

Part I

Public-key cryptosystems II. Other cryptosystems and cryptographic primitives

CHAPTER 6: OTHER CRYPTOSYSTEMS and BASIC CRYPTOGRAPHY PRIMITIVES

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Finally, we will discuss, in some details, such very important cryptography primitives as pseudo-random number generators and hash functions .

STORY of SQUARE ROOTS and QUADRATIC RESIDUES

MODULAR SQUARE ROOTS PROBLEM

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No polynomial time algorithm is known to solve the modular square root problem for arbitrary modulus n .

However, in case n is a prime or a product of two odd primes, such a polynomial squaring algorithm exists to solve modular square root problem..

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$Z_n = \{0, 1, \dots, n-1\}$ is a group under the operation $+_n$

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for some $y \in Z_n^*$, otherwise x is called a **quadratic nonresidue**.

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For any prime p the set $QR(p)$ has $\frac{p-1}{2}$ elements.

So called **Euler criterion** says that c is a quadratic residue modulo prime p iff

$$c^{(p-1)/2} \equiv 1 \pmod{p}.$$

EXAMPLES of Z_N^* SETS and THEIR MULTIPLICATION TABLES

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$$\begin{aligned} \text{If } n = 15 \text{ then } Z_{15}^* &= \{1, 2, 4, 7, 8, 11, 13, 14\} \\ 1^2 &\equiv 1 \pmod{15}, \quad 2^2 \equiv 4 \pmod{15}, \quad 4^2 \equiv 1 \pmod{15}, \\ 7^2 &\equiv 4 \pmod{15}, \quad 8^2 \equiv 4 \pmod{15}, \\ 11^2 &\equiv 1 \pmod{15}, \quad 13^2 \equiv 4 \pmod{15}, \quad 14^2 \equiv 1 \pmod{15} \end{aligned}$$

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- The inverse function is $f^{-1}(x) = x^{((p-1)(q-1)+4)/8} \pmod{n}$

EXAMPLE

For $n = 21 = 3 \times 7$

$$Z_{21}^* = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$$

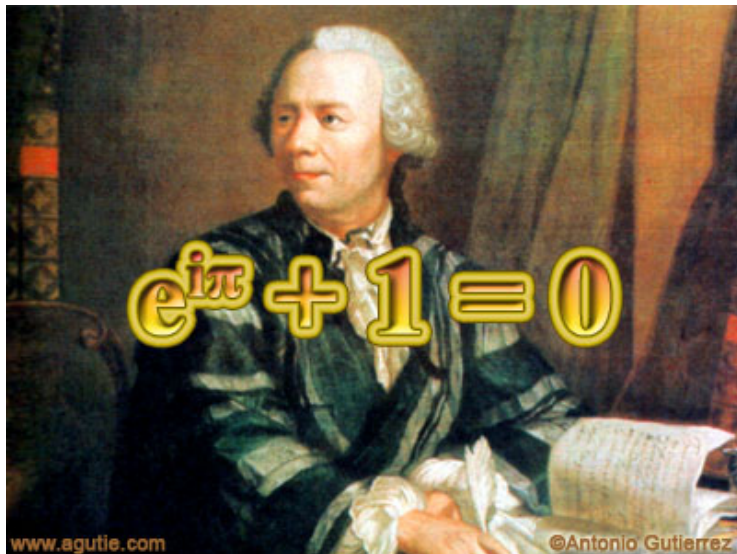
$$QR(21) = \{1, 4, 16\}$$

and

$$1^2 = 1 \bmod 21 \quad 4^2 = 16 \bmod 21 \quad 16^2 = 4 \bmod 21$$

DISCRETE SQUARE ROOTS CRYPTOSYSTEMS

Leonhard EULER - picture



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That is, likely, why Rabin did not propose this system as a practical cryptosystem.

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In case of Blum primes p and q and Blum integer $n = pq$, in order to solve the equation $x^2 \equiv a \pmod{n}$, one needs to compute squares of a modulo p and modulo q and then to use the Chinese remainder theorem to solve the equation $x^2 = a \pmod{pq}$.

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Using the Chinese Remainder Theorem we then get

$$x \equiv \pm 15, \pm 29 \pmod{77}.$$

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Example If $m_1 = 2, m_2 = 3, m_3 = 5$, then $(1, 0, 2)$ represents integer 27.

Advantage: With such a modular representation addition, subtraction and multiplication can be done component-wise and therefore in parallel time.

GENERALIZED RABIN CRYPTOSYSTEM

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Trapdoor: Blum primes p, q ($n = pq$)

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It is easy to verify that if ω is a nontrivial square root of 1 modulo n , then there are four decryptions of $e(x)$:

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Therefore decryption can be done by factoring n and solving congruences

$$x_1^2 \equiv c \pmod{p} \qquad x_1^2 \equiv c \pmod{q}$$

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Therefore computation of $\gcd(x_1 + r, n)$ or $\gcd(x_1 - r, n)$ must yield factors of n .

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Note: Security of the ElGamal cryptosystem is based on infeasibility of the discrete logarithm computation.

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then the ciphertext is (R_{n+1}, L_{n+1})

Decryption of (R_{n+1}, L_{n+1}) is done by computing, for $i = n, n-1, \dots, 0$

$$R_i = L_{i+1}, \quad L_i = R_{i+1} \oplus F(L_{i+1}, K_i)$$

and (L_0, R_0) is the plaintext

FEISTEL ENCRYPTION/DECRYPTION SCHEME

This is a general scheme for design of cryptosystems that was used at the design of several important cryptosystems, such as **Lucifer** and **DES**.

Its main advantage is that encryption and decryption are very similar, and even identical in some cases, and then the same hardware can be used for both encryption and decryption.

Let F be a so-called **round function** and K_0, K_1, \dots, K_n be sub-keys for rounds $0, 1, 2, \dots, n$.

Encryption is as follows:

- Split the plaintext into two equal size parts L_0, R_0 .
- For rounds $i \in \{0, 1, \dots, n\}$ compute

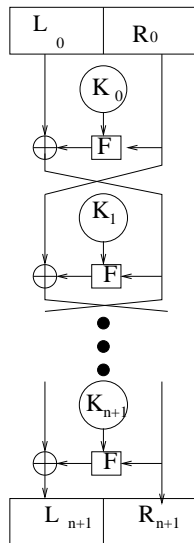
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WHEN ARE ENCRYPTIONS PERFECTLY SECURE?

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encryption: $e_k : P \times R \rightarrow C$

decryption: $d_k : C \rightarrow P$ or $C \rightarrow 2^P$ such that for any p, r :

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- d_k and e_k should be easy to compute.
- Given e_k , it should be unfeasible to determine d_k .

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- **It has been shown that perfectly secure cryptosystems have to use randomized encryptions.**

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Definition – computational distinguishability Let $X = \{X_n\}_{n \in N}$ and $Y = \{Y_n\}_{n \in N}$ be **probability ensembles** such that each X_n and Y_n ranges over strings of length n . We say that X and Y are **computationally indistinguishable** if for every feasible algorithm A the difference

$$d_A(n) = |Pr[A(X_n) = 1] - Pr[A(Y_n) = 1]|$$

is a negligible function in n .

SECURE ENCRYPTION – FIRST DEFINITION

Definition – semantic security of encryption A cryptographic system with an encryption function e is **semantically secure** if for every feasible algorithm A , there exists a feasible algorithm B so that for every two functions

$$f, h : \{0, 1\}^* \rightarrow \{0, 1\}^n$$

and all probability ensembles $\{X_n\}_{n \in \mathbb{N}}$, where X_n ranges over $\{0, 1\}^n$

$$\Pr[A(e(X_n), h(X_n)) = f(X_n)] < \Pr[B(h(X_n)) = f(X_n)] + \mu(n),$$

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RSA cryptosystem is not secure in the above sense. However, **randomized versions of RSA are semantically secure**.

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Quantum processes can generate perfect randomness and on this basis quantum (almost perfect) generators of randomness are already commercially available.

STORY of RANDOMNESS

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Epikurus (341-270 BC)

By Epikurus, there exists a true randomness that is independent of our knowledge.

Einstein also accepted the notion of randomness only in the relation to incomplete knowledge.

VIEWS on RANDOMNESS in 19th CENTURY

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There are only two possibilities, either a big chaos conquers the world, or order and law.

Marcus Aurelius

God does not roll dice.

EINSTEIN versus BOHR

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Famous reply by Niels Bohr - one of the fathers of quantum mechanics.

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- 2 Quantum measurement yields, in principle, random outcomes.

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- Attempts to formalize **chance** by mathematical laws is somehow paradoxical because, a priory, chance (randomness) is the subject of no law.
- There is no proof that perfect randomness exists in the real world.
- More exactly, there is no proof that quantum mechanical phenomena of the microworld can be exploited to provide a perfect source of randomness for the macroworld.

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Take an arbitrary integer x as the "seed" and repeat the following process:

 compute x^2 and take a sequence of the middle digits of x^2 as a new "seed" x .

SIMPLE PSEUDORANDOM GENERATORS

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Example. Linear congruential generator

One chooses n -bit numbers m , a , b , X_0 and generates an n^2 element sequence

$$X_1 X_2 \dots X_{n^2}$$

of n -bit numbers by the iterative process

$$X_{i+1} = (aX_i + b) \bmod m.$$

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- 2 The other side of the relation is more complex. It is clear that perfect randomness together with ONE-TIME PAD cryptosystem produces perfect secrecy. The price to pay: a key as long as plaintext is needed.

The way out seems to be to use an encryption algorithm with a pseudo-random generator to generate a long pseudo-random sequence from a short seed and to use the resulting sequence with ONE-TIME PAD.

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Perfect encryption should therefore produce (quite) perfect (pseudo)randomness.

- 2 The other side of the relation is more complex. It is clear that perfect randomness together with ONE-TIME PAD cryptosystem produces perfect secrecy. The price to pay: a key as long as plaintext is needed.

The way out seems to be to use an encryption algorithm with a pseudo-random generator to generate a long pseudo-random sequence from a short seed and to use the resulting sequence with ONE-TIME PAD.

Basic question: When is a pseudo-random generator good enough for cryptographic purposes?

CRYPTOGRAPHICALLY STRONG PSEUDORANDOM GENERATORS

In cryptography **random sequences** can usually be replaced by **pseudorandom sequences** generated by **(cryptographically perfect/strong) pseudorandom generators**.

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Definition. Let $I(n) : N \rightarrow N$ be such that $I(n) > n$ for all n . A **(cryptographically strong) pseudorandom generator with a stretch function I** , is an efficient deterministic algorithm which on the input of a random n -bit **seed** outputs a $I(n)$ -bit sequence which is computationally indistinguishable from any random $I(n)$ -bit sequence.

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Candidate for a cryptographically strong pseudorandom generator:

A very fundamental concept: A predicate b is a **hard core predicate** of the function f if b is easy to evaluate, but $b(x)$ is hard to predict from $f(x)$. (That is, it is unfeasible, given $f(x)$ where x is uniformly chosen, to predict $b(x)$ substantially better than with the probability $1/2$.)

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Definition. Let $l(n) : N \rightarrow N$ be such that $l(n) > n$ for all n . A **(cryptographically strong) pseudorandom generator with a stretch function l** , is an efficient deterministic algorithm which on the input of a random n -bit **seed** outputs a $l(n)$ -bit sequence which is computationally indistinguishable from any random $l(n)$ -bit sequence.

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Theorem Let f be a one-way function which is length preserving and efficiently computable, and b be a **hard core predicate** of f , then

$$G(s) = b(s) \cdot b(f(s)) \cdot \dots \cdot b\left(f^{l(|s|)-1}(s)\right)$$

is a (cryptographically strong) pseudorandom generator with stretch function $l(n)$.

Theorem A cryptographically strong (perfect) pseudorandom generator exists if one-way functions exist.

PSEUDORANDOM GENERATORS and ENCRYPTIONS

If two parties share a pseudorandom generator g ,

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for one-time pad for encoding and decoding.

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It has been shown that if integer factoring is intractable, then the so-called *BBS* pseudo-random generator, discussed below, is unpredictable to the left.

(We make use of the fact that if factoring is unfeasible, then for almost all quadratic residues $x \bmod n$, coin-tossing is the best possible way to estimate the least significant bit of x after seeing $x^2 \bmod n$.)

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Let n be a Blum integer. Choose a random quadratic residue x_0 (modulo n).

For $i \geq 0$ let

$$x_{i+1} = x_i^2 \bmod n, \quad b_i = \text{the least significant bit of } x_i$$

For each integer i , let

$$BBS_{n,i}(x_0) = b_0 \dots b_{i-1}$$

be the first i bits of the pseudo-random sequence generated from the seed x_0 by the *BBS* pseudo-random generator.

PERFECTLY SECURE CIPHERS - EXAMPLES

RANDOMIZED VERSION of RSA-LIKE CRYPTOSYSTEM

The scheme works for **any trapdoor function** (as in case of RSA),

$$f : D \rightarrow D, D \subset \{0, 1\}^n,$$

for any **pseudorandom generator**

$$G : \{0, 1\}^k \rightarrow \{0, 1\}^l, k \ll l$$

and any **hash function**

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where **$n = l + k$** .

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Encryption of a message $m \in \{0, 1\}^l$ is done as follows:

- 1 A random string $r \in \{0, 1\}^k$ is chosen.
- 2 Set $x = (m \oplus G(r)) \parallel (r \oplus h(m \oplus G(r)))$. (If $x \notin D$ go to step 1.)
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Decryption of a ciphertext c .

- Compute $f^{-1}(c) = a \parallel b$, $|a| = l$ and $|b| = k$.
- Set $r = h(a) \oplus b$ and get $m = a \oplus G(r)$.

Comment: Operation " \parallel " stands for a concatenation of strings.

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Encryption of $x \in \{0, 1\}^m$.

- 1 Randomly choose $s_0 \in \{0, 1, \dots, n\}$.
- 2 For $i = 1, 2, \dots, m + 1$ compute

$$s_i \leftarrow s_{i-1}^2 \bmod n$$

and $\sigma_i = \text{lsb}(s_i)$. — {lsb – least significant bit}

The cryptotext is then (s_{m+1}, y) , where $y = x \oplus \sigma_1 \sigma_2 \dots \sigma_m$.

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Decryption: of the ciphertext (r, y) :

Let $d = 2^{-m} \bmod \phi(n)$.

- Let $s_1 = r^d \bmod n$.
- For $i = 1, \dots, m$, compute $\sigma_i = \text{lsb}(s_i)$ and $s_{i+1} \leftarrow s_i^2 \bmod n$

The plaintext x can then be computed as $y \oplus \sigma_1 \sigma_2 \dots \sigma_m$.

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- there is no method known, in spite of many years of many attempts, to show that that problem can be solved in a reasonable length of time.

HASH FUNCTIONS

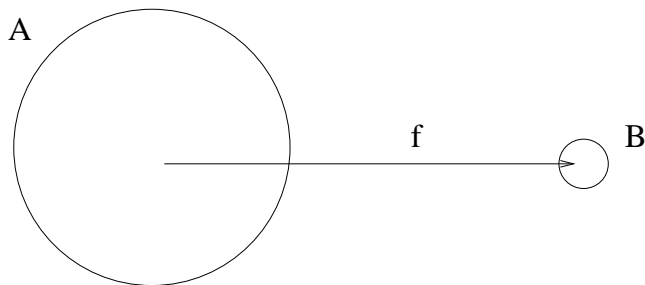
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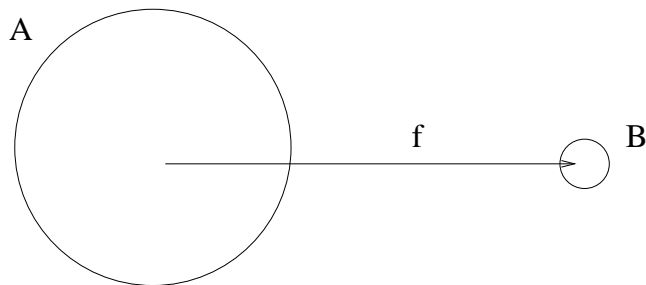
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Hash functions have a variety of applications, especially in the design of efficient algorithms and in cryptography.

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In cryptographic practice "**difficult**" generally means "**almost certainly beyond the reach of any adversary who must be prevented from breaking the system for as long as the security of the system is considered to be very important**".

SOME APPLICATIONS

- **To verify integrity of messages:** To determine whether a change was made to a message during a transmission, can be done by comparing message digests calculating before, and after, the transmission.

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In 2013 a long-term **Password Hashing Competition** was announced to choose a new, standard algorithm for password hashing.

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The receiver, familiar also with the hash function h that is being used, can then verify the integrity of the message w' he receives by computing $h(w')$ and comparing

$$h(w) \text{ and } h(w') .$$

EXAMPLES

Example 1 For a vector $a = (a_1, \dots, a_k)$ of integers let

$$H(a) = \sum_{i=0}^k a_i \bmod n$$

where n is a product of two large primes.

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This function is one-way, but it is not weakly collision resistant.

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For example such good properties have these two functions:

$$f(x_i, g_{i-1}) = e_{g_{i-1}}(x_i) \oplus x_i$$

$$f(x_i, g_{i-1}) = e_{g_{i-1}}(x_i) \oplus x_i \oplus g_{i-1}$$

PRACTICALLY USED HASH FUNCTIONS

A variety of hash functions has been constructed. Very often used hash functions were MD4, MD5 (created by Rivest in 1990 and 1991 and producing 128 bit message digest).

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Observe that every cryptographic hash function is vulnerable to a collision attack using so called **birthday attack**. Due to the **birthday problem** a hash of n bits can be broken in $\sqrt{2^n}$ evaluations of the hash function - much faster than the brute force attack.

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- In order to ensure long-term robustness of applications that use hash functions a public competition was announced by NIST to replace SHA-2.
- On October 2012 Keccak was selected as the winner and a version of this algorithm is expected to be a new standard (since 2014) under the name SHA-3.

Often used in practise has been hash function MD5 designed in 1991 by Rivest. It maps any binary message into 128-bit hash.

The input message is broken into 512-bit blocks, divided into 16 words-states (of 32 bits) and padded if needed to have final length divisible by 512. Padding consists of a bit 1 followed by so many 0's as required to have the length up to 64 bits fewer than a multiple of 512. Final 64 bits represent the length of the original message modulo 2^{64} .

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The main MD5 algorithm operates on 128-bits words that are divided into four 32-bits words A, B, C, D initialized to some fixed constants. The main algorithm then operates on 512 bit message blocks in turn - each block modifying the state.

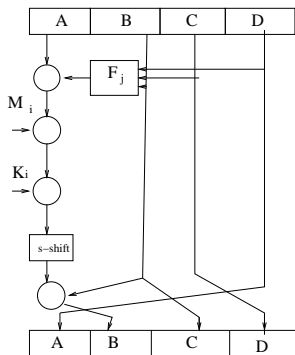
The processing of a message consists of four rounds. j -th round is composed of 16 similar operations using non-linear functions F_j and left rotations by s_j places where s_j varies for each round - see next figure.

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- In 2006 Vladimír Klima published an algorithm to find a collision for MD5 within one minute on a notebook.
- In 2010 T. Xie, O. Feng published single-block MD5 collision.

HOW to FIND COLLISIONS of HASH FUNCTIONS

The most basic method is based on so-called birthday paradox related to so-called the birthday problem.

BIRTHDAY PROBLEM and its VARIATIONS

It is well known that if there are 23 (29) [40] {57} $< 100 >$ people in one room, then the probability that two of them have the same birthday is more than 50% (70%)[89%] {99%} $< 99.99997\% >$ — this is called a Birthday paradox.

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More generally, if we have n objects and r people, each choosing one object (so that several people can choose the same object), then if $r \approx 1.177\sqrt{n}$ ($r \approx \sqrt{2n\lambda}$), then probability that two people choose the same object is 50% ($(1 - e^{-\lambda})\%$).

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Another version of the birthday paradox: Let us have n objects and two groups of r people. If $r \approx \sqrt{\lambda n}$, then probability that someone from one group chooses the same object as someone from the other group is $(1 - e^{-\lambda})$.

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For the probability $\bar{p}(n)$ that all $n < 366$ people in a room have birthday in different days, it holds

$$\bar{p}(n) = \prod_{i=1}^{n-1} \left(\frac{365-i}{365} \right) = \frac{\prod_{i=1}^{n-1} (365-i)}{365^{n-1}} = \frac{365!}{365^n (365-n)!}$$

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Probability $p(n)$ that at least two person have the same birthday is therefore

$$p(n) = 1 - \bar{p}(n)$$

This probability is larger than 0.5 first time for $n = 23$.

FINDING COLLISIONS USING BIRTHDAY PARADOX

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To find collisions, that is two x_1 and x_2 such that $h(x_1) = h(x_2)$ is easier, thanks to the birthday paradox and can be done by the following algorithm:

ALGORITHM

Input: A hash function h onto a domain of size n , a real θ and an empty hash table.

Output: A pair (x_1, x_2) such that $x_1 \neq x_2$ and $h(x_1) = h(x_2)$

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1. for $\theta\sqrt{n}$ different x do
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4. yield (x, x') and stop
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Theorem If we pick the numbers x with uniform distribution in $\{1, 2, \dots, n\}$ $\theta\sqrt{n}$ times, then we get at least one number twice with probability converging (for $n \rightarrow \infty$) to

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For $n = 365$ we get triples: $(\theta, \theta\sqrt{n}, \text{probability})$ as follows: $(0.79, 15, 25\%); (1.31, 25, 57\%); (2.09, 40, 89\%)$

WHY CURRENTLY BROADLY USED HASHES HAVE 160 BITS?

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For example, a 40-bit hashes would be insecure because a collision could be found with probability 0.5 with just over 40^{20} random guesses.

Minimum acceptable size of hashes seems to be 128 and therefore 160 are used in such important systems as **DSS – Digital Signature Schemes (a standard)**.