOGSPACE} \subseteq \mathbf{NC^2} \subseteq \mathbf{NC^3} \subseteq \ldots \subseteq \mathbf{NC} \subseteq \mathbf{P}.$$

None of these inclusions is known to be strict and the open problem

$$\mathbf{NC{=}P}$$

is considered to be a parallel analog of the **P=NP** problem.

From the practical point of view, it is of special interest to find out which problems are in $\mathbf{NC^1}$ and $\mathbf{NC^2}$. These classes represent problems that can be computed very fast using parallel computers. The following two lists present some of them:

$\mathbf{NC^1}$

1. Addition of two binary numbers of length m $(n = 2m)$.
2. Multiplication of two binary numbers of length m $(n = 2m)$.
3. Sum of m binary numbers of length m $(n = m^2)$.
4. Matrix multiplication of two $m \times m$ matrices of binary numbers of length l $(n = 2m^2l)$.
5. Prefix sum of m binary numbers of length l $(n = ml)$.
6. Merging of two sequences of m binary numbers of length l $(n = 2ml)$.
7. Regular language recognition.

$\mathbf{NC^2}$

1. Division of two binary numbers of length m $(n = 2m)$.[8]
2. Determinant of an $m \times m$ matrix of binary numbers of length l $(n = m^2 l)$.
3. Matrix inversion of an $m \times m$ matrix of binary numbers of length l $(n = m^2 l)$.
4. Sorting of m binary numbers of length l $(n = ml)$.

Remark 5.10.1 Also in the case of parallel computing one can often reduce a function problem, for $f : \mathbf{N} \rightarrow \mathbf{N}$, to a decision problem with small $(\lg f(n))$ multiplicative procesor overhead. One of the technique how to design the corresponding well parallelized decision problem is for the case that a good upper bound b for $f(n)$ can be easily determined. The corresponding decision problem is one to decide, given an n and an integer $1 \leq i \leq \lceil \lg b \rceil$, whether the i-th least significant bit of $f(n)$ is 1.

The investigation of the class NC motivated an introduction of two new concepts of reducibility: **NC-many-to-one reducibility**, in short $\leq_{NC}^{m}$, and **NC-Turing reducibility**, in short $\leq_{NC}^{T}$, that are defined analogously as many-to-one and Turing reducibility.The only difference is that reductions have to be performed in polylogarithmic time and with polynomial number of processors. This leads naturally to two new concepts of **P**-completeness, with respect to $\leq_{NC}^{m}$ and $\leq_{NC}^{T}$ reducibilities.

Exercise 5.10.2 *Show that* $L_1 \leq_{lg} L_2 \Rightarrow L_1 \leq_{NC}^{m} L_2 \Rightarrow L_1 \leq_{NC}^{T} L_2$*, for any languages* L_1 *and* L_2.

The advantage of **P**-completeness based on **NC**-reductions is that they bring important insights into the power of parallelism and a methodology to show that a problem is inherently sequential. For example, as it is easy to see, it holds:

Theorem 5.10.3 *If any* **P***-complete problem is in* **NC***, then* $\mathbf{P} = \mathbf{NC}$.

Theorem 5.10.3 implies that **P**-complete problems are the main candidates for inherently sequential problems. If one is able to show that a problem is **P**-complete, then it seems to be hopeless to try to design for it an algorithm working in polylogarithmic time on polynomially many processors. Similarly, if a problem withstands all effort to find a polylogarithmic parallel algorithm, then it seems the best to change the strategy and to try to show that the problem is **P**-complete; what is currently seen as an "evidence" that no fast parallel algorithm to solve the problem can exist. Circuit value problem, introduced in Section 5.3, is perhaps the most important **P**-complete problem also for $\leq_{NC}^{T}$ reduction.

Exercise 5.10.4 *Argue that if an* **P***-complete problem is in the class* $\mathbf{NC^i}, i > 1$, *then* $\mathbf{P} = \mathbf{NC^i}$.

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[8] It is an open problem whether division is in $\mathbf{NC}^1$.

5.11 Beyond NP — November 8, 2005

There are several important complexity classes beyond **NP**. For example: **PH, PSPACE, EXP** and **NEXP**. In spite of the fact that they seem to be much larger than **P**, there are plausible views of polynomial time computations that coincide with them and therefore we should not rule out them as potential candidates for one of the main goal of complexity theory: to find out what is **feasible**.

The class **PH**, the so-called polynomial hierarchy class, seems to be between **NP** and **PSPACE** and contains a variety of naturally defined algorithmic problems. As it has been shown in Section 4.4.6, the class **PSPACE** corresponds to the polynomial time computations on the second class machines. In Chapter 9 it will be shown that **PSPACE** corresponds to interactive proof systems with one prover and a polynomial number of interactions and **NEXP** to interactive proof systems with two provers and a polynomial number of interactions. There are therefore good theoretical and also practical reasons to pay attention to these classes.

5.11.1 Between NP and PSPACE—Polynomial Hierarchy

With the help of oracles we can use **P** and **NP** to design various infinite sequences of potentially richer and richer complexity classes. Perhaps the most important one is the following simplified version of the so-called **polynomial hierarchy**:

$$\mathbf{\Sigma_0^p} = \mathbf{P}, \qquad \mathbf{\Sigma_{k+1}^p} = \mathbf{NP}^{\mathbf{\Sigma_k^p}}, \quad k \geq 0$$

$$\mathbf{\Delta_0^p} = \mathbf{P}, \qquad \mathbf{\Delta_{k+1}^p} = \mathbf{P}^{\mathbf{\Sigma_k^p}}, \quad k \geq 0$$

and the so-called **cumulative polynomial hierarchy**:

$$\mathbf{PH} = \bigcup_{k=1}^{\infty} \mathbf{\Sigma_k^p}.$$

In other words, $\mathbf{\Sigma_{k+1}^p}$ ($\mathbf{\Delta_{k+1}^p}$) is the family of languages that can be accepted by polynomially bounded oracle NTM (DTM) with an oracle from $\mathbf{\Sigma_k^p}$. The following inclusions clearly hold

$$\mathbf{P} = \mathbf{\Sigma}_0^p \subseteq \mathbf{NP} = \mathbf{\Sigma_1^p} \subseteq \mathbf{\Sigma_2^P} \subseteq \mathbf{\Sigma_3^p} \subseteq \ldots \subseteq \mathbf{PH} \subseteq \mathbf{PSPACE}. \tag{5.6}$$

Exercise 5.11.1 *Show: (a)* $\mathbf{\Sigma_{k+1}^p} = \mathbf{NP}^{\mathbf{\Delta_{k+1}^p}}$, *for* $k \geq 0$*; (b)* $\mathbf{\Delta_k^p}$ *is closed under complement, for* $k \geq 0$*; (c)* $\mathbf{P}^{\mathbf{\Delta_k^p}} = \mathbf{\Delta_k^p}$, *for* $k \geq 0$.

Exercise 5.11.2 *Denote:* $\mathbf{\Pi_0^p} = \mathbf{P}$, $\mathbf{\Pi_{k+1}^p} = \mathbf{co}-\mathbf{NP}^{\mathbf{\Sigma_k^p}}$. *Show: (a)* $\mathbf{\Sigma_{k+1}^p} = \mathbf{NP}^{\mathbf{\Pi_k^p}}$, *for* $k \geq 0$*; (b)* $\mathbf{\Sigma_k^p} \cup \mathbf{\Pi_k^p} \subseteq \mathbf{\Delta_{k+1}^p}$, *for* $k \geq 0$*; (c)* $\mathbf{\Delta_k^p} \subseteq \mathbf{\Sigma_k^p} \cap \mathbf{\Pi_k^p}$, *for* $k \geq 0$*; (d) if* $\mathbf{\Sigma_k^p} \subseteq \mathbf{\Pi_k^p}$, *then* $\mathbf{\Sigma_k^p} = \mathbf{\Pi_k^p}$*; (e)* $\mathbf{\Sigma_k^p} \cup \mathbf{\Pi_k^p} \subseteq \mathbf{\Delta_{k+1}^p} \subseteq \mathbf{\Sigma_{k+1}^p} \cap \mathbf{\Pi_{k+1}^p}$, *for* $k \geq 0$.

In spite of the fact that polynomial hierachy classes look like being introduced artificially by a pure abstraction, they seem to be very reasonable complexity classes. This can be concluded from the observation that they have naturally defined complete problems.

One complete problem for $\mathbf{\Sigma_k^P}, k > 0$ is the following modification of the bounded halting problem:

$$L_{\mathbf{\Sigma_k^P}} = \{\langle M\rangle\langle w\rangle\#^t \mid M \text{ is a TM with an oracle from } \mathbf{\Sigma_{k-1}^P} \text{ accepting } w \text{ in } t \text{ steps}\}$$

Another complete problem for $\mathbf{\Sigma_k^P}$ is the so-called QSAT$_k$ problem—(QSAT$_k$ stands for "quantified satisfiability problem with k alternations of quantifiers")—defined as follows:

Given a Boolean formula B with Boolean variables partitioned into k sets $X_1, \ldots, X_k$. Is it true that there is a partial assignment to the variables in X_1 such that for all partial assignments to variables in X_2, there is such a partial assignment to variables in $X_3, \ldots$, that B is true by the overall assignment. An instance of QSAT$_k$ is usually presented as

$$\exists X_1 \,\forall X_2 \,\exists X_3 \,\forall X_4 \ldots Q X_k \, B$$

where Q is either the quantifier $\exists$, if k is odd, or $\forall$, if k is even, and B is a Boolean formula.

It is an open problem whether inclusions in (5.6) are proper. Observe that if $\mathbf{\Sigma_i^P} = \mathbf{\Sigma_{i+1}^P}$ for some i, then $\mathbf{\Sigma_i^P} = \mathbf{\Sigma_k^P}$ for all $k \geq i$, and in such a case we say that the polynomial hierarchy collapses.

It is not known, whether the polynomial time hierarchy collapses. There are, however, various results of the type "if ..., then the polynomial hierarchy collapses". For example, the polynomial,hierarchy collapses if:

1. **PH** has a complete problem;
2. the graph isomorphism problem is **NP**-complete;
3. SAT has a polynomial size Boolean circuit.

In Section 5.8 we have mentioned that the relation between **BPP** and $\mathbf{NP} = \mathbf{\Sigma_1^P}$ is unclear. However, it is clear that **BPP** is not too high in the polynomial hierarchy

Exercise 5.11.3 * *Show that* $\mathbf{BPP} \subseteq \mathbf{\Sigma_2^P}$.

The class **PH** is the first major deterministic complexity class we have considered so far that is not known to have complete problems and, moreover, it is actually very unlikely that it has complete problems.

An interesting/important task in complexity theory is to determine more exactly the relation between the class **PH** and other complexity classes. For example, the so-called **Toda's theorem** says that $\mathbf{PH} \subsetneq \mathbf{P^{PP}}$. This result is sometimes interpreted as $\mathbf{PH} \subsetneq \mathbf{P^{\#P}}$ (what would mean that counting is very powerful), even though we can not directly compare the class **HP** (of decision problems) and the class **#P** (of function problems). However, the class **PP** is "close enough" to **#P**. (Indeed, problems in **PP** can be seen as asking for the most significant bit concerning the number of accepting computations and the problems on **#P** can be seen as asking for all bits of the number of accepting computations.)

5.11.2 PSPACE-complete Problems

There is a variety of natural computational problems that are **PSPACE**-complete. For example, variants of the halting, tiling and satisfiability problems:

Theorem 5.11.4 (PSPACE-completeness of IN-PLACE-ACCEPTANCE problem) *The following problem is* **PSPACE***-complete: given a DTM $\mathcal{M}$ and an input w, does $\mathcal{M}$ accept w without having the head ever leaving w (the part of tape on which w is written)?*

Proof. Given $\mathcal{M} = \langle \Gamma, Q, q_0, \delta \rangle$ and $w \in \Gamma^*$, we simulate $\mathcal{M}$ on w and keep account of the number of steps. w is rejected if $\mathcal{M}$ rejects, or the head of $\mathcal{M}$ attempts to leave cells in which the input w was written, or if $\mathcal{M}$ makes more than $|\Gamma|^{|w|}|Q||w|$ steps. In order to store the number of steps, $\mathcal{O}(|w|)$ bits are needed, and this can be done in the space $|w|$ using a proper positional number system. Hence, the problem is in **PSPACE**.

Assume now that L can be accepted in space n^k by a machine $\mathcal{M}$. Clearly $\mathcal{M}$ accepts an input w, if and only if $\mathcal{M}$ accepts w "in place" $w\sqcup^{n^k}$. Thus $w \in L$, if and only if $(\mathcal{M}, w\sqcup^{n^k})$ is a "yes" instance of the IN-PLACE-ACCEPTANCE. □

PSPACE-completeness of a problem can be shown either directly or using the reduction method. For example, by reduction from the following modifications of **NP**-complete problems:

Example 5.11.5 (CORRIDOR-TILING) *Given a finite set T of Wang tiles, a pair of tiled horizontal strips U and D of the length n, does there exist an integer m such that it is possible to tile an $m \times n$ rectangle, with U as the top row and D as the bottom row, and with the left sides of the tiles of the first column, and the right sides of the tiles of the last column, having the same color? (m is not given.)*

Example 5.11.6 (QUANTIFIED-SATISFIABILITY (QSAT)) *Given a Boolean formula B with variables $x_1, \ldots, x_n$, is the following formula valid*

$$\exists x_1 \,\forall x_2 \,\exists x_3 \,\forall x_4 \ldots Q x_n \,B,$$

where $Q = \forall$ if n is even, and $Q = \exists$, otherwise?

A variety of game problems have been shown to be **PSPACE**-complete. For example:

Example 5.11.7 (GENERALIZED-GEOGRAPHY game problem:)
Given a directed graph $G = \langle V, E \rangle$ and a vertex v_0. Does the Player 1 have a winning strategy in the following game? Players alternate choosing new arcs from the set E. Player 1 starts by choosing an arc, whose tail is v_0. Thereafter each player must choose an arc whose tail equals the head of the previous chosen arc. The first player unable to choose a new arc looses.

Some other examples of **PSPACE**-complete problems: word problem for context-sensitive languages, reachability problem for cellular automata (given two configurations c_1 and c_2, is c_2 reachable from c_1?), the existence of the winning strategy for a generalization of the game GO to arbitrarily large grids,...

5.11.3 Exponential Complexity Classes

There are various ways exponential complexity classes can be defined. The most useful seem to be the following definitions, see Section 5.2:

$$\mathbf{EXP} = \bigcup_{k=1}^{\infty} \mathrm{Time}(2^{n^k}), \qquad \mathbf{NEXP} = \bigcup_{k=1}^{\infty} \mathrm{NTime}(2^{n^k}),$$

and the open problem $\mathbf{EXP} = \mathbf{NEXP}$ is an "exponential version" of the $\mathbf{P} = \mathbf{NP}$ problem. Interesting enough, these two problems are related.

Theorem 5.11.8 *If* $\mathbf{P} = \mathbf{NP}$*, then* $\mathbf{EXP} = \mathbf{NEXP}$*.*

Proof. Let $L \in \mathbf{NEXP}$, $L \subseteq \Sigma^*$, and $\mathbf{P} = \mathbf{NP}$. By definition, there is a NTM $\mathcal{M}$ that accepts L in time 2^{n^k} for some k. Consider now an "exponentially padded" version of L

$$L' = \{wa^{2^{|w|^k} - |w|} \mid w \in L\},$$

where a is a symbol not in Σ. We show how to design a polynomial time bounded NTM $\mathcal{M}'$ that decides L'. For an input $y = wa^{2^{|w|^k} - |w|}$, $\mathcal{M}'$ first checks whether a w is followed exactly by $2^{|w|^k} - |w|$ a's, and then simulates $\mathcal{M}$ on y treating a's as blanks. $\mathcal{M}'$ works in time $\mathcal{O}(2^{|w|^k})$—and therefore in polynomial time with respect to the length of the input. Thus L' is in $\mathbf{NP}$ and also in $\mathbf{P}$, due to our asumption $\mathbf{P} = \mathbf{NP}$. This implies that there is a DTM $\mathcal{M}''$ deciding L' in time ln^l for some l. We can assume, without loss of generality, that $\mathcal{M}''$ is an off-line TM that never writes on its input tape. The construction $\mathcal{M}'$ from $\mathcal{M}$ can now be reversed and we can design a DTM $\mathcal{M}'''$ that accepts w in time $2^{|w|^{l'}}$, for some l'. $\mathcal{M}'''$ simulates, on an input w, $\mathcal{M}''$ on the input $wa^{2^{|w|^k} - |w|}$. Since $\lg(2^{|w|^k} - |w|) \doteq |w|^k$, $\mathcal{M}'''$ can easily keep track about the head of $\mathcal{M}''$ by writing down its position in binary. $\Box$

As a corollary we get that $\mathbf{EXP} \neq \mathbf{NEXP} \Longrightarrow \mathbf{P} \neq \mathbf{NP}$. This indicates that to prove $\mathbf{EXP} \neq \mathbf{NEXP}$ can be even harder than to prove $\mathbf{P} \neq \mathbf{NP}$.

Various natural complete problems are known also for the classes **EXP** and **NEXP**. Many of them are again modifications of known **NP**-complete problems. For example, the following version of the tiling problem is **EXP**-complete.

Given a finite set of tiles, a string of colors w, a number n in binary, does there exists a tiling of an $n \times n$ square with one-color side (except for the leftmost part of the top row where the string w of colors has to be)?

Many **EXP**- and **NEXP**-complete problems can be obtained from **P**- and **NP**-complete problems just by taking "exponentially more succinct description" of their inputs (graphs, circuits, formulas).

For example, a succinct description of a graph with $n = 2^k$ nodes is a Boolean circuit C with $2k$ inputs. The graph $G_C = \langle V, E \rangle$ represented by C has $V = \{1, \ldots, n\}$ and $(i, j) \in E$ if and only if C accepts the binary representation of i and j on its inputs. Clearly, such a circuit representation of a graph is exponentially shorter than the usual one.

For example, the following problem is **NEXP**-complete: given a succinct description of a graph (in terms of a circuit), decide whether the graph has a Hamilton cycle.

Remark 5.11.9 There is also another way how to define exponential complexity classes:

$$\mathbf{E} = \bigcup_{k=1}^{\infty} \mathrm{Time}(k^n) \qquad \mathbf{NE} = \bigcup_{k=1}^{\infty} \mathrm{NTime}(k^n).$$

Even though these classes seem to capture better our intuition how exponential complexity classes should look like, they do not have actually such nice properties. For example, they are not closed under polynomial reductions.

The overall map of main complexity classes is depicted in Figure 5.10.

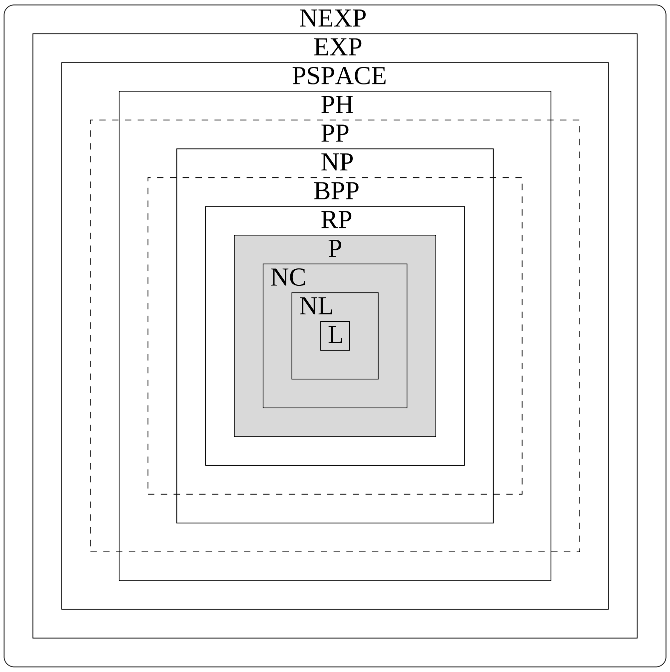

Figure 5.10: A map of complexity classes-60%

Exercise 5.11.10 *Show that for any language* $L \in \mathbf{NEXP}$ *there is a language* $L' \in \mathbf{NE}$ *such that* $L \leq_p^m L'$.

Exercise 5.11.11 * *Show that* $\mathbf{P} = \mathbf{NP}$ *implies* $\mathbf{E} = \mathbf{NE}$.

5.11.4 Far Beyond NP—with Regular Expressions only

What is beyond **NEXP**? Two interesting classes have been defined and investigated:

$$\mathbf{EXPSPACE} = \bigcup_{k=1}^{\infty} \mathit{Space}(2^{n^k}), \qquad \mathbf{ELEMENTARY} = \bigcup_{k=1}^{\infty} \mathit{Time}_k(2^{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}),$$

where Time_k denotes that there are k levels of exponentiation.

The name **ELEMENTARY** for such a huge class can be seen as ironical. This may be true from the point of view of "**P**-inhabitants", but quite justified from the point of view of recursion theory considered in the next chapter. However, in spite of that it is natural to ask whether it has a sense, from computational point of view, to pay attention to such esoteric complexity classes? Can one encounter them in down-to earth computing? Unfortunately, yes.

The following theorem summarizes results showing how far we can get in the space of complexity classes with such a very simple and down-to-earth problem as the equivalence problem for generalized regular expressions.

Theorem 5.11.12 (Stockmeyer-Meyer's Theorem) *The equivalence problem for generalized regular expressions is:*

1. **co-NP**-*complete for ordinary regular expressions without iteration;*
2. **PSPACE**-*complete for ordinary regular expressions;*
3. **co-NEXP**-*complete for regular expressions with squaring but without iteration;*
4. **EXPSPACE**-*complete for regular expressions with squaring;*
5. *Outside the class* **ELEMENTARY** *for regular expressions with negation.*

Exercise 5.11.13 * *Show* **NP**-*completeness of the inequivalence problem for regular expressions without iteration (the so-called star-free expressions). (Hint: use a reduction from SAT problem.)*

Exercise 5.11.14 * *Show that a regular language can be described by a star-free regular expression iff it is accepted by a NFA with acyclic state-diagram.*

Exercise 5.11.15 *Explain how it can happen that the last problem of Theorem 5.11.12 is computationally so hard.*

Remark 5.11.16 Two very opposite streams are noticeble in computing at the search for **feasible**. In the precomputing era the concept of feasibility was identified with the concept of recursive functions discussed in the next chapter. The more powerful computers we have and the more computation-intensive problems are attacked, the more grows frustration with what is really computable. The concept of feasibility was first restricted to the class **ELEMENTARY** that was popular for a while, then soon to **P** and nowadays there are even feelings that one should go deeper into **P**, to the class **L** or **NC**. Interesting enough, at the same time an opposite direction is getting momentum to extend the concept of feasibility beyond **P**, even into **NEXP**—this will be discussed in Chapter 9.

Moral. An experience suggests that once something is shown as impossible, then the next step to do is to start a search how to do it—from a different point of view. A good rule of thumb at complexity analysis is therefore as in real life, to do your best to solve problems you need to solve and when this turns out to be infeasible, then change your problems.

5.12 Exercises —November 8, 2005

1. Suppose that the running time of a NTM is defined to be the length of the longest computation for a given input. Show that if **NP** is defined accordingly, then it is the same class as defined in the text.
2. Show that if a language L is recognized by a $t(n)$-time bounded TM, then L is recognized by infinitely many $t(n)$-time bounded TMs.
3. * Show that 2-CNF problem is in **P**.
4. Show that the problem to decide whether a Boolean formula in DNF is satisfiable is in **P**.
5. Show that the problem to decide whether a Boolean formula in CNF is tautology is in **P**.
6. Show that the problem to decide whether a Boolean formula in DNF is a tautology is **co-NP**-complete.
7. Show that a language $L \in \mathbf{NP}$, if and only if there is a language $L_0 \in P$ (called also a witness of L), a separator \$, and a polynomial p such that $x \in L \Leftrightarrow \exists y\, |y| \leq p(|x|), x\$y \in L_0$.
8. A clause is called monotone if it consists entirely of variables (e.g. $x \vee y \vee z$), or entirely of negations of variables. Show the **NP**-completeness of the following language MONOTONE-CNFF: a set of satisfiable Boolean formulas all clauses of which are monotone.
9. Show that the following HITTING-SET problem is **NP**-complete: given a family $\mathcal{F}$ of finite sets and a $k \in \mathbf{N}$, is there a set with at most k elements intersecting every set in $\mathcal{F}$?

10. Show that the problem of coloring a graph with two colors is in **P**.
11. Show that CIRCUIT-SAT problem is polynomially reducible to the SAT problem, where CIRCUIT-SAT problem is the problem to decide whther a given Boolen circuithas a satisfying asignment.
12. Show that DOMINATING-SET problem is **NP**-complete even for bipartite graphs.
13. * Show that the VERTEX-COVER problem for a graph $G = \langle V, E\rangle$ and an integer k can be solved in time $\mathcal{O}(|E| + k|V|^{k+1})$.
14. * Show the **NP**-completeness of the following ANAGRAM problem: Given a finite multiset $\mathcal{S}$ of strings and a string w. Is there a sequence $w_1, \ldots, w_s$ of strings from $\mathcal{S}$ such that w is a permutation of the string $w_1 \ldots w_s$?
15. Show that MAX2SAT problem is **NP**-complete. (It is the following problem: given a set of clauses, each with 2 literals, and an integer k, decide whether there is a satisfying assignment that satisfies at least k clauses.) (Hint: consider the clauses x, y, z, w, $\bar{x} \vee \bar{y}$, $\bar{y} \vee \bar{z}$, $\bar{z} \vee \bar{x}$, $x \vee \bar{w}$, $y \vee \bar{w}$, $z \vee \bar{w}$, and show that an assignment satisfies $x \vee y \vee z$ if and only if it satisfies 7 of the above clauses.)
16. Show, for example, by a reduction from the CLIQUE problem, that the VERTEX-COVER problem is **NP**-complete.
17. Show, for example by a reduction from the VERTEX-COVER problem, that the SET-COVER problem is **NP**-complete. (It is the problem to decide, given a family of finite sets $S_1, \ldots, S_n$ and an integer k whether there is a family of k subsets $S_{i_1}, \ldots, S_{i_k}$ such that $\bigcup_{j=1}^{k} S_{i_j} = \bigcup_{j=1}^{n} S_j$.)
18. Denote by NAE3-CNFF the problem to decide, given a Boolean formula in 3-CNF form whether there is such a satisfying assignment that in none of the clauses all three literals have the same truth value (NAE stands for "not-all-equal"). (a) Show that the problem NAE3-CNFF is **NP**-complete; (b) Use the **NP**-completeness of NAE3-CNFF problem to show the **NP**-completeness of 3-COLORABILITY problem (to decide whether a given graph is colorable with three colours.)
19. Show that the INDEPENDENT-SET problem is **NP**-complete, for example, by a reduction from 3-CNF. (It is the following problem: Given a graph $G = \langle V, E\rangle$ and $I \subseteq V$, I is said to be independent, if for no $i, j \in I, i \neq j$, $(i, j) \in E$. Given, in addition, an integer k, the task is to decide whether G has an independent set of size at least k.)
20. Use the **NP**-completeness of the INDEPENDENT-SET problem to show that: (a) the CLIQUE problem is **NP**-complete; (b) the VERTEX-COVER problem is **NP**-complete.
21. Show, for example, by a reduction from 3-CNFF, that the TRIPARTITE-MATCHING problem is **NP**-complete. (Given three sets B (boys), G (girls), and H (homes), each containing n elements, and a ternary relation $T \subseteq B \times G \times H$, find a set of n triples

from T such that no two have a component in common. (That is, each boy is matched with a different girl, and each couple has a home of its own.)

22. Use the **NP**-completeness of the TRIPARTITE-MATCHING problem to show the **NP**-completeness of the SET-COVER problem. (Hint: Show that for those graphs nodes of which can be partitioned into disjoint triangles.)

23. Show the **NP**-completeness of the SET-PACKING problem. (Given a family of subsets of a finite set U and an integer k. Decide whether there are k pairwise disjoint sets in the family.)

24. Show that the BIN-PACKING problem is **NP**-complete. (For example, by a reduction from the TRIPARTITE-MATCHING problem.) (BIN-PACKING problem: Given n positive integers $a_1, \ldots, a_n$ (the objects), and two additional integers c (the capacity) and b (the number of bins), decide whether the set of integers can be partitioned into b subsets in such a way that each of them has the total sum at most c.)

25. Show, for example by a reduction from the VERTEX-COVER problem, the **NP**-completeness of the DOMINATING-SET problem: given a directed graph $G = \langle V, E \rangle$, and an integer k. Is there a set D of k or fewer nodes such that for each $v \in V - D$ there is a $u \in D$ such that $(u, v) \in E$?

26. Show that the SUBGRAPH-ISOMORPHISM problem is **NP**-complete (for example by a reduction from the CLIQUE problem).

27. Show: (a)3-COLORABILITY problem is **NP**-complete; (b) 2-COLORABILITY problem is in **P**.

28. Show that the following Diophantine equation problem is **NP**-complete: decide whether a system of equations $Ax \leq b$ (with A being an integer matrix and b an integer vector) has an integer solution. (Hint: Use a reduction from 3-CNFF problem.)

29. Show that if f is a polynomially computable and honest function, then there is a polynomial time NTM $\mathcal{M}$ which accepts exactly the range of f and such that for an input y, every accepting computation of $\mathcal{M}$ outputs a value x such that $f(x) = y$.

30. * Design a linear time algorithm to construct an optimal vertex-cover for a tree.

31. Consider the MINIMUM-VERTEX-COLORING problem (to determine the chromatic number of the graph). (a) Show that unless $\mathbf{P} = \mathbf{NP}$ the approximation threshold of the problem can not be larger than $\frac{1}{4}$ (Hint: Use the fact that 3-COLORING is **NP**-complete.); (b) show that the asymptotic approximation threshold (see Exercise 5.7.10) can not be smaller than $\frac{1}{4}$ (Hint: Replace each node by a clique.)

32. (a)* Show that the heuristic for the VERTEX-COVER problem from Section 5.7.2 never produces a solution that is more than $\lg n$ times the optimum; (b) find the family of graphs for which the $\lg n$ bound can be achieved in the limit.

33. Show that the approximation threshold for the minimization version of the BIN PACKING problem is at least $\frac{1}{3}$.

34. * Show that the greedy approximation algorithm for the VERTEX COVER problem presented in Section 5.7 never produces a solution which is more than $(\lg n)$-times the optimum.

35. Design an approximation algorithm for the SET COVER problem, as good as you can, and estimate its approximation treshold.

36. ** Show that a language $L \in \mathbf{RP}$ if and only if there is a language $L_0 \in P$, called also a witness language for L, and a polynomial p such that $x \in L \Leftarrow \exists y\, |y| \leq p(|x|)$, $x\$y \in L_0$ and if $x \in L$, then at least half of the words of length $f(|x|)$ are such that $x\$y \in L_0$.

37. Show that the following problem is in **BPP**: Given polynomials $p_i(x_1, \dots, x_n), 1 \leq i \leq m, g_j(x_1, \dots, x_n), 1 \leq j \leq n$, decide whether $\prod_{i=1}^{m} p_i(x_1, \dots, x_n) = \prod_{j=1}^{n} g_j(x_1, \dots, x_n)$.

38. Show that $\mathbf{NP} \subseteq \mathbf{BPP}$ implies $\mathbf{NP} = \mathbf{RP}$.

39. Show that the class **PP** is closed under: (a) complement; (b) symmetric difference; (c) $\leq_p^m$ reducibility.

40. Show that the class **BPP** is closed under (a) complement; (b) union; (c) intersection.

41. Show that the class **RP** is closed under: (a) union; (b) intersection.

42. Show that the classes **RP** and **BPP** are closed unded $\leq_p^m$ reducibility.

43. A **Boolean randomizeed circuit** is one which has, in addition to its standard inputs also the so-called random inputs. If these random inputs are drawn from a uniform distribution, the output of a circuit is a random variable. A random Boolean circuit with standard input $x_1, \dots, x_n$ computes a Boolean function $f(x_1, \dots, x_n)$ if $Pr(C(x_1, \dots, x_n) = f(x_1, \dots, x_n)) \geq \frac{3}{4}$ for all $x_1, \dots, x_n$. Show that if a Boolean function can be computed by a randomized circuit of size $s(n)$, then it can be computed by a usual (deterministic) circuit of size $\mathcal{O}(ns(n))$.

44. Show that any problem in NC^k can be solved by an EROW PRAM in $\mathcal{O}(\lg n)^k$ time using $n^{\mathcal{O}(1)}$ processors.

QUESTIONS

1. Why is nondeterministic space not essentially more powerful as deterministic space?

2. How can we introduce nondeterminism into circuits?

3. Main computational complexity classes have been defined in terms of Turing machines. Could they be defined also using other models of computers?

4. Can the reducibilities introduced in Section 5.3 be ordered with respect to their power?

5. Is there an essential difference betwee direct and indirect methods to proove **NP**-completeness?
6. Which ways to attack the $\mathbf{P} = \mathbf{NP}$ problem seem to be feasible?
7. How can the existence of complete problems contribute to the design of efficient algorithms?
8. How to describe informally main randomized complexity classes?
9. What does it imply when a complexity class has no complete problems?
10. Why it is uneasy to define complexity classes $\mathbf{NC}^i$ in terms of PRAM?

5.13 Historical and Bibliographical References

Foundations of computational complexity and a systematic investigation of time and space complexity of Turing machine computations were laid out in papers by Hartmanis and Stearns (1965) and Hartmanis, Lewis and Stearns (1965). For these contributions and further development of computational complexity Hartmanis and Stearns received the Turing award in 1993. For a personal account of the first developments in computational complexity see Hartmanis (1979).

From the earlier attempts to pose and solve computational complexity problems of a special interest is a letter of Gödel to von Neumann in 1956, see Sipser (1992), in which Gödel asked von Neumann about his views on a possibility to solve in linear or quadratic time, on a TM, a problem known today to be **NP**-complete. From others less formal and direct attempts to deal with computational complexity issues see Rabin (1960) and Trakhtenbrot (1964).

Observe that it is very natural that a systematic investigation of computational complexity started in 1960s. The borderline between algorithmically solvable and unsolvable problems has already been quite well understood at that time and so were methods to show unsolvability. On the other hand the growing use of more and more powerful computers for more and more complex tasks and to larger and largewr inputs naturally brought the question of computational complexity and efficiency as being of prime interest and importance.

There is a variety of books where computational complexity problems are presented in detail, especially Papadimitriou (1994), Reischuk (1990), Balcázar, Díaz and Gabáro (1988), Wagner and Wechsung (1986), Papadimitriou and Steiglitz (1982), Lewis and Papadimitriou (1981). See also Cormen, Leiserson and Rivest (1990), Aho, Hopcroft and Ullman (1974), Hopcroft and Ullman (1969) and an extensive survey of complexity classes by Johnson (1990a). A less formal but very illuminating discussion of complexity problems and results is due to Harel (1987).

Theorem 5.1.7 is due to Savitch (1970). Time and space hierarchies summarized in Theorem 5.2.2 were established for deterministic computations in the above mentioned pioneering papers by Hartmanis et al and for nondeterministic computations by Cook (1973) and Seiferas, Fischer and Meyer (1973)—see also Aho, Hopcroft and Ullman (1974) for a presentation of proofs. The result that space is strictly more powerful than time, presented on page 350, is due to Hopcroft, Paul and Valiant (1975). The gap theorem is due to Trakhtenbrot (1964)—see also Papadimitriou (1994). For Blum's Speed-up Theorem (in much stronger form) see Blum (1971).

There are many concepts of reducibility. Some of them have been very intensively studied already in the theory of recursive functions, see Rogers (1967). For the study of resource bounded reductions see Ladner, Lynch and Selman (1975) and Balcázar, Díaz and Gabárro (1988).

Another key idea of computational complexity, namely to consider the class **P** as the one that is both of theoretical and practical importance and the class **NP** as its natural extention for the study of computational complexity, appeared informally in several papers, especially in Cobham (1965), Edmonds (1965, 1966) and Rabin (1966). Several citations from those and other papers, including the already mentioned letter of Gödel to von Neumann, collected by Sipser (1992) well demonstrate how an awarenes emerged of the importance of the class **P** and also of the needs to try to solve efficiently those problems that seem to be solvable only by a brute force—by an exhaustive search.

The concept of **NP**-completeness, another very basic concept of computational complexity was introduced by Cook (1971) and it was Karp (1972) who convincingly demonstrated the true wealth of **NP**-completeness. (Both of them received Turing awards, Cook in 1982 and Karp in 1985.) See also Levin (1973). The book by Garey and Johnson (1979) containing a compendium of **NP**-complete problems and the on-going Johnson's column on the **NP**-completeness in the Journal of Algorithms have also significantly contributed to a broad study of **NP**-completeness. A variety of **NP**-complete problems can be found in the above references and in many books on computational complexity and foundations of computing. Cook (1971) was the first one to precisely formulate the $\mathbf{P} = \mathbf{NP}$ problem.

NP-completeness of the satisfiability problem is due to Cook (1971)—the proof presented here is due to Aho, Hopcroft and Ullman (1974). Satisfiability became the first problem that has been heavily used to derive other **NP**-completeness results. The idea to use the bounded tiling problem and other versions of tiling problems to prove **NP**-completenss and other types of completeness came from Lewis (1978) and was further developed by Lewis and Papadimitriou (1981) and especially by van Emde Boas (1982) and Savelsbergh and van Emde Boas (1984).

Strong **NP**-completeness is usually defined in a more general way: if a problem remains **NP**-complete even if any instance of length n is restricted to contain integers of size at most $p(n)$, where p is a fixed polynomial. For an analysis of this concept see, for example, Garey and Johnson (1988) and Papadimitriou (1994). For results how a compressed (succinct) description of inputs can essentially change complexity of **NP**-complete problems, see Papadimitriou (1994), Galperin and Wigderson (1983), and Balcázar, Lozano and Toran (1992).

The concept of the average-case complexity is due to Levin (1986). For a more detailed study of this concept see Gurevitch (1991), Ben-David, Chor and Goldreich (1992) and Impagliazzo (1995). Our presentation is derived from Papadimitriou (1994) and Gurevitch (1991).

Also in the RRAM framework we can define in a natural way the concept of the complexity class **P**, and using the idea of certificates also of the complexity class **NP** (for computation over reals). Moreover, one can also define **NP**-completeness. **NP**-completeness for computations over reals was introduced by Blum, Shub and Smale (1989). (An example of an **NP**-complete problem for computation over reals: to decide whether a given polynomial of n variables and of a degree at most 4, has real roots.)

There are also several fast randomized prime recognition algorithms: Rabin's algorithm (Section 8.3.4), Solovay-Strassen's algorithm (1978) (both algorithms are actually Monte Carlo algorithms for integer factorization), Rumley-Adleman's Las Vegas algorithm(see Adleman (1980)) that terminates in time $\mathcal{O}((\lg n)^{\lg\lg\lg n})$ and either provides no solution or a correct one, and a polynomial time Las Vegas algorithm due to Adleman-Hung's (1984).

The concept of certificates for **NP**-complete problems and Theorem 5.6.2 are implicit in Edmonds' papers. For a rich history of the $\mathbf{P} = \mathbf{NP}$ problem see Sipser (1992), and for the proofs and references of various statements of the type $\mathbf{P} = \mathbf{NP}$ iff ... see Papadimitriou (1994) and Balcázar, Díaz, and Garbárro (1988). The idea of the class **FP** goes also back to Cobham (1964) and Edmonds (1965). Our presentation of this subject is based on Papadimitriou (1994). The class **UP** was introduced by Valiant (1976) and Theorem 5.6.20 is due to Grollman and Selman (1988). Theorem 5.6.6 on relativization of the $\mathbf{P} = \mathbf{NP}$ problem is due to Baker, Gill and Solovay (1975) and a large body of results on relativization currently available, including those from Remark 5.6.8, is presented by Balcázar, Díaz and Gabárro (1988).

A compendium of **P**-complete problems and also an analysis of the rôle **NP**-completeness play for seeing the limitations of parallel computing is due to Greenlaw, Hoover and Ruzzo (1993). **P**-completeness of the rational linear programming problem is due to Dobkin, Lipton and Reiss (1979); **NP**-completeness of integer linear programming is in Garey and Johnson (1979). **P**-completeness of the CIRCUIT-VALUE problem is due to Ladner (1975). The connection between **P**-completeness and inherently sequential problems was first explored by Goldschlager (1977).

The class of unambiguous Turing machines was introduced by Valiant (1976) and their relation to one-way functions was estblished by Grollman and Selman (1988). The class **#P** was introduced by Valiant (1979) and the proof of **#P**-completeness for PERMANENT is due to him and Zankó (1991). For Toda's Theorem see Toda (1989). For more on **#P**-completeness see Papadimitriou (1994).

A systematic presentation of primality testing algorithms is by Kranakis (1986). The proof that prime recognition is in **P** provided the Generalized Riemann Hypothesis holds is due to Miller (1976) and that it is in **NP** is due to Pratt (1975). The fastest known deterministic algorithm for prime recognition is due to Adleman, Pomerance and Rumely (1983). For numerical verification of Rieman Hypothesis see Odlyzko (1988) and for numerical verification of Generalized Rieman Hypothesis see Rumely (1993).

Attempts to solve **NP**-complete problems by approximation are about as old as **NP**-completeness itself. Johnson (1974) and Garey and Johnson (1979) provide the first systematic presentation of approximation algorithms. For a more detailed presentation and analysis of some approximation problems and newer results on nonapproximability see Cormen, Leiserson and Rivest (1990), Papadimitriou (1994) and Babai (1995).

The vertex-cover randomized algorithm presented on page 385 is due to Gavril (1977) and the TSP approximation algorithm sketched in Exercises 5.7.13 and 5.7.14 is due to Rosenkrantz, Stearns and Lewis (1977). Proofs of Theorems 5.7.6 and 5.7.12 follow Papadimitriou (1994). The first result, for the SET-COVER problem, showing that approximation threshold is 1 unless $\mathbf{P} = \mathbf{NP}$ is due to Lund and Yannakakis (1993). Theorem 5.7.16 is due to Arora, Lund et al (1992). For an overview of intensive experimental work on solving and approximation of TSP of large size see Johnson and McGeoch (1996) and Johnson, McGeoch and Rothberg (1996).

Randomization was formally introduced by Rabin (1976) and by Solovay and Strassen (1977) as a tool to get more efficient algorithms. However, they were randomized algorithms on primality testing, due to Rabin (1976), Miller (1976) and Soloway with Strassen (1977), all invented in 1975, what made randomization a central feature in complexity theory. The formal study of randomized complexity classes started with Gill (1977), who introduced the classes **RP, PP, ZPP** and **BPP**. For more on randomized complexity classes see Balcázar, Díaz and Gabárro (1988), Papadimitriou (1994) and Johnson (1990a). Theorem 5.9.10 is due to Kotsoupias and Papadimitriou (1992). For a detailed treatment of randomized algorithms see Montwani and Raghavan(1995)

The class **NC** was introduced by Pippenger (1979) and in detail studied by Cook (1981, 1985). For more about parallel complexity classes see Reischuk (1990) and Papadimitriou (1994). For a summary of evidences that $\mathbf{NC} \neq \mathbf{P}$ see Greenlaw, Hoover and Ruzzo (1995).

The polynomial hierarchy was introduced by Stockmeyer (1976). The proof that QSAT_k is a complete problem for $\mathbf{\Sigma_k}$ is due to Wrathall (1976) and so is the result that if **PH** has a complete problem then it collapses. The collapse of the polynomial hierarchy if the graph isomorphism problem is **NP**-complete is due to Goldwasser and Sipser and for the case that SAT has a polynomial size circuit is due to Karp and Lipton (1980).

The first **PSPACE**-completeness results, for the quantified satisfiability and the equivalence of regular expressions, were due to Stockmeyer and Meyer (1973). Schäfer (1978) showed the **PSPACE**-completeness of some games including GENERAL GEOGRAPHY. The **PSPACE**-completeness of GO is due to Lichtenstein and Sipser (1980). Papadimitriou (1985, 1994) showed the **PSPACE**-completeness of various problems of decision making under uncertainity.

For a survey of results on exponential complexity classes and their complete problems see Papadimitriou (1994) and Johnson (1990a). Results on the completeness of various regular expression equivalence problems are due to Stockmeyer and Meyer (1973).

Chapter 6

COMPUTABILITY

INTRODUCTION

The search for the ultimate limitations of mind and machines brought a variety of concepts, methods and results of fundamental philosophical, theoretical and practical importance. This chapter deals with some of them. Firstly, it explores basic concepts and methods concerning limitations of effective or algorithmic methods—that is, ultimate limitations of machines. We discuss which problems are algorithmically unsolvable or undecidable (and how to show that), which functions are computable and which sets are recognizableor acceptable. Secondly, we explore one of the main products of mind, formal systems, and show their limitations to proof correct statement and to show randomness of strings. Finally, a magic "number of wisdom" is defined and analysed, that contains in a very compressed form a huge amount of knowledge.

In this search for limitations of mind and machines we present various concepts and methods of broader importance for computing and also outside of it. For instance, Kolmogorov and Chaitin concepts of descriptional complexity.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. basic concepts and relations concerning recursive and recursively enumerable sets, as well as concerning partial and primitive recursive functions,
2. the fundamental concept of undecidability and main examples of undecidable problems: halting problem, tiling problems, Post Correspondence Problem, Thue Problem and Hilbert's Tenth Problem,
3. a general concept of formal systems, Kolmogorov/Chaitin complexity and results, including Gödel Incompleteness Theorem, showing surprising limitations of formal systems to prove correctness of theorems and randomness of strings,
4. magic "number of wisdom" and its implications.

> When you have eliminated the impossible, whatever remains, however improbable, must be the truth.
>
> *Arthur Conan Doyle (1859-1930)*

The search for its own limitations is one of the main characteristics of science of this century. Discovering the power and limitations of physical systems, machines, mind, knowledge and formal systems is an intellectually exciting and practically important task.

The concept of unsolvability is one of the cornerstones of modern science, computing, and mathematics and its investigations lead to the questions of basic philosophical importance.

In this chapter we deal with several fundamental problems concerning unsolvability and other limitations of machines, formal systems and knowledge. Which problems are ultimately solvable by machines or within a framework of a formal system and which are not? How to prove that some problems are beyond the power of machines or formal systems—no matter how powerful or good they are? Where are the borderlines between solvability and unsolvability?

On a more technical level these fundamental philosophical problems can be reduced to problems concerning sets, functions, numbers, strings and theorems. Which sets are decidable or enumerable? Which functions are computable? Which numbers are computable? Which theorems are provable? Which strings are random?

All these problems are analysed in this chapter. We deal with basic concepts, methods and results concerning unsolvability, solvability and their borderlines, including basic theorems of Gödel and Chaitin concerning the limitations of formal systems.

We shall also learn several surprising methods and results. For example, we shall see that self-reference is a basic tool not only for design of algorithms but also for determining limitations of computing. We shall also learn that a computer independent view of computations is possible.

The results and methods presented in this chapter demonstrate that there are simple methods, in principle, to grasp what is graspable and to show that one cannot embrace inembracable.

The results of this chapter imply that there are algorithmic problems that are not algorithmically solvable, correct theorems that are not provable and random strings randomness of which cannot be proven, no matter how much computer resources and brain mankind has.

6.1 Recursive and Recursively Enumerable Sets — November 8, 2005

The concepts of recursive and recursively enumerable sets play the key rôle in the search for borderlines between solvability and unsolvability. To start with we describe and analyse these concepts for languages—sets of strings. We discuss also how to generalize these concepts to deal with recursivity of other objects.

As defined in Section 4.1.2, a language L is recursive (recursively enumerable) if there is a TM that recognizes/decides (accepts) L. This immediately implies that each recursive language is also recursively enumerable.

Theorem 6.1.1 shows that there is an alternative way to see recursively enumerable sets—through a generation/enumeration. We say that an off-line TM $\mathcal{M}$ generates (enumerates) a language $L \subseteq \Sigma^*$, if $\mathcal{M}$, starting with the empty input tape, writes on its output tape, symbol by symbol, a word $w_L = w_1\#w_2\#w_3\#\ldots$ where $w_i \in \Sigma^*$, $\# \notin \Sigma$, $w_i \neq w_j$, for $i \neq j$, and $L = \{w_i \,|\, 1 \leq i\}$. (In short, $\mathcal{M}$ generates all words of L, none twice, and no other words.)

Theorem 6.1.1 *A language L is recursively enumerable if and only if L is generated by an off-line Turing machine.*

Proof. Let L be a recursively enumerable language and $L = L(\mathcal{M}) \subseteq \Sigma^*$ for a TM $\mathcal{M}$. We show how to design an on-line TM $\mathcal{M}'$ generating words of L, each of them exactly once.

$\mathcal{M}'$ produces, starting with the pair $(0,0)$, all pairs $(i,j) \in \mathbf{N}^2$, one after another, with respect to the ordering $\ll$ defined as follows:

$$(i_1,j_1) \ll (i_2,j_2) \Longleftrightarrow i_1 + j_1 < i_2 + j_2 \vee (i_1 + j_1 = i_2 + j_2) \wedge (j_1 < j_2).$$

For each newly created pair (i,j), $\mathcal{M}'$ generates the i-th word w_i from Σ^* (with respect to the strict ordering), and then simulates exactly j steps of $\mathcal{M}$ on w_i. If $\mathcal{M}$ accepts w_i in exactly j steps, then $\mathcal{M}'$ generates w_i by writing $w_i\#$ on its output tape. If not, $\mathcal{M}'$ does not write anything on its output tape. In both cases, as the next step, $\mathcal{M}'$ generates the next pair (i',j'), with respect to the ordering $\ll$ and the whole process continues. In this way $\mathcal{M}'$ simulates, sooner or later, each terminating computation of $\mathcal{M}$, and generates all those words $\mathcal{M}$ accepts, and each one exactly once.

In order to prove the opposite implication let us assume that $L \subseteq \Sigma^*$ is a language generated by a k-tape off-line TM $\mathcal{M}$. We show how to design a TM $\mathcal{M}'$ that accepts L. $\mathcal{M}'$ will use k tapes to simulate k work tapes of $\mathcal{M}$ and two additional work tapes to simulate the input and output tapes of $\mathcal{M}$. For a given input word w, $\mathcal{M}'$ begins simulation of a computation of $\mathcal{M}$ that starts with the empty input tape. Each time $\mathcal{M}$ writes on its output tape a word of the type $\#w'\#$, with $w' \in \Sigma^*$, $\mathcal{M}'$ compares w' with w. If they agree, $\mathcal{M}'$ accepts w, otherwise $\mathcal{M}'$ keeps simulating $\mathcal{M}$. □

Exercise 6.1.2 *Show that if an off-line TM outputs on the empty tape an infinite word $w = w_1\#w_2\#w_3\ldots$, where $w_i \in \Sigma^*, \# \notin \Sigma$, then the language $L = \{w_i \,|\, 1 \leq i\}$ is recursively enumerable. (The condition that each word can be generated only once is relaxed.)*

Exercise 6.1.3 *Show that the set of all TM that halt on at least two inputs is recursively enumerable.*

There is the following close relation between recursive and recursively enumerable sets.

Theorem 6.1.4 *A language $L \subseteq \Sigma^*$ is recursive if and only if L and its complement L^c are recursively enumerable.*

Proof. If L is a recursive language, there is a TM $\mathcal{M}$ that recognizes L. By exchanging the accepting and rejecting states in the description of $\mathcal{M}$ we get a TM that recognizes the complement L^c. Hence both L and L^c are recursive, and therefore also recursively enumerable.

On the other side, if both L and L^c are recursively enumerable, then there are TMs $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $L(\mathcal{M}_1) = L$ and $L(\mathcal{M}_2) = L^c$. From these two TMs we can design in the following way a TM $\mathcal{M}$ that recognizes L. $\mathcal{M}$ simulates, for a given input w, both $\mathcal{M}_1$ and $\mathcal{M}_2$ in parallel—that is, $\mathcal{M}$ computes, for $i = 1, 2, \ldots$, the i-th step of $\mathcal{M}_1$ and immediately the i-th step of $\mathcal{M}_2$. One of these two machines has to terminate and accept. If $\mathcal{M}_1$ accepts first, then so does $\mathcal{M}$; otherwise, if $\mathcal{M}_2$ accepts first, $\mathcal{M}$ rejects w. □

Exercise 6.1.5 *Show that the family of recursive languages is closed under Boolean set operations (union, intersection and complement).*

As one can expect, not every recursively enumerable set is recursive and there are sets that are not even recursively enumerable. We demonstrate this by two examples.

Let us consider a fixed Gödel encoding of all TMs into an alphabet Γ, see Section 4.1, and let $\mathcal{M}_1, \mathcal{M}_2, \ldots$ be an enumeration of all TMs over the alphabet Γ ordered with respect to the strict ordering of their codes. Moreover, let $w_1, w_2, \ldots$ be the strict ordering of all words over Γ. We use these orderings to define two languages:

$$K = \{w_i \,|\, w_i \in L(\mathcal{M}_i)\}, \qquad\qquad K^c = \{w_i \,|\, w_i \notin L(\mathcal{M}_i)\}.$$

Theorem 6.1.6 (Post's Theorem) *The language K is recursively enumerable but not recursive and the language K^c is not recursively enumerable.*

Proof. We show first that K is recursively enumerable but K^c is not. By Theorem 6.1.4, this implies that K is not recursive.

To show that K is recursively enumerable we design a TM $\mathcal{M}$ that accepts K. For a given input w, $\mathcal{M}$ computes i such that $w = w_i$, then designs $\mathcal{M}_i$ and simulates $\mathcal{M}_i$ on w_i. If $\mathcal{M}_i$ accepts w_i so does $\mathcal{M}$.

In order to show that the language K^c is not recursively enumerable we use the diagonalization method and consider the infinite Boolean matrix M such that $b_{i,j} = M(i,j) = 1$ if and only if $w_i \in L(\mathcal{M}_j)$. Hence the language K^c corresponds to 0's in the diagonal of M:

$$\begin{array}{c|c|c|c|c|c}
 & \mathcal{M}_1 & \mathcal{M}_2 & \mathcal{M}_3 & \mathcal{M}_4 & \ldots \\
\hline
w_1 & b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & \ldots \\
\hline
w_2 & b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & \ldots \\
\hline
w_3 & b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & \ldots \\
\hline
w_4 & b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & \ldots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

If K^c is recursively enumerable, then there must exist a j_0 such that $L(\mathcal{M}_{j_0}) = K^c$. There are now two possibilities for w_{j_0}: either $w_{j_0} \in K^c$ or $w_{j_0} \notin K^c$. In both cases we derive a contradiction as follows:

$$w_{j_0} \in K^c \Rightarrow w_{j_0} \in L(\mathcal{M}_{j_0}) \Rightarrow w_{j_0} \notin K^c,$$

$$w_{j_0} \notin K^c \Rightarrow w_{j_0} \notin L(\mathcal{M}_{j_0}) \Rightarrow w_{j_0} \in K^c.$$

This contradiction implies that our assumption, namely that K^c is recursively enumerable, is false. □

The concepts of recursivity and recursive enumerability are often generalized and used also to characterize sets of other objects than numbers, elements of which can be encoded in a natural way by strings. The basic idea of these generalizations is that a set is recursive (recursively enumerable) if so is the set of all encodings of its elements. For example, in this sense we can speak of recursive and recursively enumerable relations. As another example, one can show that the set of all minimal solutions of the Firing Squad Synchronization Problem is not recursively enumerable.

Exercise 6.1.7 *Prove that a set $L \subseteq \mathbf{N}$ is recursive iff it can be enumerated in an increasing order by some TM.*

Exercise 6.1.8 *Show that a language L is recursively enumerable if and only if there is a recursive relation R such that $x \in L \equiv \exists y[(x, y) \in R]$.*

Remark 6.1.9 There are several other types of sets that one encounters quite often in the theory of computing. For example, productive, creative, immune and simple sets. Since they are easy to define and one should have at least a basic knowledge about them, we introduce these concepts even though we shall not explore them.

In order to define these sets let us observe that an effective enumeration of all TM induces an effective enumeration of all recursively enumerable sets (accepted by these TM). Therefore once a fixed encoding and ordering of TM is taken we can talk about the i-th recursively enumerable set S_i.

A set S is called **productive** if there is a recursive function g such that whenever $S_i \subsetneq S$, then $g(i) \in S - S_i$. A set $S \subseteq \Sigma^*$ is **creative** if S is recursively enumerable and its complement S^c is productive. (For example, the set K is creative.) A set $S \subsetneq \Sigma^*$ is **immune** if S is infinite and it has no recursively enumerable infinite subset. A set $S \subseteq \Sigma^*$ is **simple** if it is recursively enumerable and its complement is an immune set.

Exercise 6.1.10 *Show that every infinite recursively enumerable set has an infinite recursive subset.*

Exercise 6.1.11 *Show that if A and B are recursively enumerable sets, then there are recursively enumerable subsets $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cap B' = \emptyset$ and $A' \cup B' = A \cup B$.*

6.2 Recursive and Primitive Recursive Functions

Two families of functions, that can be defined inductively and in a machine independent way, play a special rôle in computing: primitive recursive and partial recursive functions. They are usually defined as functions from integers to integers. However, this can be generalized, for example, to strings-to-string functions, as shown later. The most important outcome of this approach is the knowledge that all computable functions have a closed form and the method how to obtain this closed form.

6.2.1 Primitive Recursive Functions

The family of primitive recursive functions contains practically all functions we encounter and can expect to encounter in practical computing. The basic tool to define these functions is the operation of primitive recursion—a generalization of recurrences we considered in Chapter 1.

Definition 6.2.1 *The family of primitive recursive functions is the smallest family of integers-to-integers functions with the following properties:*

1. *It contains the following* **base functions***:*

   0, (**nullary constant**),
   $S(x) = x + 1$, (**successor function**),
   $U_i^n(x_1, \ldots, x_n) = x_i$, (**projection functions**), *for* $1 \leq i \leq n$.

2. *It is closed under the following operations:*

   - **composition***: if* $h : \mathbf{N}^m \mapsto \mathbf{N}$, $g_1 : \mathbf{N}^n \mapsto \mathbf{N}, \ldots, g_m : \mathbf{N}^n \mapsto \mathbf{N}$ *are primitive recursive functions, then so is the function* $f : \mathbf{N}^n \mapsto \mathbf{N}$ *defined as follows:*

   $$f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)).$$

   - **primitive recursion***: if* $h : \mathbf{N}^n \mapsto \mathbf{N}$, $g : \mathbf{N}^{n+2} \mapsto \mathbf{N}$ *are primitive recursive functions, then so is the function* $f : \mathbf{N}^{n+1} \mapsto \mathbf{N}$ *defined as follows:*

   $$\begin{aligned} f(0, x_1, \ldots, x_n) &= h(x_1, \ldots, x_n), \\ f(z+1, x_1, \ldots, x_n) &= g(z, f(z, x_1, \ldots, x_n), x_1, \ldots, x_n), \qquad z \geq 0. \end{aligned}$$

The following examples illustrate how to construct a primitive recursive function using operations of composition and primitive recursion.

Example 6.2.2 *Addition:* $a(x,y) = x + y$*:*

$$\begin{array}{rcl} a(0,y) & = & U_1^1(y); \\ a(x+1,y) & = & S(U_2^3(x, a(x,y), y)). \end{array}$$

Example 6.2.3 *Multiplication:* $m(x,y) = x \cdot y$

$$\begin{array}{rcl} m(0,y) & = & 0; \\ m(x+1,y) & = & a(m(x,y), U_1^2(x,y)) \end{array}$$

Example 6.2.4 *Predecessor* $P(x) = x \dot{-} 1:$

$$\begin{array}{rcl} P(0) & = & 0; \\ P(x+1) & = & U_1^1(x). \end{array}$$

Example 6.2.5 *Nonnegative subtraction:* $\alpha(x,y) = x \dot{-} y:$

$$\begin{array}{rcl} \alpha(x,0) & = & U_1^1(x); \\ \alpha(x,y+1) & = & P(x \dot{-} y). \end{array}$$

Exercise 6.2.6 *Determine for Examples 6.2.2— 6.2.5 what are the functions* h *and* g *and explain why we have used the function* $U_1^1(y)$ *in Examples 6.2.2 and 6.2.4 and the function* $U_1^2(x,y)$ *in Example 6.2.3.*

Exercise 6.2.7 *Show that the following functions are primitive recursive:*

(a) exponentiation; (b) factorial.

Exercise 6.2.8 *Show that if* $f : \mathbf{N}^{n+1} \mapsto \mathbf{N}$ *is a primitive recursive function, then so are the following functions of arguments* $x_1, \ldots, x_n$ *and* z*:*

$$(a) \quad \sum_{y \leq z} f(x_1, \ldots, x_n, y) \qquad (b) \quad \prod_{y \leq z} f(x_1, \ldots, x_n, y).$$

The concept of primitive recursivity can be extended to deal also with other objects than integers-to-integers functions. For example, a set of integers or a predicate on integers are called primitive recursive if their characteristic functions are primitive recursive. We can also talk about primitive recursivity of other types of functions.

Exercise 6.2.9 *Show that the following predicates are primitive recursive:*

(a) $x \leq y$; *(b)* $x = y$.

Exercise 6.2.10 *Show that the family of primitive recursive predicates is closed under Boolean operations.*

Example 6.2.11 (Primitive recursivity of strings-to-strings functions) *There are two ways how to generalize the concept of primitive recursivity for strings-to-strings functions. An indirect one, in which a simple bijection between strings over an alphabet and integers is used. A direct one that we now use for strings-to-strings functions over the alphabet* $\{0,1\}$.

Base functions*:* $E(x) = \varepsilon$*—the empty string function, two successor functions* $S_0(x) = x0$ *and* $S_1(x) = x1$*, and the projection functions* $U_i^n(x_1, \ldots, x_n) = x_i$*,* $1 \leq i \leq n$.

Operations*: composition and the primitive recursion defined as follows:*

$$\begin{aligned} f(\varepsilon, x_1, \ldots, x_n) &= h(x_1, \ldots, x_n), \\ f(y0, x_1, \ldots, x_n) &= g_0(y, f(y, x_1, \ldots, x_n), x_1, \ldots, x_n), \ , \\ f(y1, x_1, \ldots, x_n) &= g_1(y, f(y, x_1, \ldots, x_n), x_1, \ldots, x_n). \end{aligned}$$

where h, g_0, g_1 *are primitive recursive strings-to-strings functions.*

Exercise 6.2.12 *Show that the following strings-to-strings functions over the alphabet* $\{0,1\}$ *are primitive recursive: (a)* $f(w) = ww$; *(b)* $f(w) = w^R$;

(c) $f(x,y) = xy$.

There is a powerful and elegant theory of computation based much on primitive recursive functions. This is to a large extent due to the fact that we can use primitive recursive pairing and coding functions to reduce the theory of primitive recursive functions of more variables to theory of primitive recursive functions of one-variable.

Example 6.2.13 (Pairing and depairing) *We describe now three primitive recursive bijections*

$$pair : \mathbf{N} \times \mathbf{N} \mapsto \mathbf{N} \quad \textit{and} \quad \pi_1, \pi_2 : \mathbf{N} \mapsto \mathbf{N},$$

with the property

$$\pi_1(pair(x,y)) = x, \ \pi_2(pair(x,y)) = y \ \textit{and}\ pair(\pi_1(z), \pi_2(z)) = z$$

In order to do that let us consider the mapping of pairs of integers into integers shown in Figure 6.1:

$$
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 2 & 5 & 9 & 14 & \nearrow \\
1 & 1 & 4 & 8 & 13 & \nearrow & \\
2 & 3 & 7 & 12 & \nearrow & & \\
3 & 6 & 11 & \nearrow & & & \\
4 & 10 & \nearrow & & & & \\
5 & \nearrow & & & & &
\end{array}
$$

Figure 6.1: Pairing function—matrix representation

Observe first that the i-th counterdiagonal (counting starts with 0), contains numbers corresponding to pairs (x, y) with $x + y = i$. Hence,

$$pair(x, y) = 1 + 2 + \cdots + (x + y) + y.$$

In order to define the "depairing functions" π_1 and π_2, let us introduce an auxiliary function $cd(n)$ = "the number of the counterdiagonal on which the n-th pair lies". Clearly, n and $n+1$ lie on the same counterdiagonal if and only if $n + 1 < pair(cd(n) + 1, 0)$. Therefore, we have

$$
\begin{array}{rcl}
cd(0) & = & 0; \\
cd(n+1) & = & cd(n) + ((n+2) \dot{-} pair(cd(n) + 1, 0)).
\end{array}
$$

Since $\pi_2(n)$ is the position of the n-th pair on the $cd(n)$-th counterdiagonal, and $\pi_1(n) + \pi_2(n) = cd(n)$, we get

$$\pi_2(n) = n \dot{-} pair(cd(n), 0), \qquad \pi_1(n) = cd(n) \dot{-} \pi_2(n).$$

Exercise 6.2.14 *Show, formally, using the definition of primitive recursive functions that the pairing and depairing functions* pair, *π_1 and π_2 are primitive recursive.*

It is now easy to extend the pairing function introduced in the Example 6.2.13 to a function that map, in one-to-one way, n-tuples of integers into integers, for $n > 2$. For example, we can define inductively, for any $n > 2$,

$$pair(x_1, \ldots, x_n) = pair(x_1, pair(x_2, \ldots, x_n)).$$

Moreover, we can use the depairing functions π_1 and π_2 to defined depairing functions $\pi_{n,i}, 1 \leq i \leq n$, such that $\pi_{n,i}(pair(x_1, \ldots, x_n)) = x_i$. This implies that in the study of primitive recursive functions we can restrict ourselves without loss of generality to one argument functions.

Exercise 6.2.15 *Let* $pair(x, y, z, u) = v$. *Show how to express* x, y, z *and* u *as functions of* v, *using the depairing functions* π_1 *and* π_2.

Exercise 6.2.16 *Let us consider the following total ordering in* $\mathbf{N} \times \mathbf{N}$: $(x, y) \prec (x', y')$ *if and only if either* $\max\{x, y\} < \max\{x', y'\}$ *or* $\max\{x, y\} = \max\{x', y'\}$ *and either* $x + y < x' + y'$ *or* $x + y = x' + y'$ *and* $x < x'$. *Denote* $pair_m(x, y)$ *the position of the pair in the ordering defined above. Show that such a pairing function is primitive and so are the depairing functions* π_1^m, π_2^m *such that* $\pi_1^m(pair_m(x, y)) = x$ *and similarly for* π_2^m.

Remark 6.2.17 Primitive recursive functions can also be characterized syntactically in terms of programming constructs. For example, they are exactly functions computable by programs written using the following statements: assignment statements, **for**-statements of the form **for** N **do** S (iterate S for N times), and composed statements.

6.2.2 Partial Recursive and Recursive Functions

Partial recursive functions were introduced in Definition 4.1.2, as functions computed by Turing machines. There is also an alternative way, inductive and machine independent, how to define partial recursive functions and this is now presented.

Theorem 6.2.18 *The family of partial recursive functions is the smallest family of integers-to-integers functions with the following properties:*

1. *It contains the following base functions:*

   0, (**nullary constant**),
   $S(x) = x + 1$, (**successor function**),
   $U_i^n(x_1, \ldots, x_n) = x_i$, (**projection functions**), $1 \leq i \leq n$.

2. *It is closed under the operations of* **composition, primitive recursion** *and* **minimalization**, *defined as follows:*

   if $h : \mathbf{N}^{n+1} \mapsto \mathbf{N}$ *is a partial recursive function, then so is the function* $f : \mathbf{N}^n \mapsto \mathbf{N}$, *where* $f(x_1, \ldots, x_n)$ *is the smallest* $y \in \mathbf{N}$ *such that* $h(x_1, \ldots, x_n, y) = 0$ *and* $h(x_1, \ldots, x_n, z)$ *is defined for all integers* $0 \leq z \leq y$. *Otherwise,* $f(x_1, \ldots, x_n)$ *is undefined.* f *is usually written in the form*

   $$f(x_1, \ldots, x_n) = \mu y\,[h(x_1, \ldots, x_n, y) = 0].$$

To prove Theorem 6.2.18 in one way is pretty easy. All functions constructed from the base functions using composition and minimization are clearly computable and therefore, by

Church's Thesis, partial recursive. In a more formal way, one can design a TM for any of the base functions and one can show how to design for any of the operations involved (composition, primitive recursion and minimization), a Turing machine computing the resulting function under the assumption that component functions are Turing machine computable.

To prove the theorem in the opposite direction is in principle also easy and straightforward but this time a tedious task. One has to show that all concepts concerning Turing machine computations can be arithmetized and expressed using the base functions and the operations of composition, primitive recursion and minimization—in an analogical way as it has been done in the proof of **NP**-completeness of the satisfiability problem for Boolean functions where "Booleanization" of Turing machine computations was used. In doing so the key rôle is played by the generalized pairing and depairing functions. For example, we can assume without loss of generality that states and also tape symbols of Turing machines are integers and that moves (left, right or none), are represented by integers 0, 1 or 2. In such a case each TM-instruction can be represented by a 5-tuple of integers (q, a, i, b, q'), and using the pairing function $pair(q, a, i, b, q') = x$, by a single integer x. In such a case $\pi_{5,1}(x) = q, \pi_{5,2}(x) = a$ and so on. This way one can express a sequence of TM-instructions by one number and to show that the predicate $TM\text{-}Program(x)$, that determines whether x corresponds to a valid TM-program, is primitive recursive. On this base one can express all functions and predicates specifying a Turing machine computation as recursive functions and predicates. A detailed proof can be found, for example, in Smith (1994).

Remark 6.2.19 Observe that the only effective way of computing $f(x_1, \ldots, x_n)$ for a function f defined by minimalization from h is to compute first $h(x_1, \ldots, x_n, 0)$, then $h(x_1, \ldots, x_n, 1)$,... until the desired value of y is found. Consequently, there are two ways f can be undefined for arguments $x_1, \ldots, x_n$. Firstly, there is no y such that $h(x_1, \ldots, x_n, y) = 0$. Secondly, in case that $h(x_1, \ldots, x_n, y) = 0$ for some y but $h(x_1, \ldots, x_n, z)$ is undefined for some z smaller than the smallest y for which $h(x_1, \ldots, x_n, y) = 0$.

Exercise 6.2.20 * *Show that there is no primitive recursive function* $U : \mathbf{N} \times \mathbf{N} \mapsto \mathbf{N}$ *such that for each primitive recursive function* $h : \mathbf{N} \mapsto \mathbf{N}$ *there is an integer* i_h *such that* $U(i_h, n) = h(n)$.

It is interesting that in the process of arithmetization of Turing machine computations it is enough to use the operation of minimization only once. We can even obtain through such arithmetization the following normal form for partial recursive functions (that represents also another way to show the existence of a universal computer):

Theorem 6.2.21 (Kleene's Theorem) *There exist primitive recursive functions* g *and* h *such that for each partial recursive function* f *of one variable there is an integer* i_f *such that*

$$f(x) = g(\mu y\,[h(x, i_f, y) = 0]).$$

Kleene's theorem shows that the family of partial recursive functions has a universal function. However, this is not the case for primitive recursive functions (see Exercise+6.2.20).

Exercise 6.2.22 * *Show that the following predicates are primitive recursive:*

(a) TM-program(x)—*x is an encoding of a TM; (b)* configuration(x, t)—*x is encoding of a configuration of the Turing machine with encoding t; (c)* comp-step(x, y, t)—*x and y are encodings of configurations of the Turing machine encoded by t and the configuration encoded by y can be obtained from the configuration encoded as x by one step of the TM encoded by t.*

With two examples we illustrate how to use minimization.

Example 6.2.23 $\lfloor\sqrt{x}\rfloor = \mu y\{(y+1)^2 \dot{-} x \neq 0\}$.

Example 6.2.24 $\lfloor\frac{x}{y}\rfloor = \mu i\{i \leq x \wedge (x+1) \leq (i+1)y\}$.

It is the operation of minimization that has the power to create recursive functions that are not primitive recursive. On the other hand bounded minimization discussed in exercises below is a convenient tool to design primitive recursive functions.

Exercise 6.2.25 (Bounded minimization) *Show that if $f : \mathbf{N}^{n+1} \mapsto \mathbf{N}$ is a primitive recursive function, then so is the function $\mu z \leq y\,[f(x_1, \dots, x_n, z) = 0]$ defined to be the smallest $z \leq y$ such that $f(x_1, \dots, x_n, z) = 0$ and $y+1$ if such a z does not exists.*

Exercise 6.2.26 (Bounded minimization) *Show that if $f : \mathbf{N}^{n+1} \mapsto \mathbf{N}$ and $b : \mathbf{N}^n \mapsto \mathbf{N}$ are primitive recursive functions, then so is the function $\mu z \leq b(x_1, \dots, x_n)[f(x_1, \dots, x_n, z) = 0]$ defined to be the smallest $z \leq b(x_1, \dots, x_n)$ such that $f(x_1, \dots, x_n, z) = 0$ and $b(x_1, \dots, x_n) + 1$ otherwise.*

Exercise 6.2.27 *Show that the following functions are primitive recursive:*

(a) the number of divisors of n; (b) number of primes $\leq n$; (c) n-th prime.

One of the main sources of difficulties in dealing with partial recursive functions is due to the fact that partial functions may be undefined for an argument and there is no effective way to know it beforehand. The following technique, called dovetailing, can be helpful to overcome these difficulties in some cases.

Example 6.2.28 (Dovetailing) *Suppose we are given a partial recursive function $f : \mathbf{N} \mapsto \mathbf{N}$, and we wish to find an n such that $f(n)$ is defined. One cannot do that by computing first $f(0)$, then $f(1)$ and so on, because it may happen that $f(0)$ is undefined even* *if* $f(1)$ *is defined and computation of* $f(0)$ *never stops. (Note also that in this case an application of the minimization operation to find the smallest* x *such that* $f(x) = 0$ *fails.) We can overcome this problem using the following approach known as* **dovetailing***:*

1. *Perform one step of computation of* $f(0)$*.*
2. *For* $i = 1, 2, \dots$*, until a computation terminates, perform one next step in computing* $f(0)$*,* $f(1), \dots, f(i-1)$*, and the first step in computing* $f(i)$*—that is if* $i = k$*, then the* $k+1$*-th step of computation of* $f(0)$*,* k*-th step of computation of* $f(1)$*, … and, finally, the first step of computation of* $f(k)$ *are performed.*

Exercise 6.2.29 *Show that a function* $f : \mathbf{N} \mapsto \mathbf{N}$ *is recursive if and only if its graph* $\{(x, f(x) \mid x \in \mathbf{N}\}$ *is recursively enumerable.*

Exercise 6.2.30 *Show that every infinite recursively enumerable set contains an infinite recursive set.*

Ackermann function

As already defined in Section 4.1, a total partial recursive function is called recursive. An example of a recursive function that is not primitive recursive is the so-called Ackermann function defined as follows:

$$\begin{array}{rcll} A(1,j) & = & 2^j, & \text{if } j \geq 1; \\ A(i,1) & = & A(i-1,2), & \text{if } i \geq 2; \\ A(i,j) & = & A(i-1, A(i, j-1)) & \text{if } i \geq 2,\ j \geq 2. \end{array}$$

Note, that the so-called **double recursion** is used to define $A(i,j)$. This is perfectly O.K., because the arguments of A on the right-hand sides of the above equations are always in at least one component smaller than on the left-hand side. The Ackermann function is therefore computable and by Churh's thesis recursive. Surprisingly, this double recursion has the effect that the Ackermann function grows faster than any primitive recursive function, as stated in the theorem below. Figure 6.2 shows the values of the Ackermann function for several small arguments. Already $A(2,j) = 2^{2^{\cdot^{\cdot^{\cdot 2}}}\quad \{j-times\}}$ is an enormously fast growing function and, for $i > 2$, $A(i,j)$ grows even much faster.

Surprisingly, this exotic function has its firm place in computing. More exactly, in the analysis of algorithms we encounter often the following "inverse" to the Ackermann function:

$$\alpha(m,n) = \min\{i \geq 1 \mid A(i, \lfloor m/n \rfloor) > \lg n\}.$$

In contrast to the Ackermann function, its inverse grows very slowly. For all feasible m and n, we have $\alpha(m,n) \leq 4$, and therefore, from the analysis of algorithms point of view, one can see $\alpha(m,n)$ as an "almost constant function". The following theorem summarizes the relation of the Ackermann function to primitive recursive functions.

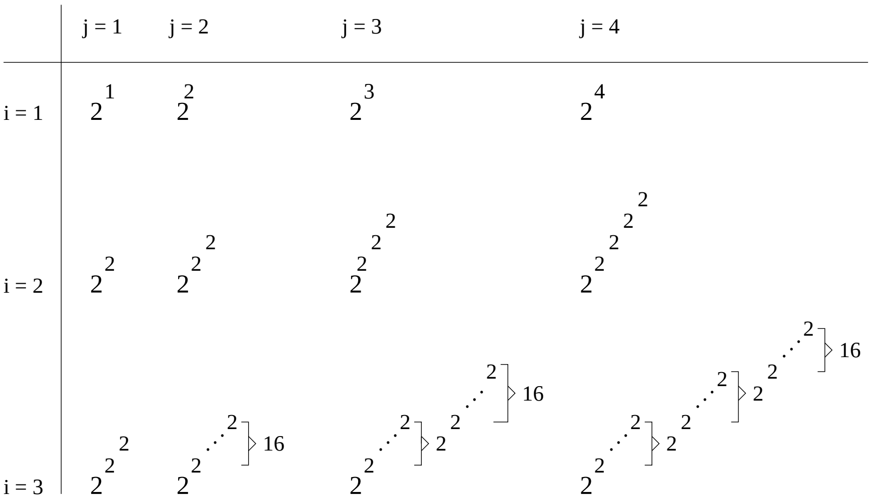

Figure 6.2: Ackermann function

Theorem 6.2.31 *For each primitive recursive function $f(n)$ there is an integer n_0 such that $f(n) \leq A(n,n)$, for all $n \geq n_0$.*

> **Exercise 6.2.32** *Show that for any fixed i the function $f(j) = A(i,j)$ is primitive recursive. (Even the predicate $k = A(i,j)$ is primitive recursive, but this is much harder to show.)*

There are also simple relations between the concepts of recursivity for sets and functions that follow easily from previous results and are now summarized for integer functions and sets.

Theorem 6.2.33

1. *A set S is recursively enumerable if and only if S is the domain of a partial recursive function.*
2. *A set is recursively enumerable if and only if S is the range of a partial recursive function.*
3. *A set S is recursively enumerable (recursive) if and only if its characteristic function is partial recursive (recursive).*

There are also nice relations between recursivity of a function and its graph.

> **Exercise 6.2.34 (Graph theorem)** *Show: (a) A function is partial recursive iff its graph is recursively enumerable; (b) a function f is recursive iff its graph is a recursive set.*

Interesting enough, the origins of the recursion theory, back to 1930-ies, predate the first computers. This theory actually provided the first basic understanding of what is computable and of basic computational principles. It also created an intellectual framework for the design and utilization of universal computers and for the understanding that, in principle, universal computers can be very simple.

6.3 Recursive reals

The concept of recursive functions can also be used to formalize the informal concept of computable real number.

It is intuitively clear that such numbers as π, e are computable in the sense that given an integer n we can potentially compute the first n digits of its decimal expansion[1].

There are various ways how to define "recursive real numbers". We present and discuss two of them. A real number α is called **recursive** if there are recursive functions f and g and an integer n_0 such that $||\alpha| - \frac{f(n)}{g(n)}| \leq \frac{1}{n+1}$, for $n \geq n_0$.

Exercise 6.3.1 *Show that a real number* $0 \leq \alpha \leq 1$ *is recursive if and only if there is a recursive function* $f : \mathbf{N} \rightarrow \{0, 1, \ldots, 9\}$ *such that* $|\alpha| = \sum_{i=0}^{\infty} \frac{f(i)}{10^i}$.

A real number α is called **limiting recursive** if there are recursive functions f, g, h and k such that the sequence $\{\alpha_n\}_{n=1}^{\infty}$, where

$$\alpha_n = \frac{f(n) - g(n)}{h(n)},$$

effectively converges to α in the sense that for each $m > 0$ and $n, n' > k(m)$,

$$|\alpha_n - \alpha_{n'}| \leq \frac{1}{m}.$$

It can be shown that each recursive number is limiting recursive but not vice versa.

The set of limiting recursive numbers is clearly countable. This implies that there are real numbers that are not limiting recursive.

The number of wisdom introduced in Section 6.5.5 is an example of a limiting recursive but not recursive real number.

6.4 Undecidable Problems

We have already seen in Section 4.1.6 that the halting problem is undecidable. This result certainly does not sound as positive but, at the first glance, it does not seem to be either a result to be too much bothered about. Who actually needs in practice to deal with the halting problem for Turing machines? Practically nobody. Cannot we therefore take such

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[1] So far π has been computed to $2 \cdot 10^9$ digits.

an undecidability result merely as an intellectual curiosity that does not really influence our life?

Unfortunately, such a conclusion would be very wrong. In this section we demonstrate that there are indeed theoretically deep and practically important reasons to be much bothered by the existence of undecidable and unsolvable problems. First of all, such problems are much more frequent than one would expect. Secondly some of the most important practical problems are undecidable. Thirdly, borderlines between decidability and undecidability are sometimes unexpectedly sharp.

In this section we present some key undecidable problems and methods how to show undecidability.

6.4.1 Rice's Theorem

We start with a very general result, contra-intuitive and quite depressing, saying that on the most general level of all Turing machines nothing interesting is decidable. Namely, we show first that no nontrivial property of recursively enumerable sets is decidable. This implies that the number of undecidable problems is not only surprisingly large but, at this general level, there are only undecidable problems.

In order to show the main result let us fix a Gödel self-delimiting encoding $\langle \mathcal{M} \rangle_\rho$ of Turing machines $\mathcal{M}$ into the alphabet $\{0,1\}$ and the corresponding encoding $\langle w \rangle_\rho$ of input words of $\mathcal{M}$. The language

$$L_u = \{\langle \mathcal{M} \rangle_\rho \langle w \rangle_\rho \,|\, \mathcal{M} \text{ accepts } w\}$$

is called the **universal language**. It follows from Theorem 4.1.23 that the language L_u is not decidable.

Definition 6.4.1 *Each family $\mathcal{S}$ of recursively enumerable languages over the alphabet $\{0,1\}$ is said to be a property of recursively enumerable languages. A property $\mathcal{S}$ is called nontrivial if $\mathcal{S} \neq \emptyset$ and $\mathcal{S}$ does not contain all recursively enumerable languages (over $\{0,1\}$).*

A nontrivial property of recursively enumerable languages is therefore characterized only by the requirement that there are recursively enumerable languages that this property have and also those that this property do not have. For example, to be a regular language is such a property.

Theorem 6.4.2 (Rice's theorem) *Each nontrivial property of recursively enumerable languages is undecidable.*

Proof. We can assume without loss of generality that $\emptyset \notin \mathcal{S}$; otherwise we can take the complement of $\mathcal{S}$. Since $\mathcal{S}$ is a nontrivial property there is a recursively enumerable language $L' \in \mathcal{S}$ (i.e. one with the property $\mathcal{S}$), and let $\mathcal{M}_{L'}$ be a Turing machine that accepts L'.

Assume that the property $\mathcal{S}$ is decidable and therefore there is a Turing machine $\mathcal{M}_\mathcal{S}$ such that $L(\mathcal{M}_\mathcal{S}) = \{\langle \mathcal{M} \rangle_\rho, \,|\, L(\mathcal{M}) \in \mathcal{S}\}$. We use now $\mathcal{M}_{L'}$ and $\mathcal{M}_\mathcal{S}$ to show that the universal language is decidable. This contradiction proves the theorem.

We describe first an algorithm to design, given a Turing machine $\mathcal{M}_0$ and its input w, a Turing machine $\mathcal{M}_{\mathcal{M}_0,w}$ such that $L(\mathcal{M}_{\mathcal{M}_0,w}) \in \mathcal{S}$ if and only if $\mathcal{M}_0$ accepts w—see

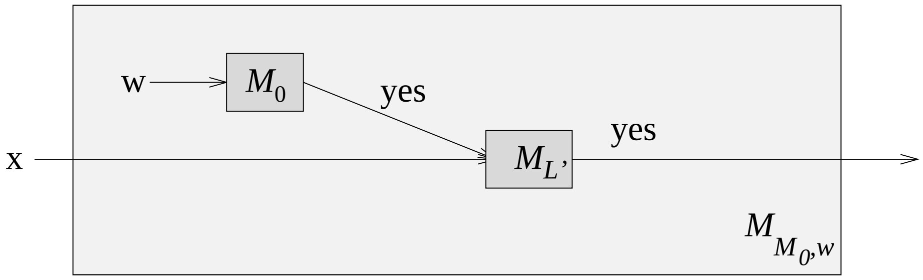


Figure 6.3: Turing machine $\mathcal{M}_{\mathcal{M}_0,w}$ – 80%.

Figure 6.3. $\mathcal{M}_{\mathcal{M}_0,w}$ ignores first its input x and simulates $\mathcal{M}_0$ on w. If $\mathcal{M}_0$ does not accept w, then $\mathcal{M}_{\mathcal{M}_0,w}$ does not accept x. On the other hand, if $\mathcal{M}_0$ accepts w, and therefore terminates, $\mathcal{M}_{\mathcal{M}_0,w}$ starts to simulate $\mathcal{M}_{L'}$ on x and accepts if and only if $\mathcal{M}_{L'}$ accepts. Thus, $\mathcal{M}_{\mathcal{M}_0,w}$ accepts either the empty language (not in $\mathcal{S}$), or L' (in $\mathcal{S}$), depending on whether w is not accepted by $\mathcal{M}_0$, or it is. We can now use $\mathcal{M}_{\mathcal{S}}$ to decide whether or not $L(\mathcal{M}_{\mathcal{M}_0,w}) \in \mathcal{S}$. Since $L(\mathcal{M}_{\mathcal{M}_0,w}) \in \mathcal{S}$ if and only if $\langle\mathcal{M}_0\rangle_\rho\langle w\rangle_\rho \in L_u$, we have an algorithm to decide the universal language L_u. Hence the property $\mathcal{S}$ is undecidable. ▯

Corollary 6.4.3 *It is undecidable whether a given recursively enumerable language is (a) empty, (b) finite, (c) regular, (d) context-free (e) context-sensitive, (f) in* **P**, *(g) in* **NP**,*...*

It is important to realize that for the Rice's theorem it is crucial that all recursively enumerable languages are considered. If this is not the case, decidability can result. For example, it is decidable, see Theorem 3.2.4, given a DFA $\mathcal{A}$, whether the language accepted by $\mathcal{A}$ is finite.

In the rest of this section we deal with several specific undecidable problems. Each of them plays an important rôle in showing the undecidability of other problems using the reduction method discussed next.

6.4.2 Halting Problem

There are two basic ways how to show undecidability of a decision problem.

1. Reduction to a paradox. For example, along the lines of the Russell's paradox (see Section 2.1.1), or its modification known as the **barber's paradox**: *In a small town there is a barber that shaves those and only those that do not shave themselves. Does he shave himself?* This approach is also behind the diagonalization arguments used in the proof of Theorem 6.1.6.

Example 6.4.4 (Printing problem) *It is the problem to decide, given an off-line Turing machine $\mathcal{M}$ and an integer i, whether $\mathcal{M}$ outputs i when starting with the empty input tape. Consider an enumeration $\mathcal{M}_1, \mathcal{M}_2, \ldots$ of all off-line Turing machines generating sets of natural numbers and consider the set $S = \{i \mid i$ is not in the set generated by $\mathcal{M}_i\}$. This set cannot be recursively enumerable because otherwise there would exist a Turing machine*

$\mathcal{M}_S$ generating S, and therefore $\mathcal{M}_S = \mathcal{M}_{i_0}$ for some i_0. Now comes the question: Is $i_0 \in S$? and we get a variant of the barber paradox.

2. Reduction from another problem undecidability of which was already shown. In other words, to prove that a decision problem $\mathcal{P}_1$ is undecidable, it is sufficient to show that the decidability of $\mathcal{P}_1$ would imply the decidability of another decision problem, say $\mathcal{P}_2$, undecidability of which was already shown. All what is required is that there is an algorithmic way of transforming (with no restriction on resources such a transformation needs), an $\mathcal{P}_2$-input into an $\mathcal{P}_1$-input in such a way, that $\mathcal{P}_2$'s yes/no answer is exactly the same as $\mathcal{P}_1$'s answer to the transformed input.

Example 6.4.5 *We can use undecidability of the printing problem to show undecidability of the halting problem as follows. To each off-line Turing machine $\mathcal{M}$ we can easily construct a Turing machine $\mathcal{M}'$ such that $\mathcal{M}'$ halts for an input w if and only if $\mathcal{M}$ prints w. The decidability of the halting problem would therefore imply the decidability of the printing problem.*

Exercise 6.4.6 *Show that the following decision problems are undecidable:*

(a) Does a given Turing machine halt on the empty tape?; (b) Does a given Turing machine halt for all inputs?

The main reason for the importance of undecidability of the halting problem is the fact that undecidability of many decision problems can be shown by reduction from the halting problem.

It is also interesting to notice that the decidability of the halting problem could have an enormous impact on mathematics and computing. To see that let us consider again perhaps the most famous problem of mathematics in the last two centuries, the so-called Fermat's last theorem, claiming that there are no integers x, y, z and w such that

$$(x+1)^{w+3} + (y+1)^{w+3} = (z+1)^{w+3} \tag{6.1}$$

Given x, y, z, w it is easy to verify whether (6.1) holds. It is therefore simple to design a Turing machine that checks for all possible quadruples (x, y, z, w) whether (6.1) holds and that halts if such a quadruple is found. Would we have the proof that this Turing machine never halts we would have proven Fermat's last theorem. In a similar way we can show that many important open mathematical problems can be reduced to the halting problem for some specific Turing machine.

As we could see in the Chapter 5, various bounded versions of the halting problem are complete problems for important complexity classes.

Exercise 6.4.7 *Show that decidability of the halting problem could be used to solve the famous Goldbach's conjecture (1742) that each even number greater than 2 is the sum of two primes.*

Remark 6.4.8 At the beginning of this century a belief in the total power of formalization has been the main driving force in mathematics. One of the key problems formulated by the leading mathematician of that time, David Hilbert, was the so-called Entscheidungsproblem: is there a general mechanical procedure which could, in principle, solve all problems of mathematics, one after another? It has actually been the Entscheidunsproblem problem that led Turing to develop his concept of machine and decidability and it was the reduction to the halting problem by which he showed the undecidability of the Entscheidungsproblemin his seminal paper

> *A. Turing: On computable numbers, with an application to the Entscheidungsproblem,*

written in 1937, considered by some as the single most important paper in the modern history of computing.

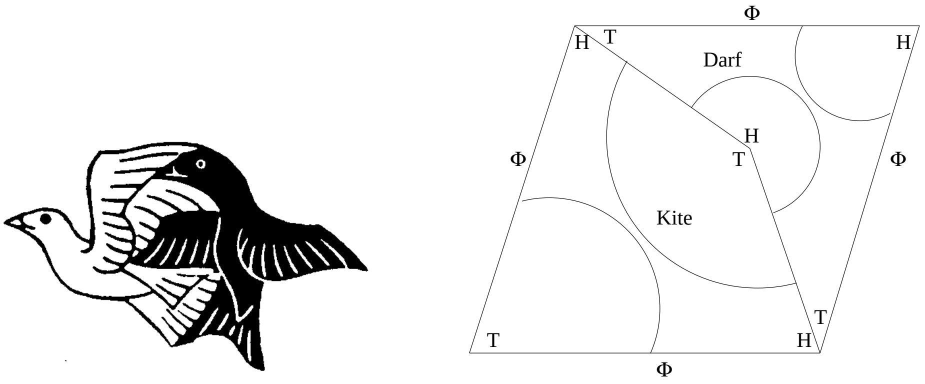
Example 6.4.9 (Program verification) *The fact that program equivalence and program verification are undecidable even for very simple programming languages has practically very negative consequences. These results practically rule out automatic program verification and reduce the hope for having fully optimizing compilers capable transforming a given program into an optimal one.*

It is readily seen that the halting problem for Turing machines can be reduced to the program verification problem. Let us sketch the idea. Given a Turing machine $\mathcal{M}$ *and its input* w *we can transform the pair* $\langle \mathcal{M}, w\rangle$*, which is the input for the halting problem, to a pair* $\langle \mathcal{P}, \mathcal{M}\rangle$ *as an input to the program verification problem. The algorithm (TM)* $\mathcal{M}$ *remains the same and* $\mathcal{P}$ *is the algorithmic problem described by specifying that* w *is the only legal input for which* $\mathcal{M}$ *should terminate and that the output for this input is not of importance.* $\mathcal{M}$ *is now correct with respect to this simple algorithmic problem* $\mathcal{P}$ *if and only if* $\mathcal{M}$ *terminates for input* w*. Consequently, the verification problem is undecidable.*

6.4.3 Tiling Problems

Tiling of the plane or space by tiles from various finite sets of (proto)tiles—especially of polygonal or polyhedral shapes—that is, a covering of the plane or space completely, without gaps and overlaps and with matching colors on contiguous vertices, edges or faces (if they are colored) is an old and much investigated mathematical problem with a variety of applications. For example, it was known already to Pythagorian school (6th century BC) that there is only 1 regular polyhedra that can tile the space completely. However, there are infinitely many sets with more than one tile that can tile plane (space). The facts that tiling can simulate Turing machine computation and various variants of tiling problem are complete for main complexity classes, show the importance of tiling for theory of computing.

A tiling of the plane (space) is called periodic if one can outline its finite region such that the whole tiling can be obtained by its translation, that is, by shifting the position of the region without rotating it. M. C. Escher got famous for his pictures obtained by periodic tilings with shapes that resemble living creatures—see Figure 6.4a for a shape (tile) consisting of a while and black bird that can be used to tile the plane periodically.



(a) (b)

Figure 6.4: Escher's figure and Penrose's tiles–70%

A tiling of the plane that is not periodic is called **aperiodic**. The problem of finding a (small) set of tiles that can be used to tile plane only aperiodically (with rotation and reflection of tiles allowed) has turned out to be intriguing and with surprising results and consequences.

Our main interest now is the following decision problem: given a set of polygon (proto)tiles with colored edges, is there a tiling of the plane with the given set of tiles?

Of a special interset for computing is the problem of tiling the plane with unit square tiles with colored edges, called Wang tiles or dominoes, when neither rotation nor reflection of tiles is allowed. This problem is closely related to decision problems of logic. Berger (1966) has shown that such a tiling problem is undecidable and his complicated proof implied that there is a set of Wang tiles that can tile the plane but only aperiodically. Moreover, he actually exhibited a set of 20406 tiles with such a property. This number has been consequently reduced and currently the smallest set of Wang tiles with such a property, due to K. Culik,

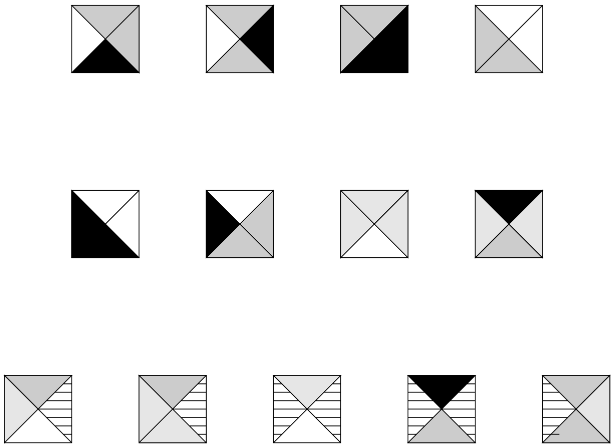


Figure 6.5: Culik's tiles–70%

is shown in Figure 6.5.[2]

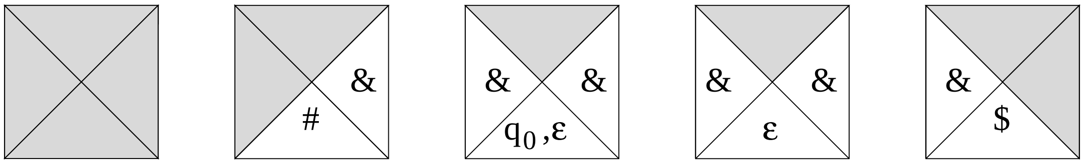
The following theorem shows undecidability of a special variant of the tiling problem with Wang dominoes:

Theorem 6.4.10 *It is undecidable, given a finite set T of Wang tiles with colored edges that includes a tile with all edges of the same color (say white), whether there is such a tiling of the plane that uses only finitely many, but at least one other than the completely white tile.*

Proof. We show that if such a tiling problem is decidable then the halting problem is decidable for one-tape Turing machines that satisfy conditions 1 to 3 on page 356 in Section 5.4.2; that is for Turing machines which have one-way infinite tape, a unique accepting configuration and no state in which the machine can move both left and right, and, moreover, start their computation with the empty tape.

To each such a Turing machine $\mathcal{M} = \langle \Gamma, Q, q_0, \delta \rangle$ we construct a set of square tiles as follows. We take all those tiles as in the proof of Theorem 5.4.2 in Section 5.4.1 and, in addition, the following sets of tiles:

1. Tiles of the form



   that will form the topmost row containing a not white tile. (Observe that the second of these tiles is the only one that can form the "top" leftmost not-completely-white tile for a tiling with not all tiles white.) This set of tiles will be used to encode the initial configuration of $\mathcal{M}$, long enough to create space for all configurations in the computation starting with the empty tape. & represents here a special color not used in tiles of other sets of tiles.

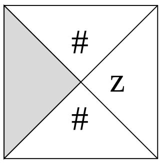
2. A set of tiles, two for each $z \in \Gamma$, of the form

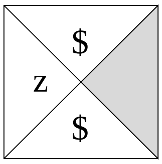
---

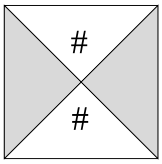
[2] Roger Penrose designed around 1975 a set of two simple polygon tiles, see Figure 6.4b, called Kite and Darf, with colored vertices (by colors H and T), that can tile the plane but only aperiodically (rotation and reflection of tiles is allowed). These two tiles are derived from a rhombus with edges of length $\Phi = (1+\sqrt{5})/2$ and 1 and angles 72° and 108° by a cut shown in Figure 6.4b. (Observe that the common "internal vertex" is colored differently in both tiles and therefore tiling as shown in Figure 6.4b is not allowed. Note also that it is easy to change such a set of tiles with colored vertices into polygonal tiles that are not colored and tile the plane only aperiodically. Indeed, it is simply enough to put bumps and dents on the edges to make jigsaw pieces that fit only in the manner prescribed by colors of vertices.) Penrose patented his tiles in UK, USA and Japan because of potential to make commercial puzzles. Especially, if two colored arcs are added in a way indicated in Figure 6.4b, then one can create from Penrose's tiles tilings with fascinating patterns.

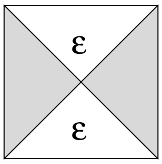
Tilings of plane with Penrose's tiles have also many surprising properties. For example, the number of different tilings of the plane is uncountable and at the same time any two tilings are alike in a special way—every finite subtiling of any tiling of the plane is contained infinitely many times inside of every other tiling.

In addition R. Ammann discovered in 1976 a set of two rhombohedra which, with suitable face-matching rules, can tile the space only aperiodically. This made Penrose to hypothesise the existence of aperiodic structures in nature. This has later been confirmed. First, by D. Schechlman in 1984 and afterwards by many discoveries of physicists, chemists and crystalographers.



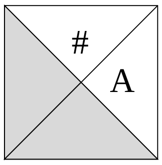


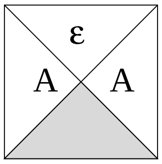


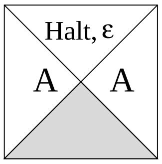


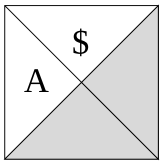
that keep the left and the right border of a computation fixed. Here $z \in \Gamma$.

3. Tiles of the form









that will be used to form the last row with not all tiles white. Semantically, they will be use to encode the last row after a halting configuration of the Turing machine is reached. The symbol A denotes here a new color not used in tiles of other sets of tiles.

It is now straightforward to see that there is a tiling of the plane with the set of tiles designed as above that uses only finitely many non white tiles, and at least one such tile, if and only if the corresponding Turing machine halts on the empty tape—what is undecidable. □

Exercise 6.4.11 *Consider the following modification of the tiling problem with Wang tiles: the set of tiles has a "starting tile" and only such tilings are considered that use at least once this starting tile.*

Is it true the plane is tileable by such a set of Wang tiles if and only if for each $n \in \mathbf{N}$, *the* $(2n+1) \times (2n+1)$ *square board is tileable with such a set of Wang tiles with the starting tile in the centre.*

There are many variants of the tiling problem that are undecidable and they are of interest by themselves. In addition, undecidability of many decision problems can be shown easily and transparently by a reduction to one of the undecidable tiling problems.

Exercise 6.4.12 *Consider the following modifications of the tiling problem (as formulated in Exercise 6.4.11):*

P1 *Tiles are allowed to be rotated 180 degree.*

P2 *Flipping around a vertical axis is allowed.*

P3 *Flipping around the main diagonal axis is allowed.*

Show: (a) problem P1 has always a solution; (b) problem P2 is decidable; (c)** problem P3 is undecidable.*

6.4.4 Thue Problem

The most basic decision problem in the area of rewriting, with many variations, is the word problem for Thue systems, considered in Section 7.1. This problem is often presented in the following form:

With any alphabet Σ and two lists of words over Σ

$$(E) \qquad A = (x_1, \ldots, x_n), \qquad B = (y_1, \ldots, y_n),$$

the following relation $\equiv_E$ on Σ^* is associated:

$x \equiv_E y$ if there are u, $v \in \Sigma^*$ and $1 \leq i \leq n$ such that either $x = ux_iv$ and $y = uy_iv$ or $x = uy_iv$ and $y = ux_iv$.

In other words, y can be obtained from x by rewriting a subword x_i from A by the corresponding y_i from B or vice versa. In this way one can see the lists in (E) as a set of equations

$$x_i = y_i, \qquad 1 \leq i \leq n \tag{6.2}$$

in the free monoid over Σ. Let $\equiv_E^*$ be the transitive and reflexive closure of the relation $\equiv_E$. The **Thue problem for semigroups**,[3] for the lists (E), is the problem to decide, given two words $x, y \in \Sigma^*$ whether $x \equiv_E^* y$, i.e., whether y can be obtained from x using equations from (6.2). In the case that equations reflect the property that the underlying set of words is an abelian semigroup (there is an equation $ab = ba$ for each two symbols of the underlying alphabet), or a group (there is an "inverse" letter a^{-1} to each letetr a and equations $aa^{-1} = \varepsilon$ and $a^{-1}a = \varepsilon$) or an abelian group, we speak of a Thue problem for abelian semigroups, or groups or abelian groups. It holds:

Theorem 6.4.13 *The Thue problem is undecidable for semigroups and for groups and decidable for abelian semigroups and groups.*

---

[3] The word problem is another often used name for the Thue problem.

Example 6.4.14 *For equations:*

$$(E_1)\qquad \begin{array}{lll} EAT = AT, & LATER = LOW, & CARP = ME, \\ ATE = A, & PAN = PILLOW, & \end{array}$$

we have, as it is easy to verify,

$$LAP \equiv^*_{E_1} LEAP \qquad MAN \equiv^*_{E_1} CATERPILLAR$$

but

$$ARORA \equiv^*_{E_1} APPLE \tag{6.3}$$

does not hold.

There is an algorithm to decide the word problem for (E_1). On the other hand the word problem for the equations (E_2):

$$\begin{array}{llll} \mathrm{AH} = \mathrm{HA}, & \mathrm{AT} = \mathrm{TA}, & \mathrm{TAI} = \mathrm{IT}, & \mathrm{THAT} = \mathrm{ITHT}, \\ \mathrm{OH} = \mathrm{HO}, & \mathrm{OT} = \mathrm{TO}, & \mathrm{HOI} = \mathrm{IH}. & \end{array}$$

is undecidable. (This is an example of a seemingly very simply looking specific decision problem that is undecidable.)

6.4.5 Post Correspondence Problem

A reduction to the Post[4] Correspondence Problem (PCP) is an important method to show the undecidability of many decision problems, especially those concerning rewritings—this will be illustrated in the next chapter. The reason behind is the fact that PCP captures the essence of rewriting.

An **instance** of the PCP is given by two lists

$$(u_1, \dots, u_n), \qquad (v_1, \dots, v_n) \tag{6.4}$$

of words over an alphabet Σ, and the problem is to decide whether there is a sequence of integers $i_1, \dots, i_m$, where $1 \le i_k \le n$ and $1 \le k \le m$, such that

$$u_{i_1} \dots u_{i_m} = v_{i_1} \dots v_{i_m}.$$

If such a sequence $(i_1, \dots, i_m)$ exists, then it is called a **solution** of (6.4). Clearly, if an instance of the PCP has a solution, then it has infinitely many solutions.

Example 6.4.15 *For the instance of the PCP defined by the lists:*

$$(a^2, b^2, ab^2) \text{ and } (a^2b, ba, b)$$

the PCP has a solution $(1, 2, 1, 3)$. *For the instance defined by lists*

$$(a, b^2) \text{ and } (a^2, b)$$

the PCP has no solution. (Show why.)

---

[4] Emil Post (1887-1954), an American mathematician and logician.

Exercise 6.4.16 *Determine for the following instances of the PCP whether they have a solution and if yes, show one: (a)* (a^2b, a^2) *and* (a, ba^2)*; (b)* (a^2b, a) *and* (a, ba^2)*; (c)* (a, ba, b, ab) *and* (aaa, b, bb, ba)*; (d)* $(ab, a, c, abca)$ *and* $(ca, abac, bb, ab)$*.*

Exercise 6.4.17 *Show that the PCP with the list of words over one-letter alphabet is decidable.*

The Post Correspondence Problem is the problem to decide, given an instance of PCP, whether this instance has a solution. The following theorem can be shown by a reduction to the halting problem.

Theorem 6.4.18 *The Post Correspondence Problem is undecidable.*

Example 6.4.19 *In order to illustrate how the PCP can be used to show the undecidability of problems concerning rewriting systems, similar to those used on page 96,let us associate two rewriting systems with each instance* $u = (u_1, \ldots, u_n)$ *and* $v = (v_1, \ldots, v_n)$ *of the PCP over the alphabet* $\{0,1\}$*:* $S \rightarrow c$ *in both cases and*

$$G_1 : S \rightarrow 0^i 1 S u_i,\ 1 \leq i \leq n, \qquad G_2 : S \rightarrow 0^i 1 S v_i,\ 1 \leq i \leq n.$$

For the languages $L(G_1)$ *and* $L(G_2)$ *of words over the alphabet* $\{0,1\}$*, generated by these two systems we have*

$$\begin{array}{rcl} L(G_1) & = & \{0^{i_1}1 \ldots 0^{i_k}1 c u_{i_k} \ldots u_{i_1} \,|\, 1 \leq i_j \leq n, 1 \leq j \leq k\} \\ L(G_2) & = & \{0^{i_1}1 \ldots 0^{i_k}1 c v_{i_k} \ldots v_{i_1} \,|\, 1 \leq i_j \leq n, 1 \leq j \leq k\} \end{array}$$

and therefore $L(G_1) \cap L(G_2) \neq \emptyset$ *if and only if the PCP for the instance* u *and* v *has a solution.*

Exercise 6.4.20 *There is another way, more compact, using morphisms, how to see the Post Correspondence Problem: Let* g *and* h *be two morphisms mapping* Δ^* *into* Σ^**. Show that the pair* (g, h) *can be considered as an instance of PCP that has a solution if there is an nonempty word* $w \in \Delta^*$ *such that* $g(w) = h(w)$*.*

Remark 6.4.21 The following bounded version of PCP is **NP**-complete: Given an instance $(u_1, \ldots, u_n)$, $(v_1, \ldots, v_n)$ of PCP and an integer b. Is there a sequence of integers $i_1, \ldots, i_m$, such that m/leb, $1 \leq i_k \leq n$, $1 \leq k \leq m$ and $u_{i_1} \ldots u_{i_m} = v_{i_1} \ldots v_{i_m}$?

6.4.6 Hilbert's Tenth Problem

The decision problem, known as Hilbert's Tenth Problem, was formulated in 1900 by David Hilbert.[5] It is perhaps the most famous decision problem and can be formulated as follows: decide, given an arbitrary polynomial $\mathcal{P}(x_1, \ldots, x_n)$ with coefficients from $\mathbf{Z}$, whether the following so-called Diophantine equation

$$\mathcal{P}(x_1, \ldots, x_n) = 0$$

has a solution in the whole numbers.

Diophantine equations (named after Diophantus of Alexandria) represent one of the oldest area of science where deeper results have been obtained already in the third century in Greece. The following examples illustrate how difficult it may be to find a solution of seemingly simple Diophantine equations:

Example 6.4.22 *(1) In the smallest nonnegative solution (x, y) of the Pell's equation*

$$x^2 - 991y^2 - 1 = 0$$

x has 30 and y 29 decimal digits.

(2) The smallest nonnegative solution, for x, of the Pell's equation

$$x^2 - 4729494y^2 - 1 = 0,$$

which can be obtained by a reduction from the system of equations, seven linear of 8 variables and two quadratic conditions, formulated by Archimedes in a letter to Eratosthenes of Cyrene, in 3rd century, has over 206 500 digits.

(3) Factorization of an integer n is as difficult as solution of the equation $(x+2)(y+2) = n$.

There are two additional reasons for a special importance of the Hilbert's Tenth Problem. Since this problem concerns integer arithmetic, it is a good starting point to show that everyday considerations in mathematics can lead to undecidable problems. Secondly, the very difficult proof of its undecidability is actually based on a new and important characterization of recursively enumerable sets and relations presented in the following theorem.

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[5] David Hilbert (1862-1943) a German mathematician and logician. Of fundamental importance are his contributions in algebra, number theory, geometry and calculus (with relations to physics and functional analysis). Hilbert was perhaps the most influential mathematician of his period. A representative of formalism in logic and his school of thought brought up such founders of modern computing theory as Gödel, von Neumann and Turing. For his opening talk at the World Congress of Mathematicians in 1900 in Paris, Hilbert prepared a list of 23 problems he considered as being of key importance for mathematics. Hilbert's Tenth Problem was solved in 1970 by a Russian mathematician Ju. Matiyasevich. (Actually it seems that Hilbert assumed that this problem is decidable and it is only a question of time when an algorithm to decide it will be found. Anyway, the concept of undecidability was unknown at his time.) English translation of the original formulation of hilbert's Tenth Problem: "Given a diphantine equation with any number of unknown quantities and with rational integral numerical coefficients. To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers."

Theorem 6.4.23 (Matiyasevich's theorem) *A set $S \subset \mathbf{N}^n$ is recursively enumerable if and only if there is a polynomial $P(a_1,\dots,a_n,y_1,\dots,y_m)$ with coefficients in $\mathbf{Z}$ such that*

$$S = \{(x_1,\dots,x_n) \mid (\exists y_1 \in \mathbf{N}),\dots,(\exists y_m \in \mathbf{N}) P(x_1,\dots,x_n,y_1,\dots,y_m) = 0\}.^6$$

Matijasevich's theorem implies that decidability of the Hilbert's Tenth Problem would imply decidability of the membership problem for recursively enumerable sets and that is an undecidable problem.

As Theorem 6.4.23 says, polynomial with coefficients in $\mathbf{Z}$ have a surprising power to represent all recursively enumerable sets. Some examples:

Example 6.4.24 *The set C of composite numbers has the following polynomial representation:*

$$x \in C \Leftrightarrow \exists y \exists z (x = (y+2)(z+2)).$$

The second example deals with primes.

Example 6.4.25 *The set of primes P has the following polynomial representation:*[7]

$$\begin{aligned}
x' \in P \quad \Leftrightarrow \quad & \exists(a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z) \\
& x' = (k+2)\{1 - [wz+h+j-q]^2 - [(gk+2g+k+1)(h+j)+h-z]^2 \\
& \quad -[2n+p+q+z-e]^2 - [16(k+1)^3(k+2)(n+1)^2+1-f^2]^2 \\
& \quad -[e^3(e+2)(a+1)^2+1-o^2]^2 - [(a^2-1)y^2+1-x^2]^2 \\
& \quad -[16r^2y^4(a^2-1)+1-u^2]^2 \\
& \quad -[((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2]^2 - [n+1+v-y]^2 \\
& \quad -[(a^2-1)l^2+1-m^2]^2 - [ai+k+1-l-i]^2 \\
& \quad -[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2 \\
& \quad -[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2 \\
& \quad -[z+pl(a-p)+t(2ap-p^2-1)-pm]^2\}.
\end{aligned}$$

Exercise 6.4.26 *(a) Design polynomial representations for the set of integers that are not powers of* 2.

Exercise 6.4.27 *Design polynomial representation of the predicates: (a) $a \leq b$; (b) $a \equiv b \pmod n$; (c) $a \equiv b \pmod c$.*

Exercise 6.4.28 *Show that the set of polynomials P with integer coefficients for which the equation $P = 0$ has a solution in natural numbers is undecidable. (Hint: you can make use of the fact that each integer is a sum of four squares of integers.)*

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[6] It has also been shown that in Theorem 6.4.23 it is sufficient to consider polynomial of the degree at most 4 or at most 9 variables.

[7] There is another representation of the set of all primes through a polynomial with only 10 variables due to Matiyasevich.

As another consequence of Theorem 6.4.23 we get the following result showing the existence of a universal polynomial.

Theorem 6.4.29 *There is a polynomial P_u with coefficients in $\mathbf{Z}$ such that for any recursively enumerable set S there is an integer i_S such that*

$$x \in S \Leftrightarrow (\exists y_1 \in \mathbf{N}), \ldots, (\exists y_n \in \mathbf{N})(P_u(i_S, x, y_1, \ldots, y_n) = 0).$$

Proof. Let $S_i, i = 1, 2, \ldots$ be an enumeration of all recursively enumerable sets of integers. The binary relation $\{(x, i) \,|\, x \in S_i\}$ is clearly recursively enumerable. This implies the existence of the universal polynomial.[8] □

For example, the last theorem implies that there is an integer i_p such that x is prime if and only if $(\exists y_1 \in \mathbf{N}), \ldots, (\exists y_n \in \mathbf{N}) P_u(i_p, x, y_1, \ldots, y_n) = 0$.

Exercise 6.4.30 *(a) Show that each Diophantine equation can be transformed into an equaivalent system of Diophantine equations of the type $x = y + 2$ or $x = yz$, where x, y, z are either natural numbers or unknowns. (b) Illustrates the method on the Diophantine equation: $5x^2y^2 - 6x^3z + 6xy^3 - 5xz = 0$.*

Exercise 6.4.31 *Show that in order to solve the Hilbert's Tenth Problem positively it would be sufficient to find a method for deciding whether Diophantine equations of degree 4 have a solution.*

6.4.7 Borderlines between Decidability and Undecidability

In few cases quite sharp borderlines between decidability and undecidability are known. In most of other cases it is an open problem to determine such sharp borderlines. The following table summarizes the current state of the knowledge.

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[8] Once the existence of a universal polynomial has been shown the problem arose to find such a universal polynomial with smallest number of variables and the smallest degree. By Jones (1982), the following pairs (# of variables, degree) for a universal polynomial are known: $(58, 4)$, $(38, 8)$, $(29, 16)$, $(24, 36)$, $(19, 2668)$, $(13, 6.6 \cdot 10^{43})$, $(9, 1.6 \cdot 10^{45})$.

Problem	Decidable cases	Undecidable cases
halting prob.—one-dim. TM	state-symbol prod. ≤ 6	state-symbol prod. ≥ 24
halting prob.—two-dim. TM	1 instruction	8 instructions
Equiv. prob.—register mach.	7 instructions	8 instructions
Halt. prob.—register mach.	8 instructions	29 instructions
tiling of plane	1 polyomino[1]	nonperiodic for 35 tiles
Word problem for groups[2]	1 equation	3 equations
Post Corr. Prob.	two-letter alphabet lists of length ≤ 2	two-letter alphabet list of length ≥ 7
Diophantine equations	equations of degree ≤ 2	equations of degree ≥ 4

Remark 6.4.32 Simplicity of presentation is the main reason why only decision problems are considered in this section. Each of the undecidable problems discussed here has a computational version that requires output and it is not computable. For example, such property has the problem to compute, for a Turing machine $\mathcal{M}$ and an input w, the function $f(\mathcal{M}, w)$, defined to be zero if $\mathcal{M}$ does not halt for w and to be the number of steps of $\mathcal{M}$ on w, otherwise.

6.4.8 Degrees of Undecidability

It is natural to ask whether all undecidable problems are "equally" undecidable. The answer is negative, and we approach this problem from two points of view.

Let us first take again a formal view of decision problems as membership problems for sets. To classify undecidable problems several types of reductions have been used.

For example, we say that a set A is (many-to-one) **reducible** to a set B (notation $A \leq_m B$), if there exists a recursive function f such that

$$x \in A \Leftrightarrow f(x) \in B,$$

and we say that the sets A and B belong to the same degree of unsolvability (with respect to the many-to-one reduction), if $A \leq_m B$ and $B \leq_m A$. It can be shown that there are infinitely many degrees of unsolvability. Some of them are comparable (that is each problem from one class can be reduced to a problem from other class), some are incomparable.

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[1] A polyomino is a polygon formed by a union of unit squares.

[2] A word problem over an alphabet Σ is said to be a word problems for groups when for any $a \in \Sigma$ there is also an equation $ab = \varepsilon$ available.

Exercise 6.4.33 *Show that if* $A \leq_m B$ *and* B *is recursive, then* A *is recursive, too.*

Exercise 6.4.34 *Let us fix a Gödel numbering* $\langle \mathcal{M} \rangle_\rho$ *of TM* $\mathcal{M}$ *into the alphabet* $\{0,1\}$ *and the corresponding encoding* $\langle w \rangle_\rho$ *of input words of* $\mathcal{M}$. *Let us consider the languages*

$$\begin{aligned} K_u &= \{\langle \mathcal{M} \rangle_\rho \langle w \rangle_\rho \,|\, \mathcal{M} \textit{ accepts } w\}, & (6.5) \\ K_d &= \{\langle \mathcal{M} \rangle_\rho \,|\, \mathcal{M} \textit{ accepts } \langle \mathcal{M} \rangle_\rho\}, & (6.6) \\ K_p &= \{\langle \mathcal{M} \rangle_\rho \,|\, \mathcal{M} \textit{ halts on at least one input}\}, & (6.7) \\ K_e &= \{\langle \mathcal{M} \rangle_\rho \,|\, \mathcal{M} \textit{ halts on the empty tape}\}. & (6.8) \end{aligned}$$

Show: (a) $K_d \leq_m K_u$; *(b)* $K_u \leq_m K_e$; *(c)* $K_e \leq_m K_p$; *(d)* $K_p \leq_m K_d$; *(e) a set* L *is recursively enumerable if and only if* $L \leq_m K'$, *where* K' *is any of the sets* K_u, K_d, K_p *and* K_e.

There are two natural ways how to form infinite hierarchies of more and more undecidable problems. The so-called **jump method** is based on the concept of **Turing reducibility**: a set A is Turing reducible to B, notation $A \leq_T B$, if there is an oracle TM with oracle B accepting A.

Definition 6.4.35 *The language*

$$K^B = \{\langle \mathcal{M} \rangle_\rho \,|\, \mathcal{M} \textit{ is an oracle TM with oracle } B \textit{ that halts on } \langle \mathcal{M} \rangle_\rho\},$$

is called the halting problem for TM with the oracle B.

Exercise 6.4.36 *Show: (a)* $A \leq_T B$ *iff* $A \leq_m K^B$; *(b)* $K^B \leq_T B$ *does not hold.*

Using the jump operator we can define the following infinite sequence of more and more undecidable languages.

$$K, K^K, K^{K^K}, K^{K^{K^K}}, \ldots$$

where K is any of the languages K_u, K_d, K_p, K_e, defined above.

In order to introduce the second very natural hierarchy of undecidable problems two new operators on families of languages have to be introduced. If $\mathcal{L}$ is a family of languages, then

$$\Sigma\mathcal{L} = \{L \,|\, \exists R \in \mathcal{L}, x \in L \Leftrightarrow \exists y[(x,y) \in R]\},$$

$$\Pi\mathcal{L} = \{L \,|\, \exists R \in \mathcal{L}, x \in L \Leftrightarrow \forall y[(x,y) \in R]\},$$

where (x,y) is a recursive encoding of the tuple x and y.

Definition 6.4.37 (Arithmetical hierarchy) *The families* Σ_i *and* Π_i, $i \in \mathbf{N}$, *are defined as follows:*

(a) $\Sigma_0 = \Pi_0 =$ *the class of recursive languages.*

(b) $\Sigma_{i+1} = \Sigma\Pi_i$ *and* $\Pi_{i+1} = \Pi\Sigma_i$, *for* $i \in \mathbf{N}$.

Note, that Σ_1 is the class of recursively enumerable languages. It can be shown that the families $\Sigma_0,\ \Sigma_1 - \Sigma_0,\ \Sigma_2 - \Sigma_1, \ldots$ and $\Pi_0,\ \Pi_1 - \Pi_0,\ \Pi_2 - \Pi_1, \ldots$ are nonempty and contain more and more undecidable languages.

The importance of the Arithmetical hierarchy is also due to the fact that classes Σ_i and Π_i contains languages that can be defined in a very natural way as the following exercise demonstrates:

Exercise 6.4.38 *Show, for example by induction,*

(a) $L \in \Sigma_i$, *iff there exists a recursive language* R *such that*

$$x \in L \Leftrightarrow \exists y_1 \forall y_2 \exists y_3 \ldots Q y_i [(x, y_1, y_2, \ldots, y_i) \in R],$$

where $Q = \exists\,(\forall)$, *if is odd (even).*

(b) $L \in \Pi_i$, *iff there exists a recursive language* R *such that*

$$x \in L \Leftrightarrow \forall y_1 \exists y_2 \forall y_3 \ldots Q y_i [(x, y_1, y_2, \ldots, y_i) \in R],$$

where $Q = \exists\,(\forall)$, *if is even (odd).*

It can be shown that such decision problems as the halting problem, Post Correspondence Problem, the tiling problem considered in this section, are of the same degree of undecidability. However, the program verification problem is "more" undecidable, and there are even "much more" undecidable problems. A simple example of two closely related problems that have different degrees of undecidability: the problems to decide whether a given Diophantine equation has at least one solution and the problem to decide whether it has infinitely many solutions.

Another way to look at decision problems is through certificates, see Section 5.6.1, and we come to another analogy with **NP**-complete problems in Chapter 5.

It is clear that "yes" answers to such problems as the halting problem, PCP, Hilbert's Tenth Problem, have finite certificates that can be verifiable in finite time. In the case of the tiling problem from this section it is the "no" answer that has a finite certificate, because if the plane cannot be tiled, then there is its finite portion that cannot be tiled and that can be verified in finite time.

Can it happen that both "yes" and "no" answers to an undecidable problem have finite certificates? No, as it is easy to verify. This would imply decidability of the problem. Are there undecidable problems for which neither "yes" nor "no" answers have certificates? Yes, for example, the problem to decide whether a given Turing machine halts for all inputs.

The space of algorithmic problems has therefore the structure shown in Figure 6.6.

Interesting enough in each class there is an example of the domino problem. (**Fixed-width bounded domino problem** is the problem to decide given a finite set of colored tiles, and an integer n, whether an rectangle of size $n \times c$, where c is fixed, can be tiled with tiles from T. **Recurring domino problem** is the problem to decide whether one can tile the whole plane with a given finite set of tiles in such a way that a particular tile occurs infinitely often.)

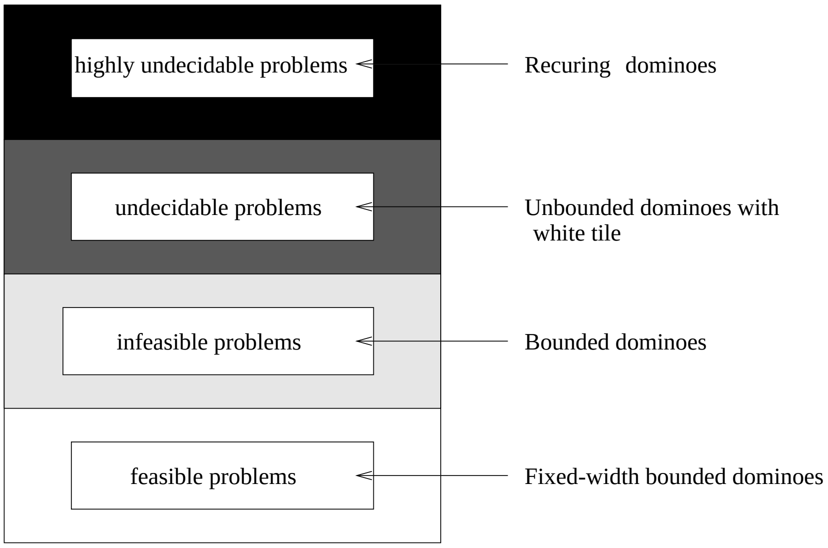

Figure 6.6: Structure of the space of algorithmic problems-70%

6.5 Limitations of Formal Systems

The proof of the existence of undecidable problems was historically the second of two main results that demonstrated striking limitations of formal systems and algorithmic methods. The first one, which says that in each rich enough formal system there are true statements that are unprovable within the given formal system, was due to Gödel[9] in 1931. In this section we start with Gödel's result—with a presentation of the simple idea of his ingenious proof. We show then how a more specific version of Gödel's theorem can be proven easily using arguments of information-theoretic flavor. The result implies that in no formal system we can prove *a theorem that contains more information than a given set of axioms.* In order to show that, two new complexity concepts are introduced: Chaitin and Kolmogorov descriptional complexity. They lead also to a new way how randomness can be seen. Finally, an intriguing "number of wisdom" is analyzed that encodes very compactly the cornerstones of undecidability and can be used to show how deeply randomness is rooted even in arithmetic.

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[9] Kurt Gödel (1886-1978), an Austrian mathematician and logician of Moravian origin.

6.5.1 Gödel's Incompleteness Theorem

The following result shows unexpected limitations of formal systems and at the time of its discovery it was a shock for scientists and caused uncertainty and depression among those believing in enormous power of formalization. It shaked the foundations of mathematics and thereby all science depending on mathematics.

Theorem 6.5.1 (Gödel's Incompleteness Theorem) *Each formal mathematical system which contains arithmetic is either inconsistent (i.e. one can prove in this system a theorem and also its negation) or contains theorems which are true but which cannot be proven within the system.*

Gödel's proof of Theorem 6.5.1 is tricky and difficult. Nowadays Gödel's result is considered as almost obvious—as one that easily follows from the undecidability of the halting problem and from the existence of sets that are recursively enumerable but not recursive.

Gödel's proof is based on a variant of the *paradox of Liar*: "This statement is false"[10]. Gödel obtained from that a paradox he needed as follows: "This statement is unprovable". Indeed, if such a statement is unprovable, then it is true and we have a true unprovable statement; if such a statement is provable, then it is false and we have inconsistency.

The basic idea of Gödel's proof is simple and we make a sketch of it. Let us consider a fixed formal system in which theorems (statements) are formulas of the predicate calculus, over the domain of integers, extended by basic arithmetic operations $\{+, -, \times\}$ and with variables being the words of the type $x\#^i, i \geq 0$, where $x \in \{a, b, c, \ldots, z\}$, $\#$ is a special symbol, and where axioms are basic identities of the predicate calculus (for example,

$$(P \cap Q) \Rightarrow P, \quad \neg(\neg P) \Leftrightarrow P, \quad \neg\exists x(R(x)) \Leftrightarrow \forall x(\neg R(x)),$$

where capital letters denote statements), and of arithmetics (commutativity, associativity, distributivity, ...). Derivation rules are basic derivation rules of the predicate calculus as modus ponens (from P and $P \Rightarrow Q$ we can deduce Q) and so on.

Such a formal system has infinitely many formulas of one variable and they can be enumerated in many ways. Let us fix such an enumeration and let us denote by $F_i(x)$ the i-th formula. In this way $F_i(x)$ is a perfectly defined arithmetical statement of two variables.

Strings of propositions that constitute proofs in such a formal system can also be enumerated and let P_i denote the i-th proof.

Consider now the following statement, perfectly well defined, that depends on the variable x:

$$\neg\exists y[P_y \text{ proves } F_x(x)], \tag{6.9}$$

which says that there is no proof of $F_x(x)$ in the formal system under consideration.

It is the most ingenious part of the Gödel's proof that shows that the predicate (6.9) can also be expressed in the considered formal system, i.e. that there is an integer k such that

$$F_k(x) \Leftrightarrow \neg\exists y[P_y \text{ proves } F_x(x)].$$

Details of his proof depend much on details of formalization, coding and enumeration of statements and proofs. However, once this is done we are ready with the proof. Indeed,

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[10] This is a modification of the paradox of Epimenides, the Cretan, who said: *"Cretans always lie."*

- if $F_k(k)$ is true, then $F_k(k)$ is not provable and we have incompletnes;
- if $F_k(k)$ is false, then $F_k(k)$ is provable and we have inconsistency.

This means that we have an example of a specific statement that is true but not provable within the considered formal system. Gödel's result concerns a particular formal system. A more modern approach is to show limitations of all formal systems.

6.5.2 Kolmogorov Complexity: Unsolvability and Randomness

Data compression is an important practical problem. In 1960's Kolmogorov and Chaitin initiated theoretical study of compressibility in order to lay out foundations for such key concepts of probability theory as randomness of strings.[11] Their simple ideas turned out to be surprisingly powerful and currently represent an important methodology to show limitations and lower bounds and to establish foundations of concepts in many areas, also outside of computing (modeling, learning, statistics, physics).

How much can a given string be compressed? How much information contains a given string? How random is a given string? All these questions are basic and closely related. They can be approached rigorously using the concepts of Kolmogorov and Chaitin complexity.

Example 6.5.2 *The string*

$$001001001001001001001001001001001001001001001001001001001001001$$

is intuitively much less random than the string

$$011011100110110011100101111001010010010000111010100010011 0.$$

*since the first string can be written as "*21 *times string* 001*".*

The basic idea of **information complexity** of Kolmogorov and Chaitin is simple. The amount of information of a string w is the length of the shortest string p (called program), from which w can be reconstructed (by a chosen computer C).

The above idea of information-theoretic complexity sounds natural on one side, but too dependent on the choice of C, on the other side. Fortunately, this problem can be overcomed quite satisfactorily, as we shall see soon, taking a universal computer for C.

Let us fix an alphabet Σ and consider a two tape universal Turing machine $\mathcal{U}$ with the tape alphabet $\Sigma_0 = \Sigma - \{\sqcup\}$ that can simulate any one tape TM with the tape alphabet Σ_0. The **Kolmogorov complexity** of an $x \in \Sigma_0^*$, with respect to $\mathcal{U}$, notation $K_U(x)$, is the length of the shortest program p such that if U starts with ε on first tape and p on second tape, then it halts with x on first tape. In other words,

$$K_U(x) = \min\{|p| \,|\, p \in \Sigma_0^*, U(p) = x\}.$$

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[11] By the way, a development of the mathematical foundations of probability was another of the famous Hilbert's problems stated in 1900 at the International Congress of Mathematicians.

A small technical condition allows to make the concept of Kolmogorov complexity only unessentially dependent (by an additive constant factor) on a particular choice of the universal computer. We assume that each TM can be simulated by a program that does not contain the 4-symbol string `DATA` and that if U gets on its second tape an input word p`DATA`x, then U simulates TM p on input x. It is straightforward to show that each universal TM can be modified to meet these conditions and therefore let us assume that so do universal TM indexTuring machine!universalwe consider.

Lemma 6.5.3 $K_U(x) \leq |x| + c_U$*, where the constant* c_U *depends on* U *only.*

Proof. Since U is universal, a one tape TM that does nothing else only immediately halts can be simulated by a program p_0 (not containing `DATA`). Hence, for the input p_0`DATA`x, U writes x on the first tape and halts. Thus $c_U = |p_0| + 4$. □

The following fundamental result shows to which extend is Kolmogorov complexity independent of the choice of the universal computer. This result also justifies that we can fix an arbitrary universal computer U and instead of $K_U(x)$, we can write $K(x)$.

Theorem 6.5.4 (Invariance Theorem) *Let* U *and* S *be universal Turing machine over* Σ_0*. Then there is a constant* $c_{U,S}$ *such that* $|K_U(x) - K_S(x)| \leq c_{U,S}$*, for all* $x \in \Sigma_0^*$*.*

Proof. We can consider S to be simulated by a one-tape TM M in such a way that if a program p produces x on S, then also M produces x for the input p. Assumne that M is simulated on U by a program p_M that does not contain `DATA`. Let now $x \in \Sigma^*$ be an arbitrary string and q_x be the shortest program producing x on S. Then the program p_M`DATA`q_x produces x on U and its length is $|p_M| + |q_x| + 4 = c_{U,S}$. A similar inequality in the opposite direction can be shown analogously. □

Unfortunately, Kolmogorov complexity is not computable.

Theorem 6.5.5 *The function* $K(x)$ *is not recursive.*

Proof. The reasoning is based on the Berry's paradox: *Let n be the smallest positive integer that can not be described with less than 99 symbols.* n is both well defined and cannot exist.

Assume $K(x)$ is computable. Let c be an integer (to be chosen later), and x_k be the k-th word of Σ_0^* in the strict ordering, and y be the first string such that $K(y) \geq c$. Consider the following algorithm:

```
begin $k \leftarrow 0$;
    while $K(x_k) < c$ do $k \leftarrow k + 1$
    write($x_k$)
end
```

The above algorithm produces y such that $K(y)/gec$. The length of the algorithm is $\lg c +$ a constant. If c is large enough, then this algorithm contains less than c symbols and produces x_0. A contradiction. □

Exercise 6.5.6 *Use Theorem 6.5.5 to produce a new proof of undecidability of the halting problem. (Hint: show that given an x, we can consider in strict ordering all possible strings as programs and test each of them whether it halts and if yes, then simulate it, until we come to a program producing x.)*

Exercise 6.5.7 * *Show that for no recursive function f there is a recursive function g such that $K(x) \leq g(x) \leq f(K(x))$. (That is, K cannot be approximated even in a very weak sense.)*

Exercise 6.5.8 ** *Show that there is no algorithm to construct, for any integer n, a binary string x of length n such that $K(x) > 2\lg n$.*

Kolmogorov complexity allows to deal rigorously with the concept of randomness and with the degree of randomness. This is based on the intuition that a string is random if it can not be compressed.

For finite words randomness is a matter of degree. A word w can be seen as **random** if the shortest program describing w is roughly of the same length as w. For example, if the difference $|w| - K(w)$ is small. Depending on the universal computer chosen, the additive constant can be larger or smaller. We can also say, for example, that a word is **fairly random** if $K(w) \geq |w| - 10$.

Almost all words are fairly random. This follows from the following simple result:

Lemma 6.5.9 *The number of binary strings x of length n with $K(x) \leq n - k$ is less than 2^{n-k+1}.*

Proof. The number of binary stings of length at most $n-k$ is $1 + 2 + \ldots + 2^{n-k} < 2^{n-k+1}$. □

As a consequence, Kolmogorov complexity of 99.9% of binary words of length n is at least $n - 10$. In addition, this lemma also provides, in a sense, a "counterexample" to the Church's Thesis. Indeed, let us consider the following problem. For a given n construct a binary string of lenth n whose Kolmogorov complexity is greater than $\frac{n}{2}$. According to Exercise 6.5.8 this problem is unsolvable. On the other hand, Lemma 6.5.9 shows that a randomly chosen string will be appropriate with a large probability.

For infinite words (strings) there is a sharp difference between randomness and non-randomness. An ω-word w is called random if there is a constant c such that

$$K(w_i) \geq i - c,$$

where w_i is the prefix of w of length i.

It has been shown that this concept of randomness of ω-strings is equivalent to those defined by statistical tests.

6.5.3 Chaitin Complexity: Algorithmic Entropy and Information

Seemingly a minor modification of Kolmogorov complexity, independently discovered by Chaitin, leads to various concepts of fundamental importance.

In the definition of Kolmogorov complexity we implicitly assume that a universal computer that simulates programs over an alphabet Σ_0 actually needs to work also with the blank in order to be able to recognize ends of programs. Surprisingly, this fact has quite a few consequences. For example, we cannot append one program after another and to expect that our universal computer can simulate their serial composition unless it is denoted somehow, for example, by a delimiter, where one programs ends and second starts. Similarly, we need a way to separate programs and data. This can result, as we shall see, in adding a logarithmic factor in some considerations.

Chaitin complexity or **self-delimiting complexity** $H_U(x)$, of a string x, with respect to a universal computer U, in short $H(x)$, is the length of the shortest self-delimiting program p that makes U to produce x. More formally,

$$H_U(x) = \min\{|p| \mid U(p) = x, p \text{ is a self-delimiting program}\}.$$

The term self-delimiting means that it is possible to recognize the end of the program by reading all its symbols and nothing else. As a consequence the set of self-delimiting programs has to be prefix-free.

Exercise 6.5.10 Self-delimiting encodings of words (SD). *For a binary word w let $L(w)$ be the length of w written in binary with the leading 1 and $M(w)$ be the word obtained from w by writing 0 between any two bits of w and 1 at the end. (For example, $L(01) = 10, M(01) = 0011, L(0^{15}) = 1111, M(1111) = 10101011$.) $SD(w) = M(L(w))w$ is called the self-delimiting version of w. Show: (a) $|SD(w)| \leq |w| + 2\lg|w|$ for every w and show the case where the equality holds; (b) Show that if $w = SD(w_1)SD(w_2)$, then the decomposition of w into $SD(w_1)$ and $SD(w_2)$ is unique—this property is behind the adjective "self-delimiting".*

Exercise 6.5.11 Kraft's inequality. *A language L is* **prefix-free** *if no word of L is a prefix of another world in L. (a) Show that words in a prefix-free language can be used to encode symbols from another alphabet; (b) Show Kraft's inequality: if $L \subseteq \Sigma^*$ is a prefix-free language and $|\Sigma| = p$, then $\sum_{w \in L} p^{-|w|} \leq 1$.*

In addition, it has turned out useful to define the concept of a **conditional Chaitin complexity** of a string x, with respect to another string t and a universal computer U, as the length of the shortest program p that makes U to compute x, given, in addition, t^*, the smallest program, with respect to strict ordering, to compute t on U. Formally,

$$H(x/t) = \min\{|p| \mid U(p, t^*) = x, p \text{ is a self-delimiting program}\}.$$

(To introduce the concepts of Chaitin complexity and conditional Chaitin complexity we can consider, for example, a 3-tape universal Turing machines that gets, on the second tape, ε in the first case and t^* in the second case, and p on the third tape and produce x on the first tape.)

Similarly, as for Kolmogorov complexity, we can show validity of the Invariance Theorem for both Chaitin and conditional Chaitin complexity and therefore we can fix a universal Turing machine U and write $H(x)$ instead of $H_U(x)$.

Exercise 6.5.12 *Show the Invariance Theorem for: (a) Chaitin complexity; (b) conditional Chaitin complexity.*

We denote briefly $H(w) = H_U(w)$, and call it the **algorithmic entropy** of w or **information-theoretic complexity** of w. Similarly, we denote $H(w/t) = H_U(w/t)$ as the **conditional Chaitin complexity** of w with respect to t. Using a pairing function on strings, that is a bijection $\phi : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$, we can define how to compress a pair of strings. Let

$$H(w,t) = H(\phi(w,t))$$

be the length of the shortest program that outputs w and t in a way that tells them apart.

The amount by which the conditional Chaitin complexity $H(w/t)$ is less than $H(w)$ can be seen as the amount of information t contains about w. This leads to the following definition of the **algorithmic information in t about w**

$$I(t : w) = H(w) - H(w/t).$$

Basic properties of the algorithmic entropy, information and the conditional Chaitin complexity are summarized in Table 6.1. These properties can be shown in a straightforward way using the basic definitions only and therefore their proofs are left as exercises.

$$\begin{array}{ll}
H(w/w) = \mathcal{O}(1), & I(t : w) = \Omega(1), \\
H(w) = H(w,t) + \mathcal{O}(1), & I(t : w) = H(t) + H(w) - H(t,w) + \mathcal{O}(1), \\
H(w/t) = H(w) + \mathcal{O}(1), & I(w : w) = H(w) + \mathcal{O}(1), \\
H(w,t) = H(w) + H(t/w) + \mathcal{O}(1), & I(w : \varepsilon) = \mathcal{O}(1), \\
H(w,t) = H(w) + H(t) + \mathcal{O}(1), & I(\varepsilon : w) = \mathcal{O}(1).
\end{array}$$

Table 6.1: Properties of algorithmic entropy and information

Exercise 6.5.13 *Show the following properties of the algorithmic entropy:*
(a) $H(w/w) = \mathcal{O}(1)$; (b) $H(w) = H(w,t) + \mathcal{O}(1)$; (c) $H(w/t) \leq H(w) + \mathcal{O}(1)$.

Exercise 6.5.14 *Show the following properties of algorithmic information:*
(a) $I(t : \varepsilon) = \Omega(1)$; (b) $I(t : w) = H(t) + H(w) - H(t,w) + \mathcal{O}(1)$.

The following lemma shows that the difference between $K(x)$ and $H(x)$ is not too big and the following exercise shows that it is essential.

Lemma 6.5.15 *$K(x) \leq H(x) \leq K(x) + 2\lg_m K(x) + \mathcal{O}(1)$ for the case of strings x over an m element alphabet.*

Proof. It follows from Exercise 6.5.10 that each program of length n can be transformed into a self-delimiting program of length $n + \lg_m n$.[12] □

Exercise 6.5.16 *Consider Kolmogorov and Chaitin complexity of strings over an m symbol alphabet Σ. Use Lemma 6.5.3 and the Kraft's inequality to show: (a) $\sum_{x\in\Sigma^*} m^{-K(x)} = \infty$; (b) $\sum_{x\in\Sigma^*} m^{-H(x)} \leq 1$.*

Exercise 6.5.17 *Show, considering the function $f(x,\varepsilon) = x$ and the computer $C(SD(x),\varepsilon) = x$ that there is a constant c such that for all $x \in \{0,1\}^*$, $H(x) \leq |x| + 2\lg(|x|) + c$.*

Exercise 6.5.18 *Show that there is a constant c such that for all natural numbers n: (a) $H(n) \leq 2\lg n + c$; (b) $H(n) \leq \lg n + \lg\lg n + c$.*

Remark 6.5.19 On the basis of the Chatin computer and complexity two important probability distributions can be defined.

If $\mathcal{U}$ is a universal Chaitin computer, then the **universal apriori probability** on $\{0,1\}^*$ is defined as

$$Q_{\mathcal{U}}(x) = \sum_{\mathcal{U}(p)=x} 2^{-|p|}$$

and **algorithmic probability** is defined by

$$R(x) = 2^{-H(x)}.$$

Observe that by Kraft's inequality $Q_{\mathcal{U}}(x) < 1$ for any x. It will be shown in Section 6.5.4 that $\sum_{x\in\{0,1\}^*} Q_{\mathcal{U}}(x) < 1$. Note also that the algorithmic probability is larger for regular strings and smaller for random strings. For example, if $H(x) \leq \lg|x| + 2\lg\lg|x| + c$, then $R(x) \geq \frac{1}{cn\lg^2|x|}$ and if $H(x) \geq n$, then $R(x) \leq \frac{1}{2^n}$.

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[12] A more precise relation between Kolmogorov and Chaitin complexity: $H(x) = K(x) + K(K(x)) + \mathcal{O}(K(K(K(x)))$, $K(x) = H(x) - H(H(x)) - \mathcal{O}(H(H(H(x)))$.

6.5.4 Limitations of Formal Systems to Prove Randomness

A general setting will now be presented in which one can introduce in a simple way such basic concepts as formal systems, computer, Kolmogorov and Chaitin complexity and in which it can be easily shown that within each formal system it is possible to prove randomness only of finitely many strings.

The key requirement for a **formal axiomatic system** is that there is an objective criterion for deciding if a proof written in the language of the system is valid or not. In other words, there must be an algorithm for checking proofs. This leads to a very general definition of a formal axiomatic system as a recursively enumerable set.

Definition 6.5.20 *A formal system is a pair* $\langle C, p_0 \rangle$*, where*

$$C : \{0,1\}^* \times \mathbf{N} \mapsto \{S \subseteq \{0,1\}^* \,|\, S \text{ is a finite set}\}$$

is a recursive mapping and $\forall p, t[C(p,t) \subseteq C(p,t+1)]$*.* C *is called an* **inference rule** *and* p *an* **axiom***.*

Interpretation: $C(p,t)$ is the set of theorems (statements) that one can prove from the axiom p through proofs of the length $\leq t$.

In formal systems as defined above we work with binary strings only. However, the i-th strings will be interpreted in the following, depending on the context, either as the i-th string, or the integer i or the i-th statements—an encoding of a theorem.

The following notation will be used to introduce an abstract definition of computer using the following notation. If $f : \{0,1\}^* \times \{0,1\}^* \mapsto \{0,1\}^*$ is a partial function, then for each $y \in \{0,1\}^*$ the projection of f on the second component y is defined by

$$f_y(x) = f(x,y).$$

Definition 6.5.21 *A* **computer** *is a partial recursive function*

$$C : \{0,1\}^* \times \{0,1\}^* \mapsto \{0,1\}^*.$$

Moreover, if the domain of C_y *is a prefix-free language for any* $y \in \{0,1\}^*$*, we speak about a Chaitin computer.*

Interpretation: A computer C produces, for a program p and an input i, a set $C(p,i)$. The additional requirement for Chaitin computers is that if $C(p,i)$ is defined and p is a prefix of p', then $C(p',i)$ is not defined. In other words no program for a successful computation of C can be a prefix of another program for a successful computation of C for the same input—what captures the idea of having only self-delimiting programs. This special requirement will later be relaxed and its importance analyzed.

Also in such a general setting there is the concept of the universal (Chaitin) computer.

Definition 6.5.22 *A (Chaitin) computer* U *is universal if and only if for every (Chaitin) computer* C*, there is a simulation constant* $sim(C)$ *such that whenever* $C(u,v)$ *is defined, then there is a program* u' *such that*

$$U(u',v) = C(u,v) \text{ and } |u'| \leq |u| + sim(C).$$

Theorem 6.5.23 *There exists a universal (Chaitin) computer.*

Proof. Let us take a fixed enumeration $C_1, C_2, \ldots$ of all computers. The function $F : \mathbf{N} \times \{0,1\}^* \times \{0,1\}^* \mapsto \{0,1\}^*$ defined by[13]

$$F(i,u,v) = C_i(u,v), \quad i \in \mathbf{N},\, u,v \in \{0,1\}^*$$

is partial recursive and so is the function $U : \{0,1\}^* \times \{0,1\}^* \mapsto \{0,1\}^*$ defined by

$$U(0^i 1u, v) = C_i(u,v).$$

In addition, the domain of $U(u,v)$ is prefix-free for each v in case of Chaitin computer, since projections of all C_i have this property. This implies that U is a universal computer with $sim(C_i) = i+1$, for each i. $\square$

Also in this general setting we can define analogously the concepts of Kolmogorov and Chaitin complexity (using the concept of Chaitin computerin the second case). Also the Invariance theorem can be shown analogously.

Kolmogorov complexity can be used to show limitations of formal systems to prove randomness, as we shall see now. In order to do that let us first introduce a new general concept of a generating computer.

Definition 6.5.24 *A* **generating computer** *is a pair $\langle C, h_C \rangle$ of recursive functions*

$$\begin{array}{rcl} C : \{0,1\}^* \times \mathbf{N} & \mapsto & \{S \subseteq \{0,1\}^* \,|\, S \textit{ is finite}\}, \\ h_C : \{0,1\}^* \times \mathbf{N} & \mapsto & \{0,1\}, \end{array}$$

such that $\forall p,t[C(p,t) \subseteq C(p,t+1)]$ and if $h_C(p,t') = 1$, then also $h_C(p,t'+1) = 1$ and $C(p,t') = C(p,t'+1)$.

Interpretation: p can be seen as a program and t as time. $C(p,t)$ is the set of words C generates till time t. If $h_C(p,t') = 1$, then C, with the program p halts at time t'.

Observe that there is a close relation between the concept of a formal system and of a generating computer. Our abstract computer can therefore play the rôle of a formal system and its program the rôle of an axiom.

The concept of universality can also be defined for generating computers. A generating computer U is called universal if for each other generating computer C there is a constant $sim(C)$ such that for every program p for C there is a program p' for U such that $|p'| \leq |p| + sim(C)$, $U(p')$ halts if and only if $C(p)$ does, and $U(p') = C(p)$.

Exercise 6.5.25 *Show the existence of universal generating computers.*

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[13] In the following we assume the following way of encoding a pair (i,w) of an integer and a binary sring, by a binary string $0^i 1w$.

In the following it is assumed that there is a universal generating computer U. With respect to U we can define Kolmogorov complexity of finite sets as follows:

$$K(S) = \min\{|p| \,|\, U(p) = S \text{ and } U \text{ halts for } p\}.$$

If $S = \{s\}$ where s is a string, then we usually write $K(s) = K(\{s\})$.

A universal computer can therefore be seen as a universal formal system.

Let us now assume that we have a fixed natural encoding "$K(s) > n$" of statements $K(s) > n$ in the binary alphabet. (Details of the encoding will not be of importance.) The following theorem implies that in any formal system one can prove randomness of only finitely many strings.

Theorem 6.5.26 (Chaitin's Theorem) *For any universal computer (formal system) U there is a constant c such that for all programs p the following holds: if for every integer n an encoding of the statement "$K(s) > n$" (as a string), is in $U(p)$ if and only if $K(s) > n$, then "$K(s) > n$" is in $U(p)$ only if $n < |p| + c$.*

Proof. Let C be a generating computer such that, for a given program p', C tries first to make the decomposition $p' = 0^k1p$. If this is not possible C halts generating the empty set. Otherwise, C simulates U on p, generates $U(p)$ and searches $U(p)$ to find an encoding "$K(s) > n$" for some $n \geq |p'| + k$. If the search is successful, C halts with s as the output.

Let us now consider what happens if C gets the string $0^{sim(C)}1p$ as the input. If $C(0^{sim(C)}1p) = \{s\}$, then from the definition of a universal generating computer it follows:

$$K(s) \leq |0^{sim(C)}1p| + sim(C) = |p| + 2sim(C) + 1. \tag{6.10}$$

On the other hand, the fact that C halts with the output $\{s\}$ implies that

$$n \geq |p'| + k = |0^{sim(C)}1p| + sim(C) = |p| + 2sim(C) + 1$$

and we get

$$K(s) > n > |p| + 2sim(C) + 1$$

what contradicts the inequality (6.10). The assumption that C can find an encoding of an assertion $K(s) > n$ leads therefore to a contradiction. Since "$K(s) > n$" if and only if $K(s) > n$, this implies that for the assertions (theorems) $K(s) > n$, $n \geq |p| + 2sim(C) + 1$ there is no proof in the formal system $\langle U, p\rangle$. □

Observe that the proof is again based on the Berry's paradox and its modification: "Find a binary string that can be proven to be of Kolmogorov complexity greater than the number of bits in the binary version of this statement."

6.5.5 The Number of Wisdom

We discuss now a special number that encodes very compactly the halting problem.

Definition 6.5.27 *The* **number of wisdom**, *or the* **halting probability of the universal Chaitin computer** U, *is defined by*

$$\Omega = \sum_{\substack{u\in\{0,1\}^* \\ U(u,\varepsilon) \text{ is defined}}} 2^{-|u|}$$

The following lemma is a justification why we use the term "probability" for Ω.

Lemma 6.5.28 $0 < \Omega < 1$.

Proof. Since the domain of U_ε is a prefix-free language the Kraft's inequality (see Exercise 6.5.11), implies $\Omega \le 1$. Since U is a universal computer there exists a u such that $U(u, \varepsilon)$ converges and such a u that it does not. This implies $0 < \Omega < 1$. □

Let us now analyze Ω in order to see whether its catchy name is justified. In order to do that let us assume that

$$\Omega = 0.b_1b_2b_3\dots$$

be the binary expansion of Ω. (As shown later this expansion is unique.)

We first show that Ω encodes the halting problem of Turing machines very compactly and that bits of Ω have the properties that justify to call them "magic bits".

The domain of U_ε, that is the set

$$dom(U_\varepsilon) = \{w \mid w \in \{0,1\}^*, U(w,\varepsilon) \text{ is defined}\},$$

is recursively enumerable. Let $g : \mathbf{N} \mapsto dom(U_\varepsilon)$ be a bijection—a fixed enumeration of $dom(U_\varepsilon)$; (such an enumeration can be obtained, for example, by dovetailing). Denote

$$\omega_n^g = \sum_{j=1}^{n} 2^{-|g(j)|}, \text{ for } n \ge 1.$$

The sequence ω_n^g is clearly increasing and converges to Ω. Moreover, it holds

Lemma 6.5.29 *Whenever* $\omega_n^g > \Omega_i = 0.b_1b_2\dots b_i$, *then*

$$\Omega_i < \omega_n^g < \Omega \le \Omega_i + 2^{-i}.$$

Moreover, given any i, *if we know the first* i*-bits of* Ω, *then we can decide the halting problem of any program with length* $\le i$.

Proof. The inequality $\Omega \le \Omega_i + 2^{-i}$ is a consequence of the inequality $2^{-i} \ge \sum_{j=i+1}^{\infty} b_j 2^{-j}$. In order to prove the second assertion of the lemma let us assume that we know the first i-bits of Ω and therefore Ω_i. We can then compute the numbers $\omega_1^g, \omega_2^g, \dots$ until we find an n such that $\Omega_i < \omega_n^g$. It follows from the definition of the sequence $\omega_n^g, n = 1, 2, \dots$ that this is always possible.

Let $u \in \{0,1\}^*, |u| = i_1 \le i$ and n be such that $\Omega_i < \omega_n^g$. We show that $U(u, \varepsilon)$ is defined if and only if u is one of the words $g(1), \dots, g(n)$. The "if" claim is trivially true. In order to prove the "only-if" claim let us assume that $u = g(m)$, for an $m > n$. In such a case the following inequalities lead to a contradiction:

$$\Omega > \omega_n^g + 2^{-i_1} \ge \omega_n^g + 2^{-i} > \Omega_i + 2^{-i} \ge \Omega.$$

□

It follows now from the discussion at the beginning of Section 6.4.2 that knowledge of sufficiently many bits of Ω could be used to solve the halting problems of all Turing machines to a certain size and thereby to find an answer to many open problems of mathematics (and therefore, for example, also of the PCP of the reasonable size). The question how many bits of Ω would be needed, depends on the formal system used and also on how the universal computer is programmed. We have used programs of the type 0^i where i represents a computer C_i. A more compact programming of U is possible, for example, using the technique of the Exercise 6.5.10 to make words to be self-delimited. A more detailed analysis reveals that knowing 10 000 bits of Ω would be sufficient to deal with the halting problem of Turing machines looking for counter-examples of practically all famous open problems of discrete mathematics.

Ω could also be used to decide whether a well-formed formula of a formal theory is a theorem, a negation of a theorem or it is independent (i.e., it is unprovable within the given formal system). Indeed, let us consider a formal system F with an axiom and rules of inference. For any well-formed formula α design a Turing machine $TM(F,\alpha)$ that checks systematically all proofs of F and halts if U finds one for α. Similarly, for $\neg\alpha$. Knowing sufficiently large portion of Ω we could decide whether α is provable, refutable or independent. Ω therefore deserves the name "number of wisdom"—it can help to solve many problems. Unfortunately, "*Nichts ist vollkommen*" as a German proverb and the following Theorem say.

Theorem 6.5.30 *If $\Omega = 0.b_1b_2b_3\ldots$, then the ω-word $b_1b_2b_3\ldots$ is random.*

Proof. We use the same notation as in the proof of Lemma 6.5.29. It was shown there that if $U(u_1,\varepsilon)$ is defined, $\omega_n^g \geq \Omega_i$, and $|u_1| \leq i$, then u_1 is one of the words $g(1),\ldots,g(n)$. Therefore

$$\{U(g(j),\varepsilon)\,|\,1 \leq j \leq n \text{ and } |g(j)| \leq i\} = \{w\,|\,H(w) \leq i\} \tag{6.11}$$

because if $H(w) \leq i$, then w has a program of length at most i.

Let now $f : \{0,1\}^* \mapsto \{0,1\}^*$ be defined as follows: if $x = x_1\ldots x_t$, $x_j \in \{0,1\}$, for $1 \leq j \leq t$, and m is the smallest integer such that $\omega_m^g > \sum_{j=1}^{t} x_j 2^{-j}$, then $f(x)$ is the first word, in the strict ordering, not in the set

$$\{g(j)\,|\,1 \leq j \leq m\}.$$

Let C be the computer defined by $C(x,\varepsilon) = f(U(x,\varepsilon))$. For each $B_i = b_1\ldots b_i$ we get consequently

$$\begin{aligned}
H(f(B_i)) &\leq H_C(f(B_i)) + sim(C)\\
&= \min\{|u|\,|\,C(u,\varepsilon) = f(B_i)\} + sim(C)\\
&= \min\{|u|\,|\,f(U(u,\varepsilon)) = f(B_i)\} + sim(C)\\
&\leq \min\{|u|\,|\,U(u,\varepsilon) = B_i\} + sim(C)\\
&= H(B_i) + sim(C).
\end{aligned}$$

Since $f(B_i)$ is the smallest word not in the set $\{g(j)\,|\,1 \leq j \leq m\}$, where $\omega_m^g \geq \Omega_i$, we get from (6.11) that $H(f(B_i)) > i$ and therefore $H(B_i) \geq i - sim(C)$ what implies that B_i is random. □

It follows from Theorem 6.5.30 that we are able to determine B_i only for finitely many i in any formal system. It can also be shown that in any formal system we can determine only finitely many bits of Ω.

Remarkable properties of Ω were illustrated on exponential Diophantine equations. Chaitin (1987) proved that there is a particular "exponential" Diophantine equation

$$P(i, x_1, \ldots, x_n) = Q(i, x_1, \ldots, x_n), \tag{6.12}$$

where P and Q are functions built from variables and integers by the operations of addition, multiplication and exponentiation and such that for each integer i the equation (6.12) has infinitely many solutions, if and only if $b_i = 1$, i.e. if and only if the i-th bit of the binary expansion of Ω equals 1. This implies, in the light of the previous discussions, that in any formal theory we can decide only for finitely many i whether the equation (6.12) has infinitely many solutions. This implies that randomness is deeply rooted already in elementary arithmetic.

Remark 6.5.31 Limitations of computers and formal systems that we have derived in this chapter would have really very strong implications would our mind work algorithmically. This, however, does not seem to be the case. Understanding of mind is currently one of the main problems of science in general. This seems to be a very difficult task (and there are respectable scientists that believe that this may require, for example, a new quantum physics to develop.

Moral. A search for borderlines between posssible and impossible is one of the main aims and also tools of science. A discovery of such limitations and borderlines is often the very beginning of a long chain of very fruitful contributions of science. A good rule of thumb by a search for limitations in computing is, as in life, to eliminate impossible and to take whatever remains as the truth.

6.6 Exercises

1. Let $A, B \subseteq \{0,1\}^*$. Which of the following claims are true? (Prove your statements.)
    (a) If A is recursively enumerable, then so is $Prefix(A)$.
    (b) If the set $\{0\}A \cup \{0\}B$ is recursive, then so are A and B.
    (c) If the set $A \cup B$ is recursive, then so are A and B.

2. Show that the following families of languages are closed under operations of iteration and shuffle: (a) recursive languages; (b) recursively enumerable languages.

3. Given a function $f : \mathbf{N} \mapsto \mathbf{R}$, construct, using the diagonalization method, a real number that is not in the range of f.

4. Show that the following sets are not recursive: (a) $\{\langle\mathcal{M}\rangle_\rho \,|\, \mathcal{M}$ halts on an odd number of inputs$\}$; (b) $\{\langle\mathcal{M}\rangle_\rho \,|\, \mathcal{M}$ halts on all inputs$\}$.

5. Show that if $\mathcal{U}$ is a universal Turing machine for all k-tape Turing machines, then the set of inputs for which $\mathcal{U}$ halts is recursively enumerable but not recursive.

6. Let f be a recursive nondecreasing function. Show that there is a primitive recursive function g such that $g(n) \leq f(n)$ for all n and $lim_{n \to \infty} = \infty$..

7. * Give an example of a partial recursive function that is not extendable to a recursive function and whose graph is recursive. (Hint: consider the running time of a universal TM.)

8. Show that if the argument functions of operations of composition and minimization are Turing machine computable, then so are the resulting functions.

9. Show that the following predicates (functions from $\mathbf{N}$ into $\{0,1\}$) are primitive recursive: (a) $sg(x) = 1$ iff $x \neq 0$; (b) $\overline{sg}(x) = 0$ iff $x \neq 0$; (c) $eq(x,y) = 1$ iff $x = y$; (d) $neq(x,y) = 1$ iff $x \neq y$.

10. Show that primitive recursive predicates are closed under Boolean operations.

11. Show that the following functions are primitive recursive: (a) the remainder of dividing x by y; (b) integer division; (c) $g(n) = n \dot{-} \lfloor \sqrt{n} \rfloor^2$.

12. Show that the function $f(x) =$ "the largest integer less than or equal to $\sqrt{x}$" is primitive recursive.

13. Show, for the pairing function $pair(x,y)$ and the depairing functions π_1 and π_2 introduced in Section 6.2 that: (a) $pair(x,y) = \frac{(x+y)^2+x+3y}{2}$; (b) $\pi_1(x) = x \dot{-} \frac{1}{2} \lfloor \frac{\lfloor \sqrt{8x+1} \rfloor + 1}{2} \rfloor \cdot \lfloor \frac{\lfloor \sqrt{8x+1} \rfloor - 1}{2} \rfloor$; (c) $\pi_2 =$?.

14. Let $pair(x_1,x_2,x_3,x_4,x_5) = pair(x_1, pair(x_2, pair(x_3, pair(x_4,x_5))))$ and let us define $\pi_{5,i}(x) = x_i$ for $1 \leq i \leq 5$. Express functions $\pi_{5,i}$ using functions π_1, π_2.

15. Show that the following general pairing function $prod : \mathbf{N}^3 \mapsto \mathbf{N}$ is primitive recursive: $prod(n,i,x) = x_i$, where $pair(x_1, \ldots, x_n) = x$.

16. $*$ Define a primitive recursive function f as follows:

$$f(x,y) = prod(y+1, \pi_1(x)+1, \pi_2(x))$$

and a sequence $d_i, 1 \leq i$, of partial recursive functions

$$d_x(y) = \begin{cases} f(x,y) - 1, & \text{if } 0 < f(x,y) \text{ and } y < \pi_1(x) + 1 \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Show that the sequence of functions $d_0, d_1, \ldots$ contains all partial recursive functions with the finite domain and each exactly once. Show that if $f : \mathbf{N} \to \mathbf{N}$ is a primitive recursive, then so is the function $g(n,x) = f^{(n)}(x)$.

17. Show that the Ackermann function has the following properties for any $i, j \in \mathbf{N}$:
(a) $A(i, j+1) > A(i, j)$; (b) $A(i+1, j) > A(i, j)$.

18. There are various modifications of the Ackermann function introduced in Section 6.2.2. For example, the function A' defined as follows: $A'(0, j) = j + 1$ for $j \geq 0$, $A'(i, 0) = A'(i-1, 1)$ for $i \geq 1$ and $A'(i, j) = A'(i-1, A'(i, j-1))$ for $i \geq 1, j \geq 1$. Show that $A'(i+1, j) \geq A'(i, j+1)$, for all $i, j \in \mathbf{N}$.

19. $*$ (**Fixed-point Theorem**) Let f be a recursive function that maps TM into TM. Show that there is a TM $\mathcal{M}$ such that $\mathcal{M}$ and $f(\mathcal{M})$ compute the same function.

20. Show that for every recursive function $f(n)$ there is a recursive language that is not in the complexity class $Time(f(n))$.

21. Determine for each of the following instances of the PCP whether they have a solution and if they do find one: (a) $A = (abb, a, bab, baba, aba)$, $B = (bbab, aa, ab, aa, a)$; (b) $A = (bb, a, bab, baba, aba)$, $B = (bab, aa, ab, aa, a)$; (c) $A = (1, 10111, 10)$, $B = (111, 10, 0)$; (d) $A = (10, 011, 101)$, $B = (101, 11, 011)$; (e) $A = (10, 10, 011, 101)$, $B = (101, 010, 11, 011)$; (f) $A = (10100, 011, 01, 0001)$, $B = (1010, 101, 11, 0010)$; (g) $A = (abba, ba, baa, aa, ab)$, $B = (baa, aba, ba, bb, a)$; (h) $A = (1, 0111, 10)$, $B = (111, 0, 0)$; (i) $A = (ab, ba, b, abb, a)$, $B = (aba, abb, ab, b, bab)$.

22. Show that the PCP is decidable for lists with: (a) one element; (b) two elements.

23. * Show that the PCP with lists over a two-letter alphabet is undecidable.

24. Show that the following modification of PCP is decidable. Given two lists of words $A = (x_1, \ldots, x_n)$, $B = (y_1, \ldots, y_n)$ over the alphabet $\Sigma, |\Sigma| \geq 2$, are there $i_1, \ldots, i_k$ and $j_1, \ldots, j_l$ such that $x_{i_1} \ldots x_{i_k} = y_{j_1} \ldots, y_{j_l}$?

25. Show that the following modifications of the Post Correspondence Problem are undecidable: (a) Given two lists $(u, u_1, \ldots, u_n)$, $(v, v_1, \ldots, v_n)$, is there a sequence of integers $i_1, \ldots, i_n$, $1 \leq i_k \leq m$, $1 \leq k \leq m$, such that $uu_{i_1} \ldots u_{i_m} = vv_{i_1} \ldots v_{i_m}$; (b) given lists $(u, u_1, \ldots, u_n, u')$, $(v, v_1, \ldots, v_n v')$, is there a sequence of integers $i_1, \ldots, i_m$, $1 \leq i_k \leq n$, $1 \leq k \leq m$, such that $uu_{i_1} \ldots u_{i_m} u' = vv_{i_1} \ldots v_{i_m} v'$.

26. ** An affine transformation on $\mathbf{N} \times \mathbf{N}$ is a mapping $f(x, y) = (ax + by + c, dx + ey + f)$, where a, b, c, d, e, f are fixed whole numbers. Show, for example by a reduction from the modified Post Correspondence Problem, from Exercice 25, that it is undecidable, given a pair $(x_0, y_0) \in \mathbf{N}$ and a finite set $\mathcal{S}$ of affine transformations whether there is a sequence $f_1, \ldots, f_k$ of affine transformations from $\mathcal{S}$ such that $f_1(f_2(\ldots f_k(x_0, y_0) \ldots)) = (x, x)$ for some x.

27. Given a set S of Wang tiles, we can assume that colors used are numbered and a set of tiles is represented by a word over the alphabet $\{0, 1, \#\}$—by writing in a clock-wise manner numbers of colors of all tiles, one after another, and separating them with #. Denote by TIL the set of those words that describe sets of tiles (with the initial tile), for which the plane is tileable. Show that the language TIL is recursively enumerable.

28. Show that the following modification of the tiling problem from Theorem 6.4.10 is also undecidable: tiles are colored only at corners and all tiles meeting in a corner have to have corners colored by the same colour.

29. Let us use quadruples $(up, right, down, left)$ to denote coloring of Wang tiles. Which of the following sets of Wang tiles can be used to tile the plane:

    (a) $(a,w,w,w),(w,w,b,c),(b,c,a,w)$; (b) $(a,w,w,w),(w,w,a,c),(b,c,b,w)$.

30. Show that the following modification of the tiling problem is also undecidable: all unit square tiles have marked corners and a tiling is consistent if colour of all four tiles that meet at a point are identical.

31. Show that the Hilbert's Tenth Problem is equivalent to the problem to decide for an arbitrary polynomial $P(x_1,\dots,x_n)$ with integer coefficients whether the equation $P(x_1,\dots,x_n)=0$ has a solution in integers.

32. Show, using a result presented in this chapter, that primes have short certificates.

33. Suppose that $L=\{x\,|\,\exists x_1,\dots,x_n[f(x,x_1,\dots,x_n)=0]\}$, where f is a polynomial. Construct a polynomial g such that $L=\{g(x_1,\dots,x_n)\,|\,g(x_1,\dots,x_n)\geq 0\}$.

34. Suppose that $L=\{x\,|\,\exists x_1,\dots,x_n\in\mathbf{Z}[f(x,x_1,\dots,x_n)=0]\}$, where f is a polynomial. Construct a polynomial g such that $L=\{x\,|\,\exists x_1,\dots,x_n\in\mathbf{N}[g(x,x_1,\dots,x_n)=0]\}$.

35. Reduce the problem of finding rational solutions to polynomial equations to the problem of finding integer solutions to polynomial equations.

36. **Show that there is an algorithm for solving Diophantine equations over **N** if and only if there is an algorithm for solving four-degree Diophantine equations over **N**. (Hint: using the distributive law each polynomial can be written as a sum of terms, where each term is a product of a multiset of variables. To each such a multiset S associate a new variable,...)

37. A function $f(x_1,\dots,x_n)$ is called Diophantine, if there is a Diophantine equation $D(a_1,\dots,a_n,x_1,\dots,x_n)$ such that

    $$x=f(x_1,\dots,x_n)\equiv\exists a_1\dots a_n D(a_1,\dots,a_n,x,x_1,\dots,x_n)=0.$$

    Show that the following functions are Diophantine: (a) gcd; (b) lcm.

38. Let A,B and C be any sets such that $A\leq_m C$ and $B\leq_m C$. Is it true that $A\oplus B\leq_m C$?

39. A recursively enumerable set S is said to be m-complete if and only if $S'\leq_m S$ for any recursively enumerable set S'. Show that if a recursively enumerable set S is productive and $S\leq_m S'$, then S' is also productive and that every m-complete set is creative.

40. (**Properties of the arithmetical hierarchy**). Show: (a) if $A\leq_m B$ and $B\in\Sigma_i$, then $A\in\Sigma_i$; (b) if $A\leq_m B$ and $B\in\Pi_i$, then $A\in\Pi_i$; (c) $\Sigma_i\subseteq\Sigma\Sigma_i$, $\Sigma_i\subseteq\Pi\Sigma_i$, $\Pi_i\subseteq\Sigma\Pi_i$ and $\Pi_i\subseteq\Pi\Pi_i$ for $i\in\mathbf{N}$; (d) $\Sigma_i\subseteq\Pi_{i+1}$ and $\Pi_i\subseteq\Sigma_{i+1}$, for $i\in\mathbf{N}$; (e) $\Sigma_i\subseteq\Sigma_{i+1}$, $\Pi_i\subseteq\Pi_{i+1}$, for $i\in\mathbf{N}$; (f) Σ_i and Π_i are closed under union and intersection.

41. Show, for each $i \in \mathbf{N}$, that the language $\emptyset^i$, obtained by applying the jump operation to $\emptyset$ i-times, is complete for the class Σ_i, in the sense that it is in Σ_i and all languages in Σ_i are m-reducible to it.

42. (a) Show an example of a prefix-free language such that Kraft's inequality is strict; (b) Show an example of a prefix-free language such that the Kraft's inequality is an equality.

43. Show that the language $S = \{SD(x) \,|\, x \in \{0,1\}^*\}$ is prefix-free and every natural number n has in S a representation with $\lg n + 2\lg\lg n$ bits.

44. Let A be a finite alphabet. Show: (a) If $S \subseteq A^*$, then the following statements are equivalent: (1) S is prefix-free, (2) $S \cap S\Sigma^+ = \emptyset$; (b) for all prefix-free languages S, T it holds: if $S\Sigma^* = T\Sigma^*$, then $S = T$.

45. Show the following properties of the algorithmic entropy: (a) $H(w,t) = H(w) + H(t/w) + \mathcal{O}(1)$; (b) $H(w,t) = H(w) + H(t) + \mathcal{O}(1)$.

46. Show the following properties of the algorithmic information: (a) $I(w : w) = H(w) + \mathcal{O}(1)$; (b) $I(w : \varepsilon) = \mathcal{O}(1)$; (c) $I(\varepsilon : w) = \mathcal{O}(1)$.

47. * Let $CP = \{x^* \,|\, x \in \{0,1\}^*\}$. (That is, CP is the set of minimal programs. Show: (a) there is a constant c such that $H(y) \geq |y| - c$ for all $y \in CP$; (b) The set CP is immune.

48. * Show that $H(x) \leq |x| + 2\lg\lg|x| + c$, for a constant c, for all $x \in \{0,1\}^*$.

49. Show that the set $I_{KC} = \{x \in \Sigma^* \,|\, K(x) \geq |x|\}$ is immune and its complement is recursively enumerable.

QUESTIONS

1. How can one transfer such concepts as recursivity and recursive enumerability to sets of graphs?

2. Ackermann function grows faster than any primitive recursive function. It would therefore seem that its inverse grows slower than any other nondecreasing primitive recursive function. Is it true? Justify your claim.

3. What type of problems would be solvable would the halting problem be decidable?

4. Is there a set of tiles that can tile the plane both periodically and also aperiodically?

5. Which variants of PCP are decidable?

6. Is the problem to solve a system of Diophantine equations computationally more difficult than the problem to solve one Diphantine equation?

7. Whyis the inequality $K(x) \leq |x|$ not valid in general?

8. How to define conditional Kolmogorov complexity?
9. How to define random languages?
10. Is the number of wisdom unique?

6.7 Historical and Bibliographical References

Papers by Gödel (1931) and Turing (1937) that showed in an indisputable way limitations of formal systems and algorithmic methods can be also seen as marking a new era in mathematics, computing and science in general.

Turing's model of computability based on his concept of machine has finally turned out more inspiring then the computationally equivalent model of partial recursive functions introduced by Kleene (1936). However, it was the theory of partial recursive, recursive and primitive recursive functions that has developed first very fast, due to its elegance and more traditional mathematical framework. This theory, that has since then a firm place in the theory of computing was at the beginning considered as a part of number theory and logic.

The origin of recursive function theory can be traced far into the history of mathematics. For example, Herrmann Grassmann (1809-1877) in his textbook in 1861 used primitive recursive definitions for addition and multiplication. Richard Dedekind (1831-1916), known also for his saying: *Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden*, proved in 1881 that primitive recursion uniquely defines a function. A systematic development of recursive functions is due to Skolem (1887-1963) and especially Rózsa Péter (1906-1977) with her book in 1951.

The results on recursively enumerable and recursive sets are from Post (1944). The exposition of pairing and depairing functions is from Engeler (1973). Exercise 16 is from Smith (1994).

There are nowadays numerous books on recursive functions. For example: Péter (1951), Malcev (1965), Davis (1958, 1965), Rogers (1967), Minsky (1967), Machtey and Young (1978), Cohen (1987), Odifredi (1989) and Smith (1994). The characterization of primitive recursive functions in terms of **for**-programs is due to Meyer and Ritchie (1967).

Various concepts of computable real numbers form bases for recursive function based approaches to calculus. See Weihrauch (1987) for a detailed exposition. The concept of limiting reccursive real number was introduced by Korec (1986).

Undecidability is also dealt with in many books. For a systematic presentation see, for example, Davis (1965) and Rozenberg and Salomaa (1994), where also philosophical and other broader aspects of undecidability and unsolvability are discussed in an illuminating way.

Theorem 6.4.2 is due to Rice (1953). The undecidability of the halting problem is due to Turing (1937). The first undecidable result on tiling is due to Berger (1966). A very thorough presentation of various tiling problems and results is due to Grünbaum and Shephard (1987). Their book and also Gardner's (1989) contain also a detailed presentation of Penrose's tilings and their properties. Aperiodic tiling of the plane with 13 Wang dominoes is described by Culik (1996). For the importance of tiling for proving undecidability results see van Emde

Boas (1982). The Post Correspondence Problem is due to Post (1946) and for the proof see Hopcroftand Ullman (1969), Salomaa (1973) and Rozenberg and Salomaa (1994), where a detailed discussion of the problem can be found. The undecidability of the Thue problem for semigroups was shown by Post (1947) and Markov(1947), for groups by Novikov (1955) and the decidability of the Thue problem for Abelian semigroups is due to Malcev (1958). The Thue problem (E1) from page 440 is from Penrose (1990) and Thue problem (E2) is, Penrose's modification of the problem due to G.S. Tseitin and D. Scott—see Gardner (1958). Hilbert's Tenth Problem—Hilbert (1935), was solved with large effort and contributions by many authors (J. Robinson and M. Davis). Final step was done by Matiyasevich (1971). For the history of the problem and related results see Davis (1980) and Matiyasevich (1993). For another presentation of the problem see Cohen (1978) and Rozenberg and Salomaa (1994). The first part of Example 6.4.22 is from Rozenberg and Salomaa (1994), the second from Babai (1990)—for the solution of the second one see Archibald (1918). For the borderlines between decidability and undecidability of the halting problem for one-dimensional one-tape Turing machines see Rogozhin (1996); for two dimensional Turing machines see Priese (1979b), for undecidability of the equivalence problem for register machines see Korec (1977) and for undecidability of the halting problem for register machines see Korec (1996).

For a more readible presentation of the Gödel's incompleteness theorem see also Rozenberg and Salomaa (1994). Limitations of formal systems to prove randomness are due to Chaitin (1987a, 1987b). See also Rozenberg and Salomaa (1994) for another presentation of these results as well as results concerning the magic number of wisdom. Two concepts of descriptional complexity based on the length of the shortest description are due to Solomonoff (1960), Kolmogorov (1965) and Chaitin (1966). For a comprehensive presentation of Kolmogorov/Chaitin complexity and its relation to randomness, as well as for proofs that new concept of randomness agree with that defined using statistical tests see Li Ming and Vitányi (1993) and Calude (1994).There is several names and notations used for Kolmogorov and Chaitin complexities. For example, Li and Vitanyi (1993) use terms plain Kolmogorov complexity ($C(x)$) and prefix Kolmogorov complexity ($K(x)$). A more precise relation between these two types of complexity from page 455 was established by R. M. Solovay.

Chapter 7

REWRITING

INTRODUCTION

Formal grammars and, more generally, rewriting systems, are as indispensable tools for description and recognition of complex objects, their structure and semantics, as grammars of natural languages are for our capability to communicate exactly. The main concepts, methods and results concerning string and graph rewriting systems are presented and analyzed in this chapter. In the first part the concentration is on Chomsky grammars and related automata, especially context-free grammars and languages are discussed in detail. Basic properties and surprising applications of parallel rewriting systems are then demonstrated. Finally, several main techniques describing how to define rewriting in graph grammars are introduced and illustrated.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. basic rewriting systems and their applications;
2. main relations between rewriting systems and automata;
3. basics of context-free grammars and languages;
4. a general method of recognition and parsing of context-free languages;
5. Lindenmayer systems and their use for graphical modeling;
6. main types of graph grammar rewritings.

> To change your language
> you must change your life.
> *Derek Walcott*

Rewriting is a technique for defining or designing/generating complex objects by successive replacing parts of a simple initial object using a set of rewriting rules. The main advantage of rewriting systems is that they also assign a structure and a derivation history to objects they generate. This can be utilized to recognize and manipulate objects and to assign a semantics to them.

String rewriting systems, usually called grammars, have their origin in the mathematical logic (due to A. Thue (1906) and E. Post (1943)), especially in the theory of formal systems. The American linguist Noam Chomsky showed in 1957 how to use formal grammars to describe and study natural languages. The fact that context-free grammars turned out to be a useful tool to describe programming languages, and to design compilers, was another strong stimulus for an explosion of interest of computer scientists in rewriting systems. Biological motivation was behind the development of the so-called Lindenmayer systems. Nowadays also rewriting systems for more complex objects, as terms, arrays, graphs and pictures, are of growing interest and importance.

Rewriting systems have turned out also to be good tools to investigate the objects they generate, i.e. string and graph languages. Basic rewriting systems are closely related to the basic models of automata.

7.1 String Rewriting Systems — November 8, 2005

Basic ideas of sequential string rewriting were introduced and well formalized by semi-Thue systems[1].

Definition 7.1.1 *A* **production system** $\mathcal{S} = \langle \Sigma, P \rangle$, *over an* **alphabet** Σ, *is defined by a finite set* $P \subseteq \Sigma^* \times \Sigma^*$ *of* **productions**. *A production* $(u, v) \in P$ *is usually written as* $u \xrightarrow[P]{} v$ *or* $u \longrightarrow v$, *if* P *is clear from the context.*

There are many ways one can use a production system to define a rewriting relation (rule) and thereby to create a rewriting system.

A production system $\mathcal{S} = \langle \Sigma, P \rangle$ is called a **semi-Thue system** if the following **rewriting relation (rule)** $\xRightarrow[P]{}$ on Σ^* is used:

$$w_1 \xRightarrow[P]{} w_2 \quad \text{if and only if} \quad w_1 = xuy, w_2 = xvy, \text{ and } (u, v) \in P.$$

A sequence of strings $w_1, w_2, \ldots, w_n$ such that $w_i \xRightarrow[P]{} w_{i+1}$ for $1 \leq i < n$, is called a **derivation**. The transitive and reflexive closure $\xRightarrow[P]{*}$ of the relation $\xRightarrow[P]{}$ is called a **derivation**

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[1] Axel Thue (1863-1922), a Norwegian mathematician.

relation. If $w_1 \overset{*}{\underset{P}{\Longrightarrow}} w_2$, we say that the string w_2 can be derived from w_1 by a sequence of rewriting step defined by P. A semi-Thue system $\mathcal{S} = \langle \Sigma, P \rangle$ is called a **Thue system** if the relation P is symmetric.

Example 7.1.2 *$\mathcal{S}_1 = \langle \Sigma_1, P_1 \rangle$, where $\Sigma_1 = \{a, b, S\}$ and*

$$P_1 : \qquad S \longrightarrow aSb, \qquad S \longrightarrow ab,$$

is a semi-Thue system.

Example 7.1.3 *$\mathcal{S}_2 = \langle \Sigma_2, P_2 \rangle$, where $\Sigma_2 = \{A, C, E, I, L, M, N, O, P, R, T, W\}$ and*

$$P_2 : \begin{array}{rcl rcl rcl} EAT & \longrightarrow & AT & AT & \longrightarrow & EAT & ATE & \longrightarrow & A \\ A & \longrightarrow & ATE & LATER & \longrightarrow & LOW & LOW & \longrightarrow & LATER \\ PAN & \longrightarrow & PILLOW & PILLOW & \longrightarrow & PAN & CARP & \longrightarrow & ME \\ ME & \longrightarrow & CARP & & & & & & \end{array}$$

is a Thue system.

Two basic problems for rewriting systems $\mathcal{S} = \langle \Sigma, P \rangle$ are:

- **Word problem**: Given $x, y \in \Sigma^*$, is it true that $x \overset{*}{\underset{P}{\Longrightarrow}} y$?
- **Characterization problem**: For which strings $x, y \in \Sigma^*$ does the relation $x \overset{*}{\underset{P}{\Longrightarrow}} y$ hold?

For some rewriting systems the word problem is decidable, for others not.

Example 7.1.4 *For the semi-Thue system $\mathcal{S}_1$ from Example 7.1.2 we have*

$$S \overset{*}{\Longrightarrow} w \text{ if and only if } w = a^i S b^i \text{ or } w = a^i b^i, \text{ for some } i \geq 1.$$

using this result we can easily design an algorithm to decide the word problem for $\mathcal{S}_1$.

Exercise 7.1.5 * *(a) Show that the word problem is decidable for the Thue system $\mathcal{S}_2$ from Example 7.1.3; (b) show that if $x \overset{*}{\underset{P_2}{\Longrightarrow}} y$, then both x and y have to have the same number of occurrences of symbols from the set $\{A, W, M\}$. (This implies, for example, that $MEAT \overset{*}{\underset{P_2}{\not\Longrightarrow}} CARPET$—see Section 6.4.4.)*

Exercise 7.1.6 *Show that there is no infinite derivation, no matter with which word we start, in the semi-Thue system with the alphabet $\{A, B\}$ and the productions $BA \longrightarrow AAAB$, $AB \longrightarrow B$, $BBB \longrightarrow AAAA$, $AA \longrightarrow A$.*

A production system $S = \langle \Sigma, P \rangle$ is called a **Post normal system** if:

$$w_1 \underset{P}{\Longrightarrow} w_2, \quad \text{if and only if} \quad w_1 = uw, w_2 = wv, \text{ and } (u \longrightarrow v) \in P.$$

In other words, in a Post normal system in each rewriting step a prefix u is removed from a given word uw and a word v is added, provided $(u \longrightarrow v)$ is a production of S.

Exercise 7.1.7 *Design a Post normal system that generates longer and longer prefixes of the Thue ω-word.*

If left sides of all productions of a Post normal system $S = \langle \Sigma, P \rangle$, have the same length, and the right side of each production depends only on the first symbol of the left side, we speak of a **tag system**. Observe that a tag system can be alternatively specified by a morphism $\phi : \Sigma \rightarrow \Sigma^*$, $(a \rightarrow \phi(a)$, $a \in \Sigma$, is again called a production), an integer k, and the rewriting rule defined by

$$w_1 \Longrightarrow w_2 \quad \text{if and only if} \quad w_1 = axv, a \in \Sigma, |ax| = k, w_2 = v\phi(a).$$

In such a case we speak of a ***k*-tag system**.

Example 7.1.8 *In the 2-tag system with the productions*

$$a \longrightarrow b, \qquad b \longrightarrow bc, \qquad c \longrightarrow \varepsilon.$$

we have, for example, the following derivation:

$$\underbrace{bb}\, b \Longrightarrow \underbrace{bb}\, c \Longrightarrow \underbrace{cb}\, c \Longrightarrow c.$$

Example 7.1.9 *A 3-tag system with productions $0 \longrightarrow 00$ and $1 \longrightarrow 1101$ was investigated by Post in 1921. The basic problem that interested Post was to find an algorithm to decide, given an initial string $w \in \{0,1\}^*$, whether a derivation from w terminates or gets periodic after a certain number of steps. This problem seems to be still open.*

It can be shown that both semi-Thue and tag systems are as powerful as Turing machines, namely they generate exactly recursively enumerable sets.

Exercise 7.1.10 * *Show that each one-tape Turing machine can be simulated by a 2-tag system.*

The basic idea of string rewriting has been extended in several interesting/important ways. For example, the idea of a **parallel string rewriting** is well captured by the so-called context-independent **Lindenmayer systems** $\mathcal{S} = \langle \Sigma, P \rangle$, where $P \subseteq \Sigma \times \Sigma^*$, and the rewriting rule is defined by

$$w_1 \Longrightarrow w_2 \text{ if and only if } w_1 = u_1 \dots u_k, w_2 = v_1 \dots v_k \text{ and } u_i \Longrightarrow v_i \in P,\ 1 \leq i \leq k.$$

In other words, $w_1 \Longrightarrow w_2$ means that w_2 is obtained from w_1 by replacing, all symbols of w_1, in parallel, using the productions from P.

Example 7.1.11 *If* $\mathcal{S}_3 = \langle \Sigma_3, P_3 \rangle$, $\Sigma_3 = \{a\}$, $P_3 = \{a \longrightarrow aa\}$, *we get a derivation*

$$a \Longrightarrow aa \Longrightarrow aaaa \Longrightarrow a^8 \Longrightarrow a^{16} \Longrightarrow \ldots$$

i.e., in n *derivation steps we obtain* a^{2^n}.

The basic motivation for Lindenmayer systems came from biology. With Lindenmayer systems we deal in more detail in Section 7.4. Another natural and powerful idea is to consider graph rewriting systems; they have an interesting theory and various applications. In Section 7.5 we look at them in more detail. Other approaches are mentioned in the historical and bibliographical references.

7.2 Chomsky Grammars and Automata

Noam Chomsky introduced three simple modifications of semi-Thue systems, crucial for both applications and theory: a specification of a **start symbol**; a partition of the alphabet into **nonterminals** (or **variables**—they correspond also to the **syntactical category** in natural languages) and **terminals**; and, finally, consideration of only those words that can be derived from the start symbol and that contain only terminal symbols. This allowed to use such rewriting systems, usually called **grammars**, to specify and to study formal languages and to investigate natural languages.

7.2.1 Chomsky Grammars

Chomsky also introduced four basic types of grammars. As we shall see all of them are closely related to the basic types of automata.

Definition 7.2.1 *A* **phrase structure grammar**, *or* **type-0 grammar**, $G = \langle V_N, V_T, S, P \rangle$ *is specified by*

- V_N — *a finite* **alphabet of nonterminals**;
- V_T — *a finite* **alphabet of terminals** (*with* $V_N \cap V_T = \emptyset$);
- S — *the* **start symbol** *from* V_n;
- $P \subseteq (V_N \cup V_T)^* V_N (V_N \cup V_T)^* \times (V_N \cup V_T)^*$—*a finite set of* **production**.

(Observe that the left side of each production has to contain at least one nonterminal.)

Definition 7.2.2 *A type-0 grammar* $G = \langle V_N, V_T, S, P \rangle$ *is called*

1. *A* **context-sensitive grammar (CSG)**, *or* **type-1 grammar**, *if* $u \longrightarrow v \in P$ *implies*
    - *either* $u = \alpha A \beta$, $v = \alpha w \beta$, $A \in V_N$, $w \in (V_N \cup V_T)^+$, $\alpha, \beta \in (V_N \cup V_T)^*$.

- *or* $u = S$, $v = \varepsilon$ *and* S *does not occur on the right hand side of a production in* P.

2. *A* **context-free grammar (CFG)**, *or* **type-2 grammar**, *if*

   $u \longrightarrow v \in P$ *implies* $u \in V_N$.

3. *A* **regular**, *or* **type-3 grammar** *if*

   either $P \subseteq V_N \times (V_N V_T^* \cup V_T^*)$ *or* $P \subseteq V_N \times (V_T^* V_N \cup V_T^*)$.

With each Chomsky grammar a language is associated in the following way:

Definition 7.2.3 *The language generated by a Chomsky grammar* $G = \langle V_N, V_T, S, P\rangle$ *is defined by*

$$L(G) = \{w \in V_T^* \mid S \underset{P}{\overset{*}{\Longrightarrow}} w\}.$$

Moreover, when $S \underset{P}{\overset{*}{\Longrightarrow}} x \in (V_T \cup V_N)^*$, *then* x *is said to be a* **sentential form** *of* G. *The family of languages generated by type-i Chomsky grammars will be denoted by* $\mathcal{L}_i, i = 0, 1, 2, 3$. *On a more general level we assign a language* $L(s)$ *to any* $s \in (V)T \cup V_N)^*$ *defined by* $L(s) = \{w \in V_T^* \mid s \underset{P}{\overset{*}{\Longrightarrow}}\}$.

Two Chomsky grammars are said to be equivalent if they generate the same language.

Example 7.2.4 *Consider the grammar* $G = \langle\{S, L, K, W, B, C\}, \{a\}, S, P\rangle$, *where* P *contains productions:*

$$\begin{array}{llll llll llll}
(1) & S & \longrightarrow & LaK, & (2) & aK & \longrightarrow & WCCK, & (3) & aW & \longrightarrow & WCC, \\
(4) & LWC & \longrightarrow & LaB, & (5) & LWC & \longrightarrow & aB, & (6) & BC & \longrightarrow & aB, \\
(7) & BK & \longrightarrow & K, & (8) & BK & \longrightarrow & \varepsilon. & & & &
\end{array}$$

An example of a derivation in G:

$$\begin{array}{lllll}
S & \Longrightarrow LaK & \Longrightarrow LWCCK & \Longrightarrow LaBCK & \Longrightarrow LaaBK \\
 & \Longrightarrow LaaK & \Longrightarrow LaWCCK & \Longrightarrow LWCCCCK & \Longrightarrow aBCCCK \\
 & \Longrightarrow aaBCCK & \Longrightarrow aaaBCK & \Longrightarrow aaaaBK & \Longrightarrow aaaa.
\end{array}$$

We now show that $L(G) = \{a^{2^i} \mid i \geq 1\}$. The production (1) generates LaK, (2) and (3) generate for each "a" two "C" and, in addition, using the production (3), "W moves left". (4) and (6) exchange a "C" into an "a", and with (6) "B moves right". (7) allows a new application of (2). A derivation leads to a terminal word if and only if the production (5) removes "L" and (8) removes "BK". Observe also that $La^iK \overset{*}{\Longrightarrow} LWC^{2i}K \overset{*}{\Longrightarrow} a^{2i}$ and $LWC^{2i}K \overset{*}{\Longrightarrow} La^{2i}K$.

Exercise 7.2.5 *Which language is generated by the grammar* $G = \langle V_N, V_T, S, P\rangle$ *with* $V_N = \{S, X, Y, Z\}$, $V_T = \{a\}$ *and the productions:*

$S \longrightarrow YXY$, $YX \longrightarrow YZ$, $ZX \longrightarrow XXZ$, $ZY \longrightarrow XXY$, $X \longrightarrow a$, $Y \longrightarrow \varepsilon$*?*

Exercise 7.2.6 *Which language is generated by the grammar* $G = \langle V_N, V_T, S, P\rangle$, *with* $V_N = \{S, A, B, L, R\}, V_T = \{a, b\}$ *and the productions:*

$$\begin{array}{llll} S \longrightarrow LR, & L \longrightarrow LaA, & L \longrightarrow LbB, & L \longrightarrow \varepsilon, \\ AR \longrightarrow Ra, & BR \longrightarrow Rb, & R \longrightarrow \varepsilon, & Xx \longrightarrow xX, \end{array}$$

where $x \in \{a, b\}$ *and* $X \in \{A, B\}$*?*

Remark 7.2.7 (1) Productions of a CSG of the form $uAv \longrightarrow uwv$, $A \in V_N$, $w \neq \varepsilon$ can be interpreted as follows: the nonterminal A may be replaced, in the context (u, v), by w. This is also the reason for the attribute "context-sensitive" for such productions and those grammars all productions of which are of such a type.

(2) Productions of a CFG have the form $A \longrightarrow w$, $A \in V_N$ and one can interpret them as follows: each occurrence of the nonterminal A may be replaced by w independently of the context of A. This is the reason for the attribute "context-free" for such productions and grammars with such productions.

(3) Each Chomsky grammar of the type-i is also of the type-$(i-1)$, for $i = 1, 3$. This is not, formally, true for $i = 2$, because a context-free grammar can have rules of the type $A \longrightarrow \varepsilon$, even if A is not the start symbol). However, it is possible to show that for each type-2 grammar G one can effectively construct another type-2 grammar G_1 that is already of type-1 and such that $L(G_1) = L(G)$—see Section 7.3.2.

(4) If all productions of a regular grammar have the form

$$A \longrightarrow u \text{ or } A \longrightarrow Bu, \text{ where } A, B \in V_T, u \in V_N^*$$

we speak of a **left-linear grammar**. Similarly, if all productions of a regular grammar are of the form

$$A \longrightarrow u \text{ or } A \longrightarrow uB, \text{ where } A, B \in V_N, u \in V_T^*$$

we speak of a **right-linear grammar**.

In the following we demonstrate that, for $i = 0, 1, 2, 3$, the type-i grammars are closely related to basic models of automata.

7.2.2 Chomsky Grammars and Turing Machines

We show first that Chomsky type-0 grammars have exactly the same generating power as Turing machines.

Theorem 7.2.8 *A language is recursively enumerable if and only if it is generated by a Chomsky grammar.*

Proof. 1. Let $L = L(G)$, where $G = \langle V_N, V_T, S, P\rangle$ is a Chomsky grammar. We describe the behaviour of a two-tape nondeterministic Turing machine $\mathcal{M}_G$ that simulates derivations of G and accepts exactly L. The first tape will be used to store the input word, the second tape to store words α generated by G. At the start of each simulation we have $\alpha = S$. $\mathcal{M}_G$ simulates one derivation step of G by the following sequence of steps:

(a) $\mathcal{M}_G$ chooses, in a nondeterministic way, a position i in α, $1 \leq i \leq |\alpha|$, and a production $u \longrightarrow v \in P$.

(b) If u is a prefix of $\alpha_i \dots \alpha_{|\alpha|}$, $\mathcal{M}_G$ replaces $\alpha_i \dots \alpha_{i+|u|-1} = u$ by v and starts to perform step (c); otherwise $\mathcal{M}_G$ goes to step (a).

(c) $\mathcal{M}_G$ compares the contents of the second and the first tape. If they are identical, $\mathcal{M}_G$ accepts w; if not $\mathcal{M}_G$ goes to step (a).

Clearly, $\mathcal{M}_G$ accepts w if and only if G generates w.

2. Let $L = L(\mathcal{M})$ for a one-tape Turing machine $\mathcal{M} = \langle \Sigma, \Gamma, Q, q_0, \delta\rangle$. We show how to construct a Chomsky grammar $G_{\mathcal{M}} = \langle V_N, \Sigma, q_0, P\rangle$ generating L. The productions of $G_{\mathcal{M}}$ fall into three groups.

(a) The first group contains productions that generate from q_0 the set of all words of the form $\$q_0w|w\#$ where $w \in \Sigma^*$ and $\{\$, |, \#\}$ are markers not in $\Gamma \cup Q$. (It is not difficult to design such productions—see, for example, the grammar from Exercise 7.2.6 generating the language $\{ww \mid w \in \{a,b\}^*\}$.

(b) Productions of the second group are used to simulate computations of $\mathcal{M}$ on the first w. For each transition $\delta(q,a) = (q', b, \rightarrow)$ of $\mathcal{M}$, $G_{\mathcal{M}}$ contains productions

$$qac \longrightarrow bq'c \text{ for all } c \in \Gamma, \qquad \text{and} \quad qa| \longrightarrow bq' \sqcup |.$$

(The last production is for the case that $\sqcup$, standing for the blank on the tape, is in Γ.) For each transition $\delta(q,a) = (q', b, \leftarrow), G_{\mathcal{M}}$ contains productions

$$cqa \longrightarrow q'cb, \text{ for each } c \in \Gamma, \quad \text{and} \quad \$qa \longrightarrow \$q' \sqcup b.$$

Finally, for each transition $\delta(q,a) = (q', b, \downarrow)$, there is a production $qa \longrightarrow q'b$.

(c) Productions of the third group transform each word "$\$w_1qw_2|w\#$", $w_1, w_2 \in \Gamma^*$, $q = ACCEPT\}$ into the word w. (If $\mathcal{M}$ does not halt for a $w \in \Sigma^*$, or does not accept w, $G_{\mathcal{M}}$ does not generate any word from $\$q_0w|w\#$.) The generation of w from $\$w_1qw_2|w\#$ can be done, for example, by the the following productions, where $\{F, F_1, F_2\}$ are new nonterminals):

$$\begin{array}{rcllcl}
ACCEPT & \longrightarrow & F, & aF & \longrightarrow & Fa,\, a \in \Gamma, \\
\$F & \longrightarrow & F_1 & F_1a & \longrightarrow & F_1,\, a \in \Gamma, \\
F_1| & \longrightarrow & F_2, & F_2a & \longrightarrow & aF_2,\, a \in \Sigma, \\
F_2\# & \longrightarrow & \sqcup & & &
\end{array}$$

□

Chomsky type-0 grammars may have many nonterminals and complicated productions. It is therefore natural to ask whether this is necessary. The following theorem summarizes several such results and shows that not all available means are surprisingly needed.

Theorem 7.2.9 *(1) For every Chomsky grammar an equivalent Chomsky grammar with only two nonterminals can be effectively constructed.*

(2) Chomsky grammars with only one nonterminal generate a proper subset of recursively enumerable languages.

(3) For every Chomsky grammar an equivalent Chomsky grammar with only one non-context-free production can be constructed effectively.

7.2.3 Context-sensitive Grammars and Linearly Bounded Automata

The basic idea of context-sensitive grammars is very simple, beautiful, and has a good linguistic motivation: a production $uAv \longrightarrow uwv$ replaces the nonterminal A by w, in the context (u, v). (Indeed, in natural languages the meaning of a part of sentence may much depend on the context.)

The so-called **monotone Chomsky grammars** have the same generative power as the context-sensitive grammars. Their main advantage is that it is often more easy to design a monotone than a context-sensitive grammar to generate a given language.

Definition 7.2.10 *A Chomsky grammar $G = \langle V_N, V_T, S, P\rangle$ is called* **monotone** *if, for each production $u \longrightarrow v$ in P, either $|u| \leq |v|$ or $u = S, v = \varepsilon$, and S does not occur on the right-hand side of any production. (The last condition is just to allow to generate the empty word, too.)*

Theorem 7.2.11 *A language is generated by a context-sensitive grammar if and only if it is generated by a monotone Chomsky grammar.*

Proof. Each context-sensitive grammar is monotone and therefore in order to prove the theorem, let us assume that we have a monotone grammar G. At first we transform G into an equivalent grammar that has only nonterminals on the left sides of their productions. This is easy. For each terminal a we take a new nonterminal X_a, add the production $X_a \longrightarrow a$ and replace a by X_a in the left sides of all productions. Now it is enough to show that to each production of a monotone Chomsky grammar $G = \langle V_N, V_T, S, P\rangle$, with only nonterminals on its left side, there is an equivalent context-sensitive Chomsky grammar. In order to do that let us assume a fixed ordering of the productions of P, and let us consider an extended set of nonterminals

$$V' = \left\{ \binom{A}{k} \mid A \in V_T \cup V_N,\ 1 \leq k \leq |P| \right\} \cup V_N$$

where $\binom{A}{i}$ denotes a pair of symbols.

The following set of context-sensitive productions corresponds to the k-th production $A_1 \dots A_n \longrightarrow B_1 \dots B_m$,$n \leq m$:

$$\begin{aligned}
A_1 \dots A_n &\longrightarrow \tbinom{A_1}{k} A_2 \dots A_n \\
\tbinom{A_1}{k} A_2 \dots A_n &\longrightarrow \tbinom{A_1}{k}\tbinom{A_2}{k} \dots A_n \\
\dots\dots\dots \quad &\dots \quad \dots\dots\dots \\
\tbinom{A_1}{k}\tbinom{A_2}{k} \dots \tbinom{A_{n-1}}{k} A_n &\longrightarrow \tbinom{A_1}{k} \dots \tbinom{A_n}{k}\tbinom{B_{n+1}}{k} \dots \tbinom{B_m}{k} \\
\tbinom{A_1}{k} \dots \tbinom{A_n}{k}\tbinom{B_{n+1}}{k} \dots \tbinom{B_m}{k} &\longrightarrow B_1 \tbinom{A_2}{k} \dots \tbinom{A_n}{k}\tbinom{B_{n+1}}{k} \dots \tbinom{B_m}{k} \\
\dots\dots\dots \quad &\dots \quad \dots\dots\dots \\
B_1 B_2 \dots B_{m-1} \tbinom{B_m}{k} &\longrightarrow B_1 B_2 \dots B_m .
\end{aligned}$$

In this way to any two different productions of P two different sets of context-sensitive productions correspond such that productions of these two sets contain different new nonterminals. As a consequence, if all rules of G are replaced by the set of context-sensitive productions designed as shown above, then we get a set of context-sensitive productions generating the same language as G. □

Example 7.2.12 *Consider the grammar G with $V_N = \{S, X, Y\}$, $V_T = \{a, b, c\}$, and the start symbol S and the rules:*

$$\begin{array}{llllllll}
(1)\ S & \longrightarrow & aSX, & (2)\ YX & \longrightarrow & bYc, & (3)\ Y \longrightarrow bc, \\
S & \longrightarrow & aY, & cX & \longrightarrow & Xc, &
\end{array}$$

We show now that $L(G) \subseteq \{a^n b^n c^n \mid n \geq 1\}$. Indeed, with productions (1) we derive $S \overset{*}{\Longrightarrow} a^n Y X^{n-1}$, for each $n > 1$. By induction we can prove that for each $1 \leq i < n$,

$$YX^{n-1} \overset{*}{\Longrightarrow} b^i Y c^i X^{n-i-1}$$

and using the productions (3) we get that for each $n \in \mathbf{N}$,

$$S \overset{*}{\Longrightarrow} a^n b^{n-1} Y c^{n-1} \Longrightarrow a^n b^n c^n .$$

Hence $L(G) \subseteq \{a^n b^n c^n \mid n \geq 1\}$.

Exercise 7.2.13 * *Show that the grammar from Example 7.2.12 generates exactly the language $\{a^n b^n c^n \mid n \geq 1\}$.*

Exercise 7.2.14 *Design a monotone grammar generating the languages:*

(a) $\{w \mid w \in \{a, b\}^*, \#_a w = \#_b w\}$; (b) $\{a^n b^{2n} a^n \mid n \geq 1\}$; (c) $\{a^p \mid p \text{ is a prime}\}$.

The following relation between context-sensitive grammars and linearly bounded automata (see Section 3.8.5), justifies the use of the attribute "context-sensitive" for languages generated by LBA.

Theorem 7.2.15 *Context-sensitive grammars generate exactly those languages linearly bounded automata accept.*

Proof. The proof of this theorem is similar to that of Theorem 7.2.8 and therefore we concentrate only on the points where the differences lie.

Let G be a monotone grammar. Similarly as in Theorem 7.2.8 we design a Turing machine $\mathcal{M}_G$ that simulates derivations of G. However, instead of two tapes, as in the proof of Theorem 7.2.8, $\mathcal{M}_G$ uses only one tape, but with two tracks. In addition, $\mathcal{M}_G$ checks, each time a production should be applied, whether the newly created word is not longer than the input word w (stored in the first track). If it is the case, such a rewriting is not performed—here we are making use of the fact that in a monotone grammar a rewriting never shortens a sentential form. It is now easy to see that $\mathcal{M}_G$ can be changed in such a way that its head actually never gets outside of the tape squares occupied by the input word and therefore it is actually a linearly bounded automaton.

Similarly, the proof that we can construct for each LBA an equivalent monotone grammar can be done also by a modification of the proof of Theorem 7.2.8. Only a special trick has to be used to ensure the monotonicity of the resulting grammar.

Let $\mathcal{A} = \langle \Sigma, Q, q_0, Q_F, \spadesuit, \#, \delta \rangle$ be a LBA. The productions of the equivalent monotone grammar fall into three groups:

Productions of the first group have the form:

$$S \longrightarrow A \begin{bmatrix} x \\ x \\ \# \\ - \end{bmatrix}, \quad A \longrightarrow A \begin{bmatrix} x \\ x \\ - \\ - \end{bmatrix}, \quad A \longrightarrow \begin{bmatrix} x \\ x \\ \spadesuit \\ q_0 \end{bmatrix}$$

where $x \in \Sigma$, and each 4-tuple is considered to be a new nonterminal. These productions generate the following representation of "a two track-tape", with the initial content $w = w_1 \ldots w_n, w_i \in \Sigma$:

$$\begin{bmatrix} w_1 \\ w_1 \\ \spadesuit \\ q_0 \end{bmatrix} \begin{bmatrix} w_2 \\ w_2 \\ - \\ - \end{bmatrix} \cdots \begin{bmatrix} w_n \\ w_n \\ \# \\ - \end{bmatrix},$$

Productions of the second group, that are now easy to design, simulate $\mathcal{A}$ on the "first track". For each transition of $\mathcal{A}$ there is again a new set of productions.

Finally, productions of the third group transform each nonterminal word with the accepting state into that terminal word that is on the "second track". These productions can also be designed in a quite straight-forward way. $\square$

The family of context-sensitive languages contains practically all languages one encounters in computing. The following theorem shows the relation between context-sensitive and recursive languages.

Theorem 7.2.16 *Each context-sensitive language is recursive. On the other hand, there are recursive languages that are not context-sensitive.*

Proof. Recursivity of context-sensitive languages follows from Theorem 3.8.27. In order to define a recursive language that is not context-sensitive, let $G_0, G_1, \ldots$ be a strict enumeration of encodings of all monotone grammars in $\{0,1\}^*$. In addition, let $f : \{0,1\}^* \to \mathbf{N}$ be a computable bijection. (For example, $f(w) = i$ if and only if w is the i-th word in the strict ordering.)

The language $L_0 = \{w \in \{0,1\}^* \,|\, w \notin L(G_{f(w)})\}$ is decidable. Indeed, for a given w one computes $f(w)$, designs $G_{f(w)}$, and tests membership of w in $L(G_{f(w)})$.

The diagonalization method will now be used to show that L_0 is not a context-sensitive language. Indeed, assuming that L_0 is context-sensitive, there has to exist a monotone grammar G_{n_0} such that $L_0 = L(G_{n_0})$.

Now let w_0 be such that $f(w_0) = n_0$. A contradiction can be derived as follows:

If $w_0 \in L_0$, then, according to the definition of L_0, $w_0 \notin L(G_{n_0})$ and therefore (by the assumption) $w_0 \notin L_0$. If $w_0 \notin L_0$, then, according to the definition of $L_0, w_0 \in L(G_{n_0})$, and therefore (again by the assumption) $w_0 \in L_0$. ▯

On the other hand, the following theorem shows that the difference between recursively enumerable and context-sensitive languages is actually very subtle.

Lemma 7.2.17 *If $L \subseteq \Sigma^*$ is a recursively enumerable language and $\$, \#$ are symbols not in Σ, then there is a context-sensitive language L_1 such that*

1. $L_1 \subseteq \{\#^i\$w \,|\, w \in L,\ i \geq 0\}$;
2. *for each $w \in L$ there is an $i \geq 0$ such that $\#^i\$w \in L_1$.*

Proof. Let $L = L(G)$, $G = \langle V_N, \Sigma, S, P\rangle$, and $\$, \#$ be symbols not in Σ. We introduce two new variables $\{S_0, Y\}$ and define three sets of productions:

$P_1 = \{u \longrightarrow v \,|\, u \longrightarrow v \in P,\ |u| \leq |v|\}$;
$P_2 = \{u \longrightarrow Y^i v \,|\, u \longrightarrow v \in P,\ |u| > |v|,\ i = |v| - |u|\}$;
$P_3 = \{S_0 \longrightarrow \$S,\ \$Y \longrightarrow \#\$\} \cup \{\alpha Y \to Y\alpha,\ \alpha \in V_N \cup \Sigma\}$.

The grammar

$$G_1 = \langle V_N \cup \{S_0, Y\},\ \Sigma \cup \{\$, \#\}, S_0, P_1 \cup P_2 \cup P_3\rangle$$

is monotone and the language $L(G)$ satisfies both conditions of the theorem. ▯

As a corollary we get:

Theorem 7.2.18 *For each recursively enumerable language L there is a context-sensitive language L_1 and a homomorphism h such that $L = h(L_1)$.*

Proof. Take $h(\$) = \varepsilon$, $h(\#) = \varepsilon$ and $h(a) = a$ for all $a \in \Sigma$. ▯

7.2.4 Regular Grammars and Finite Automata

In order to show relations between regular grammars and finite automata we make use of the fact that the family of regular languages is closed under the operation of reversal.

Theorem 7.2.19 *Regular grammars generate exactly those languages finite automata accept.*

Proof. (1) Let $G = \langle V_N, V_T, S, P\rangle$ be a right-linear grammar, i.e., a grammar with productions of the form

$$C \longrightarrow w \quad \text{or} \quad C \longrightarrow wB, \quad B \in V_N, w \in V_T^*.$$

We design a transition system (see Section 3.8.1), $\mathcal{A} = \langle V_N \cup \{E\}, V_T, S, \{E\}, \delta\rangle$, with a new state $E \notin V_N \cup V_T$, and with the transition relation:

$$\begin{aligned} E \in \delta(C, w) &\quad \text{if and only if } C \longrightarrow w \in P; \\ B \in \delta(C, w) &\quad \text{if and only if } C \longrightarrow wB \in P. \end{aligned}$$

By induction it is straightforward to show that $L(G) = L(\mathcal{A})$.

(2) Now let $G = \langle V_N, V_T, S, P\rangle$ be a left-linear grammar, that is a grammar with productions of the form $C \longrightarrow w$ and $C \longrightarrow Bw$, where $C, B \in V_N$, $w \in V_T^*$. Then $G^R = \langle V_N, V_T, S, P^R\rangle$ with $P^R = \{u \longrightarrow v \,|\, u \longrightarrow v^R \in P\}$ is a right-linear grammar. According to (1), the language $L(G^R)$ is regular. Since $L(G) = L(G^R)^R$ and the family of regular languages is closed under reversal, the language $L(G)$ is also regular.

(3) If $\mathcal{A} = \langle Q, \Sigma, q_0, Q_F, \delta\rangle$ is a DFA, then the grammar $G = \langle Q, \Sigma, q_0, P\rangle$ with productions

$$\begin{array}{lll} q \longrightarrow w \in P & \text{if} & w \in \Sigma, \delta(q, w) \in Q_F; \\ q \longrightarrow wq_i \in P & \text{if} & w \in \Sigma, \delta(q, w) = q_i, q_i \in Q - Q_F \\ q_0 \longrightarrow \varepsilon & \text{if} & q_0 \in Q_F; \end{array}$$

is right-linear. Clearly, $q_0 \overset{*}{\Longrightarrow} w'q_i$, $q_i \in Q$, if and only if $\delta(q_0, w') = q_i$, and therefore $L(G) = L(\mathcal{A})$. □

Exercise 7.2.20 *Design: (a) a right-linear grammar generating the language* $\{a^i b^j \,|\, i, j \geq 0\}$*; (b) a left-linear grammar generating the language* $L \subset \{0,1\}^*$ *consisting of words that are normal forms of the Fibonacci representations of integers; (c) perform in detail the induction proof mentioned in the part (1) of the theorem 7.2.19.*

7.3 Context-free Grammars and Languages

There are several reasons why context-free grammars are of special interest. From the practical point of view, they are deeply related to the basic techniques of description of the syntax of programming languages and to translation methods. The corresponding pushdown automata are also closely related to basic methods of recursion handling. In addition, context-free grammars are of interest also for description of natural language. From the theoretical point of view, the corresponding family of context-free languages plays an important rôle in the formal language theory—next to the family of regular languages.

7.3.1 Basic Concepts

Three rewriting (or derivation) relations are considered for context-free grammars $G = \langle V_N, V_T, S, P\rangle$.

Rewriting (derivation) relation $\underset{P}{\Longrightarrow}$

$$w_1 \underset{P}{\Longrightarrow} w_2 \quad \text{if and only if} \quad w_1 = uAv, w_2 = uwv, A \longrightarrow w \in P.$$

Leftmost rewriting (derivation) relation $\underset{P}{\Longrightarrow}_L$

$$w_1 \underset{P}{\Longrightarrow}_L w_2 \quad \text{if and only if} \quad w_1 = uAv, w_2 = uwv, A \longrightarrow w \in P, u \in V_T^*.$$

Rightmost rewriting (derivation) relation $\underset{P}{\Longrightarrow}_R$

$$w_1 \underset{P}{\Longrightarrow}_R w_2 \quad \text{if and only if} \quad w_1 = uAv, w_2 = uwv, A \longrightarrow w \in P, v \in V_T^*.$$

A **derivation** in G is a sequence of words from $(V_N \cup V_T)^*$

$$w_1, w_2, \ldots, w_k$$

such that $w_i \underset{P}{\Longrightarrow} w_{i+1}$, for $1 \leq i < k$. If always $w_i \underset{P}{\Longrightarrow}_L w_{i+1}$ ($w_i \underset{P}{\Longrightarrow}_R w_{i+1}$) holds, we speak of a leftmost (rightmost) derivation. In each step of a derivation a nonterminal A is replaced using a production $A \longrightarrow u$ from P. In case of the leftmost (rightmost) derivation, always the leftmost (rightmost) nonterminal is rewritten.

A language L is called a **context-free language (CFL)** if there is a CFG generating L.

Each derivation assigns a **derivation tree** to the string it derives, see the figures on pages 483 and 484. The internal nodes of such a tree are labeled by nonterminals, leaves by terminals or ε. If an internal node is labeled by a nonterminal A and its children, by $x_1, \ldots, x_k$, counting from left, then $A \longrightarrow x_1 \ldots x_k$ has to be a production of the grammar.

Now we present two examples of context-free grammars. In doing so we describe a CFG, as it is usual, by a list of productions, with the start symbol on the left hand side of the first production. In addition, to describe a set of productions

$$A \longrightarrow \alpha_1, A \longrightarrow \alpha_2, \ldots, A \longrightarrow \alpha_k,$$

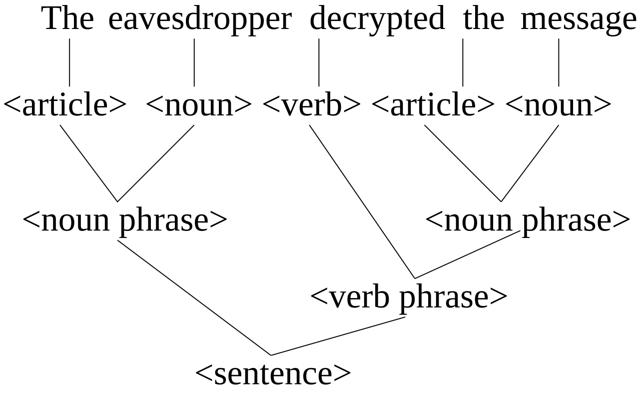
with the same symbol on the left hand side, we use, as usual, the following concise description

$$A \longrightarrow \alpha_1 \mid \alpha_2 \mid \ldots \mid \alpha_k.$$

Example 7.3.1 (Natural language description) *The original motivation behind introducing CFG was to describe derivations and structures of sentences of natural languages with such productions as, for example:*

$$\begin{array}{ll}
\langle \textit{sentence} \rangle \rightarrow \langle \textit{noun phrase} \rangle \ \langle \textit{verb phrase} \rangle, & \langle \textit{article} \rangle \rightarrow \textit{The, the} \\
\langle \textit{noun phrase } \rangle \rightarrow \langle \textit{article } \rangle \ \langle \textit{noun} \rangle, & \langle \textit{noun} \rangle \rightarrow \textit{eavesdropper} \mid \textit{message,} \\
\langle \textit{verb phrase} \rangle \rightarrow \langle \textit{verb} \rangle \ \langle \textit{noun phrase} \rangle, & \langle \textit{verb} \rangle \rightarrow \textit{decrypted,}
\end{array}$$

where the syntactical categories of the grammar (nonterminals) are denoted by words between the symbols "⟨ " and "⟩ " and the words like "eavesdropper" are single terminals. An example of a derivation tree:



In spite of the fact that context-free grammars are not powerful enough to describe natural languages in a completely satisfactory way, they, and their various modifications, play an important rôle as useful tools in (computational) linguistics.

The use of CFG to describe programming and other formal languages has been much more successful. With CFG one can significantly simplify descriptions of the syntax of programming languages. Moreover, CFG allowed to develop a successful theory and practice of compilation. The reason behind lies to a large extent in the natural way how many constructs of programming languages can be described by CFG.

Example 7.3.2 (Programming language description) *The basic arithmetical expressions can be described, for example, using the productions of the form*

$$\begin{array}{rcl}
\langle \textit{expression} \rangle & \longrightarrow & \langle \textit{expression} \rangle \ \langle \ \pm \ \rangle \ \langle \textit{expression} \rangle \\
\langle \textit{expression} \rangle & \longrightarrow & \langle \textit{expression 1} \rangle \\
\langle \textit{expression 1} \rangle & \longrightarrow & \langle \textit{expression 1} \rangle \ \langle \textit{mult} \rangle \ \langle \textit{expression 1} \rangle \\
\langle \textit{expression 1} \rangle & \longrightarrow & (\langle \textit{expression} \rangle) \\
\langle \pm \rangle & \longrightarrow & + \mid - \\
\langle \textit{mult} \rangle & \longrightarrow & \times \mid / \\
\langle \textit{expression}1 \rangle & \rightarrow & a \,|\, b \,|\, c \,|\, \ldots \,|\, y \,|\, z
\end{array}$$

and they can be used to derive, for example, $a/b+c$*, as in Figure 7.1.*

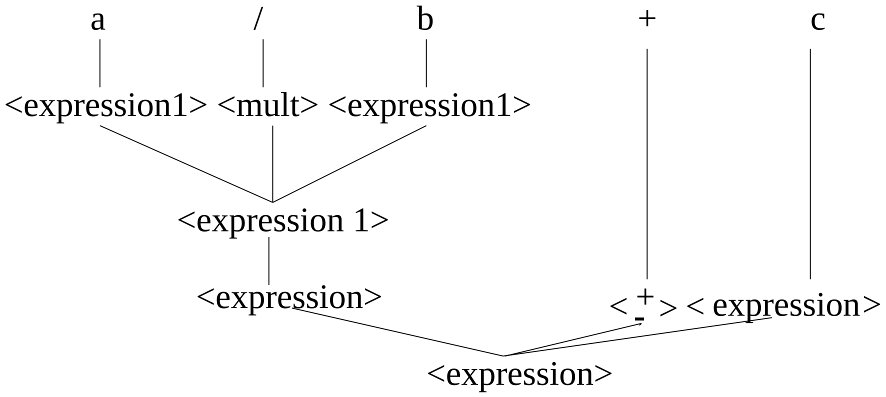


Figure 7.1: A derivation tree-80%

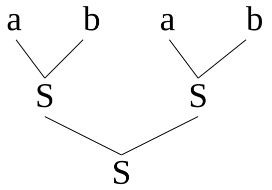
Exercise 7.3.3 *Design CFG generating languages: (a) the language of all Boolean expressions; (b) the language of Lisp expressions; (c)* $\{a^ib^{2i} \mid i,j \geq 1\}$*; (d)* $\{ww^R \mid w \in \{0,1\}\}$*; (e)* $\{a^ib^jc^k \mid i \neq j \text{ or } j \neq k\}$.

It can happen that a word $w \in L(G)$ has two different derivations in a CFG G but the corresponding derivation trees are identical. For example, for the grammar with two productions, $S \longrightarrow SS \mid ab$, we have the following two derivations of the string $abab$:

$$d_1: \qquad S \Longrightarrow SS \Longrightarrow abS \Longrightarrow abab,$$

$$d_2: \qquad S \Longrightarrow SS \Longrightarrow Sab \Longrightarrow abab,$$

and both of them correspond to the derivation trees:



Exercise 7.3.4 *Show that there is a bijection between derivation trees and leftmost derivations (rightmost derivations).*

It can also happen that a word $w \in L(G)$ has two derivations in G such that the corresponding derivation trees are different. For example, in the CFG with productions $S \longrightarrow Sa \mid a \mid aa$, the word aaa has two derivations that correspond to the derivation trees in Figure 7.2.

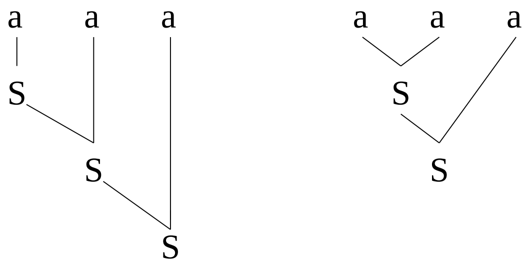


Figure 7.2: Derivation trees

A CFG G with the property that some word $w \in L(G)$ has two different derivation trees is called **ambiguous**. A context-free language is called **(inherently) ambiguous** if each context-free grammar for L is ambiguous. For example, the language

$$L = \{a^i b^i a^j \,|\, i,j \geq 1\} \cup \{a^i b^j a^j \,|\, i,j \geq 1\}$$

is ambiguous. It can be shown that in each CFG for L some words of the form $a^k b^k a^k$ have to have two essentially different derivation trees.

Exercise 7.3.5 *Which of the following CFG is ambiguous:*

(a) $S \longrightarrow a \,|\, abSb \,|\, aAb,\ A \longrightarrow bS \,|\, aAAb;$

(b) $S \longrightarrow aSbc \,|\, c \,|\, bA,\ A \longrightarrow bA \,|\, a;$

(c) $S \longrightarrow aS \,|\, Sa \,|\, bAb,\ A \longrightarrow aAb \,|\, acb.$

Exercise 7.3.6 *Consider CFG G with the productions*

$$S \longrightarrow bA \,|\, aB,\ A \longrightarrow a \,|\, aS \,|\, bAA,\ B \longrightarrow b \,|\, bS \,|\, aBB.$$

Show that G is ambiguous, but $L(G)$ is not.

Even in case of context-free grammars it is in general not easy to show which language is generated by a given CFG, or to design a CFG generating a given language.

Example 7.3.7 *The CFG $G = \langle V_N, V_T, S, P\rangle$ with the productions*

$$S \longrightarrow aB \,|\, bA, \quad A \longrightarrow bAA \,|\, aS \,|\, a, \quad B \longrightarrow aBB \,|\, bS \,|\, b,$$

generates the language

$$L(G) = \{w \,|\, w \in \{a,b\}^+, \text{ w contains as many a's as b's}\}.$$

In order to show that, it is sufficient to prove, for example by induction on the length of w what is straightforward, the following assertions:

1. $S \overset{*}{\Longrightarrow} w$, *if and only if* $\#_a w = \#_b w$,

2. $A \overset{*}{\Longrightarrow} w$, *if and only if* $\#_a w = \#_b w + 1$,

3. $B \overset{*}{\Longrightarrow} w$, *if and only if* $\#_b w = \#_a w + 1$.

(In fact, it suffices to prove the first claim, but it is helpful to show the assertions 2 and 3 to get 1.)

Some languages are context-free even though one would not expect that. For example, the set of all satisfiable Boolean formulas is **NP**-complete (and therefore no polynomial time recognizing algorithm seems to exist for them), but, as it follows from the next example, the set of all satisfiable Boolean formulas over a fixed set of variables is context-free (and, as discussed later, it can be recognized in linear time).

Example 7.3.8 *Denote by* $\mathcal{F}_n$ *the set of Boolean formulas over the variables* $x_1, \ldots, x_n$ *and Boolean operations* $\vee$ *and* $\neg$. *Moreover, denote by* $\mathcal{A}_n$ *the set of all assignments* $\alpha : \{x_1, \ldots, x_n\} \to \{0,1\}$. *For* $\alpha \in \mathcal{A}_n$ *and* $F \in \mathcal{F}_n$, *let* $\alpha(F)$ *denote the value of* F *at the assignment* α. *The set of all tautologies over the variables* $\{x_1, \ldots, x_n\}$ *is defined by* $\mathcal{T}_n = \{F \in \mathcal{F}_n \,|\, \forall \alpha \in \mathcal{A}_n, \alpha(F) = 1\}$. *We show that* $\mathcal{T}_n$ *is a context-free language.*

Let $G_n = \langle V_N^n, V_T^n, S_{\mathcal{A}_n}, P_n \rangle$ *be a CFG such that* $V_N^n = \{S_A \,|\, A \subseteq \mathcal{A}_n\}$ *is the set of nonterminals, one for each subset of* $\mathcal{A}_n$; $V_T^n = \{x_1, \ldots, x_n\} \cup \{\vee, \neg, (,)\}$ *and* P_n *be the set of productions:*

$$\begin{array}{rcllr} S_A & \longrightarrow & \neg(S_{\mathcal{A}_n - A}), & \textit{if } A \in \mathcal{A}_n; & (i) \\ S_{A \vee B} & \longrightarrow & S_A \vee S_B, & \textit{if } A, B \in \mathcal{A}_n; & (ii) \\ S_{\{\alpha \,|\, \alpha(x) = 1\}} & \longrightarrow & x, & \textit{if } x \in \{x_1, \ldots, x_n\}. & (iii) \end{array}$$

In order to show that $L(G_n) = \mathcal{T}_n$, *it is sufficient to prove, what can be done in a straightforward way by induction, that if* $A \in \mathcal{A}_n$, *then* $S_A \overset{*}{\Longrightarrow} F$ *if and only if* $A = \{\alpha \,|\, \alpha(F) = 1\}$. *Let* $A \in \mathcal{A}_n$ *and* $S_A \overset{*}{\Longrightarrow} F$. *Three cases for* ϕ *are possible: If* $F = x \in \{x_1, \ldots, x_n\}$, *then* x *can be derived only by rule (iii). If* $S_A \Rightarrow \neg(S_B) \overset{*}{\Longrightarrow} F$, *then* $F = \neg(F')$ *and, by the induction hypothesis* $B = \{\alpha \,|\, \alpha(F') = 0\}$ *and therefore, by (i),* $S_B \overset{*}{\Longrightarrow} F'$ *and* $A = \mathcal{A}_n - B = \{\alpha \,|\, \alpha(\neg(F')) = 1\}$. *Last case to consider is that* $S_A \Rightarrow S_B \vee S_C \overset{*}{\Longrightarrow} F$. *Then* $F = F_1 \vee F_2$ *and* $A = B \cup C$. *By (ii),* $B = \{\beta \,|\, \beta(F_1) = 1\}$, $C = \{\gamma \,|\, \gamma(F_2) = 1\}$ *and therefore* $A = \{\alpha \,|\, \alpha(F_1 \cup F_2) = 1\}$.

In a similar way we can prove that if $A = \{\alpha \,|\, \alpha(F) = 1\}$, *then* $S_A \overset{*}{\Longrightarrow} F$ *and from that* $L(G_n) = \mathcal{T}_n$ *follows.*

Exercise 7.3.9 *Show that the language of all satisfiable Boolean formula over a fixed set of nonterminals is context-free.*

Exercise 7.3.10 *Design a CFG generating the language* $\{w \in \{0,1\}^* \,|\, w$ *contains three times more* 1*'s as* 0*'s*$\}$.

7.3.2 Normal Forms

In many cases it is desirable that a CFG we use has a "nice form". The following three normal forms for CFG are of such a type.

Definition 7.3.11 *Let* $G = \langle V_N, V_T, S, P\rangle$ *be a CFG.*

G **is in the reduced normal form** *if the following conditions are satisfied:*

1. *Each nonterminal of* G *occurs in a derivation of* G *from the start symbol and each nonterminal generates a terminal word.*
2. *No production has the form* $A \longrightarrow B$, $B \in V_N$.
3. *If* $\varepsilon \notin L(G)$, *then* G *has no production of the form* $A \longrightarrow \varepsilon$ *(no* ε*-production), and if* $\varepsilon \in L(G)$, *then* $S \longrightarrow \varepsilon$ *is the only* ε*-production.*

G **is in the Chomsky normal form** *if each production has either the form* $A \longrightarrow BC$ *or* $A \longrightarrow u$, *where* $B, C \in V_N$, $u \in V_T^*$, *or the form* $S \longrightarrow \varepsilon$ *(and* S *not occuring on the right-hand side of any other production);*

G **is in the Greibach normal form** *if each production has either the form* $A \longrightarrow a\alpha$, $a \in V_T$, $\alpha \in V_N^*$ *or the form* $S \longrightarrow \varepsilon$ *(and* S *not occuring on the right-hand side of any other production).*

Theorem 7.3.12 *(1) To each CFG one can construct an equivalent reduced CFG.*

(2) To each CFG one can construct an equivalent CFG in the Chomsky normal form and an equivalent CFG in the Greibach normal form.

Proof. The assertion (1) is easy to verify. For example, it is sufficient to use the results of the following Exercise.

Exercise 7.3.13 *Let* $G = \langle V_N, V_T, S, P\rangle$ *be a CFG and* $n = |V_T \cup V_N|$.

(a) Consider the recurrence: $X_0 = \{A \mid A \in V_N, \exists(A \longrightarrow \alpha) \in P, \alpha \in V_T^*\}$, *and for* $i > 0$, $X_i = \{A \mid A \in V_N, \exists(A \longrightarrow \alpha) \in P, \alpha \in (V_T \cup X_{i-1})^*\}$.

Show that $A \in X_n$ *iff* $A \overset{*}{\Longrightarrow} w$ *for some* $w \in V_T^*$.

(b) Consider the recurrence: $Y_0 = \{S\}$ *and for* $i > 0$,

$Y_i = Y_{i-1} \cup \{A \mid A \in V_N,\ \exists(B \longrightarrow uAv) \in P,\ B \in Y_{i-1}\}$.

Show that $A \in Y_n$ *iff there are* $u', v' \in (V_T \cup V_N)^*$ *such that* $S \overset{*}{\Longrightarrow} u'Av'$.

(c) Consider the recurrence; $Z_0 = \{A \mid (A \longrightarrow \varepsilon) \in P\}$ *and, for* $i > 0$ $Z_i = \{A \mid \exists(A \longrightarrow \alpha) \in P, \alpha \in Z_{i-1}^*\}$. *Show that* $A \in Z_n$ *iff* $A \overset{*}{\Longrightarrow} \varepsilon$.

We show now how to design a CFG G' in Chomsky normal form equivalent to a given reduced CFG $G = \langle V_N, V_T, S, P\rangle$.

For each terminal c let X_c be a new nonterminal. G' is constructed in two phases.

1. In each production $A \longrightarrow \alpha$, $|\alpha| \geq 2$, each terminal c is replaced by X_c and all productions $X_c \longrightarrow c$, $c \in V_T$ are added into the set of productions.
2. Each production $A \longrightarrow B_1 \dots B_m$, $m \geq 3$, is replaced by the following set of productions:

$$A \longrightarrow B_1D_1, D_1 \longrightarrow B_2D_2, \dots, D_{m-3} \longrightarrow B_{m-2}D_{m-2}, D_{m-2} \longrightarrow B_{m-1}B_m,$$

   where $\{D_1, \dots, D_{m-2}\}$ is, for each production, a new set of nonterminals. The resulting CFG is in Chomsky normal form and evidently equivalent to G.

Transformation of a CFG into Greibach normal form is more involved—see references. □

Example 7.3.14 (Construction of a Chomsky normal form) *For the CFG with the productions* $S \longrightarrow aSbbSa \mid ab$, *we get, after the first step,*

$$S \longrightarrow X_aSX_bX_bSX_a \mid X_aX_b, \qquad X_a \longrightarrow a, \qquad X_b \longrightarrow b$$

and after step 2

$$\begin{array}{llll} S \longrightarrow X_aD_1, & D_1 \longrightarrow SD_2, & D_2, \longrightarrow X_bD_3, & D_3 \longrightarrow X_bD_4, \\ D_4 \longrightarrow SX_a, & X_a \longrightarrow a, & X_b \longrightarrow b. & \end{array}$$

Exercise 7.3.15 *Design a CFG in Chomsky normal form equivalent to the grammar from Example 7.3.7. (Observe that this grammar is already in Greibach normal form.)*

Transformation of a CFG into a normal form does not only take time but usually leads to an increase of the size. In order to specify quantitatively how big such an increase can be in the worst case, let us define the size of a CFG G as

$$Size(G) = \sum_{A \to u \in P} (|u| + 2).$$

It can be shown that to each reduced CFG G there exists an equivalent CFG G' in Chomsky normal form such that $Size(G') \leq 7Size(G)$ and an equivalent CFG G'' in Greibach normal form such that $Size(G'') = \mathcal{O}(Size^3(G))$. It is not clear whether the upper bound is tight but for some CFG G it holds that $Size(G'') \geq Size^2(G)$, for each CFG G'' which is in Greibach normal form and equivalent to G.

Exercise 7.3.16 *Show that for each CFG G there is a CFG G' in the Chomsky normal form such that $Size(G') \leq 7Size(G)$.*

In the case of type-0 grammars it has been possible to show that grammars with only two nonterminals are sufficient to generate all recursively enumerable languages. It is therefore natural to ask whether all available resources of CFG, namely potentially infinite pools of nonterminals and productions are really necessary to generate all CFL. For example, is it not enough to consider only CFG with a fixed number of nonterminals or productions? No, as the following result says.

Theorem 7.3.17 *For every integer $n > 1$ there is a CFL $L_n \subseteq \{a,b\}^*$ ($L'_n \subseteq \{a,b\}^*$) such that L_n (L'_n) can be generated by a CFG with n nonterminals (productions) but not by a CFG with $n-1$ nonterminals (productions).*

7.3.3 Context-free Grammars and Pushdown Automata

Historically, pushdown automata (PDA) played an important rôle in the development of programming and especially compiling techniques. Nowadays they are of broader importance for computing.

Informally, a PDA is an automaton with finite control, a (potentially infinite) input tape, a potentially infinite pushdown tape, the input tape head (read-only), and the pushdown head—see Figure 7.3. The input tape head may move only to right. The pushdown tape is a "first-in-last-out" list. The pushdown head can read only the topmost symbol of the pushdown tape and can write only on the top of the pushdown tape. More formally:

Definition 7.3.18 *A* **(nondeterministic) pushdown automaton (PDA or NPDA)** $\mathcal{A} = \langle Q, \Sigma, \Gamma, q_0, Q_F, \gamma_0, \delta \rangle$ *has the set of* **states** Q, *with the* **initial state** q_0 *and a subset* Q_F *of* **final states**, *an* **input alphabet** Σ, *a* **pushdown alphabet** Γ, *with* $\gamma_0 \in \Gamma$ *being a* **starting pushdown symbol**, *and the* **transition function** δ *defined by*

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}.$$

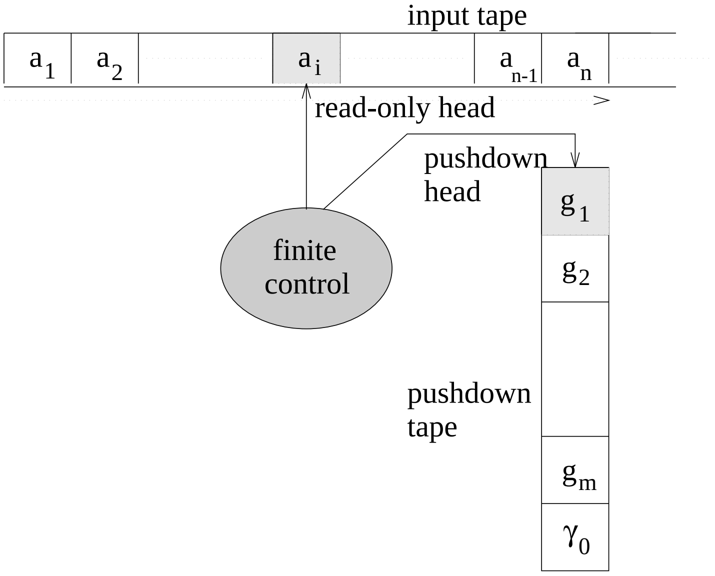


Figure 7.3: A pushdown automaton–70%

A **configuration** of $\mathcal{A}$ is a triple (q, w, γ). We say that $\mathcal{A}$ is in a configuration (q, w, γ), if $\mathcal{A}$ is in the state q, w is the not-yet-read portion of the input tape with the head on the first symbol of w, and γ is the current contents of the pushdown tape (with the leftmost symbol of γ on the top of the pushdown tape. (q_0, w, γ_0) is, for any input word w, an initial configuration.

Two types of computational steps of $\mathcal{A}$ are considered. Both of them can be seen as a relation $\vdash_{\mathcal{A}} \subseteq (Q \times \Sigma^* \times \Gamma^*) \times (Q \times \Sigma^* \times \Gamma^*)$ between configurations. The so-called **Σ-step** is defined by

$$(p, v_1 v, \gamma_1 \gamma) \vdash_{\mathcal{A}} (q, v, \bar{\gamma}\gamma) \Leftrightarrow (q, \bar{\gamma}) \in \delta(p, v_1, \gamma_1),$$

where $v_1 \in \Sigma, \gamma_1 \in \Gamma, \bar{\gamma} \in \Gamma^*$. The so-called **$\varepsilon$-step** is defined by

$$(p, v, \gamma_1 \gamma) \vdash_{\mathcal{A}} (q, v, \bar{\gamma}\gamma) \Leftrightarrow (q, \bar{\gamma}) \in \delta(p, \varepsilon, \gamma_1).$$

In a Σ-step, the input head moves to the next input symbol; in a ε-step the input head does not move. In both steps the top-most pushdown symbol γ_1 is replaced by a string $\bar{\gamma} \in \Gamma^*$. If $|\bar{\gamma}| = 0$, this results in removing of γ_1.

There are also two natural ways how to define an acceptance by a PDA.

Acceptance by a final state:

$$L_f(\mathcal{A}) = \{w \mid (q_0, w, \gamma_0) \vdash^*_{\mathcal{A}} (p, \varepsilon, \gamma), p \in Q_F, \gamma \in \Gamma^*\}.$$

Acceptance by the empty pushdown tape:

$$L_e(\mathcal{A}) = \{w \mid (q_0, w, \gamma_0) \vdash^*_{\mathcal{A}} (p, \varepsilon, \varepsilon), p \in Q\}.$$

However, these two acceptance modes are not essentially different.

Exercise 7.3.19 *Show that for each pushdown automaton* $\mathcal{A}$ *one can easily construct a pushdown automaton* $\mathcal{A}'$ *such that* $L_e(\mathcal{A}) = L_f(\mathcal{A}')$*, and vice-versa.*

Example 7.3.20 *PDA* $\mathcal{A}_1 = \langle\{q_1,q_2\},\{0,1,c\},\{B,0,1\},q_1,\emptyset,B,\delta\rangle$*, with the transitions*

$$\begin{array}{rclrclrcl}
\delta(q_1,0,B) & = & (q_1,0B), & \delta(q_1,1,B) & = & (q_1,1B), & \delta(q_1,c,B) & = & (q_2,B),\\
\delta(q_1,0,0) & = & (q_1,00), & \delta(q_1,1,0) & = & (q_1,10), & \delta(q_1,c,0) & = & (q_2,0),\\
\delta(q_1,0,1) & = & (q_1,01), & \delta(q_1,1,1) & = & (q_1,11), & \delta(q_1,c,1) & = & (q_2,1),\\
\delta(q_2,0,0) & = & (q_2,\varepsilon), & \delta(q_2,1,1) & = & (q_2,\varepsilon), & \delta(q_2,\varepsilon,B) & = & (q_2,\varepsilon).
\end{array}$$

accepts, through the empty pushdown tape, the language $\{wcw^R \mid w \in \{0,1\}^*\}$*. This is easy to see. Indeed,* $\mathcal{A}_1$ *first stores* w *in the pushdown tape. After reading* c*,* $\mathcal{A}_1$ *goes into the state* q_2 *and starts to compare input symbols with those on the top of the pushdown tape. Each time they agree, the input head moves to the next symbol and the topmost symbol from the pushdown tape is removed. If they do not agree once,* $\mathcal{A}$ *does not accept.*

Example 7.3.21 *PDA* $\mathcal{A}_2 = \langle\{q_1,q_2\},\{0,1\},\{B,0,1\},q_1,\emptyset,B,\delta\rangle$ *with the transitions*

$$\begin{array}{rclrcl}
\delta(q_1,0,B) & = & (q_1,0B), & \delta(q_1,0,0) & = & \{(q_1,00),(q_2,\varepsilon)\},\\
\delta(q_1,1,B) & = & (q_1,1B), & \delta(q_1,1,1) & = & \{(q_1,11),(q_2,\varepsilon)\},\\
\delta(q_1,0,1) & = & (q_1,01), & \delta(q_2,1,1) & = & (q_2,\varepsilon),\\
\delta(q_1,1,0) & = & (q_1,10), & \delta(q_2,0,0) & = & (q_2,\varepsilon),\\
\delta(q_1,\varepsilon,B) & = & (q_2,\varepsilon), & \delta(q_2,\varepsilon,B) & = & (q_2,\varepsilon),
\end{array}$$

accepts, again through the empty pushdown tape, the language $\{ww^R \mid w \in \{0,1\}^*\}$*. Indeed, the basic idea is the same as in the previous example. In the state* q_1*,* $\mathcal{A}_2$ *stores* w *in the pushdown tape.* $\mathcal{A}_2$ *compares, in the state* q_2*, the next input symbol with the symbol on the top of the pushdown tape and if they agree, the input tape head makes a move and the topmost pushdown symbol is removed. However,* $\mathcal{A}_2$ *switches from the state* q_1 *to* q_2 *nondeterministically only.* $\mathcal{A}_2$ *"guesses" when it is in the middle of the input word.*

Exercise 7.3.22 *Let L be the language generated by the PDA shown in Figure 7.4 through the empty pushdown tape. Determine L and design a CFG for L. (In Figure 7.4 transitions are written in the form* (a,z,z') *and mean that if the input symbol is* a *and* z *is on the top of the pushdown tape, than* z *is replaced by* z'*.)*

Exercise 7.3.23 * *Show that to each PDA* $\mathcal{A}$ *with* $2n$ *states there is a PDA* $\mathcal{A}'$ *with* n *states such that* $L_e(\mathcal{A}) = L_e(\mathcal{A}')$

PDA has a very simple normal form (see Exercise 7.3.23):

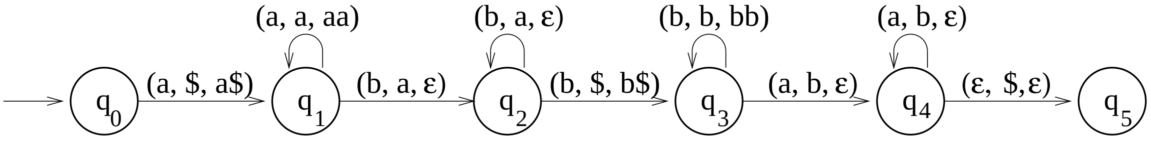


Figure 7.4: A pushdown automaton –70%

Theorem 7.3.24 *To every PDA $\mathcal{A}$ there is a one-state PDA $\mathcal{A}'$ such that $L_e(\mathcal{A}) = L_e(\mathcal{A}')$.*

Now we are ready to show the basic relation between the context-free grammars and pushdown automata.

Theorem 7.3.25 *A language is generated by a CFG if and only if it is accepted by a PDA.*

Proof. Let $G = \langle V_N, V_T, S, P\rangle$ be a CFG. We design a one-state PDA

$$\mathcal{A} = \langle \{q\}, V_T, V_N \cup V_T \cup \{\gamma_0, \#\}, q, \emptyset, \gamma_0, \delta\rangle,$$

with the transition function

$$\begin{array}{rclrcl} \delta(q,\varepsilon,\gamma_0) & = & (q, S\#), & \delta(q,\varepsilon,A) & = & \{(q,w) \mid A \longrightarrow w \in P\}, \\ \delta(q,a,a) & = & (q,\varepsilon), & \delta(q,\varepsilon,\#) & = & (q,\varepsilon), \end{array}$$

where $a \in V_T$. $\mathcal{A}$ first replaces the initial symbol γ_0 of the pushdown tape by the initial symbol of the grammar and by a special marker. $\mathcal{A}$ then simulates the leftmost derivation of G. Whenever the leftmost symbol of the pushdown tape is a terminal of G, then the only way to proceed is to compare this terminal with the next input symbol. If they do agree the top pushdown symbol is removed and the input head moves to the next symbol. If they do not agree, the computation stops. In this way, at any moment of computation, the already consumed part of the input and the contents of the pushdown tape forms a sentential form of a leftmost derivation. Finally, $\mathcal{A}$ empties its pushdown tape if the marker $\#$ is reached. (A more detailed proof can be done by induction.)

Let now $\mathcal{A}$ be a PDA. By Theorem 7.3.24 there is a one-state PDA $\mathcal{A}' = \langle \{q\}, \Sigma, \Gamma, q, \emptyset, z_0, \delta\rangle$, $\Sigma \cap \Gamma = \emptyset$, such that $L_e(\mathcal{A}) = L_e(\mathcal{A}')$.

Let $G = \langle \{S\} \cup \Gamma, \Sigma, S, P\rangle$ be a CFG with the following set of productions

$$S \longrightarrow z_0, \qquad A \longrightarrow xB_1B_2\dots B_m \quad \text{iff} \quad (q, B_1\dots B_m) \in \delta(q,x,A),$$

where x is is a terminal symbol or $x = \varepsilon$. (If $m = 0$, then $A \longrightarrow aB_1\dots B_m$ has the form $A \longrightarrow a$.)

A derivation in G is clearly a simulation of a computation in $\mathcal{A}'$ and this derivation results in a terminal word w if this input empties the pushdown tape of $\mathcal{A}'$. $\Box$

Exercise 7.3.26 *Design a PDA accepting the languages:*
(a) $\{w \mid w \in \{a,b\}^*, |\#_a w - \#_b w| \bmod 4 = 0\}$*; (b)* $\{0^i 1^j \mid 0 \leq i \leq j \leq 2i\}$.

Exercise 7.3.27 *Design a PDA equivalent to CFG with the productions:*
$S \longrightarrow BC \mid s$*,* $B \longrightarrow CS \mid b$*, and* $C \longrightarrow SB \mid c$.

7.3.4 Recognition and Parsing of Context-free Grammars

Algorithms for recognition and/or parsing of context-free grammars form important subprograms of many programs that receive their inputs in a natural or formal language form. In particular, they are key elements of most of translators and compilers. Efficient recognition and parsing of CFG is therefore an important practical task, but also an interesting theoretical problem.

Recognition problem — for a CFG G, is the problem to decide, given a word w, whether $w \in L(G)$.

Parsing problem — for a CFG G, is the problem to construct, given a word $w \in L(G)$, a derivation tree for w.

The following general and beautiful recognition algorithm CYK (due to Cocke, Younger and Kasami), one of the pearls of the algorithm design, assumes that $G = \langle V_N, V_T, S, P\rangle$ is a CFG in Chomsky normal form and $w = w_1 \ldots w_n, w_i \in V_T$ is an input word. The algorithm designs an $n \times n$ upper-triangular recognition matrix T, the elements $T_{i,j}, 1 \leq i \leq j \leq n$, of which are subsets of V_N.

```
begin for $1 \leq i,j \leq n$ do $T_{i,j} \leftarrow \emptyset$;
      for $i \leftarrow 1$ to $n$ do $T_{i,i} \leftarrow \{A \mid A \rightarrow w_i \in P\}$;
      for $d \leftarrow 1$ to $n-1$ do
          for $i \leftarrow 1$ to $n-d$ do
              begin $j \leftarrow d+i$;
                    $T_{i,j} \leftarrow \{A \mid \exists k,\ i < k \leq j$ such that
                              $A \longrightarrow BC \in P,\ B \in T_{i,k-1}, C \in T_{k,j}\}$
              end
      if $S \in T_{1,n}$ then accept else reject
end
```

In order to show the correctness of this algorithm, it is sufficient to prove by induction, that

$$T_{i,j} = \{A \mid A \overset{*}{\underset{P}{\Longrightarrow}} w_i \ldots w_j\}, \quad \text{if } 1 \leq i \leq j \leq n,$$

and therefore $w \in L(G)$ if and only if $S \in T_{1,n}$.

The following table shows the values of the matrix T when the CYK-algorithm is used for a CFG with the productions

$$\begin{array}{lll} (1)\ S \longrightarrow SS, & (2)\ S \longrightarrow AA, & (3)\ S \longrightarrow b, \\ (4)\ A \longrightarrow AS, & (5)\ A \longrightarrow AA, & (6)\ A \longrightarrow a, \end{array}$$

and the word "*abbaa*" is recognized:

A	A	A	-	A,S
	S	S	-	S
		S	-	S
			A	A,S
				A

To design the derivation tree, we can use the matrix T. Let us assume a fixed enumeration of productions of G and let $\pi_i, 1 \leq i \leq |P|$, denote the i-th production. To design the derivation tree for an $w = w_1 \ldots w_n$, $w_i \in V_T$, we can use the program

if $S \in T_{1,n}$ **then** $parse(1, n, S)$ **else** output "error"

and the following procedure *parse*:

```
procedure $parse(i, j, A)$
begin
    if $j = i$ then output($m$) such that $\pi_m = A \rightarrow w_i \in P$
        else if k is the least integer such that $i < k \leq j$ and there exist
            $\pi_m = A \rightarrow BC \in P$ with $B \in T_{i,k-1}$, $C \in T_{k,j}$
        then begin output($m$);
                $parse(i, k-1, B)$; $parse(k, j, C)$
            end
end
```

This procedure designs in the so-called "top-down manner" (i.e., from S to w) the leftmost derivation of w. For example, for the grammar given above and $w = abbaa$, we get as the output the sequence of productions: $2, 4, 6, 1, 3, 3, 5, 6, 6$ and the derivation tree

When implemented on a RAM, the time complexity of the CYK-algorithm is clearly $\Theta(n^3)$ and $\Theta(n)$ for the parsing algorithm. It can be shown that both algorithms can be implemented also on Turing machines with asymptotically the same time performance. It therefore holds

Theorem 7.3.28 *Recognition and parsing of general CFG can be done in* $\Theta(n^3)$ *time on RAM and Turing machines.*

Exercise 7.3.29 *Design a CFG G' in Chomsky normal form that generates the language $L(G) - \{\varepsilon\}$, where G is the grammar $S \longrightarrow aS \mid aSbS \mid \varepsilon$, and design the upper-triangular matrix that is created by CYK-algorithm when recognizing the word "aabba".*

Exercise 7.3.30 *Implement the CYK recognition algorithm on a multitape Turing machine in such a way that recognition is done in $\mathcal{O}(n^3)$-time.*

Parsing algorithms belong to the most often used algorithms (they are part of any text processing system), and therefore parsing algorithms with time complexity $\Theta(n^3)$ are unacceptably slow. It is therefore of importance to find out whether there are faster parsing algorithms and to develop parsing algorithms that are as fast as possible.

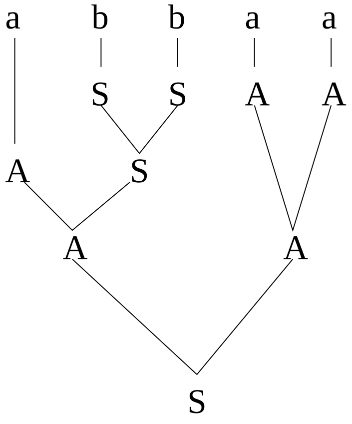
Surprisingly, the problem of fast recognition of context-free grammars seems to be still far from being solved and, even more surprisingly, this problem has turned out to be closely related to such seemingly very different problems, as Boolean matrix multiplication. It holds:

Theorem 7.3.31 (Valiant's Theorem) *Let $\mathcal{A}$ be a RAM algorithm to multiply two Boolean matrices of degree n with time complexity $M(n)$ (with respect to the logarithmic time complexity). Then there is an $\mathcal{O}(M(n))$-time RAM algorithm to recognize an arbitrary context-free grammar.*

Since there is an $\mathcal{O}(n^{2.37})$ RAM algorithm for Boolean matrix multiplication, see Section 4.2.4, we have

Corollary 7.3.32 *There is an $\mathcal{O}(n^{2.37})$ RAM algorithm to recognize an arbitrary CFG.*

Recognition of general CFG has been intensively investigated also with respect to other complexity measures and computational models. $\mathcal{O}(\lg^2 n)$ is the best known result for space complexity on Turing machines and also for time complexity on CRCW$^+$ PRAM (and on hypercubes—see Section 10.1— with $\mathcal{O}(n^6)$ processors).



Since parsing is so massively used a linear parsing algorithm is practically the only acceptable solution. This does not seem to be possible for arbitrary CFG, and therefore the way out is to consider parsing for special classes of CFG that would be, however, rich enough for most of the applications.

Restrictions to unambiguous CFG or even to the so-called **linear CFG**—with productions of the form $A \longrightarrow uBv$ or $A \longrightarrow w$, where A, B are nonterminals and u, v, w are terminal words—do not seem to help. The fastest known recognition algorithms for such grammars have time complexity $\Theta(n^2)$. On the other side, and this has turned out practically a satisfactory solution, a restriction to recognition of only deterministic context-free languages leads to $\Theta(n)$ algorithms.

Definition 7.3.33 *A CFL L is a* **deterministic context-free language (DCFL)** *if* $L = L_f(\mathcal{A})$ *for a deterministic PDA* $\mathcal{A}$. *A PDA* $\mathcal{A} = \langle Q, \Sigma, \Gamma, q_0, \emptyset, z, \delta\rangle$ *is a* **deterministic PDA (DPDA)** *if it holds:*

1. $q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$ *and* $\gamma \in \Gamma$ *implies* $|\delta(q, a, \gamma)| \leq 1$*;*
2. *if* $q \in Q$, $\gamma \in \Gamma$ *and* $\delta(q, \varepsilon, \gamma) \neq \emptyset$, *then* $\delta(q, a, \gamma) = \emptyset$, *for all* $a \in \Sigma$.

In other words in a DPDA at any global state at most one transition is possible. For example, the PDA in Example 7.3.20 is deterministic, but the PDA from Example 7.3.21 is not.

Exercise 7.3.34 *Show that the following languages are DCFL:*

(a) $\{w \mid w \in \{a, b\}^*, \#_a w = \#_b w\}$*; (b)* $\{a^n b^m \mid 1 \leq n \leq 3m\}$.

Caution. In the case of DPDA the acceptance by a final state and an acceptance by an empty pushdown tape are not equivalent.

Exercise 7.3.35 *Show that every language accepted by a DPDA with respect to an empty pushdown tape can be accepted by a DPDA with respect to a final state.*

Exercise 7.3.36 ** *Show that the following language is acceptable by a deterministic pushdown automaton with respect to a final state but not with respect to the empty pushdown tape:* $\{w \in \{a, b\}^* \mid w$ *contains the same number of occurrences of a and b*$\}$.

Due to ε-moves a DPDA may need more than $|w|$ time steps to recognize an input word w. In spite of that DCFL can be recognized in linear time.

Theorem 7.3.37 *For each DPDA* $\mathcal{A}$ *there is a constant* $c_{\mathcal{A}}$ *such that each word* $w \in L_f(\mathcal{A})$ *can be recognized in* $c_{\mathcal{A}}|w|$ *steps.*

Proof idea. From the description of $\mathcal{A}$ one can determine an upper bound $c_{\mathcal{A}}$ on the number of steps $\mathcal{A}$ can make without being in a cycle. $\Box$

There are also several well defined subclasses of CFG that generate exactly deterministic CFL. For example, the so-called $LR(k)$ grammars. They are dealt with in any book on parsing and compilation.

7.3.5 Context-free Languages

Context-free languages play an important rôle in formal language theory and computing, next to the regular languages. We deal here with several basic questions concerning CFL. How to determine that a language is not context-free? Which closure properties has the family of CFL? Which decision problems are decidable (undecidable) for CFL? What is the overall rôle of CFL in the formal language theory? Are there some especially important (or "complete") context-free languages?

The most basic way to determine that a language L is context-free, is to design a CFG or a PDA for L. It is much less clear how to show that a language is not context-free (and therefore it has no sense to try to design a CFG or a PDA for it). One way of doing that is to use the following result.

Lemma 7.3.38 (Bar-Hillel's Pumping Lemma) *To every CFG G one can compute an integer n_G such that for each $z \in L(G)$, $|z| > n_G$ there are words x, u, w, v, y such that*

1. $z = xuwvy, |uv| \geq 1, |uwv| \leq n_G,$
2. $xu^iwv^iy \in L(G)$, *for all* $i \geq 0$.

Example 7.3.39 *We show that the language $L = \{a^ib^ic^i \,|\, i \geq 1\}$ is not context-free by deriving a contradiction, using the Pumping Lemma, from the assumption that L is context-free.*

Assume L is context-free. Then there is a CFG G generating L. Let $n = n_G$ be an integer satisfying the conditions of the Pumping Lemma for G. In such a case $z = a^nb^nc^n$ can be split as $z = xuwvy$ such that the conditions of Pumping Lemma are fulfilled and therefore $|uwv| \leq n$. However, then the string uwv can not contain both an "a" and a "c" (any two occurrences of such symbols have to be at least $n + 1$ symbols apart). Hence the word xu^2wv^2y, that should also be in L by the Pumping Lemma, does not contain the same number of a's, b's and c's. A contradiction. $\Box$

Exercise 7.3.40 *Show that the following languages are not context-free:*
(a) $L = \{ww \,|\, w \in \{a,b\}^*\}^c$; *(b)* $\{a^nb^ncb^na^n \,|\, n \geq 1\}$;
(c) $\{a^ib^ja^ib^j \,|\, i,j \geq 1\}$; *(d)* $\{a^{n^2} | n \geq 1\}$; *(e)* $\{a^i \,|\, i$ *is composite*$\}$.

Bar-Hillel's Pumping Lemma is an important but not niversal tool to show that a language is not context-free. For example the language $\{a^*bc\} \cup \{a^pba^nca^n \mid p \text{ is prime}, n \geq p\}$ is not context-free but this can not be shown using Bar-Hillel's Pumping Lemma. (Show why?)

Since each CFL is clearly also context-sensitive, , it follows from Example 7.3.39 that the family of context free languages is a proper subfamily of context-sensitive languages. Similarly, it is evident that each regular language is context-free. Since the syntactical monoid of the language $L_0 = \{ww^R \mid w \in \{0,1\}^*\}$ is infinite—see Section 3.3, L_0 is an example of a context-free language that is not regular.

It can be shown that each deterministic context-free language is unambiguous and therefore $L = \{a^ib^ja^k \mid i = j \text{ or } j = k\}$ is an example of a CFL that is not deterministic. Hence we have the hierarchy

$$\mathrm{L}(NFA) \; (\; \mathrm{L}(DPDA) \; (\; \mathrm{L}(NPDA) \; (\; \mathcal{L}(LBA).$$

Another method how to prove that a language L is context-free is to show that L can be designed from another languages, already known to be context-free, using operations under which the family of CFL is closed.

Theorem 7.3.41 *The family of CFL is closed under operations of union, concatenation, iteration, homomorphism, intersection with regular sets and difference with regular sets. It is not closed with respect to the intersection, deletion and complement.*

The family of deterministic CFL is closed under complement, intersection with regular sets and difference with regular sets, but it is not closed with respect to union, concatenation, iteration, homomorphism and difference.

Proof. Closure of $\mathcal{L}(CFL)$ under union, concatenation, iteration and homomorphism is easy to show using CFG. Indeed, let G_1 and G_2 be two CFG with disjoint sets of nonterminals and not containing the symbol "S", and let S_1, S_2 be their start symbols. If we take the productions from G_1 and add productions $\{S \longrightarrow SS_1 \mid \varepsilon\}$, we get a CFG generating, from the new start symbol S, the language $L(G_1)^*$. Moreover, if we take the productions from G_1 and G_2 and add productions $\{S \longrightarrow S_1 \mid S_2\}$ (or $S \longrightarrow S_1S_2$) we get a CFG generating the language $L(G_1) \cup L(G_2)$ (or $L(G_1)L(G_2)$). In order to get a CFG for the language $h(L(G_1))$, where h is a morphism, we replace, in productions of G_1, each terminal a by $h(a)$.

In order to show the closure of the family $\mathcal{L}(CFG)$ under intersection with regular languages, let us assume that L is a CFL and R a regular set. By Theorem 7.3.24, there is a one-state PDA $\mathcal{A} = \langle \{q\}, \Sigma, \Gamma, q, \emptyset, y_0, \delta_p \rangle$ which has the form as on page 492 and $L_e(\mathcal{A}) = L$. Let $\mathcal{A}' = \langle Q, \Sigma, q_0, Q_F, \delta_f \rangle$ be a DFA accepting R. A PDA $\mathcal{A}'' = \langle Q, \Sigma, \Gamma, q_0, \emptyset, z, \delta \rangle$ with the following transition function, where q_1 is an arbitrary state from Q and $A \in \Gamma$,

$$\begin{aligned} \delta(q_0, \varepsilon, z) &= \{(q_0, S\#)\}; && (7.1) \\ \delta(q_1, \varepsilon, A) &= \{(q_1, w) \mid (q, w) \in \delta_p(q, \varepsilon, A)\}; && (7.2) \\ \delta(q_1, a, a) &= \{(q_2, \varepsilon) \mid q_2 \in \delta_f(q_1, a)\}; && (7.3) \\ \delta(q_1, \varepsilon, \#) &= \{(q_1, \varepsilon) \mid q_1 \in Q_F\}; && (7.4) \end{aligned}$$

accepts $L \cap R$. Indeed, when ignoring states of $\mathcal{A}'$ we recover $\mathcal{A}$, and therefore $L_e(\mathcal{A}'') \subseteq L_e(\mathcal{A})$. On the other side, each word accepted by $\mathcal{A}''$ is also in R, due to the transitions in (7.2) and (7.3). Hence $L_e(\mathcal{A}'') = L \cap R$.

Since $L - R = L \cap R^c$, we get that the family $\mathcal{L}(CFG)$ is closed also under the difference with regular languages.

For the non context-free language $L = \{a^ib^ic^i \,|\, i \geq 1\}$ we have $L = L_1 \cap L_2$, where $L = \{a^ib^ic^j \,|\, i,j \geq 1\}$, $L_2 = \{a^ib^jc^j \,|\, i,j \geq 1\}$. This implies that the family $\mathcal{L}(CFG)$ is not closed under intersection and since $L_1 \cap L_2 = (L_1^c \cup {L_2}^c)^c$ it is not closed under the complement either. Moreover, since the language $L_3 = \{a^ib^jc^k \,|\, i,j,k \geq 1\}$ is regular and therefore context-free and also $L_4 = \{a^ib^jc^k \,|\, i \neq j \text{ or } j \neq k\}$ is a context-free language and $L = L_3 - L_4$, we get that the family $\mathcal{L}(CFG)$ is not closed under set difference.

Proofs for closure and non-closure properties of deterministic context-free languages are more involved and can be found in the literature—see the references. □

Concerning decision problems news are not good. Unfortunately, most of the basic decision problems for CFL are undecidable.

Theorem 7.3.42 *(1) The following decision problems are decidable for a CFG G: (a) Is $L(G)$ empty? (b) Is $L(G)$ infinite?*

(2) The following decision problems are undecidable for CFG G and G' with the terminal alphabet Σ:

(c) Is $L(G) = \Sigma^*$? *(d) Is $L(G)$ regular?*
(e) Is $L(G)$ unambiguous? *(f) Is $L(G)^c$ infinite?*
(g) Is $L(G)^c$ context-free? *(h) Is $L(G) = L(G')$?*
(i) Is $L(G) \cap L(G')$ empty? *(j) Is $L(G_1) \cap L(G_2)$ context-free?*

Sketch of the proof Let us consider at first the assertion (1). It is easy to decide whether $L(G) = \emptyset$ for a CFG G. Indeed, if $G = \langle V_N, V_T, S, P\rangle$, we construct the following sequence of sets:

$$X_0 = V_T,\ X_{i+1} = X_i \cup \{A \,|\, A \longrightarrow u \in P, u \in X_i^*\}, i \geq 1.$$

Clearly, $L(G)$ is empty if and only if $S \notin X_{|V_N|}$.

We now show that the question whether $L(G_1)$ is finite is decidable. By the Pumping Lemma we can compute n such that $L(G)$ is infinite if and only if $L(G)$ contains a word longer than n. Now let $L_n(G)$ be the set of words of $L(G)$ of length $\leq n$. Since the recognition problem is decidable for CFG, $L_n(G)$ is computable. Therefore, by Theorem 7.3.41, $L(G) - L_n(G)$ is a CFL and one can design effectively a CFG G_0 for $L(G) - L_n(G)$. Now $L(G)$ is infinite if and only if $L(G_0)$ is nonempty, what is decidable, by (a).

It has actually been shown in Example 6.4.19 that it is undecidable whether the intersection of two context-free languages is empty. Let us now present a technique that can be used to show various other undecidability results for CFG.

Let $A = (u_1, \ldots, u_k)$, $B = (v_1, \ldots, v_k)$ be two lists of words over the alphabet $\Sigma = \{0, 1\}$, $K = \{a_1, \ldots, a_k\}$ be a set of distinct symbols not in Σ, and $c \notin \Sigma$ be an additional new symbol.

Let

$$L_A = \{u_{i_1} \dots u_{i_m} a_{i_m} \dots a_{i_1} \,|\, 1 \leq i_s \leq k, 1 \leq s \leq m\}$$

and let L_B be a similarly defined language for the list B. The languages

$$R_{AB} = \{ycy^R \,|\, y \in \Sigma^* K^*\}, \qquad S_{AB} = \{ycz^R \,|\, y \in L_A, z \in L_B\},$$

are clearly DCFL and therefore, by Theorem 7.3.41, also their complements R^c_{AB}, S^c_{AB} are CFL. Hence $L_{AB} = R^c_{AB} \cup S^c_{AB}$ is a CFL. It is now easy to see that

$$L_{AB} = (\Sigma \cup K \cup \{c\})^* \text{ if and only if the PCP for } A \text{ and } B \text{ has no solution.} \tag{7.5}$$

The language L_{AB} is regular and therefore (7.5) implies not only that the equivalence problem for CFG is undecidable, but that it is also undecidable, for a CFG G and a regular language R (in particular $R = \Sigma^*$), whether $L(G) = R$.

Using the Pumping Lemma we can show that the language $R_{AB} \cap S_{AB}$ is context-free if and only if it is empty. On the other hand, $R_{AB} \cap S_{AB}$ is empty if and only if the Post Correspondence Problem for A and B has no solution. Thus, it is undecidable whether the intersection of two CFL is a CFL.

Undecidability proofs for the remaining problems can be found in references. □

Exercise 7.3.43 *Show that it is decidable, given a CFG G and a regular language R, whether $L(G) \subseteq R$.*

Exercise 7.3.44 * *Show that the question whether a given CFG is (un)ambiguous is undecidable.*

Interesting enough, several basic decision problems for CFG, as membership, emptiness and infiniteness problems, are **P**-complete, and therefore belong to the inherently sequential problems.

Finally, we discuss the overall rôle context-free language play in the formal language theory. We shall see that they can be used, together with the operations of intersection and homomorphism to describe any recursively enumerable language. This will also illustrate the power that the operations of intersection and homomorphism have. For example, it can be shown:

Theorem 7.3.45 *For any alphabet Σ there are two fixed DCFL, L_1^Σ and L_2^Σ, and a fixed homomorphism h^Σ, such that for any recursively enumerable language $L \subseteq \Sigma^*$ there is a regular language R_L such that*

$$L = h^\Sigma(L_1^\Sigma \cup (L_2^\Sigma \cap R_L)).$$

Languages L_1^Σ and L_2^Σ seem to capture fully the essence of "context-freeness". It would seem therefore that they have to be very complicated. Interestingly enough, this is not the case.

Let $\Sigma_k = \{a_1, \overline{a_1}, a_2, \overline{a_2}, \ldots, a_k, \overline{a_k}\}$ be an alphabet of k pairs of symbols. They will be used to play the rôle of pairs of brackets: $a_i (\overline{a_i})$ will be the i-th left (right) bracket. The alphabet Σ_k is used to define the so-called **Dyck language** D_k, $k \geq 1$. It is the language generated by the grammar

$$S \longrightarrow \varepsilon \,|\, S a_i S \overline{a_i}, \qquad 1 \leq i \leq k.$$

Observe that if $a_1 =$ (and $\overline{a_1} =$), then D_1 is just the set of all well-parenthesized strings.

Exercise 7.3.46 *Let D_k be the Dyck language with k parentheses. Design a homomorphism h such that $D_k = h^{-1}(D_2)$.*

The following theorem shows that Dyck languages reflect the structure of all CFL.

Theorem 7.3.47 (Chomsky-Schützenberger's Theorem) *(1) For each CFL L there is an integer r, a regular language R and a homomorphism h such that*

$$L = h(D_r \cap R).$$

(2) For each CFL L there is a regular language R and two homomorphisms h_1, h_2 such that

$$L = h_2(h_1^{-1}(D_2) \cap R).$$

In addition, D_2 can be used to define "the" context-free language. It is the so-called **Greibach language**

$$\begin{aligned} L_G &= \{\varepsilon\} \cup \{x_1 c y_1 c z_1 d \ldots x_n c y_n c z_n d \,|\, n \geq 1, y_1 \ldots y_n \in \gamma D_2, \\ &\quad x_i, z_i \in \Sigma^*,\ 1 \leq i \leq n,\ y_i \in \{a_1, a_2, \overline{a_1}, \overline{a_2}\}^*,\ i \geq 2\}, \end{aligned}$$

where $\Sigma = \{a_1, a_2, \overline{a_1}, \overline{a_2}, \gamma, c\}$ and $d \in \Sigma$. (Note that words x_i, z_i may contain symbols "c" and "γ".)

A PDA that recognizes the language L_G works as follows: it reads input, guesses the beginnings of $y_1, y_2, \ldots, y_n$ and recognizes whether $y_1 \ldots y_n \in \gamma D_2$.

There are two reasons why L_G has a very special rôle among CFL.

Theorem 7.3.48 (Greibach's Theorem) *(1) For every CFL L there is a homomorphism h such that $L = h^{-1}(L_G)$ if $\varepsilon \in L$ and $L = h^{-1}(L_G - \{\varepsilon\})$ if $\varepsilon \notin L$.*

(2) L_G is the hardest to recognize CFL. In other words, if L_G can be recognized in time $p(n)$ on a TM (or on a RAM or on a CA), then each CFL can be recognized in time $\mathcal{O}(p(n))$ on the same model.

This means that in order to improve the $\mathcal{O}(n^{2.37})$ upper-bound for time complexity of recognition of CFL it is enough to find a faster recognition algorithm for a single CFL L_G—the Greibach language.

7.4 Lindenmayer Systems

Lindenmayer[2] systems—L-systems for short—were introduced to create a formal theory of plant development. Their consequent intensive investigation was mainly due to their generality and elegance and the fact that they represent the basic model of parallel context-free string rewriting systems.

7.4.1 0L-systems and Growth Functions

There is a variety of L-systems. The most basic are 0L-, D0L- and PD0L-systems.

Definition 7.4.1 *A* **0L-system** $G = \langle \Sigma, \omega, h \rangle$ *is given by a finite* **alphabet** Σ, *an* **axiom** (*or an* **initial string**) $\omega \in \Sigma^*$, *and a* **finite substitution** $h : \Sigma \rightarrow 2^{\Sigma^*}$ *such that* $h(a) = \emptyset$ *for no* $a \in \Sigma$. *(If* $u \in h(a)$, $a \in \Sigma$, *then* $a \longrightarrow u$ *is called a* **production** *of L.) The* **0L-language** *generated by G is defined by*

$$L(G) = \bigcup_{i \geq 0} \{h^i(\omega)\}.$$

If h is a morphism, i.e, $h(a) \in \Sigma^*$, *for each* $a \in \Sigma$, *we talk about a* **D0L-system**, *and if h is a nonerasing morphism, i.e.,* $h(a) \in \Sigma^+$, *for each* $a \in \Sigma$, *then G is said to be a* **PD0L-system**.

An 0L-system can be seen as given by an initial word and a set of context-free productions $a \longrightarrow u$, at least one for each $a \in \Sigma$. This time, however, there is no partition of symbols into nonterminals and terminals. A derivation step $w \Longrightarrow w'$ consists of rewriting each symbol of w using a production with that symbol on the left hand side. 0L-systems can be seen as nondeterministic versions of D0L-systems; in D0L-systems there is only one production $a \longrightarrow u$ for each $a \in \Sigma$. In a PD0L-system (or a "propagating D0L-system"), if $(a \longrightarrow u$, then $|a| \leq |u|$. Each derivation in an 0L-system can be depicted, similarly as for CFG, by a derivation tree.

Example 7.4.2 *In the D0L system*[3] *with the axiom* $\omega = a_r$ *and productions*

$$a_r \longrightarrow a_l b_r, \qquad a_l \longrightarrow b_l a_r, \qquad b_r \longrightarrow a_r, \qquad b_l \longrightarrow a_l$$

we have a derivation (see Figure 7.5 for the corresponding derivation tree):

$$a_r \Longrightarrow a_l b_r \Longrightarrow b_l a_r a_r \Longrightarrow a_l a_l b_r a_l b_r \Longrightarrow b_l a_r b_l a_r a_r b_l a_r a_r \Longrightarrow \ldots.$$

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[2] Aristid Lindenmayer (1922-1990), a Dutch biologist, introduced L-systems in 1968.

[3] This system is taken from modeling the development of a fragment of a multicellular filament such as that found in the blue-green bacteria *Anabaena catenula* and various algae. The symbols a and b represent cytological stages of the cell (their size and readiness to divide). The subscript r and l indicate cell polarity specifying positions in which daughter cells of type a and b will be produced.

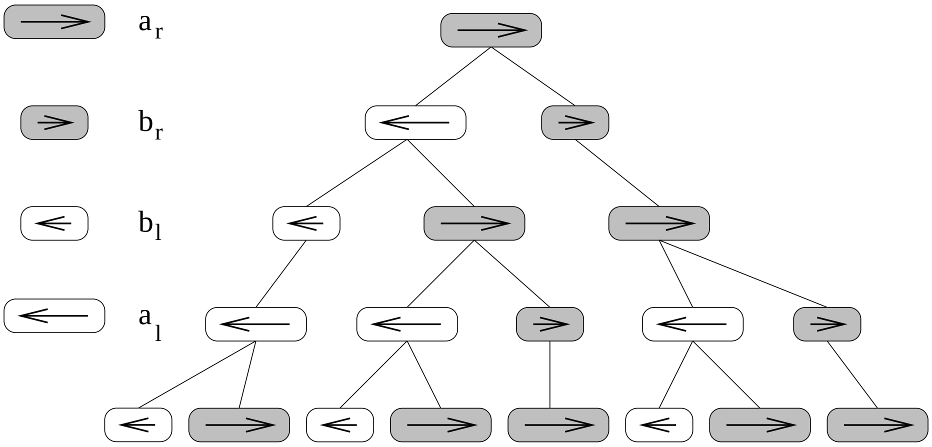


Figure 7.5: Development of a filament simulated using a D0L-system—70%

Exercise 7.4.3 *Show that the PD0L-system* G_1 *with the axiom "a" and the only one production* $a \longrightarrow aaa$, *generates the language*
$L(G_1) = \{a^{3^n} \mid n \geq 0\}$.

Exercise 7.4.4 *Show that the D0L-system* G_2 *with the axiom* ab^3a *and productions* $P = \{a \longrightarrow ab^3a,\ b \longrightarrow \varepsilon\}$, *generates the language* $L(G_2) = \{(ab^3a)^{2^n} \mid n \geq 0\}$.

Exercise 7.4.5 *Show that the 0L-system* $G_3 = \langle\{a,b\}, a, h\rangle$ *with* $h(a) = h(b) = S = \{aabb, abab, baab, abba, baba, bbaa\}$, *generates the language* $L(G) = \{a\} \cup \{a,b\}^{4^n} \cap S^*$.

A **derivation** in a D0L-system $G = \langle\Sigma, \omega, h\rangle$ can be seen as a sequence

$$\omega = h^0(\omega), h^1(\omega), h^2(\omega), h^3(\omega), \ldots$$

and the function

$$f_G(n) = |h^n(\omega)|$$

is called the **growth function** of G.

With respect to the original motivation, growth functions capture the development of the size of the simulated biological system. On a theoretical level growth functions represent an important tool to investigate various problems concerning languages.

Example 7.4.6 *For the PD0L-system* G *with axiom* $\omega = a$ *and morphism* $h(a) = b, h(b) = ab$ *we have the only possible derivation*

$$a, b, ab, bab, abbab, bababbab, abbabbababbab, \ldots$$

and for the derivation sequence $\{h^n(\omega)\}_{n\geq 0}$, *we have, for* $n \geq 2$

$$\begin{aligned} h^n(a) &= h^{n-1}(h(a)) = h^{n-1}(b) = h^{n-2}(h(b)) = h^{n-2}(ab) \\ &= h^{n-2}(a)h^{n-2}(b) = h^{n-2}(a)h^{n-1}(a), \end{aligned}$$

and therefore

$$\begin{aligned} f_G(0) &= f_G(1) \;=\; 1, \\ f_G(n) &= f_G(n-1) + f_G(n-2) \text{ for } n \geq 2. \end{aligned}$$

This implies that $f_G(n) = F_n$*—the* n*-th Fibonacci number.*

Exercise 7.4.7 *Show, for the PD0L-system with the axiom "a" and the productions*

$$a \longrightarrow abcc, \quad b \longrightarrow bcc, \quad c \longrightarrow c,$$

for example, using the same technique as in the previous example, that $f_G(n) = f_G(n-1) + 2n + 1$ *and therefore* $f_G(n) = (n+1)^2$.

The growth functions of D0L-systems have a useful matrix representation. This is based on the observation that the growth function of a D0L-system does not depend on the ordering of symbol in axioms, productions and derived words.

Let $G = \langle \Sigma, \omega, h \rangle$ and $\Sigma = \{a_1, \ldots, a_k\}$. The growth matrix for G is defined by

$$M_G = \left| \begin{array}{lll} \#_{a_1}h(a_1) & \ldots & \#_{a_k}h(a_1) \\ \ldots & & \ldots \\ \#_{a_1}h(a_k) & \ldots & \#_{a_k}h(a_k) \end{array} \right|.$$

If $\pi_\omega = (\#_{a_1}\omega, \ldots, \#_{a_k}\omega)$ and $\eta = (1, \ldots, 1)^T$ are row and column vectors, then clearly

$$f_G(n) = \pi_\omega M_G^n \eta.$$

Theorem 7.4.8 *The growth function* f_G *of a D0L-system* G *satisfies the recurrence*

$$f_G(n) = c_1 f_G(n-1) + c_2 f_G(n-2) + \ldots + c_k f(n-k) \tag{7.6}$$

for some constants $c_1, \ldots, c_k$, *and therefore each such function is a sum of exponential and polynomial functions.*

Proof. It follows from linear algebra that M_G satisfies its own characteristic equation

$$M_G^k = c_1 M_G^{k-1} + c_2 M_G^{k-2} + \ldots + c_{k-1} M_G^1 + c_k M_G^0 \tag{7.7}$$

for some coefficients $c_1, \ldots, c_k$. By multiplying both sides of (7.7) with π_ω from the left, and by η from the right, we get (7.6). Since (7.6) is a homogeneous linear recurrence, the second result follows from theorems in Section 1.2.3. $\Box$

There is a modification of 0L-systems, the so-called E0L-systems, in which symbols are partitioned into nonterminals and terminals.

An **E0L-system** is defined by $G = \langle \Sigma, \Delta, \omega, h \rangle$, where $G' = \langle \Sigma, \omega, h \rangle$ is an 0L-system and $\Delta \subseteq \Sigma$. The language generated by G is defined by

$$L(G) = L(G') \cap \Delta^*.$$

In other words, only strings from Δ^*, derived by the underlying 0L-system G' are taken into $L(G)$. Symbols from $\Sigma - \Delta$ (Δ) play the rôle of nonterminals (terminals).

Exercise 7.4.9 *Show that the E0L-system with the alphabets* $\Sigma = \{S, a, b\}$, $\Delta = \{a, b\}$, *the axiom* SbS, *and productions* $S \longrightarrow S \,|\, a$, $a \longrightarrow aa$, $b \longrightarrow b$ *generates the language* $\{a^{2^i} b a^{2^j} \mid i, j \geq 0\}$.

The family $\mathcal{L}(E0L)$ of languages generated by E0L-systems has nicer properties than the family $\mathcal{L}(0L)$ of languages generated by 0L-systems. For example, $\mathcal{L}(0L)$ is closed neither under union, nor concatenation, iteration or intersection with regular sets, but $\mathcal{L}(E0L)$ is closed under all these operations. On the other side, the equivalence problem, that is undecidable for E0L- and 0L-systems, is decidable for D0L-systems

7.4.2 Graphical Modeling with L-systems

The idea to use L-systems to model plants has been questioned for a long time. L-systems do not seem to include enough details to model well higher plants. Emphases in L-systems are only on neighbourhood relations between cells and geometrical interpretations seemed to be beyond the scope of the model. However, once various geometrical interpretations and modifications of L-systems were discovered, L-systems turned out to be a versatile tool for plant modeling.

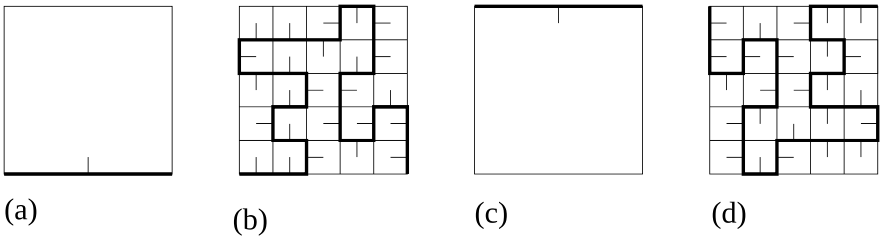
We discuss here several approaches to the graphical modeling with L-systems. They also illustrate, what is often the case, that simple modifications, twistings and interpretations, of basic theoretical concepts can lead to highly complex and useful systems. For example, it has been demonstrated that there are various D0L-systems G over the alphabets $\Sigma \supseteq \{f, +, -\}$ with the following property: if the morphism $h : \Sigma \rightarrow \{F, f, +, -\}$ defined by $h(a) = F$ if $a \notin \{f, +, -\}$ and $h(a) = a$ otherwise, is applied to strings generated by G, one gets strings over the turtle alphabet $\{F, f, +, -\}$ such that their turtle interpretation (see Section 2.5.3), produces interesting fractal or space filling curves. This is illustrated in Figure 7.6, which contains for each curve also a description of the corresponding D0L-system (an axiom and productions), the number n of derivation steps and the degree δ of the angle of turtle‘s turns.

No well developed methodology is known to design, given a family $\mathcal{C}$ of similar curves, a D0L-systems that generates strings whose turtle interpretation provides exactly curves from $\mathcal{C}$. For this problem, the so-called **inference problem,** only some intuitive techniques are available. One of them, called edge rewriting, specifies how an edge is to be replaced by a curve, and this is then expressed by productions of an D0L-system. For example, Figures 7.7b

(a)	(b)	(c)
$n = 5$, $\delta = 90°$	$n = 6$, $\delta = 60°$	$n = 2$, $\delta = 90°$
axiom $= F - F - F - F$	axiom $= F_r$	axiom $= -F_r$
$F \to F - FF - -F - F$	$F_l \to F_r + F_l + F_r$, $F_r \to F_l - F_r - F_l$	productions as in Fig. 7.7e

Figure 7.6: Fractal and space-filling curves generated by the turtle interpretation of strings generated by D0L-systems

and 7.7d show a way how an F_l-edge (Figure 7.7a) and an F_r-edge (Figure 7.7c) can be replaced by square grid filling curves and also the corresponding D0L-system (Figure 7.7e). The resulting curve, for the axion "F_l", $n = 2$ and $\delta = 90°$ is shown in Figure 7.6c.


(a) (b) (c) (d)

$$F_l \longrightarrow F_l\ F_l\ +F_r\ +F_r\ -F_l\ -F_l\ +F_r\ +F_r\ F_l\ -F_r\ -F_l\ F_l\ F_r\ +$$
$$F_l\ -F_r\ -F_l F_l\ -F_r\ +F_l\ F_r\ +F_r +F_l -F_l\ -F_r F_r +$$

$$F \longrightarrow -F_l\ F_l + F_r\ +F_r\ -F_l -F_l F_r - F_l + F_r F_r +F_l +F_r -$$
$$F_l\ F_r\ F_r\ +F_l\ +F_r\ F_l\ -F_l\ -F_l\ +F_r\ +F_l\ -F_l\ -F_r\ F_r$$

(e)

Figure 7.7: Construction of a space filling curve on the square grid using an edge rewriting with the corresponding PD0L-system and its turtle interpretation –70%

The turtle interpretation of a string results always in a single curve. This curve may intersect with itself, may have invisible lines (more exactly, interruptions caused by f-statements for turtle), and may have segments drawn several times, but it is always only a single curve. However, this is not the way plants and nature develop. A branching recursive structure is more characteristic for their development. To model that, a slight modification of L-systems, the so-called tree 0L-systems, and/or of string interpretations, turned out to be more appropriate.

A **tree 0L-system** $\mathcal{T}$ is determined by three components: a set of edge labels E, an initial (axial) tree T_0, with edges labeled by labels from E, see Figure 7.8a, and a set P of tree productions, see Figure 7.8b, at least one for each edge label, in which a labeled edge is replaced by a finite, edge labeled axial tree with a specified begin-node (denoted by small black circle) and an end-node (denoted by a small empty circle). (By an axial tree we mean here any rooted tree in which any internal node has at most three ordered successors (left, right and straightforward—some may be missing).)

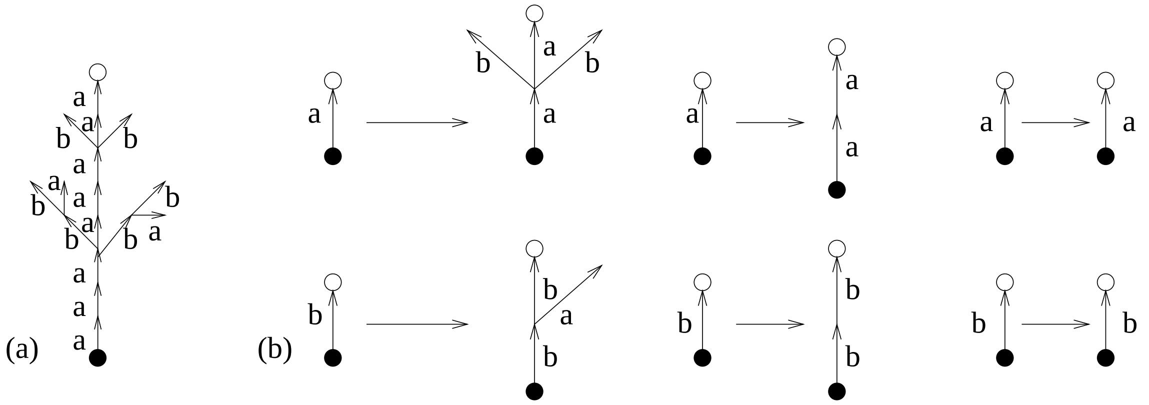


Figure 7.8: A tree 0L-system, axiom and production–70%

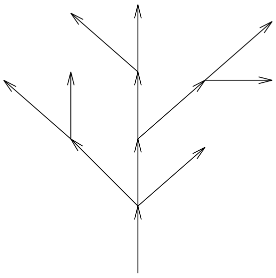
An axial tree T_2 is said to be directly derived from an axial tree T_1 using a tree 0L-system $\mathcal{T}$, notation $T_1 \underset{P}{\Longrightarrow} T_2$, if T_2 is obtained from T_1 by replacing each edge of T_1 by an axial tree given by a tree production of $\mathcal{T}$ for that edge; and identifying the begin-node (end-node) of the axial tree with the starting (ending)node of the edge that is being replaced. A tree T is generated from the initial tree T_0 by a derivation (notation $T_0 \overset{*}{\underset{P}{\Longrightarrow}} T$), if there is a sequence of axial trees $T_0, T_1, \ldots, T_n$ such that $T_i \underset{P}{\Longrightarrow} T_{i+1}$, for $i = 0, 1, \ldots, n-1$, and $T = T_n$.

Exercise 7.4.10 *Show how the tree in Figure 7.8a can be generated using the tree 0L-system shown in Figure 7.8b from a simple tree with two nodes and the edge labeled by a.*

Axial trees have a simple linear "bracket representation" that allows to use ordinary 0L-systems to generate them. The left bracket "[" represents the beginning of a branching and the right bracket "]" its end. Figure 7.9 shows an axial tree and its bracket interpretation.

In order to draw an axial tree from its bracket representation the following interpretation of brackets is used:

[— push the current state of the turtle into the pushdown memory.



F[+F[-F]F][-F]F[-F[-F]F]F[+F]F

Figure 7.9: An axial tree and its bracket representation for $\delta = 45°$

] — pop the pushdown memory and make the turtle's state obtained this way its current state.

(In applications the current state of the turtle may contain various other information in addition to the turtle's position and orientation. For example, width, length and colour of lines.)

Figures 7.10a, b, c show several L-systems that generate bracket representations of axial trees and the corresponding trees (plants).

(a)	(b)	(c)
$n = 5$, $\delta = 25.7°$	$n = 5$, $\delta = 20°$	$n = 4$, $\delta = 22.5°$
F	F	F
$f \to F[+F]F[-F]F$	$F \to F[+F]F[-F][F]$	$F \to FF - [-F + F + F] + [+F - F - F]$

Figure 7.10: Axial trees generated by tree L-systems

There are various other modifications of L-systems that can be used to generate a variety of branching structures, plants, figures. For examples, stochastic and context-sensitive L-systems.

A **stochastic** 0**L-system** $G_\pi = \langle \Sigma, \omega, P, \pi \rangle$, is formed from an 0L-system $\langle \Sigma, \omega, P \rangle$ by adding a mapping $\pi : P \to (0, 1]$, called a probability distribution, such that for any $a \in \Sigma$, the sum of "probabilities" of all productions with "a" on its left side is 1. A derivation $w_1 \underset{P}{\Longrightarrow} w_2$ is called stochastic in G_π, if for each occurrence of the letter a in the word w_1

the probability of applying a production $p = a \longrightarrow u$ is equal to $\pi(p)$. Using stochastic 0L-systems various families of quite complex but similar branching structures have been derived.

Context-sensitive L-systems (IL-system). The concept of "context-sensitiveness" can be applied also to L-systems. Productions are of the form $uav \longrightarrow uwv$, $a \in \Sigma$ and such a production can be applied to rewrite a particular occurrence of a by w only if (u, v) is the context of that occurrence of a. (It may therefore happen that a symbol can not be replaced in a derivation step if it has no suitable context—this can be used also to handle the problem of endmarkers.)

It seems to be intuitively clear that IL-systems could provide richer tools to generate figures and branching structures. One can also show that they are actually necessary in the following sense. Growth functions of 0L-systems are linear combinations of polynomial and exponential functions. However, many growth processes observed in the nature do not have growth function of such a type. On the other side, IL-systems may exhibit growth functions not achievable by an 0L-systems.

Exercise 7.4.11 *The IL-system with the axiom "xuax" and productions*

$$\begin{array}{rclrclrclrcl}
u\underline{a}a & \longrightarrow & uua, & u\underline{a}x & \longrightarrow & udax, & a\underline{a}d & \longrightarrow & add, & x & \longrightarrow & x, \\
x\underline{a}d & \longrightarrow & xud, & u & \longrightarrow & a, & d & \longrightarrow & a, & & &
\end{array}$$

has a derivation

$$\begin{array}{rcccccccc}
xuax & \Longrightarrow & xadax & \Longrightarrow & xuaax & \Longrightarrow & xauax & \Longrightarrow & xaadax \\
& \Longrightarrow & xadaax & \Longrightarrow & xuaaax & \Longrightarrow & xauaax & \Longrightarrow & xaauax \\
& \Longrightarrow & xaaadax & \Longrightarrow & xaadaax & \Longrightarrow & xadaaax & \Longrightarrow & xuaaaax.
\end{array}$$

Show that its growth function is $\lfloor \sqrt{n} \rfloor + 4$*—not achievable by an 0L-system.*

7.5 Graph Rewriting

Graph rewriting is a commonly used method to design larger and more complicated graphs from simpler ones. Graphs are often used to represent relational structures and these structures are then extended and refined. For example, this is done in software development processes, in specifications of concurrent systems, in database specifications and so on. It is therefore desirable to formalize and understand the power of various graph rewriting methods.

The basic idea of graph rewriting systems is basically the same as for the string rewriting systems. A graph rewriting system is given by an initial graph G_0 (axiom), and a finite set P of rewriting productions $G_i \longrightarrow G'_i$, where G_i and G'_i are graphs. A direct rewriting relation $\underset{P}{\Longrightarrow}$ between graphs is also defined analogically: $G \underset{P}{\Longrightarrow} G'$, if G' can be obtained from the (host) graph G by replacing a subgraph, say G_i (a mother graph), of G, by G'_i (a daughter graph), where $G_i \longrightarrow G_i'$ is a production of P.

To state this very natural idea more precisely and formally is, however, far from simple. Several basic problems arise. How to specify when G_i occurs in G? How to replace G_i by G_i'? The difficulty lies in the fact that if no restriction is made, G_i' may be very different from G_i, and therefore it is far from clear how to embed G_i' into the graph obtained from G by removing G_i.

There are several general approaches to graph rewriting, but complexity and sophistication of their basic concepts and high computational complexity of basic algorithms to deal with them (for example for parsing), make these methods hard to use. More manageable are simpler approaches based, in various ways, on an intuitive idea of "context-free replacements". Two of them will be now introduced.

7.5.1 Node Rewriting

The basic idea of a node rewriting is that all productions are of the form $A \longrightarrow G_i'$, where A is a one-node graph. Rewriting by such a production consists of removing A and all incident edges, adding G_i', and connecting (gluing) its nodes with the rest of the graph. The problem is now how to define such a connection (gluing). The approach presented here is called "node-label-controled graph grammars"—in short NLC graph grammars.

Definition 7.5.1 *An* **NLC graph grammar** $\mathcal{G} = \langle V_N, V_T, C, G_0, P\rangle$ *is given by a* **nonterminal alphabet** V_N*, a* **terminal alphabet** V_T*, an* **initial graph** G_0 *with nodes labeled by elements from* $V = V_N \cup V_T$*, a finite set* P *of* **productions** *of the form* $A \longrightarrow G$*, where* A *is a nonterminal (interpreted as a single-node graph with the node labeled by* A*), and* G *is a graph with nodes labeled by labels from* V*. Finally,* $C \subseteq V \times V$ *is a* **connection relation**.

Example 7.5.2 *Let* $\mathcal{G}$ *be an NLC graph grammar with* $V_T = \{a, b, c, d, a', b', c', d'\}$, $V_N = \{S, S'\}$*, the initial graph* G_0 *consisting of a single node labelled by* S*, the productions shown in Figure 7.11:*

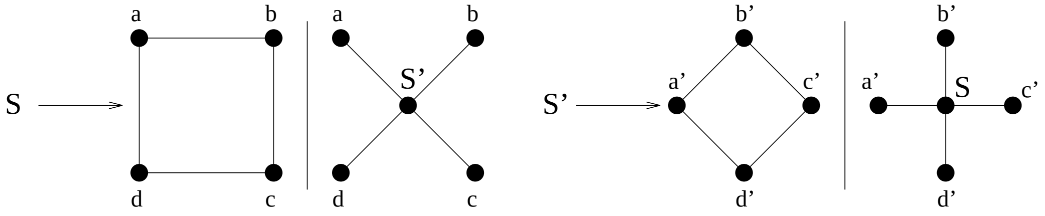


Figure 7.11: Graph grammar productions

and the connecting relation

$$\begin{aligned} C \;=\; & \{(a,a'),(a',a),(a,b'),(b',a),(b,b'),(b',b),(b,c'),(c',b), \\ & (c,c'),(c',c),(c,d'),(d',c),(d,d'),(d',d),(a',d),(d,a')\}. \end{aligned}$$

The graph rewriting relation $\underset{P}{\Longrightarrow}$ is now defined as follows: A graph G' is obtained from a graph G by a production $A \longrightarrow G_i$, if, in the graph G, a node $\bar{N}$ labeled by A is removed, together with all incident edges, G_i is added to the resulting graph (denote it by G') and a node N of $G' - \{\bar{N}\}$ is connected to a node N' in G_i if and only if N is a direct neighbour of $\bar{N}$ in G and $(n, n') \in C$, where n is the label of N and n' of N'.

Example 7.5.3 *In the NLC graph grammar from Example 7.5.2 we have, for instance, the derivation shown in Figure 7.12:*

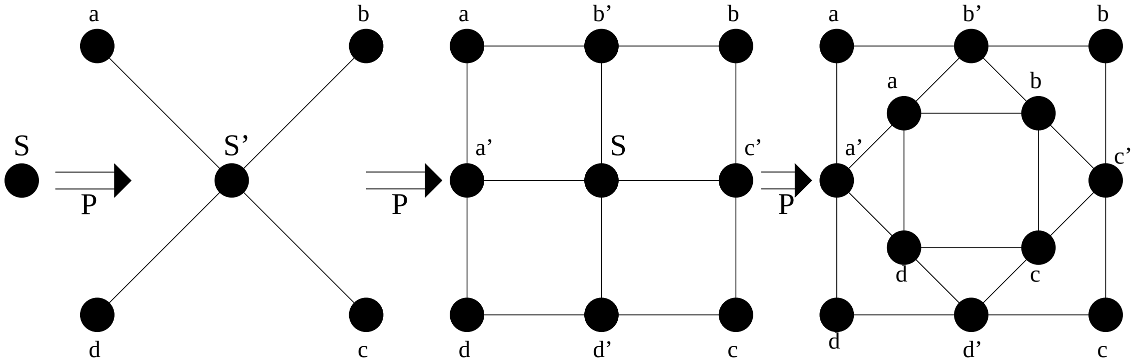


Figure 7.12: Derivation in a NLC–70%

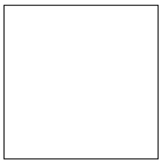
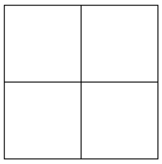
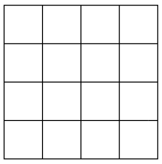
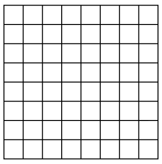
With an NLC grammar $\mathcal{G} = \langle V_N, V_T, C, G_0, P \rangle$ we can associate several sets of graphs (called "graph languages"):

- $L_e(\mathcal{G}) = \{G \mid G_0 \overset{*}{\underset{P}{\Longrightarrow}} G\}$ — a set of all generated graphs;
- $L(\mathcal{G}) = \{G \mid G_0 \overset{*}{\underset{P}{\Longrightarrow}} G,$ and all nodes of G are labeled by terminals$\}$ — a set of all generated "terminal graphs";
- $L_u(\mathcal{G}) = \{G \mid G_0 \overset{*}{\underset{P}{\Longrightarrow}} G',$ where G is obtained from G' by removing all node labels$\}$— a set of all generated unlabeled graphs.

In spite of their apparent simplicity, NLC graph grammars have strong generating power. For example, they can generate **PSPACE**-complete graph languages. This motivated investigations of various subclasses of NCL graph grammars. For example, the so-called **boundary NLC graph grammars**, where neither the initial graph nor graphs on the right hand side of productions have nonterminals on two incident nodes. Graph languages generated by these grammars are in **NP**. Other approaches lead to graph grammars for which parsing can be done in low polynomial time.

Results concerning decision problems for NLC graph grammars also indicate their power. It is decidable, given an NLC graph grammar G, whether the language $L(G)$ is empty, or whether it is infinite. However, other interesting decision problems are undecidable. For example, the equivalence problem or the problem to decide whether the language $L(G)$ contains a planar, a hamiltonian, or a connected graph.

It is also natural to ask where are the limits of NLC graph grammars and how to show that a graph language is outside the power of NLC graph grammars. This can be shown using a **pumping lemma** for NLC graph grammars and languages. With such a lemma it can be shown, for example, that there is no NLC graph grammar such that $L_u(G)$ contains exactly all finite square grid graphs (as those in the following figure):

7.5.2 Edge and Hyperedge Rewriting

The second natural idea to do a "context-free graph rewriting" is an edge rewriting. This has been generalized to hyperedge rewriting.

The intuitive idea of edge rewriting is formalized in several ways. For example, by the so-called **handle NLC graph grammars** (HNLC graph grammars for short). They are defined in a similar way as NLC graph grammars, only the left sides of all productions have to be edges with both nodes labeled by nonterminals (such edges are called "handles"). The embedding mechanism is the same as for NLC graph grammars.

Interesting enough, this simple and natural modification of NLC graph grammars provides graph rewriting systems with maximal generative power. Indeed, it has been shown that each recursively enumerable graph language can be generated by an HNLC graph grammar.

Another approach along the same line, presented below, is less powerful, but it is often, especially for applications, more handy.

A **hyperedge** is specified by a name (label) and sequences of incoming and outgoing "tentacles"—see Figure 7.13a. In this way an hyperedge may connect more than two nodes. The label of a hyperedge plays the rôle of a nonterminal in the hyperedge rewriting. A hyperedge replacement will be done within hypergraphs. Informally, hypergraphs consist of nodes and hyperedges.

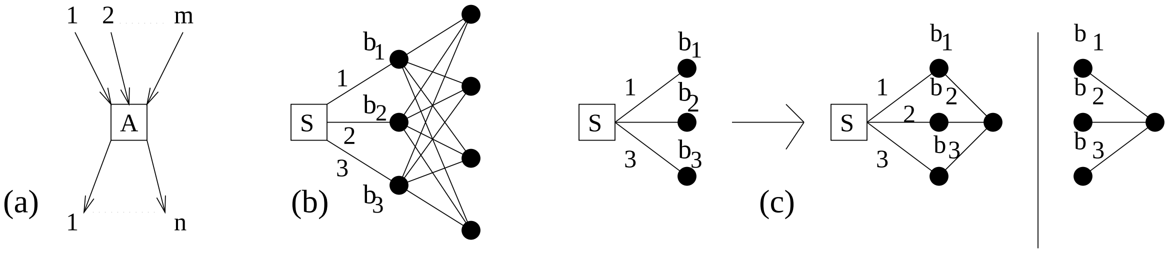


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Figure 7.13: A hyperedge and hyperedge productions–75%

Definition 7.5.4 *A* **hypergraph** $G = \langle V, E, s, t, l, \Lambda\rangle$ *is given by a set* V *of* **nodes**, *a set* E *of* **hyperedges**, *two mappings* $s : E \longrightarrow V^*$, *and* $t : E \longrightarrow V^*$, *assigning* **a sequence of source nodes** $s(e)$ *and a* **sequence of target nodes** $t(e)$ *to each hyperedge* e, *and a* **labelling mapping** $l : E \rightarrow \Lambda$, *where* Λ *is a set of labels.*

A hyperedge e *is called an* (m, n)**-hyperedge**, *or of type* (m, n), *if* $|s(e)| = m, |t(e)| = n$. *A* $(1, 1)$*-hyperedge is an ordinary edge.*

A **multipointed hypergraph** $\mathcal{H} = \langle V, E, s, t, l, \Lambda, \texttt{begin}, \texttt{end}\rangle$ *is given by an hypergraph* $\langle V, E, s, t, l, \Lambda\rangle$, *and two strings* `begin` *and* `end` $\in V^*$. *A multipointed hypergraph is an* (m, n)*-hypergraph or a multigraph of the type* (m, n), *if* $|\texttt{begin}| = m, |\texttt{end}| = n$. *The set of* **external nodes** *of* $\mathcal{H}$ *is the set of all symbols in strings* `begin` *and* `end`.

Let $\mathcal{H}_\Lambda$ *denote the set of all multipointed hypergraphs with labels in* Λ.

A multipointed hypergraph has two sequences of external nodes represented by the strings `begin` and `end`.

Definition 7.5.5 *A* **hyperedge rewriting graph grammar** *(HR (graph) grammar in short)* $\mathcal{G} = \langle V_N, V_T, G_0, P\rangle$ *is given by a set of nonterminals* V_N, *a set of terminals* V_T, *an initial multipointed hypergraph* $G_0 \in \mathcal{H}_\Lambda$, $\Lambda = V_N \cup V_T$, *and a set* P *of productions. Each production of* P *has the form* $A \longrightarrow R$, *where* A *is a nonterminal,* $R \in \mathcal{H}_\Lambda$, *and* $type(e) = type(R)$.

Example 7.5.6 *A HR grammar is depicted in Figure 7.13. The axiom is shown in Figure 7.13b and the productions are in Figure 7.13c. (Terminal labels of* $(1, 1)$*-hyperedges are not depicted). The grammar has two productions. In both cases* `begin`$= b_1b_2b_3$ *and* `end` $= \varepsilon$, $V_N = \{A, S\}$ *and* $V_T = \{t\}$.

In order to define the rewriting relation $\underset{P}{\Longrightarrow}$ for HR grammars one needs to describe how an (m, n)-hyperedge e is replaced by an (m, n)-hypergraph R, in a hypergraph. This is done in two steps.

1. Remove the hyperedge e.
2. Add the hypergraph R, except its external nodes, and connect each tentacle of a hyperedge of R which is connected to an external node of R, to the corresponding source or target node of e.

Exercise 7.5.7 *If we use, in HR grammar shown in Figure 7.13b,c, the first production* n *times and then the second production once, we derive from the initial graph* G_0 *the complete bipartite graph* $K_{3,n+5}$. *Show in detail how to do it.*

Example 7.5.8 *Starting with the axiom shown in Figure 7.14a, and using productions given in Figures 7.14b,c, various flow-diagram graphs can be generated.*

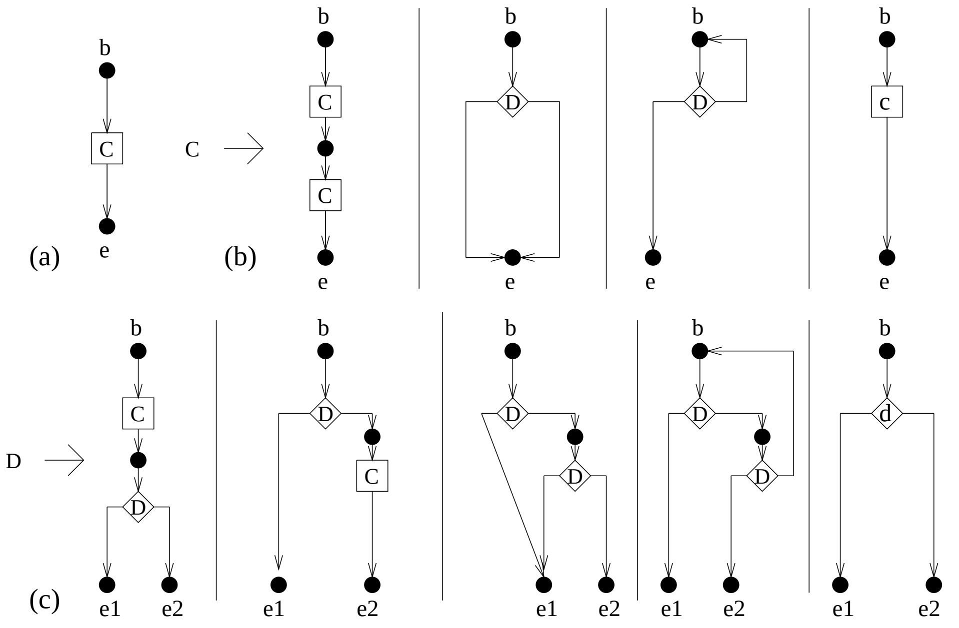

Figure 7.14: A HRG grammar to generate flowcharts–65%

The graph language $L(G)$ generated by an HR grammar G is the set of those graph generated from the initial multipointed hypergraph that contain only $(1, 1)$-hyperedges labeled by terminals.

Also for HR grammars there is a pumping lemma that can be used to show for some languages that they are outside the power of HR grammars.

Concerning decision problems, the more restricted power of HR graph grammars brings more decidability. In contrast to NLC graph grammars, it is decidable for HR grapg grammars whether $L(G)$ contains a planar, hamiltonian or connected graph.

Exercise 7.5.9 *Show that the HR graph grammar with two nonterminals* S, T *and the productions* $\{(S, T_0), (T, T_1), (T, T_2)\}$ *as depicted in Figure 7.15a, b, c generates an approximation of the Sierpiński triangle—see, for example, Figure 7.15b.*

Remark 7.5.10 The idea of string-rewriting, especially context-free rewriting is simple and powerful and allowed to get deeper insights and interesting results. The main motivation for considering more complex rewriting systems, such as graph rewriting systems, comes primarily from applications and naturally leads to less elegant and more complex (but useful) systems.

Moral. There have been many formal languages developed and many rewriting systems

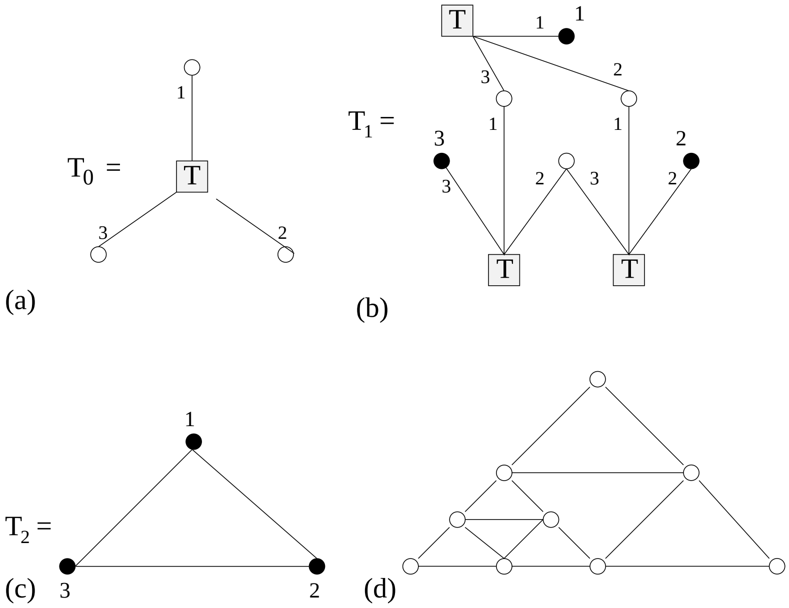

Figure 7.15: Graph grammar generating an approximation of Sierpiński triangle– 75%

designed and investigated. A good rule of thumb by rewriting is, as in real life, to learn as many languages as you can but master at least one of them.

7.6 Exercises

1. Design a Post normal system that generates longer and longer prefixes of the Thue ω-word.

2. * Show that each one-tape Turing machine can be simulated by a Post normal system.

3. A group can be represented by a Thue system over an alphabet $\Sigma \cup \{a^{-1} \mid a \in \Sigma\}$, with the set P of productions that includes the productions $aa^{-1} \to \varepsilon$, $a^{-1}a \to \varepsilon$, $\varepsilon \to aa^{-1}$ and $\varepsilon \to a^{-1}a$. Show that the world problem for groups, namely to decide, given x and y, whether $x \overset{*}{\Longrightarrow} y$, is polynomial time reducible to the problem to decide, given a z, whether $z \overset{*}{\Longrightarrow} \varepsilon$.

4. Design a type-0 grammar generating the language: (a) $\{a^n b^{n^2} \mid n \geq 1\}$; (b) $\{a^{F_i} \mid i \geq 0\}$; (c) $\{a^{n^2} \mid n \geq 1\}$; (d) $\{a^p \mid p \text{ is a prime}\}$

5. Describe the language generated by the Chomsky grammars: (a) $S \longrightarrow aS \mid aSbS \mid \varepsilon$;

   (b) $S \longrightarrow aSAB$, $S \longrightarrow abB$, $BA \longrightarrow AB$, $bA \longrightarrow bb$, $bB \longrightarrow bc$, $cB \longrightarrow cc$;

   (c) $S \longrightarrow abc$, $S \longrightarrow aAbc$, $Ab \longrightarrow bA$, $Ac \longrightarrow Bbcc$, $bB \longrightarrow Bb$, $aB \longrightarrow aaA$, $aB \longrightarrow aa$.

6. Given two Chomsky grammars G_1 and G_2, show how to design a Chomsky grammar generating the language: (a) $L(G_1) \cup L(G_2)$; (b) $L(G_1) \cap L(G_2)$; (c) $L(G_1)^*$.

7. Show that to each type-0 grammar there exits an equivalent one all rules of which have the form $A \longrightarrow \varepsilon$, $A \longrightarrow a$, $A \longrightarrow BC$, or $AB \longrightarrow CD$, where A, B, C, D are nonterminals, a is a terminal and there is at most one ε-rule.

8. Show that each context-sensitive grammar can be transformed into a similar normal form as in the previous exercise.

9. Show that to each Chomsky grammar there is an equivalent Chomsky grammar that uses only two nonterminals.

10. Show that Chomsky grammars with one nonterminal generate a proper subset of recursively enumerable languages.

11. ** (Universal Chomsky grammar) A Chomsky grammar $G_u = \langle V_T, V_N, P, \sigma \rangle$ is called universal if for every recursively enumerable language $L \subset V_T^*$ there exists a string $w_L \in (V_N \cup V_T)^*$ such that $L(w_L) = L$. Show that there exists a universal Chomsky grammar for every terminal alphabet V_T.

12. Design a CSG generating the language: (a) $\{w \mid w \in \{a, b, c\}^*, w \text{ contains the same number of } a\text{'s}, b\text{'s and } c\text{'s}\}$; (b) $\{0^n 1^m 0^n 1^m \mid n, m \geq 0\}$; (c) $\{a^{n^2} | n \geq 0\}$; (d) $\{a^p \mid p \text{ is prime}\}$.

13. Determine languages generated by the grammar:

    (a) $S \longrightarrow aSBC \mid aBC$, $CB \longrightarrow BC$, $aB \longrightarrow ab$, $bB \longrightarrow bb$, $bC \longrightarrow bc$, $cC \longrightarrow cc$;

    (b) $S \longrightarrow SaBC \mid abC$, $aB \longrightarrow Ba$, $Ba \longrightarrow aB$, $aC \longrightarrow Ca$, $Ca \longrightarrow aC$, $BC \longrightarrow CB$, $CB \longrightarrow BC$, $B \longrightarrow b$, $C \longrightarrow c$.

14. Show that the family of context-sensitive languages is closed under operations: (a) union; (b) concatenation; (c) iteration; (d) reversal.

15. Show that the family of context-sensitive languages is not closed under homomorphism.

16. Show, for example by a reduction to the PCP, that the emptiness problem for CSG is undecidable.

17. Design a regular grammar generating the language: (a) $(01+101)^* + (1+00)^*01^*0$; (b) $((a+bc)(aa^*+ab)^*c+a)^*$; (c) $((0^*10+((01)^*100)^*+0)^*(101(10010)^*+(01)^*1(001)^*)^*)^*$. (In the last case one nonterminal should be enough!)

18. Show that there is a Chomsky grammar that has only productions of the type $A \longrightarrow wB$, $A \longrightarrow Bw$, $A \longrightarrow w$, where A and B are nonterminals and w is a terminal word, that generates a non-regular language.

19. Show that an intersection of two CSL is again a CSL.

20. A nonterminal A of a CFG G is cyclic if there is in G a derivation $A \overset{*}{\Longrightarrow} uAv$ for some u, v with $uv \neq \varepsilon$. Let G be a CFG in the reduced normal form. Show that the language $L(G)$ is infinite if and only if G has a cyclic nonterminal.

21. Describe a method to design to each CFG G an equivalent CFG such that all its nonterminals, with perhaps the exception of the initial symbol, generate an infinite language.

22. Design a CFG generating the language

    $L = \{a^{i_1}ba^{i_2}b \ldots a^{i_k}b \,|\, k \geq 2, \exists X \subset \{1, \ldots, k\}, \sum_{j \in X} i_j = \sum_{j \notin X} i_j\}$.

23. Design a CFG in the reduced normal form equivalent to the grammar:

    $S \longrightarrow Ab, A \longrightarrow Ba \,|\, ab \,|\, B,\ B \longrightarrow bBa \,|\, aA \,|\, C,\ C \longrightarrow \varepsilon \,|\, Sa.$

24. Show that for every CFG G with a terminal alphabet Σ and each integer n, there is a CFG G' generating the language $L(G') = \{u \in \Sigma^* \,|\, |u| \leq n, u \in L(G)\}$ and such that $|v| \leq n$, for each production $A \longrightarrow v$ of G'.

25. A CFG G is **self-embedded** if there is a nonterminal A such that $A \overset{*}{\Longrightarrow} uAv$, where $u \neq \varepsilon \neq v$. Show that the language $L(G)$ is regular for every non-self-embedding CFG G.

26. A PDA $\mathcal{A}$ is said to be unambiguous if for each word $w \in L_f(\mathcal{A})$ there is exactly one sequence of moves by which $\mathcal{A}$ accepts w. Show that a CFL L is unambiguous if and only if there is an unambiguous PDA $\mathcal{A}$ such that $L = L_f(\mathcal{A})$.

27. Show that for every CFL L there is a PDA with two states that accept L with respect to a final state.

28. Which of the following problems is decidable for CFG G_1 and G_2, nonterminals X and Y and a terminal a? (a) $Prefix(L(G_1)) = Prefix(L(G_2))$; (b) $L_X(G_1) = L_Y(G_1)$; (c) $|L(G_1)| = 1$; (d) $L(G_1) \subseteq a^*$; (e) $L(G_1) = a^*$.

29. Design the upper-triangular matrix the CYK algorithm uses to recognize the string "*aabababb*" generated by the grammar with the productions: $S \longrightarrow CB$, $S \longrightarrow FB$, $S \longrightarrow FA$, $A \longrightarrow a$, $B \longrightarrow FS$,$E \longrightarrow BB$, $B \longrightarrow CE$, $A \longrightarrow CS$, $B \longrightarrow b$, $C \longrightarrow a$, $F \longrightarrow b$.

30. Implement the CYK algorithm on a one-tape Turing machine in such a way that recognition is done in $\mathcal{O}(n^4)$-time.

31. Design a modification of the CYK algorithm that does not require to have CFG in some special form.

32. Make a proof of correctness for CYK algorithm.

33. Show that the following context-free language is not linear: $\{a^n b^n a^m b^m \,|\, n \geq 1\}$.

34. Find another example of a CFL that is not generated by a linear CFG.

35. * Show that the language $\{a^ib^jc^ka^i \mid i \geq 1, j \geq k \geq 1\}$ is a DCFL that is not acceptable by a DPDA that does not make an ε-move.

36. Show that if L is a DCFL, then so is the complement of L.

37. Which of the following languages is context-free: (a) $\{a^ib^jc^k \mid i, j \geq 1, k > \max\{i, j\}\}$; (b) $\{ww \mid w \in \{0,1\}^*\}$.

38. Show that the following languages are context-free:

    (a) $L = \{w_1cw_2c \ldots cw_nccw_i^R \mid \ 1 \leq i \leq n, w_j \in \{0,1\}^* \text{ for } \ 1 \leq j \leq n\}$;

    (b) $\{0^{i_1}1^{j_1}0^{i_2}1^{j_2} \ldots 0^{i_n}1^{j_n} \mid \ n \text{ is even and for n/2 pairs it holds } i_k = 2j_k\}$;

    (c) (Goldstine language) $\{a^{n_1}ba^{n_2}b \ldots ba^{a_p}b \mid \ p \geq 1, n_i \geq 0, n_j \neq j \text{ for some } 1 \leq j \leq p\}$;

    (d) the set of those words over $\{a, b\}^*$ that are not prefixes of the ω-word $x = aba^2ba^3ba^4 \ldots a^nba^{n+1} \ldots$

39. Show that the following languages are not context-free: (a) $\{a^nb^na^m \mid m \geq n \geq 1\}$; (b) $\{a^i \mid i \text{ is prime}\}$; (c) $\{a^ib^jc^k \mid 0 \leq i \leq j \leq k\}$; (d) $\{a^ib^jc^k \mid i \neq j, j \neq k, i \neq k\}$.

40. Show that if a language $L \subseteq \{0,1\}^*$ is regular, $c \notin \{0,1\}$, then the language $L' = \{ucu^R | u \in L\}$ is context-free.

41. Show that every CFL over one-letter alphabet is regular.

42. Show that if L is a CFL, then also the following language is context-free:

    $L_1 = \{a_1a_3a_5 \ldots a_{2n+1} \mid a_1a_2a_3 \ldots a_{2n}a_{2n+1} \in L\}$.

43. * Show: (a) any family of languages closed under concatenation, homomorphism, inverse homomorphism and intersection with regular sets is closed also under union; (b) any family of languages closed under iteration, homomorphism, inverse homomorphism, union and intersection with regular languages is closed also under concatenation.

44. Show that if L is a CFL, then the set $S = \{|w| \mid w \in L\}$ is an ultimately periodic set of integers (i.e. there are integers n_0 and p such that if $x \in S, x > n_0$, then $(x + p) \in S$).

45. Design a PDA accepting the Greibach's language.

46. * Show that the Dyck language can be accepted by a Turing machine with space complexity $\mathcal{O}(\lg n)$.

47. * Show that every context-free language is a homomorphic image of a deterministic CFL.

48. Show that the family of 0L-languages does not contain all finite languages and it is not closed under operations: (a) union; (b) concatenation; (c) intersection with regular languages.

49. Show that every language generated by a 0L-system is context-sensitive.

50. Determine the growth function for the following 0L-systems: (a) With axiom S and productions: $S \longrightarrow Sbd^6, b \longrightarrow bcd^{11}, c \longrightarrow cd^6, d \longrightarrow d$; (b) With axiom a and productions $a \longrightarrow abcc, b \longrightarrow bcc, c \longrightarrow, a \longrightarrow abc, b \longrightarrow bc, c \longrightarrow c.$

51. Design a 0L-system with the growth function $(n+1)^4$.

52. The so-called ET0L-systems have especially nice properties. An ET0L-system is defined by $G = \langle \Sigma, \mathcal{H}, \omega, \Delta \rangle$, where $\mathcal{H}$ is a finite set of substitutions $h : \Sigma \rightarrow 2^{\Sigma^*}$ and for every $h \in \mathcal{H}$, (Σ, h, ω) is an 0L-system, and $\Delta \subseteq \Sigma$ is a terminal alphabet. The language L generated by G is defined by $L(G) = \{h_1(h_2(\ldots(h_k(\omega))\ldots)) \,|\, h_i \in \mathcal{H}\} \cap \Delta^*$. (In other words, an ET0L-systems consists of a finite set of 0L-systems and at each step of a derivation one of them is used. Finally, only those of the generated words go to the language that are in Δ^*.)

    (a) Show that the family of languages $\mathcal{L}$(ET0L) generated by ET0L-systems is closed under operations: (i) union, (ii) concatenation, (iii) intersection with regular languages, (iv) homomorphism and (v) inverse homomorphism; (b) design an ET0L system generating the language $\{a^ib^ia^i \,|\, i \geq 0\}$.

53. (Array rewriting) Similarly as we have string rewritings and string rewriting grammars we can consider array rewritings and array rewriting grammars. An array will now be seen as a mapping $\mathcal{A} : \mathbf{Z} \times \mathbf{Z} \rightarrow \Sigma \cup \{\#\}$, such that $\mathcal{A}(i,j) \neq \#$ only for finitely many pairs. Informally, an array rewriting production gives a rule describing how a connected subarray (pattern) can be rewritten by another one geometrically of the same shape. An extension or a shortening can be achieved by rewriting the surrounding $\varepsilon' s$, or by replacing a symbol from the alphabet Σ by #. The following "context-free" array productions generate "T"s of "a"s from the start array S:

$$\begin{array}{cccccccccccc} \overset{\#\#\#}{S} & \longrightarrow & \overset{LaR}{D}, & \overset{D}{\#} & \longrightarrow & \overset{a}{D}, & \overset{D}{\#} & \longrightarrow & \overset{a}{a}, & & & \\ \#L & \longrightarrow & La, & L & \longrightarrow & a, & R\# & \longrightarrow & aR, & R & \longrightarrow & a. \end{array}$$

    Construct context-free array grammars generating: (a) rectangles of "a"s; (b) squares of "a"s.

54. (Generation of strings by graph grammars) A string $a_1 \ldots a_n$ can be seen as a *string graph* with $n+1$ nodes and n edges labeled by $a_1 \ldots a_n$, respectively, connecting the nodes. Similarly, each string graph G can be seen as representing a string G_s of labels of its edges. Show that a (context-free) HR graph grammar $\mathcal{G}$ can generate a non-context-free string language $L = \{w \,|\, w \in \{0,1\}^*\}$ in the sense that $L = \{G_s \,|\, G \in L(\mathcal{G})\}$.

55. Design a HR graph grammar $\mathcal{G}$ that generates string graphs such that $\{G_s \,|\, G \in L(\mathcal{G})\} = \{a^nb^nc^n \,|\, n \geq 1\}$.

56. A NLC graph grammar $\mathcal{G} = \langle V_N, V_T, C, G_0, P \rangle$ is said to be context-free if for each $a \in V_T$ either $(\{a\} \times V_T) \cap C = \emptyset$ or $(\{a\} \times V_T) \cap C = \{a\} \times V_T$. Show that it

is decidable, given a context-free NLC graph grammar $\mathcal{G}$, whether $L(\mathcal{G})$ contains a discrete graph (no two nodes of which are connected by an edge).

57. * Design a handle NLC graph grammar to generate all rings with at least three nodes. Can this be done by a NLC graph grammar?

58. * Show that if we do not use a global gluing operation in the case of handle NLC graph grammars, but for each production a special one of the same type, then this does not increase the generative power of HNLC grammars.

59. Show that for every recursively enumerable string language L there is HNLC graph grammar $\mathcal{G}$ generating string graphs such that $L = \{G_s | G \in L(\mathcal{G})\}$. (Hint: design a HNLC graph grammar simulating a Chomsky grammar for L.)

QUESTIONS

1. Production systems, as introduced in Section 7.1 deal with rewriting of one-dimensional strings. Can this be generalized to deal with rewriting of two-dimensional strings? If yes, how? If not, why?

2. Equivalence of Turing machinesand Chomsky grammars implies that problems stated in terms of one of these models of computation can be rephrased in terms of another model. Is this always true? If not, when is this true?

3. Can every regular language be generated by an unambiguous CFG?

4. What does the undecidability of the halting problem implies for the type-0 grammars?

5. What kind of English sentences can not be generated by a context-free grammar?

6. How much can it cost to transform a given CFG into: (a) Chomsky normal form; (b) Greibach normal form?

7. What is the difference between the two basic acceptance modes for (deterministic) pushdown automata?

8. What kind of growth functions have different types of D0L systems?

9. How can one show that context-sensitive L-systems are more powerful than D0L systems?

10. What is the basic idea of: (a) node rewriting; (b) edge rewriting, for graphs?

7.7 Historical and Bibliographical References

Two papers by Thue (1906, 1914) can be seen as the first contributions to rewriting systems and formal language theory. However, it was actually Noam Chomsky (1956, 1957, 1959) who introduced the concept of formal grammar and basic grammar hierarchy, brought vigorously new research paradigms into linguistics. Chomsky with Schützenberger (1963) introduced basic aims, tools and methods of the formal language theory. The importance of context-free languages for the description of the syntax of programming languages and for compiling was another stimulus for the very fast development of the area in 70-ties and 80-ties. Books by Ginsburg (1966), Hopcroft and Ullman (1969) and Salomaa (1973) much contributed to that development. Nowadays there is a variety of other books available. For example, Harrison (1978), and Floyd and Beigel (1994).

Post (1943) introduced systems called nowadays by his name. Example 7.1.3 is due to Penrose (1990) and credited to G. S. Tseitin and D. Scott. Basic relations between type-0, and type-3 grammars and automata are due to Chomsky (1957, 1959), Chomsky and Schützenberger (1963). The first claim of Theorem 7.2.9 is a folklore, for the second one see Exercise 10, due to Geffert, and for the third see Geffert (1991). Example 7.3.8 is due to Bertol and Reinhardt (1995). Greibach (1965) introduced the normal form that now carries her name. The formal notion of a PDA and their equivalence with CFG are due to Chomsky (1962) and Evey (1963).

The normal form for PDA is from Maurer (1969). Kuroda (1964) has shown that NLBA and context-sensitive grammars have the same power.

Methods to transform a given CFG into a Greibach normal form can be found in Salomaa (1973), Harrison (1978) or Floyd and Beigel (1994).The original sources for the CYK parsing algorithm are Kasami (1965) and Younger (1967). This algorithmis one of those that have been often studied from various points of view (correctness and complexity). There are many book on parsing, for example Aho and Ullman (1972) and Sippu and Soisalon-Soininen (1990). Reduction of parsing to Boolean matrix multiplication is due to Valiant (1975)—see Harrison (1978) for a detailed exposition. A parsing algorithm for CFG with the space complexity $\mathcal{O}(\lg^2 n)$ on MTM is due to Lewis, Stearns and Hartmanis (1965); with $\mathcal{O}(lg^2 n)$ time complexity on PRAM is due to to Ruzzo (1980) and on hypercubes with $\mathcal{O}(n^6)$ processors is due to Rytter (1985). $\mathcal{O}(n^2)$ algorithm for syntactical analysis of unambiguous CFG is due to Kasami and Torii (1969). Deterministic pushdown automata and languages are dealt with in many books, especially by Harrison (1978).

The pumping lemma for context free languages presented in Section 7.3 is due to Bar-Hillel (1964). There are several other pumping lemmas discussed in details by Harrison (1978) and Floyd and Beigel (1994). Characterization results are presented by Salomaa (1973) and Harrison (1978). For results, and the corresponding references, concerning closure properties, undecidability and ambiguity for context-free grammars and languages see Ginsburg (1966). For **P**-completenss results for CFG see Jones and Laaser (1976) and Greenlaw, Hoover and Ruzzo (1995). The hardest CFL is due to Greibach (1973) and so is Theorem 7.3.48. Theorem 7.3.17 is due to Gruska (1969).

The concept of L-system was introduced by Aristid Lindenmayer (1968). The formal theory of L-systems is presented by Rozenberg and Salomaa (1980), where one can also find results concerning closure and undecidability properties, as well as references to the earlier works in this area. The study of growth functions was introduced by Paz and Salomaa (1973). For basic results concerning E0L systems see Rozenberg and Salomaa (1986). The decidability of D0L-systems is due to Culik and Friš (1977).

There have been various attempts to develop graphical modeling of L-systems with the one developed by Prusinkiewicz as perhaps the most successful so far. For a detailed presentation of this approach see Prusienkewicz and Lindenmayer (1990), a well illustrated book, rich on references. Section 7.4.2 is derived from this book and so are examples and pictures—drawn by the system due to H. Fernau and using specifications from the book by Prusienkewicz and Lindenmayer. Also Example 7.4.2 and Figure 7.5 are due to them.

There is also a variety of other modifications of L-systems than those discussed in this chapter that have been successfully used to model plants and nature scenes. Much more refined and sophisticated implementations bring and use additional parameters and features, for example colour, and provide interesting visual results. See Prusinkiewicz and Lindenmayer (1990) for a comprehensive treatment of the subject.

There is nowadays a numerous literature on graph-grammars, presented especially in proceedings of the Graph Grammar Workshops, see LNCS 153, 291, 532. NLC graph grammars were introduced by Janssensand Rozenberg (1980, 1980a) and intensively developed since then. These papers deal also with a pumping lemma for graph grammar and its applications as well as with decidability results. For an introduction to NLC graph grammars see Rozenberg (1987), from which our presentation and examples were derived. Edge rewriting has been introduced by H.-J. Kreowski (1977). The pumping lemma concerning the edge rewriting is due to Kreowski (1979). Hyperedge rewriting was introduced by Habel and Kreowski (1977) and Bauderon and Courcelle (1987). The pumping lemma for HR graph grammars is due to Habel and Kreowski (1987). Decidability results are due to Habel, Kreowski and Vogler (1989). For an introduction to the subject see Habel and Kreowski (1987a)—our presentation and examples are derived from it— and Habel (1990,1990a). For recent surveys on node and hyperedge replacement grammars see Engelfriet and Rozenberg(1996) and Drewews, Habel and Kreowski (1996).

From a variety of other rewriting ideas let us mention briefly three. For some other approaches and references see Salomaa (1973, 1985). **Term rewiting**, credited usually to Evans (1951), deals with methods to transform complex expressions/terms into simpler ones. It is an intensively developed idea with various applications, especially in the area of formal methods for software development. For a comprehensive treatment see Dershowitz and Jouannaud (1990) and Kirchner (1996). **Array grammars**, used to rewrite two dimensional arrays (array-pictures), were introduced by Milgram and Rosenfeld (1971). For an interesting presentation of various approaches and results see Wang (1989) and the Exercise 53 is due to R. Freund. For array grammars generating squares see Freund (1994). **Cooperating grammars** were introduced by Meersman and Rozenberg (1978). The basic idea is that several rewriting systems, of the same type, participate, using various rules for cooperation, in rewriting. In a rudimentary form this is true also for T0L systems. For a survey see Paŭn (1995). For a combination of both approaches see Dassow, Freund and Paŭn (1995).

Chapter 8

CRYPTOGRAPHY

INTRODUCTION

A successful, insightful and fruitful search for borderlines between possible and impossible, that was highlighted since 1930s by the computability theory developing an understanding what is effectively computable, continued since 1960's by the complexity theory developing an understanding what is efficiently computable, and continues with the modern cryptography developing an understanding what can be securely communicated. Cryptography started as a very old art, became deep science and aims to be one of the key technology of information era.

Modern cryptography can be seen as an important dividend of complexity theory that is bringing in turn important stimuli not only for complexity theory, foundations of computing, but for the whole science. Cryptography is rich on deep ideas, interesting applications and contrasts. An area with very close relation between theory and applications.

In this chapter the main ideas of classical and especially modern cryptography are presented, illustrated, analyzed and their applications demonstrated.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. the basic aims, concepts, cryptosytems and methods of the classical and modern cryptography;
2. main approaches and applications of public-key cryptography;
3. knapsack and RSA cryptosystems and their analysis;
4. the key concepts of the trapdoor one-way functions and predicates and cryptographically strong pseudo-random genarators;
5. main approaches to randomized encryptions and their security;
6. methods of digital signatures, including the DSS system.

> Secret de deux, secret de Dieu,
> secret de trois, secret de tous.
> *French proverb*

For thousands of years, **cryptography** has been an art of providing secure communication over insecure channels, and **cryptoanalysis** has been a dual art of breaking into such communications. Up to the advent of computers and information driven society, **cryptology**—a combined art of cryptography and cryptoanalysis—was almost exclusively in the hands of diplomats and militaries. Nowadays, cryptography is a technology without which public communications can hardly exist, and also a science with deep contributions to the foundations of computing. A short modern history of cryptography—amazing and impressive—has three milestones.

First, they were the needs of cryptoanalysis that led, in 1940-ies, in the Bletchey Park, to the development of Colossus—the first very powerful electronic computer, used to speed-up breaking of the ENIGMA code. This contributed significantly to the success of Allies in Second World War and has much influenced the postwar recognition of the potential of science and technology for the society.

Secondly, it was the cryptography's capability to extend its goals and "go public", in order to create the efficient and secure communication and information storing without which the modern society could hardly function. The idea of the public-key cryptography, digital signatures and cryptographical communication protocols has changed our views what is possible concerning secure communications.

Finally, the ideas coming from the cryptography led to such new and deep concepts as one-way functions, zero-knowledge proofs, interactive proof systems, holographic proofs, program checking. They led in general to significantly new developments concerning the understanding of the power of randomness and interactions for computing.

The first theoretical approach to cryptography, due to Shannon (1949), was based on **information theory**—developed in turn by Shannon on the base of his work in cryptography—and the belief that the cryptoanalysts should not have enough information to decrypt messages. The current approach is based on **complexity theory** and the belief that the cryptoanalysts should not have enough time or space to decrypt messages. There are also promising attempts to develop quantum cryptography security of which is based on the **laws of quantum physics**.

There are various peculiarities and paradoxes connected with modern cryptology. Once nation's most closely held secret has gone recently more public more important it has became. Positive results of cryptography are based on negative results of complexity theory—on the existence of infeasible computational problems.[1] Computers, that were originally developed

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[1] The idea to use infeasible problems for a protection of communication is actually very old and goes back at least to Archimedes. He used to send lists of his recent discoveries, stated without proofs, to his colleagues in Alexandria. In order to prevent statements "We have discovered all that by ourselves" as a response, Archimedes inserted occasionally false statements or practically unsolvable problems among them.

to help cryptoanalysts, seem nowadays much more useful for cryptography. Surprisingly, too perfect cryptography also causes problems. Once developed to protect before "bad forces", can now well serve to protect "bad forces".

There are also very few areas of computing with such a close interplay between a deep theory and an important practice, and where this relation is so complicated as in the modern cryptography.

The cryptography has also a quite unique view on what does it mean that an integer is "practically large enough". In some cases only numbers at least 512 bit long, far exceeding total life time of universe, are considered large enough. The practical cryptography has also developed a special view on what is computationally unfeasible. If something can be done with one million supercomputers in a couple of weeks, then this is not considered as completely unfeasible. As a consequence, mostly only toy examples can be presented in any book on cryptology.

In this chapter we deal with two perhaps the most basic problems of cryptography, secure encryptions and secure digital signatures. In the next chapter, more theoretical concepts developed from the cryptographical considerations are discussed.

8.1 Cryptosystems and Cryptology - November 8, 2005

The cryptology can be seen as an ongoing battle, in the space of cryptosystems, between the cryptography and the cryptoanalysis. So far with no indications which side is going to win. It is also an ongoing search for proper tradeoffs between security and efficiency.

Applications of cryptography are numerous and there is no problem to entertain with impressive examples. One can even say, without exaggeration, that an information era is impossible without cryptography. For example, it is true that electronic communications are paper-less. However, we still need electronic versions of envelops, signatures and company letterheads and they can hardly exists meaningfully without cryptography.

8.1.1 Cryptosystems

The cryptography deals with the problem of sending an (intelligible) message (usually called **plaintext** or **cleartext**), through an unsecure channel that may be tapped by an enemy (usually called **eavesdropper**, **adversary**, or simply **cryptoanalyst**), to a legal receiver. In order to increase the security that the message will not be learned by some unintended receiver, the sender **encrypts (enciphers)** the plaintext to produce an (unintelligable) **cryptotext (ciphertext, cryptogram)** and sends the cryptotext through the channel. The encryption has to be done in such a way that the legal receiver is able to **decrypt** the cryptotext to obtain the plaintext. However, an eventual eavesdropper should not be able to do so (see Figure 8.1).

Encryption and decryption always take place within a specific **cryptosystem**. Each cryptosystem has the following components:

For example, the problem mentioned in Example 6.4.22 that has a solution with more than 206 500 digits.

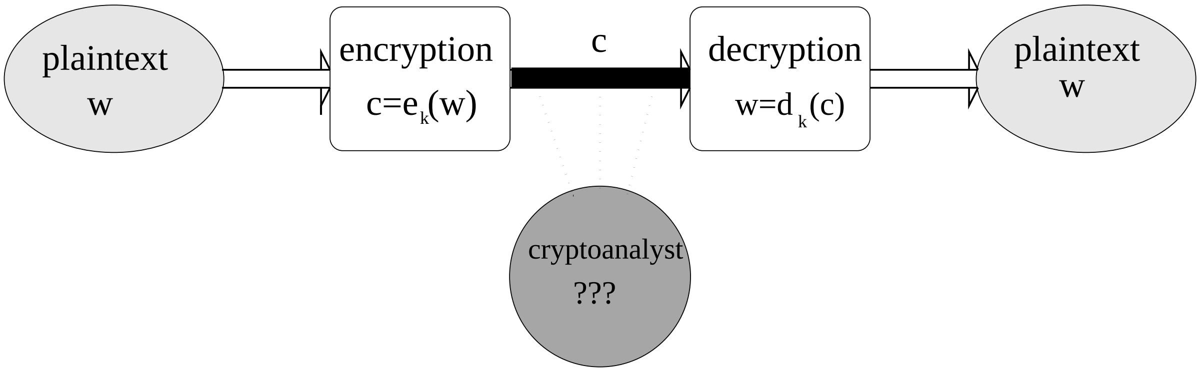

Figure 8.1: Cryptosystem-75%

Plaintext-space $\mathcal{P}$—a set of words over an alphabet Σ, called plaintexts, or sentences in a natural language.

Cryptotext-space $\mathcal{C}$—a set of words over an alphabet Δ, called cryptotexts.

Key-space $\mathcal{K}$—a set of keys.

Each key k determines, within a cryptosystem, an **encryption algorithm** (function) e_k and a **decryption algorithm (function)** d_k such that, for any plaintext w, $e_k(w)$ is the corresponding cryptotext and $w \in d_k(e_k(w))$. A decryption algorithm is therefore a sort of inverse of the encryption algorithm. Encryption algorithms can be probabilistic, that is neither encryption nor decryption has to be unique. However, for practical reasons, unique decryptions are preferable.

Encryption and decryption are often specified by a general encryption algorithm e and a general decryption algorithm d such that $e_k(w) = e(k, w), d_k(c) = d(k, c)$ for any plaintext w, cryptotext c, and any key k.

We start a series of examples of cryptosystems with one of the best known classical cryptosystems.

Example 8.1.1 (CAESAR cryptosystem) *We illustrate this cryptosystem, described by Julius Caesar (100–42 BC), in his letter to Cicero, on encrypting words of the English alphabet with 26 capital letters. The key space consists of 26 integers* $0, 1, \ldots, 25$. *The encryption algorithm* e_k *substitutes any letter by the one occurring* k *positions ahead (cyclically) in the alphabet; the decryption algorithm* d_k *substitutes any letter by that occurring* k *position backwards (cyclically) in the alphabet. For* $k = 3$ *the substitution has the following form*

```
Old:  A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
New:  D E F G H I J K L M N O P Q R S T U V W X Y Z A B C
```

Some encryptions:

$$e_{25}(IBM) = HAL, \quad e_{11}(KNAPSACK) = VYLADLNV, \quad e_{20}(PARIS) = JULCM.$$

The history of cryptography can be seen as being about 4000 years old, if one counts cryptographic transformations on tomb inscriptions. The following cryptosystem is perhaps the oldest one among the so-called substitution cryptosystems.

Example 8.1.2 (POLYBIOS cryptosystem) *It is the cryptosystem described by the Greek historian Polybios (200-118 BC). It uses as keys the so-called Polybios checkerboards. For example, the one shown in Figure 8.2a with the English alphabet of 25 letters, ('J' is omitted)*[2]*. Each symbol is substituted by the pair of symbols, for the row and the column of the checkerboard in which the symbol is placed. For example, the plaintext "INFORMATION" is encrypted as "BICHBFCIDGCGAFDIBICICH".*

	F	G	H	I	J
A	A	B	C	D	E
B	F	G	H	I	K
C	L	M	N	O	P
D	Q	R	S	T	U
E	V	W	X	Y	Z

(a) Polybios checkerboard

S	D	Z	I	U
H	A	F	N	G
B	M	V	Y	W
R	P	L	C	X
T	O	E	K	Q

(b) Playfair square

Figure 8.2: Classical cryptosystems

The cryptosystem presented in the next example was probably never used. In spite of that, this cryptosystem played an important rôle in the history of cryptography. It initiated the development of algebraic and combinatorial methods in cryptology and attracted mathematicians to cryptography.

Example 8.1.3 (HILL cryptosystem) *In this cryptosystem, based on linear algebra and invented by L. S. Hill (1929), an integer n is fixed first. The plaintext and cryptotext space consists of words of length n. For example, over the English alphabet of 26 letters. Keys are matrices M of degree n, elements of which are integers from the set $\Delta = \{0, 1, \ldots, 25\}$, such that the inverse matrix M^{-1} modulo 26 exists. For a word w let C_w be the column vector of length n consisting of codes of n symbols in w—each symbol is replaced by its position in the alphabet.*

To encrypt a plaintext w of length n, the matrix-vector product $C_c = MC_w \bmod 26$ is computed, and then, in the resulting vector, integers are decoded—replaced by the corresponding letters. To decrypt a cryptotext c, at first the product $M^{-1}C_c \bmod 26$ is computed and then the numbers are replaced by letters. A longer plaintext has to be first broken into the words of length n and then each of them is encrypted separately .

For an illustration, let us consider the case $n = 2$ and

$$M = \begin{pmatrix} 4 & 7 \\ 1 & 1 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} 17 & 11 \\ 9 & 16 \end{pmatrix}.$$

---

[2] It is not by chance that the letter "J" is omitted—it is the last letter introduced to the current English alphabet. PLAYFAIR cryptosystem, with keys in form of "Playfair squares", see Figure 8.2b,will be discussed later.

For the plaintext $w = LONDON$ *we have* $C_{LO} = (11, 14)^T$, $C_{ND} = (13, 3)^T$, $C_{ON} = (14, 13)^T$, *and therefore,*

$$MC_{LO} = (12, 25)^T, \qquad MC_{ND} = (21, 16)^T, \qquad MC_{ON} = (17, 1)^T.$$

The corresponding cryptotext is then "MZVQRB". It is easy to check that for the cryptotext "WWXTTX" the plaintext "SECRET" is obtained. Indeed,

$$M^{-1}C_{WW} = \begin{pmatrix} 17 & 11 \\ 9 & 16 \end{pmatrix} \begin{pmatrix} 22 \\ 22 \end{pmatrix} = \begin{pmatrix} 18 \\ 4 \end{pmatrix} = \begin{pmatrix} S \\ E \end{pmatrix},$$

and so on.

In most of the practical cryptosystems, as in the HILL cryptosystem, the plaintext-space is finite and much smaller than the space of messages that need to be encrypted. To encrypt a longer message, it has to be broken into pieces and each encrypted separately. This brings additional problems discussed later. In addition, if a message to be encrypted is not in the plaintext-space alphabet, it has first to be encoded into such an alphabet. For example, if the plaintext-space is the set of all binary strings of certain length, what is often the case, then in order to encrypt an English alphabet text, its symbols have to be first replaced (encoded) by some, fixed-length, binary codes.

Exercise 8.1.4 *Encrypt the plaintext "A GOOD PROOF MAKE US WISER" using: (a) CAESAR cryptosystem with* $k = 13$; *(b) POLYBIOS cryptosystem with some checkerboard; (c) HILL cryptosystem with some matrix.*

Already Sir Francis R. Bacon (1561-1626), formulated requirements an idel cryptosystem should have. Currently we require from a **good cryptosystem** the following properties:

1. Given e_k and a plaintext w, it should be easy to compute $c = e_k(w)$.
2. Given d_k and a cryptotext c, it should be easy to compute $w = d_k(c)$.
3. A cryptotext $e_k(w)$ should be not much longer than the plaintext w.
4. It should be unfeasible to determine w from $e_k(w)$ without knowing d_k.
5. The so-called **avalanche effect** should hold. A small change in the plaintext, or in the key, should lead to a big change of the cryptotext (i.e. a change of one bit of a plaintext should result in a change of all bits of the cryptotext with the probability close to 0.5).

The requirement (4) is the minimum we require from a cryptosystem to be considered secure. However, as discussed later, even cryptosystems with this property may not be secure enough under special circumstances.

8.1.2 Cryptoanalysis

The aim of the **cryptoanalysis** is to get as much information as possible about the plaintext or the key. In doing so it is usually assumed that it is known which cryptosystem was used, or at least a small set of the potential cryptosystems one of which was used. The main types of the cryptoanalytic attacks are:

1. **Cryptotexts-only attack.** The cryptoanalysts get cryptotexts $c_1 = e_k(w_1), \ldots, c_n = e_k(w_n)$ and aim to infer the key k or as many from plaintexts $w_1, \ldots, w_n$ as possible.
2. **Known-plaintexts attack.** The cryptoanalysts know some pairs $(w_i, e_k(w_i)), 1 \leq i \leq n$, and aim to infer k, or at least to determine w_{n+1}, for a new cryptotext $e_k(w_{n+1})$.
3. **Chosen-plaintexts attack.** The cryptoanalysts choose plaintexts $w_1, \ldots, w_n$, get cryptotexts $e_k(w_1), \ldots, e_k(w_n)$, and aim to infer k or at least w_{n+1} for a new cryptotext $c_{n+1} = e_k(w_{n+1})$.
4. **Known-encryption-algorithm attack.** The encryption algorithm e_k is given and the cryptoanalyst tries to get the decryption algorithm d_k before actually receiving any samples of the cryptotext.
5. **Chosen-cryptotext attack.** The cryptoanalysts know some pairs $(c_i, d_k(c_i))$, $1 \leq i \leq n$, where the cryptotexts c_i have been chosen by cryptoanalysts. The task is to determine the key.

Exercise 8.1.5 *A spy group got info about the arrival of a new member. The secret police succeeded to learn the message and knows that it was encrypted using the HILL cryptosystem with a matrix of the degree* 2. *It also learned that the code "10 3 11 21 19 5" stands for the name of the spy and "24 19 16 19 5 21" for the city, TANGER, the spy should come from. What is the name of the spy?*

One of the standard techniques to increase security of a cryptosystem is to do encryption twice. The first time the plaintext is encrypted with one key and then the resulting cryptotext is encrypted with another key. In other words $e_{k_2}(e_{k_1}(w))$ is computed for the plaintext w and keys k_1, k_2. A cryptosystem is **closed under composition** if for every two encryption keys k_1, k_2, there is a single encryption key having the effect of these two keys applied consecutively. That is $e_k(w) = e_{k_2}(e_{k_1}(w))$, for all w. A closure under the composition therefore means that a consecutive application of two keys does not increase security. CAESAR is clearly composite. POLYBIOS is clearly not composite.

Exercise 8.1.6 * *Show that the HILL cryptosystem is composite.*

There are two basic types of cryptosystems: **secret-key** cryptosystems and **public-key cryptosystems**. We deal with them in the next two sections.

8.2 Secret-key Cryptosystems

A cryptosystem is called a secret-key cryptosystem if some secret piece of information—the key—has to be agreed upon ahead of time between any two parties that want, or have to, communicate through the cryptosystem. CAESAR, POLYBIOS and HILL are examples of such cryptosystems.

There are two basic types of secret-key cryptosystems. Those based on **substitutions**—each letter of the plaintext is replaced by another letter or word—and those based on **transpositions**—letters of the plaintext are permuted.

8.2.1 Monoalphabetic Substitution Cryptosystems

Cryptosystems based on a substitution are either **monoalphabetic** or **polyalphabetic**. In a monoalphabetic substitution cryptosystem the substitution rule remains unaltered during the encryption, while in a polyalphabetic substitution cryptosystems this is not the case. CAESAR and POLYBIOS are examples of monoalphabetic cryptosystems.

A monoalphabetic substitution cryptosystem, with letter-by-letter substitution and with the alphabet of plaintexts equal to that of cryptotexts, is uniquely specified by a permutation of letters in the alphabet. Various cryptosystems differ in the way how such a permutation is specified—the main aim is usually that the permutation is easy to remember and use.

In the **AFFINE** cryptosystem (for English) a permutation is specified by two integers $1 \leq a, b \leq 25$ such that a and 26 are relatively prime and the x-th letter of the alphabet is substituted by the $((ax + b) \bmod 26)$-th letter. (The condition that a and 26 are relatively prime is necessary in order for the mapping

$$f(x) = (ax + b) \bmod 26$$

to be a permutation.)

Exercise 8.2.1 *Determine the permutation of letters of the English alphabet obtained when AFFINE cryprosystem with $a = 3$ and $b = 5$ is used.*

Exercise 8.2.2 *For the following pairs plaintext-cryptotext determine which cryptosystem was used.*

(a) `COMPUTER`—`HOWEVER THE REST UNDERESTIMATES ZANINESS YOUR JUDICIOUS WISDOM`;

(b) `SAUNA AND LIFE`—`RMEMHCZZTCEZTZKKDA`;

Exercise 8.2.3 *Decrypt the following cryptotexts that have been encrypted using one of the above described cryptosystems, or some of their modifications. (Caution, not all plaintexts are in English.)*

(a) `WFLEUKZFEKZFEJFWTFDGLKZEX`*;*

(b) `DANVHEYD SENHGKIIAJ VQN GNULPKCNWLDEA;`

(c) `DHAJAHDGAJDI AIAJ AIAJDJEH DHAJAHDGAJDI AIDJ AIBIAJDJ`

`DHAJAHDGAJDI AIAJ DIDGCIBIDH DHAJAHDGAJDI AIAJ DICIDJDH`*;*

(d) `KLJPMYHUKV LZAL ALEAV LZ TBF MHJPS.`

The decryption of a longer cryptotext obtained by a monoalphabetic encryption from a meaningful English text is fairly easy using a frequency table for letters, for example the one in Figure 8.3.[3]

Indeed, it is often enough to use a frequency table to determine the most frequently used letters and then to guess the rest. (One can also use the frequency tables for pairs (digrams) and triples (trigrams) that were published for various languages.) In case of an AFFINE cryptosystem a frequency table can help to determine coefficients a and b.

	%		%		%		%		%		%		%
E	13.04	N	7.07	H	5.28	C	2.79	G	1.99	V	0.92	Q	0.12
T	10.45	R	6.77	D	3.78	U	2.49	P	1.99	K	0.42	Z	0.08
A	8.56	S	6.27	L	3.39	M	2.49	W	1.49	X	0.17		
O	7.97	I	6.07	F	2.89	Y	1.99	B	1.39	J	0.13		

Figure 8.3: Frequency table for English letters due to A. Konheim (1981)

Exercise 8.2.4 *On the base of frequency analysis it has been guessed that the most common letter in a cryptotext, Z, corresponds to O and the second most frequent letter I correpsonds to T. If you know that the AFFINE cryptosystem was used, determine its coefficients.*

Exercise 8.2.5 *Suppose the encryption is done using the AFFINE cryptosystem with* $c(x) = (ax + b) \bmod 26$*. Determine the decryption function.*

The fact that we can, with large probability, guess the rest of the plaintext, once several letters of the cryptotext have been decrypted, is based on the following result. In case of

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[3] The most frequently used symbols in some other languages, by Gaines (1939): French: E–15.87%, A–9,42%, I–8.41%, S–7.90%, T–7.26%, N–7.15%; German: E–18.46%, N–11.42%, I–8.02%, R–7.14%, S–7.04%; Spanish: E–13.15%, A–12.69%, O–9.49%, S–7.60%.

monoalphabetic substitution cryptosystems the expected number of potentially meaningful plaintexts for a cryptotext of lenght n is $2^{H(K)-nD}-1$, where $H(K)$ is the so-called entropy of the key space (for example, $\lg 26!$ for English), and D is a measure of the redundancy of the plaintext language in bits per letter (for example, 3.5 for English). This means, for example, that for a cryptotext of length $n > 25$ only one meaningful English plaintext is expected.

Finally, let us illustrate on monoalphabetic substitution cryptosystems differences between the first three cryptoanalytic attacks described on page 529.

We have already indicated how frequency tables can be used to make cryptoanalytic attacs under the "cryptotexts only" condition fairly easy—though it may require some work. Monoalphabetic substitution cryptosystems are trivial to break under the "known plaintext" attack as soon as the known plaintexts used all symbols of the alphabet. These cryptosystems are even more trivial to break in case of the "chosen plaintext" attack—choose ABCDEFGHIJKLMNOPQRSTUVWXYZ as the plaintext.

Exercise 8.2.6 * *Assume that the most frequent trigrams in a cryptotext obtained using the HILL cryptosystem are LME, WRI and XYC and they are THE, AND and THA in the plaintext. Determine the* 3×3 *matrix that was used.*

8.2.2 Polyalphabetic Substitution Cryptosystems

The oldest idea for a polyalphabetic cryptosystem was to divide the plaintext into blocks of 2 letters and to use a mapping $\phi : \Sigma \times \Sigma \mapsto \Sigma^*$ described usually by a table. The oldest of such cryptosystem is due to Giovanni Polleste Porta (1563). The cryptosystem shown in the next example, due to Charles Wheatsone (1854) and named by baron Lyon Playfair was first used in the Crimean War, then intensively in the field in the First World War and also in Second World War by Allies.

Example 8.2.7 (PLAYFAIR cryptosystem) *To illustrate the idea we restrict ourselves again to 25 letters of the English alphabet arranged into a* 5×5 *table—Figure 8.2b—called the Playfair square. To encrypt a plaintext its letters are grouped into blocks of two and it is assumed that no block contains two identical letters. (If this is not the case the plaintext has to be modified. For example, by introducing some trivial spelling errors.)*

The encryption of a pair of letters, X *and* Y*, is done as follows. If* X *and* Y *are neither in the same row nor in the same column, then the smallest rectangle containg* X, Y *is taken and* X, Y *are replaced by the pair of symbols in the remaining corners and the corresponding columns. If* X *and* Y *are in the same row (column), then they are replaced by the pair of symbols to the right (below) of them—in a cyclic way, if necessary. An illustration: using the square from Figure 8.2b the plaintext PLAYFAIR is encrypted as LCMNNFCS.*

Various polyalphabetic cryptosystems are created as a modification of the CAESAR cryptosystem using the following scheme—illustrated again on English texts. A 26×26 table is first designed with the first row containing all symbols of the alphabet and all columns representing CAESAR shifts starting with the symbol of the first row. Secondly, for a plaintext

w a key k—a word of the same length as w—is chosen. In the encryption the i-th letter of the plaintext—$w(i)$—is replaced by the letter in the $w(i)$-row and the column with $k(i)$ as the first symbol. Various such cryptosystems differ in the way the key is determined.

In the cryptosystems **VIGENERE**, named by the French cryptographer Blaise de Vigenère (1523-1596), the key k for a plaintext w is created from a keyword p as $Prefix_{|w|}\{p^{\omega}\}$. In the cryptosystem called **AUTOCLAVE**, credited to the Italian mathematician Geronimo Gardono (1501-1576), the key k is created from a keyword p as $Prefix_{|w|}\{pw\}$—in other words, the plaintext itself is used, together with the keyword p, to form the key. For example, for the keyword HAMBURG we get

Plaintext:	`INJEDEMMENSCHENGESICHTESTEHTSEINEGESCHICHTE`
Key in VIGENERE	`HAMBURGHAMBURGHAMBURGHAMBURGHAMBURGHAMBURGH`
Key in AUTOCLAVE:	`HAMBURGINJEDEMMENSCHENGESICHTESTEHTSEINEGES`
Cryptotext in VIGENERE:	`PNVFXVSTEZTWYKUGQTCTNAEEUYYZZEUOYXKZCTJWYZL`
Cryptotext in AUTOCLAVE:	`PNVFXVSURWWFLQZKRKKJLGKWLMJALIAGINXKGFVGNXW`

A popular way of specifying a key used to be to fix a place in a well-known book, as bible, and to take the text starting at that point, of the length of plaintext, as a key.

Exercise 8.2.8 *Encrypt the plaintext "EVERYTHING IS POSSIBLE ONLY MIRACLES TAKE LONGER" using the key word OPTIMIST and:*

(a) VIGENERE cryptosystem; (b) AUTOCLAVE cryptosystem.

In case of polyalphabetic cryptosystems the cryptoanalysis is much more difficult. There are some techniques to guess the size of the keywordthat was used. Polish-British advances in breaking ENIGMA, that performed polyalphabetic substitutions, belong to the most spectacular and important successes of the cryptoanalysis.

In spite of their apparent simplicity polyalphabetic substitution cryptosystems are not to underestimate. In addition, they can provide perfect secrecy, as shown soon.

8.2.3 Transposition Cryptosystems

The basic idea is very simple and powerful. Permute plaintext. Less clear is how to specify and perform efficiently permutations.

The history of transposition encryptions and devices for it goes back to Sparta in about 475 BC.

To write the plaintext backwards is a simple example of an encryption by a transposition. Another simple method is to choose an integer n, write the plaintext into rows, with n symbols in each, and then read it by columns to get the cryptotext. This can be made more tricky by choosing a permutation of columns, or rows, or both.

Cryptotexts obtained by transpositions, called anagrams, were popular among scientists of 17th century. They were used also to encrypt scientific findings. For example, Newton wrote to Leibnitz:

$$a^7c^2d^2e^{14}f^2i^7l^3m^1n^8o^4q^3r^2s^4t^8v^{12}x^1$$

what stands for "data aequatione quodcumque fluentes quantitates involvente, fluxiones invenire et vice versa".

Exercise 8.2.9 *Decrypt anagrams: (a) INGO DILMUR, PEINE; (b) KARL SURDORT PEINE; (c)* $a^2cdef^3g^2i^2jkmn^3o^5prs^2t^2u^3z$*; (d)* $ro^4b^2t^3e^2$.

Exercise 8.2.10 *Consider the following transposition cryptosystem. An integer* n *and a permutation* π *on* $\{1,\ldots,n\}$ *are choosen. The plaintext is divided into blocks of length* n *and in each block the permutation* π *is applied. Show that the same effect can be obtained by a suitable HILL cryptosystem.*

Practical cryptography often combines and modifies basic encryption techniques. For example, by adding, in various ways, a garbage text into the cryptotext.

Exercise 8.2.11 *Decrypt: (a) OCORMYSPOTROSTREPXIT; (b) LIASHRYNCBXOCGNSGXC.*

8.2.4 Perfect Secrecy Cryptosystems

According to Shannon[4], a cryptosystem is perfect if the knowledge of the cryptotext provides no information whatsoever about the plaintext—with the possible exception of its length. It also follows from Shannon's result that perfect secrecy is possible only if the key space is as large as the plaintext space. This implies that the key has to be at least as long as the plaintext and the same key can not be used more than once.

A perfect cryptosystem is the so-called **ONE-TIME PAD** cryptosystem, invented by Gilbert S. Vernam (1917). When used to encode an English plaintext, it is simply a polyalphabetic substitution cryptosystem of VIGENERE type, with the key a randomly chosen English alphabet word of the same length as the plaintext. Each symbol of the key specifies a CAESAR shift that is to be performed on the corresponding symbol of the plaintext.

More straightforward to implement is its original bit-version due to Vernam, who also constructed a machine and got a patent. In this case both plaintext and key are binary words of the same length. Both encryption and decryption are perfomed simply by the bitwise XOR operation.

The proof of perfect secrecy is very simple. By the proper choice of the key any plaintext of the same lenght could lead to the given cryptotext.

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[4] Claude E. Shannon (1917-) from MIT, Cambridge, with his seminal paper "Communication theory of secrecy systems" (1949), has started the scientific era of cryptography.

At the first glance it seems that nothing has been achieved with ONE-TIME PAD cryptosystem.The problem of a secure communication of a plaintext has been transformed to the problem of secure communication of an equally long key. However, this is not completely so.

First of all, the ONE-TIME PAD cryptosystem is indeed used when perfect secrecy is really necessary. For example, for some hot lines. Secondly, and this is perhaps the most important, ONE-TIME PAD cryptosystem provides an idea how to design practically secure cryptosystems. The idea goes as follows: use a pseudo-random generator to generate, from a small random seed, a long pseudo-random word, and then use this pseudo-random word as the key for the ONE-TIME PAD cryptosystem. In such a case two parties need to agree only on a much smaller random seed than is the key really used. This idea is actually behind various modern cryptosystems.[5]

Exercise 8.2.12 *The following example illustrates unbreakability of the ONE-TIME PAD cryptosystem. Consider the extended English alphabet with 27 symbols—also a space character. Given the cryptotext*

`ANKYODKYUREPFJBYOJDSPLREYIUNOFDOIUERFPLVYTS`

find: (a) the key that yields the plaintext

`MR MUSTARD WITH THE CANDLESTICK IN THE HALL;`

(b) the key that yields the plaintext

`MISS SCARLET WITH THE KNIFE IN THE LIBRARY;`

(c) find another examples of this type.

8.2.5 How to Make Cryptoanalysts' Task Harder

Two simple but powerful methods to increase security of a not perfect cryptosystem are called, according to Shannon, **diffusion** and **confusion**.

The aim of diffusion is to dissipate the source language redundancy found in the plaintext by spreading it out over the cryptotext. For example, a permutation of the plaintext can rule out possibility to use frequency tables for digrams, trigrams and so on.

Another way to achieve diffusion is to make each letter of the cryptotext to depend on so many letters of the plaintext as possible. Consider, for example, the case that letters of the English alphabet are represented by integers from 0 to 25 and a key $k = k_1, \dots, k_s$, a sequence of such integers, is used. Let $m = m_1 \dots m_n$ be a plainext. Define, for $0 \leq i < s, m_{-i} = k_{s-i}$.

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[5] In addition, modern technology allows that ONE-TIME PAD cryptosystem can be seen as fully practical. It is enough to take optical discs, with thousands gigabytes, fill it with random bits, make a copy of it and deliver it, through a secure channel. Such a source of random bits can last quite a while.

The letters of the cryptotext are then defined by

$$c_i = (\sum_{j=0}^{s} m_{i-j}) \bmod 26,$$

for each $1 \leq i \leq m$. (Observe that the decryption is easy when the key is known.)

The aim of the confusion is to make the relation between the cryptotext and the plaintext as complex as possible. Polyalphabetic substitutions, as a modification of monoalphabetic substitutions, are examples how confusion helps. Additional examples of diffusion and confusion will be shown in Section 8.2.6.

There is also a variety of techniques how to improve security of encryption of long plaintext, for the case they have to be decomposed into fixed size blocks. The basic idea is that two identical blocks should not be encrypted twice in the same way because this already gives some information to cryptoanalysts. One of the techniques that can be used is that the encryption of each block depends on the encryption of previous blocks as it has been shown above for single letters. This again will be illustrated in Section 8.2.6, where so far the most often used cryptosystem is discussed.

8.2.6 DES Cryptosystem

A revolutionary step in the secret-key cryptography has been the acceptance, in 1977, by the National Bureau of Standards in USA, of the cryptosystem DES (Data Encryption Standard), developed by IBM. Especially revolutionary was the fact that both encryption and decryption algorithms were made public. DES became the most widely used cryptosystem of all times.

To use DES, a **user first chooses a secret 56-bit long key** k_{56} and then adds, after each group of 7 bits, a parity bit, to create a new 64-bit key k_{64}. This key is then preprocessed using the following algorithm.

Preprocessing. 1. A fixed and publicly known permutation π_{56} is applied to k_{56}, to get a 56-bit string $\pi_{56}(k_{56})$. The first (second) part of the resulting string is then taken to form a 28-bit block C_0 (D_0).

2. Using a fixed and publicly known sequence $s_1, \ldots, s_{16}$ of integers (each is 1 or 2), 16 pairs of block $(C_i, D_i), i = 1, \ldots, 16$, each of 28 bits, are created as follows: C_i (D_i) is obtained from C_{i-1} (D_{i-1}) by s_i left cyclic shifts.

3. Using a fixed and publicly known order (bits' numbers; 14, 17, 11,...), 48 bits are chosen from each pair of blocks (C_i, D_i) to form a new block K_i.

The aim of this preprocessing is to make, from k_{56}, a more random sequence of bits.

Encryption. 1. A fixed and publicly known permutation π_{64} is applied to a 64-bit plaintext w to get a new plaintext $w' = \pi_{64}(w)$. (This is a diffusion step in the Shannon sense.) w' is then written in the form $w' = L_0R_0$, with each L_0 and R_0 consisting of 32 bits.

2. 16 pairs of 32-bit blocks $L_i, R_i, 1 \leq i \leq 16$ are constructed using the recurrence

$$L_i = R_{i-1}, \tag{8.1}$$
$$R_i = L_{i-1} \oplus f(R_{i-1}, K_i), \tag{8.2}$$

where f is a fixed mapping, publicly known and easy to implement, both in hardware and software. (Computation of each pair of blocks actually represents one confusion step—computation of f—followed by one diffusion step—an XOR operation.)

3. The cryptotext is obtained as $\pi_{64}^{-1}(L_{16}R_{16})$.

Decryption. Given a cryptotext c, $\pi_{64}(c) = L_{16}R_{16}$ is first computed, then blocks L_i, R_i, $i = 15, 14, \ldots, 0$, using the recurrence

$$\begin{aligned} R_{i-1} &= L_i, && (8.3) \\ L_{i-1} &= R_i \oplus f(L_i, K_i), && (8.4) \end{aligned}$$

and, finally, the plaintext $w = \pi_{64}^{-1}(L_0R_0)$ is obtained.

This means that the same algorithm is used for encryption and decryption. In addition, this algorithm can be implemented fast in both hardware and software. As a consequence, at the time this book went to print, DES could be used to encrypt more than 200 megabits per second using a special hardware.

Since the permutations π_{56} and π_{64}, the sequence $s_1, \ldots, s_{16}$, the order to choose 48 bits out of 56 and the function f are fixed and made public, it would be no problem to present them here. However, and this is the point, they have been designed so carefully to make the cryptoanalysis very hard, that one can hardly learn more about DES from knowing these permutations, than from knowing that they exist and are easily available.

Since its adoption as a standard, there have been concerns about the level of security provided by DES. They fall into two categories: key size and the nature of the algorithm. Various estimations were given how much it would cost to build a special hardware to do decryption by a exhaustive searching through the space of 2^{56} keys. For example, it has been estimated that a molecular computer could be built to break DES in 3 months. On the other hand, none of the cryptoanalytic attacks has turned out successful so far. It has also been demonstrated that the avalanche effect holds for DES. There is an example of a key and two 64-bit long plaintexts that differ by one bit but the corresponding cryptotexts differ in 34 bits.

There are also various techniques how to increase security when using DES. The basic idea is to use two keys and to use the second one to encrypt the cryptotext obtained after the encryption with the first key. Since the cryptosystem DES is not composite, this increases security. Another idea, that has been shown to be powerful, is to use three independent keys k_1, k_2, k_3 and to compute the cryptotext c from the plaintext w using DES three times as follows:

$$c = DES_{k_1}(DES_{k_2}^{-1}(DES_{k_3}(w))).$$

Various ideas were also developed how to increase security when encrypting long plaintexts. Let a plaintext w be divided into n 64-bit blocks $m_1, \ldots, m_n$, i.e. $w = m_1 \ldots m_n$. Choose a 56-bit key k and a 64-bit block c_0. The cryptotext c_i of the block m_i can then be defined as $c_i = DES(m_i \oplus c_{i-1})$. Clearly, the knowledge of k and c_0 makes the decryption easy.

Exercise 8.2.13 *Show that if in DES all bits of the plaintext and those of the key are replaced by their complements, then also in the resulting cryptotext every bit changes to its complement.*

8.2.7 Public Distribution of Secret Keys

The need to secure the secret key distribution ahead a transmission has been an unpleasant but not impossible problem in old times when only few parties needed secure communications (and time did not have such a value). This idea is, however, unfeasible for the current communication needs, when not only the number of parties that need to communicate securely, has enormously increased, but there is often a need for secure communication between two totally unacquainted parties that get a need to communicate all of a sudden. Diffie and Hellman (1976) solved this problem by designing a protocol for communication between two parties to achieve a secure key distributionover a public channel. This result has started a new era in cryptography. The belief in security of this protocol is based on the assumption that modular exponentiation is a one-way function—see Section 2.3.3.

Let two parties—call them from now on Alice and Bob, as it became traditional in cryptography, want to agree on a secret key. First they agree on a large integer n and a $1 < g < n$. They can do that over an insecure channel, or n and g may be even fixed for all users of an information system. Then Alice chooses, randomly, some large integer x and computes $X = g^x \bmod n$. Similarly, Bob chooses, again randomly, a large y and computes $Y = g^y \bmod n$. After that Alice and Bob exchange X and Y, but keep x and y secret. (In other words, only Alice knows x and only Bob knows y). Finally, Alice computes $Y^x \bmod n$ and Bob computes $X^y \bmod n$. Both these values are $g^{xy} \bmod n$ and therefore equal. This value is then the key they wish to agree on.

Observe that an eavesdropper seems to need, in order to be able to determine x from X, g and n, or to determine y from Y, g and n, to have a capability to compute discrete logarithms. (However, no proof is known that such a capability is really needed in order to break the system. Since it is believed that modular exponentiation is a one-way function, the above problem is believed to be unfeasible. Currently the fastest known algorithm to compute the discrete logarithm has complexity $e^{((\ln n)^{1/3} \ln \ln n)^{2/3}}$.)

Remark. Not all values of n and g are equally good. If n is a prime than there exists an g such that $g^x \bmod n$ is a permutation on $\{1, \ldots, n-1\}$ and such g is preferable.

Exercise 8.2.14 *Consider the Diffie-Hellmann's key exchange system with* $q = 1709$, $n = 4079$ *and the secret numbers* $x = 2344$ *and* $y = 3420$. *What is the key Alice and Bob agree on?*

Exercise 8.2.15 * *Extend the Diffie-Hellman's key exchange system: (a) to three users; (b) to more users.*

There is also a way how to have a secure communication with the secret-key cryptosystems without agreeing beforehand on a key, i.e. with no need for a key distribution.

Let each user X have its secret encryption function e_X and a secret decryption function d_X and assume that any two such functions, no matter of which user, are commutative. (In such a case we say that we have a **commutative cryptosystem**.) Consider the following communication protocol in which Alice wants to sent a plaintext w to Bob.

1. Alice sends $e_A(w)$ to Bob.
2. Bob sends $e_B(e_A(w))$ to Alice.
3. Alice sends $d_A(e_B(e_A(w))) = e_B(w)$ to Bob.
4. Bob decrypts $d_B(e_B(w)) = w$.

This has, however, a clear disadvantage of three communication rounds needed. Much better seems to be the idea of the public-key cryptosystems discussed in the next section.

8.3 Public-key Cryptosystems

The key observation that leads to the public-key cryptography is that whoever encrypts a plaintext does not need to be able to decrypt the resulting cryptotext. Therefore, if it is unfeasible from the knowledge of an encryption algorithm e_k to construct the corresponding decryption algorithm d_k, then the encryption algorithm e_k can be made public! As a consequence, in such a case each user U can choose its private key k_U, make the encryption algorithm e_{k_U} public and keep secret the decryption algorithm d_{k_U}. In such a case anybody can send messages to U and U is the only one capable to decrypt them.

This basic idea can be illustrated on the following toy cryptosystem.

Example 8.3.1 (Telephone directory encryption) *Each user make public which telephone directory should be used to encrypt messages for her. The general encryption algorithm is to take the directory, the key, of the intended receiver and to encrypt the plaintext by replacing each its letter by a telephone number of a person whose name starts with this letter. To decrypt, each user is supposed to have one reverse telephon directory, sorted by numbers, and therefore can easily replace numbers by letters to get the plaintext. For example, using the telephone directory of Philadelphia the plaintext CRYPTO can be encrypted using the following entries:*

Carden Frank	*3381276,*	*Roberts Victoria*	*7729094,*
Yeats John	*2890399,*	*Plummer Donald*	*7323232,*
Turne Helen	*4389705,*	*Owens Eric*	*3516765,*

as 338127677290942890399732323243897053516765.

There is also a mechanical analogy to illustrate the difference between the secret-key and the public-key cryptography. Assume that information is sent in boxes. In a secret-key cryptosystem information is put into a box, locked with a padlock and sent, for example by post, and the key is sent by some secure channel. In the public-key modification, anyone can get a padlock for any user U, say at the post office, put information into a box, lock it with the padlock and sent it. U is the only one having the key to open it—no key distribution is needed.

8.3.1 Trapdoor One-way Functions

The basic idea of the public-key cryptosystems is simple, but do such cryptosystems exist? We know that there are strong candidates for one-way functions that can be easily computed but to compute their inverse seems to be unfeasible. This is, however, too much. Nobody, not even the sender, would be able to decrypt a cryptotext encrypted by a one-way function. Fortunately, there is a modification of the concept of one-way functions, the so-called trapdoor one-way functions, that seems to be the appropriate to make public-key cryptosystems.

A function $f : X \mapsto Y$ is a **trapdoor one-way function** if f and also its inverse can be computed efficiently, yet even the complete knowledge of the algorithm to compute f does not make it feasible to determine a polynomial time algorithm to compute its inverse. The secret needed to obtain an efficient algorithm for the inverse is known as the **trapdoor information**.

There is no proof that such functions exist, but there are several strong candidates for them.

Candidate 8.3.2 (Modular exponentiation with a fixed exponent and modulus) *It is the function* $f_{n,x} : \mathbf{Z} \mapsto \mathbf{Z}$ *defined by* $f_{n,x}(a) = a^x \bmod n$*. As already mentioned in Chapter 1, it is known that for any fixed* n *and* x *there is an efficient algorithm to compute the inverse operation of taking* x*-th root modulo* n*. However, all known algorithms to compute the* x*-th root modulo* n *require knowledge of the prime factors of* n*—and such a factoring is precisely the trapdoor information. A public-key cryptosystem based on this trapdoor one-way function will be discussed in Section 8.3.3.*

Candidate 8.3.3 (Modular squaring with fixed modulus) *This is another example of a trapdoor one-way function. As already mentioned in Section 1.7.3, modular squaring seems to be in general unfeasable but easy, if the decomposition of the modulus into primes is known.*

Our second example has a special cryptographical significance because, by Theorem 1.8.15, computation of square roots is exactly as difficult as factoring of integers.

8.3.2 Knapsack Cryptosystems

The first public-key cryptosystem, based on the knapsack problem, was developed by Ralp C. Merkle and Martin Hellman (1978). The cryptosystem has been patented in 10 countries, played an important rôle in the history of the public-key cryptography and so did the exciting attempts to break it.

In spite of the fact that the KNAPSACK public-key cryptosystem is not much used, it has several features that make out of it a good example to illustrate how to design public-key cryptosystems, difficulties one can encounter by that, and ways to overcome them.

There is the following simple and general idea how to design a trapdoor function and a public-key cryptosystem based on it, that will be illustrated in this and the following sections.

1. Choose an algorithmic problem P that is provably intractable, or there is at least a strong evidence for it.
2. Find a key space $\mathcal{K}$, a plaintext space $\mathcal{P}$, and a general encryption algorithm e that maps $\mathcal{K} \times \mathcal{P}$ into instances of P in such a way that p is the solution of the instance $e(k,p)$ of P.
3. Using the chosen (trapdoor) data t, design and make public a specific key k_t such that knowing t it is easy to solve any instance $e(k_t,p)$ of the problem P, but without knowing t this appears to be unfeasable. (One way of doing this is to choose a key k such that anybody can easily solve any instance $e(k,p)$ of P, and then to transform k, using some combination of diffusion and confusion steps (as the trapdoor information), into another key k' in such a way that whenever $e(k',p)$ is known this can be easily transformed, using the trapdoor information, into an easily solvable instance of P.

Let us illustrate now this idea on the KNAPSACK cryptosystem. Let K be the knapsack problem with the instances (X, s) where X is a knapsack vector and s an integer. The key-space $\mathcal{K}$ will be $\mathbf{N}^n$ for a fixed integer n, i.e. the space of n-dimensional vectors.[6]The plaintext-space $\mathcal{P}$ will be the set of n-dimensional bit vectors. (This means, that whenever such a cryptosystem is to be used, the original plaintext has to be divided into blocks and then each encoded by an n-bit vector.) The encryption function e is designed to map any knapsack vector X and any plaintext p—a binary column vector—both of the same length, into the instance of the knapsack problem

$$(X, Xp),$$

where Xp is the scalar product of two vectors. Since the general knapsack problem is **NP**-complete, no polynomial time algorithm seems to exist, to compute p from X and Xp, i.e. for decryption.

Exercise 8.3.4 *Assume that letters of the English alphabet are encodded by binary vectors of 5 bits (space—00000, A—00001, B—00010,...) and that we use* $(1,2,3,5,8,21,34,55,89)$ *as the kanpasack vector.*

(a) Encrypt the plaintext "TOO HOT SUMMER"; (b) determine in how many ways one can decrypt the cryptotext $(128,126,124,122)$.

Exercise 8.3.5 *Consider knapsack vectors* $X = (x_1, \ldots, x_n)$*, where* $x_i = P/p_i$*,* p_i *are distinct primes, and* P *is their product. Show that knapsack problems with such vectors can be solved efficiently.*

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[6] Merkle and Hellman suggested to use 100-dimensional vectors.

On the other hand, and this is the key to the KNAPSACK cryptosystem, any instance (X, s) of the knapsack problem can be solved in linear time (that is, one can find a $p \in \mathcal{P}$ such that $s = Xp$, if it exists, or to show that it does not exists), if $X = (x_1, \ldots, x_n)$ is the so-called **super-increasing vector**, that is $x_i > \sum_{j=1}^{i-1} x_j$ holds, for each $1 < i \leq n$. Indeed, the following algorithm does it:

Algorithm 8.3.6 (Knapsack problem with a super-increasing vector)

Input: a superincreasing knapsack vector $X = (x_1, \ldots, x_n)$ and an $s \in \mathbf{N}$.

```
for $i \leftarrow n$ downto 2 do
    if $s \geq 2x_i$ then terminate—no solution
                  else if $s > x_i$ then $p_i \leftarrow 1; s \leftarrow s - x_i$;
                           else $p_i \leftarrow 0$;
if $s = x_1$ then $p_1 \leftarrow 1$
            else if $s = 0$ then $p_1 \leftarrow 0$
                     else terminate—no solution.
```

Example 8.3.7 *For the super-incresing vector*

$$(1, 2, 4, 8, 16, 32, 64, 128, 256, 512) \tag{8.5}$$

and $s = 999$ *we get the solution* $(1, 1, 1, 0, 0, 1, 1, 1, 1, 1)$. *For* $X = (1, 3, 5, 10, 20, 41, 94, 199)$ *and* $s = 242$ *there is no solution.*

Following the general idea described above, we show now how any super-increasing vector X can be transformed (using a carefully chosen trapdoor information) into another vector X' in such a way that using the trapdoor information we can solve easily any instance of the knapsack problem with X', but this does not seem to be so without this trapdoor information.

Let $X = (x_1, \ldots, x_n)$ be a super-increasing vector. Choose an integer $m > 2x_n$ and an integer u relatively prime to m. Compute, for example using the extended Euclid's algorithm, an u^{-1} such that $uu^{-1} \equiv 1 \pmod{m}$—$u$ and m will form the trapdoor information.

Construct now the knapsack vector $X' = (x'_1, \ldots, x'_n)$ by computing

$$x'_i = ux_i \bmod m,$$

for $1 \leq i \leq n$. (Multiplication by u makes a diffusion, taking values modulo m makes a confusion and also scales the values.)

For example, taking $m = 1250, u = 41$ we get $u^{-1} = 61$, and from the super-increasing vector (8.5) we get, using the above method, the vector

$$X' = (41, 82, 164, 328, 656, 62, 124, 248, 496, 992).$$

Vectors X' obtained this way usually look pretty random, and therefore it seems that to solve the knapsack problem for them can be as hard as in the general case. The following lemma guarantees that this is not so—if the trapdoor information is available. This way a public-key cryptosystem can be created.

Lemma 8.3.8 *Let X, m, u, X' be defined as above, i.e. X is a super-increasing vector, and for $c \in \mathbf{N}$, $c' = u^{-1}c \bmod m$. Then the knapsack problem instances (X, c') and (X', c) have at most one solution, and if one of them has a solution, then the second has the same solution.*

Proof. It follows from Algorithm 8.3.6 that if (X, c') has a solution, then it is unique. Let now p' be a solution of the knapsack instance (X', c), that is $X'p' = c$. Then,

$$c' \equiv u^{-1}c = u^{-1}X'p' \equiv u^{-1}uXp' \equiv Xp' \pmod{m}.$$

Since the vector X is super-increasing and $m > 2x_n$, we have

$$Xp' \bmod m = Xp',$$

and therefore,

$$c' = Xp'.$$

This means that each solution of a knapsack instance (X', c) is also a solution of the knapsack instance (X, c'). Since this knapsack instance has at most one solution, the same has to hold for the instance (X', c). □

KNAPSACK cryptosystem design. A super-increasing vector X and numbers m, u are chosen, X' is computed and made public as the key. X, u and m are kept secret as the trapdoor information.

Encryption. A plaintext w is first divided into blocks and then each block is encoded by a binary vector p_w of lenght $|X'|$. Encryption is then done by computing the scalar product $X'p_w$.

Decryption. $c' = u^{-1}c \bmod m$ is first computed for the cryptotext c and then the instance (X, c') of the knapsack problem is solved using Algorithm 8.3.6.

Example 8.3.9 *Choosing* $X = (1, 2, 4, 9, 18, 35, 75, 151, 302, 606)$, $m = 1250$ *and* $u = 41$, *we design the public key* $X' = (41, 82, 164, 369, 738, 185, 575, 1191, 1132, 1096)$.

To encrypt an English text we first encode its letters by 5-bit numbers: space—00000, *A*—00001, *B*—00010, … *and then divide the binary string into blocks of 10 bits. For the plaintext "AFRICA" we get three plainext vectors* $p_1 = (0000100110)$, $p_2 = (1001001001)$, $p_3 = (0001100001)$ *and they will be encrypted as*

$$c_1' = X'p_1 = 3061, \qquad c_2' = X'p_2 = 2081, \qquad c_3' = X'p_3 = 2203.$$

To decrypt the cryptotext $(9133, 2116, 1870, 3599)$ *we first multiply all these numbers by* $u^{-1} = 61 \bmod 1250$ *to get* $(693, 326, 320, 789)$, *and then for all of them we have to solve the knapsack problem with the vector X which yields the binary plaintext vector*

$$(1101001001, 0110100010, 0000100010, 1011100101)$$

and, consequently, the plaintext "ZIMBABWE".

Exercise 8.3.10 *Take the super-increasing vector*

$$X = (103, 107, 211, 425, 863, 1715, 3346, 6907, 13807, 27610)$$

and $m = 55207$, $u = 25236$.

(a) Design for X, m *and* u *the public knapsack vector* X'. *(b) encrypt using* X' *the plaintext: "A POET CAN SURVIVE EVERYTHING BUT A MISPRINT"; (c) decrypt the cryptotext obtained using the vector* $X' = (80187, 109, 302, 102943, 113783, 197914, 178076, 77610, 117278, 103967, 124929)$.

The Merkle-Hellman's KNAPSACK cryptosystem (called also the single-iteration knapsack) was broken by Adi Shamir (1982). Naturally a question arose whether there are other variants of the knapsack-based cryptosystems that are not breakable.

The first natural idea was to use several times the diffusion-confusion transformation that have produced non-super-increasing vectors from super-increasing. More exactly, the idea is to use **iterated knapsack cryptosystem**—to design the so-called hyper-reachable vectors and to make them public keys.

Definition 8.3.11 *A knapsack vector* $X' = (x'_1, \dots, x'_n)$ *is obtained from a knapsack vector* $X = (x_1, \dots, x_n)$ *by the so-called* **strong modular multiplication** *if* $x'_i = u.x_i \bmod m$, $i = 1, \dots, n$, *where* $m > 2\sum_{i=1}^{n} x_i$ *and* u *is relatively prime to* m. *A knapsack vector* X' *is called* **hyper-reachable**, *if there is a sequence of knapsack vectors* $X = X_0, X_1, \dots, X_n = X'$, *where* X_0 *is a super-increasing vector, and for* $i = 1, \dots, n$, X_i *is obtained from* X_{i-1} *by a strong modular multiplication.*

It has been shown that there are hyper-reacheable knapsack vectors that can not be obtained from a super-increasing vector by a single strong modular multiplication. The so-called **multiple-iterated knapsack cryptosystem** with hyper-reacheable vectors is therefore more secure. However, not enough; it was broken by E. Brickell (1985).

Exercise 8.3.12 * *Design an infinite sequence* $(X_i, s_i), i = 1, 2, \dots$ *of knapsack problems such that the problem* (X_i, s_i) *has* i *solutions.*

Exercise 8.3.13 *A knapsack vector* X *is called injective if for every* s *there is at most one solution of the knapsack proble* (X, s). *Show that each hyper-reacheable knapsack vector is injective.*

There are also variants of the knapsack cryptosystem that have not been broken yet. For example, the so-called **dense knapsack cryptosystem**, in which two new ideas are used: dense knapsack vectors and special arithmetic based on the so-called Galois fields. The density of a knapsack vector $X = (x_1, \ldots, x_n)$ is defined as

$$d(X) = \frac{n}{\lg(\max\{x_i \mid 1 \leq i \leq n\})}.$$

Density of any super-increasing vector is always smaller than $n/(n-1)$ because the largest element has to be at least 2^{n-1}. This has actually been used to break the basic, single iteration, knapsack cryptosystem.

8.3.3 RSA Cryptosystem

The basic idea of the public key cryptosystem of **R**ivest, **S**hamir and **A**dleman (1978), the most widely investigated one, is very simple: *it is easy to multiply two large primes p and q, but it appears to be unfeasible to find p, q when only the product $n = pq$ is given and n is large.*

Design of the RSA cryptosystem. Two large primes p, q are chosen. (In Section 8.3.4 we discuss how this can be done. By **large primes** are currently understood primes that have more than 512 bits.) Denote

$$n = pq, \qquad \phi(n) = (p-1)(q-1),$$

where $\phi(n)$ is Euler's totient function (see page 53). A large $d < n$ relatively prime to $\phi(n)$ is chosen and an e is computed such that

$$ed \equiv 1 \pmod{\phi(n)}.$$

(As we shall see this can also be done fast.) Then

$$n \text{ (modulus) and } e \text{ (encryption exponent)}$$

form the **public key** and

$$p, q, d$$

form the **trapdoor information**.

Encryption: To get the cryptotext c, a plaintext $w \in \mathbf{N}$ is encrypted by

$$c = w^e \bmod n. \tag{8.6}$$

Decryption:

$$w = c^d \bmod n. \tag{8.7}$$

Details and correctness: A plaintext is first encoded as a word over the alphabet $\Sigma = \{0, 1, \ldots, 9\}$, then divided into blocks of length $i - 1$, where $10^{i-1} < n < 10^i$. Each block is then taken as an integer and encrypted using the modular exponentiation (8.6).

The correctness of the decryption algorithm follows from the following result.

Theorem 8.3.14 *Let $c = w^e \bmod n$ be the cryptotext for the plaintext w, $ed \equiv 1 \bmod \phi(n)$ and d relatively prime to $\phi(n)$. Then $w \equiv c^d \pmod{n}$. Hence, if the decryption is unique, then $w = c^d \bmod n$.*

Proof. Let us first observe that since $ed \equiv 1 \pmod{\phi(n)}$, there exists an $j \in \mathbf{N}$ such that $ed = j\phi(n) + 1$. Let us now distinguish three cases.

Case 1. Neither p nor q divides w. Hence $gcd(n, w) = 1$ and by Euler's Totient Theorem,

$$c^d \equiv (w^e)^d \equiv w^{j\phi(n)+1} \equiv w \pmod{n}. \tag{8.8}$$

Case 2. Exactly one of p, q divides w—say p. This immediately implies $w^{ed} \equiv w \pmod{p}$. By Fermat's Little Theorem, $w^{q-1} \equiv 1 \pmod{q}$, and therefore,

$$w^{q-1} \equiv 1 \pmod{q} \Rightarrow w^{\phi(n)} \equiv 1 \pmod{q} \Rightarrow w^{j\phi(n)} \equiv 1 \pmod{q} \Leftrightarrow w^{ed} \equiv w \pmod{q}. \tag{8.9}$$

and therefore, by the property (1.63) of congruences on page 50, we get $w \equiv w^{ed} \equiv c^d \pmod{n}$.

Case 3. Both p and q divide w. This case can not occur because we have assumed that $w < n$. □

Example 8.3.15 *Let us try to make an example. Choosing* $p = 41, q = 61$*, we get* $n = 2501$, $\phi(n) = 2400$*. Taking* $e = 23$*, we get, using the generalized version of Euclid's algorithm* $d = 2087$*; the choice* $e = 29$ *yields* $d = 2069$*. Let us stick to* $e = 23, d = 2087$*. To encrypt the plaintext KARLSRUHE we first represent letters by their positions in the alphabet and obtain the numerical version of the plaintext as 100017111817200704. Since* $10^3 < n < 10^4$*, the numerical plaintext is divided into blocks of three digits and 6 plaintext integers are obtained*

$$100, \quad 017, \quad 111, \quad 817, \quad 200, \quad 704.$$

To encrypt the plaintext we need to compute

$$100^{23} \bmod 2501, \quad 17^{23} \bmod 2501, \quad 111^{23} \bmod 2501,$$

$$817^{23} \bmod 2501, \quad 200^{23} \bmod 2501, \quad 704^{23} \bmod 2501,$$

what yields the cryptotexts

$$2306, \quad 1893, \quad 621, \quad 1380, \quad 490, \quad 313.$$

To decrypt we need to compute

$$2036^{2087} \bmod 2501 = 100, \quad 1893^{2087} \bmod 2501 = 17, \quad 621^{2087} \bmod 2501 = 111,$$

$$1380^{2087} \bmod 2051 = 817, \quad 490^{2087} \bmod 2501 = 200, \quad 313^{2087} \bmod 2051 = 704.$$

Exercise 8.3.16 *Taking small primes and large blocks can lead to a confusion. Indeed, taking* $p = 17$*,* $q = 23$ *we get* $n = 391$*,* $\phi(n) = 352$*. For* $e = 29$*, and* $d = 85$*, the plaintexts* $100, 017, 111, 817, 200, 704$ *are encrypted as* $104, 204, 314, 154, 064, 295$*, and the decryption then provides* $100, 017, 111, 035, 200, 313$*. Where is the problem?*

Exercise 8.3.17 *Consider the RSA cryptosystem with* $p = 47$*,* $q = 71$ *and* $e = 79$*. (a) Compute d; (b) encrypt the plaintext "THE TRUTH IS MORE CERTAIN THAN PROBABLE"; (c) decrypt 3301, 1393, 2120, 1789, 1701, 2639, 895, 1150, 742, 1633, 1572, 1550, 2668, 2375, 1643, 108.*

8.3.4 Analysis of RSA

Let us first discuss several assumptions crucial for designs of RSA cryptosystems. The first assumption was that we can easily find large primes.

As already mentioned in Section 1.7, no deterministic polynomial time algorithm is known to decide whether a given number is prime. The fastest known sequential deterministic algorithm has complexity $\mathcal{O}(n^{\mathcal{O}(1)\lg\lg n})$. There are, however, several fast randomized algorithms, both of Monte Carlo and Las Vegas type, to decide primality. Rabin's Monte Carlo algorithm is based on the following result from the number theory.

Lemma 8.3.18 *Let* $n \in \mathbf{N}$*. Denote, for* $1 \leq x < n$*, by* $C(x)$ *the condition.*

> *Either* $x^{n-1} \not\equiv 1 \bmod n$*, or there is an* $m \in \mathbf{N}$*,* $m = (n-1)/2^i$ *for some* i*, such that* $gcd(n, x^m - 1) \neq 1$*.*

If $C(x)$ *holds for some* $1 \leq x < n$*, then* n *is not prime. If* n *is not prime, then* $C(x)$ *holds at least for half of* x *between* 1 *and* n*.*

Algorithm 8.3.19 (Rabin-Miller's Algorithm, 1980)

```
Choose randomly integers $x_1, \dots, x_m$ such that $1 \leq x_j < n$;
For each $x_j$ determine whether $C(x_j)$ holds;
if $C(x_j)$ holds for some $x_j$
        then $n$ is not prime
        else $n$ is prime, with the probability of error $4^{-m}$.
```

To find a large prime, a large pseudo-random sequence of bits is generated to represent an odd n, and then it is checked, using Rabin-Miller's, or some other, fast primality testing algorithm, whether n is prime. If not, primality of $n+2, n+4, \dots$ is checked until a number is found that is prime—with very large probability.

It is not obvious that this procedure provides a prime fast enough. However, it easily follows from Prime Number Theorem that there are approximately

$$\frac{2^d}{\ln 2^d} - \frac{2^{d-1}}{\ln 2^{(d-1)}}$$

d-bit primes. If this is compared with the total number of odd d-bit integers, $(2^d - 2^{d-1})/2$, we get that the probability that a 512-bit number is prime is 0.00562, and the probability that a 1024-bit number is prime is 0.002815. This shows that the above described procedure for finding large primes is reasonably fast.

To verify that chosen d is relativly prime to $\phi(n)$ the extended version of the Euclid's algorithm can be used. This procedure provides e at the same time.

Exercise 8.3.20 *A natural question is how difficult is to find, given an m, an integer that is relatively prime with m. The following results show that this is fairly easy. Denote $Pr(gcd(m,n) = 1) = P$. (a) Show that $Pr(gcd(m,n) = d) = \frac{P}{d^2}$; (b) use the previous result to show that $Pr(gcd(m,n) = 1) \approx 0.6$.*

The design of an RSA cryptosystem seems therefore quite simple. Unfortunately, this is not really so because to have the resulting cryptosystem secure enough, p, q, d and e should be chosen carefully, to satisfy various conditions. Some of them:

1. The difference $|p - q|$ should be neither too large nor too small. (It is advisable that their bit representation differs in several bits.)
2. $gcd(p - 1, q - 1)$ should not be large.
3. Neither d nor e should be small.

For example, if $|p - q|$ is small, and $p > q$, then $(p+q)/2$ is only slightly larger than $\sqrt{n}$, because $(p+q)^2/4 - n = (p-q)^2/4$ and, in addition, $(p+q)^2/4 - n$ is a square, for example y^2. To factorize n it is then enough to test numbers $x > \sqrt{n}$ until an x is found such that $x^2 - n$ is square. In such a case $p = x + y$, $q = x - y$.

Exercise 8.3.21 *Explain why in designing a RSA cryptosystem:*

(a) $gcd(p - 1, q - 1)$ should be small;

(b) both $p - 1$ and $q - 1$ should contain large prime factors.

Exercise 8.3.22 * *It is evident that d should not be too small—otherwise decryption can be done by testing all small d. In order to show that also a small e can be a security risk, let us assume that three users A, B and C use the number 3 as the encryption exponent and that they use as the modulus n_A, n_B and n_C, that are relatively prime. Assume further that they transmitt the messages $c_i = w^3 \bmod n_i$, $i = A, B, C$, $0 < w \leq \min\{n_A, n_B, n_C\}$. Show that a cryptoanalyst can compute w using Chinesse Remainder Theorem.*

Exercise 8.3.23 ** *Show that for any choice of primes p and q we can choose $e \notin \{1, \phi(pq) + 1\}$ in such a way that $w^e \equiv w \bmod n$ for all plaintexts w.*

Let us now discuss another two important questions: how hard factoring is and how important actually factoring is for the security of the RSA cryptosystem.[7]

At the time this book was put into print, the fastest algorithm to factor integers run in time $\Theta(e^{\alpha\sqrt{\ln n \ln\ln n}})$, where $\alpha > 1$ but could be made as close to one as desirable. There is no definite answer to the second question yet. This is a tricky problem what can be seen also from the fact knowledge of $\phi(n)$ or d is sufficient to break RSA.

Theorem 8.3.24 *(1) To factor a number n is as hard as to compute $\phi(n)$.*

(2) Any polynomial time algorithm to compute d can be converted into a polynomial time randomized algorithm to factor n.

The first claim of Theorem 8.3.24 follows easily from the identities

$$p + q = n - \phi(n) + 1, \qquad p - q = \sqrt{(p+q)^2 - 4n}.$$

The proof of the second statement, due to DeLaurentis (1984), is too involved to be presented here.

Finally, let us mention three results indicating how strong the security of RSA is, how little we know about it, and how easy it is to break it, if the upmost care is not made in designing it.

It has been shown that any algorithm that is able to determine **one bit** of the plaintext, for example, the rightmost one, can be converted into an algorithm that can determine the whole plaintext, and it is not of substantially larger complexity. (Actually it is sufficient that the last bit can be determined with the probability larger than $\frac{1}{2}$.)

The cryptoanalysis of any reasonable public-key cryptosystem is both in **NP** and **co-NP** and therefore unlikely **NP**-complete. (In the case of deterministic encryptions this is trivial. To find out a plaintext, one guesses it and applies the public encryption function. The same idea is used to show that the cryptoanalysis problem is in **co-NP**. A little bit more involved is to show that this is true also for nondeterministic encryptions.)

It can also be shown that if more users use the RSA cryptosystem with the same n, then they are able to determine in deterministic quadratic time another users decryption exponent—without factoring n. This setting refers to the following hypotethical case: an

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[7] Factoring of large numbers is another big challenge for computing theory, people and technology. In 1971, 40-digit numbers seemed to be the limit; in 1976, 80-digit numbers seemed to be the limit. In 1977, it was estimated that it would take 40 quadrilion years to factor 125-digit numbers. In 1990, the 155-digit number 2^{2^9+1}, the so-called 9-th Fermat number, was factored, using about 1000 computers and several hundred of colaborators, by K. Lenstra, W. Lenstra, S. Manasse and M. Pollard, into three factors— 99-digits, 49-digit and 7-digit—with 2 424 833 as the smallest one. Factoring of this number was put as the challenge in one of the first papers on RSA by Gardner (1978) in Scientific American and at that time this number was at the top of the list of "most wanted to factor numbers". In 1994 there was another spectacular cryptographical success. Using a network about 1600 computers a 96-digit cryptotext, encrypted using a 129-bit modulus and the encryption exponent $e = 9007$, and set as a challenge by the RSA founders, was decrypted—see D. Atkins, M. Graff, A. K. Lenstraand P. C. Leyland (1995). The experiences led to an estimation that it is possible to set up projects that would use 100 000 computers and require half a milion mips years. Moreover, it got quite clear that the RSA Cryptosystem with 512-bit long modulus is breakable by anybody willing to spent a few million dollars and wait few months. For a factoring of a 120-digit number in 825 MIPS years see Denny, Dodson, Lenstra, Manase (1994).

agency would like to make business out of making RSA cryptosystems and therefore, it would choose one pair p, q, sends $n = pq$ to all users, and deliver to each user a unique encryption exponent and the corresponding decryption exponent.

Taking into account simplicity, elegance, power and mystery RSA provides, it is no wonder that already in 1993 more than 10 different RSA chips were produced.

Exercise 8.3.25 **[RABIN cryptosystem] If a cryptoanalyst knows to factor efficiently, he is able to break RSA. However, it is not known if the converse of this statement is true. On the other hand, it has been shown by Rabin (1979) that for the following cryptosystem the problem of factoring is computationally equivalent to that of breaking the cryptosystem.*

In the RABIN cryptosystem each user selects a pair p, q of distinct Blum integers to be kept secret and publicizes n and a $b < n$. The encryption function is $e_{n,b}(w) = w(w + b) \bmod n$. Show that the knowledge of the trapdoor information p, q is sufficient to make decryptions effectively. (In the original Rabin's cryptosystem $b = 0$

8.4 Cryptography and Randomness*

Randomness and cryptography are deeply related. On one side, the prime purpose of encryption methods is to transform a highly non-random plaintext into a highly random cryptotext. For example, let e_k be an encryption mapping, x_0 a plaintext and $x_i, i = 1, 2, \ldots$ be a sequence of cryptotexts constructed by encryptions $x_{i+1} = e_k(x_i)$. If e_k is cryptographically "secure" enough, it is likely that the sequence $x_1, x_2, \ldots$ looks quite random. Encryptions can therefore produce (pseudo)-randomness.

The other side of the relation is more involved. On one side, it is clear that perfect randomness combined with the ONE-TIME PAD cryptosystem provides perfect cryptographical security. However, the price to be paid by the need to have keys as long as the plaintext is too big. The other idea is to use, as illustrated above, a cryptosystem, or another pseudo-random generator, to provide a long pseudo-random string from a short random seed and then to use this long sequence as the key for the ONE-TIME PAD cryptosystem. This brings us to the fundamental question when a pseudo-random generator is good enough for cryptographical purposes? The following concept has turned out to capture this intuitive idea.

A pseudo-random generator is called **cryptographically strong** if the sequence of bits it produces, from a short random seed, is so good for using with ONE-TIME PAD cryptosystem, that no polynomial time algorithm allows a cryptoanalyst to learn **any** information about the plainext from the cryptotext.

Clearly, such a pseudo-random generator would provide sufficient security in a secret-key cryptosystem if both parties agree on some short seed and never use it twice. As we shall see later, cryptographically strong pseudo-random generators could provide perfect security also for the public-key cryptography. However, do they exist?

Before going to a further discussion of these ideas let us mention that the concept of a cryptographically strong pseudo-random generator is, surprisingly, one of the key concepts of foundations of computation. This follows, for example, from the fact that a cryptographically strong pseudo-random generator exists if and only if a one-way function exists what is equivalent with $\mathbf{P} \neq \mathbf{UP}$ and what implies $\mathbf{P} \neq \mathbf{NP}$.

The key primitive to deal with this problem, and also with the problem of randomized encryptions, is that of a **(trapdoor) one-way predicate**.

Definition 8.4.1 *A* **one-way predicate** *is a Boolean function* $P : \{0,1\}^* \mapsto \{0,1\}$ *such that*

1. *For an input* $v1^k, v \in \{0,1\}$*, one can choose, randomly and uniformly, in expected polynomial time, an* $x \in \{0,1\}^*, |x| \leq k$*, such that* $P(x) = v$*. (The suffix* 1^k *is just to have meaningful the requirement of the polynomial time in the length of input.)*

2. *For all* $c > 0$*, and any sufficiently large* k*, no polynomial time algorithm can compute* $P(x)$*, given* $x \in \{0,1\}^*, |x| \leq k$*, with the probability greater than* $\frac{1}{k^c}$*. (The probability is taken over the random choices made by the algorithmnd* x *such that* $|x| \leq k$*.)*

A **trapdoor one-way predicate** *is a one-way predicate for which there exists, for every* k*, a trapdoor information* t_k*, the size of which is bounded by a polynomial in* k*, that can be used to compute, in polynomial time* $P(x)$*, for all* $|x| \leq k$*.*

Candidate 8.4.2 *Take two large primes* p, q*,* $n = pq$*, a* d *relatively prime to* $\phi(n)$ *and* e *such that* $ed \equiv 1 \bmod \phi(n)$*. Define* $P(x)$ *to be the least significant bit of* $x^d \bmod n$*, for* $x \in \mathbf{Z}_n^*$*. To select uniformly an* $x \in \mathbf{Z}_n^*$ *such that* $P(x) = v$*, take a* $y \in \mathbf{Z}_n^*$ *whose least significant bit is* v *and set* $x = y^e \bmod n$*.*

8.4.1 Cryptographically Strong Pseudo-random Generators

As usual in cryptography, the existence of such generators has not been proven yet but there are strong candidates for them.

It has been shown, for example, that cryptographically strong are all pseudo-random number generators that are **unpredictable to the left** in the sense that a cryptoanalyst that knows the generator and sees the whole generated sequence except its first bit has no better way to find out this first bit than coin-tossing It has also been proven, that if integer factoring is intractable, then the so called **BBS pseudo-random generator**, introduced in Section 1.9.3, is unpredictable to the left.

To analyse the BBS pseudo-random generator in more detail we need some basic results concerning modular squaring (see Section 1.7). Recall that computation of square roots modulo a Blum integer n is a permutation of the quadratic residues modulo n and the problem of computing principal square roots has been proven to be computationally equivalent to that of factoring of integers. In addition, it has been proven that if factoring is unfeasible, then for almost all quadratic residues x (modulo n), coin-tossing is the best possible way to estimate the least significant bit of x after seeing $x^2 \bmod n$. This fact will now be used.

Let n be a Blum integer. Choose a random quadratic residue x_0. (For example, choose a random integer x, relatively prime to n, and compute $x_0 = x^2 \bmod n$.) For $i \geq 0$ let

$$x_{i+1} = x_i^2 \bmod n \qquad \text{and} \qquad b_i \text{ be the least significant bit of } x_i.$$

For each integer i, let $BBS_{n,i}(x_0) = b_0 \dots b_{i-1}$ be the first i bits of the pseudo-random sequence generated from the seed x_0 by the BBS pseudo-random generator.

Assume that the BBS pseudo-random generator, with a Blum integer as the modulus, is not unpredictable to the left. Let y be a quadratic residue from $\mathbf{Z}_n^*$. Compute $BBS_{n,i-1}(y)$ for some $i > 1$. Let us now pretend that the last $(i-1)$ bits of $BBS_{n,i}(x)$ are actually the first $(i-1)$ bits of $BBS_{n,i-1}(y)$, where x is the unknown principal square roott significant bit of x. Hence, if the BBS pseudo-random generator is not unpredictable to the left, then there exist a better method than coin-tossing to determine the least significant bit of x what is, as mentioned above, impossible.

Observe also that the BBS pseudo-random generator has a nice property that one can determine, directly and efficiently, for any $i > 0$, the i-th bit of the sequence of bits generated by the generator. Indeed, $x_i = x_0^{2^i} \bmod n$ and using Euler's Totient Theorem,

$$x_i = x_0^{2^i \bmod \phi(n)} \bmod n.$$

There is also a general method how to design cryptographically strong pseudo-random generators. The method is based on the result that cryptographically strong is any pseudo-random generator that passes the **next-bit test**: if the generator generates the sequence $b_0, b_1, \dots$ of bits, then it is unfeasible to predict b_{i+1} from $b_0, \dots, b_i$, with probability greater than $\frac{\varepsilon+1}{2}$ in polynomial time with respect to $\frac{1}{\varepsilon}$ and the size of the seed. Here, the key rôle plays the following modification of the concept of a one-way predicate.

Let D be a finite set, $f : D \mapsto D$ a permutation. Moreover, let $P : D \mapsto \{0,1\}$ be a mapping such that it is unfeasible to predict (to compute) $P(x)$ with probability larger than $\frac{1}{2}$, given x only, but it is easy to compute $P(x)$ if $f^{-1}(x)$ is given. A candidate for such a predicate: $D = \mathbf{Z}_n^*$ where n is Blum integer, $f(x) = x^2 \bmod n$, and $P(x) = 1$ if and only if the principal square root of x modulo n is even.

To get from a seed x_0 a pseudo-random sequence of bits the elements $x_{i+1} = f(x_i)$ are first computed for $i = 0, \dots, n$, and then b_i are defined by $b_i = P(x_{n-i})$ for $i = 0, \dots, n$. (Observe the reverse order of the sequences—to determine b_0 we need to know x_n first.)

Suppose now, that the pseudo-random generator described above does not pass the next-bit test. We sketch, how we can then compute $P(x)$ from x. Since f is a permutation, there must exist x_0 such that $x = x_i$, for some i, in the sequence generated from x_0. Compute then $x_{i+1}, \dots, x_n$ and determine the sequence $b_0, \dots, b_{n-i-1}$. Suppose, we can predict b_{n-i}. Since $b_{n-i} = P(x_i) = P(x)$, we get a contradiction with the assumption that the computation of $P(x)$ is not feasible if only x is known.

8.4.2 Randomized Encryptions

The public-key cryptography with deterministic encryptions solves the key distribution problem quite satisfactory but still has significant disadvantages. Its sufficient security is questioned. For example, a cryptoanalyst who knows the public encryption function e_k and a cryptotext c can choose a plaintext w, compute $e_k(w)$ and compare it with c. In any case, this way some information is obtained about what is, or is not, a plaintext corresponding to c.

The purpose of randomized encryption, invented by S. Goldwasser, S. Micali (1984), is to encrypt messages, using randomized algorithms, in such a way that we can prove that no feasible computation on the cryptotext can provide any information whatsoever about the corresponding plaintext (except with a negligible probability).

As a consequence even a cryptoanalyst familiar with the encryption procedure can no longer guess the plaintext corresponding to a given cryptotext, and can not verify the guess by providing an encryption of the guessed plaintext.

Formally, we have again a **plaintext-space** $\mathcal{P}$, a **cryptotext-space** $\mathcal{C}$ and a **key-space** $\mathcal{K}$. In addition, there is a **random-space** $\mathcal{R}$. For any $k \in \mathcal{K}$, there is an encryption mapping $e_k : \mathcal{P} \times \mathcal{R} \rightarrow \mathcal{C}$ and a decryption mapping $d_k : \mathcal{C} \rightarrow \mathcal{P}$ such that for any plaintext p and any randomness source $r \in \mathcal{R}$ we have $d_k(e_k(p, r)) = p$. Given a k, both e_k and d_k should be easy to design and to compute. However, given e_k it should be unfeasible to determine d_k without knowing k. e_k is a public key. Encryptions and decryptions are done as in the public-key cryptography. (Observe that if a randomized encryption is used, then the cryptotext is not determined uniquelly, but the plaintext is!)

Exercise 8.4.3 ***[Quadratic residue cryptosystem—QRS] Each user chooses primes p, q such that $n = pq$ is a Blum integer and makes public n and an $y \notin QR_n$. To encrypt a binary message $w = w_1 \ldots w_r$ for a user with the public key n, the cryptotext $c = (y^{w_1}x_1^2 \bmod n, \ldots, y^{w_r}x_r^2 \bmod n)$ is computed, where $x_1, \ldots, x_r$ is a randomly choosen sequence of elements from $\mathbf{Z}_n^*$. Show that the legal receiver can decrypt the cryptotext efficiently.*

The idea of randomized encryptions has also led to various definition of security that has turned out to be equivalent to the following one:

Definition 8.4.4 *A* **randomized encryption cryptosystem is polynomial-time secure** *if, for all $c \in N$ and sufficiently large integer s (the so-called* **security parameter***), any randomized polynomial time algorithm that takes as input s (in unary) and a public key, can not distinguish between randomized encryptions, by that key, of two given messages of length c, with the probability greater than $\frac{1}{2} + \frac{1}{s^c}$.*

We describe now a randomized-encryption cryptosystem that has been proven to be polynomial-time secure and it is also from the efficiency point of view a good one. It is based on the assumption that squaring modulo a Blum integer is a trapdoor one-way function, and uses the cryptographically strong BBS pseudo-random generator described in the previous section. Informally, the BBS pseudo-random generator is used to provide the key for the ONE-TIME-PAD cryptosystem. The legal receiver capability to compute the principal square roots, using the trapdoor information, allows him to recover the pad and to obtain the plaintext.

Formally, let p, q be two large Blum integer. Their product, $n = pq$, is the public key. The random-space is QR_n of all quadratic residues modulo n. The plaintext-space is the set of all binary strings—for an encryption they will not have to be divided to blocks. The cryptotext-space is the set of pairs formed by elements of QR_n and binary strings.

Encryption: Let w be a t-bit plaintext and x_0 a random quadratic residue modulo n. Compute x_t and $BBS_{n,t}(x_0)$, using the recurrence $x_{i+1} = x_i^2 \bmod n$, as shown in the previous section. The cryptotext is then the pair $\langle x_t, w \oplus BBS_{n,t}(x_0)\rangle$.

Decryption: The legal user, who knows the trapdoor information p and q, can first compute x_0 from x_t, then $BBS_{n,t}(x_0)$ and, finally, can determine w. To determine x_0 one can use a brute force method to compute, using the trapdoor information, $x_i = \sqrt{x_{i+1}} \bmod n$, for $i = t-1, \ldots, 0$, or the following, more efficient, algorithm:

Algorithm 8.4.5 (Fast multiple modular square-rooting)

```
Compute $a, b$ such that $ap + bq = 1$;
$x \leftarrow ((p+1)/4)^t \bmod (p-1)$;
$y \leftarrow ((q+1)/4)^t \bmod (q-1)$;
$u \leftarrow (x_t \bmod p)^x \bmod p$;
$v \leftarrow (x_t \bmod q)^y \bmod q$;
$x_0 \leftarrow (bqu + apv) \bmod n$.
```

There is also the following general method to make randomized encryptions, based on the concept of the trapdoor one-way predicate, that has been shown polynomial time secure. Alice chooses a trapdoor predicate P_A, a security parameter $s \in \mathbf{N}$, and makes public the description of P_A and s. The trapdoor information, needed to compute P_A efficiently, is kept secret. Anybody wishing to send a plaintext $p = p_1 \ldots p_s$ of s bits to Alice encrypts p as follows: for $i = 1, \ldots, k$, and $p_i \in \{0, 1\}$, an x_i, $|x_i| \leq s$, is randomly chosen such that $P_A(x_i) = p_i$ (what can be done in polynomial time) and p_i is encrypted as x_i. To decrypt, Alice, who knows the trapdoor information, computes $p_i = P_A(x_i)$, for $i = 1, \ldots s$.

Observe, that such cryptotext may be up to s times longer than plaintext—s is here a security parameter.

8.4.3 Down to Earth and Up

In any secure randomized cryptosystem more cryptotexts have to correspond to one plaintext. **Data expansion**, as we have seen at the end of the last section, is therefore unavoidable. This is certainly a drawback of randomized encryptions.

On the other hand, it has been proven that breaking randomized encryptions described above is as hard as factoring. Such strong statement is not known for the RSA cryptosystem. Moreover, randomized encryptions are usually faster than deterministic ones.

None of the public-key cryptosystems seems to be able to compete in speed with DES. (In 1994 DES was in software (hardware) about 1000 (100) times faster than RSA.) Therefore, it seems currently as the best to combine both of them. To use a public key cryptosystem to transport a key and then to use this key with DES to encode messages. (And change keys often). Without too much slowing down transmissions, this can significantly increase security.

Public-key cryptography does not mean an end of the secret-key cryptography. One of the main applications of the public key encryptions are fast and secure key transmissions

There is another drawback of the public-key cryptography. Proofs of security of several public-key cryptosystems are based on unproven assumptions from the number theory. This is not the case with the quantum cryptography based on sending photons through quantum channels on which it is impossible, in principle, to eavesdrope without a high probability of disturbing the transition in such a way that it is detectable.

8.5 Digital Signatures

The number of crucial applications of cryptography, as well as the basic concepts arising from it, have already far exceeded those related to secure encryptions. Digital signatures, with which we deal in this section, are one of the most fundamental and important inventions of modern cryptography. To make digital signatures we can use various techniques similar to those used for encryptions and therefore we deal with digital signatures in this chapter. With other applications and fundamental concepts arising from cryptographical considerations we deal in the following chapter.

If Alice gets a message that claims to be from Bob, she needs to be convinced that the message is really from Bob and that nobody else, pretending being Bob, has sent the message. Moreover, for example, for legal reasons, she may need to be able to convince the third party that the message has been indeed from Bob (and not from somebody else, or even that the message is not her own invention). For modern business and financial interactions these are crucial problems.

A commutative public-key cryptosystem, in which plaintext and cryptotext spaces are the same and each user U makes his encryption function e_U public and keeps his decryption function d_U secret, can be used for this purpose.

If Bob wants to send a plaintext w with his "signature" to Alice, he sends Alice $e_A(d_B(w))$. Only Alice can get out of it $d_B(w)$, but anybody can verify, using the public key e_B that message was sent by Bob. Observe that also Bob is protected against changing the message after he has signed it.

Note also, that a "natural, alternative or symmetric, solution", to encode a message w as $c = d_B(e_A(w))$, is not good. An active enemy T, with his public key e_T, called usually **tamperer**, could intercept such a message, then compute $d_T(e_B(c)) = d_T(e_A(w))$ and sent it to A pretending that it is the message from him—without being able to decrypt the message.

Exercise 8.5.1 * *Alice wants to sign a message for Bob using a secret-key cryptosystem through Trent a trusteed arbitrator, with whom both Alice and Bob share secret keys. Design a secure signature protocol to do this.*

The RSA cryptosystem can be used for digital signature in the above way. The private key then becomes the **signing exponent**. However, not all public-key cryptosytems are equally good for that. In addition to that, there are also some signatures-only (crypto)schemes. The key point is, that various applications put very strong requirements on efficiency. In 1993, the National Institute of Standards and Technology proposed the following Digital Signature Algorithm (DSA) for use in their Digital Signature Standard (DSS).

Design of DSS

1. Global public key components are chosen:
    - p — a random l-bit prime, $512 \leq l \leq 1024$, is chosen and l is a multiple of 64;
    - q — a random 160-bit prime dividing $p-1$ is chosen;
    - $g = h^{(p-1)/q} \bmod p$ is computed, where h is a random integer $1 < h < p-1$, such that $g \neq 1$.
2. User's private key is chosen:
    - x — a random integer, $0 < x < q$, is chosen;
3. User's public key is computed:
    - $y = g^x \bmod p$.

Signing algorithm—w is a plaintext.[8]

- Choose a (pseudo)-random $0 < k < q$ (a user's "per-message" secret number);
- Compute $r = (g^k \bmod p) \bmod q$;
- Compute $s = k^{-1}(w + xr) \bmod q$, where $kk^{-1} \equiv 1 \pmod{q}$.
- (r, s) is a signature.

Verification algorithm—(r, s) is a signature.

- Compute $z = s^{-1} \bmod q$;
- Compute $u_1 = wz \bmod q$ and $u_2 = rz \bmod q$;
- Compute $v = (g^{u_1} y^{u_2} \bmod p) \bmod q$.

  Test whether $v = r$. If yes, the signature is correct.

---

[8] In practice one does not sign the original mesage w. First, a hash function h is applied to w and $h(w)$ is signed. Also for hashing a standard was developed in 1993 by National Institute of Standards and Technology, the so-called Secure Hash Algorithm (SHA). This algorithm takes as an input any message that has less than 2^{64} bits and transforms it into a 160-bit message digest.

Exercise 8.5.2 *Show that the DSA signature scheme is correct and argue that it is sufficiently secure.*

Moral. History of cryptology is full of inventions of perfect ideas and unbreakable systems that turned out soon not to be so perfect and unbreakable. A good rule of thumb by cryptology is therefore, as in real life, never overestimate yourself, never underestimate your oponents/enemies and remember that human ingenuity cannot concot an obstacle human ingenuity cannot resolve.

8.6 Exercises

1. Decrypt the following messages encrypted using the CAESAR cryptosystem:

   (a) EOXH MHDQV; (b) WHVW WAGOB; (c) JWUSBSFS;

   (d) UJQHLGYJSHZAW HJGLGUGDK.

2. Decrypt names of microcomputers depicted in Figure **??**.

3. Find an English meaningful text c, as long as you can, such that there are two meaningful plaintexts w_1 and w_2 that both can be encrypted by CAESAR as c.

4. Encrypt the plaintext POPOCATEPETL using the HILL cryptosystem and the matrix

$$(a) \qquad \begin{pmatrix} 11 & 2 & 19 \\ 5 & 23 & 25 \\ 20 & 7 & 1 \end{pmatrix}; \quad (b) \qquad \begin{pmatrix} 23 & 7 & 3 \\ 25 & 1 & 19 \\ 5 & 20 & 11 \end{pmatrix}.$$

5. Encrypt the following plaintext using the PLAYFAIR cryptosystem: EVEN IN CRYPTOGRAPHY SILENCE IS GOLDEN.

6. Apply the frequency analysis to decrypt the following cryptotexts encrypted using a CAESAR cryptosystem: (a) ESPNLETYESPSLE; (b) DRKYVDRKZTJTREYVCGJFCMVKYVNFICUJGIFSCVDJ; (c) SQUIQHICUJXETMQIJEEIYCFBUJERUKIUVKB.

7. Encrypt the plaintext ENCRYPT THE PLAINTEXT using the AUTOCLAVE cryptosystem and the key SALERNO.

8. * Find a necessary and sufficient condition for a mapping to be realized by a Playfair square. Show how this can be utilized for the cryptoanalysis.

9. The cryptotext obtained using the AFFINE cryptosystem with $c(x) = (7x+10) \bmod 26$ is LJMKG MGMXF QEXMW. Determine the plaintext.

10. Decrypt the following messages obtained using an AFFINE cryptosystem:

    (a) BIFUIMZLMJSVIZLZUUZD; (b) NXUSTUMFXUFJMCTGPOTGXWHOO.

11. Encrypt: the following plaintexts using the AUTOCLAVE cryptosystem and a keyword: (a) "THE TRUTH IS MORE CERTAIN THAN PROBABLE" with the keyword "OPTIMIST"; (b) "THERE IS NO ROYAL ROAD TO CRYPTOGRAPHY" with "EUCLID"; (c) "I CAME I SAW I CONQUER" with "CAESAR"; (d) "THOSE WHO KNOW HOW TO WIN ARE MUCH MORE NUMEROUS THAN THOSE WHO KNOW HOW TO MAKE A PROPER USE OF THEIR VICTORY" with "POLYBIOS".

12. Decrypt the following cryptotext given to the participants of EUROCRYPT'88 in Davos.

```
EXVILT  AMSYMX  EAKSSI  KIRZMS
YEKDAV  OSINAL  PVITHE  RRJMLO
OIEUSM  GPLKSM  ADAVOS  LULRVK
SIXMTA  IDAVOS
```

13. Among 10-letters English anagrams we have:

    ALGORITHMUS—LOGARITHMUS, ANTAGONIST—STAGNATION,

    COMPRESSED—DECOMPRESS, CREATIVITY—REACTIVITY.

    Find 5 others that make a nice sense.

14. Find the binary vector $(b_1, \dots, b_6)$, if it exists, such that $s = b_1 2 + b_2 5 + b_3 9 + b_4 19 + b_5 55 + b_6 91$, for: (a) $s = 155$; (b) $s = 105$; (c) $s = 77$; (d) $s = 44$.

15. * Show that the knapsack vector $(2106, 880, 1320, 974, 2388, 1617, 1568, 2523, 48, 897)$ can be obtained from a super-increasing vector by a strong modular multiplication.

16. For each n design a super-increasing vector of length n.

17. * Show that in the set of vectors $\{(a, b, c) \mid \max\{a, b, c\} = 4\}$ there are 5 super-increasing vectors.

18. Determine the following numbers: (a) 7^{-1} modulo 26; (b) 13^{-1} modulo 2436; (c) 144^{-1} modulo 233.

19. Compute: (a) 2^{340} mod 11; (b) 2^{340} mod 31; (c) 2^{340} mod 341.

20. Factor numbers: (a) 323; (b) 54053; (c) 209501; (d) 43794427;

21. * Estimate the number of possibilities to factor n if $n = pq$ and p, q are 150-bit primes.

22. Compute the secret exponent for the RSA cryptosystem with: (a) $p = 7$, $q = 11$, $e = 7$; (b) $p = 5$, $q = 17$, $e = 3$; (c) $p = 7$, $q = 17$, $e = 5$.

23. Let $p = 11$, $q = 23$. List all possibilities for encryption and decryption exponents for the RSA cryptosystem.

24. Given RSA cryptosystem with $n = 43 \times 59$ and $e = 13$. (a) Encrypt the plaintex CRYPTOGRAPHY; (b) Decrypt the cryptotext 0667 1947 0671.

25. ** Show that for every pair of primes p, q one can chose $e \notin \{1, \phi(pq)+1\}$ such that $w^e \equiv w \bmod n$ for all w and $n = pq$.

26. $*$ Show, that the number of blocks that are mapped into itself at the RSA encryption is $(1+gcd(e-1,p-1))(1+gcd(e-1,q-1)))$.

27. Factor n if n and $\phi(n)$ are given as follows: (a) $n = 5767$, $\phi(n) = 5618$; (b) $n = 4386007$, $\phi(n) = 4382136$; (c) $n = 1003968$, $\phi(n) = 1005973$; (d) $n = 3992003$, $\phi(n) = 3988008$.

28. * Show that the cryptoanalysis problem for the public-key cryptography is in **NP**$\cap$**co-NP** even for nondeterministic encryptions.

29. Consider the following modification of the RSA cryptosystem. A large prime$p$ is known to all users. Each user chooses and keeps secret encryption and decryption exponents e and d such that $ed \equiv 1 \pmod{p-1}$. Communication goes as follows. A encrypts a plaintext w by computing $c = E_A(w) = w^{e_A} \bmod p$ and sends it to B. B responds by sending $c_1 = E_B(c)$. Finaly, A sends $D_A(c_1)$ to B. Show that B is able to decrypt and analyse the security of the cryptosystem.

30. (**LUC cryptosystem and Lucas numbers**)** There is a whole family of Lucas numbers. They have remarkable properties and an interesting cryptosystem is based on them. Let p, q be integers such that $d = p^2 - 4q \neq 0$, and let α and β be such that $\alpha+\beta = p$ and $\alpha\beta = q$. Clearly, $\alpha-\beta = \sqrt{d}$. For $n \geq 1$ we define Lucas numbers $V_n(p,q)$ as follows: $V_n(p,q) = \alpha^n+\beta^n$. Show: (a) $V_n(p,q) = pV_{n-1}(p,q) - qV_{n-2}(p,q)$, for $n \geq 2$; (b) $V_n(p \bmod m, q \bmod m) = V_n(p,q) \bmod m$ for all m, n; (c) $V_{nk}(p,1) = V_n(V_k(p,1),1)$ for all n, k; (d) if p, q are primes, $n = pq$, $s(n) = lcm(p-(d|p), q-(d|p))$, where $(d|p)$ is the Legendre symbol, and e, d are relatively prime to $s(n)$, $ed \equiv 1 \pmod{s(n)}$; then $V_e(V_d(x,1),1) = V_d(V_e(x,1),1) = x$ for all $x < n$.

    LUC cryptosystem. Encryption: $c = V_e(w,1) \bmod n$; decryption: $V_d(c,1) \bmod n$.

31. List all quadratic residues modulo: (a) 23; (b) 37.

32. * In the El Gamal cryptosystem a large prime p is chosen aswell as an $a \in \mathbf{Z}_p$, and an $x \in \mathbf{Z}_p$. p, a and $y = a^x \bmod p$ form the public key and x form the trapdoor information. $\mathbf{Z}_p$ is both the plaintext- and cryptotext-space. To encrypt a plaintext w a random k is first chosen, $K = y^k \bmod p$ computed and the pair $c_1 = a^k \bmod p$ and $c_2 = Kw \bmod p$ form the cryptotext. Show how one can make decryption efficiently when the trapdoor information is available.

QUESTIONS

1. Why is PLAYFAIR cryptosystem much more secure than CAESAR cryptosystem?

2. How can one slightly modify monoalphabetic substitution cryptosystem in such a way that the frequency counting does not help too much to make decryptions?

3. What is the maximal density of a knapsack vector of length n?

4. Why the secret exponent d in the RSA cryptosystem should have no common factors with $p-1$ and $q-1$?
5. Are the numbers π and e random?
6. Can we have such one-way functions that cannot be inverted for any argument effectively?
7. Do you know some "Golden rule" for the design of public-key cryptosystems?
8. What are the main types of attacks a cryptosystem has to be designed to detect, prevent and recover from?
9. What are the advantages and disadvantages of randomized encryptions comparing with deterministic encryptions?
10. How could you formulate properties a good signature scheme should have?

8.7 Historical and Bibliographical References

Kahn (1967) is an interesting reading of an exciting 4000 years old history of cryptology. There are numerous books on classical cryptology. From the older ones see Gaines (1939). From more recent ones see Bauer (1993) and also the introductory chapter in Salomaa (1990).

Estimations on the number of meaningful plaintext presented in Section 8.2 are due to Hellman (1977). To guess the size of the keyword in case of polyalphabetic substitution cryptosystems one can use the so-called Kinski's method (see Salomaa (1990)).

For a more detailed description of DES see, for example, Salomaa (1990) and Schneier (1994). DES was developed from the encryption algorithm LUCIFER—see Feistel (1973). For a history of DES development see Smid and Branstead (1988). The proof that the cryptosystem DES is not composite is due to Campbell and Wiener (1992). $\mathcal{O}(e^{((\ln n)^{1/3}\ln\ln n)^{1/3}})$ algorithm to compute discrete logarithms is due to Adleman (1979).

For a detailed presentation of the public-key cryptography see Salomaa (1990), Brassard (1988) and also Schneier (1994) and also the survey papers by Rivest (1990), Brickel and Odlyzko (1988) and Diffie (1988). The last paper describes in details beginnings of the public-key cryptography.

The knapsack cryptosystem and its variants, including the dense knapsackand their cryptoanalysis are presented in Salomaa (1990). Chor (1986) is the basic reference on the dense knapsack. The whole story of knapsack cryptosystemis described in the book by O'Connor and Seberry (1987).

A detailed presentation and analysis of the RSA cryptosystem is in Salomaa (1990). Currently the fastest deterministic algorithm for primality testing is due to Adleman, Pomerance and Rumely (1983). Primality testing is discussed in details by Kranakis (1986). The second result of Theorem 8.3.24 is due to DeLaurentis (1984). The result that any polynomial time algorithm to determine one bit of cryptotexts encrypted by RSA can be transformed into a polynomial time algorithm to break RSA is due to Goldwasser, Micali and Tong (1982). In both cases see also Salomaa (1990) and Kranakis (1986) for a presentation of these results.

For the problem how to break RSA in case that several users use the same modulus see Salomaa (1990). LUC cryptosystem, discussed in Exercise 30, is due to Smith and Lennon (1993); see also Stallings (1995) for a presentation.

Basic results concerning relations between randomness and cryptography, cryptographically strong pseudo-random generators and randomized encryptions are presented by Rivest (1990). The result that a cryptographically strong pseudo-random generator exists if and only if a one-way function exists is implicitly in Yao (1982). The concepts of one-way predicate and polynomial time secure randomized encryption are due to Goldwasser and Micali (1984). Rabin's randomized cryptosystem is from Rabin (1979) and El Gamal cryptosystem from El Gamal (1985). Algorithm 8.4.5 is from Brassard (1988). For a general method to design cryptographically strong pseudorandom generators see Blum and Micali (1984). The randomized cryptosystem presented in Section 8.4 is due to Blum and Goldasser (1985).

The concept of the digital signature is due to Diffie and Hellman (1976) and it is discussed in details by Brassard (1988), Schneier (1994) and Mitchell, Pepper and Wild (1992). The DSS signature scheme was developed on the base of the signature schemes due to El Gamal (1985) and Schnorr (1991).

For quantum cryptography see Brassard (1988) and Bennett, Bessette, Brassard and Salvail (1992). The idea of quantum cryptography was born in the late 1960s (due to S. Wiesner). The first successful quantum exchange took place in 1989 and already in 1994 British Telecom announced the successful completion of a fully working prototype capable of implementing quantum key distribution along 10km of optical fibre.

Chapter 9

PROTOCOLS

INTRODUCTION

Attempts to prove security of cryptographical systems have brought a variety of important and deep concepts and methods with surprising practical and theoretical applications. With these concepts and techniques deals this chapter. At first some examples of cryptographical protocols are presented that solve apparently impossible communication problems. Corresponding abstract concepts of interactive protocols and proof systems are introduced that bring radicaly new view how to formalize one of the key concept of modern science—evidence. New views of interactions allow to see, in a sense, such powerful complexity classes as **PSPACE** and **NEXP** as representing class of problems having feasible solutions. The related concept of zero-knowledge proofs is also the base for dealing with perfect security of cryptographical protocols. One of the surprising applications of the ideas comming from formalizations of interactionsis in the area of program checking, testing and self-correcting—a radicaly new approach to these problems, based on randomness, is suggested.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. cryptographical protocols to solve apparently impossible communication problems;
2. basic concepts of interactive proof systems and basic complexity results showing enormous computational power of interactions;
3. the concept of zero-knowledge proofs and methods to design such proofs;
4. new randomized and interactive approach to program checking and self-correcting.

> Faith is the substance of things hoped for,
> the evidence of things not seen.
> *Hebrew II-I*

A variety of cryptographical primitives, operators and interactive protocols have been developed that allow two or more parties to develop trust that their communication, or coordination/cooperation, has desired properties in spite of the best effort of adversaries or untrusted parties. This allows to realize successfully a variety of important, though seemingly impossible, communication and cooperation tasks.

The attempts to achieve perfect secrecy, minimal disclosure of knowledge or a perfect protection of a cooperation in a large variety of communication and cooperation tasks, have also led to the development of a new methodology—the so-called interactive and zero-knowledge protocols. This has initiated development of a new approach to one of the most fundamental concepts of science—the evidence. Interactive, zero-knowledge, transparent, and other new types of proofs represent radically new ways how to formalize our intuitive concepts of evidence and security. This also brought new understanding of the power of interactions and randomness, with applications also in such seemingly remoted areas as approximation algorithms and program checking and self-correcting.

9.1 Cryptographical Protocols - November 8, 2005

Cryptographical protocols are specifications how parties should prepare themselves for a communication/interaction, and how they should behave during a communication/interaction, in order to achieve their goals and to be protected against adversaries. It is assumed that all parties involved in a protocol know the protocol and fully follow it. The parties can be friends that trust each other, or adversaries that do not trust each other. Cryptographical protocols often use some cryptographical primitives but goals of protocols are usually beyond a simple security. The parties participating in a protocol may want to share parts of their secrets to compute together a value; generate jointly random numbers, convince each other about their identity, simultaneously sign a contract or participate in a secret voting. Cryptographical protocols that accomplish such task have radically changed our views of what mutually distrusted parties can accomplish over a network.

Protocols can be described on two levels. On an abstract level, assuming the existence of basic cryptographical operators with certain security properties (secret keys, encryptions, decryptions, one-way or one-way trapdoor functions, pseudo-random generators, bit commitment schemes, and so on). On a lower level, with concentration on particular implementations of these operators. We concentrate here on the abstract level.

Randomization and interactions are two essential features of interactive protocols. In designing them it is assumed that each party has its own, private, independent source of randomness.

In order to show the potential of interactions we present first several examples of communication problems and protocols for **two party** and **multiparty communications**.

Example 9.1.1 *Let us consider the following simple protocol, that employs a public-key cryptosystem, for sending and acknowledging receipts of messages. (A and B stands there for strings identifying users.)*

1. *Alice sends the triple* $(A, e_B(w), B)$ *to Bob.*
2. *Bob decrypts* w *using his decryption algorithm and acknowledges the receipt of the message by sending back the triple* $(B, e_A(w), A)$.

Is the protocol in Example 9.1.1 secure? Actually, what kind of attacks are to consider in order to explore the problem of security of cryptographical protocols?

There are various types of attacksagains cryptographical protocols. In a **passive attack** the attacker tries to learn information being transmitted without altering the communication and the protocol; in an **active attack**, the atacker (**tamperer** or **man-in-the-middle**), destroys or changes information being transmitted or starts to send and receive messages. In the case the attacker is one of the parties involved we speak about a **cheater**.

For example, in case of the protocol from Example 9.1.1 an active eavesdroper C may intercept the triple being sent in Step 1 and forward to Bob the triple $(C, e_B(w), B)$. Without realizing the danger, Bob responds, following the protocol, by sending $(B, e_C(w), C)$ and now C is able to learn w.

Exercise 9.1.2 *Consider the following communication protocol in which Alice and Bob use a public-key cryptosystem, with the encryption and decryption functions operating on integers, to send a message* w.

1. *Alice sends Bob the pair* $(e_B(e_B(w)A), B)$.
2. *Bob uses his decryption algorithm* d_B *to find* A *and* w *and acknowledges the receipt of the message by sending Alice the pair* $(e_A(e_A(w)B), A)$.

A *and* B *are here strings identifying Alice and Bob.* $e_B(w)A$ *is the message obtained by concatenating* $e_B(w)$ *and* A. *Show how an active tamperer can intercept this protocol to learn* w.

Our first problem is a variant of the identification problem.

Protocol 9.1.3 (Friend-or-foe identification) *Alice, sharing a cryptosystem and a secret key with Bob, is engaged in a communication with somebody and wants to be sure that the person she communicates with is really Bob. To verify that, Alice uses the following* **challenge-response** *protocol:*

1. *Alice generates a random integer* r *and sends* r *to the communicating party.*

2. *The communicating party encrypts r using the shared secret key and returns the resulting cryptotext c.*
3. *Alice compares the cryptotext c with the one she gets by her encryption of r. If they agree, she gets convinced that the other party is indeed Bob.*

This protocol seems to be more secure than asking the other party to sent the shared key—an active tamperer could intercept it and later pretend to be Bob. However, it can not be used with the ONE-TIME PAD cryptosystem. (Explain why!)

Example 9.1.4 (Man-in-the-middle attack) *To protect a communication against an active tamperer is one of the hard problems of cryptography. Here is a simple way how a tamperer, called usually Mallet (or Mallory), can simulate Bob when communicating with Alice and Alice when communicating with Bob. His attacks work as follows:*

1. *Alice sends Bob her public key. Mallet intercepts the message and sends instead of it his public key to Bob.*
2. *Bob sends Alice his public key. Mallet intercepts and sends Alice his public key.*

Now whenever Alice sends a message to Bob, encrypted in "Bob's" public key, Mallet intercepts it and since the message is actually encrypted using his public key, he can decrypt it, change, and send to Bob using his public key. In a similar way Mallet can intercept and change messages sent by Bob to Alice.

Exercise 9.1.5 *The so-called* **interlock protocol** *has a good chance of fooling the man-in-the-middle attacker. Here are its first four steps:*

1. *Alice sends Bob her public key.*
2. *Bob sends Alice his public key.*
3. *Alice encrypts her message using Bob's public key and sends half of the encrypted message to Bob.*
4. *Bob encrypts his message using Alice's public key and sends half of the message to Alice.*

Finish the design of the protocol in such a way that Mallet can not get messages Alice and Bob communicate to each other. Explain why.

Bit commitment problem. Two parties, located far apart, want to agree alone, without the assistance of a trusted referee, on a randomly chosen bit. More exactly, Bob wants that **Alice chooses a bit and gets committed to it** in the following sense: Bob has no way to know what Alice has chosen, and Alice has no way to change her commitment once she made it—say after Bob announces his guess what has Alice chosen.

This is a very basic communication problem with many applications. For example, two parties, located far apart, want to agree on a random sequence. Popularly is this problem called the **coin-flipping over the telephon problem**. It was formulated by Manuel Blum (1981) with the following sad story behind. Alice and Bob got divorced, they do not trust each other any more, and want to decide, communicating only by telephone, by coin tossing, who gets the car. There are various protocols to achieve it. Two of them will now be considered.

Protocol 9.1.6 (Coin flipping by telephone - I) *Alice sends Bob encrypted messages "head" and "tail". Bob, not able to decrypt them, picks one of them and inform about his choice Alice. Alice then sends Bob the encryption procedure.*

There is a general scheme, that seems to be good enough for a protocol to solve the coin flipping by telephon problem, based on the assumption that both Alice and Bob knows a one-way function f. Alice chooses a random x and sends $f(x)$ to Bob. He guesses some "50-50 property" of x, for example, whether x is even, and informs Alice about his guess. She tells him whether the guess was correct. (Later, if necessary for some reasons, she can send x to Bob.)

Is this protocol secure? Can some of them cheat? The protocol looks secure because Bob has no way to determine x from $f(x)$. However, the situation is actually more complicated. Security of the above protocol depends much on which of the potential one-way functions is chosen. An analysis of this protocol shows well difficulties one can encounter in making communication protocols secure.

Indeed, it could happen that Alice knows two x_1 and x_2 such that $f(x_1) = f(x_2)$ and one of them is even and second odd! In such a case Alice could easily cheat! Also Bob could cheat would he be able to find out the parity of x. (Note that the fact that he cannot determine x does not imply that he can not determine the parity of x.)

The following secure protocol for the bit commitment problem is based on the fact that computation of square roots modulo the product of two primes is a trapdoor function.

Protocol 9.1.7 (Coin flipping by telephone - II)

1. *Alice chooses two large primes p, q, sends Bob $n = pq$, and keeps p, q secret.*
2. *Bob chooses a random number $y \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and sends Alice $x = y^2 \bmod n$.*
3. *Alice computes four square roots $(x_1, n - x_1)$ and $(x_2, n - x_2)$ of x. (Alice can compute them because she knows p and q.) Let $x'_1 = \min\{x_1, n - x_1\}$, $x'_2 = \min\{x_2, n - x_2\}$. Since $y \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor)$, either $y = x'_1$ or $y = x'_2$. Alice then guesses whether $y = x'_1$ or $y = x'_2$ and tells Bob her choice (for example, by reporting the position and the value of the leftmost bit in which x'_1 and x'_2 differ).*
4. *Bob tells Alice whether her guess was correct (head) or not correct (tail).*

Later, if necessary, Alice can reveal p and q, and Bob can reveal y.

Observe that Alice has no way to know y and so her guess is a real one. Would Bob be able to cheat by changing the number y after Alice's guess, then he would have both x_1' and x_2' and therefore he could factorize n. To avoisd this, Alice tells in Step 3 only one bit, rather than entire x_1' or x_2'.

Exercise 9.1.8 * *Consider the following protocol for the bit commitment problem: (1) Alice randomly chooses large primes* p, q, *computes* $n = pq$, *chooses* $x \in_{\mathcal{U}} \mathbf{Z}_n^*$, *computes* $y = x^2 \bmod n, z = y^2 \bmod n$ *and sends Bob* n, *and* z. *(2) Bob announces his guess; that is whether* y *is even or odd. (3) Alice let Bob to know* x, y *and Bob verifies that* $y = x^2 \bmod n, z = y^2 \bmod n$. *Is this protocol correct and secure? If not how to change it so it gets secure?*

There are also solutions to the bit commitment problem based on the laws of quantum physics that remain secure even if both parties have unlimited computational power.

Partial disclosure of secrets. There are k parties, $P_1, \ldots, P_k$ and they are to compute the value of a function f of k arguments. Assume that the party P_i knows only the i-th argument a_i. The task is to design a protocol that allows the parties to compute together $f(a_1, \ldots, a_k)$ in such a way that at the end of the communication each party P_i knows the value $f(a_1, \ldots, a_k)$, but no party gives away any information whatsoever concerning his/her argument a_i, except that information one can learn knowing only a_i and $f(a_1, \ldots, a_k)$.

There are two popular variants of the problem. Two millionaires want to engage in a conversation that allows them to find out who is richer (what is an understandable wish), without disclosing any information about their wealth (what is even a more understandable wish). Another variant: Alice and Bob wants to find out who is older without disclosing any other information about their age.

The following protocol, based on an arbitrary public-key cryptosystem, in which we assume, though even this may not be very realistic, that neither Alice nor Bob are older than 100 years, solves the problem. In the following we assume that e_A is the public encryption key of Alice and d_A her secret decryption key. Assume also that i is the age of Alice and j of Bob.

Protocol 9.1.9 (Age difference finding)

1. *Bob chooses a large random integer* x, *computes* $k = e_A(x)$ *and sends Alice* $s = k - j$.
2. *Alice first computes numbers*

$$y_u = d_A(s + u), \quad \textit{for} \quad 1 \le u \le 100,$$

then chooses a large random prime p, *computes numbers*

$$z_u = y_u \bmod p, \quad \textit{for} \quad 1 \le u \le 100, \tag{9.1}$$

and verifies that for all $u \neq v$

$$|z_u - z_v| \geq 2 \quad \text{and} \quad z_u \neq 0. \tag{9.2}$$

If this is not the case Alice chooses a new p and repeats computations in (9.1) and checks (9.2) again. Finally, Alice sends Bob the following sequence (the order is important!):

$$z_1, \ldots, z_i, z_{i+1} + 1, \ldots, z_{100} + 1, p. \tag{9.3}$$

3. *Bob checks whether the j-th number in this sequence, z'_j, is congruent to $x \bmod p$. If yes, he knows that $i \geq j$; otherwise he knows that $i < j$ and informs Alice.*

Correctness of the protocol: Because of the condition (9.2), any two of the first 100 numbers in the sequence (9.3) are different. Moreover,

$$\begin{array}{lll} i \geq j & \Rightarrow & z'_j = z_j \equiv y_j = d_A(k) = x \pmod p; \\ i < j & \Rightarrow & z'_j = z_j + 1 \not\equiv z_j \equiv y_j = d_A(k) = x \pmod p. \end{array}$$

Exercise 9.1.10 *Illustrate the age difference finding protocol by considering the RSA cryptosystem with $p = 11$, $q = 5$, $e = 7$, $d = 23$. Assume that the maximum age is 21 and Bob chooses $x = 39$ in Step 1.*

Example 9.1.11 *Of interest and importance is also an "inverse" problem to that of partial disclosure of secrets—a* **secret sharing problem**. *A "secret" $s \in \mathbf{N}$ should be "distributed" among n parties in such a way that any t of them, for a fixed t, can determine s but no $t-1$ of them are able to do that.*

The problem has a simple and elegant solution that is based on the well-known fact that a polynomial of degree $t-1$ is uniquelly determined given its t points.

A solution of the above secret sharing problem: choose randomly a polynomial p of degree $t-1$ such that $p(0) = s$. Choose randomly n integers $a_1, \ldots, a_n$, with $a_i \neq a_j$ for $i \neq j$ and such that $a_i \neq 0$ for all i, and sent to the party a_i the pair $(a_i, p(a_i))$.

There are many other interesting communication problems for two parties that has been solved by designing secure communication protocols. For example, **contract signing** (both parties signs the contract but no party obtains the other's party signature before giving up his own), **secret voting** (in which every voter's vote remains private and every voter can be sure that vote-counting was correct).

Various communication primitives have also been identified and protocols for them developed. They can now be used to design protocols for other communication problems. Bit commitment problem is one of them. The following, strangely looking communication problem was shown to be a sufficiently strong primitive to design protocols for **any** two party communication problem.

Oblivious transfer problem. Design a protocol for sending a message from Alice to Bob in such a way that Bob receives the message with probability $\frac{1}{2}$ and garbage with the probability $\frac{1}{2}$. Moreover, at the end Bob knows whether he got message or garbage, but Alice has no idea which of them has Bob received.

One popular interpretation is that Alice knows a secret and wants to sent it to Bob in such a way that he gets it with probability $\frac{1}{2}$, and he knows whether he got it, but Alice has no idea whether he really received it. A modified version of this problem is that Alice has several secrets and Bob wants to buy one of them in such a way that only he knows which one he has received. (Find out applications!)

The following protocol is based on the fact, see Theorem 1.8.15, that knowledge of two different square roots of an integer modulo n allows to factor n.

Protocol 9.1.12 (Oblivious transfer)

1. Alice chooses two large primes p and q and sends $n = pq$ to Bob.
2. Bob chooses a number x and sends $y = x^2 \bmod n$ to Alice.
3. Alice computes four square roots $\pm x_1, \pm x_2$ of y modulo n, and sends one of them to Bob. (Since she knows factorization of n she can do it. She has, however, no idea which of them is x.)
4. Bob checks whether the number he got is congruent to x. If yes, he has received no new information. Otherwise, Bob has two different square roots modulo n and can therefore factor n. Alice has no way of knowing whether this is the case.

Exercise 9.1.13 * *Alice knows secrets $s_1, \ldots, s_k$—each a sequence of bits—that are answers to important questions. Assume that Alice publicized these questions and Bob wants to get one of the secrets but he does not want that Alice knows which one he got. Design a protocol that achieves such a secretbuying of secrets and that is secure agains a passive adversary. (Hint: In the first step Alice sends Bob a trapdoor one-way function but keeps the trapdoor information necessary to compute its inverse secret.)*

Exercise 9.1.14 ** *There is another protocol for selling secrets $s_1, \ldots, s_k$ in which a commutative cryptosystem is used and the first two steps are*

1. *Bob sends Alice random bit sequences $y_1, \ldots, y_k$ of the same length as the secrets $s_1, \ldots, s_k$.*
2. *Alice sends Bob the bit sequence $z_j = e_A(s_j \oplus y_j)$, $j = 1, \ldots, k$.*

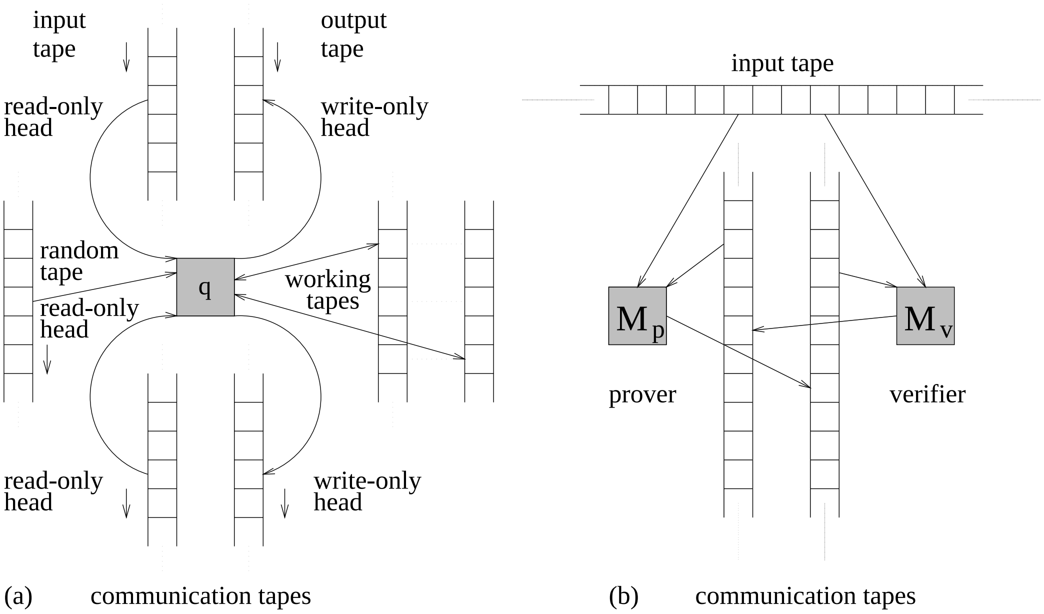
Finish the design of the protocol and analyze its security.

To show the existence of powerful methods to prove security of cryptographic protocols, several new fundamental concepts concerning interactions, protocols and proofs have been developed, with impacts far extending the original motivation. With these new concepts and related results we deal in the next sections.

Exercise 9.1.15 (Poker playing by telephone) *Alice and Bob want to play a simple variant of poker, with 5 cards, by phone, without any third party acting as an impartial judge. Both of them have to their disposal a commutative cryptosystem in which neither decryption nor encryption functions are published. Show how Alice and Bob can play so that the following conditions are satisfied: (a) All hands (sets of 5 cards) are equally likely. (b) The hands of Alice and Bob are different. (c) Both players know their own hands but have no information about the opponents' hands. (d) It is possible for each player to find out eventual cheating of the other player.*

9.2 Interactive Protocols and Proofs

Very often, when new fundamental ideas are developed in computing and basic problems investigated, Turing machines are the proper framework to set up the corresponding formal concepts. Formalization of the ideas concerning security and evidence, motivated by cryptographical problems and solutions, is also such a case.



(a) communication tapes (b) communication tapes

Figure 9.1: An interactive Turing machine and an interactive protocol 60%

Definition 9.2.1 *An* **interactive Turing machine** *(ITM) is a Turing machine with a one-way read-only* **input tape***, a read-only* **random tape***, a one-way read-only* **communication tape***, a one-way write-only* **communication tape***, a one-way write-only* **output tape** *and several* **working tapes** *(see Figure 9.1a).*

At the beginning of each computation of an ITM, the random tape is expected to be filled with a new random infinite sequence of bits. (Coin-tossing in an ITM is simulated by reading the next bit from its (private) random tape.) The content of the read-only (write-only) communication tape is considered as the message the machine received (sends). An interactive Turing machine is a building block for the following formal model of interactions.

Definition 9.2.2 *An* **interactive protocol** *is a pair (M_P, M_V) of interactive Turing machines that have a common input tape and share communication tapes as follows: the communication tape that is the read-only (write-only) tape for one of the machines is the write-only (read-only) tape for the other machine. M_P is usually called the* **prover** *or* **Peggy** *and M_V is called the* **verifier** *or* **Vic***.*

In an interactive protocol the verifier and the prover work in turns, called also **moves**. The verifier starts an interaction. At any moment of the discrete time only one of the machines is active. When one of them is active, it performs, on the contents of its accessible tapes, usual Turing machine computations. What one of the machines writes on its write-only communication tape is considered as the message for the other machine. The i-th message of M_V (of M_P) is the message M_V (M_P) writes on its communication tape during its i-th move. Any two of the consecutive moves: verifier → prover → verifier or prover → verifier → prover, are called a **round**. A machine terminates computation by sending no message. M_V accepts (rejects) the initial contents of the input tape when M_V ends the computation in the accepting (rejecting) state ACCEPT (REJECT)—this is its output. The **history** of a computation (communication) is the sequence $h_n = (v_1, p_1, \ldots, v_n, p_n)$, where v_i (p_i) denotes the message sent from the verifier (the prover) during its i-th move and $p_n = \varepsilon$.

Both M_V and M_P are probabilistic Turing machines in the sense that the way their computation proceed depends may depend also on contents of their random tapes. The prover has unlimite computational power. The verifier has to work in polynomial time with respect to the length of the input—the sum of computation times during all active steps of the verifier is counted. This, of course, implies that for the prover it has a sense to write only polynomially long messages on the common communication tape—longer messages the verifier could not even read in the polynomial time.

Let $(M_P, M_V)(x)$ denote the verifiers output for an input x. For every x, $(M_P, M_V)(x)$ is a random variable on the space of all possible computations of the protocol (M_V, M_P) with respect to private random bits.

For any interactive protocol (M_P, M_V), we denote by (M_{P^*}, M_V) the interactive protocol in which the verifier M_V behaves exactly as in the original protocol, but the prover computes the function specified by M_{P^*}—this is said to be a "cheating prover" if M_{P^*} computes differently as M_P. Similarly, we denote by (M_P, M_{V^*}) the interactive protocol in which the prover behaves exactly as in the original protocol but the (potentially cheating) verifier can behave differently. The potentially cheating prover is limited only in that the messages she sends must have polynomially bounded size, with respect to the length of input; the potentially cheating verifier is limited to randomized polynomial time computations.

9.2.1 Interactive Proof Systems

Our formal concept of an interactive protocol, expressed in terms of Turing machines, captures an intuitive idea of interactions and forms a base to investigate the power of interactions.

As usually, when a new concept of computation is developed, the most basic problem to investigate is how powerful is new concept for solving decision problems. This is usually formalized in terms of language recognition. In case of interactive protocols this leads to the following definition.

Definition 9.2.3 *The interactive protocol* (M_P, M_V) *recognizes the language* L *if, for all* $x \in L$,

$$Pr((M_P, M_V)(x) = ACCEPT) > \frac{2}{3};$$

and, for all $x \notin L$ *and all provers* M_{P^*}

$$Pr((M_P^*, M_V)(x) = REJECT) > \frac{2}{3}.$$

If there is an interactive protocol for a language L*, we say also that* L *has an interactive proof system.*

There are several ways how to view such an interactive protocol.

1. The goal of any prover can be seen as to convince the verifier that $x \in L$, no matter whether this is true. The goal of the verifier is that no prover can convince him that $x \in L$ if this is not true or that $x \notin L$ if this is not true. However, he is willing to tolerate a small probability of getting fooled.

2. The term "proof system" is used here in the following sense: By a "theorem" we understand here a truth statement concerning specific objects. For example, that a certain x is in a certain language L or that a specific graph G_0 is 3-colorable, or that it is hamiltonian.

   A classical formal proof of a concrete statement, for example $x_0 \in L_0$ for a specific x_0 and L_0, is a formal reasoning showing the validity of such a theorem that can be verified (even mechanically) without doubts. An interactive proof system for a particular language L_0 can be used for any specific x to produce statistical evidence whether $x \in L_0$ or not. More times we apply the proof system to the same x more sure we can be whether $x \in L_0$, or not.

As an example, we show now a simple and fast interactive proof system for computationally a very hard problem, the graph non-isomorphism problem, that is not known to be in **NP**.

$$\begin{aligned} NONISO \quad = \quad & \{(G_0, G_1) \mid G_0 = \langle V, E_0 \rangle \text{ and } G_1 = \langle V, E_1 \rangle \\ & \text{are non-isomorphic graphs, i.e. } \ G_0 \not\cong G_1\}. \end{aligned}$$

Protocol 9.2.4 (Graph non-isomorphism) 1. *The verifier, Vic, reads the input* (G_0, G_1), *and* s *bits* $b_1, \ldots, b_s$ *from his random tape—*s *is here a security parameter that guarantees, as shown later, a certain probability of the correctness of the outcome. Vic then designs, for* $1 \leq i \leq s$, *a graph* $H_i = \langle V, F_i \rangle$, *by deriving* F_i *from* E_{b_i} *by taking a random permutation of* V *and sends all* H_i *to the prover, Peggy, asking her to determine, for all* $1 \leq i \leq s$, *which of the two given graphs,* G_0 *and* G_1 *was used to design* H_i. *(Observe that* $G_{b_i} \cong H_i$.*)*

2. *Peggy also reads the input* (G_0, G_1) *and checks first whether these two graphs are isomorphic. (Since she has unlimited computational power, this is no problem for her.) If* G_0 *and* G_1 *are isomorphic, Peggy reads next* s *bits* $c_i, i = 1, \ldots, s$, *from her private random tape and sends them to Vic as the answer (without even looking at the inputs* H_i *she got from Vic—it has no sense for her to do it, because she has no way to find out which of the graphs* G_0 *or* G_1 *was used to design* H_i*). If* (G_0, G_1) *are not isomorphic, Peggy verifies, for each* $1 \leq i \leq s$, *whether* $H_i \cong G_0$ *or* $H_i \cong G_1$ *and reports the results to Vic as a sequence of* s *bits* $c_i, 1 \leq i \leq s$, *where* $c_i = 0$ *if and only if* $G_0 \cong H_i$.

3. *Vic compares* b_i *and* c_i *for all* i, $1 \leq i \leq s$. *If there is an* i_0 *such that* $b_{i_0} \neq c_{i_0}$, *then Vic rejects the input* (G_0, G_1), *otherwise accepts.*

The correctness proof. If $G_0 \not\cong G_1$, then Peggy can determine whether $H_i \cong G_0$ or $H_i = G_1$ and therefore $b_i = c_i$ for $1 \leq i \leq n$.

If $G_0 \cong G_1$, what Peggy can find out, there is no way for Peggy to know b_i and therefore the probability that $c_i = b_i$ is $\frac{1}{2}$. The probability that in case $G_0 \cong G_1$ there is no i such that $b_i \neq c_i$ is 2^{-s}. □

Exercise 9.2.5 * *Design an interactive proof system for the language* $Q = \{(x, y) \,|\, y \in QNR_x\}$ *in which the verifier can recognize* Q *using only one interaction with the prover. (Hint: For* $|x| = n$ *Peggy gets* n *random bits* $b_1, \ldots, b_n$ *and* n *random integers* $z_1, \ldots, z_n$, $0 < z_i < x$, $gcd(x, z_i) = 1$, ...*)*

In the following we address four basic questions about interactive protocols.

1. Which languages have interactive proofs?
2. How many interactions is needed for the prover to convince the verifier?
3. How much knowledge does the verifier get during the interaction with the prover?
4. How can we apply the idea of interactive proof system?

9.2.2 Interactive Complexity Classes and Shamir's Theorem

Protocol 9.2.4 is very simple; it "proves" in two rounds. The number of rounds, as we shall see, is an important complexity measure of proof systems. The following definition brings the basic concepts.

Definition 9.2.6 *1. Let $g : \mathbf{N} \mapsto \mathbf{N}$ be a nondecreasing function. An interactive proof system (M_P, M_V) for a language $L \subseteq \{0,1\}^*$ has a* **round complexity** *bounded by g if, for every $x \in L$, the number of rounds of any computation of (M_P, M_V) on x is bounded from above by $g(|x|)$.*

2. Denote by $IP[g(n)]$ be the family of languages that have an interactive proof system with the round complexity bounded by g and denote by

$$\mathbf{IP} = \bigcup_{i \geq 0} IP[n^i]$$

the family of languages acceptable by interactive proof systems with a polynomial number of rounds (interactions)—with respect to the length of input.

For instance, $\mathbf{IP[2]}$ is the class of languages for which there exists an interactive proof system of the following type. On an input x the verifier flips the coin, makes on the base of the outcome some computations and sends a message to the prover. The prover, after doing whatever he does, sends a message back to the verifier—this is the first round. The verifier again flips the coin, computes and accepts or rejects the input—the second round. Observe that for languages in $\mathbf{NP}$ there are certificates verifiable in polynomial time and therefore $\mathbf{NP} \subseteq \mathbf{IP[2]}$—the prover sends a cerificate and the verifier verifies it. On the other hand, clearly $\mathbf{IP[1]} = \mathbf{BPP}$.

Basic relations between these new and some old complexity classes are ready to see.

Theorem 9.2.7 *For any nondecreasing function $g : \mathbf{N} \mapsto \mathbf{N}$, with $g(n) \geq 2$, for all n, and any $1 \leq n$:*

$$\mathbf{P} \subseteq \mathbf{NP} \subseteq IP[2] \subseteq IP[g(n)] \subseteq IP[g(n+1)] \subseteq \ldots \subseteq \mathbf{IP} \subseteq \mathbf{PSPACE}$$

The validity of all except the last inclusion is trivial. Concerning the last inclusion, this is also fairly easy. Indeed, any interactive protocol with polynomially many rounds can be simulated by a PSPACE bounded machine traversing the tree of all possible interactions. (No communication between the prover and the verifier requires more than polynomial space.)

The basic problem is now to determine how powerful are the classes $IP[k], IP[n^k]$ and especially the class $\mathbf{IP}$ and what relations hold between them and with respect to other complexity classes. Observe that the graph non-isomorphism problem, that is not known to be in $\mathbf{NP}$, is already in IP[2]. We concentrate now on the power of the class $\mathbf{IP}$.

Before proving the first main result of this section—Theorem 9.2.11—we present the basic idea and some examples of the so-called "sum protocols". These protocols can be used to make the prover to compute computationally unfeasible sums and to achieve that the verifier can get convinced, by the overwhelming statistical evidence, about their correctness.

The key probabilistic argument used in these protocols concerns the roots of polynomials. If $p_1(x)$ and $p_2(x)$ are two different polynomials of degree n, and α is a randomly chosen integer in the range $\{0, \ldots, N\}$, then

$$Pr(p_1(\alpha) = p_2(\alpha)) \leq \frac{n}{N}, \tag{9.4}$$

because the polynomial $p_1(x) - p_2(x)$ has at most n roots.

Example 9.2.8 (Protocol to compute permanent) *The first problem we deal with is that of computing the permanent of a matrix* $M = \{m_{i,j}\}_{i,j=1}^{n}$, *i.e.*

$$perm(M) = \sum_{\sigma} \prod_{i=1}^{n} m_{i,\sigma(i)},$$

where σ *goes through all permutations of the set* $\{1, 2, \ldots, n\}$. *(As already mentioned in Chapter 4, there is no polynomial time algorithm known to compute the permanent.)*

In order to explain the basic idea of an interactive protocol to compute $perm(M)$, *let us first consider a "harder problem" and assume that the verifier needs to compute permanents of two matrices* A, B *of degree* n. *The verifier asks the prover to do it and the prover, with unlimited computational power, sends to the verifier two numbers* ρ_A *and* ρ_B *claiming that* $\rho_A = perm(A)$ *and* $\rho_B = perm(B)$. *The basic problem now is how can the verifier get convinced that the values* ρ_A *and* ρ_B *are correct? He can not do that by direct calculations—this is computationally unfeasible for him. The way out is that the verifier starts an interaction with the prover in such a way that the prover will be forced to make, with large probability, sooner or later, a false statement, easily checkable by the verifier, in case the prover cheated in one of the values* ρ_A *and* ρ_B. *Here is the basic trick:*

Consider the linear function $D(x) = (1-x)A + xB$ *in the space of all matrices of degree* n. $perm(D(x))$ *is then clearly a polynomial, say* $d(x)$, *of degree* n *and such that* $d(0) = perm(A)$ *and* $d(1) = perm(B)$.

Now comes the main idea. The verifier asks the prover to sent him $d(x)$. *The prover does it. However, in case the prover cheated on* ρ_A *or* ρ_B, *the prover has to cheat also on coefficients of* $d(x)$*—otherwise the verifier could immediately find out that either* $\rho(A) \neq d(0)$ *or* $\rho(B) \neq d(1)$. *In order to catch the prover, in case of cheating, the verifier chooses a random number* $\alpha \in \{0, \ldots, N\}$, *where* $N \geq n^3$, *and asks the prover to send him* $d(\alpha)$. *In case the prover cheated, either on* ρ_A *or on* ρ_B, *the chance that the prover sends the correct values of* $d(\alpha)$ *is, by (9.4), at most* $\frac{n}{N}$.

In a similar way, given k *matrices* $A_1, \ldots, A_k$ *of degree* n, *the verifier can design a single matrix* B *of degree* n, *such that if the prover has cheated on at least one of the values* $perm(A_1), \ldots, perm(A_k)$, *then the prover will have to make, with large probability, a false statement also about* $perm(B)$.

Let now A *be a matrix of degree* n, *and* $A_{1,i}, 1 \leq i \leq n$, *submatrices obtained from* A *by deleting the first row and the* i*-th column. In such a case*

$$perm(A) = \sum_{i=1}^{n} a_{1,i} perm(A_{1,i}). \tag{9.5}$$

Communication in the interactive protocol now goes as follows. The verifier asks for the values $perm(A)$, $perm(A_{1,1}), \ldots, perm(A_{1,n})$ *and uses (9.5) as the first consistency check. Would the prover cheat on* $perm(A)$, *she would have to cheat also on at least one of the values* $perm(A_{1,1}), \ldots, perm(A_{1,n})$. *The verifier can now, using the idea presented above, choose a random number* $\alpha \in \{0, \ldots, N\}$, *and design a single matrix* A', *of the degree* $n-1$, *such that if the prover cheated on perm*(A)*, she would have to cheat, with large probability, also on perm*(A')*. Now the interaction continues in an analogical way, designing matrices of smaller and smaller degree, and such that would the prover cheat on perm*(A) *she would have to cheat also on permanents of all these smaller matrices, until such a small matrix is designed that the verifier is capable to compute directly its permanent and to get this way convinced about the correctness (or incorrectness) of the first value sent by the prover. The probability that the prover can succeed in cheating without being caught, is smaller than* $\frac{n^2}{N}$*, and therefore negligible small, if* N *is large enough. (Notice also, that in this protocol the number of rounds is not bounded by a constant; it depends on the degree of the matrix.)*

Example 9.2.9 *We demonstrate now the basic ideas of the interactive protocol for the so-called* $\#SAT$ *problem. It is the problem to determine the number of satisfying assignments to a Boolean formula* $F(x_1,\ldots,x_n)$ *of* n *variables.*

As the first step, using the arithmetization:

$$x \wedge y \rightarrow xy, \quad x \vee y \rightarrow 1-(1-x)(1-y), \quad \bar{x} \rightarrow 1-x, \tag{9.6}$$

(see Section 2.3.2), a polynomial $p(x_1,\ldots,x_n)$ *approximating* $F(x_1,\ldots,x_n)$ *can be constructed in linear time (in length of* F*), and the problem gets reduced this way to that of computing the sum*

$$\#SAT(F) = \sum_{x_1=0}^{1}\sum_{x_2=0}^{1}\cdots\sum_{x_n=0}^{1} p(x_1,\ldots,x_n). \tag{9.7}$$

For example, if $F(x,y,z) = (x \vee y \vee z)(x \vee \bar{y} \vee z)$*, then*

$$p(x,y,z) = (1-(1-x)(1-y)(1-z))(1-(1-x)y(1-z)).$$

We show now the first rounds of the protocol that reduces computation of the expression of the type (9.7) with n *sums to a computation of another expression of a similar type, but with* $n-1$ *sums. The overall protocol then consists of* $n-1$ *repetitions of such rounds.*

The verifiers' aim is again to get from the prover the resulting sum (9.7). Therefore, the verifier asks the prover for the resulting sum w *of (9.7), but also for the polynomial*

$$p_1(x_1) = \sum_{x_2=0}^{1}\cdots\sum_{x_n=0}^{1} p(x_1,\ldots,x_n).$$

The verifier then first makes the consistency check, i.e., whether $w = p_1(0)+p_1(1)$*, then chooses a random* $r \in \{0,\ldots,N\}$*, where* $N \geq n^3$*, and starts another rounds, the task of which is to get from the prover the correct value of* $p_1(r)$ *and an evidence that the value supplied by the prover is correct. Observe, that the probability that the prover sends a false* w *but the correct* $p_1(r)$ *is at most* $\frac{n}{N}$*. After* n *rounds, either the verifier will catch the prover, or he get convinced, by the overwhelming statistical evidence, that* w *is the correct value.*

Exercise 9.2.10 *Show why using the arithmetization (9.6) we can always transform a Boolean formula in linear time into an approximating polynomial, but that can not do done in general in linear time if the arithmetization* $x \vee y \rightarrow x + y - xy$ *is used.*

We are now in the position to prove an important result that brings a new view on what can be seen as computationally feasible:

Theorem 9.2.11 (Shamir's Theorem) IP = PSPACE

Proof. Since **IP** $\subseteq$ **PSPACE** (see Theorem 9.2.7), in order to prove the theorem it is sufficient to show that there exists a **PSPACE**-complete language that is in **IP**, i.e. the language for which there is an interactive proof system with the number of rounds bounded by a polynomial.We show that this holds for the so-called Quantified Boolean Formulas Satisfiability Problem (see also Section 5.11.2). This is the problem to decide whether a formula

$$\underset{x_1}{Q}\underset{x_2}{Q} \ldots \underset{x_n}{Q}\; F(x_1, \ldots, x_n), \tag{9.8}$$

is valid, where F is a Boolean formula, and each Q_i is either an existential or a universal bounded quantifier—bounded to the values 0 and 1.

The basic idea of the proof is simple: to use an arithmetization to reduce the decision problem (9.8) to a "sum problem" and then to use a "sum protocol" described above.

Unfortunately, there is a problem with this idea. A "natural arithmetization" of the quantifiers, namely,

$$\forall x T(x) \rightarrow T(0)T(1), \qquad \exists x T(x) \rightarrow T(0) + T(1) - T(0)T(1),$$

can double, for each quantifier, the size of the corresponding polynomial, and can therefore produce formulas of an exponential size—and they can be, therefore, "unreadable" for the verifier.

Fortunately, there is a trick to get around this exponential explosion. The basic idea consists in introducing new quantifiers, notation $\underset{x}{R}$. If the quantifier $\underset{x}{R}$ is applied to a polynomial p, it reduces all powers of x, x^i, to x—this is equivalent to taking $p \bmod (x^2 - x)$. Since $0^k = 0$ and $1^k = 1$ for any integer k, such a reduction does not change the values of the polynomial on the set $\{0, 1\}$. Instead of the formula (9.8), we then consider the formula

$$\underset{x_1}{Q}\underset{x_1}{R}\underset{x_2}{Q}\underset{x_1}{R}\underset{x_2}{R}\underset{x_3}{Q}\underset{x_1}{R}\underset{x_2}{R}\underset{x_3}{R}\underset{x_4}{Q} \ldots \underset{x_n}{Q}\underset{x_1}{R} \ldots \underset{x_n}{R}\; p(x_1, \ldots, x_n), \tag{9.9}$$

where $p(x_1, \ldots, x_n)$ is a polynomial approximation of F that can be obtained from F in linear time.

Note that the degree of p does not exceed the length of F, say m, and that, after each group or R-quantifier is applied, the degree of each variable is down to 1. Moreover, since the arithmetization of quantifiers $\exists$ and $\forall$ can at most double the degree of each variable in the corresponding polynomials, the degree of any polynomial obtained in the arithmetization process is never more than 2 in any variable.

The protocol consists of two phases. The first phase has the number of rounds proportional to the number of quantifiers in (9.9) and in each two rounds a quantifier is removed from the formula in (9.9).

The strategy of the verifier consists in asking, from the prover, in each round a number or a polynomial of one variable, of the degree at most two, in such a way that would the prover cheat once she would have, with a large probability, keep cheating until she gets caught. To make all computations reasonable, a prime P is choosen at the very beginning, and both the prover and the verifier have to perform all computations modulo P—it will be explained later how to choose P. The first phase of the protocol starts as follows:

1. Vic ask Peggy for the value w (0 or 1) of the formula (9.9). {A stripping of the quantifier $\underset{x_1}{Q}$ begins.}

2. Peggy sends w, claiming it is correct.

3. Vic wants to be sure, and therefore asks Peggy for the polynomial equivalent to

   $$\underset{x_1}{R}\underset{x_2}{Q}\underset{x_1}{R}\underset{x_2}{R}\underset{x_3}{Q}\underset{x_1}{R}\underset{x_2}{R}\underset{x_3}{R}\underset{x_4}{Q}\dots\underset{x_n}{Q}\underset{x_1}{R}\dots\underset{x_n}{R}\,p(x_1,\dots,x_n).$$

   {Remember, calculations are done modulo P.}

4. Peggy sends Vic a polynomial $p_1(x_1)$, claiming it is correct.

5. Vic makes the consistency check by verifying whether

   - $p(0)+p(1)-p(0)p(1)=w$, if the leftmost quantifier is $\exists$;
   - $p(0)p(1)=w$ if the leftmost quantifier is $\forall$.

   In order to get more sure that p_1 is correct, Vic asks Peggy for the polynomial equivalent (congruent) to

   $$\underset{x_2}{Q}\underset{x_1}{R}\underset{x_2}{R}\underset{x_3}{Q}\underset{x_1}{R}\underset{x_2}{R}\underset{x_3}{R}\underset{x_4}{Q}\dots\underset{x_n}{Q}\underset{x_1}{R}\dots\underset{x_n}{R}\,p(x_1,\dots,x_n).$$

6. Peggy sends a polynomial $p_2(x_1)$, claiming it is correct.

7. Vic chooses a random number α_{11} and makes a consistency check by computing $(p_2(x_1) \bmod (x_1^2-x_1))|_{\alpha_{11}} = p_1(\alpha_{11})$.

   In order to get more sure that p_2 is correct, Vic chooses a random α_{12} and asks Peggy for the polynomial equivalent to

   $$\underset{x_1}{R}\underset{x_2}{R}\underset{x_3}{Q}\underset{x_1}{R}\underset{x_2}{R}\underset{x_3}{R}\underset{x_4}{Q}\dots\underset{x_n}{Q}\underset{x_1}{R}\dots\underset{x_n}{R}\,p(x_1,\dots,x_n)|_{x_1=\alpha_{12}}.$$

8. Peggy returns a polynomial $p_3(x_2)$, claiming it is correct.

9. Vic checks as in Step 5.

$\vdots$

The protocol continues until either a consistency check fails or all quantifiers are stripped off. Then the second phase of the protocol beginns with the aim to determine the value of p for already chosen values of variables. In each round p can be seen as being decomposed either into $p'p''$ or $1-(1-p')(1-p'')$ and Vic asks Peggy for the values of the whole polynomial and its subpolynomials p' and p''.

Analysis. During the first phase, until $n+(n-1)n/2$ quantifiers are removed, the prover has to supply the verifier each time with a polynomial of the degree at most 2. Since each time the chance of cheating is at most $\frac{2}{P}$, the total chance of cheating is clearly less than $\frac{2m^2}{P}$. The number of rounds in the second phase, when the polynomial itself is shrunk, is at most m and the probability of cheating is at most $\frac{m}{P}$. Therefore, the total probability that the prover could fool the verifier is at most $\frac{3m^2}{P}$. Now it is clear how large P has to be chosen in order to get an overhelming statistical evidence. $\Box$

Theorem 9.2.11 actually implies that there is a reasonable model of computation within which we can see the whole class **PSPACE** as consisting of problems having feasible solutions. This has brought a significant change in the views what is "feasible".

9.2.3 The Tale of Proofs in a Nutshell

The history of the concept of proof, one of the most fundamental concepts not only of the science but of the whole civilization, is rich and interesting. Once developed as a key tool to the search for truth, it has then been developed into the key tool to achieve security.

There used to be a very different understandings what a proof means. For example, in the middle age, a proof "by authority"was common. Even mathematicians for a long time did not really bothered too much to put their basic tool on a firm base. "Go on, the faith will come to you" used to be a response to complains of purists for the lack of exactness.[1]

Mathematicians have long been convinced that a mathematical proof, when written out in all detail, can be checked unambiguously. Already Aristotle (384-322 BC) made attempts to formalize the rules of deduction. However, the concept of a formal proof, cheakable by a machine, was developed only at the beginning of the century by Frége (1848-1923) and Russel (1872-1970). It has been a major breakthrough and proofs "within ZF"—Zermello-Fränkel axiomatic system—got a standard for "working mathematicians".

Some of the problems with such a concept of proof were discussed in Chapter 6. Another practical, but also theoretical, difficulty lies in the fact that some proofs are too complicated to be understood the proof of the classification of all finite simple groups has about 15 000 pages), some proofs are provably unfeasible (a theorem of less than 700 symbols was found any proof of which is longer than the number of particles in the universe).

The concept of the interactive proofs has been another breakthrough in the proof history and has motivated development of several other fundamental concepts concerning proofs and also concerning unexpected applications. Sections 9.3 and 9.4 deal with two of them. Two other are now briefly discussed.

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[1] For example, Fermat has stated many theorems, but proved only few.

Interactive proofs with multiple provers

The first idea, theoretically very natural, was to consider interactions between one, polynomial time bounded, verifier and several, powerful, provers. This idea seemed at first as a pure abstraction, without a deeper motivation and applications. This has turned out to be deeply wrong.

The formal scenario goes as follows. The verifier and all provers are probabilistic Turing machines. The verifier is again required to do all computations in polynomial time. All provers have unlimited power. The provers can agree on a strategy before an interaction starts, but during the protocol they are not allowed to communicate among themselves. In one move the verifier sends messages to all provers, but each of them can read only the message addressed to her. Similarly, in one move all provers simultaneously send messages to the verifier and again none of them can learn messages sent by others. The acceptance conditions for a language L is similar as before: each $x \in L$ is accepted with the probability at least $\frac{2}{3}$ and each $x \notin L$ is accepted with the probability at most $\frac{1}{3}$. The family of languages accepted by interactive protocols with multiple provers and polynomial number of rounds is denote by **MIP**.

It is evident that it is meaningless to have more than polynomially many provers. Not only that. It has been shown that two provers are always sufficient. However, the second prover can significantly increase the power of interactions as the following theorem shows:

Theorem 9.2.12 MIP = NEXP.

The extraordinary power of two provers comes from the fact that the verifier can ask simultaneously questions to both provers and they have to answer independently without learning the answer of the second one. In other words, the provers are securely separated.

If we now interpret **NP** as the family of languages admitting efficient formal proof of membership (formal in the sense that a machine can verify it), then **MIP** can be seen as the class of languages admitting efficient proofs of membership by the overwhelming statistical evidence. In this sense **MIP** is like a "randomized and interactive version" of **NP**.

The result **IP = PSPACE** can also be seen, as asserting, informally, that via an interactive proof one can verify in polynomial time any theorem admitting exponentially long formal proof, say in ZF, as long as the proof could (in principle) be presented on a "polynomial-size blackboard". The result **MIP = NEXP** asserts, similarly, that with two infinitely powerful and securely separated provers one can verify in polynomial time any theorem admitting an exponentially long proof.

Transparent proofs and limitations of approximability

Informally, a formal proof is **transparent** or **holographic** if it can be verified, with large confidence, by a small number of spot-checks. This seemingly paradoxical concept, in which randomness again play the key rôle, has also turned out to be deep and powerful.

One of the main results says that every formal proof, say in ZF, can be rewritten in transparent format (proving the same theorem in a different proof system), without increasing the length of the proof too much.

There is also a surprising tradeoff between the length of transparent proofs and the number of proof-bits that need to be checked. For a transparent proof of length n (the length of the original proof) the number of bits of the proof that need to be checked is $(\lg n)^{\mathcal{O}(1)}$. Increasing length of the transparent proof to $\mathcal{O}(n)$, the number of check-bits can be reduced to a small constant (for example, to 36, in a reasonable setting). This implies that even theorems with very long proof have easily cheakable, randomized, proofs.

On a more practical side a surprising connection has been shown between randomized proofs and the problem of approximability of **NP**-complete problems. For example, nonapproximability has been shown for a large class of approximation problems.

9.3 Zero-knowledge Proofs

A special type of interactive protocols and proof systems are zero-knowlege protocols and proofs. For cryptography they represent an elegant way to show security of cryptographical protocols. On a more theoretical level zero-knowledge proofs represent a fundamentally new way how to formalize the concept of evidence. They allow, for example, to prove a theorem that no one can claim it.

Informally, a protocol is a zero-knowledge proof protocol for a theorem if one party does not learn from communication anything more except whether theorem is true or not.

Example 9.3.1 $670592745 = 12345 \times 54321$ *is not a zero-knowledge proof of the theorem "*670592745 *is a composite integer", because the proof reveals not only that the theorem is true, but also an additional information—two factors of* 670592745.

More formally, a zero-knowlege proof of a theorem T is an interactive two-party protocol with a special property. The prover, with unlimited power, is able, following the protocol to convince the verifier, who follows the same protocol, by the overwhelming statistical evidence, that T is true, if this is really so, but has almost no chance to convince a verifier that follows the protocol that the theorem T is true if this is not so. In addition, and this is essential, during their interactions the prover does not reveal to the verifier any other information, not a single bit, except the fact whether the theorem T is true, no matter what the verifier does. This means, that for all practical purposes whatever the verifier can do after interacting with the prover, he can do just believing that the claim the prover makes is valid. Therefore "zero-knowledge" is a property of the prover—her robustness against the attempts of any verifier, working in polynomial time, to extract some knowledge from an interaction with the prover.

In other words, a zero-knowledge proof is an interactive proof that provides highly convincing (but not absolutely certain) evidence that a theorem is true and the prover knows a proof (a standard proof in a logical system that can be in principle, but not necessarily in polynomial time, checked by a machine), while providing not a single additional bit of information about the proof. In particular, the verifier who just got convinced about correctness of a theorem by a zero-knowledge protocol, can not turn around and prove the theorem to somebody else without proving it from scratch for himself.

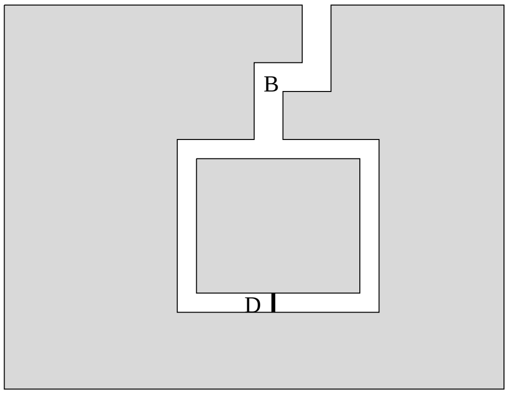


Figure 9.2: A cave with a door opening on a secret word 80%

Exercise 9.3.2 *The following simple problem has a simple solution that well illustrates the idea of zero-knowledge proofs. Alice knows a secret word that opens the door D in the cave on Figure 9.2. How can she convince Bob that she really knows this word, without telling it to Bob, when Bob is not allowed to see which path she takes going to the door and he is not allowed to go to the cave beyond the point B? (However, the cave is small and Alice can always hear Bob if she is in care and Bob is in position B.)*

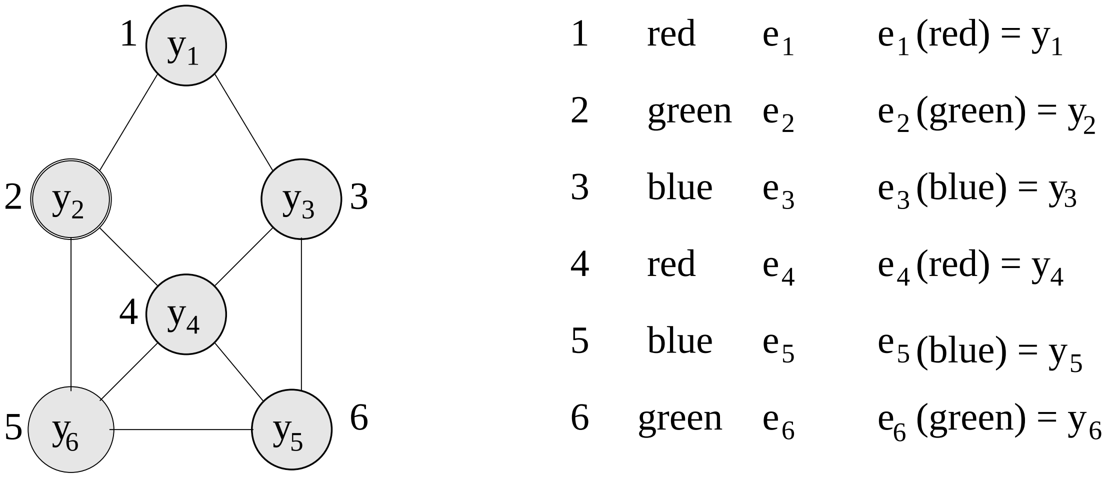
9.3.1 Examples

Using the following protocol Peggy can convince Vic that a particular graph G, which they both know, is colorable with three colors, say red, blue and green, and that Peggy knows such a coloring, without revealing to Vic any information whatsoever how such a coloring of G looks.

Protocol 9.3.3 (3-colorability of graphs)

Peggy colors $G = (V, E)$ with three colors in such a way that no two neighbouring nodes are colored by the same color. Then Peggy performs with Vic $|E|^2$-times the following interaction (where $v_1, \ldots, v_n$ are nodes of V):

1. Peggy chooses a random permutation of colours (red, blue, green), recolors correspondingly the graph and encrypts, for $i = 1, \ldots, n$, the color c_i of the node v_i by an encryption procedure e_i—for each i different. Then Peggy removes colors from nodes and labels the i-th node of G with the cryptotext $y_i = e_i(c_i)$ (see Figure 9.3a). Peggy then designs a table T_G in which, for every i, she puts the color of the node i, the corresponding encryption procedure for that node and the result of the encryption (see Figure 9.3b). Finally, Peggy shows Vic the graph with nodes labelled by cryptotexts (for example, the one from Figure 9.3a).



(a) (b)

Figure 9.3: Encryption of a 3-coloring of a graph 80%

2. Vic chooses an edge and sends Peggy request to show him the coloring of the corresponding nodes.

3. Peggy reveals to Vic the entries in the table T_G for both nodes of the edge Vic has required.

4. Vic performs encryptions to check that nodes really have colors as shown.

Vic accepts the proof if and only if all his checks agree.

The correctness proof. If G is colorable by three colors, and Peggy knows such a coloring and uses it, then all checks Vic performs have to agree. On the other hand, if this is not the case, then at each interaction there is a chance $\frac{1}{|E|}$ that Peggy gets caught. The probability that she does not get caught in $|E|^2$ interactions is $(1 - 1/|E|)^{|E|^2}$—negligibly small. □

The essence of a zero-knowledge proof, as demonstrated also by Protocols 9.3.3 and 9.3.5, can be formulated as follows: the prover breaks the proof into pieces and encrypts each piece using a new one-way function in such a way that:

1. The verifier can easily verify each piece of the proof, whether it has been properly constructed.

2. If the verifier keeps checking randomly chosen pieces of the proof and all are correctly designed, then his confidence in the correctness of the whole proof keeps increasing and, at the same time, this does not bring to the verifier any additional information about the proof itself.

3. The verifier knows that each prover that knows the proof can make its decomposition into pieces in such a way that the verifier finds all pieces correctly designed, but no prover that does not know the proof is able to do it.

The key requirement, namely that the verifier randomly picks up pieces of the proof to check, is taken care by the prover! At each interaction the prover makes a random permutation of the proof and uses for the encryption new one-way functions. As the result, no matter what kind of strategy the verifier chooses to pick up the pieces of the proof, his strategy is equivalent to a random choice.

Example 9.3.4 *With the following protocol, Peggy can convince Vic that the graph G they both know has a Hamilton cycle (without revealing any information how such a cycle looks).*

Protocol 9.3.5 (Existence of Hamilton cycles)

Given a graph $G = \langle V, E\rangle$ with n nodes, say $V = \{1, 2, \ldots, n\}$, each round of the protocol proceeds as follows.

Peggy chooses a random permutation π of $\{1, \ldots, n\}$, a one-way function e_i for each $i \in \{1, \ldots, n\}$, and also a one-way function $e_{i,j}$ for each pair $i, j \in \{1, \ldots, n\}$. Peggy then sends to Vic:

1. Pairs (i, x_i), where $x_i = e_i(\pi(i))$, for $i = 1, \ldots, n$ and all e_i are so choosen that all x_i are different.

2. Triples $(x_i, x_j, y_{i,j})$ where $y_{i,j} = e_{i,j}(b_{i,j}), i \neq j$, $b_{i,j} \in \{0, 1\}$ and $b_{i,j} = 1$ if and only if $(\pi(i), \pi(j))$ is an edge of G—e_{ij} are supposed to be so chosen that all y_{ij} are different.

Vic then gets two possibilities to choose:

1. Vic can ask Peggy to demonstrate the correctness of all encryptions—that is to reveal π and all encryption functions $e_i, e_{i,j}$. In this way Vic can get convinced that x_i and $y_{i,j}$ really represent an encryption of G.

2. Vic can ask Peggy to show a Hamilton cycle in G. Peggy can do that by revealing exactly n distinct numbers $y_{i_1,i_2}, y_{i_2,i_3}, \ldots, y_{i_n,i_1}$ such that $\{1, 2, \ldots, n\} = \{i_1, \ldots, i_n\}$. This proves to Vic, who knows all triples $(x_i, x_j, y_{i,j})$, the existence of a Hamilton cycle in whatever graph, is represented by the presented encryptions. Since the x_i are not decrypted, no information is revealed concerning the sequence of nodes defining a Hamilton cycle in G.

Vic then chooses, randomly, one of these two offers (to show either the encryption of the graph or the Hamilton cycle) and Peggy reveals the requested information.

If Peggy does not know the Hamilton cycle, then in order not to get caught, she must make always a correct guesses which possibility Vic will choose. This means, that the probability that Peggy does not get caught in k rounds, if she does not know a Hamilton cycle, is at most 2^{-k}.

Observe that the above protocol does not reveal any information whatsoever about how a Hamilton cycle for G look. Indeed, if Vic asks for the encryption of the encoding, he gets just a random encryption of G. When asking for a Hamilton cycle the verifier gets just a random cycle of length n with any such cycle being equally probable. This is due to the fact that Peggy is required to deal always with the same proof, that is with the same Hamilton cycle and π is a random permutation.

Exercise 9.3.6 * *Design a zero-knowledge proof for integer factorization.*

Exercise 9.3.7 * *Design a zero-knowledge proof for the knapsack problem.*

Exercise 9.3.8 * *Design a zero-knowledge proof for the travelling salesman problem.*

9.3.2 Theorems with Zero-knowledge Proofs*

In order to discuss in more detail when a theorem has a zero-knowledge proof, we sketch a more formal definition of what is a "zero-knowledge proof". In doing so the key concept is that of the **polynomial time indistinguishability** of two **probability ensembles** $\Pi_1 = \{\pi_{1,i}\}_{i\in\mathbf{N}}$ and $\Pi_2 = \{\pi_{2,i}\}_{i\in\mathbf{N}}$—two sequences of probability distributions on $\{0,1\}^*$, indexed by $\mathbf{N}$, where distributions $\pi_{1,i}$ and $\pi_{2,i}$ assign nonzero probabilities only to strings of length polynomial in $|bin^{-1}(i)|$.

Let T be a probabilistic polynomial time Turing machine with output from the set $\{0,1\}$—called a **test** or a **distinguisher** here—that has two inputs, $i \in \mathbf{N}$ and $\alpha \in \{0,1\}^*$. Denote, for $j = 1,2$,

$$p_j^T(i) = \sum_{\alpha\in\{0,1\}^*} \pi_{j,i}(\alpha) Pr(T(i,\alpha) = 1),$$

i.e., $p_1^T(i)$ is the probability that on inputs i and α, chosen according to the distribution $\pi_{1,i}$, the test T outputs 1. Π_1 and Π_2 are said to be **polynomial time indistinguishable** if for all probabilistic polynomial time tests T, all constants $c > 0$, and all sufficiently big $k \in I$ (k is a "confidence parameter"):

$$|p_1^T(i) - p_2^T(i)| < k^{-c}.$$

Informally, two probability ensembles are polynomial time indistinguishable, if they assign "about the same probability to any efficiently recognizable set of words over $\{0,1\}^*$".

In the following definition we use the notation $hist(M_P, M_V, x)$ for the random variable the values of which consist of the concatenated messages of the interaction of the protocol (M_P, M_V) on the input x with random bits, consumed by M_V during the interaction, attached. (Such a concatenated message is called also a **history** of communication.)

Definition 9.3.9 *The interactive protocol (M_P, M_V) for a language L is (computationally)* **zero-knowledge** *if, for every verifier M_{V^*}, there exists a probabilistic polyno-*

mial time Turing machine M_{S^*}, called **simulator**, *such that the probability ensembles $\{hist(M_P, M_{V^*}, x)\}_{x\in L}$ and $\{M_{S^*}(x)\}_{x\in L}$ are polynomial-time indistinguishable.*

We present now two main approaches how to show that an interactive proof is a zero-knowledge proof.

1. For some interactive proofs it has been shown, under the asumption that one-way functions exist, that the histories of their protocols are polynomial time indistinguishable from random strings.
2. Another method is to show that the verifier can actually simulate the prover. That is, the verifier can take also the prover's position in the interaction with the verifier, and therefore, any polynomial time randomized algorithm that enables the verifier to extract some information from the interaction with the prover, could be used to extract this information without an interaction with the prover.

Let us illustrate the last idea on the protocol proving, for a fixed graph G with n nodes, that G has a Hamilton cycle.

A verifier V first simulates the prover P. V flips coins and, according to the outcome, encrypts a random permutation of the whole graph (just how P would do), or encrypts a randomly chosen permutation of nodes. Then, acting as the prover, the verifier presents the encrypted information to the verifier, that is to himself, and takes the position of the verifier.

V uses now his algorithm, say A, to decide whether to request a graph or a cycle. Because A has no way of knowing what V did in the guise of P, there is a 50% chance that A requests exactly the option V, in the guise of P, supplies. If not, V backs up the algorithm A to the state it was at the start of this round and restarts the entire round.

This means, that in the expected two passes through each round V obtains the benefit of the algorithm A without any help of the prover. Therefore, A does not help V to do something with P in an expected polynomial time, what V could not do as well without P, in expected polynomial time.

The family of theorems that have a zero-knowledge proof seems to be surprisingly large. It holds:

Theorem 9.3.10 *If one-way functions exist, then every language in* **PSPACE** *has a zero-knowledge proof.*[2]

The proof of the theorem is too involved to be presented here and we only sketch an idea of the proof of a weaker statement for the class **NP**: One first shows for an **NP**-complete language L_0 that it has a zero-knowledge proof system. (This we have already done—see Example 9.3.4—under the assumption that one-way functions exist.) Secondly, one shows that if a language $L \subseteq$ **NP** is in polynomial time reducible to L_0, then this reducibility can be used to transform a zero-knowledge proof for L_0 to a zero-knowledge proof for L. □

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[2] On the other hand, if one-way functions do not exist, then the class of languages having zero-knowledge proofs is identical with **BPP**.

9.3.3 Analysis and Applications of Zero-knowledge Proofs*

Note first that the concept of zero-knowledge proofs brings a new view what "knowledge" is. Something is implicitly regarded as "knowledge" only if there is no polynomial time computation that can produce it.

Observe also, see the next exercise, that both randomness and interaction are essential for non-triviality of the concept of zero-knowledge proofs.

Exercise 9.3.11 *Show that zero-knowledge proofs in which the verifier either tosses no coins, or does not interact, exist only for languages in* **BPP**.

Observe also the following paradox in the concept of zero-knowledge proofs of a theorem. Such a proof can be constructed as described above, by the verifier himself, who only believes in the correctness of the theorem, and in spite of that such a proof does convince the verifier! The "paradox" is resolved by noting that it is not the text of "conversation" that convinces the verifier, but rather the fact that the conversation is held "on line".

Theorem 9.3.10 and its proofs provide a powerful tool for design of cryptographical protocols. To see that let us first discuss a general setting in which cryptographical protocols arise.

A cryptographical protocol can be seen as a set of interactive programs to be executed by nontrusting parties. Each party has a local input unknown to others that is kept secret. The protocol usually specifies actions parties should take dependening on their local secrets and previous messages exchanged. The main problem in this setting is how can a party verify that the other parties have really followed the protocol? A verification is difficult, because a verifier, say A, does not know the secrets of the communicating party, say B, who does not want to reveal its secret. The way out is to use zero-knowledge proofs. B can convince A that the message transmitted by B has been computed according to the protocol without revealing any secret.

Now comes the main idea how to design cryptographical protocols. First, design a protocol under the assumption that all parties will follow the protocol properly. Next, transform this protocol, using already well-known and mechanical methods how to make zero-knowledge proofs from "normal proofs", into a protocol in which communication is based on zero-knowledge proofs and preserves both correctness and privacy, and works even if a minority of parties display an adversary behaviour.

There are various others surprising applications of zero-knowledge proofs.

Example 9.3.12 (User identification) *The idea of zero-knowledge proofs offers a radically new approach to the user identification problem. For each user, a theorem, the proof of which only this user knows, is stored in a directory. After login, the user starts a zero-knowledge proof of the correctness of the theorem. If the proof is convincing, his/her access is guaranteed. The new important point is that even an adversary who could follow the communication fully would not get any information allowing him/her to get the access.*

The concept of a zero-knowlege proof system can be generalized in natural way to the case of multiple provers and it holds:

Theorem 9.3.13 *Every language in* **NEXP** *has a zero-knowledge, two-provers, interactive proof system.*

Observe that in the last theorem no assumption has been made about the existence of one-way functions.

9.4 Interactive Program Validation

Program validation is one of the key problems in computing. Traditional program testing is feasible but insufficient—not all input data can be tested, and therefore, on a particular run the user may have no guarantee the result is correct. Program verification, while ideal in theory, is not currently and may never be a pragmatic approach to program reliability. It is neither feasible nor sufficient. Only not too large programs may be verified, and even this does not say anything about the correctness of a particular computer run due to possible compiler or hardware failures. Interactive program (result) checking, and especially interactive program self-correcting, discussed in this section, offer an alternative approach to program validation. Program checkers may provide the bases for a debugging methodology more rigorous than program testing and more pragmatic than verification.

9.4.1 Interactive Result Checkers

The basic idea is to develop, given an algorithmic problem $\mathcal{P}$, a **result checker** for $\mathcal{P}$, that is capable to find out, with large probability correctly, given any program P for $\mathcal{P}$, and any input data d for P, whether the result P produces for d is correct—in short, whether $\mathcal{P}(d) = P(d)$. To do that, the result checker may interact with P and use P to do computations also for some other input data than d, if needed. The result checker produces the answer "PASS", if the program P is correct for all inputs, and therefore, also for d. It produces the output "FAIL", if $\mathcal{P}(d) \neq P(d)$. (The output is not specified in the remaining cases, i.e, when the program is not correct in general but $\mathcal{P}(d) = P(d)$.)

Of a special interest are the so-called **simple (result) checkers**—for an algorithmic problem $\mathcal{P}$ with the best sequential time $t(n)$. They get as an input a pair (d, y) and have to return YES (NO) if $y = \mathcal{P}(d)$ (if $y \neq \mathcal{P}(d)$); in both cases with the probability (over internal randomization) close to 1. Moreover, a simple checker is required to work in time $o(t(n))$. (The last condition requires that a simple checker for $\mathcal{P}$ is essentially different and faster than any program to solve $\mathcal{P}$.)

The idea of a (simple) checker is based on the observation that for certain mathematical functions it is much easier to determine, for inputs x and y, whether $y = f(x)$, than to determine $f(x)$, for an input x. For example, the problem to find a non-trivial factor p of an integer n is computationally infeasible, but it is easy—one division sufficies—to check whether p is a division of n. Let us now illustrate the idea of simple checkers on two examples.

Example 9.4.1 (Result checker for the generalized GCD problem) Algorithmic problem $\mathcal{P}_{GGCD}$: *Given integers* m, n, *compute* $d = gcd(m, n)$ *and* $u, v \in \mathbf{Z}$ *such that* $um + vn = d$.

Program checker $\mathcal{C}_{GGCD}$ *takes a given program* P *to solve (supposedly) the problem* $\mathcal{P}_{GGCD}$, *and makes* P *to compute, given* m *and* n, *the corresponding* d, u, v. *After that, it performs the following check:*

```
if $d$ does not divide $m$ or does not divide $n$
  then $\mathcal{C}_{GGCD}$ outputs "FAIL"
  else if $mu + nv \neq d$ then $\mathcal{C}_{GGCD}$ outputs "FAIL"
                           else $\mathcal{C}_{GGCD}$ outputs "PASS".
```

The first condition checks whether d *is a divisor of both* m *and* n. *The second whether it is the largest divisor. Observe, that the checker needs to perform only 2 divisions, 2 multiplications and one addition—the checker is far more efficient than any algorithm to compute* $gcd(m, n)$.

Example 9.4.2 (Freivald's checker for matrix multiplication) *If* A, B *and* C *are matrices of degree* n *and* $AB = C$, *then also* $A(Bv) = Cv$ *for any vector* v *of length* n. *To compute* AB *one needs between* $\mathcal{O}(n^{2.376})$ *and* $\Theta(n^3)$ *arithmetical operations; to compute* $A(Bv)$ *and* Cv, $\Theta(n^2)$ *operations suffice. Moreover, it can be shown that if* v *is a randomly chosen vector, then the probability that* $AB \neq C$ *once* $A(Bv) = Cv$ *is very small. This yields the following* $\Theta(kn^2)$ *simple checker for matrix multiplication.*[3] *Choose* k *random vectors* $v_1, \ldots, v_k$ *and compute* $A(Bv_i)$ *and* Cv_i *for* $i = 1, \ldots, k$. *If* $A(Bv_i)$ *and* Cv_i *once differ, then rejects, otherwise accept.*

Exercise 9.4.3 *Design a simple checker for multiplication of two polynomials.*

Exercise 9.4.4 *Design a simple checker for integer multiplication. (Hint: use Exercise 1.7.12.)*

The following definition deal with the general case. Probability is there considered, as usually, over the space of all internal coin-tossings of the result checker.

Definition 9.4.5 *A* **result checker** $\mathcal{C}_{\mathcal{P}}$, *for an algorithmic problem* $\mathcal{P}$, *is a probabilistic oracle Turing machine such that given any program* P *(supposedly) for* $\mathcal{P}$, *that always terminates, any particular input data* x_0 *for* $\mathcal{P}$, *and any integer* k, $\mathcal{C}_{\mathcal{P}}$ *works in the expected polynomial time (with respect to* $|x_0 1^k|$*), and produces the following result:*

1. *If* P *is correct, that is* $P(x) = \mathcal{P}(x)$ *for all possible inputs* x, *then, with the probability greater than* $1 - \frac{1}{2^k}$, $\mathcal{C}_{\mathcal{P}}(P, x_0) =$ *"PASS".*

---

[3] Provided matrix multiplication can not be done in $\Theta(n^2)$ time, what seems to be likely but not proven yet.

2. If $P(x_0) \neq \mathcal{P}(x_0)$ *then* $\mathcal{C}_{\mathcal{P}}(P, x_0) =$ *' 'FALL" with the probability greater than* $1 - \frac{1}{2^k}$.

In Definition 9.4.5, k is a **confidence parameter** that specifies the degree of confidence in the outcome. The time needed to prepare inputs for P and to process outputs of P, is counted into the overall time complexity of $\mathcal{C}_P$, but not time P needs to compute its results.

The program checker for the graph isomorphism problem presented in the following example is a modification of the zero-knowledge proof for the graph non-isomorphism problem from Section 9.2.1. This time, however, without an all-powerful prover.

Example 9.4.6 (Program checker for the graph isomorphism problem)

Input: a program P to determine an isomorphism of arbitrary graphs and two graphs G_0, G_1, the isomorphisms of which is to be determined. The protocol for an interaction between the program checker $\mathcal{C}_{GI}$ and P has the following form:

```
begin
make $P$ to compute $P(G_0, G_1)$;
if $P(G_0, G_1) =$ "YES" then
  begin use $P$ (assuming that it is "bug-free") to find out, by the method
        described bellow, an isomorphism between $G_0$ and $G_1$ and to check
        whether the isomorphism obtained is correct;
        if not correct then return "FAIL" else return "PASS";
  end;
if $P(G_0, G_1) =$ "NO" then for $i = 1$ to $k$ do
  begin get a random bit $b_i$;
        generate a random permutation $H_i$ of $G_{b_i}$ and compute $P(G_0, H_i)$;
        if $b_i = 0$ and $P(G_0, H_i) =$ " NO"
              then return "FAIL"
              else if $b_i = 1$ and $P(G_0, H_i) =$ "YES" then return "FAIL"
  end
 return "PASS"
end.
```

In order to finish the description of the protocol, we have to describe how the checker $\mathcal{C}_{GI}$ can use P to construct an isomorphism between G_0 and G_1 in case such an isomorphism exists. This can be done as follows. A node v from G_0 is arbitrarily chosen and a larger clique with new nodes is attached to it—denote by G_0' the resulting graph. The same clique is then added, step by step, to various nodes of G_1, and each time this is done, P is used to check whether the resulting graph is isomorphic to G_0'. If no node of G_1 is found such that the modified graph is isomorphic to G_0', then $\mathcal{C}_{GI}$ outputs "FAIL". If such a node, say v', is found, then v is removed from G_0 and v' from G_1, and the same method is used to build further an isomorphism between G_0 and G_1.

It is clear that the checker always produces the result "PASS" if the program P is totally correct. Consider now the case that P produces sometimes a not correct result. We show now that the probability that the checker produces "PASS" if the program P is not correct is at most $\frac{1}{2^k}$. Consider two cases:

1. $P(G_0, G_1) =$ "YES", but G_0 and G_1 are not isomorphic. Then the checker has to fail to produce an isomorphism and therefore it has to output "FAIL".
2. $P(G_0, G_1) =$ "NO" but G_0 and G_1 are isomorphic. The only way that the checker would produce "PASS" in such a case is that P produces correct answers for all k checks $P(G_0, H_i)$. That is, the checker produces the answer "YES" if H_i is a permutation of G_0 and "NO" if H_i is a permutation of G_1. However, since b_i are random and permutations H_i of G_0 and G_1 are also random, there is the same probability that H_i is a permutationof G_0 as of G_1. Therefore, P can correctly distinguish whether H_i was designed by a permutation of G_0 or of G_1 only by chance; that is, exactly for 1 out of 2^k possible sequences of k bits b_i. □

It has been shown that there are effective result checkers for all problems in PSPACE and the following lemma implies that given a result checker for a problem one can effectively construct out of it a result checker for another problem computationally equivalent with the first one. This actually implies that for all practical problems there is a result checker and can be efficiently constructed.

Lemma 9.4.7 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two polynomial time equivalent algorithmic problems. From any efficient result checker $\mathcal{C}_{\mathcal{P}_1}$ for $\mathcal{P}_1$, it is possible to construct an efficient result checker $\mathcal{C}_{\mathcal{P}_2}$ for $\mathcal{P}_2$.*

Proof. Let r_{12} and r_{21} be two polynomial time computable functions such that r_{ij} maps YES-instance (NO-instance) of $\mathcal{P}_i$ into YES-instance (NO-instance) of $\mathcal{P}_j$. Let P_2 be a program for $\mathcal{P}_2$. $\mathcal{C}_{\mathcal{P}_2}(P_2, x_2)$ works as follows.

1. $\mathcal{C}_{\mathcal{P}_2}$ computes $x_1 = r_{21}(x_2)$ and designs a program P_1 for $\mathcal{P}_1$ that works as follows: $P_1(x) = P_2(r_{12}(x))$.
2. $\mathcal{C}_{\mathcal{P}_2}(P_2, x_2)$ checks whether

$$P_2(x_2) = P_2(r_{12}(r_{21}(x_2))) \tag{9.10}$$

and whether

$$P_1(x_1) = \mathcal{P}(x_1) \tag{9.11}$$

(and therefore whether $P_2(x_2) = \mathcal{P}_2(x_2)$) by using $\mathcal{C}_{\mathcal{P}_1}(P_1, x_1)$.

If either of the conditions 9.10 and 9.11 fails, then $\mathcal{C}_{\mathcal{P}_2}$ returns NO, otherwise returns YES.

If P_2 is correct, then both checks are satisfied and therefore the checker reports correctly. On the other hand, if $P_2(x_2) \neq \mathcal{P}_2(x_2)$, then either the check 9.10 fails and $\mathcal{C}_{\mathcal{P}_2}$ reports correctly NO or 9.10 holds and then

$$\begin{aligned} P_1(x_1) &= P_1(r_{21}(x_2)) \\ &= P_2(r_{12}(r_{21}(x_2))), \text{ since } P_1(x) = P_2(r_{12}(x)) \\ &= P_2(x_2), \text{because 9.10 holds} \end{aligned}$$

$$
\begin{aligned}
&\neq \mathcal{P}_2(x_2) \text{ by assumption}\\
&= \mathcal{P}_1(r_{21}(x_2))\\
&= \mathcal{P}_1(x_1)
\end{aligned}
$$

in which case the result checker $\mathcal{C}_{\mathcal{P}_1}$ and therefore also the checker $\mathcal{C}_{\mathcal{P}_2}$ produce NO correctly, with the probability at least $1 - \frac{1}{2^k}$. ☐

Exercise 9.4.8 *Design an $\mathcal{O}(n)$-time result checker for sorting of n integers. (Hint: use Exercise 2.3.28.)*

Although a result checker $\mathcal{C}_{\mathcal{P}}$ can be used to verify, for a particular input x and program P, whether $\mathcal{P}(x) = P(x)$, it does not provide a method to compute $\mathcal{P}(x)$ in the case P is found to be faulty. From this point of view **self-correcting/tesing programs**, discussed in the next section, present an additional step forward to program validation. To simplify the presentation we deal only with programs that compute functions.

Remark 9.4.9 An interest of scientific community in developing result checking means is actually very old. One of the basic tools were various formulas as $e^x = e^{x-a}e^a$ or $\tan x = (\tan(x+a)+\tan a)/(1-\tan(x-a)\tan a)$ that related to each other the values of the function at a given point and a few other points. Such formulas allowed both result checking and self-correcting as discussed bellow. Actually, the desire to obtain such formulas was one of the main forces inspiring progress of mathematics in 18th and 19th century.

Exercise 9.4.10 *Design a formula that relates values of the function $f(x) = \frac{e^x}{x}$ at points $x, x+1$ and $x+2$.*

9.4.2 Interactive Self-correcting and Self-testing Programs

Informally, a (randomized) program $\mathcal{C}_f$ is a **self-correcting program** for f if, for any program P that, supposedly, computes f and the error probability of which is sufficiently low, and any input x, $\mathcal{C}_f$ can make a use of P to compute $f(x)$ correctly.

The idea of self-correcting programs is based on the fact that, for some function f, we can efficiently compute $f(x)$, if we know the value of f at several others, randomly looking inputs.

Example 9.4.11 *To compute the product of two matrices A and B of degree n, we choose $2k$ random matrices R_1, R_1', R_2, $R_2', \ldots, R_k$, R_k', all of degree n and take as the value of AB that value that occurs most often among values*

$$(A - R_i)(B - R_i') + R_i(B - R_i') + (A - R_i)R_i' + R_iR_i' \tag{9.12}$$

Observe that if a matrix multiplication algorithm P is correct and it is used to perform multiplications in 9.12, then all values in 9.12 are exactly AB. If P produces correct values with high probability, then again most of the values from 9.12 are equal AB.

The idea of having self-correcting programs is attractive but two basic questions immediately arise: their efficiency and correctness.

The whole idea of result checkers as well as self-correcting and self-testing programs requires, in order to be fully meaningful, that they are **efficient** in the following sense. Their **proper (incremental) running time**—that is time the program P, which they validate, spends whenever called by a self-testing program or a self-correcting program, is never counted—should be asymptotically smaller than computation time of P. On the other hand, the **total running time**—that is time P spends whenever called by a self-testing program or a self-correcting program, is also included—should be asymptotically of the same order as computation time of P. This should be true for any program P computing f. It is therefore clear that self-testing and self-correcting programs for f have to be **essentially different** than any program to compute f.

The problem how to verify result checkers and self-correcting programs, has no simple solution. However, it is believed, and experiences confirm it, that all these programs can be essentially simpler than those they have to validate, and therefore, they should be more easy to verify. In addition, in case of problems for which a large amount of time is spent to find the best algorithms, for example, for number operations and matrix operations, it should pay off to make a larger effort to verify self-testing and a self-correcting programs for them. In any case, also the verification problem requires that self-testing and self-correcting programs for a function f are **essentially different** from any program for f.

To simplify the presentation a uniform probability distribution is assumed on all sets of inputs of the same size and $error(f,P)$ is used to denote the probability that $P(x) \neq f(x)$ when x is randomly chosen from inputs of the same size.

Definition 9.4.12 *(1) Let $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 1$. An $(\varepsilon_1, \varepsilon_2)$–**self-testing program** for f is a randomized oracle program $\mathcal{T}_f$ such that for any program P for f, any integers n (the size of inputs), and k (a confidence parameter) the following holds (e denotes the base of natural logarithms):*

1. *If $error(f,P) \leq \varepsilon_1$ for inputs of size n, then $\mathcal{T}_f(P)$ outputs "PASS" with the probability at least $1 - \frac{1}{e^k}$.*
2. *If $error(f,P) \geq \varepsilon_2$, then $\mathcal{T}_f(P)$ outputs "FAIL" with the probability at least $1 - \frac{1}{e^k}$.*

*(2) Let $0 \leq \varepsilon \leq 1$. An ε-**self-correcting program** for f is a randomized oracle program $\mathcal{C}_f$ such that for any program P for f, any integer n, any input x of the size n, and any integer k, the following property holds: If $error(f,P) \leq \varepsilon$, then $\mathcal{C}_f(P,x) = f(x)$ with the probability at least $1 - \frac{1}{e^k}$.*

The main advantage of self-correcting programs is that they can be used to transform programs correct for most of the inputs to programs correct, with high probability, on any input.

Remark 9.4.13 In discussing the incremental and total time any dependence on the confidence parameter k has been ignored. This usually adds a multiplicative factor of the order $\mathcal{O}(k)$.

The basic problem in designing self-testing and self-correcting programs is how to make them to be essentially different from those programs computing f directly. One idea, that has turned out to be useful, is to compute f indirectly by computing f on random inputs.

Definition 9.4.14 (Random self-reducibility property) *Let $m > 1$ be an integer. A function f is called m-**random self-reducible** if, for any x, $f(x)$ can be expressed as an easily computable function F of $x, a_1, \ldots, a_m$ and $f(a_1), \ldots, f(a_m)$, where $a_1, \ldots a_m$ are randomly chosen from the inputs of the same size as x. (By "easily computable" is understood that the total computation time of a random self-reduction—that is of computing F from the arguments $x, a_1, \ldots, a_n$ and $f(a_1), \ldots, f(a_n)$, is smaller than that of computing $f(x)$).*

The first two examples are that of self-correcting programs. They are easier to present and to prove correct than self-testing programs. (We use here the notation $x \in_{\mathcal{U}} A$, see Section 2.1, to mean that X is randomly taken from A, with respect to the uniform probability distribution.

Example 9.4.15 (Self-correcting program for mod **function)** *Let $f(x) = x \bmod m$ with $x \in \mathbf{Z}_{m2^n}$, for some integer n. Assume that P is a program to compute f, for $0 \leq x \leq m2^n$, with $error(f, P) \leq \frac{1}{8}$. The inputs of the following $\frac{1}{8}$-self-correcting program are x, m, n, k and a program P for f and $+_m$ denotes addition modulo m.*

Protocol 9.4.16 (Self-correcting program for mod **function)**

```
begin
    $N \leftarrow 12k$;
    for $i \leftarrow 1$ to $N$ do
        call Random-split$(m2^n, x, x_1, x_2, e)$;
        $a_i \leftarrow P(x_1, m) +_m P(x_2, m)$;
    output the most common answer among $\{a_i \mid 1 \leq i \leq N\}$
end,
```

where the procedure "Random-split", with the output parameters z_1, z_2, e, is defined as follows:

```
procedure Random-split$(s, z, z_1, z_2, e)$
    choose $z_1 \in_{\mathcal{U}} \mathbf{Z}_s$;
    if $z_1 \leq z$ then $e \leftarrow 0$ else $e \leftarrow 1$;
    $z_2 \leftarrow es + z - z_1$.
```

The correctness proof. As $x_j \in_{\mathcal{U}} \mathbf{Z}_{m2^n}$ for $j = 1, 2$ we get that $P(x_i, m) \neq x_i \bmod m$ with probability at most $\frac{1}{8}$. Therefore for any $1 \leq i \leq N$ $a_i = x_1 \bmod m + x_2 \bmod m$ with probability at least $\frac{3}{4}$. The correctness of the protocol now follows from the following Lemma (with $N = 12k$), a consequence of the Chernoff's bound (see Section 1.9.2).

Lemma 9.4.17 *If* $x_1, \ldots, x_N$ *are independent* 0/1*-valued random variables such that* $Pr(x_i = 1) \geq \frac{3}{4}$, *for* $i = 1, \ldots, N$, *then*

$$Pr\left(\sum_{i=1}^{m} x_i > N/2\right) \geq 1 - e^{-N/12}.$$

Observe that the self-correcting program presented above is essentially different from any program to compute f. Its incremental time is linear in N and its total time is linear in time of P.

Example 9.4.18 (Self-correcting program for integer multiplication) *In this case* $f(x,y) = xy$ *and we assume that* $x, y \in \mathbf{Z}_{2^n}$ *for a fixed* n. *Let us also assume that there is a program* P *to compute* f *and* $error(f,P) \leq \frac{1}{16}$. *The following program is a* $\frac{1}{16}$*-self-correcting program for* f. *Inputs:* $n, x, y \in \mathbf{Z}_{2^n}, k$ *and* P.

Protocol 9.4.19 (Self-correcting program for integer multiplication)

```
begin$N \leftarrow 12k$
    for $i \leftarrow 1$ to $N$ do
        call Random-split$(2^n, x, x_1, x_2, c)$;
        call Random-split$(2^n, y, y_1, y_2, d)$;
        $a_i \leftarrow P(x_1,y_1) + P(x_1,y_2) + P(x_2,y_1) + P(x_2,y_2) - cy2^n - dx2^n + cd2^{2n}$;
    output the most common value from $\{a_i \mid 1 \leq i \leq N\}$
end
```

The correctness proof. Since $x_i, y_j, i, j = 1, 2$ are randomly chosen from $\mathbf{Z}_{2^n}$ we get, by the property of P, that $P(x_i, y_j) \neq x_k y_j$ with the probability at most $\frac{1}{16}$, and therefore the probability that all four calls to P, during a pass through the cycle, return the correct value is at least $\frac{3}{4}$. Since $x = x_1 + x_2 - c2^n$, and $y = y_1 + y_2 - d2^n$, we have

$$xy = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 - cy2^n - dx2^n + cd2^{2n}.$$

Thus, if all four calls to P are correctly answered during the i-th cycle, then $a_i = xy$. The correctness of the protocol follows again from Lemma 9.4.17. □

Exercise 9.4.20 *Show that we can construct a* $\frac{1}{16}$*-self-correcting program for modular number multiplication* $f(x,y,m) = xy \bmod m$ *to deal with any program* P *for* f *with* $error(f,P) \leq \frac{1}{16}$.

Example 9.4.21 (Self-correcting program for modular exponentiation) *Consider now the function* $f(a,x,m) = a^x \bmod m$, *where* a *and* m *are fixed and* $\gcd(a,m) = 1$. *Suppose that the factorization of* m *is known and therefore* $\phi(m)$ *is known where* ϕ *is Euler's totient function. Let* $x \in \mathbf{Z}_{\phi(m)2^n}$. *Finally, let us assume that we have a program* P *to compute* f *such that* $error(f,P) \leq \frac{1}{8}$. *The inputs to the following protocol are* $n, x, k, a, m, \phi(m)$ *and* P.

Protocol 9.4.22 (Self-correcting program for modular exponentiation)

```
begin
    $N \leftarrow 12k$;
    for $i \leftarrow 1$ to $N$ do
        call Random-split$(\phi(m)2^n, x, x_1, x_2, c)$;
        $a_i \leftarrow P(a, x_1, m)P(a, x_2, m) \bmod m$;
    output the most common value among $\{a_i \mid i = 1, \ldots, N\}$
end
```

The correctness proof. Clearly $P(a, x_i, m) \neq a^{x_i} \bmod m$ with probability at most $\frac{1}{8}$ and therefore the probability that a_i is computed correctly is at least $\frac{3}{4}$. As $x = x_1 + x_2 - c\phi(m)2^n$, for a $c \in \{0, 1\}$ and, in addition, $\gcd(a, m) = 1$, $a^{\phi(m)} \equiv 1 \pmod m$, we get $a^x \equiv a^{x_1} \cdot a^{x_2} \pmod m$. This means that if, in the i-th pass through the cycle, both calls to P are correctly answered, then $a_i = a^x \bmod m$. The correctness of the protocol now follows from Lemma 9.4.17. □

All three self-correcting programs/protocols presented above are based on a similar idea, that can be generalized as follows:

Theorem 9.4.23 *If f is a m-random self-reducible function, then there is a $(\frac{1}{4m})$-self correcting program for f (to correct any program P for f such that $error(f, P) \leq \frac{1}{4m}$).*

Proof. Consider the following program with the inputs m, x, k and P.

Protocol 9.4.24 (Generic self-correcting program)

```
begin
    $N \leftarrow 12k$;
    for $i \leftarrow 1$ to $N$ do
        randomly choose $a_1, \ldots, a_m$ of the same size as $x$;
        for $j \leftarrow 1$ to $N$ do $\alpha_j \leftarrow P(a_j)$;
        $a_i \leftarrow F(x, a_1, \ldots, a_m, \alpha_1, \ldots, \alpha_m)$;
    output the most common value among $\{a_i \mid 1 \leq i \leq m\}$
end
```

The assumption $error(f, P) \leq \frac{1}{4m}$ implies that in the i-th cycle all values α_j are computed correctly with the probability at least $\frac{3}{4}$. Moreover, due to the self-reducibility property of f, also $a_i = f(x)$ with probability at least $\frac{3}{4}$. The theorem follows, as before, from Lemma 9.4.17. □

Exercise 9.4.25 *Design a self-correcting program for integer division.*

Exercise 9.4.26 *Design a self-correcting program for polynomial multiplication.*

Remark 9.4.27 On a similar base as result checkers and self-correcting programs also the concept of the self-testing programs have been developed.

Exercise 9.4.28 ** *Construct a polynomial formula that relates values of the function* $x + \sin x$ *for several arguments and can therefore be used for result checking and self-correcting programs.*

Moral. New interactive, holographic,... proof techniques allow to prove otherwise practically unprovable. In spite of that, a good rule of thumb by dealing with the concept of evidence in computing is, as in life, to explore all feasible concepts of evidence but to rember that a good proof makes us wiser.

9.5 Exercises

1. Let us generalize the age difference protocol to the case that n friends want to find out who is the oldest, without disclosing any other information about their age. (a) Show how to use the age-difference protocol for two persons to solve the general case; (b) show the disadvantage of such a solution and find a better one.

2. Alice want to sell a computer to Bob whom she does not trust anymore. Bob wants to pay by checks but Alice has no way of knowing if the check is good or not. Alice wants Bob to clear checks before she turns the computer to him. However, Bob also does not trust Alice and therefore does not want to give checks to Alice without first receiving the computer. How to proceed?

3. (**Secret voting.**) Design a protocol for a secret balloting system satisfying the following conditions; (a) Only legitimate voters should cast a ballot. (b) The ballot should be kept secret. (c) Everybody is allowed to have only one vote. (d) Any voter should be able to verify that his/her vote was taking into the account of the electoral outcome.

4. Design a bit commitment protocol on the base of discrete logarithms.

5. Design a bit commitment protocol using only the oblivious transfer protocol.

6. Design a coin-flipping protocol on the base of the oblivious transfer protocol.

7. Construct a polynomial approximation of the following Boolean formulas:

   (a) $(x \vee y \vee \bar{z})(v \vee \bar{x} \vee \bar{y})$; (b) $(x \vee \bar{z})(y \vee \bar{x})(x \vee y \vee z)$; (c) $(x \vee p)((x \vee \bar{z}) \vee (s \wedge \bar{t})$.

8. Show that if F a 3-CNF Boolean formula with n variables and m clauses, then there is an arithmetical formula of length at most $20m$ and degree at most $3m$, that approximates F.

9. Consider the following protocol for the **NP**-complete language $L = \{(G_0, G_1) \mid G_0$ is a subgraph of $G_1\}$.
   (a) The verifier sends G_0 and G_1 to the prover.
   (b) The prover constructs k random graphs $H_1, \ldots, H_k$ such that each H_i is a subgraph of G_1 and contains G_0 as a subgraph, and sends these graphs to the verifier.
   (c) The verifier chooses k random bits $b_1, \ldots, b_k$ and asks, for $1 \leq i \leq k$, the prover to show him either that G_0 is a subgraph of H_i (if $b_i = 0$) or that H_i is a subgraph of G_1 (if $b_i = 1$).
   (d) The prover does what the verifier asks for.

   (a) Show that this protocol is an interactive protocol for the language L; (b) explain why this protocol is not a zero-knowledge protocol for L; (c) design a zero-knowledge protocol for L.

10. Design an interactive proof system for the following reachability problem for cellular automata. Given is a finite two-dimensional toroidal CA $\mathcal{A}$, (i.e. a CA whose finite automata are connected into toroidal inteconnection structure), specified by a set of states, a neighbourhood and an integer n (the size of the array), and an initial configuration. Assume that the global mapping G of $\mathcal{A}$ is injective. This means that all computational orbits $c, G(c), G^2(c), \ldots$ form a cycle for any configuration c. The problem is to decide, given two configurations, whether they are in the same orbit.
11. Design a zero-knowledge proof for the 3CNFF problem.
12. Design a zero-knowledge proof for the graph isomorphism problem.
13. (**Multiset equality test**) The following test play a useful rôle in some result checkers. Given two arrays of elements, X and Y, and an integer k, output "yes", if X and Y represent the same multiset and output "no" otherwise. Show how to make this test in such a way that the probability of error is $\frac{1}{2^k}$.
14. Design a result checker for for the discrete logarithm problem.
15. Design a result checker for the quadratic residuocity problem.
16. * Design a self-correcting program for multiplication of Boolean matrices.

QUESTIONS

1. What are the main primitives of cryptographical protocols?
2. What is the essence of the bit commitment problem and how to implement this problem?
3. What are potential applications of the oblivious transfer problem?

4. Are the numbers $\frac{2}{3}$ in Definition 9.2.3 magic or can they be replaced by some other without an essential impact?
5. Can the prover and the verifier of an interactive proof system be seen as having conflicting aims?
6. Why it is the case that more than two provers do not increase power of polynomial interactions?
7. What are the basic ideas of proofs that particular interactive proofs are zero-knowledge proofs?
8. What is the main advantage of using result checkers comparing with the other methods of program verification?
9. In which sense has a checker to be essentially different from a program to be checked?
10. What does a random self-reducibility property mean?

9.6 Historical and Bibliographical References

Manuel Blum (Turing award for 1995), in his seminal paper (1981) showed that apparently impossible communication problems are possible and gave an impedus to an explosion of papers on cryptographical protocols. A systematic presentation of such protocols is, for example, in Salomaa (1990) and Schneier (1994). For the poker-playing protocol see Shamir, Rivest and Adleman (1981) and Goldwasser and Micali (1984). The age difference finding protocol is due to Yao (1982b). The notion of oblivious transfer protocol is due to Rabin (1981) and the interlock protocol is due to Rivest and Shamir (1984).

The concept of interactive proof system was developed by Goldwasser, Micali and Rackoff (1985), with applications to cryptographical protocols as the main motivation. Closely related, and more convenient for showing complexity results is the concept of Arthur versus Merlin games due to Babai (1985). See also Babai and Moran (1988). These games are restricted cases of the Games Against Nature due to Papadimitriou (1985). A systematic description of interactive proof systems is in Balcázar, Díaz and Gabáro(1988). Graph non-isomorphism protocol from Section 9.2 is due to Goldreich, Micali and Wigderson (1986).

Protocol for the permanent is due to Lund, Fortnow, Karloff and Nissan (1990) and the protocol for #SAT problem is from Beigel (1993). **IP = PSPACE** result is due to Shamir (1990)—see Babai (1990) for the history of efforts culminating with this result. The proof presented here follows Shen (1992).

The concept of multi-prover protocols was introduced by Benn-Or, Goldwasser, Kilian and Wigderson (1989) and **MIP = NEXP** theorem is due to Babai, Fortnow and Lund (1990).

The concept of holographic proofs was developed by Babai, Fortnow, Levin and Szegedy (1991). For a detailed exposition see Babai (1995). For the relation between randomized proof and nonapproximability of **NP**-complete problems see Arora , Lund, Montwani, Sudan and Szegedy (1992). For a detailed discussion of proofs concerning the classification of finite groups see Gorenstein (1985).

The idea of zero-knowledge proofs is due to Goldwasser, Micali and Rackoff (1985) and since then has developed rapidly. See also Blum (1986), Goldreich (1988), Salomaa (1990) and Feigenbaum (1992), for surveys. Another concept of zero-knowledge proofs, that are not statistically but computationally convincing, is due to Brassard, Chaum and Crèpau (1988). Blum, Feldman and Micali (1988) showed that interaction in any zero-knowledge proof can be replaced by sharing a common, short, random string. Cave example is due to Quisquater and Guillou (1990).

Theorem 9.3.10 is the consequence of results of Impagliazzo and Jung (1988) and Naor (1989). Theorem 9.3.13 is due to Ben-Or, Goldwasser, Kilian and Wigderson (1988). The proof that the class of languages with zero-knowledge proofs is identical with **BPP** provided one-way-functions exist is due to Ostrovsky and Wigderson.

The first particular result checkers are due to Freivalds (1979). However, it was M. Blum (see Blum (1988), Blum and Raghavan (1989) and Blumand Kamman (1989)) who has developed a general approach to result checking. Basic results on checking approximate numerical computations are due to Ar, Blum, Codenotti and Gemmell (1993). The idea of self-testing/correcting programs is due to Blum, Luby and Rubinfeld (1993) and Section 9.4 follows them. See also Blum and Wasserman (1994) and (1995). The last paper analysis a possibility to build result checkers and self-testing programs into microprocessors.

Vanstein (1991) has shown that for a broad class of functions (rational functions constructed from x, e^x, $\sin(ax+b)$ and $\cos(ax+b)$ using operations $+, -, \times$ and fractional exponentiation) an algebraic equation can be derived which relates to each other the values of the function at a given point and few other points, and this equation can be then used to design a result checker and a self-correcting program.

The area of interactive protocols has developed very fast. Interesting enough this has been to a large extend due to an extraordinary interaction power of email as well analyzed by Babai (1990).

Chapter 10

NETWORKS

INTRODUCTION

Information processing and computing are in principle distributive and parallel. No matter, whether they have executed on computer networks, in society, or by nature. Many of the fundamental problems of computing are therefore, naturally, in this area.

Distribution of tasks and cooperation are the basic engines behind the power the distributiveness and parallelism offer. Communication networks and methods to make a use of them, are the basic tools to harvest this power. However, conflicting requirements such communication networks should satisfy, with respect to their simplicity, regularity and efficiency, and large complexity of many basic communication tasks on networks, are the main obstacle in harvesting this potential power. A deeper understanding of basic concepts concerning networks, their characteristics, properties, algorithms, embeddings in other networks, mutual simulations and layouts are therefore of prime importance for capturing the principles and laws of distributive and parallel computing. Fortunately, such an understanding can be done to a large degree on a graph-theoretical level and this is what is done in this chapter.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. basic communication networks, their characteristics and properties;
2. methods to solve basic information dissemination problems;
3. several embedding methods, especially for embeddings into hypercubes;
4. main routing methods, including randomized ones; their properties and limitations;
5. simulation universality of main types of bounded degree networks;
6. simulation of shared memory computer on bounded degree network computers;
7. basic layout concepts and several layout methods and results.

> The firm, the enduring, the simple
> and the modest are near to virtue.
> *Confucious* (551-479) B.C.

Information processing is by its nature distributive and parallel—no matter whether performed on computer networks, in society or by nature. Many of the basic problems of computing are therefore, naturally, in this area.

Cooperation between computing elements is the basic ingredient behind harvesting the power of distributivness and parallelism and communication between processing/memory elements is the key tool to make such a cooperation fruitful.

The ideal case would be to have a direct and fast communication lines between all cooperating elements/memories. In other words, complete graphs would be an ideal interconnection structure. Unfortunately, with the current and forseeable technologies it is not feasible to implement such an ideal interconnection structure in a case of a large number of communicating elements. On the other hand, forseeable technologies clearly prefer the nearest neigbbours interconnection structures. Namely, one-, two- and three-dimensional arrays of communicating elements. On the other hand, a strong disadvantages of such interconnection structures are long communication paths between some processing/memory elements. There is therefore a strong need to explore a variety of other interconnection structures for networks and relations between them.

The main difficulty in designing good interconnection graphs are conflicting requirements such graphs should satisfy. They should be much regular so it would be easy to design cooperating programs for computer networks on such graphs; networks on such graphs should be able efficiently simulate and be efficiently simulated by networks on other interconnection structures. It should be easy and fast to send messages from all processing/memory elements of the network to others processing/memory elements and, finally, networks with such interconnection structures should be reasonably easily implemenetable with the forseeable technologies.

In this chapter we discuss several key problems of programming, embedding, simulation and layouts of communication networks. Fortunately, all these problems can be discussed well on an abstract, graph-theoretical, level.

Main types of networks are introduced in Section 10.1, where also some of their basic properties are analysed and several techniques are demonstrated how to design algorithms for such networks of processors. Basic information dissemination problems and modes are analysed in Section 10.2. Broadcasting, accumulation and gossiping are basic subproblems for many tasks in order to utilize the power of distributivness and parallelism and their effective solution is of key importance for an effective use of particular interconnection structures. An analysis of these problems also bring some techniques of communication on networks as well as an understanding how difficult are even very simple communication tasks on commonly used networks, provided one search for optimal performance.

Many problems of efficient communications between elements of networks can be further reduced to several basic routing problems—through which paths to sent information from one communicating element to another one, especially if many such communications are to take place concurrently. A variety of routing problems and their solutions are presented and analysed in Section 10.4.

An efficient simulation of networks with one interconnection structure on networks with different interconnection structures is of large importamce for the overall portability and efficiency of distributive and parallel programming. The existence of very straigtforward embeddings of one network into another is discussed in Section 10.3. Two other key problems of this type are discussed in Section 10.5: the existence of universal networks and simulation of PRAMs on bounded-degree networks and hypercubes.

Finally, some network implementation problems, namely layouts of networks into two-dimensional grids, are discussed on the graph-theoretical level in Section 10.6. Graph decomposition and separability plays the key rôle by that.

10.1 Basic Networks — November 8, 2005

Due to its abstraction from most of the communication problems PRAM (Figure 10.1a), is an easy to use model of parallel computing. However, more realistic seem to be models depicted in Figure 10.1b, c, d, in which a particular communication network is used to manage communications. In the last model, Figure 10.1, memory moduls are parts of processors. In all these models the underlying interconnection network and its graph theoretical properties play the key rôle.

10.1.1 Networks

There are several properties a family of graphs should have in order to be a good model of interconnection networks. Two basic ones are regularity and the existence of simple enough descriptions—for example, by means of string-identifiers of nodes—that allow to determine easily the identifiers of neighbouring nodes from the identifier of a given node. This is of crucial importance for communication problems discussed in Sections 10.2 and 10.4. Other requirements good networks should fullfil are discussed in Sections 10.3 and 10.6.

Clear and concise representation of strings, sets of strings and conversion operations between strings and integers is of large importance for clear and concise description of networks and for discussion of network problems. Let us therefore summarize this notation.

For an integer n let $[n] = \{0, 1, \ldots, n-1\}$—in particular $[2] = \{0, 1\}$. $[n]^d$ is the set of all d-tuples of elements from $[n]$. A string $\bar{a} \in [n]^d$ will be denoted by $\bar{a} = (a_{d-1}, \ldots, a_0)$ or $\bar{a} = a_{d-1} \ldots a_0$ and a_0 will be called the least significant, or the right-most, symbol of $\bar{a}$. *bin* is a conversion function from binary strings to integers and $bin_d^{-1}(n), d, n \in \mathbf{N}, \lceil \lg n \rceil < d$, is the binary representation of n using exactly d bits.

To simplify descriptions of networks and reasoning about networks we often identify nodes of networks, or of the underlying graphs, with their identifiers.

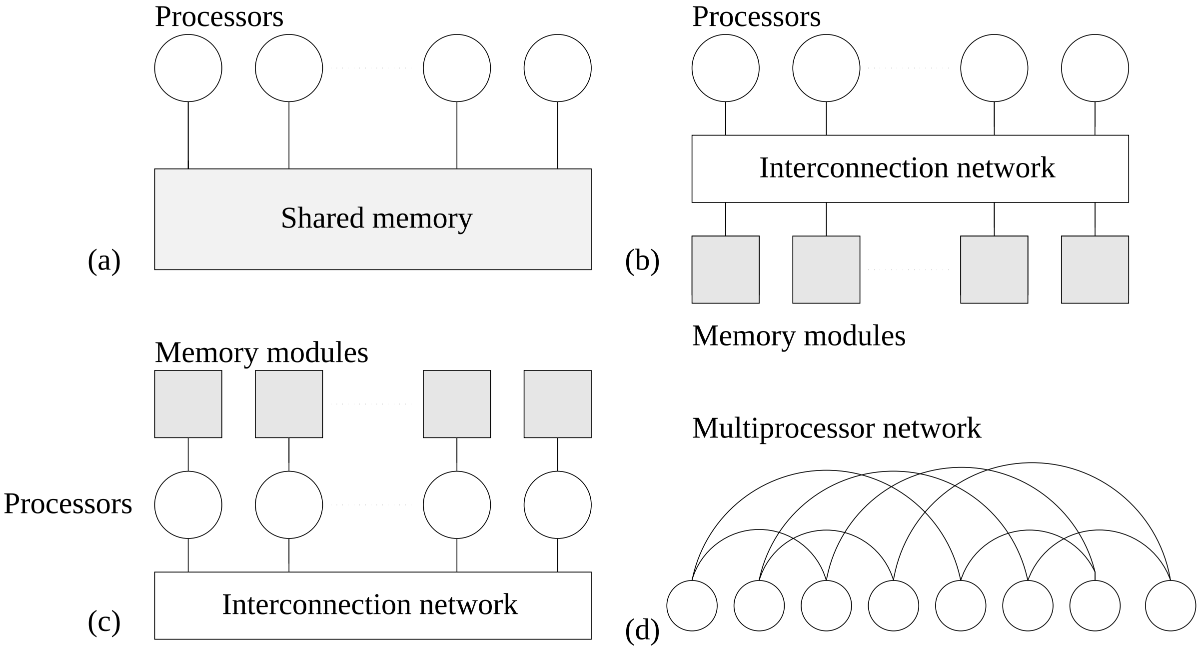


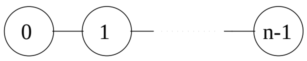
Figure 10.1: Models of parallel computing {pram-70%}

Strings $\bar{a} = (a_{d-1}, \ldots, a_0)$ and $\bar{b} = (b_{d-1}, \ldots, b_0)$ from $[n]^d$, and their **Hamming distance** $ham(\bar{a}, \bar{b}) = \sum_{i=0}^{d-1} |a_i - b_i|$ are the basic elements to describe a variety of networks. The most basic ones are **arrays** and **toroids**. If $d, n \in \mathbf{N}$, then an **(n,d)-array** $A[n,d]$ is the undirected graph $G = \langle V, E \rangle$, where $V = [n]^d$, $E = \{(\bar{a}, \bar{b}) \mid \bar{a}, \bar{b} \in [n]^d, ham(\bar{a}, \bar{b}) = 1\}$. Similarly, a **$(n,d)$-toroid** $T[n,d]$ is the undirected graph $G = \langle V, E \rangle$, where $V = [n]^d, E = \{(\bar{a}, \bar{b}) \mid \bar{a}, \bar{b} \in [n]^d, ham(\bar{a}, \bar{b}) = 1$ or $ham(\bar{a}, \bar{b}) = n - 1$ and $\bar{a}, \bar{b}$ differ only in one symbol$\}$.

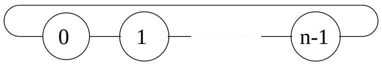
Example 10.1.1 *$A[n,1]$ is a* **one-dimensional array** *of n nodes (see Figure 10.2a); $T[n,1]$ is a* **ring (cycle)** *of n nodes (see Figure 10.2b)—also notation R_n will be used; $A[n,d]$ is a* **d-dimensional array** *of n^d nodes (see Figure 10.2c for $n = 4, d = 2$); $T[n,d]$ is a* **d-dimensional toroid** *of n^d nodes (see Figure 10.2d for $n = 4, d = 2$); and $A[2,d]$ is a* **d-dimesional hypercube**, *notation H_d (see Figure 10.2e for $d = 3$). Another way to depict H_3 is shown in Figure 10.2j.*

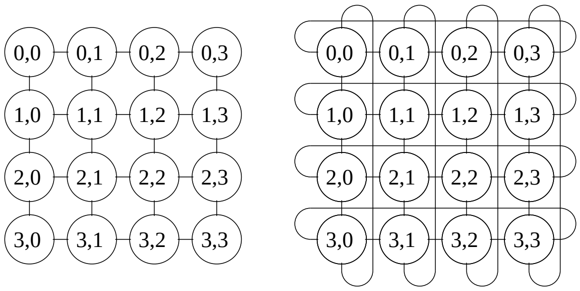
The hypercube is one of the basic networks with many natural and important properties and we shall analyze this network intensively. The following terminology will be useful by that. If two nodes of the hypercube H_d differ only in the i-th bit from the left, then they are said to be connected by an **edge in the i-th dimension**.

Exercise 10.1.2 *Show that if $\bar{a}, \bar{b} \in [2]^d$, $ham(\bar{a}, \bar{b}) = t$, $1 \leq t \leq d$, then between nodes $\bar{a}$ and $\bar{b}$ of the hypercube H_d there are: (a) t node-disjoint paths of length t; (b) $d - t$ node-disjoint paths of length $t + 2$.*

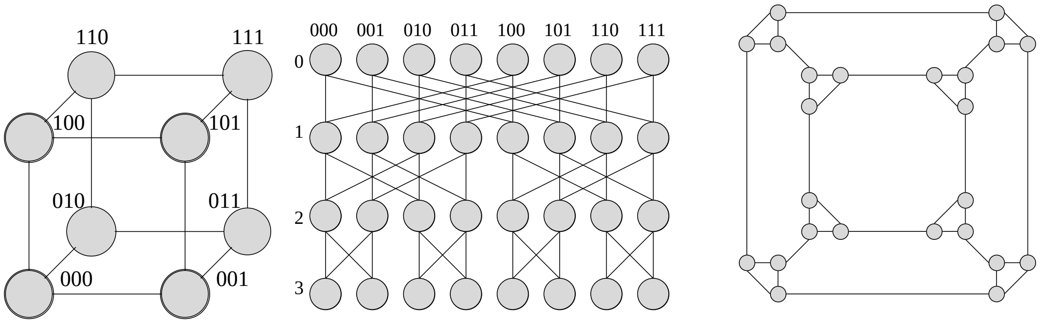


(a) Linear array A[n,1]

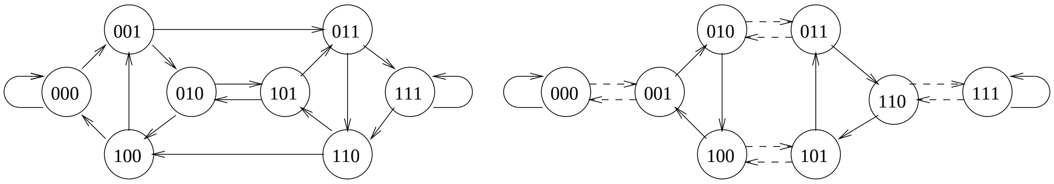




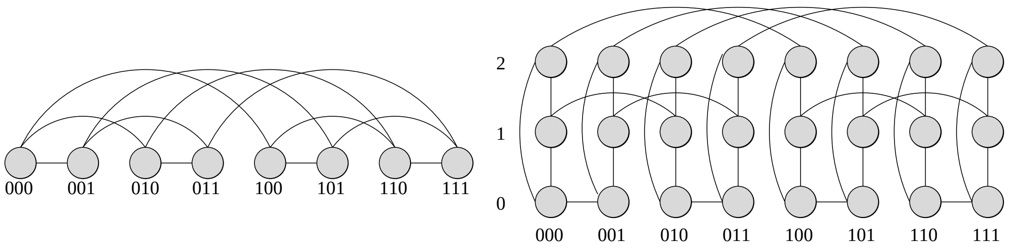
(b) Ring R_n (c) Array A[4,2] (d) Toroid T[4,2]



(e) Hypercube H_3 (f) Butterfly B_3 (g) Cube-connected cycles CCC_3



(h) De Bruijn graph DB_3 (i) Shuffle exchange graph SE_3



(j) Hypercube H_3 (k) Cube-connected cycles CCC_3

Figure 10.2: Basic networks — {bn-bn-65%}

The following networks are closely related to hypercubes H_d: the butterfly networks B_d (see Figure 10.2f for $d = 3$), the wrapped butterfly networks, and the cube-connected cycles CCC_d (see Figure 10.2g for $d = 3$).

The **butterfly network** B_d can be seen as an unfolded hypercube H_d. The i-th level of edges, $1 \leq i \leq d$, of the buterfly network B_d can be seen as an unfolding of H_d according to the edges of the i-th dimension. Formally, the butterfly network B_d is an undirected graph $B_d = \langle V_d, E_d \rangle$, where $V_d = [d+1] \times [2]^d$, and $E_d = \{((i, \bar{a}), (i+1, \bar{b})) \,|\, \text{either } \bar{a} = \bar{b} \text{ or } \bar{a} \text{ and } \bar{b} \text{ differ in the } (i+1)\text{-th leftmost bit and in no other bit}\}$. For any $s \in [2]^d$, the nodes from the set $[d+1] \times \{s\}$ form a column of the butterfly. For any $j \in [d+1]$, the nodes from the set $\{j\} \times [2]^d$ form the so-called j-th level (or j-th **rank**) of B_d. The nodes of the rank 0 (of the last rank) are sometimes called the **source** (**target**) nodes. The **wrapped butterfly** WB_d is obtained from the butterfly B_d by identifying the first and the last node of each column of B_d.

The **cube-connected cycles** CCC_d can be seen as being obtained from the hypercube H_d by first removing all nodes and then connecting all remaining edges incident with each node of H_d by a cycle (see Figure 10.2g). Formally, the cube-connected cycles is an undirected graph $CCC_d = \langle V_d, E_d \rangle$, where $V_d = [d] \times [2]^d$, $E_d = \{((i, \bar{a}), ((i+1) \bmod d, \bar{a})) \,|\, 0 \leq i < d, \bar{a} \in [2]^d\} \cup$

10.2 Dissemination of Information in Networks

The ability of a network to disseminate effectively information is an important criterion for its suitability for parallel computing.

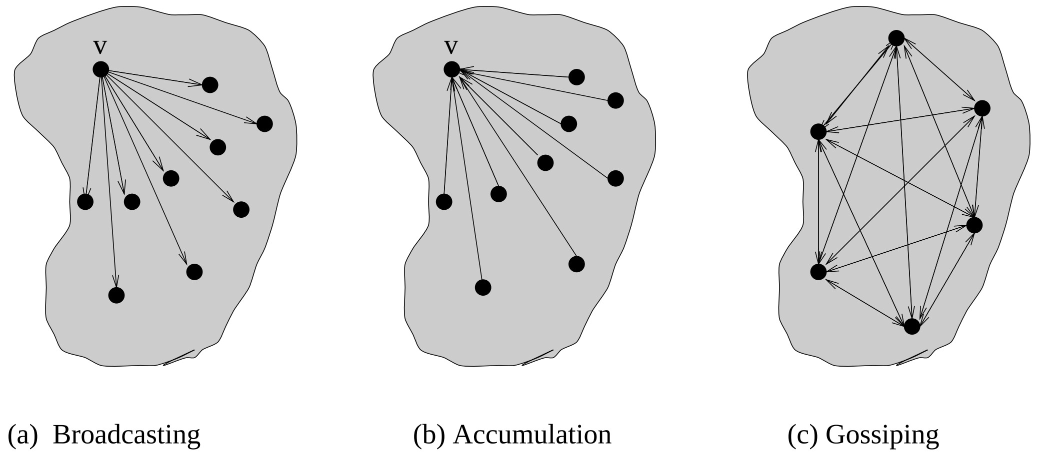
10.2.1 Information Dissemination Problems

In this section we first introduce three basic communication problems for networks—broadcasting, accumulation and gossiping.

Solution of many programming tasks for networks can be either reduced to these tasks or they at least play an important rôle in their overall solution. We discuss two basic communication modes and present both algorithms and lower bounds for these information dissemination problems for several types of communication networks. In doing that we illustrate how difficult is the task to make an optimal use of communication facilities main types of networks offer, and also some techniques how to manage information dissemination problems.

1. **Broadcasting problem** for a graph $G = \langle V, E \rangle$ and a node $v \in V$. Let v know an information $I(v)$ which is unknown to other nodes of G. The problem is to find the best communication strategy, in the sense discussed later, such that all nodes of G get $I(v)$—see Figure 10.3a.

2. **Accumulation problem** for a graph $G = \langle V, E \rangle$ and a node $v \in V$. Let each node u know some information $I(u)$ and for two different nodes u, w let their informations $I(u), I(w)$ be different. The set $I(G) = \{I(u) \,|\, u \in V\}$ is called **cumulative message**



(a) Broadcasting (b) Accumulation (c) Gossiping

Figure 10.3: Basic information dissemination problems–{bdp-65}

of G. The problem is to find the best communication strategy such that the node v learns the cumulative message $I(G)$—see Figure 10.3b.

3. **Gossiping problem** for a graph $G = \langle V, E\rangle$. Let each node u know information $I(u)$ and let $I(u)$ and $I(w)$ be different for different nodes u, w. The problem is to find the best communication strategy such that all nodes get the cumulative message $I(G)$—see Figure 10.3c.

A communication strategy, or a **communication algorithm**, is a specification of a sequence of parallel communication steps (**rounds**). Each of them specifies which nodes sent their information and to whom or which nodes exchange information. There are two basic modes in which such a communication can be carried out.

1. **Telegraph mode** (one-way communication). In each round each processor can either send or receive information and in doing so can communicate with at most one neighbouring node. This means that a one-way communication algorithm for a graph $G = \langle V, E\rangle$ is a sequence $C_1, \ldots, C_s$ where each $C_i \subseteq C = \{(v,u),(u,v) \mid (u,v) \in E\}$, such that if (x_1, y_1) and (x_2, y_2) are two different pairs from C_i, then $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$. (In other words, each C_i contains ordered pairs of nodes that communicate in the i-th round, and no node can be in two such pairs.)

Example 10.2.1 *The following communication algorithm for one-dimensional array*

$$\{(x_1,x_2),(x_9,x_8)\},\{(x_2,x_3),(x_8,x_7)\},\{(x_3,x_4),(x_7,x_6)\},\{(x_4,x_5)\}$$

is depicted in Figure 10.4a, where integers specify rounds.

2. **Telephone mode** (two-way communication). In each round each processor can exchange information with at most one neighbouring node. This means that a two-way communication algorithm for a graph $G = \langle V, E\rangle$ is a sequence $C_1, \ldots, C_s$ such that $C_i \subseteq E$ and such that if (x_1, y_1) and (x_2, y_2) are two different pairs from C_i, then $\{x_1, y_1\} \cap \{x_2, y_2\} =$

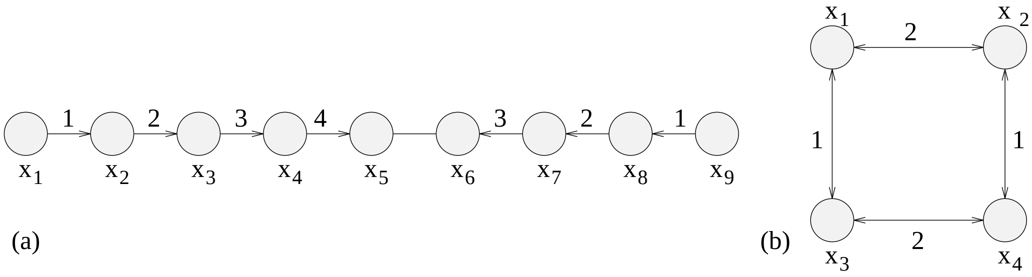


Figure 10.4: Communication algorithms—{ca-60}

$\emptyset$. (In other words, each C_i contains pairs of nodes that communicate in the i-th round and no node can be in two such pairs.)

Example 10.2.2 *A two-way gossiping algorithm* $\{(x_1,x_3),(x_2,x_4)\},\{(x_1,x_2),(x_3,x_4)\}$ *for the 4-node ring is depicted in Figure 10.4b (integers specify rounds).*

Communication complexity of the basic information dissemination problems, with respect to two communication modes, is defined for graphs as follows.

Definition 10.2.3 *Let* $G = \langle V, E\rangle$ *be a graph,* $v \in V, i = 1, 2$.

1. **Broadcasting complexity** $b_i(v,G)$ *is the minimum number of rounds of communication algorithms for broadcasting in G from the node v in the i-way communication mode;* $b_i(G) = \max\{b_i(v,G)\,|\,v \in V\}$ *is the overall broadcasting complexity of the graph G.*
2. **Accumulation complexity** $a_i(v,G)$ *is the minimum number of rounds of any communication algorithm for accumulation in G to the node v in the i-way communication mode;* $a_i(G) = \max\{a_i(v,G)\,|\,v \in V\}$ *is the overall accumulation complexity of the graph G.*
3. **Gossiping complexity** $g_i(G)$ *of the graph G is the minimum number of rounds of any communication algorithm for gossiping in G in the i-way communication mode.*

For each graph G we have introduced altogether six complexity problems of information dissemination on G. However, as the following theorem says, only three of them are essentially different from the complexity point of view: one broadcasting and two gossiping problems—for both communication modes.

Theorem 10.2.4 $a_1(G) = a_2(G) = b_1(G) = b_2(G)$, *for each graph* $G = \langle V, E\rangle$.

Proof. (1) If $E_1, \ldots, E_k$ is a broadcasting algorithm in the telephone mode for the graph G and a node $v \in V$, then $E_k, \ldots, E_1$ is an accumulation algorithm for G and v—and vice-versa. Therefore $b_2(v,G) = a_2(v,G)$ for every v and so $b_2(G) = a_2(G)$. Analogously, we can show $b_1(G) = a_1(G)$.

(2) It is trivial t

10.3 Embeddings

Parallelism brings a new dimension to the problem of simulation of one computer on another. Since the existence of (very) different networks of processors is inherent for parallel computing, the task to develop efficient simulations of one network on another is the key problem for portability in parallel computing and for an efficient coexistence of different parallel architectures. In addition, the pure fact that we have more processors increases the complexity of simulation problems.

The ideal case for a simulation of networks with an interconnection graph G_1 by networks with an interconnection graph G_2 is when G_1 is a subgraph of G_2. A simulation can then be done with no time overhead. Less easy to deal with, but still quite good, is the case that there is a small k such that nodes of G_1 can be mapped, by an injective mapping, into nodes of G_2 in such a way that any two neighbouring nodes of G_1 are mapped into nodes of G_2 connected by a path of length at most k.

With respect to the portability of algorithms for multiprocessor networks, the embedding methods can then be viewed as high level descriptions of efficient methods how to simulate algorithms for one type of network computers on a different type of network computers. As a consequence a network into which one can well embed other networks is preferable to networks without this property.

10.3.1 Basic Concepts and Results

The basic idea of embeddings is very general, simple and natural and so are the main ways how to measure a quality of embeddings.

Definition 10.3.1 *Let* $G = \langle V_1, E_1 \rangle$ *and* $H = \langle V_2, E_2 \rangle$ *be connected graphs. An* **embedding of the "guest graph"** G *into the* **"host graph"** H *is an injective mapping* $f : V_1 \rightarrow V_2$.

The quality of an embedding f is mainly expressed using a **dilation** D_f and an **expansion** E_f defined by

$$D_f = \max_{(x,y)\in E_1} d_H(f(x), f(y)), \qquad E_f = \frac{|V_2|}{|V_1|}.$$

In other words the dilation of an embedding is the maximum distance of the images, in the host graph, of the adjacent nodes of the guest graph. The expansion of an embedding is the ratio of the number of nodes in the host graph to the number of nodes in the guest graph.

We shall often discuss the problem to embedf graphs from a family $\mathcal{G}_1$ of graphs into graphs from another graph family $\mathcal{G}_2$. In such a case a graph $G' \in \mathcal{G}_2$ will be called **optimal** to a graph $G \in \mathcal{G}_1$ if $|G| \leq |G'|$ and there is no other graph $G'' \in \mathcal{G}_2$ such that $|G| \leq |G''| < |G'|$.

A natural way to extend the above concept of embedding is to consider also a mapping of edges of G into paths between the corresponding nodes of H. In such a case (embedding) **congestion**—the maximum, over all edges e of H, of the number of edges of G that are mapped into a path in H that includes e—is another important measure of embeddings.

The ideal case is when all these factors are equal to 1. Observe that a graph G is embedded into a graph H with dilation factor 1 if and only if G is isomorphic to a subgraph of H.

Example 10.3.2 *Figure 10.5a shows an embedding of a ring with* 16 *nodes into the hypercube of* 16 *nodes with dilation and expansion* 1*. Figure 10.5b shows an embedding of a ring of* 10 *nodes into the hypercube of* 16 *nodes with dilation* 1 *and expansion* 1.6*.*

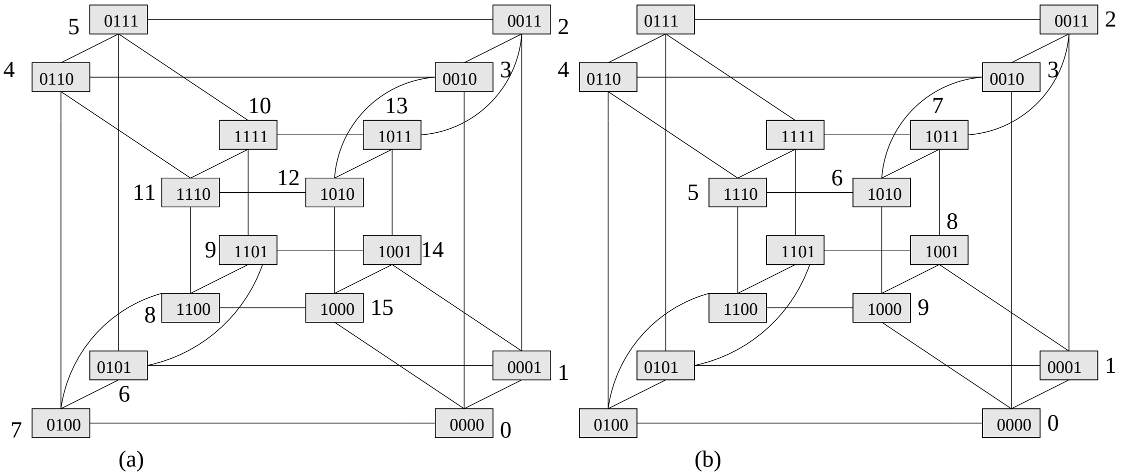


Figure 10.5: Embedding of rings into hypercube—{ring-60%}

Embedding problems are computationally inherently hard even in very restricted cases. For example, the following problems are **NP**-complete:

1. Given a binary tree T. Does there exist an embedding of T of dilation 1 into its optimal hypercube?
2. Given a graph (or even only a tree) G and an integer k. Does there exist an embedding of G into a binary tree with dilation k?

On the other hand, several important cases are easy to deal with:

Theorem 10.3.3 *The following embeddings can be done with dilation 1:*

1. *The cube connected cycles CCC_d into the wrapped butterfly WB_d.*
2. *The shuffle exchange graph USE_d into the de Bruijn graph UDB_d.*
3. *The complete binary tree T_d of depth d, into the de Bruijn graph UDB_d.*

Proof. For a word $w \in [2]^*$ let

$$k(w) = \begin{cases} 1, & \text{if } w \text{ contains an odd number of 1's;} \\ 0, & \text{otherwise.} \end{cases}$$

1. An embedding of CCC_d into WB_d of dilation 1 is given by the mapping

$$e((i,w)) = ((i + k(w)) \bmod d, w^R).$$

It follows directly from the definition of CCC_d and WB_d that the mapping e is a bijection and maps any neighbouring nodes on neighbouring nodes.

2. The mapping

$$e(w) = PS^{-k(w)}(w)$$

is an bijection and maps neighbouring nodes of USE_d into neighbouring nodes of UDB_d, for each d. The case $d = 3$ is illustrated in Figure 10.6. The proof of the general case is left as an exercise.

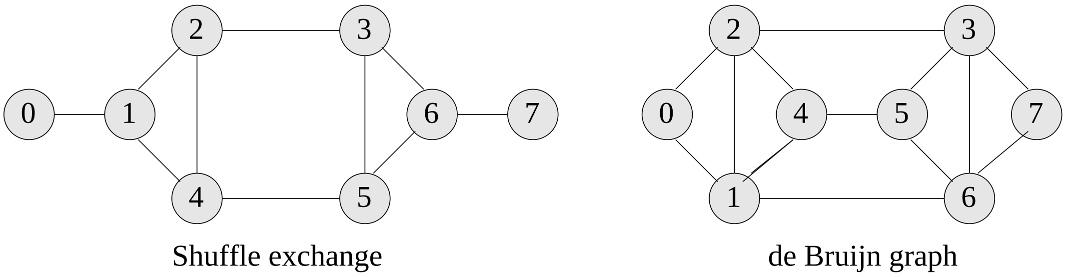


Figure 10.6: Embedding of USE_3 into UDB_3 with dilation 1-sedb 70%

3. The existence of a dilation 1 embedding of a complete binary tree T_d into the de Bruijn graph UDB_d follows from the fact that in UDB_d each node $i < 2^{d-1}$ is connected by an edge with the nodes labeled by $2i + 1$ and $2i$—see Figure 10.7.

□

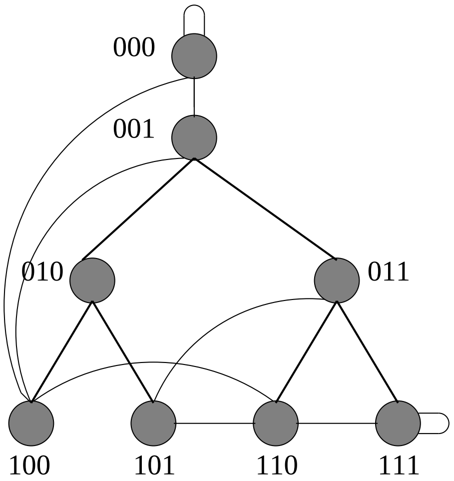


Figure 10.7: Embedding of trees into de Bruijn graphs—{tdb-85% }.

Relations between butterflies, cube-connected cycles, de Brujin and shuffle exchange graphs are so close that, with respect to ascend/descend algorithms, these networks are practically of the same quality, with respect to programming.

Exercise 10.3.4 *Show that ascend-descend algorithms for hypercubes can be implemented with only constant time overhead on butterflies, cube-connected cycles, perfect shuffle network*

10.4 Routing

Broadcasting, accumulation and gossiping can be seen as "one-to-all", "all-to-one" and "all-to-all" information dissemination problems. At the end of the dissemination one message is delivered, either to all nodes or to a particular node. A very different but also a very basic type of communication problems, the so-called routing problems, are considered in this section. They can be seen as one-to-one communication problems. Some (source) processors send messages, and each to a uniquely determined (target) processor.

There is a variety of routing problems. The most basic one is the so-called **one-packet routing problem**: how—through which path—to send a so-called packet (i, x, j) with a message x from a processor (node) P_i to a processor (node) P_j. For the one-packet routing problem it is naturally the best to send the packet along the shortest paths between P_i and P_j. All networks considered in this chapter have an important property that one-packet routing along the shortest path can be performed by a simple **greedy routing algorithm** with which each processor can easily decide, depending on the target, which way to sent a packet it has received.

For example, to send a packet from a node $u \in [2]^d$, to a node $v \in [2]^d$ in the hypercube H_d, the following algorithm can be used:

The **left-to-right routing on hypercubes**. The packet is sent first from u to that neighbour w of u, which is obtained from u by flipping in u the leftmost bit different from the corresponding bit of v, and then, recursively, the same algorithm is used to send the packet from w to v.

Example 10.4.1 *In the hypercube* H_6 *the greedy routing takes the packet from the node* $u = 010101$ *to the node* $v = 110011$ *through the following sequence of nodes*

$$u = \underline{0}10101 \rightarrow 110\underline{1}01 \rightarrow 1100\underline{0}1 \rightarrow 110011 = v,$$

where the underlined bits are those that determine the edge to go through in the given routing step.

In the shuffle exchange network SE_d in order to send a packet from a processor $P_u, u = u_{d-1} \ldots u_0$ to $P_v, v = v_{d-1} \ldots v_0$, bits of u are rotated (what corresponds to sending a packet through a shuffle-edge) and, after each shoufle-edge routing, if necessary, the last bit is changed (what corresponds to sending a packet through an exchange edge). This can be illustrated as follows:

$$
\begin{aligned}
u = {} & u_{d-1}u_{d-2}\dots u_0\\
& \overbrace{\rightarrow}^{PS} u_{d-2}u_{d-3}\dots u_0u_{d-1}\\
& \overbrace{\rightarrow}^{EX?} u_{d-2}u_{d-3}\dots u_0v_{d-1}\\
& \quad \overbrace{\rightarrow}^{PS} u_{d-3}u_{d-4}\dots u_0v_{d-1}u_{d-2}\\
& \quad \overbrace{\rightarrow}^{EX?} u_{d-3}u_{d-4}\dots u_0v_{d-1}v_{d-2}\\
& \qquad \rightarrow\\
& \qquad \rightarrow\\
& \qquad\qquad \vdots \;\; \vdots \;\; \vdots\\
& \qquad\qquad\qquad \rightarrow u_0v_{d-1}\dots v_1\\
& \qquad\qquad\qquad \overbrace{\rightarrow}^{PS} v_{d-1}\dots v_1u_0\\
& \qquad\qquad\qquad \overbrace{\rightarrow}^{EX?} v_{d-1}\dots v_1v_0 = v
\end{aligned}
$$

Exercise 10.4.2 *Describe a greedy one-packet routing for: (a) butterfly networks; (b) de Bruijn graphs; (c) mesh of trees; (d) toroids; (e) star graphs; (f) Kautz graphs.*

More difficult, but also very basic is **the permutation routing problem**: how to design a special (permutation) network or a routing protocol for a given network of processors, such that all processors (senders) can simultaneously send messages to other processors (receivers) for the case that there is a one-to-one correspondence between senders and receivers (given by a to-be-routed permutation π).

A message x from a processor P_i to a processor P_j is usually sent as a "packet" (i, x, j). The last component of such a packet is used, by a routing protocol, to route the packet on the way from the processor P_i to the processor P_j. The first component is used in case there is "a need" for a response.

The main new problem is that of (routing) **congestion**. It may happen that several packets try to pass through a processor or an edge. To handle such situations processors (and edges) have buffers and it is natural to require that only small size buffers are needed for any routing). Buffer-size of a network, with respect to a routing protocol, is the maximal size of buffers needed by particular processors or edges.

A routing protocol is an algorithm each processor executes in order to perform a routing. In one routing step each processor P performs the following operations: chooses a packet $(i, x, \pi(i))$ from its buffer, chooses a neighbourhood node P' (according to $\pi(i)$) and tries to sent the packet to P', where the packet is stored in the buffer if the buffer is not yet full. If the buffer of P' is full, the packet remains in the buffer of P.

Routing is **on-line** (or **without preprocessing**) if the routing protocol does not depend on the permutation to be routed; otherwise it is **off-line** (or **with preprocessing**).

The **permutation routing problem** for a graph G is the problem to design such a permutation routing protocol for networks with G as the underlying graph, that the routing is done as efficiently as possible. We can therefore talk about the computational complexity of the permutation routing for a graph G and also about upper and lower bounds for this complexity.

10.4.1 Permutation Networks

A permutation network connects n source nodes P_i, $1 \leq i \leq n$—for example processors—and n target nodes M_i, $1 \leq i \leq n$—for example memory modules; see Figure 10.8a. Their elements are binary switches, see Figure 10.8b, that can be in one of two states: **on** and **off**. Each setting of states of switches realizes a permutation π in the following sense: for each i there is a path from P_i to $M_{\pi(i)}$ and any two such paths are edge-disjoint. Permutation networks that can realize any permutation $\pi : \{1,\ldots,n\} \mapsto \{1,\ldots,n\}$ are called **non-blocking permutation networks** (or **permuters**).

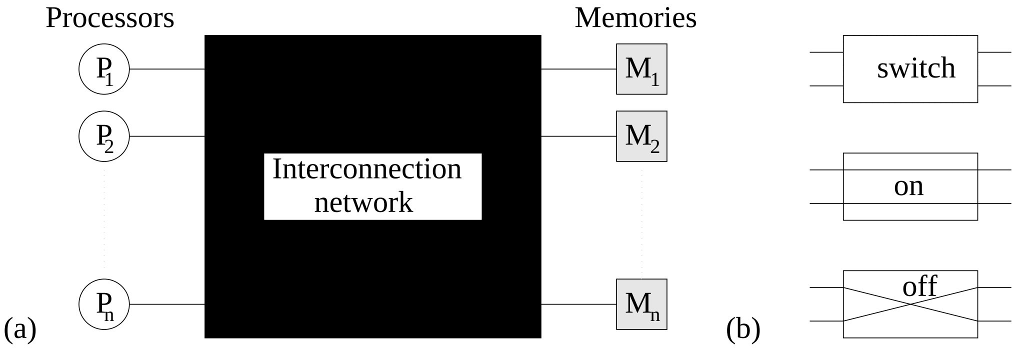


Figure 10.8: Permutation network and switches—{pn-70%}

A very simple permutation network is an $n \times n$ **crossbar switch**. On any intersection of a row and a column of an $n \times n$ grid there is a binary switch. Figure 10.9 shows a realization of the permutation $(3, 5, 1, 6, 4, 2)$ on a 6×6 crossbar switch.

An $n \times n$ crossbar switch has n^2 switches. Can we do better? Is there a permutation network which can realize all permutations and has asymptotically less than n^2 switches?

A lower bound on the number of switches can be easily established.

Theorem 10.4.3 *Each permutation network with n inputs and n outputs has $\Omega(n \lg n)$ switches.*

Proof. A permutation network with s switches has 2^s global states—each setting of switches (into an "off" or an "on" state) forms a global state. Since this network should implement any permutation of n elements it must hold (using Stirling's approximation from page 33):

$$2^s \geq n! \approx \frac{1}{\sqrt{2\pi n}} n^n e^{-(n+0.5)}$$

and therefore $s \geq n \lg n - c_1 n - c_2$, where c_1, c_2 are constants. $\Box$

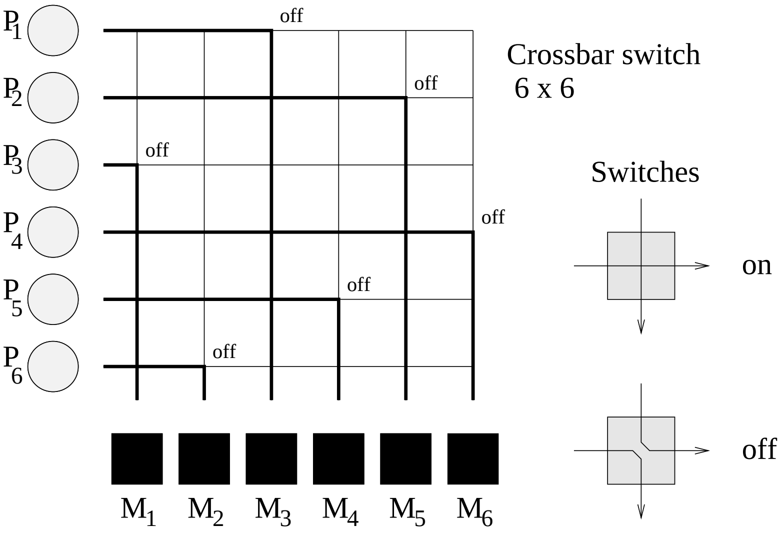
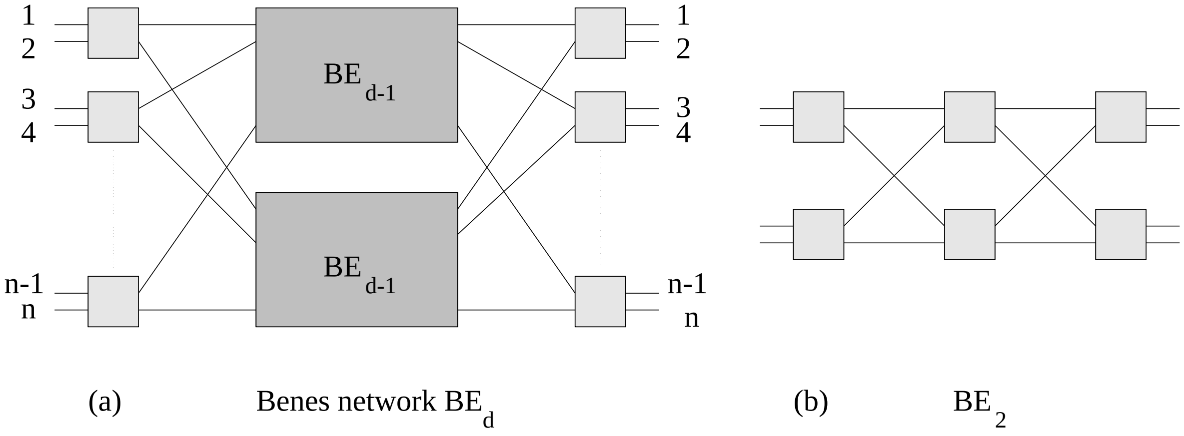


Figure 10.9: A crossbar switch and realization of a permutation on it—{cs-70%}

We show now that this asymptotic lower bound is achievable by the so-called **Beneš network** BE_d (also called the **Waksman network** or the **back-to-back butterfly**). This network consists, for $d = 1$, of a single switch and for $d > 1$ is recursively defined by the scheme in Figure 10.10a—the upper output of the i-th switch S_i of the first column of switches is connected with the i-th input of the top network BE_{d-1}; the lower output of S_i is connected with the i-th input of the lower network BE_{d-1}; for outputs of BE_{d-1} networks the connections are done in the reverse way. BE_2 is shown in Figure 10.10b.



(a) Benes network BE$_d$ (b) BE$_2$

Figure 10.10: Recursive description of the Beneš networks (with $n = 2^d$) and BE_2—{70% }

From the recursive definition of BE_d we get the following recurrence for the number $s(n)$, $n = 2^d$, of switches of the Beneš network BE_d:

$$s(2) = 1 \text{ and } s(n) = 2s\left(\frac{n}{2}\right) + n \text{ for } n > 2$$

and therefore, using methods of Section 1.2, we get $s(n) = n \lg n - \frac{n}{2}$.

The Beneš networks have an important property:

Theorem 10.4.4 (Beneš-Slepian-Duguid's Theorem) *Every Beneš network BE_d can realize any permutation of $n = 2^d$ elements.*

Proof. The proof can be done elegantly using the following theorem from combinatorics:

Theorem 10.4.5 (Hall's Theorem) *Let S be a finite set and $C = \{A_i \,|\, 1 \leq i \leq n\}$ a family of subsets (not necessarily disjoint) of S such that for any $1 \leq k \leq n$, the union of each subfamily of k subsets from C has at least k elements. Then there is a set of n elements $\{a_1, \ldots, a_n\}$ such that $a_i \in A_i, 1 \leq i \leq n$, and $a_i \neq a_j$ if $i \neq j$.*

We show now by induction on d that the Beneš network BE_d can realize any permutation of $n = 2^d$ inputs. The case $d = 1$ is trivially true. Let the inductive hypothesis be true for $d - 1 \geq 1$, and let $n = 2^d$ and $\pi : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$ be a permutation.

For $1 \leq i \leq \frac{n}{2}$ let

$$A_i = \left\{ \left\lceil \frac{\pi(2i-1)}{2} \right\rceil, \left\lceil \frac{\pi(2i)}{2} \right\rceil \right\}. \tag{10.1}$$

A_i can be seen as containing the numbers of those switches of the last column—the target level—with which the i-th switch of the first column—the source level—should be connected when the permutation π is realized. (Observe that each A_i contains one or two numbers.)

Let $A_{i_1}, \ldots, A_{i_k}$ be an arbitrary collection of k different sets of the type (10.1). The union $\bigcup_{j=1}^{k} A_{i_j}$ contains numbers of all switches of the target level that should be connected by π with $2k$ inputs of the source level switches $i_1, \ldots, i_k$. Since the corresponding number of outputs is $2k$ the union $\bigcup_{j=1}^{k} A_{i_j}$ has to contain at least k elements. This means that the family $C = \{A_i \,|\, 1 \leq i \leq \frac{n}{2}\}$ satisfies the assumptions of Hall's Theorem. Therefore there is a set of $\frac{n}{2}$ different integers $a_1, \ldots, a_{\frac{n}{2}}$, $a_i \in A_i$, such that a_i is the number of a switch of the target level with which the i-th switch of the input level is connected when the network realizes the permutation π.

It is therefore possible to choose $\frac{n}{2}$ pairs $(i_j, \pi(i_j))$, $1 \leq j \leq \frac{n}{2}$, in such a way that i_j is the input of the j-th source level switch and $\pi(i_j)$ are from different switches of the target level. This means, by the induction hypothesis, that we can realize these $\frac{n}{2}$ connections in such a way that only switches of the upper part of the internal levels of switches are used. This way the problem is reduced to two permutation routing problems for Beneš networks BE_{d-1}. In order to realize the remaining interconnections we can use, again by the induction assumption, the lower subnetwork BE_{d-1}. □

Example 10.4.6 *In order to implement the permutation*

$$\pi : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

we first use switches of the upper internal part of the network to realize the following half of the permutation

$$1 \rightarrow 8,\ 3 \rightarrow 6,\ 5 \rightarrow 4,\ 7 \rightarrow 2$$

(see Figure 10.11a), and then switches of the lower internal part of the network for the rest of the permutation (see Figure 10.11b):

$$2 \to 7,\ 4 \to 5,\ 6 \to 3,\ 8 \to 1.$$

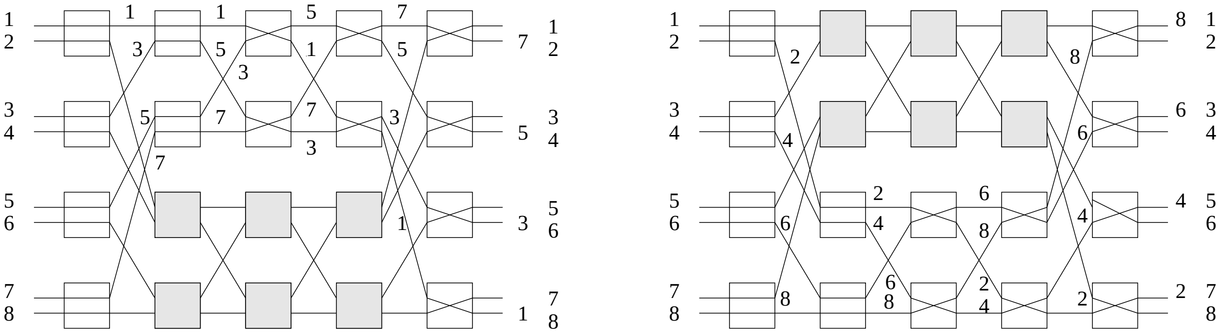


Figure 10.11: Implementation of a permutation on a Beneš network—{63%}

Exercise 10.4.7
Implement permutations: (a) $(3, 7, 8, 1, 4, 6, 5, 2)$*; (b)* $(8, 7, 6, 5, 4, 3, 2, 1)$ *on the Beneš network* BE_3*.*

Exercise 10.4.8 (Baseline network) BN_d *consists for* $d = 1$ *of one binary switch and for* $d > 1$ *is defined recursively in a similar way as the Beneš network, except that the last column of switches in the recursive definition in Figure 10.10 is missing. For the number* $S(n)$, $n = 2^d$, *of switches of* BN_d *we therefore have the recurrence* $S(n) = 2S(n/2) + n/2$*. (a) Show that* BN_d *can not implement all permutations of* $n = 2^d$ *elements; (b)* determine the upper bound on the number of permutations* BN_d *can implement.*

10.4.2 Deterministic Permutation Routing with Preprocessing

Permutation networks, such as the Beneš network, are an example of deterministic routing with preprocessing.

It is easy to see that each Beneš network BE_d actually consists of two back-to-back connected butterfly networks B_d (with the corresponding nodes of the last ranks identified)—from that comes its name "back-to-back" butterfly. In other words, the Beneš network BE_d and the network consisting of two back-to-back connected butterflies are isomorphic as graphs. Each butterfly BE_d can be seen as such an unrolling of the hypercube H_d, that edges between any two neighbouring ranks represent the edges of H_d of a fixed dimension. The previous result, namely that the Beneš networks can realize any permutation, therefore shows that one can perform permutation routing on the hypercube H_d in time $2d - 1$ and with the minimal buffer size 1 if a preprocessing is allowed. Indeed, communications between nodes of two neighbouring columns in the Beneš network correspond always to communications between nodes of the hypercubealong a fixed dimension. Hence each node of the hypercube can play the rôle of all nodes of the Beneš network in a fixed row. All $2d + 1$ communication steps of the Beneš network can therefore be realized by a proper sequence of parallel steps on the hypercube.

This holds also for other networks with logarithmic diameter that have been introduced in Section 10.1. Indeed, permutation routing on the butterfly is a fully normal algorithm and therefore, see Exercise 10.4.2, can run also on cube-connected cycles, shuffle exchange graphs and de-Bruijn graphs with only constant time overhead. In addition, preprocessing cn be done in $\mathcal{O}(d^4)$ time and therefore we have:

Theorem 10.4.9 *The permutation routing can be done on hypercubes, butterflies, cube-connected cycles, shuffle exhange and de Bruijn graphs in $\mathcal{O}(d)$ time if ($\mathcal{O}(d^4)$-time) preprocessing is allowed.*

10.4.3 Deterministic Permutation Routing without Preprocessing

In many important cases, e.g. when PRAM is simulated on multiprocessor networks—see Section **??**—permutation is not known until the very beginning a permutation routing is to be performed. In such cases the preprocessing time has to be also included into the overall routing time. Of importance is therefore to develop permutation routing protocols that are fast and do not require a preprocessing of a permutation to be routed.

Let us first realize that a permutation routing of packets $(i, x_i, \pi(i)), 1 \leq i \leq n$, corresponds to sorting of all packets according to the second key. Since sorting of 2^d elements can be done on the butterfly B_d in $\Theta(d^2)$ time—see page **??**—using a multiple ascend/descend algorithm, and each such algorithm can run with only constant time overhead on the cube-connected cycles, de Bruijn and shuffle exchange graphs we have the following result.

Theorem 10.4.10 *Permutation routing on the butterfly network B_d, hypercube H_d, cube-connected cycles CCC_d, shuffle exchange SH_d and de Bruijn graphs DB_d can be performed in time $\mathcal{O}(\lg^2 n)$ time, $n = 2^d$, without preprocessing.*

Can we do asymptotically better? The first natural idea is to consider the so-called **oblivious** routing algorithms in which the way a packet $(i, x_i, \pi(i))$ travels depends only on i and $\pi(i)$, but not on the whole permutation π. For example, can we route all packets using the greedy method for one packet routing?

It is intuitively clear that in such a case we can have congestion problems.

Example 10.4.11 *Let us consider the case that the greedy method is used in the hypercube H_{10} to realize the so-called bit-reversal permutation $\pi(a_9 \ldots a_0) = a_0 \ldots a_9$. In such a case all 32 packets from processors $P_u, u = u_1u_2u_3u_4u_5 00000$ will try to get, during the first five steps, through the node 0000000000. To route all these packets through this node at least 32 steps are needed—in spite of the fact that each two nodes i and $\pi(i)$ are at most 10 edges apart. This can be generalized to show that time $\Omega(\sqrt{2^d})$ is needed in the worst case in order to realize the bit-reversal permutation on H_d with a greedy method.*

Whenever the greedy method is used for permutation routing then the basic question arises: Is there such a strategy to solve the congestion problem that gives good routing times? In some cases the answer is yes. For example, for routing on two-dimensional arrays.

Exercise 10.4.12 *Consider an n-node linear array in which each node contains an arbitrary number of packets but from which there is at most one packet destined for each node. Show that if the edge congestion is resolved by giving priority to the packet that needs to go farthest, then the greedy algorithm routes all packets in $n-1$ steps.*

Exercise 10.4.13 * *Show that if the greedy routing is used for two-dimensional arrays, i.e. at the beginning each node contains a packet and each packet is first routed to its correct column and then to its destination within that column, then the permutation routing on an $n \times n$ array can be performed in $2n-2$ steps.*

The following general result implies that the oblivious routing cannot perform too well, in the worst case, on hypercubes and similar networks:

Theorem 10.4.14 (Borodin-Hopcroft's Theorem) *Any oblivious permutation routing requires at least $\sqrt{\frac{n}{c}}-1$ steps, in the worst case, in a network with n processors and degree c.*

Oblivious permutation routing is therefore not the way how to get under the $\mathcal{O}(\lg^2 n)$ upper bound for a routing without preprocessing on hypercubes. Is there some other way out? Surprisingly, yes.

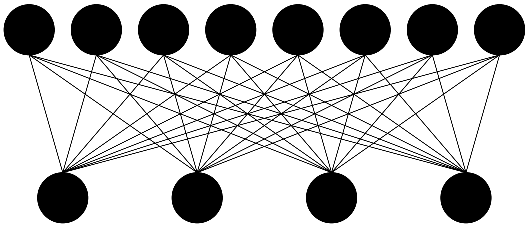
We show later that the so-called **multibutterfly network** MB_n with n input and n output processors—with $n(\lg n+1)$ processors in total—can perform any permutation routing in $\mathcal{O}(\lg n)$ time and in such a way that the buffer size of all processors is minimal (that is 1).

Multibutterfly networks MB_n are based on special bipartite graphs, the so-called **concentrators**.

Definition 10.4.15 *Let $\alpha, \beta \in \mathbf{R}^+$, $m, c \in \mathbf{N}$, and let m be even. A bipartite graph $G = \langle A \dot\cup B, E\rangle$, where A and B are disjoint, is* $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{m}, \boldsymbol{c})$**-concentrator** *if*

1. *$|A| = m$, $|B| = \frac{m}{2}$.*
2. *Degree$(v) = c$, if $v \in A$ and degree$(v) = 2c$ if $v \in B$.*
3. *(Expansion property) For all $X \subseteq A$, $|X| \leq \alpha m$, we have $|\{v \mid (x,v) \in E, x \in A\}| \geq \beta|X|$.*

That is, in an (α, β, m, c) concentrator, each set X of nodes from A, up to a certain size, αm, has many neighbours in B—at least $\beta|X|$. In other words, if $\beta > 1$, then A "expands", and β is the **expansion factor**—see Figure **??**.



10.5 Simulations

There are three main types of simulations of one network $\mathcal{N}$ on another network $\mathcal{N}'$.

Embedding — To each processor P of $\mathcal{N}$ a processor P' of $\mathcal{N}'$ is associated that simulates P.

Dynamical embedding — At each step of discrete computation time each processor of $\mathcal{N}$ is simulated by a processor of $\mathcal{N}'$. However, it can be dynamically changed, from step to step, which processor of $\mathcal{N}'$ simulates which processor of $\mathcal{N}$.

Multiple embedding — To each processor P of $\mathcal{N}$ several processors of $\mathcal{N}'$ are associated that simulate P.

The main network simulation problem: given two families of (uniformly defined) graphs, $\mathcal{G}_1$ and $\mathcal{G}_2$, develop a method how to simulate each network with the underlying graph from $\mathcal{G}_1$ by a network with the underlying graph from $\mathcal{G}_2$. For example, how to simulate networks on rings by networks on hypercubes?

The main network simulation problem consists actually of two subproblems. The first one is on the graph-theoretical level. How to map nodes of graphs from $\mathcal{G}_1$ into nodes (or sets of nodes) of graphs from $\mathcal{G}_2$ well—in such a way that the neighbouring nodes are mapped either into neighbouring nodes or at least to nodes not too far away from each other. This problem has been discussed in Section 10.3. The second subproblem consists of designing processors of the network that performs simulation. The main issue is, in case that neighbouring nodes are not mapped into neighbouring nodes, how to realize communication by routing.

As we saw in the Section 10.3, in many important cases there are good embeddings of graphs of one family of graphs into graphs of another family. This is, however, not always so and in such cases dynamical or multiple embeddings are to be used.

Example 10.5.1 (Simulation of two-directional rings by one-directional rings.) *There is no way to embed an n-node two-directional ring TR_n with self-loops—Figure*[1] *10.12a—into an n-node one-directional ring OR_n also with self-loops—Figure 10.12c—in such a way that two neighbouring nodes of TR_n would be always mapped into two nodes of OR_n that are only $\mathcal{O}(1)$ interconnections apart—in both communication directions. On the other hand, it is evident that each network over a modified one-way ring MR_n—Figure 10.12b—in which each node i is connected with nodes $(i+1) \bmod n$, $(i+2) \bmod n$ and itself, can be easily simulated by a network over OR_n with the slowdown at most by the factor 2.*

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[1] Squares stand for nodes with self-loops—self-loops themselves are not explicitly depicted.

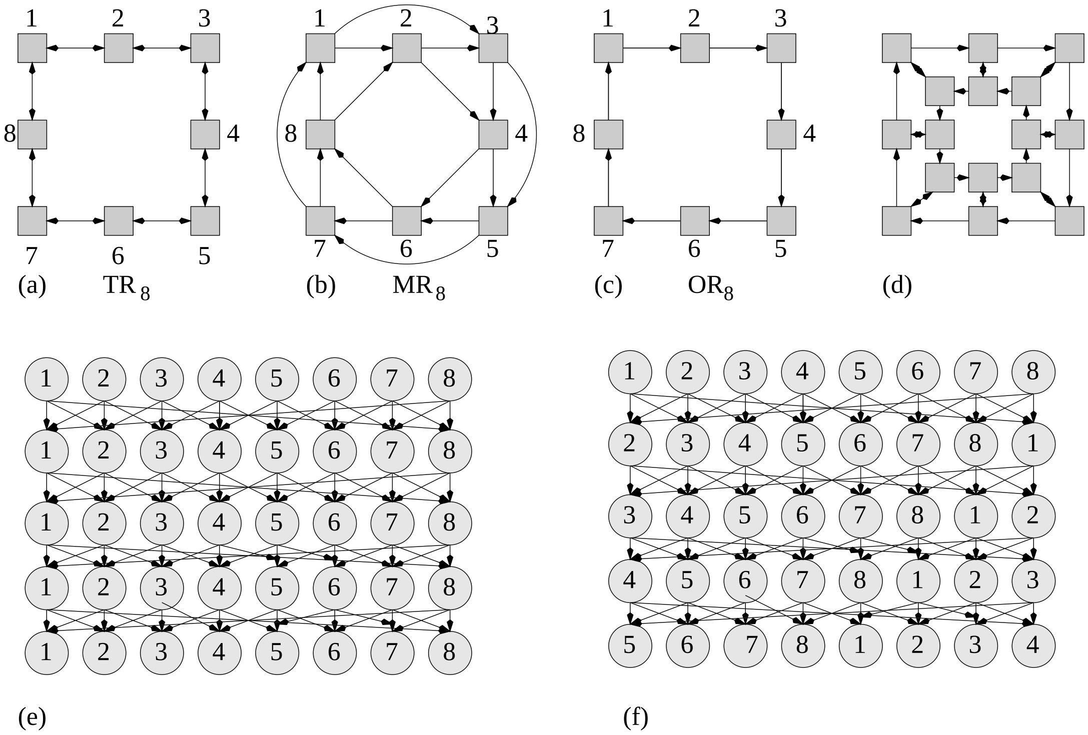


Figure 10.12: Simulation of two-directional rings on one-directional rings-60%

We show now that each network over TR_n can be easily simulated by a network over MR_n using a dynamical embedding—in the i-th simulation step the j-th node of TR_n is simulated by the $(j+i-1) \bmod n$-th node of MR_n. This means that at this simulation processors "travel around the ring". This fact, together with the existence of self-loops, allows to simulate two-directional ring communications on one-directional rings. Figures 10.12e, f show the space-time unrolling of TR_8 and MR_8 and the isomorphism of those two graphs that corresponds exactly to the dynamical embedding. (A generalization to rings with an arbitrary number of nodes is now straightforward.)

Example 10.5.2 *Figure 10.12d shows a graph consisting of two one-directional rings with opposite orientation of edges and with corresponding processors connected by undirected edges. Each network over a two-directional ring can be simulated by a network over such graph using a multiple embedding.*

10.5.1 Universal Networks

Two simulation problems are considered in this section: the existence of universal interconnection graphs and simulation of PRAMs on bounded-degree networks.

We shall see that the shuffle exchange graphs and the cube-connected cycles are, in a reasonable sense, universal communication structures for networks. The same can be shown also for several others graphs introduced in Section 10.1: de Bruijn graphs, wrapped butterflies, hypercubes. These results again show that graphs introduced in Section 10.1 are reasonably well chosen—to model network-computers interconnections.

Definition 10.5.3 *A family of graphs* $\mathcal{G} = \{G_1, G_2, \ldots\}$ *is a family of bounded-degree graphs if there is a constant* c *such that* $degree(G_i) \leq c$ *for all* $G_i \in \mathcal{G}$. *A graph* G_0 *is* k*-universal,* $k \in \mathbf{N}$*, for a family* $\mathcal{G}$ *of bounded-degree graphs, if each network on a graph from* $\mathcal{G}$ *can be simulated by a network on* G_0*, with the time overhead* $\mathcal{O}(k)$*; if* $\mathcal{G}$ *is the class of all graphs of degree* c *and* n *nodes, then we say that* G_0 *is* $\boldsymbol{(c,n,k)}$*-universal. If a graph is* (c,n,k)*-universal for*

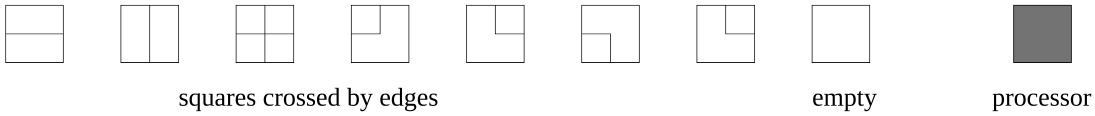
10.6 Layouts

Of large importance are special embeddings, called layouts, of graphs into two-dimensional or, potentially, three-dimensional grids, that model layouts of integrated circuits. Layouts of nodes model processors (transistors) and layouts of edges model interconnections (wires).

Enormous number of nodes of circuits that often need to be laid out, together with very strong requirements on minimality, efficiency and regularity of layouts, make layout problems difficult and complex. We can therefore present here only very basic insights, concepts, problems, methods and results concerning layouts, and in doing so we use the simplest model of layouts.

10.6.1 Basic Model, Problems and Layouts

A layout of a graph G of a degree at most 4 into the two-dimensional grid is a mapping of nodes of G into unit-squares of the grid and edges of G into disjoint paths that cross squares of the grid in one of the following ways only:



We shall deal mainly with layouts of graphs of degree at most 4. There are several reasons for doing that. Two-dimensional grids that model layout space have degree 4; main bounded-degree networks introduced in Section 10.1 have degree at most 4; methods of layouts discussed here are easily and naturally extendible to deal also with graphs of higher degrees.

Two layouts of the graph from Figure 10.13a are shown in Figures 10.13b, c; full circles denote contacts.

Remark 10.6.1 We can consider this model as representing layouts with two physical layers to run wires—what corresponds well to main layout technologies. All vertical segments of wires run in one layer and all horizontal segments in the second layer. This means that whenever a wire changes direction there must be a contact there. We shall not draw contacts—Figure 10.13b is the only case where they are depicted.

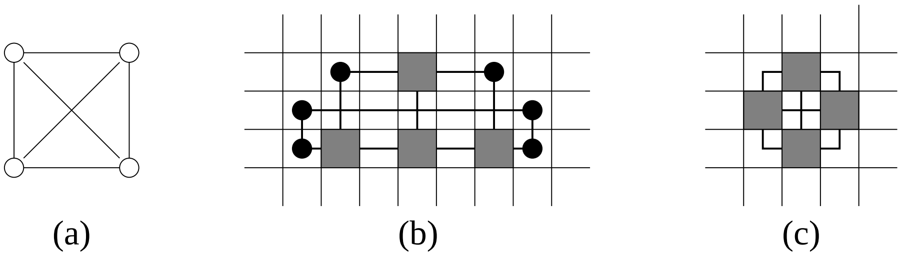


Figure 10.13: Layouts—{be-??}

The most important complexity measure of layouts is **area complexity**: the size of the smallest rectangle that contains all nodes and edges of the layout. For a graph G, let $area(G)$ be the size of the smallest rectangle containing a layout of G. For example, two layouts of the graph from Figure 10.13a, shown in Figures 10.13b,c, have area complexity $3 \times 7 = 21$ and $3 \times 3 = 9$, respectively. It is easy to see that 9 is also the area complexity of the graph from Figure 10.13a.

Exercise 10.6.2 *Design a layout with as small area as possible for : (a) mesh of trees MT_2; (b) de Bruijn graph DB_4.*

The fact that the error probability, and consequently also the production cost, grows rapidly with the size of the layout is the main reason why the area complexity of layouts is practically of such great importance.

Remark 10.6.3 In practice one actually needs more layers to layout a circuit; to run power, ground, clock, signals, and so on. Our model is easy to modify to capture also layouts with more layers; for example, 2k layers—k "horizontal" and k "vertical"—for an arbitrary but fixed k; see Figure 10.14a, for $k = 3$. Indeed, let us represent each processor by a $k \times k$ square and its i-th vertical (horizontal) layer by the i-th vertical (horizontal) edge leaving or entering a processor square. This way the area complexity can be increased at most by the factor k^2. A similar layout technique can be used to model layouts of graphs with larger degree than 4. Indeed, if the degree of a graph G is k, it is enough to model a layout of a processorby a square of the size $\lceil k/4 \rceil \times \lceil k/4 \rceil$—see Figure 10.14b for a layout of the complete graph with 6 nodes.

Two most important problems concerning layouts are:

- to design the best layouts for main interconnection graphs;
- to develop general layout methods.

Well understood are layouts of binary trees.

Example 10.6.4 *The most natural layout of the complete binary tree T_d was illustrated in the very first example of this book—Figure 1.1d. Since T_d has $n = 2^{d+1} - 1$ nodes, the area*

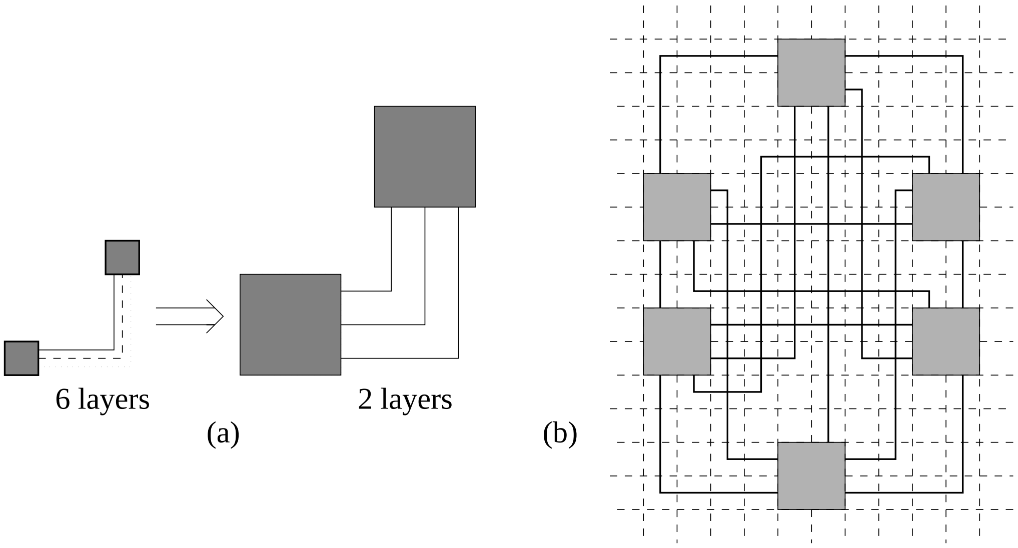


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Figure 10.14: Layout of circuits with more layers or a larger degree—{ml-70%}

complexity of that layout is clearly $\Theta(n \lg n)$*. As illustrated in Figure 1.1b,c, the so-called* H*-layout of complete binary trees has asymptotically the best possible area complexity*—$\Theta(n)$.

The minimal area complexity of a layout is not the only requirement a good layout should satisfy. If a layout of a circuit is to be made, it is often very desirable and even necessary that input nodes are on the boundaries and not inside of the layout-rectangle. For example, this is not so in case of the H-layout of binary trees if all leaves are input nodes. Inputs on boundaries are better accessible; "chips" with inputs on boundaries are more easy to pack, combine, ... Unfortunately, as the following result shows, such a requirement can asymptotically increase the area complexity of layouts.

Theorem 10.6.5 *Each layout of the complete binary tree* T_d *with* $n = 2^d$ *leaves and all leaves on the layout-rectangle boundaries, has the area complexity* $\Omega(n \lg n)$.

Two lemmas will be used to prove the last theorem.

Lemma 10.6.6 *Suppose that a complete binary tree* T *can be laid out in an area* A *with leaves on the boundaries of the layout-rectangle. Then there is a layout of* T *in area* $8A$ *with all leaves on one side of the layout-rectangle.*

Proof. By enlarging the length of each side of the layout by two length units, and therefore by at most doubling the layout area, we can achieve that no edge runs along the boundaries.

Assume now that a new layout-rectangle for T has sides h and w with $h \leq w$, and h even. We rearrange this layout as follows:

1. By extending the horizontal side of the rectangle by h, in the way shown in Figure 10.15, we can shift all nodes representing leaves—they are of the degree 1—to two horizonta

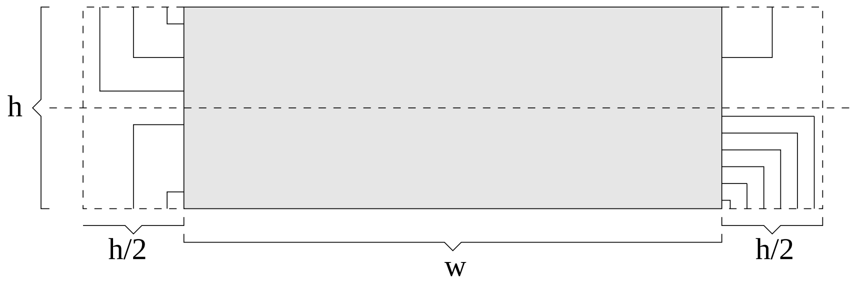

Figure 10.15: Shifting leaves of the tree to horizontal boundaries

10.7 Exercises

1. (A card trick.) The following card trick is based on magicians capability to remember exactly where into a deck of cards a volunteer inserts a chosen card, as well as on their capability to perform fast routing on the shuffle exchange graph networks.

   A volunteer is asked to pick an arbitrary card from a deck of 2^d cards and to insert it back into an arbitrary position in such a way that the magician can not see which card has been chosen. The magician then performs a certain number of outshuffle and inshuffle operations and as the result the chosen card appears at the first position in the deck (or in the k-th position where k has been announced in advance).

   Explain the trick. (An outshuffle (inshuffle) operation gets each card from a binary position $a_{d-1}\dots a_0$ into the position $a_{d-2}\dots a_0 a_{d-1}(a_{d-2}\dots a_0\overline{a_{d-1}})$.

2. Prove Lemma **??**.

3. Draw nicely: (a) DB_4; (b) CCC_4; (c) SE_4.

4. Show that the following graphs are Cayley graphs: (a) wrapped butterflies; (b) toroids; (c) star graphs.

5. (Fast Fourier Transform)** The Discrete Fourier Transform of a sequence $a_0,\dots,a_{n-1}$ is the sequence $b_0,\dots,b_{n-1}$, where $b_j = \sum_{i=0}^{n-1} a_i\omega^{ij}$ and ω is n-th primitive root of 1. Show how to compute the Discrete Fourier Transform, for $n = 2^d$, on the butterfly B_d in time $\Theta(\lg n)$, if we assume that the node (i,α) of B_d knows $\omega^{exp(i,\alpha)}$, where for $\alpha = w_1\dots w_d$, $exp(i,\alpha) = w_i w_{i-1}\dots w_1 0\dots 0$.

6. (Fibonacci cube.) For an integer i let i_F denote the unique Fibonacci representation of i— see Exercise 2.1.8. The Fibonacci cube of degre d, notation FC_d, is a graph $\langle V_d, E_d\rangle$, where $V_d = \{0,1,\dots,F_d - 1\}$ and $(i,j)\in E_d$ if and only if $ham(i_F, j_F) = 1$. (a) Draw FC_2, FC_3, FC_4, FC_5; (b) Determine for FC_d: number of edges, degree of nodes and diameter.

7. The Fibonacci cube FC_d can be decomposed in various ways into Fibonacci cubes of smaller degrees. Find such decompositions.

8. Determine the number of nodes for hypercubes, de Bruijn, star and Kautz graphs of degree and diameter $2, 4, 6, 8, 10$. (You will see that de Bruijn graphs, star graphs and Kautz graphs come very favourable comparing with the hypercubes with respect to the number of nodes that can be connected in networks of the same degree and diameter.)

9. * A set S of nodes of the de Bruijn graph DB_d forms a **necklace** if S is the set of all those nodes that can be obtained from one of them using the perfect shuffle operation repeatedly. (a) Determine the number of necklaces; (b) show that there is a linear time algorithm to produce all necklaces.

10. Show: (a) Each Euler tour for a shuffle exchange graph SE_d uniquely specifies a Hamilton cycle for DB_{d-1}; (b) each de Bruijn graph has a Hamilton cycle.

11. The problem to determine exactly the bisection-width for de Bruijn graphs and shuffle exchange graphs is still open. (a) Determine the bisection-width for SE_d and DB_d for $d = 2, 3, 4, 5$. (b) DB_7 is possible to bisect by removing 30 edges. Show that. Can you do better?

12. (Generalized de Bruijn and Kautz graphs.) Let $m, d \in \mathbf{N}$. Generalized de Bruijn graphs $GDB(m, d) = \langle V, E\rangle, V = [m]^d$, $E = \{(a_{d-1} \dots a_0, a_{d-2} \dots a_0 x) \mid a_{d-1} \dots a_0 \in [m]^d, x \in [m]\}$. Generalized Kautz graphs $GK(m, d) = \langle V, E\rangle$, $V = \{\bar{a} | \bar{a} \in [m+1]^d$ and no two consecutive symbols of $\bar{a}$ are the same$\}$, $E = \{(a_{d-1} \dots a_0, a_{d-2} \dots a_0 x) \mid a_{d-1} \dots a_0 \in V, a_0 \neq x\}$. Determine the number of nodes, edges, degree and diameter for generalized de Bruijn and Kautz graphs.

13. Show that in the generalized de Bruijn graphs $GDB(m, d)$ there is exactly one path of length d between any two nodes and therefore $M(m, d)^d = I$, where $M(m, d)$ is the adjancency matrix for $GDB(m, d)$.

14. * Show that in the generalized Kautz graphs $GK(m, d)$ there is exactly one path of length d or $d-1$ between any two nodes. Show that $M(m, d)^{d-1} + M(m, d)^d = I$, where $M(m, d)$ is the adjancency matrix for $GK(m, d)$.

15. Describe greedy routing methods for the generalized de Bruijn and Kautz graphs.

16. (Fault tolerance.) Fault tolerance of a graph is the minimum number of nodes or edges that can be removed to disconnect the graph. It is usually defined through node- and edge-connectivity. **Node-connectivity** $k(G)$ of a graph G is the minimum number of nodes whose removal disconnects G. **Edge-connectivity** $\lambda(G)$ is the minimum number of edges whose removal disconnects G. (a)* Show that $k(G) =$ maximum of the number of node-disjoint paths $p(u, v)$ over all nodes u, v of G; (b)** $\lambda(G) =$ maximum of the number of edge-disjoint paths $p(u, v)$ over all nodes u, v of G; (c) determine node- and edge-connectivity for the following graphs: (c) arrays; (d) toroids; (e) hypercubes; (f) cube-connected cycles; (g) de Bruijn graphs; (h) shuffle exchange graphs; (i) star graphs; (j) Kautz graphs.

17. (Möbius graphs.) A 0-Möbius graph, notation 0-M$_d$ is defined by: 0-M$_d = \langle V, E\rangle, V = [2]^d, E = E_1 \cup E_2$, where $E_1 = \{(a_{d-1} \dots a_0, a_{d-1} \dots a_{i+1} \bar{a}_i a_{i-1} \dots a_0)$, $a_{i+1} = 0$ or $i =$

$d-1\}$ $E_2 = \{(a_{d-1} \ldots a_0, a_{d-1} \ldots a_{i+1}\bar{a}_i\bar{a}_{i-1} \ldots \bar{a}_0),\ a_{i+1} = 1\}$ (a) Depict 0-M$_2$, 0-M$_3$, 0-M$_4$; (b) show that the diameter of 0-M$_d$ is $\lceil \frac{d+2}{2} \rceil$ (therefore smaller than for the hypercube H_d.)

10.8 Historical and Bibliographical References

The communication network theory has several roots and their offsprings have merged recently.

The oldest root is in the graph theory. Cayley (1889), connected the graph and group theory and created grounds for a general and abstract treatment of regular graphs. In addition, graph theory in general forms an important theoretical base for communication network theory. Design of switching networks, especially in connection with the very first very large scale project in parallel and distributed communication—telephone networks—was the first significant technology rooted motivation. Permutation network and sorting network problems, well defined, motivated, and intriguing, with pioneering works of Beneš (1964, 1965) and Batcher (1968), created an important area of research. Emerging ideas of parallel computing brought another motivation and led to the study of hypercubes and bounded-degree regular networks—see Schwartz (1980) for a survey of earlier works. Experiences with the first designs and utilizations of parallel computers and a search for good models of parallel computing turned attention to such problems as routing, embeddings, and simulations. VLSI technology brought the layout problem. The last root of modern network theory is within computational complexity theory. Attempts to understand power and limitations of parallel computing soon revealed communication problems as the key ones and led to investigation of a variety of networks—with aims to derive new upper and lower bounds.

The most comprehensive treatment of hypercubes and also bounded-degree networks—properties, routing, simulations, and embeddings—is in the book by Leighton (1992). One can find also detailed historical and bibliographical references there. Information dissemination problems are well surveyed by Hromkovič, Klasing, Monien, and Peine (1995). Layout problems are systematically presented by Ullman (1984) and Lengauer (1990, 1990a). Complexity approaches to networks are surveyed by Pippenger (1990).

Hypercubes and their properties are discussed in details in Leighton (1992), Harary, Hayes and Wu (1988), Lakshmivarahan and Dhall (1990). The origin of the butterfly network is in the early works on Fast Fourier Transform—invented in 1920's—see Press, Flannery, Teukolsky and Vetterling (1986). The cube-connected cycles network was introduced by Preparata and Vuillemin (1981); the shuffle exchange graphs by Stone (1971); the de Bruijn graphs were reinvented by many, especially by de Bruijn (1946). Kautz graphs, star graphs and Fibonacci cubes were introduced by Kautz (1968), Akers and Krishnamurthy (1986), and Hsu (1993), respectively. Meshes of trees were introduced by Muller and Preparata (1975) and trees of meshes by Leighton (1981).

Complex plane layout for the shuffle exchange graph is due to Hoey and Leiserson (1980). Full analysis of the bisection-width for the shuffle-exchange graphs is in Leighton (1992) and our presentation follows it. A variety of algorithms for the hypercube and butterfly networks is presented by Akl (1989), Ranka and Sahni (1990), Lakshmivarahan and Dhall (1990), and Ja'Ja (1992). For Exercise 9 in Section 10.7 see Liu, Hilderandt and Caven (1989).

There is a rich literature on broadcasting and gossiping. Our presentation, and also Table **??**, is an updated version from the survey by Hromkovič, Klasing, Monien and Peine (1990), where one can find many bibliographical references on the subject—also to the results presented here. Some newer bounds are due to S. Perennes (1996). Another survey on this subject is due to Fraigniaud and Lazard (1996).

Embeddings are covered by Leighton (1990), Monien and Sudborough (1992), Lakhshmivarahan and Dhall (1990).

NP-completeness results concerning embeddings on page 612 are due to Wagner and Corneil (1990) and Monien (1985). The first three claims in Theorem 10.3.3 are due to Greenberg, Heath and Rosenberg (1990) and Feldman and Unger (1992)-the fourth one is a folklore. Gray-code embeddings are discussed in details in Lakshmivarahan and Dhall (1990). Array embeddings are surveyed by Monien and Sudborough (1990) and Leighton (1990). Theorem **??** is due to Chan (1988, 1989). Embeddings of complete binary trees into hypercubes are first discussed by Havel and Liebl (1973)—see also Wu (1985) for Theorems **??**, **??** and **??**. Results on embedding of arbitrary trees into hypercube, presented in Theorem **??**, are due to Monien and Sudborough (1988) and Wagner (1987).

Table **??** is due to Ralf Klasing. The third entry in that ta

Chapter 11

COMMUNICATIONS

INTRODUCTION

Communication is clearly of large importance for distributive, parallel and concurrent computing and communication costs often dominate computation costs. Less obvious, but of equal importance, is that communications dominate also sequential computing and that communication problems have inherent communication complexity and the discovery of this complexity brings deep theoretical insights.

In the previous chapter we have explored a variety of communication structures (graphs) per se—no matter what they are used to communicate. In this chapter, on the other hand, we deal with communications per se—no matter how powerfull, computationally, are communicating parties, and which way they communicate. We explore power of various communication modes, especially deterministic and randomized ones, of various partitions of input data and also basic relations between communication and computational complexity.

LEARNING OBJECTIVES

The aim of the chapter is to learn:

1. basic concepts concerning communication protocols and their complexity;
2. lower bounds methods for fixed partitions and their relative power;
3. a method to determine the AT^2-complexity of VLSI circuits;
4. several types of nondeterministic and randomized protocols: Las Vegas, Monte Carlo and BPPC protocols;
5. relations between the power of different types of communication protocols;
6. basic relations between communication and computational complexity.

> Genuine poetry can communicate
> before it is understood.
>
> *Thomas Stearns Elliot (1929-1965)*

One of the main discoveries of the modern search for the essence of difficulties in computing is that it is actually the inherent communication complexity of a problem, independent of a particular communication structure, that is the main factor contributing to the inherent computational complexity of the problem.

At first it was in parallel and distributed computing where this understanding came up. Especially after the VLSI technology had been developed, where the cost of communication is the dominating factor—both from the point of view of chips production and performance. The following investigations of the so-called AT^2-complexity of VLSI circuits and Boolean functions revealed clearly the central rôle of communication complexity. Problems with the performance and programming of networks of processors also pointed out communication as the critical issue. Interactive protocols have been another area where the surprising power of communications and interactions came up clearly, on a deep theoretical level, and with important applications.

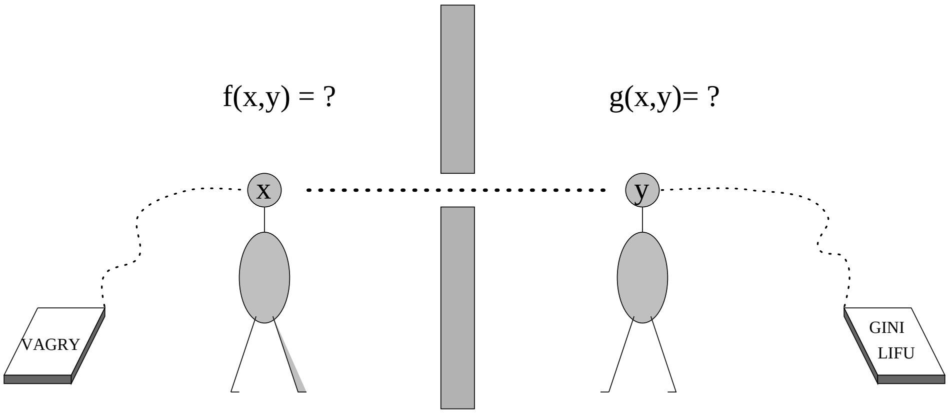

Figure 11.1: Communications between two parties to solve a problem - 70%

In the PRAM model of parallel computing communication problems are ignored and only computations are considered. On the other hand, in the study of communication complexity, what is the aim of this chapter, computations are completely ignored and only communications are considered. Two communicating parties, each of them possessing only a part of the input data, are allowed to have unlimited computational power. We are only interested in how many bits they have to exchange in order to solve a given problem—see Figure 11.1.

It has turned out, surprisingly, that this idea to isolate communication from computation, and to study it per se, on the first sight almost absurd, is actually very fruitful and brings deep insights into the nature of computation and also into the nature of computational complexity. It has also turned out that communication complexity has deep relation to such computational problems that seem to have, at the first sight, nothing to do with communication.

11.1 Examples and Basic Model — November 8, 2005

Given a computational problem P and its input data partitioned into two sets X and Y, the basic question is how much information two parties, A and B, both with unlimited computational power, have to exchange in order to solve P, provided that, at the beginning of a communication, the party A knows only the input data from X and B only the input data from Y.

11.1.1 Examples

Example 11.1.1 *(Parity function.) Let us assume that B is to compute the* **parity function** $z = \sum_{i=1}^{n} a_i \bmod 2$, *where* $a_i \in \{0,1\}$, *but B knows only $n-k+1$ bits $a_k, \ldots, a_n$, for some $k > 1$, and A knows the rest, $a_1, \ldots, a_{k-1}$, of the input bits. It is easy to see that it is enough for A to send to B a single bit, namely* $\sum_{j=1}^{k-1} a_j \bmod 2$. *With this information B can determine z. Observe also that for this communication algorithm—protocol—it is of no importance how large k is. Also in case that the input data are partitioned between A and B in some other way, it is always for A sufficient to send to B a single bit (provided B knows what this one bit mean). In other words, for the communication during a parity function computation by two parties it is not important how the input data are partitioned between two parties.*

Example 11.1.2 *(Palindrome recognition.) Let us again assume that the party A knows a prefix $x = a_1 \ldots a_{k-1}$, and B knows the suffix $y = a_k \ldots a_n$ of a binary string $a_1 \ldots a_n$. This time, however, the task for B is to determine whether $z = xy$ is a palindrome. Intuitively, it seems to be clear that in the worst case it may happen that A needs to send to B the whole x and therefore the number of bits that need to be exchanged depends on how big x is. In other words, the amount of bits that need to be exchanged seems to depend much on how the input data are partitioned between two communicating parties. However, is our intuition correct?*

Exercise 11.1.3 *The party A knows bits $x_1, \ldots, x_n$, B knows bits $y_1, \ldots, y_n$. A has to compute $z_1 = \bigvee_{i=1}^{n}(x_i \vee y_i)$; B has to compute $z_2 = \bigwedge_{i=1}^{n}(x_i \wedge y_i)$. How many bits do they have to exchange?*

Example 11.1.4 *(Addition of binary numbers.) Assume that parties A and B are to compute the sum of two n-bit numbers $x = a_n \ldots a_1$, $y = b_n \ldots b_1$, where n is even, and each of them knows exactly half of the input bits. Assume also that B is to compute $\frac{n}{2}$ of the least significant bits of the sum and A the rest. How many bits do they need to exchange? Th*

11.2 Lower Bounds

There are several methods to show lower bounds for communication complexity of Boolean functions with respect to fixed input and output partitions. The basic underlying concept is that of the communication matrix for a Boolean function and its input partition.

Definition 11.2.1 *A* **communication matrix** M_f *for a Boolean function* $f \in \mathcal{B}_n^m$, *and a partition* $\pi_{in} = (A_{in}, B_{in})$ *is a* $2^{|A_{in}|} \times 2^{|B_{in}|}$ *matrix with rows labeled by values of inputs* x_A *from* A_{in} *and columns by values of inputs* x_B *from* B_{in}, *such that*

$$M_f[x_A, x_B] = f(x),$$

where x *is the total input composed of* x_A *and* x_B.

Analogously, we define matrices M_f^A *and* M_f^B *as having the same dimension and the same labeling of rows and columns as* M_f, *and such that* $M_f^A[x_A, x_B] = f_A(x)$, $M_f^B[x_A, x_B] = f_B(x)$. *(Of course,* M_f^A *(*M_f^B*) is meaningful only if* $A_{ou} \neq \emptyset$ *(*$B_{ou} \neq \emptyset$*).)*

Example 11.2.2 *(Identity function.) For* $x, y \in \{0,1\}^n$, *let*

$$IDEN_n(x,y) = \begin{cases} 1, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

Consider now the input partition $\pi_{in} = (\{1, \ldots, n\}, \{n+1, \ldots, 2n\})$, *and labeling of the rows and columns of* M_{IDEN_n} *with an arbitrary but the same labeling for rows and columns.* $M_{IDEN_n} = I_{2^n}$, *where* I_{2^n} *is the* $2^n \times 2^n$ *unit matrix (the matrix with* 1 *in the main diagonal and* 0 *outside of it).*

Example 11.2.3 *(Comparison function.) For* $x, y \in \{0,1\}^n$ *let*

$$COMP_n(x,y) = \begin{cases} 1, & \text{if } x \preceq y; \\ 0, & \text{otherwise,} \end{cases}$$

where $\preceq$ *denotes the lexicographical ordering of binary strings. In case* π_{in} *is the same as in Example 11.2.2 and the labeling of rows and columns of* M_{COMP_n} *by strings from* $\{0,1\}^n$ *is in lexicographical order, the communication matrix* M_{COMP_n} *is the upper-triangular matrix*

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$

with 0 *below the main diagonal and* 1 *otherwise.*

Exercise 11.2.4 *Design communication matrices for Boolean functions:*

(1) $f(x_1,\dots,x_8)=1$*, if and only if* $bin(x_1\dots x_8)$ *is a prime,*

(2) $f(x_1,\dots,x_{2n})=1$*, if and only if* $x_1\dots x_{2n}$ *is a palindrome,*

(3) $f(x_1,\dots,x_n,y_1,\dots,y_n)=\bigvee_{i=1}^n(x_i\wedge y_i)$

for input partitions $(\{x_1,x_2,x_3,x_4\},\ \{x_5,x_6,x_7,x_8\})$ *in the first case,* $(\{x_1,\dots,x_n\},\ \{x_{n+1},\dots,x_{2n}\})$ *in the second case and* $(\{x_1,\dots,x_n\},\{y_1,\dots,y_n\})$ *in the third case.*

Exercise 11.2.5 *Show that the following communication problems can not be solved with fewer than the trivial number bits (n). Two parties know a subset* X *and* Y *of an* n *element set. They have to decide whether; (a)* $X\cap Y=\emptyset$*; (b)* $|X\cap Y|$ *is odd.*

Note that the communication matrix M_f completely describes the communication problem for computing f.

Observe also that a communication protocol actually describes a process of recursive subdivision of M_f into smaller and smaller submatrices. The process ends with all submatrices being **monochromatic**, i.e. having all elements the same. Indeed, the protocol determines for each communication bit, which of the parties sends it. Let us assume that the party whose inputs label the rows of the communication matrix starts a communication. In such a case the protocol specifies for the first bit of communication a partition of rows of M_f into two sets, creating thereby two submatrices and the particular bit A sends only specifies to which of these submatrices the input belong. Similarly, for each of next exchange bits the protocol specifies either a partition of rows of all current submatrices (if the bit is sent by A) or a partition of columns of all current submatrices (if the bit is sent by B).

The communication complexity of the problem is therefore the smallest number of such partitions of M_f and of the resulting submatrices that ends with all submatrices being monochromatic. More about such partitions see in Section 11.2.2.

11.2.1 Fooling Set Method

The basic idea is simple. If any two inputs from a set F of inputs "require" different communications, then there must exist a communication of length $\lceil\lg|F|\rceil$. The following definition makes more precise what does it mean that "two inputs require different communications":

Definition 11.2.6 *Let* $f\in\mathcal{B}_n^m$ *and* (π_{in},π_{ou}) *be partitions for* f*. Consider a set of positions in the communication matrix* M_f*:*

$$F=\{(u_1,v_1),\dots,(u_k,v_k)\}$$

such that no two positions are in the same row or column. F *is said to be a* **fooling set** *for* f*, with respect to* (π_{in},π_{ou})*, if for every two elements* $(u_i,v_i),(u_j,v_j),i\neq j$ *from* F*, at least one of the following conditions holds (where* $f_A(u,v)=M_f^A[u,v]$ *and* $f_B(u,v)=M_f^B[u,v]$*):*

(i) $f_A(u_i, v_i) \neq f_A(u_i, v_j)$;
(ii) $f_A(u_j, v_i) \neq f_A(u_j, v_j)$;
(iii) $f_B(u_i, v_i) \neq f_B(u_j, v_i)$;
(iv) $f_B(u_i, v_j) \neq f_B(u_j, v_j)$.

Remark 11.2.7 Definition 11.2.6 captures the following reasoning: if inputs (u_i, v_i) and (u_j, v_j) yield the same (communication) history H at the given protocol, then (u_i, v_j) and (u_j, v_i) yield the history H, too. But if the history is the same for (u_i, v_i), (u_j, v_j), (u_i, v_j) and (u_j, v_i), then $f_A(u_i, v_i) = f_A(u_i, v_j)$, $f_A(u_j, v_i) = f_A(u_j, v_j)$, $f_B(u_i, v_i) = f_B(u_j, v_i)$ and $f_B(u_i, v_j) = f_B(u_j, v_j)$. In more detail this is explored in the proof of the next theorem.

In the following two examples we assume that the party A knows $x = x_1, \ldots, x_n$, B knows $y = y_1, \ldots, y_n$, and B produces the output.

Example 11.2.8 *For the identity function $IDEN_n$ from Example 11.2.2, with the unit matrix as the communication matrix, the set*

$$F = \{(x, x) \mid x \in \{0,1\}^n\} \tag{11.1}$$

is a fooling set. Indeed, for any pairs (x, x) and (y, y), with $x \neq y$ and $x, y \in \{0,1\}^n$, both conditions (iii) and (iv) in the Definition 11.2.6 are satisfied.

Example 11.2.9 *Also for the comparison function $COMP_n$ from Example 11.2.3, the set F from (11.1) is a fooling set. Indeed, for any $(x, x), (y, y), x \neq y$, one of the inequalities (iii) and (iv) in Definition 11.2.6 is satisfied because either $x \preceq y$ or $y \preceq x$.*

The concept of a fooling set derives its importance from the following result.

Theorem 11.2.10 *Let $f \in \mathcal{B}_n^m$ and (π_{in}, π_{ou}) be its partitions. Let F be a fooling set for f and partitions (π_{in}, π_{ou}). Then*

$$C(f, \pi_{in}, \pi_{ou}) \geq \lceil \lg |F| \rceil.$$

Proof. We are done as soon as we show that we get two different communications for every two different inputs $(u_i, v_i), (u_j, v_j) \in F$. Indeed, that implies the existence of at least $|F|$ different communications and so of a communication of the length at least $\lceil \lg |F| \rceil$.

Assume that two different inputs

$$(u_i, v_i), (u_j, v_j) \text{ result in the same communication history } H = m_1 \ldots m_p. \tag{11.2}$$

We show that in such a case H is the communication history for the inputs (u_i, v_j) and (u_j, v_i), too.

According to our definition of communication, A starts a communication. Communication then goes in the following way—remember that A does not know the input of B and vice versa:

Step 1 Because of (11.2) and the prefix-freeness property, A has to send the message m_1 no matter whether it has as input u_i or u_j.

Step 2 B sees m_1 and, because of (11.2) and the prefix-freeness property, it has to respond with m_2, no matter whether its input is v_i or v_j.

Step 3 A sees $m_1 m_2$ and, again because of (11.2) and the prefix-freeness property, it has to respond with m_3, no matter whether its input is v_i or v_j.

This goes on and in general the following steps are done for $k \geq 1$:

Step $2k$: B sends the message m_{2k} as a function of the previous messages $m_1 \dots m_{2k-1}$ and either of the inputs v_i or v_j. (The fact, that in both cases is m_{2k} the same, follows from the prefix-freeness property.)

Step $2k+1$: A sends the message m_{2k+1} as a function of the messages $m_1, \dots, m_{2k}$ and either of the inputs u_i or u_j.

Therefore all possible combinations of inputs—(u_i, v_i), (u_i, v_j), (u_j, v_i) and (u_j, v_j)—result in the same communication history and therefore in the same outputs by both parties. However, this contradicts the definition of fooling sets. ▯

Corollary 11.2.11 *For partitions* $\pi_{in} = (\{1, \dots, n\}, \{n+1, \dots, 2n\})$ *and* $\pi_{ou} = (\emptyset, \{1\})$, *we have*

1. $C(IDEN_n, \pi_{in}, \pi_{ou}) = n$,
2. $C(COMP_n, \pi_{in}, \pi_{ou}) = n$

Proof. Upper bounds follows from Theorem **??**; lower bounds from Examples 11.2.8 and 11.2.9, as well as from Theorem 11.2.10. ▯

Exercise 11.2.12 *Consider the function*

$$DISJ_n(x_1, \dots, x_n, y_1, \dots, y_n) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} x_i y_i = 0; \\ 0, & \text{otherwise,} \end{cases}$$

and the partitions $\pi_{in} = (\{1, \dots, n\}, \{n+1, \dots, 2n\})$ *and* $\pi_{ou} = (\emptyset, \{1\})$.

(a) Design a communication matrix for f; (b) design a fooling set for f; (c) show that $C(DISJ_n, \pi_{in}, \pi_{ou}) = n$.

11.2.2 Matrix Rank Method

This method is based on the concept of the rank of matrices as defined in linear algebra. we show that it is enough to compute the rank of the communication matrix M_f in order to get a lower bound for the communication complexity of a Boolean function f. This method and the one in Section 11.3 can be used directly to get lower bounds for communication complexity only for functions from $\mathcal{B}_n$.

Theorem 11.2.13 *Let $f \in \mathcal{B}_n$ and (π_{in}, π_{ou}) be partitions for f. Then*

$$C(f, \pi_{in}, \pi_{ou}) \geq \lceil \lg rank(M_f) \rceil .$$

Proof. Theorem clearly holds if $rank(M_f) = 0$. Assume therefore that this is not the case and let us analyse an arbitrary protocol for f and partitions (π_{in}, π_{ou}), in terms of the communication matrix M_f. Assume that party A starts a communication by sending a bit. For some inputs, that is for strings by which rows of M_f are labeled, A sends 1, for others 0. On this base we can partition rows of M_f and consider two submatrices of M_f: M_f^0 and M_f^1. The first one with rows for the inputs for which A sends 0 as the first bit. The remaining rows go to M_f^1. Since

$$rank(M_f) \leq rank(M_f^0) + rank(M_f^1),$$

one of the submatrices M_f^0 and M_f^1 has to have rank at least $\frac{1}{2}rank(M_f)$. Let us assume that this holds for M_f^0. The other case can be treated similarly.

There are again two possibilities. The first one is that A sends also the second bit. In this case rows of M_f^0 can be again partition to get submatrices M_f^{00} and M_f^{01}, with rows producing 0 or 1 as the next communication bit. In the second case, when B is to produce the next bit, we partition columns of M_f^0 to get submatrices M_f^{00} and M_f^{01}. In both cases it holds

$$rank(M_f^0) \leq rank(M_f^{00}) + rank(M_f^{01})$$

and one of these submatrices has to have rank at least $\frac{1}{2}rank(M_f^0) \geq \frac{1}{2^2}rank(M_f)$. This process of partitioning of M_f can be continued according to the chosen protocol and after k bits are exchanged, we get matrices $M_f^{b_1 \ldots b_k}, b_i \in \{0,1\}, 1 \leq i \leq k$, such that at least for one of them

$$rank(M_f^{b_1 \ldots b_k}) \geq \frac{1}{2^k} rank(M_f).$$

At the end of the communication either all rows of $M_f^{b_1 \ldots b_k}$ have to be monochromatic[1] (when B is responsible for the output), or all columns have to be monochromatic (if A is responsible for the output). In such a case the matrix $M_f^{b_1 \ldots b_k}$ has rank 1 and therefore we have

$$1 \geq \frac{1}{2^k} rank(M_f).$$

This implies $k \geq \lceil \lg rank(M_f) \rceil$ and therefore

$$C(f, \pi_{in}, \pi_{ou}) \geq \lceil \lg rank(M_f) \rceil .$$

□

Example 11.2.14 *For the identity and the comparison functions, $IDEN_n$ and $COMP_n$, from Examples 11.2.2 and 11.2.3, their communication matrices have clearly rank 2^n. Theorem 11.2.13 therefore in both cases provides the optimal lower*

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[1] A matrix is called monochromatic if all its elemnts are the same; it is called α-matrix if all its elements are α.

11.3 Communication Complexity

As indicated in Examples 11.1.2 and 11.1.4, the ways the inputs and the outputs are partitioned may have a large impact on the communication complexity of a problem. Of main interest is the worst case, when we have "almost balanced" partitions of inputs and outputs.

A partition $X = A \dot{\cup} B$ of a set X into two disjoint subsets A and B, is said to be an **almost balanced partition** if $\frac{1}{3}|X| \leq |A| \leq \frac{2}{3}|X|$ and also $\frac{1}{3}|X| \leq |B| \leq \frac{2}{3}|X|$. It is called a **balanced partition** if $||A| - |B|| \leq 1$.

11.3.1 Basic Concepts and Examples

We start with the main definition of communication complexity:

Definition 11.3.1 *Let $f \in \mathcal{B}_n$ be a Boolean function. The communication complexity of f, with respect to an arbitrary almost balanced, or balanced, partition of inputs, is defined by*

$$\begin{aligned} C_a(f) &= \min\{C(f, \pi_{in}, \pi_{ou}) \,|\, \pi_{in} \textit{ is an almost balanced partition of the inputs and} \\ &\qquad \pi_{ou} \textit{ is a partition of the outputs}\}, \\ C(f) &= \min\{C(f, \pi_{in}, \pi_{ou}) \,|\, \pi_{in} \textit{ is a balanced partition of the inputs and} \\ &\qquad \pi_{ou} \textit{ is a partition of the outputs}\}. \end{aligned}$$

The restriction to at least almost balanced partitions of inputs captures the most important and hardest case—two communicating parties have almost balanced amount of inputs. In addition, would we consider communication complexity of a function as the minimum of communication complexity with respect to any non-trivial partition —that is with respect to any such partition that each communicating party gets a non-empty portion of inputs—each Boolean function would have communication complexity equal 1. Moreover, since we consider mainly functions from $\mathcal{B}_n$, that is with one output bit only, it is mostly not of large importance which party is charged with producing the output bit. All Boolean functions considered in Section 11.2, that is functions $IDEN_n$, $COMP_n$ and MOD_n, have communication complexity 1, with respect to some balanced partition. This is, however, not always so.

There are Boolean functions with substantially higher communication complexity. For example, this is true for the following functions:

1. $MULT_n(x, y) = bin_{2n}^{-1}(bin(x) \cdot bin(y))$, where $x, y \in \{0,1\}^n$—multiplication of integers $bin(x)$ and $bin(y)$.
2. $SORT_n(x_1, \ldots, x_n) = (x_{\alpha(1)}, \ldots, x_{\alpha(n)}), x_i \in \{0,1\}^k, 1 \leq i \leq n$, and $bin(x_{\alpha(i)}) \leq bin(x_{\alpha(i+1)})$, for $1 \leq i < n$ and a permutation α on $\{1, \ldots, n\}$; k is a constant here.
3. $CONN_n(x) = 1$, if and only if $x \in \{0,1\}^{n(n+1)/2}$ describes a connected oriented graph of n nodes, namely the upper triangular part of its incidence matrix.

We now apply the concept of communication complexity with respect to an arbitrary partition of inputs to computations on VLSI circuits; the aim is to derive basic results concerning lower bounds on AT^2-complexity. Our starting point is the following claim:

Lemma 11.3.2 *Let Λ be a layout of a graph G of degree 4 into a $w \times h$ rectangle R, $h \leq w$, in the two-dimensional plane. For any selection of n nodes, $N_1, \ldots, N_n$, of G, there is a cut of R, by a vertical line of length h, or a vertical-with-one-zig-zag line—see Figure 11.2a,b—of length $h+1$, that makes a balanced partition of the images of nodes $N_1, \ldots, N_n$.*

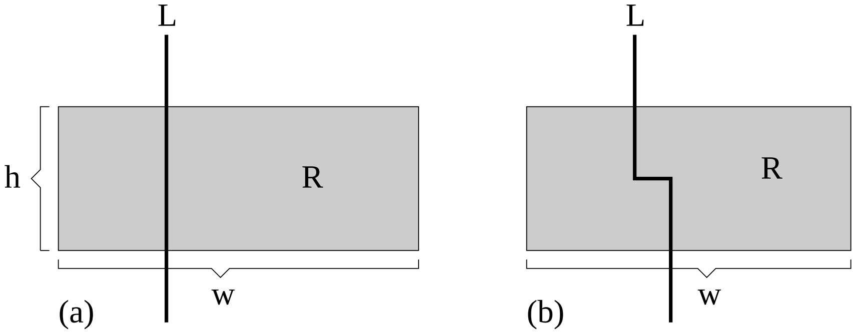

Figure 11.2: Cuts to make balanced partitions of inputs of VLSI-circuit-75%

Proof. Consider the right-most vertical cut L of R, along a grid line, such that the left part of the rectangle R contains at most $\lfloor \frac{n}{2} \rfloor$ images of selected nodes—see Figure 11.2a. This implies that the cut along the next to the right from L vertical grid line would put more than $\lfloor \frac{n}{2} \rfloor$ nodes to the left part. Therefore there has to be a zig-zag modification of the line L—see Figure 11.2b—satisfying the requirements of the Lemma. ▯

We are now in the position to show our main result relating VLSI complexity and communication complexity:

Theorem 11.3.3 *(AT^2-Theorem) If $f \in \mathcal{B}_n^m$ is a Boolean function, then for any VLSI circuit $\mathcal{C}$ for f, such that different inputs of f enter different inputs of C, we have:*

$$Area(\mathcal{C})Time^2(\mathcal{C}) = \Omega(C^2(f)).$$

In short, $AT^2 = \Omega(C^2(f))$.

Proof. Let $\mathcal{C}$ be laid out in a $w \times h$ layout-rectangle R, $h \leq w$. By Lemma 11.3.2, there is a vertical or a vertical with one zig-zag cut L of R that yields a balanced partition of input nodes of $\mathcal{C}$. A computation of $\mathcal{C}$ has to lead to an exchange of at least $C(f)$ bits through L. By Example **??**, this implies $Area(\mathcal{C})Time^2(\mathcal{C}) = \Omega(C^2(f))$. Hence the Theorem.[2] ▯

Corollary 11.3.4 *$Area(\mathcal{C}) = \Omega(C^2(f))$ for every Boolean circuit $\mathcal{C}$ computing a Boolean function f.*

Proof. Each input node of a Boolean circuit $\mathcal{C}$ corresponds exactly to one input variable. Each edge of $\mathcal{C}$ transfers exactly one bit during the whole computation, because $\mathcal{A}$ has no cycle. Thus the number of edges crossing the vertical cut of Figure 11.2 must be at least $C(f)$. ▯

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[2] Theorem 11.3.3 holds also for the case that more input can enter the circuit through the same input node.

Exercise 11.3.5 ** *We can generalize in a natural way the concept of layouts and VLSI circuits to deal with 3-dimensional layouts and circuits. Instead of the area complexity we then have* volume complexity—*the volume of the smallest rectangular parallelepiped containing a 3-dimensional layout. Show that* $V^2T^3 = \Omega(C^3(f))$ *for any 3-dimensional layout of a Boolean function* f.

Remark 11.3.6 The concept of communication complexity with respect to an arbitrary (almost) balanced partition of inputs was substantially generalized to the so-called **strong communication complexity**. The basic motivation is natural and simple: the strong communication complexity

11.4 Nondeterministic and Randomized Communications

Nondeterminism and randomization may substantially decrease resources needed also in case of communications.

In order to develop an understanding of the rôle of nondeterminism and randomization in communications, it is again useful to consider the computation of Boolean functions $f : \{0,1\}^n \rightarrow \{0,1\}$, but this time to interpret such functions as language acceptors that accept the languages $L^n_f = \{x \,|\, x \in \{0,1\}^n, f(x) = 1\}$. The reason behind such a point of view is that both nondeterminism and randomization may have very different impacts on recognition of a language L_f and its complement $\bar{L}_f = L_{\bar{f}}$.

11.4.1 Nondeterministic Communications

Nondeterministic protocols are defined analogically as deterministic ones. However, there are two essential differences. The first difference is that each party may have, at each move, a finite number of messages to choose and send. A nondeterministic protocol $\mathcal{P}$ accepts an input x if there is a communication that leads to an acceptance. The second essential difference is that into the communication complexity of a function we take into considerations only those communications that lead to an acceptance (and we do not care how much messages have to exchange other communications).

The **nondeterministic communication complexity** of a protocol $\mathcal{P}$, for a function $f \in \mathcal{B}_n$, with respect to partitions (π_{in}, π_{ou}) and an input x such that $f(x) = 1$, that is,

$$NC(\mathcal{P}, \pi_{in}, \pi_{ou}, x),$$

is the minimum number of bits of communications that lead to an acceptance of x.

The **nondeterministic communication complexity** of $\mathcal{P}$, with respect to partitions π_{in} and π_{ou}, in short $NC(\mathcal{P}, \pi_{in}, \pi_{ou})$, is defined by

$$NC(\mathcal{P}, \pi_{in}, \pi_{ou}) = \max\{NC(\mathcal{P}, \pi_{in}, \pi_{ou}, x)) \,|\, f(x) = 1, x \in \{0,1\}^n\}$$

and the **nondeterministic communication complexity of f, with respect to partitions** (π_{in}, π_{ou}), is defined by

$$NC(f, \pi_{in}, \pi_{ou}) = \min\{NC(\mathcal{P}, \pi_{in}, \pi_{ou}) \,|\, \mathcal{P} \text{ is a nondeterministic protocol for } f, \pi_{in}, \pi_{ou}\}.$$

The following example shows that nondeterminism can exponentially decrease the amount of communication needed.

Example 11.4.1 *For the complement* $\overline{IDEN}_n$ *of the identity function* $IDEN_n$*, that is for the function*

$$\overline{IDEN_n}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \begin{cases} 1, & \textit{if } (x_1, \ldots, x_n) \neq (y_1, \ldots, y_n); \\ 0, & \textit{otherwise,} \end{cases}$$

$F = \{(x, x) \,|\, x \in \{0,1\}^n\}$ *is a fooling set of* 2^n *elements and therefore, by Theorem 11.2.10,* $C(\overline{IDEN_n}, \pi_{in}, \pi_{ou}) \geq n$*, for partitions* $\pi_{in} = (\{1, \ldots, n\}, \{n+1, \ldots, 2n\}), \pi_{ou} = (\emptyset, \{1\})$.

We show now that $NC(\overline{IDEN}_n, \pi_{in}, \pi_{ou}) \leq \lceil \lg n \rceil + 1$*. Indeed, consider the following nondeterministic protocol the party A chooses one of the bits of the input and sends to B the choosen bit and its position. To describe such a position* $\lceil \lg n \rceil$ *bits are sufficient. B compares this bit with the one in its inputs in the same position and accepts if these two bits are different.*

On the other hand, $NC(IDEN_n, \pi_{in}, \pi_{ou}) = C(IDEN_n, \pi_{in}, \pi_{ou}) = n$, as it will be shown soon, and therefore nondeterminism does not help a bit in this case. Moreover, as it follows from Theorem 11.4.6, nondeterminism can never bring more than an exponential decrease of communications.

As we could see in Example 11.4.1, nondeterminism can bring exponential gain in communications when computing Boolean functions. However, and this is both interesting and important to know, if nondeterminism brings an exponential gain in communication complexity when computing a Boolean function f, then it can not bring exponential gain also when computing the complement of f. It can be shown, that it holds:

Lemma 11.4.2 *For any Boolean function* $f : \{0,1\}^n \mapsto \{0,1\}$ *and any partitions* π_{in}, π_{ou}*,*

$$C(f, \pi_{in}, \pi_{ou}) \leq (NC(f, \pi_{in}, \pi_{ou}) + 1)(NC(\bar{f}, \pi_{in}, \pi_{ou}) + 1).$$

On the other hand, it may happen that nondeterminism brings a decrease, but not exponential, of the communication complexity for both a function and its complement—see Exercise 11.4.3.

Exercise 11.4.3 * *(Nondeterminism may help to compute a function and also its complement.) Consider the function* $IDEN_n^*(x_1,\dots,x_n,y_1,\dots,y_n)$, *where* $x_i, y_i \in \{0,1\}^n$ *and*

$$IDEN_n^*(x_1,\dots,x_n,y_1,\dots,y_n) = \begin{cases} 1, & \text{if there is } 1 \le i \le n \text{ with } x_i = y_i; \\ 0, & \text{otherwise.} \end{cases}.$$

Show for $\pi_{in} = (\{x_1,\dots,x_n\},\{y_1,\dots,y_n\}), \pi_{ou} = (\emptyset,\{1\})$:

(a) $NC(IDEN_n^*,\pi_{in},\pi_{ou}) \le \lceil \lg n \rceil + n$;

*(b)** $NC(\overline{IDEN}_n^*,\pi_{in},\pi_{ou}) \le n\lceil \lg n \rceil + n$;

*(c)*** $C(IDEN_n^*,\pi_{in},\pi_{ou}) = \Theta(n^2)$.

Note that in the nondeterministic protocol of Example 11.4.1 only the party A sends a message. Communication is therefore one-way. This concept of "one-way communication" will now be generalized and the next result justifies this generalization—one-way communications are sufficient in case of nondeterministic communications.

Definition 11.4.4 *If* $f \in \mathcal{B}_n$ *and* (π_{in},π_{ou}) *are partitions for* f, *then the* **one-way nondeterministic communication complexity for** f, *with respect to partitions* π_{in} *and* $\pi_{ou} = (\emptyset,\{1\})$, *is defined by*

$$NC_1(f,\pi_{in},\pi_{ou}) = \min\{NC(\mathcal{P},\pi_{in},\pi_{ou}) \,|\, \mathcal{P} \text{ is a protocol for } f \text{ and only } A \text{ sends a message}\}.$$

Theorem 11.4.5 *If* $f \in \mathcal{B}_n$, *and* (π_{in},π_{ou}) *are partitions for* f, *then*

$$NC(f,\pi_{in},\pi_{ou}) = NC_1(f,\pi_{in},\pi_{ou}).$$

Proof. The basic idea of the proof is very simple. To every nondeterministic protocol $\mathcal{P}$ for f, π_{in}, and π_{ou}, we design a one-way nondeterministic protocol $\mathcal{P}_1$ that simulates $\mathcal{P}$ as follows. A guesses the whole communication between A and B on the base of the input and the protocol for a two-way communication. In other words, A guesses a communication history $H = m_1m_2\dots m_k$ as a communication according to $\mathcal{P}$ as follows. Depending on the input A chooses m_1, then guess m_2, chooses m_3 on the assumption that the guess m_2 was correct, then guess m_4, and so on. A then sends the whole message H to B. B checks whether guesses $m_2, m_4, \dots$ were correct under the assumption that the choices A made were correct. If all guesses of A are correct, and it is an accepting communication, then B accepts; otherwise rejects. Clearly, in this one-way protocol the necessary amount of bits to exchange is the same as in case of $\mathcal{P}$. Moreover, only one message is sent. □

As already mentioned, nondeterminism can not bring more than an exponential decrease of communication:

Theorem 11.4.6 *For each* $f \in \mathcal{B}_n$ *and partitions* (π_{in},π_{ou}) *we have*

$$C(f,\pi_{in},\pi_{ou}) \le 2^{NC(f,\pi_{in},\pi_{ou})}.$$

Proof. According to the previous theorem it is enough to show that a one-way nondeterministic communication that sends only one message can be simulated by a deterministic one, with at most an exponential increase in the size of communications.

If $NC_1(f, \pi_{in}, \pi_{ou}) = m$, then there is a nondeterministic one-way protocol $\mathcal{P}$ for f, π_{in} and π_{ou} such that the number of possible nondeterministic communications for all inputs is at most 2^m. Let us order lexicographically all words of length m and let m_i denote the i-th word.

The following deterministic protocol can now be used to compute f. The party A sends to B the message $H = c_1 \dots c_{2^m}$, where $c_i = 1$ if and only if A could send m_i according to the protocol $\mathcal{P}$ and $c_i = 0$ otherwise. B accepts if and only if there is such an i that $c_i = 1$ and B would accept, according to the nondeterministic protocol, if it would receive m_i. □

As we have already seen in Example 11.4.1, the previous upper bound is asymptotically the best possible.

A good news is that there is an exact method to determine the nondeterministic communication complexity, in terms of the communication matrix M_f—what has not been the case for deterministic communication complexity. The key concept for this method is that of a covering of the communication matrix. A bad news is that a computation of $cover(M_f)$ is computationally a hard optimization problem.

Definition 11.4.7 *Let M be a Boolean matrix. A* **covering** *of M is a set of 1-monochromatic submatrices of M such that each 1-element of M is at least in one of these submatrices. The number of such submatrices is the size of the covering. $cover(M)$ is the minimum of the sizes of all possible coverings of M.*

Example 11.4.8 *For an $n \times n$ matrix M with 1 on and above the main diagonal, and with 0 below the main diagonal, we have $cover(M) = n$. Figures 11.3a, b show two such coverings for $n = 8$. Matrix M in Figure 11.3c has $cover(M) = 2$. In this case it is essential that submatrices that form a minimal covering can overlap.*

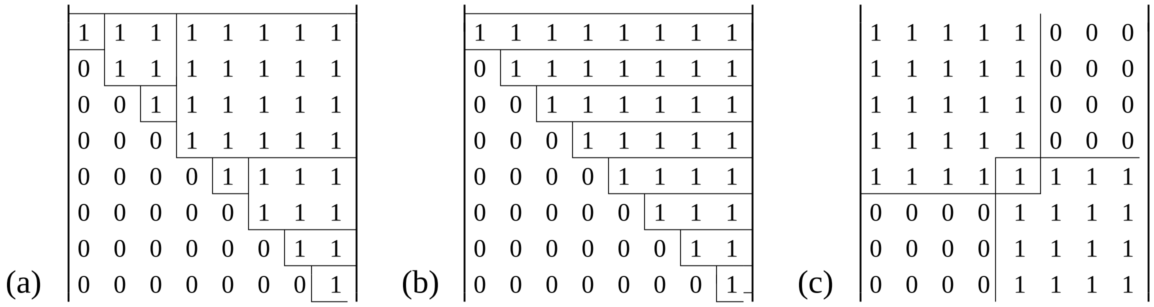

Figure 11.3: Coverings of matrices - 75%

Theorem 11.4.9 *If $f : \{0,1\}^n \mapsto \{0,1\}$, and (π_{in}, π_{ou}) are partitions for f, then*

$$NC(f, \pi_{in}, \pi_{ou}) = \lceil \lg(cover(M_f)) \rceil .$$

Proof. Due to Theorem 11.4.5, it is enough to show that $NC_1(f, \pi_{in}, \pi_{ou}) = \lceil \lg(cover(M_f)) \rceil$.

(1) Let $M_1, \ldots, M_s$ be a covering of M_f. In order to show the inequality $NC_1(f, \pi_{in}, \pi_{ou}) \leq \lceil \lg(cover(M_f)) \rceil$, it is sufficient to design a one-way nondeterministic protocol $\mathcal{P}$ for f that exchanges at most $\lceil \lg s \rceil$ bits. This property has, for example, the protocol $\mathcal{P}$ that works as follows:

For an input (x_A, x_B), such that $f(x) = 1$, A chooses an element i_0 from the set

$$\{i \mid \text{ the row labeled by } x_A \text{ belong to rows of } M_i\}$$

that has to be nonempty, and sends to B the binary representation of i_0. B checks whether x_B belongs to columns of M_{i_0}. If yes, B accepts, otherwise rejects. The protocol is correct because

$$\begin{array}{lcl} B \text{ accepts} & \Leftrightarrow & \text{there is an } i \in \{1, \ldots, s\} \text{ such that } x_A \text{ is a row and } (x_B) \text{ is a column of } M_i \\ & \Leftrightarrow & \text{there is an } i \text{ such that } M_i \text{ covers the entry } (x_A, x_B) \\ & \Leftrightarrow & M_f(x_A, x_B) = 1 \\ & \Leftrightarrow & f(x_A, x_B) = 1. \end{array}$$

(2) Let $\mathcal{P}$ be a one-way nondeterministic protocol for f. In order to prove the inequality $NC_1(f, \pi_{in}, \pi_{ou}) \geq \lceil \lg cover(M_f) \rceil$, it is sufficient to show that if $NC_1(\mathcal{P}, \pi_{in}, \pi_{ou}) = s$, then there is a covering of M_f with at most 2^s 1-matrices.

Let $H_1, \ldots, H_k, k \leq 2^s$, be all possible messages (or histories—in the case of one-way communications this is the same) during nondeterministic communications, according to $\mathcal{P}$—for all possible inputs that are accepted. For $1 \leq i \leq k$, let R_i be the set of those rows of M_f, and C_i the set of those columns of M_f, for which communication between parties A and B produce the history H_i and B accepts. For $1 \leq i \leq k$, let us define the submatrix M_i of M_f as follows

$$M_i = M_f[l, m]_{l \in R_i, m \in C_i}.$$

In order to finish the proof of the inequality $\geq$, it is now sufficient to show that matrices $M_i, 1 \leq i \leq k$, are 1-monochromatic and form a cover of M_f.

Indeed, if M_i covers an entry (x_A, x_B), then $x_A \in R_i$ and $x_B \in C_i$. This means that if A sends H_i to B, this message will be accepted by B, because B knows x_B. Since $\mathcal{P}$ computes f, we have $f(x_A, x_B) = 1$. Hence the matrix M_i is 1-submatrix. On the other side, let x_A, x_B be such that $f(x_A, x_B) = 1$. Then there is a message H_i that is sent by A and which B accepts (if its input is x_B). Thus, M_i covers $f(x_A, x_B)$. ◻

Corollary 11.4.10 *$NC(IDEN_n, \pi_{in}, \pi_{ou}) = n$ for $\pi_{in} = (\{x_1, \ldots, x_n\}, \{x_{n+1}, \ldots, x_{2n}\})$.*

Indeed, the matrix M_{IDEN_n} has 1's only in the main diagonal and therefore it is easy to see that $cover(M_{IDEN_n}) = 2^n$.

Exercise 11.4.11 *Show that $NC(\overline{IDEN}_n, \pi_{in}, \pi_{ou}) = \lceil \lg n \rceil + 1$ for $\pi_{in} = (\{x_1, \ldots x_n\}, \{y_1, \ldots, y_n\}), \pi_{ou} = (\emptyset, \{1\})$.*

In order to determine the nondeterministic communication complexity

11.5 Communication versus Computational Complexity

In this section we show that communication and computational complexity are surprisingly closely related.

As we saw in Chapter 4, the most basic model of computing, which actually abstracts completely from all communications, are Boolean circuits. Main complexity measures for Boolean circuits, and consequently also for Boolean functions, are *Size* and *Depth*. We saw that these measures are closely related to time and space on other models of computers.

We shall see in this Section that with any Boolean function f we can associate an easily defined relation R_f such that the $Depth(f)$ is exactly the communication complexity of R_f. In order to show that, the concepts of a "search communication problem" and a "communication game" are introduced.

11.5.1 Communication Games

There are many ways how to generalize the concept of communication complexity. For example, one can consider communications between more than two parties cooperating on the solution of a problem; with each party having a unique portion of input data. Another possibility, that will be introduced now, is to consider the communication complexity of relations.

Definition 11.5.1 *A* **search communication problem** $\mathcal{S} = \langle X, Y, Z, R\rangle$ *is given by three sets* X, Y, Z, *and a relation* $R \subseteq X \times Y \times Z$. *The task is, for two parties* A *with an input* $x \in X$ *and* B *with an input* $y \in Y$, *to find, if it exists, a* $z \in Z$ *such that* $(x, y, z) \in R$.

Example 11.5.2 $X = Y = \{0,1\}^n$, *Z is a set of primes smaller than* $2n$ *and* $R = \{(x, y, p) | x \not\equiv y \pmod p\}$*—see Example* **??**.

A **(deterministic) communication protocol** $\mathcal{P}$ for a search communication problem $\mathcal{S} = \langle X, Y, Z, R\rangle$ is an algorithm which two players follow: A, who knows an $x \in X$, and B, who knows a $y \in Y$—with the aim to find a $z \in Z$, such that $(x,y,z) \in R$. The **communication complexity of the protocol** $\mathcal{P}$ is defined by

$$C(\mathcal{P}, X, Y, Z, R) = \max_{(x,y)\in X\times Y}\{\text{the number of bits exchanged between the parties } A \text{ and } B \text{ for inputs } x \text{ and } y \text{ and the protocol } \mathcal{P}\},$$

and the **communication complexity of the search communication problem** $\mathcal{S} = \langle X, Y, Z, R\rangle$ is defined by

$$C(\mathcal{S}) = \min\{C(\mathcal{P}, X, Y, Z, R) \,|\, \mathcal{P} \text{ is a protocol for } \mathcal{S}\}.$$

Definition 11.5.3 *Let* $f \in \mathcal{B}_n$ *be a Boolean function. The* **communication game for** f, *in short game*(f), *is the search problem* $\mathcal{S} = \langle X, Y, Z, R\rangle$ *such that*

- $X = f^{-1}(0)$, $Y = f^{-1}(1)$,
- $Z = \{1, \dots, n\}$,
- $(x, y, i) \in R \Leftrightarrow x^{(i)} \neq y^{(i)}$,

where $w^{(i)}$ *denotes the* i*-th bit of* w.

In other words, the task in a communication game, in which the party A knows an x such that $f(x) = 0$ and B knows a y such that $f(y) = 1$, is to determine, by a communication, an integer i such that x and y differ in the i-th bit.

11.5.2 Complexity of Communication Games

We shall deal now with the depth complexity $Depth(f)$ of Boolean functions f, with respect to Boolean circuits over the base $\mathcal{B} = \{\vee, \wedge, \neg\}$.

In order to simplify the exposition, we consider only Boolean circuits all internal nodes of which are either $\vee$-nodes or $\wedge$-nodes—and therefore negations are not allowed—but we allow for each variable v two input nodes, one for v itself, and one for $\bar{v}$.

It is easy to see that each Boolean circuit over the base $\mathcal{B} = \{\vee, \wedge, \neg\}$ can be transformed, using de Morgan rules, into a Boolean circuit without internal nodes with negation, but with possibly two input nodes per a variable and its negation, in such a way that the depth of the circuit is not increased.[3]

Surprisingly, the depth of a Boolean function f, which represents parallel time needed to compute f, is exactly the communication complexity of the corresponding communication game.

Theorem 11.5.4 *For any function* $f \in \mathcal{B}_n$, *we have*

$$C(game(f)) = Depth(f).$$

Proof. (1) At first we show the inequality $C(game(f)) \leq Depth(f)$. In order to do that, let us assume that $\mathcal{C}$ is a circuit for f, over the base $\{\vee, \wedge, \neg\}$, with all negations in inputs.

Let the party A get as input an x such that $f(x) = 0$ and B an y such that $f(y) = 1$. We describe now recursively a communication protocol $\mathcal{P}$ that finds a position in which x and y differ. Actually, the protocol will do more; it will search for a path in $\mathcal{C}$, from the root to a leaf, such that all nodes on that path compute functions that produce different values for x and y. Starting with the root, A and B can design such a path as follows:

First observe that, by definition, the root has the above property. It produces different values for x and y. Assume now that such a path has already been built from the root to a node w and we show now how to make a continuation of the path by one additional node. Once this is shown we know how to come to a leaf. Let us consider two cases concerning w.

(a) w is a $\wedge$-node with two children u and v (Figure 11.4a). Let $f_u \in \mathcal{B}_n$, and $f_v \in \mathcal{B}_n$, be functions computed by nodes u and v, respectively. For each $z \in \{0,1\}^n$, the node w

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[3] Using de Morgan rules we just push negation to leaves. In doing so the size of circuits can increase, but not more than by the factor of two—to each node N of the original circuit we may need to have now two, one computing the same function as N and the other one computing its complement.

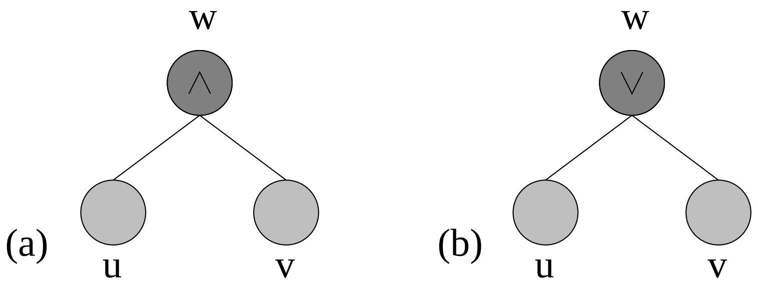

Figure 11.4: Two cases for the communication game-90%

computes the function $f_w(z) = f_u(z) \wedge f_v(z)$. Since f_w has to produce different values for x and y, we can assume, without loss of generality, that $f_w(x) = 0$ (and therefore either $f_u(x) = 0$ or $f_v(x) = 0$) and $f_w(y) = 1$ (and therefore both $f_u(y) = 1$ and $f_v(y) = 1$). This means that at least one of the nodes u and v computes different values for x and y. Parties A and B, who know the circuit and the path chosen so far, can therefore choose, by exchanging one bit, one of such nodes as a continuation of the path.

(b) w is a $\vee$-node. Assume that $f_w(x) = 0$. This implies that both $f_u(x)$ and $f_v(x)$ equal to 0 and if $f_w(x) = 0$, then at least one of $f_u(y)$ and $f_v(y)$ equals to 1. Parties can therefore a

11.6 Exercises

1. For a set X of n elements determine: (a) the number of partitions of X; (b) the number of balanced partitions of X; (c) the number of almost balanced partitions of X.

2. * Given is an n node graph. One party knows a complete subgraph, the other party an independent set of vertices. Design a protocol to decide whether these two subgraphs have a vertex in common. (This can be done by exchanging $\mathcal{O}(\lg^2 n)$ bits.)

3. Consider the same problem as in Exercise **??** with $X = \{x \mid x \in \{0,1\}^*$ and the number $\#_1 x$ even$\}$, $Y = \{x \mid x \in \{0,1\}^*, \#_1 x$ is odd$\}$. Show that in this case the problem can be solved by exchanging $2\lceil \lg n \rceil$ bits.

4. Consider the problem to decide whether an undirected graph of n nodes, given by an adjacency matrix, is connected. Show that for any partition π_{in} of inputs there is a protocol $\mathcal{P}$ such that $C(\mathcal{P}, \pi_{in}, \pi_{ou}) = \mathcal{O}(n \lg n)$.

5. For the function

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \bigwedge_{i=1}^{n} (x_i \Rightarrow y_i)$$

find an almost balanced partition π_{in} of the inputs such that the communication complexity of f, with respect to π_{in}, is: (a) maximum; (b) minimum.

6. Let $f \in \mathcal{B}_{3n}^3$, $f(x,y,z) = (u_1,u_2,u_3)$, where $x,y,z \in \{0,1\}^n$, $u_1 = \bigwedge_{i=1}^n (x_i \equiv y_i)$, $u_2 = \bigvee_{i=1}^n (x_i \vee y_i \vee z_i)$ and $u_3 = \bigvee_{i=1}^n y_i$. Find the best possible communication protocol to compute f for the input partition $\pi_{in} = (\{x, y_1, y_2, y_3\}, \{y_4, \dots, y_n, z\})$ and:

   (a) $\pi_{ou} = (\{u_1\}, \{u_2, u_3\})$; (b) $\pi_{ou} = (\{u_1, u_2\}, \{u_3\})$.

7. Design, for any balanced input partition, a communication protocol to recognize the language
$$L = \{w \mid w \in \{0,1\}^*, \#_1(w) = \#_0(w)\},$$
with the communication complexity $\lceil \lg(n+1) \rceil$ for input words of length $2n$.

8. Find the fooling set for the function $f \in \mathcal{B}_n^2$, n even, where $f(x_1, \dots, x_n, y_1, \dots, y_n) = (z_1, z_2)$,
$$z_1 = \bigoplus_{i=1}^{n} (x_i \wedge y_i), \quad z_2 = \bigwedge_{i=1}^{n/2} (x_i \equiv x_{i+n/2}),$$
and π_{in}, π_{ou} are partitions with
$$\pi_{in} = (\{x_1, \dots, x_{\frac{n}{2}}, y_{\frac{n}{2}+1}, \dots, y_n\}, \{x_{\frac{n}{2}+1}, \dots, x_n, y_1, \dots, y_{\frac{n}{2}}\}),$$
$\pi_{ou} = (\{z_1\}, \{z_2\})$, and show that $C(f, \pi_{in}, \pi_{ou}) \geq n+1$.

9. Consider the Boolean function $f(x_1, \dots, x_{2n}) = (y_1, \dots, y_n)$, where
$$y_i = f_i(x_1, \dots, x_{2n}) = \bigwedge_{j=1}^{n} (x_j \equiv x_{(n+i+j-1) \bmod n}).$$
Show: (a) $C(f_i) \leq 1$;, for each $1 \leq i \leq n$, (b) for each balanced partition π_{in} of $\{x_1, \dots, x_{2n}\}$ there exists a $1 \leq j \leq n$ such that $C(f_j, \pi_{in}, \pi_{ou}) \geq \frac{n}{4}$.

10. Show that for the function $f_k \in \mathcal{B}_n$: $f_k(x_1, \dots, x_{2n}) = 1$ if and only if $x_1, \dots, x_{2n}$ has exactly k 1's, and the partition $\pi_{in} = (\{1, \dots, n\}, \{n+1, \dots, 2n\})$, we have $C(f, \pi_{in}, \pi_{ou}) \geq \lceil \lg k \rceil$.

11. Show $C(MULT_n, \pi_{in}, \pi_{ou}) = \Omega(n)$, for the multiplication function defined by $MULT_n(x,y,z) = 1$, where $x, y \in \{0,1\}^n$ and $z \in \{0,1\}^{2n}$, iff $bin(x) \cdot bin(y) = bin(z)$, and for partitions $\pi_{in} = (\{x,y\}, \{z\})$, $\pi_{ou} = (\emptyset, \{1\})$.

12. Design Boolean functions for which the matrix rank method provides optimal lower bounds.

13. (Tiling complexity.) The concept of a communication matrix of a communication problem and its tiling complexity motivated the following definition of a characteristic matrix of a language $L \subset \Sigma^*$ and its tiling complexity. It is the infinite Boolean matrix M_L with rows and columns labeled by strings from Σ^* and such that $M_L[x,y] = 1$ iff $xy \in L$. For any $n \in N$, let M_L^n be the submatrix of M_L whose rows and columns are labeled by strings from $\Sigma^{\leq n}$ and let $T(M_L^n)$ be the minimal number of 1-submatrices of M_L^n that

cover all 1-entries of M_L^n. Tiling complexity t_L of L is the mapping $t_L(n) = T(M_L^n)$. (a) Design M_L^3 for the language of binary palindromes of length at most 8; (b)* Show that a language L is regular iff $t_L(n) = \mathcal{O}(1)$.

14. Show the lower bound $\Omega(n)$ for the communication complexity of the problem to determine whether a given undirected graph with n nodes is connected. (Hint: you can use Exercise 11.2.5 from the text.)

15. Show that the communication complexity equals n for the problem to determine whether $X \cup Y = Z$, where Z is an n element set, providing the party A knows X and Z, and the party B knows Y and Z.

16. * Find a la

11.7 Historical and Bibliographical References

The idea to consider communication complexity as a method to prove lower bounds came up in various papers on distributed and parallel computing, especially in theoretical approaches to complexity of VLSI computing. The most influential were Thompson's paper (1979) and his PhD thesis (1980); papers by Lipton and Sedgewick (1981), Yao (1979, 1981) and Ullman's book (1984). There is nowadays numerous literature on this subject well overviewed by Lengauer (1990a) and Hromkovič (1996) .

A formal definition of protocols and communication complexity—deterministic and nondeterministic—was introduced by Papadimitriou and Sipser (1982, 1984). Randomized communications were introduced by Yao (bounded-error—Monte Carlo and BPPC—protocols—1979,1981,1983), Melhorn and Schmidt (Las Vegas—1982), Paturi and Simon (unbounded error protocols—1986). The concept of multiparty protocols was introduced by Chandra, Furst and Lipton (1983). The concept of communication games is due to Karchmer and Wigderson (1988); from there also Theorem 11.5.4 comes.

A systematic presentation of communication complexity concepts, methods and results can be found in the survey paper by Orlitsky and El Gamal (1988), lecture notes by Schnitger and Schmetzer (1994) and forthcoming book by Hromkovič (1996). The last two resources much influenced the presentation of this chapter and are due to most of the exercises.

The concept of communication matrix and tiling method is due to Yao (1981); the matrix rank method is due to Mehlhorn and Schmidt (1982). The fooling set concept and method were developed by various authors and explicitely formulated by Aho, Ullman and Yannakakis (1983), where also several basic relations between methods for proving lower bounds at fixed partition were established, including, essentially, Theorems **??** and **??**. The exponential gap between the fooling set and tiling method, mentioned on page **??** is due to Dietzfelbinger, Hromkovič and Schnitger (1994) as well as the result showing that the folling set method can be much weaker as the matrix rank method—both gaps are shown in an existential way—and that the fooling set method can not be much better than the rank method. The exponential gap between rank method and fooling set method was established by Aho, Ullman and Yannakakis (1983).

An application of communication complexity to proving bounds on AT^2-complexity of circuits presented in Section 11.3.3, that follows Schnitger and Schmetzer (1994), is due to Vuillemin (1980) and tradeoffs mentioned on page **??** between area and time complexity for sorting and integer multiplication are due to Bilardi and Preparata (1985) and Melhorn and Preparata (1983), respectively.

The lower bounds method for nondeterministic communications from Section 11.4.1 is due to Aho, Ullman and Yannakakis (1983), where also Lemma 11.4.2 was shown. An exponential gap between deterministic and nondeterministic communication complexity was established by Papadimitriou and Sipser (1982). Relations between various types of randomized protocols are summarized in Hromkovič (1996) as well as by Schnitger and Schmetzer (1994). An exponential gap between communication complexity and Monte Carlo complexity and between nondeterministic communication complexity and BPPC-complexity was shown by Ja'Ja,Prassana, Kamar and Simon (1984). (Another approach to randomized communications (see for example Hromkovič (1996), is to consider BPPC communication (probability of the correct result is at least $\frac{1}{2}$), one-sided Monte Carlo communication (probability of error in case of acceptance is an $\varepsilon > 0$) and two-sided Monte Carlo communication (similar as BPPC but the probability of the correct answer is at least $\frac{1}{2} + \varepsilon$ with $\varepsilon > 0$).

The study of communication complexity classes was initiated by Papadimitriou and Sipser (1982, 1984) and basic hierarchy results from Section **??** are due to them. The result that $m + 1$ bits deterministically communicated can be more powerful than m bits used by nondeterministic protocols is due to Ďuriš, Galil and Schnitger (1984). The fact that almost all Boolean function have the worst possible communication complexity is also due to Papadimitriou and Sipser (1982,1984). Strong communication complexity was introduced by Papadimitriou and Sipser (1982) and worked out by Hromkovič (1996), with an example that strong communication complexity can be exponentially higher than ordinary communication complexity. Hromkovič's bookneukoncene

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