Part I

Cyclic, stream and channel codes. Speccial decoding

CHAPTER 3: CYCLIC, STREAM and CHANNEL CODES - SPECIAL DECODINGS

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4. Locally decodable codes can be seen as a theoretical extreme of coding theory with deep theoretical implications.

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codeword of length n - a generator codeword of the code C.

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- (i) C is a linear code;
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- (iii) The binary linear code $\{0000, 1001, 0110, 1111\}$ is not cyclic, but it is equivalent to a cyclic code. – to get a cyclic code exchange first two symbols in all codewords.
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- (a) cyclic?
- (b) or at least equivalent to a cyclic code?

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For some cases, for example for $n = 19$ and $F = GF(2)$, the above four trivial cyclic codes are the only cyclic codes.

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and it is cyclic because the right shifts have the following impacts

$$c_1 \rightarrow c_2, \qquad c_2 \rightarrow c_3, \qquad c_3 \rightarrow c_1 + c_3$$
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A codeword of a cyclic code is usually denoted by

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Division of polynomials For every pair of polynomials $a(x)$, $b(x) \neq 0$ in $F_q[x]$ there exists a unique pair of polynomials $q(x)$, $r(x)$ in $F_q[x]$ such that

$$a(x) = q(x)b(x) + r(x), deg(r(x)) < deg(b(x)).$$

Example Divide $x^3 + x + 1$ by $x^2 + x + 1$ in $F_2[x]$.

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Definition Let $f(x)$ be a fixed polynomial in $F_q[x]$. Two polynomials $g(x)$, $h(x)$ are said to be **congruent modulo** $f(x)$, notation

$$g(x) \equiv h(x) \pmod{f(x)},$$

if $g(x) - h(x)$ is divisible by $f(x)$.

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The word starting with 2^{124} zeros and followed by one 1 has the polynomial representation:

$$x^{124}$$

In the alphabet $\{0, 1, 2\}$ $2x^2$ represents the string 002

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APPENDIX - III.

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Example Which of the following sets is an (Abelian) group:

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- What happens if we consider only matrices with determinants not equal zero?

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A non-zero element g is a **primitive element** of a field F if all non-zero elements of F are powers of g.

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If $f(x)$ is not reducible, then it is said to be **irreducible** in $F_q[x]$.

Theorem The ring $F_q[x]/f(x)$ is a field if $f(x)$ is irreducible in $F_q[x]$.

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is in C by (i) because all summons above are cyclic shifts of $a(x)$.

(2) Let (i) and (ii) hold

- Taking $r(x)$ to be a scalar the conditions (i) and (ii) imply linearity of C.
- Taking $r(x) = x$ the conditions (i) and (ii) imply cyclicity of C.

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$$C = \{000, 110, 101, 011\}$$

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Indeed, all we need to do is to find all factors (**in GF(q)**) of

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HOW TO DESIGN CYCLIC CODES?

The last claim of the previous theorem gives **a recipe to get all cyclic codes of the given length n in GF(q)**

Indeed, all we need to do is to find all factors (**in GF(q)**) of

$$x^n - 1.$$

Problem: Find all binary cyclic codes of length 3.

Solution: Make decomposition

$$x^3 - 1 = \underbrace{(x-1)(x^2+x+1)}_{\text{both factors are irreducible in GF(2)}}$$

Therefore, we have the following generator polynomials and cyclic codes of length 3.

Generator polynomials	Code in R_3	Code in $V(3,2)$
1	R_3	$V(3,2)$
$x+1$	$\{0, 1+x, x+x^2, 1+x^2\}$	$\{000, 110, 011, 101\}$
x^2+x+1	$\{0, 1+x+x^2\}$	$\{000, 111\}$
$x^3-1\ (=0)$	$\{0\}$	$\{000\}$

DESIGN of GENERATOR MATRICES for CYCLIC CODES

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Hence

$$\begin{aligned} q(x)g(x) &= (q_0 + q_1 x + \ldots + q_{n-r-1} x^{n-r-1}) g(x) \\ &= q_0 g(x) + q_1 x g(x) + \ldots + q_{n-r-1} x^{n-r-1} g(x). \end{aligned}$$

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EXAMPLE - II

In order to determine all binary cyclic codes of length 7, consider decomposition

$$x^7 - 1 = (x - 1)(x^3 + x + 1)(x^3 + x^2 + 1)$$

Since we want to determine binary codes, all computations should be modulo 2 and therefor all minus signs can be replaced by plus signs. Therefore

$$x^7 + 1 = (x + 1)(x^3 + x + 1)(x^3 + x^2 + 1)$$

Therefore generators for 2^3 binary cyclic codes of length 7 are

$$1, \quad a(x) = x + 1, \quad b(x) = x^3 + x + 1, \quad c(x) = x^3 + x^2 + 1$$
$$a(x)b(x), \quad a(x)c(x), \quad b(x)c(x), \quad a(x)b(x)c(x) = x^7 + 1$$

ENCODING with CYCLIC CODES I

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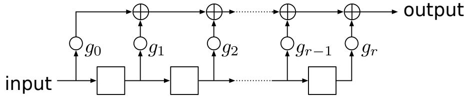

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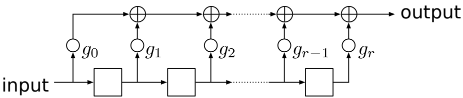

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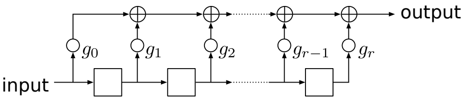

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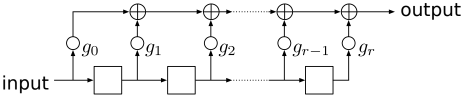

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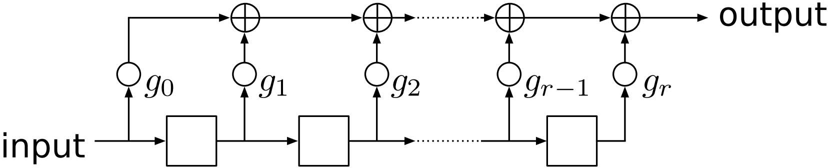

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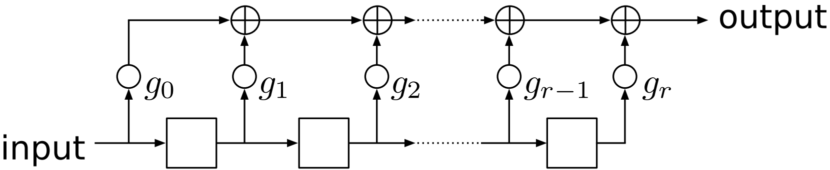

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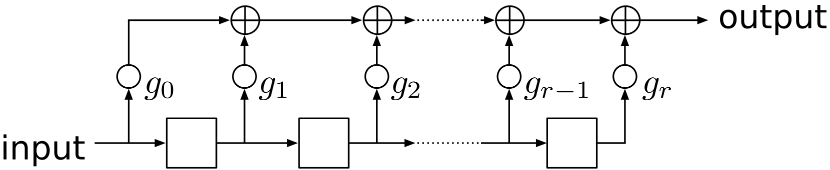

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MULTIPLICATION of POLYNOMIALS by SHIFT-REGISTERS

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Let us compute

$$(m_0 + m_1 x + \dots m_{k-1} x^{k-1}) \times (g_0 + g_1 x + g_2 x^2 \dots g_{r-1} x^{r-1})$$

$$=$$

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EXAMPLES of CYCLIC CODES

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\end{pmatrix}$$

G_{24} is $(24, 12, 8)$-code and the weights of all codewords are multiples of 4. G_{23} is obtained from G_{24} by deleting last symbol of each codeword of G_{24}. G_{23} is $(23, 12, 7)$-code. It is a perfect code.

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This code can be constructed via factorization of $x^{23} - 1$.

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Golay codes are named to honour Marcel J. E. Golay - from 1949.

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A **Polynomial code**, with codewords of length n, **generated by a (generator) polynomial** $g(x)$ of degree $m < n$ over a GF(q) is the code whose codewords are represented exactly by those polynomials of degree less than n that are divisible by $g(x)$.

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Example: For the binary polynomial code with $n = 5$ and $m = 2$ generated by the polynomial $g(x) = x^2 + x + 1$ all codewords are of the form:

$$a(x)g(x)$$

where

$$a(x) \in \{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$$

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what results in the code with codewords

$$00000, 00111, 01110, 01001,$$

$$11100, 11011, 10010, 10101.$$

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Each Reed-Muller code $RM(r, m)$ is the code of k codewords of length $n = 2^m$, to encode k messages, and distance 2^{m-r}, where

$$k = \sum_{s=0}^{r} \binom{d}{s}.$$

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$RM(r,m)$ code is generated by the set of all up to r inner products of the codewords v_i, $0 \leq i \leq d$, where $v_0 = 1^{2^d}$ and v_i are prefixes of the word $\{1^i 0^i\}^*$.

Example 1: $RM(1,3)$ code is generated by the codewords

$$v_0 = 11111111$$

$$v_1 = 10101010$$

$$v_2 = 11001100$$

Example 2: $RM(2,3)$ code is generated by the codewords

$$v_0, v_1, v_2, v_3, v_1 \cdot v_2, v_1 \cdot v_3, v_2 \cdot v_3$$

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Reed-Muller codes are closely related to Polar codes. David E. Muller discovered them in 1954 and Irving S. Reed was first to propose for them efficient decoding algorithm.

BCH CODES and REED-SOLOMON CODES

[1] BHC stands for Bose and Ray-Chaudhuri and Hocquenghem who discovered these codes in 1959.

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Reed-Solomon codes found many important applications from deep-space travel to consumer electronics.

They are very useful especially in those applications where one can expect that errors occur in bursts - such as ones caused by solar energy.

CHANNELS (STREAMS) CODING

CHANNEL CODING - BASICS

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A breakthrough came when D. Forney, in his PhD thesis in 1972, showed that so called concatenated codes could be used to achieve exponentially decreasing error probabilities at all data rates less than the Shannon channel capacity, with decoding complexity increasing only polynomially with the code length.

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The code rate express the amount of redundancy in the code - the lower is the code rate, the more redundancy is in the codewords.

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By the **noisy-channel Shannon coding theorem**, the channel capacity of a given channel is the limiting code rate (in units of information per unit time) that can be achieved with arbitrary small error probability.

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The joint distribution $P_{X,Y}(x,y)$ is then defined by

$$P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x),$$

where $P_X(x)$ is the marginal distribution.

CHANNEL CAPACITY - FORMAL DEFINITION

Let X and Y be random variables representing the input and output of a channel.

Let $P_{Y|X}(y|x)$ be the conditional probability distribution function of Y given X, which can be seen as an inherent fixed probability of the communication channel.

The joint distribution $P_{X,Y}(x,y)$ is then defined by

$$P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x),$$

where $P_X(x)$ is the marginal distribution.

The channel capacity is then defined by

$$C = \sup_{P_X(x)} I(X,Y)$$

where

$$I(X,Y) = \sum_{y\in Y}\sum_{x\in X} P_{X,Y}(x,y)\log\left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}\right)$$

is the mutual distribution - a measure of variables mutual distribution.

SHANNON NOISY CHANEL THEOREM

For every discrete memoryless channel, the channel capacity

$$C = \sup_{P_X} I(X, Y)$$

has the following properties:
1. For every $\varepsilon > 0$ and $R < C$, for large enough N there exists a code of length N and code rate R and a decoding algorithm, such that the maximal probability of the block error is $\leq \varepsilon$.
2. If a probability of the block error p_b is acceptable, code rates up to $R(p_b)$ are achievable, where

$$R(p_b) = \frac{C}{1 - H_2(p_b)}$$

and $H_2(p_b)$ is the binary entropy function.
3. For any p_b code rates greater than $R(p_b)$ are not achievable.

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For example,

$$G_1 = [x^2 + 1, x^2 + x + 1]$$

is the generator matrix for a $(2, 1)$ convolution code, denoted CC_1, and

$$G_2 = \begin{pmatrix} 1 + x & 0 & x + 1 \\ 0 & 1 & x \end{pmatrix}$$

is the generator matrix for a $(3, 2)$ convolution code denoted CC_2

ENCODING of FINITE POLYNOMIALS

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An (n,k) convolution code with a k x n generator matrix G can be used to encode a k-tuple of **message-polynomials** (polynomial input information)

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EXAMPLES

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That is the output streams C_0 and C_1 are obtained by convoluting the input stream with polynomials of G_1.

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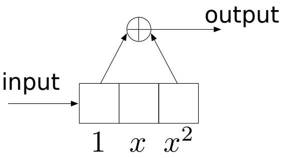

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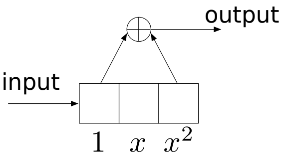

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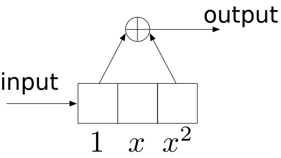

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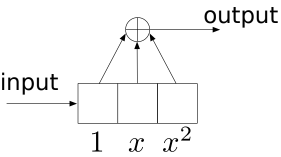

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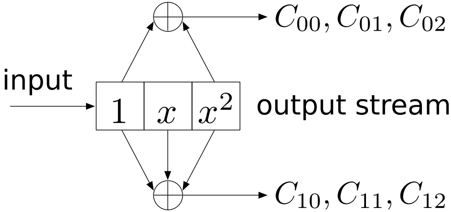

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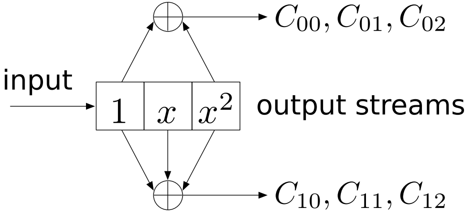

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Given $(x, y) \in \{-1, 1\} \times R$, the noise $y - x$ is distributed according to the Gaussian distribution of zero mean and standard derivation σ of the channel

$$Pr(y|x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2\sigma^2}}$$

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Concatenated codes and Turbo codes, discussed later, have such a Shannon capacity approaching property.

CONCATENATED CODES - I

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The basic idea of concatenated codes is extremely simple. A given message is first encoded by the first (outer) code C_1 (C_{out}) and C_1-output is then encoded by the second code C_2 (C_{in}). To decode, at first C_2 decoding and then C_1 decoding are used.

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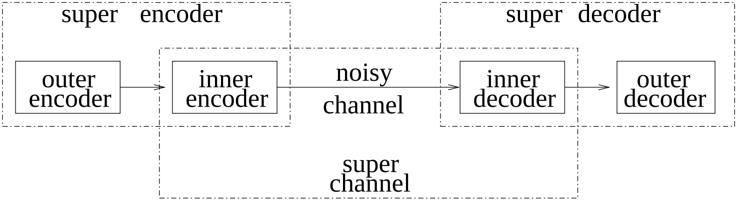

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In 1965 concatenated codes were considered as unfeasible. However, already in 1970s technology has advanced sufficiently and they became standardize by NASA for space applications.

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A code concatenated codes C_{out} and C_{in} maps a message

$$m = (m_1, m_2, \dots, m_K),$$

as follows: At first C_{out} encoding is applied to get

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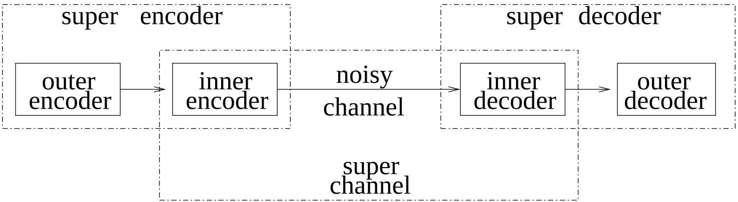

- **Outer code:** - (n_2, k_2) code
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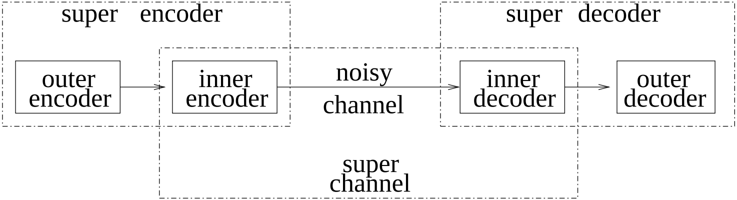

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- **length** of such a concatenated code is $n_1 n_2$
- **dimension** of such a concatenated code is $k_1 k_2$
- if **minimal distances** of both codes are d_1 and d_2, then resulting concatenated code has minimal distance $\geq d_1 d_2$.

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- In such a case we can use an exponential time but optimal maximum likelihood decoder for the inner code.

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- Concatenated codes are used also on Compact Disc.
- The best concatenated codes for many applications were based on outer Reed-Solomon codes and inner Viterbi-decoded short constant length convolution codes.

EXAMPLE from SPACE EXPLORATION

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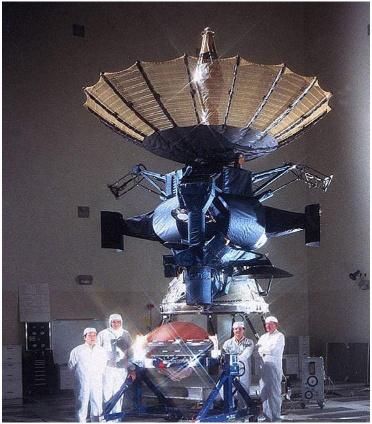

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GALILEO MISSION - SOLUTION

The primary antenna was designed to send 100, 000 b/s. Spacecraft had also another antenna, but that was capable to send only 10 b/s. The whole mission looked as being a disaster.
A heroic engineering effort was immediately undertaken in the mission center to design the most powerful concatenated code conceived up to that time, and to program it into the spacecraft computer.

The inner code was a 2^{14} convolution code, decoded by the Viterbi algorithm.

The outer code consisted of multiple Reed-Solomon codes of varying length.

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Nowadays when so called iterative decoding is used concatenation of even very simple codes can yield superb performance.

TURBO CODES

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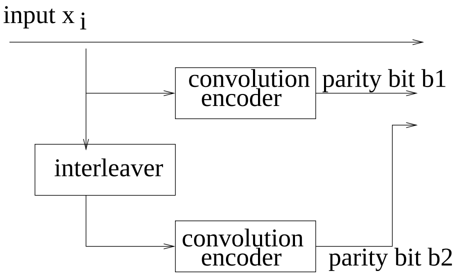
A Turbo encoder is formed from the parallel composition of two (convolution) encoders separated by an interleaver.



EXAMPLES of TURBO and CONVOLUTION ENCODERS

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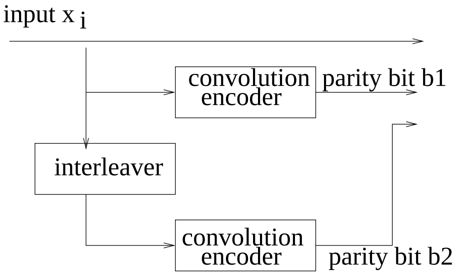

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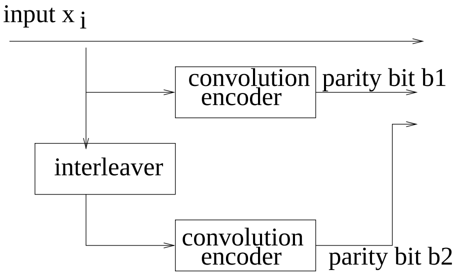
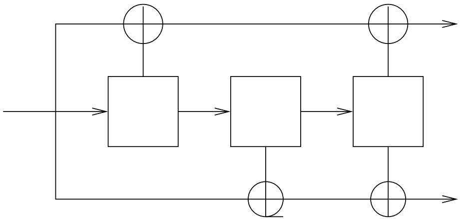


and a convolution encoder



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However, after the inverse permutation the output actually will be

$$c.n.j.200k.$$

which is quite easy to decode correctly!!!!

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- Literature: M.C. Valenti and J.Sun: Turbo codes - tutorial, Handbook of RF and Wireless Technologies, 2004 - reachable by Google.

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A decibel is a relative measure. If E is the actual energy and E_{ref} is the theoretical lower bound, then the relative energy increase in decibels is

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- For code rate $\frac{1}{2}$ the relative increase in energy consumption is about 4.8 dB for convolution codes and 0.98 for Turbo codes.

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- For sufficiently large size of interleavers, the correcting performance of turbo codes, as shown by simulations, appears to be close to the theoretical Shannon limit.
- Permutations performed by interleaver can often by specified by simple polynomials that make one-to-one mapping of some sets $\{0, 1, \ldots, q-1\}$.

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- A big advantage of Turbo encoders is that they reduce the number of low-weight codewords because their output is the sum of the weights of the input and two parity output bits.
- A turbo code can be seen as a refinement of concatenated codes plus an iterative algorithm for decoding.

LIST DECODING

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List decoding seems to be a stronger error-correcting mode than unique decoding.

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Theorem let $q \geq 2$, $0 \leq p \leq 1 - 1/q$ and $\varepsilon \geq 0$ then for large enough block length n if the code rate $R \leq 1 - H_q(p) - \varepsilon$, then there exists a $(p, O(1/\varepsilon))$-list decodable code.
[$H_q(p) = p \log_q(q-1) - p \log_q p - (1-p) \log_q(1-p)$ is q-ary entropy function.]
Moreover, if $R > 1 - H_q(p) + \varepsilon$, then every (p, L)-list-decodable code has $L = q^{\Omega(n)}$

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APPENDIX - I.

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- Modern versions of concatenated Reed-Solomon/Viterbi decoder convolution coding were and are used on the Mars Pathfinder, Galileo, Mars exploration Rover and Cassini missions, where they performed within about 1-1.5dB of the ultimate limit imposed by the Shannon theorem.

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- New computation tools are developed - for example special types of parallelization,....

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Locally decodable codes have a variety of applications in cryptography and theory of fault-tolerant computation.

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Moreover, this can be done by picking at random only three bits of the received message and combining them in a right way.