Part I

Linear codes

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- All next sets of Exercises will be put on my web page and into IS, always at 18.00 on Thurdays after my lecture and solutions should be delvered in 2 weeks to you.

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Many practically important linear codes have also an efficient decoding.

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Comment. To design linear codes we will use Galois fields $GF(q)$ with q being a prime. One can also use Galois fields $GF(q^k)$, $k > 1$, but their structure and operations are defined in a more complex way, see the Appendix.

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Encoding (code) is called systematic if for any $m \in M \subset \Sigma^*$

$$e(m) = mc_m \text{ for some } c_m \in \Sigma^*$$

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A code is called systematic if its encoder transmit a message (an input dataword) w into a codeword of the form wc_w, or (w, c_w). That is if the codeword for the message w consists of two parts: the message w itself (called also information part) and a redundancy part c_w

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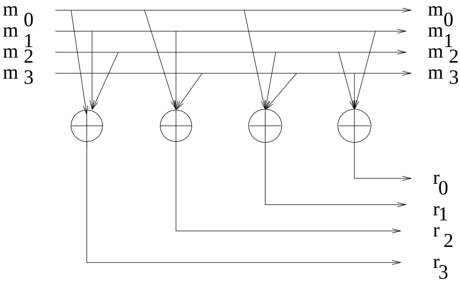
Nowadays most of the stream codes that are used in practice are systematic.

An example of a systematic encoder, that produces so called extended Hamming $(8, 4, 1)$ code is in the following figure.



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In general, does it has a sense to look for such codes that some important sum of any two codewords is again a codeword?

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Lemma A subset $C \subseteq F_q^n$ is a linear code iff one of the following conditions is satisfied

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Theorem A binary linear code of dimension k has

$$\frac{1}{k!}\prod_{i=0}^{k-1}(2^k - 2^i)$$

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$$C_4 \text{ is } \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

ADVANTAGES and DISADVANTAGES of LINEAR CODES I.

Advantages - are big.

1. Minimal distance $h(C)$ is easy to compute if C is a linear code.
2. Linear codes have simple specifications.

- To specify a non-linear code usually all codewords have to be listed.
- To specify a linear $[n, k]$-code it is enough to list k codewords (of a basis).

Definition A $k \times n$ matrix whose rows form a basis of a linear $[n, k]$-code (subspace) C is said to be the generator matrix of C.

Example One of the generator matrices of the binary code

$$C_2 = \left\{\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}\right\} \text{ is the matrix } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

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3. There are simple encoding/decoding procedures for linear codes.

ADANTAGES and DISADVANTAGES of LINEAR CODES II.

Disadvantages of linear codes are small:

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2. The restriction to linear codes might be a restriction to weaker codes than sometimes desired.

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(e) multiplication of a column by a non-zero scalar

Proof Operations (a) - (c) just replace one basis by another. Last two operations convert a generator matrix to one of an equivalent code.

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Theorem Let G be a generator matrix of an $[n, k]$-code. Rows of G are then linearly independent .By operations (a) - (e) the matrix G can be transformed into the form: $[I_k|A]$ where I_k is the $k \times k$ identity matrix, and A is a $k \times (n-k)$ matrix.

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$$0 = \sum_{i=1}^k u_{1,i} w_i - \sum_{i=1}^k u_{2,i} w_i = \sum_{i=1}^k (u_{1,i} - u_{2,i}) w_i$$

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And, therefore, since w_i are linearly independent, $u_1 = u_2$.

LINEAR CODES as SYSTEMATIC CODES

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Since to each linear $[n, k]$-code C there is a generator matrix of the form $G = [I_k|A]$ an encoding of a dataword w with G has the form

$$wG = w \cdot wA$$

Each linear code is therefore equivalent to a systematic code.

DECODING of LINEAR CODES - BASICS

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Definition Suppose C is an $[n, k]$-code over F_q^n and $u \in F_q^n$. Then the set

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is called a **coset** **(u-coset) of C in F_q^n.**

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In practice, this decoding method is too slow and requires too much memory.

How good are particular linear codes?

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SYNDROMES APPROACH to DECODING

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Answer: All words of S have at the end the same symbol as at the beginning.

EXAMPLE

For the $[n, 1]$-repetition (binary) code C, with the generator matrix

$$G = (1, 1, \ldots, 1)$$

the dual code $C^\perp$ is $[n, n-1]$-code with the generator matrix $G^\perp$, described by

$$G^\perp = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ & \ldots & & & & \\ 1 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

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The rows of a parity check matrix are parity checks on codewords. They actually say that certain linear combinations of elements of every codeword are zeros modulo 2.

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KEY OBSERVATION for SYNDROM COMPUTATION

When preparing a "syndrome decoding" it is sufcient to store only two columns: one for **coset leaders** and one for **syndromes**.

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Example If $y = 1111$, then $S(y) = 01$ and the above decoding procedure produces

$$1111–0100 = 1011.$$

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In general, the problem of finding the nearest neighbour in a linear code is NP-complete. Fortunately, there are important linear codes with really efficient decoding.

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Properties of binary Hamming codes Coset leaders are precisely words of weight ≤ 1. The syndrome of the word $0 \ldots 010 \ldots 0$ with 1 in j-th position and 0 otherwise is the transpose of the j-th column of H.

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Hamming codes were originally used to deal with errors in long-distance telephon calls.

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In case $q = 0.9$ the probability of correct transmission is 0.6561 in the case no error correction is used and 0.8503 in the case Hamming code is used - an essential improvement.

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G_{24} is $(24, 12, 8)$-code and the weights of all codewords are multiples of 4. G_{23} is obtained from G_{24} by deleting last symbols of each codeword of G_{24}. G_{23} is $(23, 12, 7)$-code.

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Rows of the last 11 columns are cyclic permutations of the first row which has 1 at those positions that are squares modulo 11, that is

$$0,\ 1,\ 3,\ 4,\ 5,\ 9.$$

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$$M_{2^n} = \begin{bmatrix} M_{2^{n-1}} & M_{2^{n-1}} \\ M_{2n-1} & \bar{M}_{2^{n-1}} n \end{bmatrix}$$

where $\bar{M}_n$ is the complementary matrix to M_n (with 0 and 1 interchanged).

EXAMPLE

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Hadamard code

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- Matrix Q_r is obtained by considering all combinations of r rows of $G_{1,m}$ and by obtaining products of these rows/vectors, component by component. The result of each of such a multiplication constitues a row of Q_r.

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$$G_{1,4} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

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Plotkin bound implies that q-nary error-correcting codes with $d \geq n(1 - 1/q)$ have only polynomially many codewords and hence are not very interesting.

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If C is a q-ary linear $[n, k, d]$-code, then

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When $d = 1$, then E is an $[n-1, k, 1]$ code, if C has no codeword of weight 1 whose nonzero entry is in last coordinate; otherwise, if $k > 1$, then E is an $[n-1, k-1, d^*]$ code with $d^* > 1$

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They are codes a generator matrix of which has rows labelled by polynomials X^i, $0 \leq i \leq k-1$, columns labeled by elements $0, 1, \ldots, q-1$ and the element in the row labelled by a polynomial p and in the column labelled by an element u is $p(u)$.

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Reed-Solomon codes are used in digital television, satellite communication, wireless communication, barcodes, compact discs, DVD,... They are very good to correct burst errors - such as ones caused by solar energy.

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In case one has to bet for 13 games, then one can usually have two games with pretty sure outcomes and for the rest one can use the above ternary Golay code.

APPENDIX

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3. Some LDPC codes are well suited for implementations that make heavy use of parallelism.

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2. Good LDPC codes can be decoded in time linear to their block length using special (for example "iterative belief propagation") approximation techniques.
3. Some LDPC codes are well suited for implementations that make heavy use of parallelism.

LDPC (Low-Density Parity Check) - CODES

A LDPC code is a binary linear code whose parity check matrix is very sparse - it contains only very few 1's.

A linear $[n, k]$ code is said to be a **regular** $[n, k, r, c]$ **LDPC code** if $r << n, c << n - k$ and its parity-check matrix has exactly r 1's in each row and exactly c 1's in each column.

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Parity-check matrices for LDPC codes are often (pseudo)-randomly generated, subject to sparsity constrains. Such LDPC codes are proven to be good with a high probability.

DISCOVERY and APPLICATION of LDPC CODES

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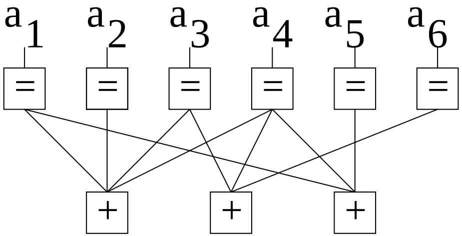
BI-PARTITE (TANNER) GRAPHS REPRESENTATION of LDPC CODES

An $[n, k]$ LDPC code can be represented by a bipartite graph between a set of n top "variable-nodes (v-nodes)" and a set of bottom $(n - k)$ "parity check nodes (pc-nodes)".
Variable nodes:



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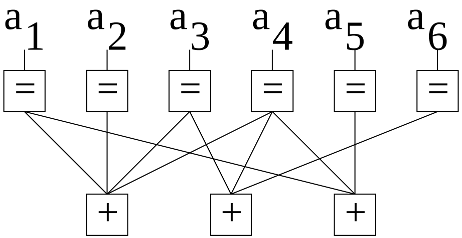

Parity check nodes:

The corresponding parity check matrix has $n-k$ rows and n columns and i-th column has 1 in the j-th row exactly in case if i-th v-node is connected to j-th c-node.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

TANNER GRAPHS - CONTINUATION

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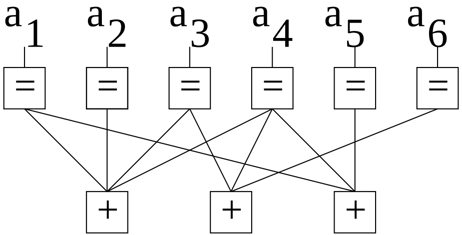
The LDPC-code with the Tanner bipartite graph for $(6, 3)$ LDPC-code.



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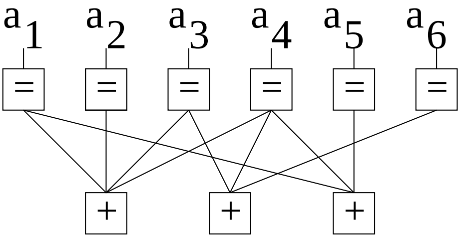

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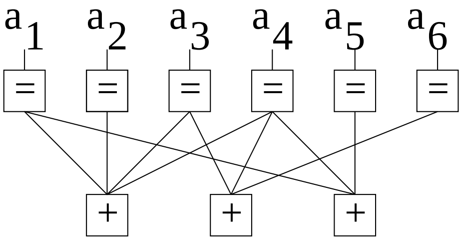

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and therefore the following constrains have to be satisfied:

$$a_1 + a_2 + a_3 + a_4 = 0$$

$$a_3 + a_4 + a_6 = 0$$

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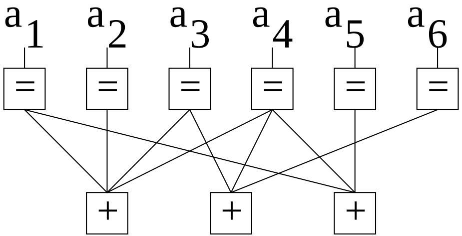

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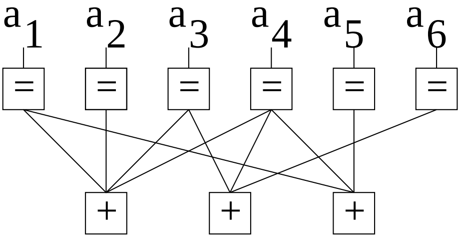

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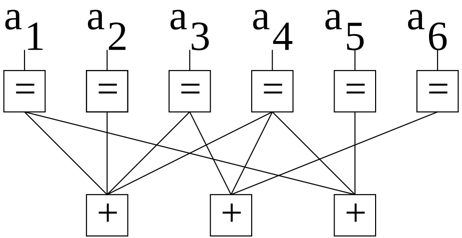

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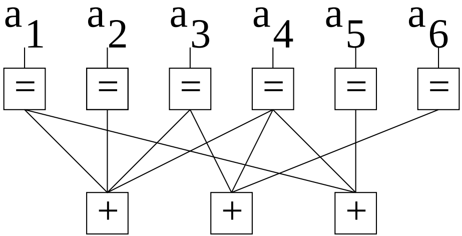

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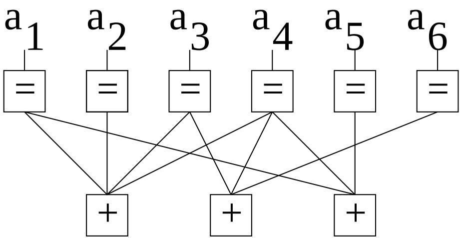

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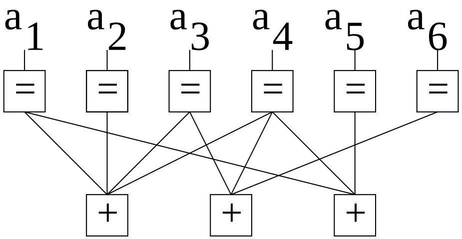

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Using so called **iterative belief propagation techniques**, LDPC codes can be decoded in time linear to their block length.

DESIGN of LDPC codes

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- Some good LDPC codes were designed through randomly chosen parity check matrices.
- Some LDPC codes are based on Reed-Solomon codes, such as the RS-LDPC code used in the 10-gigabit Ethernet standard.