Part I

Quantum cryptography

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A new and important feature of quantum cryptography is that security of quantum cryptographical protocols is based on the laws of nature – of quantum physics, and not on the unproven assumptions of computational complexity.

Quantum cryptography is the first area of information processing and communication in which quantum physics laws were directly exploited to bring an essential advantage in information processing.

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- Unconditionally secure basic quantum cryptography primitives, such as bit commitment and oblivious transfer, are impossible.
- Quantum teleportation and pseudo-telepathy are possible.
- Quantum cryptography and quantum networks are already in the developmental stages. Quantum communication between satellites and ground stations were already demonstrated for 2000 km in 2019 in China. That indicates that quantum internet seems possible.

BASICS of QUANTUM INFORMATION PROCESSING

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will be presented in the next few slides.

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This is very likely to have important consequences for 21th century.

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  Quantum physics is full of counter-intuitive, weird, mysterious and even paradoxical events.

FEYNMAN's VIEW

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However, do not keep saying to yourself, if you can possibly avoid it,

BUT HOW CAN IT BE LIKE THAT?

Because you will get "down the drain" into a blind alley from which nobody has yet escaped

NOBODY KNOWS HOW IT CAN BE LIKE THAT

Richard Feynman (1965): The character of physical law.

CLASSICAL versus QUANTUM INFORMATION

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Main properties of quantum information:

1. It is difficult to store, transmit and process quantum information
2. There is no way to copy perfectly unknown quantum information
3. Measurement of quantum information destroys it, in general.

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In quantum computers, information is represented on microscopic level using qubits, (quantum bits) which can take on any from the following uncountable many values

$$\alpha|0\rangle + \beta|1\rangle$$

where α, β are arbitrary complex numbers such that

$$|\alpha|^2 + |\beta|^2 = 1.$$

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This enormous massive parallelism is one reason why quantum computing can be so powerful.

BASIC EXPERIMENTS

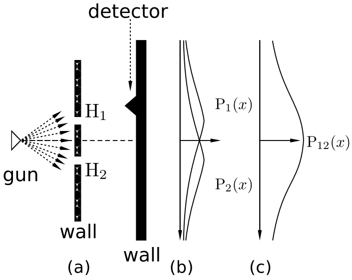


Figure 1: Experiment with bullets

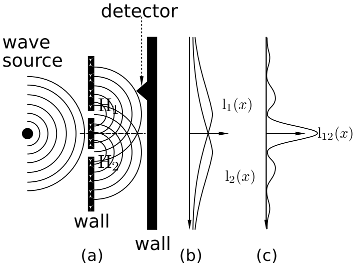
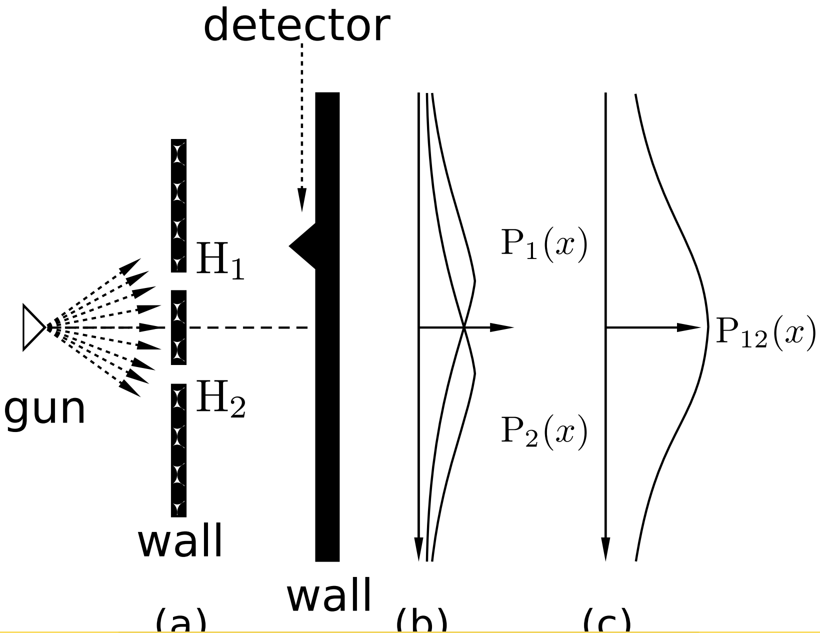


Figure 2: Experiments with waves

CLASSICAL EXPERIMENT with bullets



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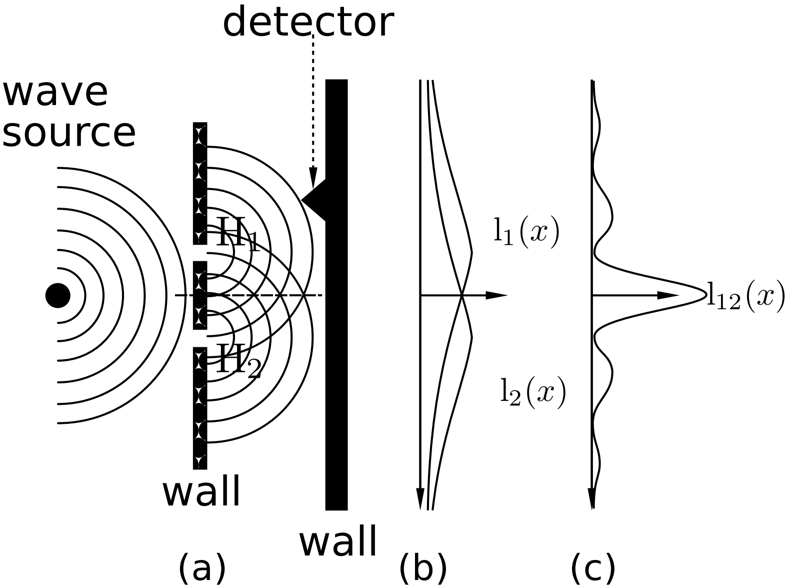


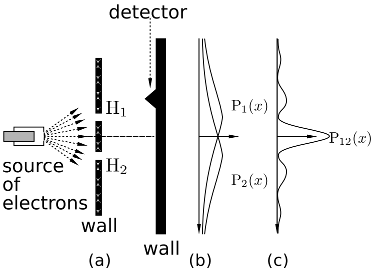


Figure 3: Two-slit experiment

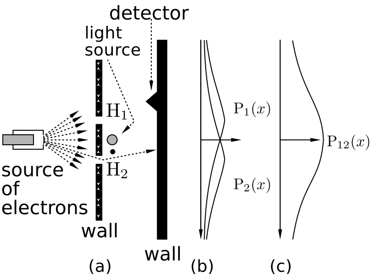
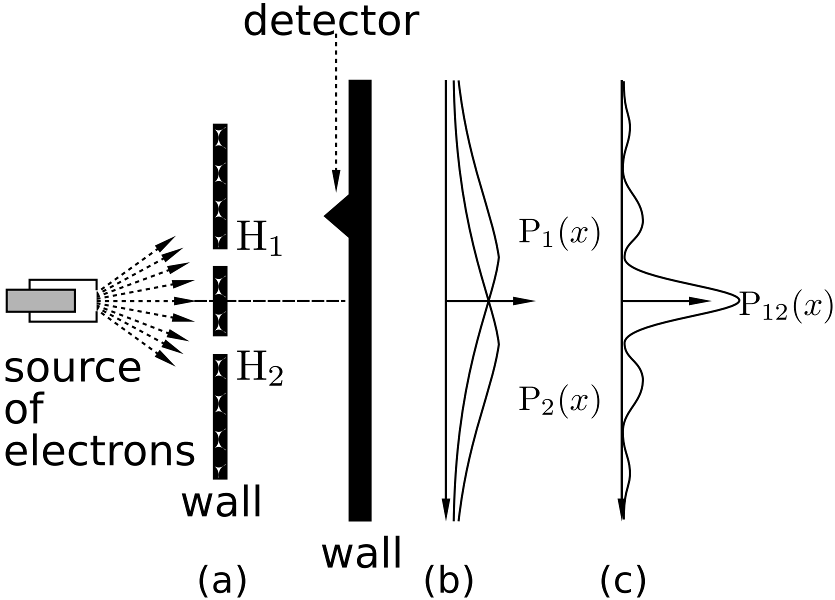
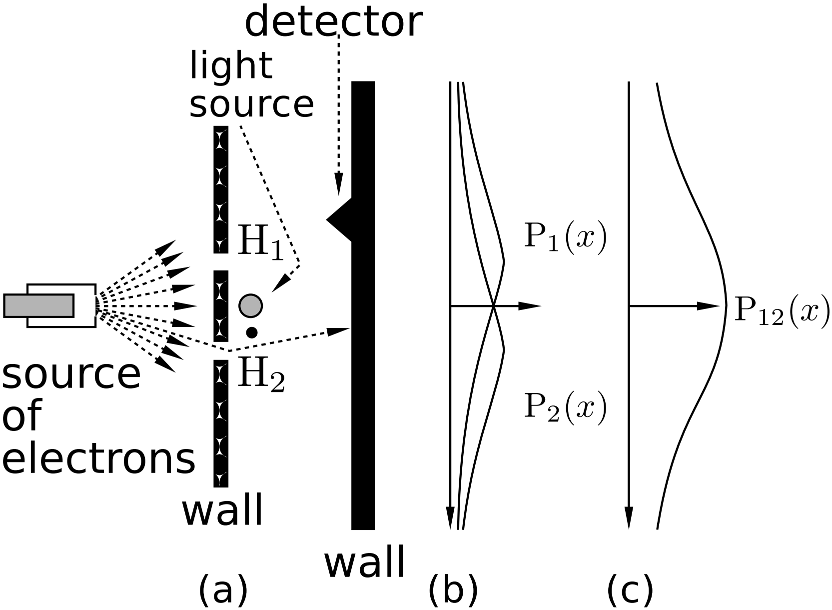


Figure 4: Two-slit experiment with an observation

TWO-SLIT EXPERIMENT



TWO-SLIT EXPERIMENT with OBSERVATION



THREE BASIC PRINCIPLES of QUANTUM WORLD

P1 To each transfer from a quantum state ϕ to a state ψ a complex number

$$\langle\psi|\phi\rangle$$

is associated. This number is called the probability amplitude of the transfer and

$$|\langle\psi|\phi\rangle|^2$$

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P2 If a transfer from a quantum state ϕ to a quantum state ψ can be decomposed into two subsequent transfers

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P3 If a transfer from a state ϕ to a state ψ has two independent alternatives

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QUANTUM SYSTEMS = HILBERT SPACE

Hilbert space H_n is an n-dimensional complex vector space with

scalar product

$$\langle\psi|\phi\rangle = \sum_{i=1}^{n} \phi_i \psi_i^* \text{ of vectors } |\phi\rangle = \begin{vmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{vmatrix}, |\psi\rangle = \begin{vmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_n\end{vmatrix},$$

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Given a basis $B = \{|b_i\rangle\}_{i=1}^n$, any vector $|\psi\rangle$ from H_n can be uniquely expressed in the form:

$$|\psi\rangle = \sum_{i=1}^{n} \alpha_i |b_i\rangle.$$

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$\langle\psi|$ – bra-vector (a row vector) a linear functional on H

such that $\langle\psi|(|\phi\rangle) = \langle\psi|\phi\rangle$

EXAMPLES

Example For states $\phi = (\phi_1, \ldots, \phi_n)$ and $\psi = (\psi_1, \ldots, \psi_n)$ we have

$$|\phi\rangle = \begin{pmatrix} \phi_1 \\ \ldots \\ \phi_n \end{pmatrix}, \langle\phi| = (\phi_1^*, \ldots, \phi_n^*); \langle\phi|\psi\rangle = \sum_{i=1}^{n} \phi_i^* \psi_i;$$

$$|\phi\rangle\langle\psi| = \begin{pmatrix} \phi_1\psi_1^* & \ldots & \phi_1\psi_n^* \\ \vdots & \ddots & \vdots \\ \phi_n\psi_1^* & \ldots & \phi_n\psi_n^* \end{pmatrix}$$

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Schrödinger linear equation

$$ih\frac{\partial|\Phi(t)\rangle}{\partial t} = H(t)|\Phi(t)\rangle$$

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If the Hamiltonian is time independent then the above Schrödinger equation has solution

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is the evolution operator that can be represented by a **unitary matrix**. **A step of such an evolution is therefore a multiplication of a "unitary matrix" A with a vector $|\psi\rangle$, i.e. A $|\psi\rangle$**

UNITARY MATRICES

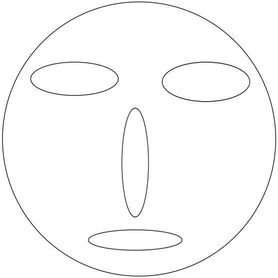
A matrix A is **unitary** if

$$A \cdot A^{\dagger} = A^{\dagger} \cdot A = I$$

where the matrix $A^{\dagger}$ is obtained from the matrix A by revolving A around the main diagonal and changing all elements by their complex conjugates.

QUANTUM (PROJECTION) MEASUREMENTS

A quantum state is always observed (measured) with respect to an **observable** O – a decomposition of a given Hilbert space into orthogonal subspaces (where each vector can be uniquely represented as a sum of vectors of these subspaces).



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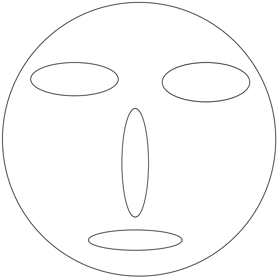

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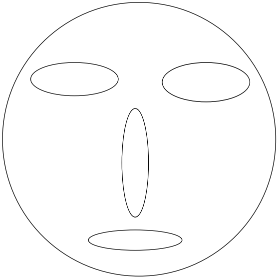

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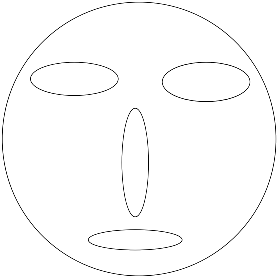

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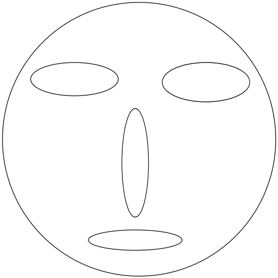

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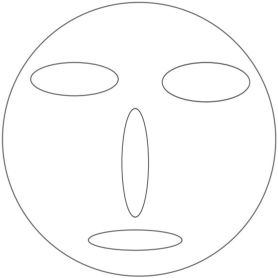

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The subspace into which projection is made is chosen **randomly** and the corresponding probability is uniquely determined by the amplitudes at the representation of $|\phi\rangle$ as a sum of states of the subspaces.

QUANTUM STATES and PROJECTION MEASUREMENT

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$$|\phi\rangle = \sum_{i=1}^{n} a_i|\beta_i\rangle, \quad \sum_{i=1}^{n} |a_i|^2 = 1$$

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that if the state $|\phi\rangle$ is measured with respect to the basis $\{\beta_i\}_{i=1}^n$, then the state $|\phi\rangle$ collapses into the state $|\beta_i\rangle$ with probability $|a_i|^2$.

The classical "outcome" of the measurement of the state $|\phi\rangle$ with respect to the basis $\{\beta_i\}_{i=1}^n$ is the index i of that state $|\beta_i\rangle$ into which the state $|\phi\rangle$ collapses.

QUBITS

A **qubit** is a quantum state in H_2

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha, \beta \in C$ are such that $|\alpha|^2 + |\beta|^2 = 1$ and

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EXAMPLE: Representation of qubits by

(a electron in a Hydrogen atom

(b a spin-1/2 particle

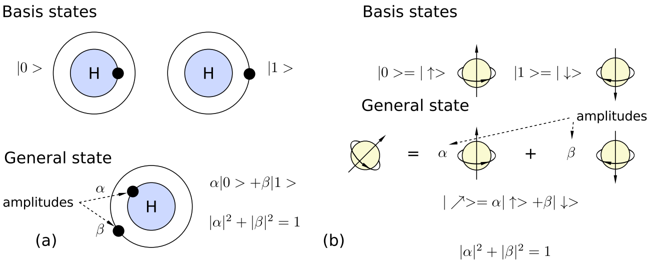

Figure 5: Qubit representations by energy levels of an electron in a hydrogen atom and by a spin-1/2 particle. The condition $|\alpha|^2 + |\beta|^2 = 1$ is a legal one if $|\alpha|^2$ and $|\beta|^2$ are to be the probabilities of being in one of two basis states (of electrons or photons).

HILBERT SPACE H_2

STANDARD BASIS	DUAL BASIS
$\|0\rangle, \|1\rangle$	$\|0'\rangle, \|1'\rangle$
$\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$

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Hadamard matrix

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$H|0\rangle = |0'\rangle \qquad H|0'\rangle = |0\rangle$$
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General form of a unitary matrix of degree 2

$$U = e^{i\gamma}\begin{pmatrix}e^{i\alpha} & 0\\ 0 & e^{-i\alpha}\end{pmatrix}\begin{pmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^{i\beta} & 0\\ 0 & e^{-i\beta}\end{pmatrix}$$

PAULI MATRICES

Very important one-qubit unary operators are the following Pauli operators, expressed in the standard basis as follows;

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Observe that Pauli matrices transform a qubit state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ as follows

$$\begin{aligned} \sigma_x(\alpha|0\rangle + \beta|1\rangle) &= \beta|0\rangle + \alpha|1\rangle \\ \sigma_z(\alpha|0\rangle + \beta|1\rangle) &= \alpha|0\rangle - \beta|1\rangle \\ \sigma_y(\alpha|0\rangle + \beta|1\rangle) &= \beta|0\rangle - \alpha|1\rangle \end{aligned}$$

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Operators σ_x, σ_z and σ_y represent therefore a bit error, a sign error and a bit-sign error.

of a qubit state

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By a **measurement** of the qubit state

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with respect to the basis

$$\{|0\rangle, |1\rangle\}$$

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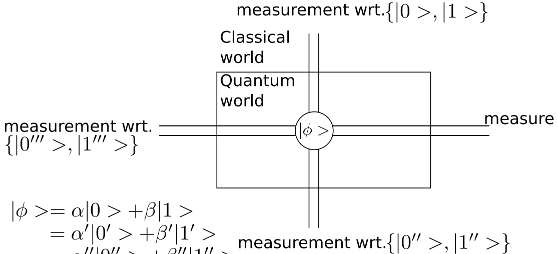
0 with probability $|\alpha|^2$ 1 with probability $|\beta|^2$



$$\begin{aligned}|\phi> &= \alpha|0> + \beta|1> \\ &= \alpha'|0'> + \beta'|1'> \\ &= \alpha''|0''> + \beta''|1''> \\ &= \alpha'''|0'''> + \beta'''|1'''>\end{aligned}$$

MIXED STATES – DENSITY MATRICES

A probability distribution $\{(p_i, |\phi_i\rangle)\}_{i=1}^{k}$ on pure states is called a mixed state to which it is assigned a density operator

$$\rho = \sum_{i=1}^{n} p_i |\phi\rangle\langle\phi_i|.$$

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To two different mixed states can correspond the same density matrix.

Two mixes states with the same density matrix are physically undistinguishable.

MAXIMALLY MIXED STATES

To the maximally mixed state,

$$\left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |1\rangle\right)$$

representing a random bit, corresponds the density matrix

$$\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}(1,0) + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}(0,1) = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \frac{1}{2}I_2$$

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Surprisingly, many other mixed states have density matrix that is the same as that of the maximally mixed state.

QUANTUM ONE-TIME PAD CRYPTOSYSTEM

CLASSICAL ONE-TIME PAD cryptosystem

plaintext	an n-bit string p
shared key	an n-bit string k
cryptotext	an n-bit string c
encoding	$c = p \oplus k$
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plaintext:	an n-qubit string $\|p\rangle = \|p_1\rangle \dots \|p_n\rangle$
shared key:	two n-bit strings k,k'
cryptotext:	an n-qubit string $\|c\rangle = \|c_1\rangle \dots \|c_n\rangle$
encoding:	$\|c_i\rangle = \sigma_x^{k_i} \sigma_z^{k_i'} \|p_i\rangle$
decoding:	$\|p_i\rangle = \sigma_z^{k_i'} \sigma_x^{k_i} \|c_i\rangle$

where $|p_i\rangle = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$ and $|c_i\rangle = \begin{pmatrix} d_i \\ e_i \end{pmatrix}$ are qubits and $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices.

UNCONDITIONAL SECURITY of QUANTUM ONE-TIME PAD

In the case of encryption of a qubit

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

by QUANTUM ONE-TIME PAD cryptosystem, what is being transmitted is the mixed state

$$\left(\frac{1}{4}, |\phi\rangle\right), \left(\frac{1}{4}, \sigma_x|\phi\rangle\right), \left(\frac{1}{4}, \sigma_z|\phi\rangle\right), \left(\frac{1}{4}, \sigma_x\sigma_z|\phi\rangle\right)$$

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This density matrix is identical to the density matrix corresponding to that of a random bit, that is to the mixed state

$$\left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |1\rangle\right)$$

SHANNON's THEOREMS

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Quantum version of Shannon encryption theorem says that 2n classical bits are necessary and sufficient to encrypt securely n qubits.

COMPOSED QUANTUM SYSTEMS (1)

Tensor product of vectors

$(x_1, \ldots, x_n) \otimes (y_1, \ldots, y_m) = (x_1 y_1, \ldots, x_1 y_m, x_2 y_1, \ldots, x_2 y_m, \ldots, x_2 y_m, \ldots, x_n y_1, \ldots, x_n y_m)$

Tensor product of matrices $A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \ldots & a_{nn}B \end{pmatrix}$

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Example $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix}$

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Tensor product of Hilbert spaces $H_1 \otimes H_2$ is the complex vector space spanned by tensor products of vectors from H_1 and H_2 . That corresponds to the quantum system composed of the quantum systems corresponding to Hilbert spaces H_1 and H_2.

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An important difference between classical and quantum systems

A state of a compound classical (quantum) system can be (cannot be) always composed from the states of the subsystem.

QUANTUM REGISTERS

A general state of a 2-qubit register is:

$$|\phi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

where

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

and $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are vectors of the "standard" basis of H_4, i.e.

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

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An important unitary matrix of degree 4, to transform states of 2-qubit registers:

$$CNOT = XOR = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix}$$

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It holds:

$$\text{CNOT} : |x, y\rangle \Rightarrow |x, x \oplus y\rangle$$

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However, CNOT can make copies of the basis states $|0\rangle, |1\rangle$: Indeed, for $x \in \{0, 1\}$,

$$CNOT(|x\rangle|0\rangle) = |x\rangle|x\rangle$$

BELL STATES

States

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

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Theoretically, there is an observable for this basis. However, no one has been able to construct a device for Bell measurement using linear elements only.

QUANTUM n-qubit REGISTERS

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and, in general, for $x \in \{0,1\}^n$

$$H_n|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{x\cdot y} |y\rangle.$$ [1]

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then the mapping

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OBSERVE THAT IN A SINGLE COMPUTATIONAL STEP 2^n VALUES OF f ARE COMPUTED!

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This means that subsequent measurement of other particle (on another planet) provides the same result as the measurement of the first particle. This indicate that in quantum world non-local influences, correlations, exist.

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- To increase capacity of quantum channels

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  Quantum key generation, on the other hand, needs to be designed only to be secure against technology available at the moment of key generation.

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another term is

quantum key distribution (QKD)

where one can expect the first

transfer from the experimental to the application stage.

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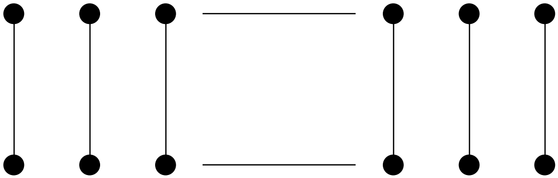

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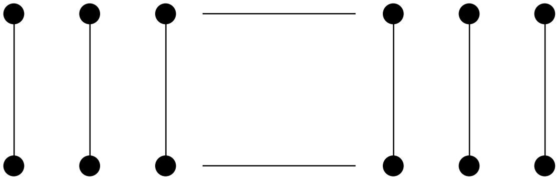

n pairs of particles in EPR state

If both of them measure their particles in the standard basis, then they get, as the classical outcome of their measurements the same random, shared and secret binary key of length n.

POLARIZATION of PHOTONS

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An important property of photons is polarization – it refers to the bias of the electric field in the electromagnetic field of the photon.

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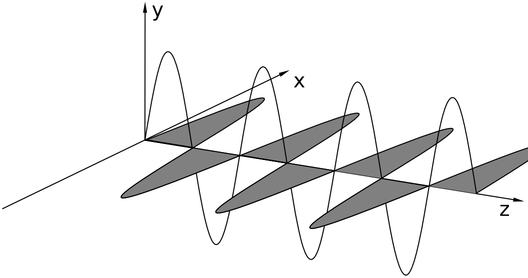

Figure: Linearly polarized photons - visualization

Both vertical and horizontal polarizations are examples of " linear polarizations".

CIRCULAR POLARIZATION

If the free end of the rope is moved around in a circle, then we would get a wave that looks like a corkscrew. This would visualize circular polarization"

Generation of orthogonally polarized photons.

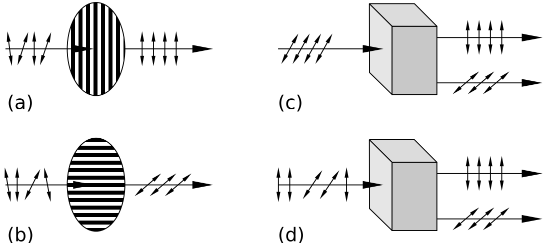

Figure: Photon polarizers and measuring devices

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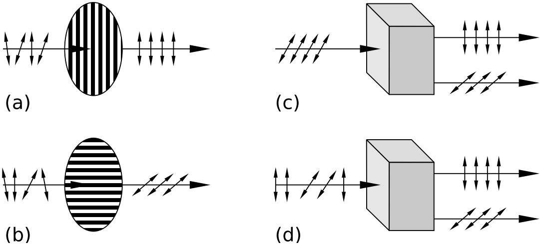

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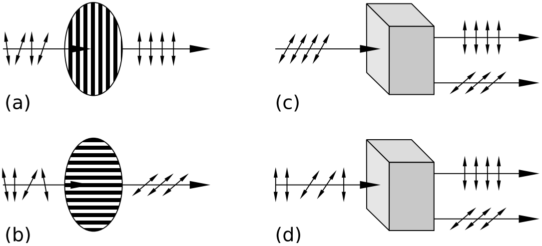

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Fig. d – a calcite crystal can be used to separate horizontally and vertically polarized photons.

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BB84 QUANTUM GENERATION of CLASSICAL RANDOM KEY

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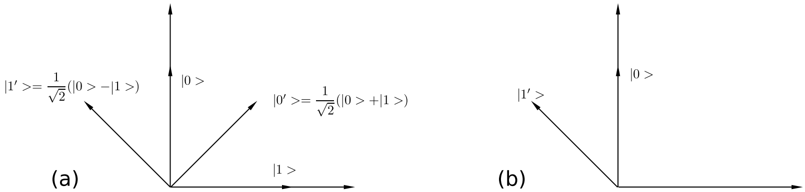

Figure 8: Polarizations of photons for BB84 and B92 protocols

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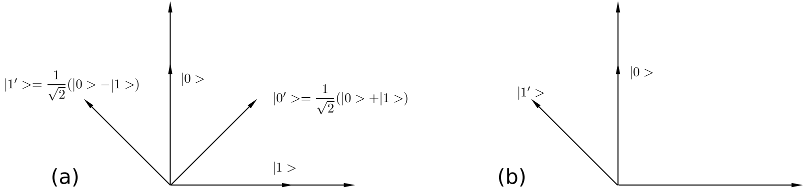

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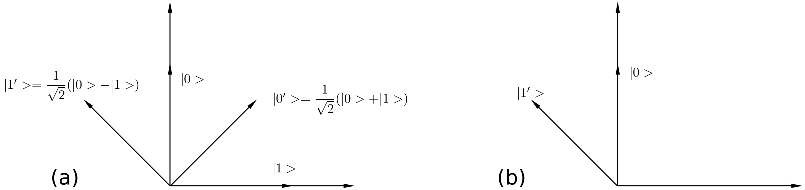

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Bob has a detector that can be set up to distinguish between rectilinear polarizations (0 and 90 degrees) or can be quickly reset to distinguish between diagonal polarizations (45 and 135 degrees).

BB84 QUANTUM KEY GENERATION PROTOCOL III

An example of an encoding – decoding process is in the Figure 10.

Raw key extraction

Bob makes public the sequence of bases he used to measure the photons he received – but not the results of the measurements – and Alice tells Bob, through a classical channel, in which cases he has chosen the same basis for measurement as she did for encoding. The corresponding bits then form the basic raw key.

BB84 QUANTUM KEY GENERATION PROTOCOL III

An example of an encoding – decoding process is in the Figure 10.

Raw key extraction

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1	0	0	0	1	1	0	0	0	1	1	Alice's random sequence
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0	1	1	1	0	0	1	0	0	1	0	Bob's random sequence
B	D	D	D	B	B	D	B	B	D	B	Bob's observable
1	0	R	0	1	R	0	0	0	R	R	outcomes

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Figure 10: Quantum transmissions in the BB84 protocol – R stands for the case that the result of the measurement is random.

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Alice and Bob agree on a sequence of indices of the raw key and make the corresponding bits of their raw keys public.

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A way out is to use a special error correction techniques and at the end of this stage both Alice and Bob share identical keys.

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Privacy amplification is a method how to select a short and very secret binary string s from a longer but less secret string s'. The main idea is simple. If $|s| = n$, then one picks up n random subsets $S_1, \ldots, S_n$ of bits of s' and let s_i, the i-th bit of S, be the parity of S_i. One way to do it is to take a random binary matrix of size $|s| \times |s'|$ and to perform multiplication Ms'^T, where s'^T is the binary column vector corresponding to s'.

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The point is that even in the case where an eavesdropper knows quite a few bits of s', she will have almost no information about s.

More exactly, if Eve knows parity bits of k subsets of s', then if a random subset of bits of s' is chosen, then the probability that Eve has any information about its parity bit is less than $\frac{2^{-(n-k-1)}}{\ln 2}$.

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3. To move from the experimental to the developmental stage.

QUANTUM TELEPORTATION - BASIC SETTING

Quantum teleportation allows to transmit unknown quantum information to a very distant place in spite of impossibility to measure or to broadcast information to be transmitted.

Alice and Bob share two particles in the EPR-state

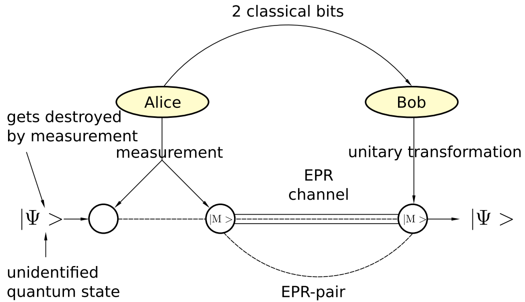
$$|EPR_{pair}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and then Alice receives another particle in an unknown qubit state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Alice then measure her two particles in the Bell basis.

QUANTUM TELEPORTATION - BASIC SETTING I



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |EPR - pair\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Total state

$$|\psi\rangle|EPR - pair\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

Alice measures her two qubits with respect to the "Bell basis":

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

QUANTUM TELEPORTATION II

Since the total state of all three particles is:

$$|\psi\rangle|EPR - pair\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

and can be expressed also as follows:

$$|\psi\rangle|EPR - pair\rangle = |\Phi^+\rangle\frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle) + |\Psi^+\rangle\frac{1}{\sqrt{2}}(\beta|0\rangle + \alpha|1\rangle) + |\Phi^-\rangle\frac{1}{\sqrt{2}}(\alpha|0\rangle - \beta|1\rangle) + |\Psi^-\rangle\frac{1}{\sqrt{2}}(-\beta|0\rangle + \alpha|1\rangle)$$

then the Bell measurement of the first two particles projects the state of Bob's particle into a "small modification" $|\psi_1\rangle$ of the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

$$|\Psi_1\rangle = \text{either } |\Psi\rangle \text{ or } \sigma_x|\Psi\rangle \text{ or } \sigma_z|\Psi\rangle \text{ or } \sigma_x\sigma_z|\psi\rangle$$

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The unknown state $|\psi\rangle$ can therefore be obtained from $|\psi_1\rangle$ by applying one of the four operations

$$\sigma_x, \sigma_y, \sigma_z, I$$

and the result of the Bell measurement provides two bits specifying which of the above four operations should be applied.

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These four bits Alice needs to send to Bob using a classical channel (by email, for example).

QUANTUM TELEPORTATION III.

If the first two particles of the state

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are measured with respect to the Bell basis then Bob's particle gets into the mixed state

$$\left(\frac{1}{4},\alpha|0\rangle+\beta|1\rangle\right)\oplus\left(\frac{1}{4},\alpha|0\rangle-\beta|1\rangle\right)\oplus\left(\frac{1}{4},\beta|0\rangle+\alpha|1\rangle\right)\oplus\left(\frac{1}{4},\beta|0\rangle-\alpha|1\rangle\right)$$

to which corresponds the density matrix

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The resulting density matrix is identical to the density matrix for the mixed state

$$\left(\frac{1}{2}, |0\rangle\right) \oplus \left(\frac{1}{2}, |1\rangle\right)$$

Indeed, the density matrix for the last mixed state has the form

$$\frac{1}{2}\binom{1}{0}(1,0) + \frac{1}{2}\binom{0}{1}(0,1) = \frac{1}{2}I$$

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- QIPC has been shown to be more efficient in interesting/important cases.

UNIVERSAL SETS of QUANTUM GATES

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A simple universal set of quantum gates consists of gates.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \sigma_z^{\frac{1}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{\pi}{4}i} \end{pmatrix}$$

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Theorem 0.2 CNOT gate and elementary rotation gates

$$R_\alpha(\theta) = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2}\sigma_\alpha \qquad \text{for } \alpha \in \{x, y, z\}$$

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- Measurement.

EXAMPLES of QUANTUM ALGORITHMS

Deutsch problem: Given is a black-box function f: $\{0,1\} \rightarrow \{0,1\}$, how many queries are needed to find out whether f is constant or balanced:

Classically: 2

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Deutsch-Jozsa Problem: Given is a black-box function $f : \{0,1\}^n \rightarrow \{0,1\}$ and a promise that f is either constant or balanced, how many queries are needed to find out whether f is constant or balanced.

Classically: n

Quantumly 1

EXAMPLES of QUANTUM ALGORITHMS

Deutsch problem: Given is a black-box function f: $\{0,1\} \to \{0,1\}$, how many queries are needed to find out whether f is constant or balanced:

Classically: 2

Quantumly: 1

Deutsch-Jozsa Problem: Given is a black-box function $f : \{0,1\}^n \to \{0,1\}$ and a promise that f is either constant or balanced, how many queries are needed to find out whether f is constant or balanced.

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Search of an element in an unordered database of n elements:

Classically n queries are needed in the worst case

Lov Grover showed that quantumly $\sqrt{n}$ queries are enough

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Quantum computers works with superpositions of basic quantum states on which very special (unitary) operations are applied and and very special quantum features (non-locality) are used.

Quantum computers work not with bits, that can take on any of two values 0 and 1, but with qubits (quantum bits) that can take on any of infinitely many states $\alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$.

REDUCTIONS

Shor's polynomial time quantum factorization algorithm is based on an understanding that factorization problem can be reduced

1. first on the problem of solving a simple modular quadratic equation;
2. second on the problem of finding periods of functions $f(x) = a^x \bmod n$.

FIRST REDUCTION

Lemma If there is a polynomial time deterministic (randomized) [quantum] algorithm to find a nontrivial solution of the modular quadratic equations

$$a^2 \equiv 1 \pmod{n},$$

then there is a polynomial time deterministic (randomized) [quantum] algorithm to factorize integers.

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then there is a polynomial time deterministic (randomized) [quantum] algorithm to factorize integers.

Proof. Let $a \neq \pm 1$ be such that $a^2 \equiv 1 \pmod n$. Since

$$a^2 - 1 = (a+1)(a-1),$$

if n is not prime, then a prime factor of n has to be a prime factor of either $a+1$ or $a-1$. By using Euclid's algorithm to compute

$$gcd(a+1, n) \quad \text{and} \quad gcd(a-1, n)$$

we can find, in $O(\lg n)$ steps, a prime factor of n.

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AN ALGORITHM TO SOLVE EQUATION $x^2 \equiv 1 \pmod{n}$.

1. *Choose randomly* $1 < a < n$.
2. *Compute* $gcd(a, n)$. *If* $gcd(a, n) \neq 1$ *we have a factor.*
3. *Find period* r *of function* $a^k \bmod n$.
4. *If* r *is odd or* $a^{r/2} \equiv \pm 1 \pmod{n}$,*then go to step 1; otherwise stop.*

If this algorithm stops, then $a^{r/2}$ is a non-trivial solution of the equation

$$x^2 \equiv 1 \pmod{n}.$$

EXAMPLE

Let $n = 15$. Select $a < 15$ such that $gcd(a, 15) = 1$.
$\{$The set of such a is $\{2, 4, 7, 8, 11, 13, 14\}\}$

Choose $a = 11$. Values of 11^x mod 15 are then

$$11, 1, 11, 1, 11, 1$$

which gives $r = 2$.

Hence $a^{r/2} = 11 \pmod{15}$. Therefore

$$gcd(15, 12) = 3, \qquad gcd(15, 10) = 5$$

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For $a = 14$ we get again $r = 2$, but in this case

$$14^{2/2} \equiv -1 \pmod{15}$$

and the following algorithm fails.

1. *Choose randomly $1 < a < n$.*
2. *Compute $gcd(a, n)$. If $gcd(a, n) \neq 1$ we have a factor.*
3. *Find period r of function $a^k \bmod n$.*
4. *If r is odd or $a^{r/2} \equiv \pm 1 \pmod{n}$,then go to step 1; otherwise stop.*

EFFICIENCY of REDUCTION

Lemma If $1 < a < n$ satisfying $gcd(n, a) = 1$ is selected in the above algorithm randomly and n is not a power of prime, then

$$Pr\{r \text{ is even and } a^{r/2} \not\equiv \pm 1\} \geq \frac{9}{16}.$$

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2. *Compute* $gcd(a, n)$. *If* $gcd(a, n) \neq 1$ *we have a factor.*
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4. *If* r *is odd or* $a^{r/2} \equiv \pm 1 \pmod{n}$,*then go to step 1; otherwise stop.*

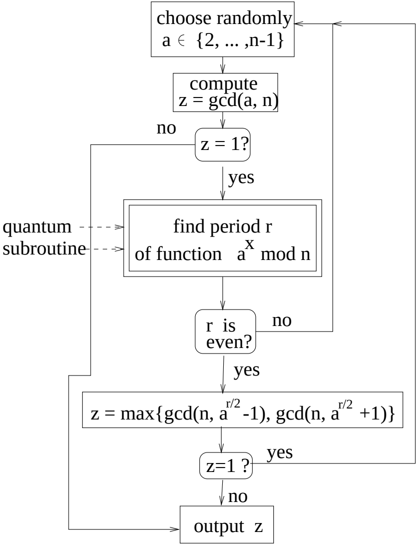
Corollary If there is a polynomial time randomized [quantum] algorithm to compute the period of the function

$$f_{n,a}(k) = a^k \bmod n,$$

then there is a polynomial time randomized [quantum] algorithm to find non-trivial solution of the equation $a^2 \equiv 1 \pmod{n}$ (and therefore also to factorize integers).

A GENERAL SCHEME for Shor's ALGORITHM

The following flow diagram shows the general scheme of Shor's quantum factorization algorithm



SHOR's QUANTUM FACTORIZATION ALGORITHM I.

1. For given $n, q = 2^d, a$ create states

$$\frac{1}{\sqrt{q}}\sum_{x=0}^{q-1}|n,a,q,x,\mathbf{0}\rangle \text{ and } \frac{1}{\sqrt{q}}\sum_{x=0}^{q-1}|n,a,q,x,a^x \bmod n\rangle$$

2. By measuring the last register the state collapses into the state

$$\frac{1}{\sqrt{A+1}}\sum_{j=0}^{A}|n,a,q,jr+l,y\rangle \text{ or, shortly } \frac{1}{\sqrt{A+1}}\sum_{j=0}^{A}|jr+l\rangle,$$

where A is the largest integer such that $l + Ar \leq q$, r is the period of a^x mod n and l is the offset.

$$\sqrt{\frac{r}{q}}\sum_{j=0}^{\frac{q}{r}-1}|jr+l\rangle$$

3. By applying quantum Fourier transformation we get then the state

$$\frac{1}{\sqrt{r}}\sum_{j=0}^{r-1}e^{2\pi ilj/r}|j\frac{q}{r}\rangle.$$

4. By measuring the resulting state we get $c = \frac{jq}{r}$ and if $gcd(j,r) = 1$, what is very likely, then from c and q we can determine the period r.

SHOR's QUANTUM FACTORIZATION ALGORITHM II.

Indeed, since

$$c = \frac{jq}{r}$$

for randomly chosen j and still unknown period r and very likely $gcd(j, r) = 1$
we have

$$\frac{c}{j} = \frac{q}{r}$$

and therefore

$$r = \frac{q}{gcd(c, q)}$$