ITI Online Seminar

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Toroidal grid minors and stretch in embedded graphs

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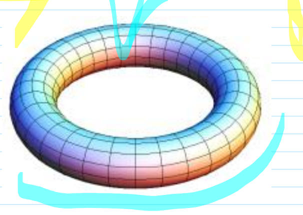
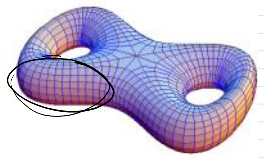
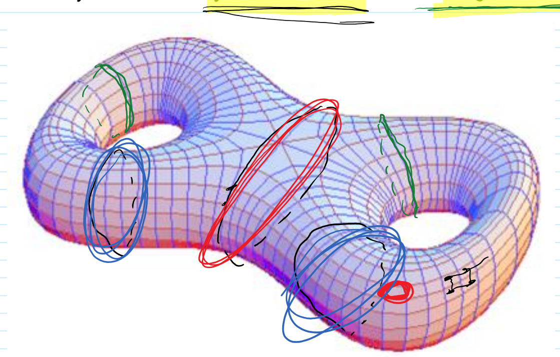




- orientable surfaces only!

Basic definitions: face-width and edge-width



~~separating~~ / non-separating *primal / dual*

dual edge-width *vs.* *face-width*

touched by edges *≙* *touched by vert.*

chains of faces

Having large face-width...

Robertson and Seymour [29]

For any graph H embedded on a surface Σ, there exists a constant $c := c_\Sigma(H)$ such that every graph G that embeds in Σ with face-width at least c contains H as a minor.

Brunet, Mohar, and Richter

Large face-width >c => various collections of cycles of ord. $\Omega(c)$...

de Graaf and Schrijver [9]

Let G be a graph embedded in the torus with face-width $fw(G) = r \geq 5$. Then G contains the toroidal $\lfloor 2r/3 \rfloor \times \lfloor 2r/3 \rfloor$ -grid as a minor.

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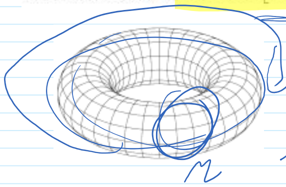
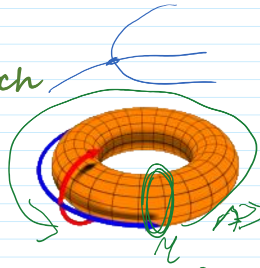
- best possible!



$\leadsto \Omega(n \times n)$

New definitions – stretch

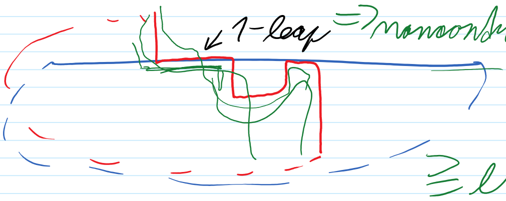
- One-crossing position (of two loops)
  --> one-leap (cf. crossing numb.)



$n \cdot n >> n^2$

Definition 2.7 Stretch

Let G be a graph embedded in an orientable surface Σ. The *stretch* $Str(G)$ of G is the minimum value of $\|A\| \cdot \|B\|$ over all pairs of cycles $A, B \subseteq G$ that are in a one-leap position in Σ.

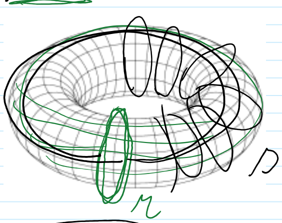


1-leap => monoonlu!

$\geq ew^2$

Highlights of stretch

- efficiently computable (in P) [S. Cabello, M. Chimani, PH, 2014] ✓
- two-dimensional analogue of edge-width ✓
- relation of dual stretch to toroidal grids... (on Σ_1)



Definition 1.3 Toroidal expanse

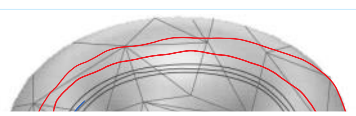
The *toroidal expanse* of a graph G, denoted by $Tex(G)$, is the larges value of $p \cdot q$ over all integers $p, q \geq 3$ such that G contains a toroidal $p \times q$-grid as a minor.

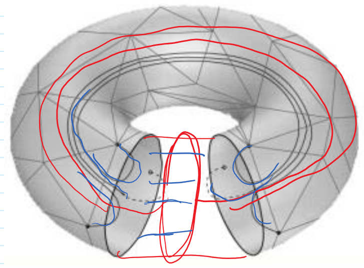
≥ 3

Highlights of stretch cont.

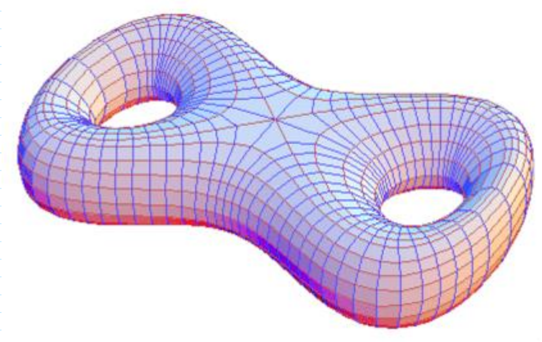
- relation of dual stretch to the crossing number... (on Σ_1)

$cr(G) \leq Str(G^*)$





M^*D = stretch



New results

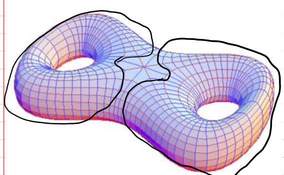
Theorem 1.4 Main theorem

Let Σ be an orientable surface of fixed genus $g > 0$, and let Δ be an integer. There exist constants $r_0, c_0, c_1, c_2 > 0$, depending only on g and Δ, such that the following two claims hold for any graph G of maximum degree Δ embedded in Σ:

(a) If G is embedded in Σ with face-width at least r_0, then

$$c_0 \cdot cr(G) \leq Tex(G) \leq c_1 \cdot cr(G).$$

(b) There is a polynomial time algorithm that outputs a drawing of G in the plane with at most $c_2 \cdot cr(G)$ crossings.



- r_0 is exponential in g (of order 2^g) and linear in Δ,
- $1/c_0$ is quadratic in Δ and exponential in g (of order 8^g),
- c_1 is independent of g, Δ, and
- c_2 is quartic in Δ and exponential in g (of order 16^g).

$$c_0 \cdot cr(G) \leq Tex(G) \leq c_1 \cdot cr(G).$$

HARD,
with important
role of (dual) stretch

EASY
$cr(G) \geq \frac{1}{72} \cdot Tex(G)$

$7(g-2)$

Breakdown of the proof of Theorem 1.4

"planarization cost"

$$Str(G^*) \preceq_{g,\Delta} Tex(G) \preceq_1 cr(G) \preceq_g Pcost(G^*) \preceq_g Str(G_1^*)$$

I II III

plus...

$$Str(G_1^*) \preceq_{g,\Delta} Tex(G_1) \leq Tex(G)$$

I "$Str(G^*) \preceq_{g,\Delta} Tex(G)$"

Theorem 3.1

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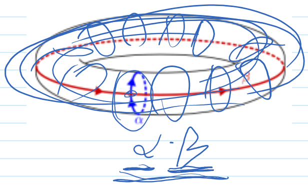
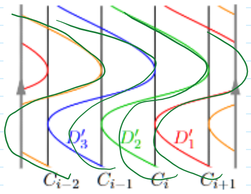
Let G be a graph embedded in the torus. Suppose that G contains a collection $\{C_1, \ldots, C_p\}$ of

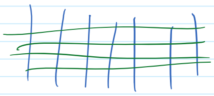
I “$Str(G^*) \preceq_{g,\Delta} Tex(G)$”

Theorem 3.1

Let G be a graph embedded in the torus. Suppose that G contains a collection $\{C_1, \ldots, C_p\}$ of $p \geq 3$ pairwise disjoint, pairwise homotopic cycles, and a collection $\{D_1, \ldots, D_q\}$ of $q \geq 3$ pairwise disjoint, pairwise homotopic cycles. Further suppose that the pair (C_1, D_1) is a basis. Then G contains a $p \times q$-toroidal grid as a minor.







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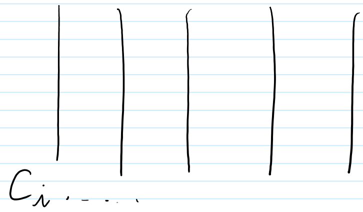
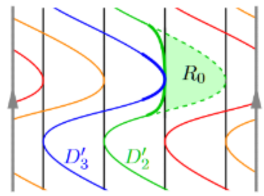




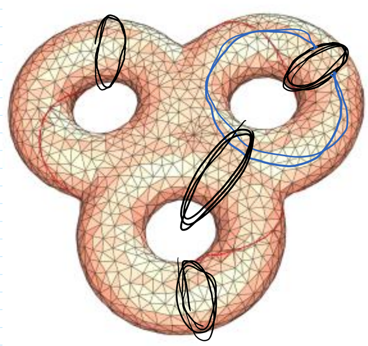
II “$cr(G) \preceq_g Pcost(G^*)$ ($\preceq_g Str(G_1^*)$)”

Theorem 3.6

Let G be a graph embedded in Σ_g. Let $(G_1^*, \gamma_1), \ldots, (G_g^*, \gamma_g)$ be any good planarizing sequence for the topological dual G^* with associated lengths $\{k_i, \ell_i\}_{i=1,\ldots,g}$ (Definition
Then

"planarization cost"

$$cr(G) \leq 3 \cdot (2^{g+1} - 2 - g) \cdot Pcost(G^*) = 3 \cdot (2^{g+1} - 2 - g) \cdot \max\{k_i \cdot \ell_i\}_{i=1,2,\ldots,g}.$$



TODO

III “($cr(G) \preceq_g$) $Pcost(G^*) \preceq_g Str(G_1^*)$”

Lemma 3.7

Let H be a graph embedded in the surface Σ_g. Let $k := ewn^*(H)$ and assume $k \geq 2^g$. Let ℓ be the largest integer such that there is a cycle γ of length k in H^* whose shortest γ-switching ear has length ℓ. Then there exists an integer g', $0 < g' \leq g$, and a subgraph H' of H embedded in $\Sigma_{g'}$ such that

$$ewn^*(H') \geq 2^{g'-g} k \qquad \text{and} \qquad \underbrace{Str^*(H')}_{= G_1} \geq 2^{2g'-2g} \cdot k\ell.$$

Lemma 3.7

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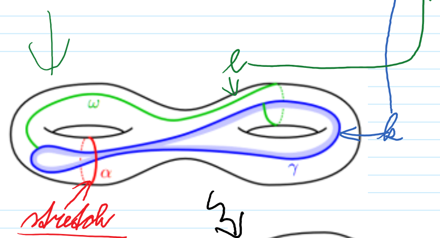

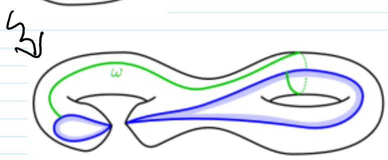

$$Pcost(G^*) \preceq_g Str(G_1^*) \preceq_{g,\Delta} Tex(G_1) \leq Tex(G)$$

Lemma 3.8

Let G be a graph embedded in Σ_g. Let $(G_i^*, \gamma_i)_{i=1,\ldots,g}$ be a good planarizing sequence of G^*, with associated lengths $\{k_i, \ell_i\}_{i=1,\ldots,g}$. Suppose that $ewn^*(G) \geq 5 \cdot 2^{g-1} \lfloor \Delta(G)/2 \rfloor$. There exists g', $0 < g' \leq g$, and a subgraph H' of G embedded in $\Sigma_{g'}$ such that $ewn^*(H') \geq 5 \cdot 2^{g'-1} \lfloor \Delta(G)/2 \rfloor$ and

$$Tex(G) \geq \tfrac{1}{7} 2^{3-2g'} \lfloor \Delta(G)/2 \rfloor^{-2} \cdot Str^*(H') \geq \tfrac{1}{7} 2^{3-2g} \lfloor \Delta(G)/2 \rfloor^{-2} \cdot \max\{k_i \cdot \ell_i\}$$

G embedded in $\Sigma_{g'}$ such that $ewn^*(H') \geq 5 \cdot 2^{g'-1}\lfloor \Delta(G)/2 \rfloor$ and

$$Tex(G) \geq \frac{1}{7} 2^{3-2g'} \lfloor \Delta(G)/2 \rfloor^{-2} \cdot Str^*(\underset{\substack{\| \\ G_1}}{H'}) \geq \frac{1}{7} 2^{3-2g} \lfloor \Delta(G)/2 \rfloor^{-2} \cdot \max\{k_i \cdot \ell_i\}$$

Theorem 3.10

Let $g > 0$ and Δ be integer constants. There are constant $c_0', c_1' > 0$ and $c_0'', c_1'' > 0$, depending on g and Δ, such that the following holds for any graph G of maximum degree Δ embedded in Σ_g with nonseparating dual edge-width at least $5 \cdot 2^{g-1}\lfloor \Delta/2 \rfloor$: There exists g', $0 < g' \leq g$, and a subgraph H' of G embedded in $\Sigma_{g'}$ such that $ewn^*(H') \geq 5 \cdot 2^{g'-1}\lfloor \Delta/2 \rfloor$ and

$$c_0' \cdot cr(G) \leq Str^*(H') \leq c_1' \cdot cr(G). \qquad H' = G_1 \tag{6}$$

Consequently,

$$c_0'' \cdot Tex(G) \leq Str^*(H') \leq c_1'' \cdot Tex(G). \tag{7}$$

IV Algorithmic part (simplified)

Weaker version of Theorem 1.4(b)

Let $g > 0$ and Δ be integer constants. Assume G is a graph of maximum degree Δ embeddable in the surface Σ_g with $ewn^*(G) \geq 5 \cdot 2^{g-1}\lfloor \Delta/2 \rfloor$. There is an algorithm that, in time $\mathcal{O}(n \log\log n)$ where $n = |V(G)|$, outputs a drawing of G in the plane with at most $c_2' \cdot cr(G)$ crossings, where $c_2' > 0$ depends only on g and Δ.

BREAK?