Limits of sequences of Latin squares

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Limit theories of discrete structures

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Graphs: Borgs-Chayes-Lovász-Sós-Szegedy-Vesztergombi (2007+)
Permutations: Hoppen-Kohayakawa-Moreira-Ráth-Sampaio (2013)

Limits of permutations

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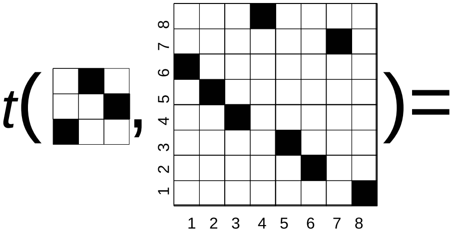
$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 4 & 8 & 3 & 2 & 7 & 1 \end{pmatrix}$$



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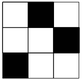
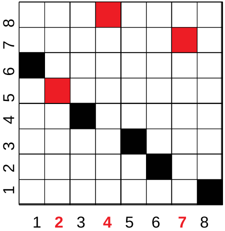
$t($  , 

$) = \frac{3}{\binom{8}{3}}$

Limits of permutations

A permuton P is a probability measure on $[0,1]^2$ that has uniform marginals, i.e. $P(A \times [0,1]) = P([0,1] \times A) = \lambda(A)$, where λ is the Lebesgue measure.

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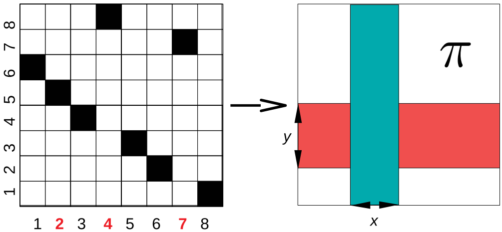

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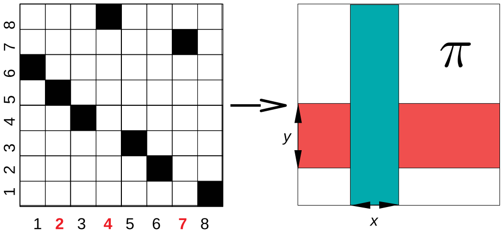

For this theory a compactness result, a cut-norm, and counting lemmas were developed by Hoppen, Kohayakawa, Moreira, Ráth and Sampaio ('13, JCTB).

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One can think of a Latin square as a **2-dimensional permutation**.

Densities in Latin squares

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$$L[r_i, c_j] \leq L[r_{i'}, c_{j'}] \text{ iff } A_{i,j} \leq A_{i',j'} \quad \forall i, i' \in [k], j, j' \in [\ell] \, .$$

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Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad L = \begin{matrix} 1 & 5 & 3 & 6 & 4 & 2 \\ 4 & 3 & 6 & 1 & 2 & 5 \\ 3 & 4 & 5 & 2 & 1 & 6 \\ 6 & 2 & 4 & 3 & 5 & 1 \\ 2 & 6 & 1 & 5 & 3 & 4 \\ 5 & 1 & 2 & 4 & 6 & 3 \end{matrix}$$

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$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad L = \begin{matrix} \color{red}{1} & 5 & \color{red}{3} & 6 & 4 & 2 \\ \color{red}{4} & 3 & \color{red}{6} & 1 & 2 & 5 \\ 3 & 4 & 5 & 2 & 1 & 6 \\ 6 & 2 & 4 & 3 & 5 & 1 \\ 2 & 6 & 1 & 5 & 3 & 4 \\ 5 & 1 & 2 & 4 & 6 & 3 \end{matrix}$$

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Limit objects - Motivational examples

Standard Cyclic Example

$$L_n(i,j) := i + j \mod n$$

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0	1	2	3
1	2	3	0
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0	1	2	3	4
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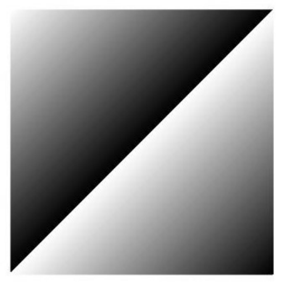

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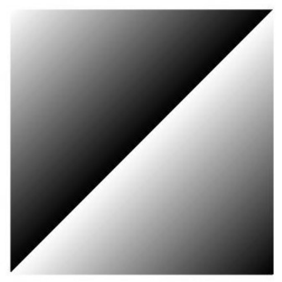

So should the limit object be a function

$$L : [0,1]^2 \to [0,1] \ ?$$

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Probabilistic example

$$P_n(i,j) := \begin{cases} i+j \mod n & \text{with probability } 1/2, \\ -i-j \mod n & \text{with probability } 1/2. \end{cases}$$

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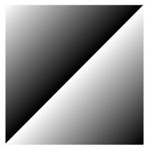

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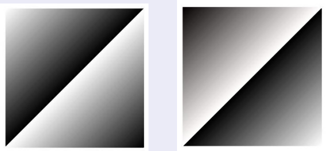

$\mathcal{P}$ is the space of probability distributions on $[0,1]$ and

$$L : [0,1]^2 \to \mathcal{P} \ ?$$

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Odd-even example

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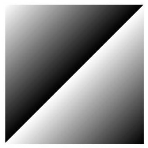

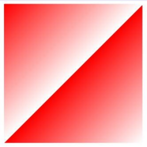

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Equivalently: (ν_W, f), where f is as above and ν_W is a probability measure on $\Omega^2 \times [0, 1]$ with uniform marginals related to the above definition via

$$\nu_W(S \times T \times V) = \int_S \int_T W(x,y)(V) dxdy \, .$$

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Limit of the cyclic example

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$\Omega = [0,1]$, f is the identity and $W : [0,1]^2 \to \mathcal{P}$ is defined by

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Limit of the probabilistic example

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$$W(x,y) := \tfrac{1}{2} Dirac(x+y \mod 1) + \tfrac{1}{2} Dirac(-x-y \mod 1)$$

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Limit of the odd-even example

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$$\Omega = [0,1] \times \{\text{odd, even}\}\ ,$$

$$f : [0,1] \times \{\text{odd, even}\} \to [0,1], (x,a) \mapsto x$$

$$W : ([0,1] \times \{\text{odd, even}\})^2 \to \mathcal{P} \text{ is defined by}$$

$$W((x,a),(y,b)) := \begin{cases} Dirac(x+y \mod 1) & \text{if } a = b, \\ Dirac(-x-y \mod 1) & \text{if } a \neq b. \end{cases}$$

Densities in Latinons

For a Latinon $L = (W, f)$, define

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Sampling: Select $(x_1, \ldots, x_k) \in \Omega^k$ and $(y_1, \ldots, y_\ell) \in \Omega^\ell$ with

$$f(x_1) < f(x_2) < \cdots < f(x_k) \text{ and } f(y_1) < f(y_2) < \cdots < f(y_\ell) \text{ u.a.r.}$$

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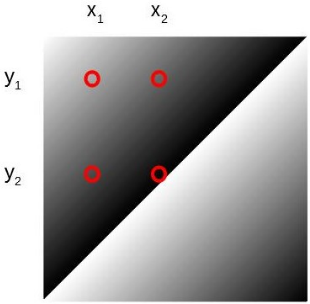

Example

For $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$,

$t(A, (W, f)) = \frac{1}{6}$.

Densities in Latinons

For a matrix $A \in \mathbb{R}^{k\times\ell}$ we set

$$\mathcal{R}^A([0,1]) := \{M \in [0,1]^{k\times\ell} \mid M \equiv A\} .$$

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Let (W, f) be a Latinon and $A \in \mathbb{R}^{k\times\ell}$. We denote by $t(A,(W,f))$ the *density* of the *pattern* A in (W,f) and define it to be

$$t(A,(W,f)) :=$$

$$k!\ell! \int_{\mathbf{x}\in[0,1]^k_{<_f}} \int_{\mathbf{y}\in[0,1]^\ell_{<_f}} \left(\bigotimes_{(i,j)\in[k]\times[\ell]} W(x_i, y_j) \right) (\mathcal{R}^A([0,1])) dy dx \ .$$

Limit theories of discrete structures

(F1) **Finite discrete structures** and substructures together with a notion of density $t(\cdot, \cdot)$. (E.g. homomorphism density in graphs.)

(F2) **Left-convergence**: A sequence of structures (S_n) is left-convergent if $(t(H, S_n))$ converges for every finite substructure H.

(F3) **Limit objects**: We define a space of analytic limits objects and the notion of density is extended to those limit objects. (Graphons.)

(F4) **Compactness**: Every sequence of structures contains a subsequence converging to a limit object. (Lovász-Szegedy '06.)

(F5) **Denseness**: For every limit object there exists a converging sequence of discrete structures. (W-random graphs.)

(F6) **Equivalence of local and global**: There is another 'global' metric generating the same topology as left-convergence. (Cut-distance.)

Compactness theorem

Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of Latin squares or Latinons, and let (W, f) be a Latinon. We say $L_n \to (W, f)$ if $\lim_{n\to\infty} t(A, L_n) = t(A, (W, f))$ for every $k, \ell \in \mathbb{N}$ and $k \times \ell$ pattern A.

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Compactness for Latinons (G., Hancock, Hladký, Sharifzadeh, 20^+)

Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of Latinons. There exists a subsequence $(L_{n_i})_{i\in\mathbb{N}}$ and a Latinon (W, f) such that

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Proof of compactness: preparation

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Continuous image of compact space is compact, hence $\iota^{-1}(\iota(\mathcal{L})) = \mathcal{L}$ is compact.

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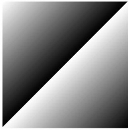
- Define compression $\iota((W,f)) := (O^f, W_{1,1}, W_{1,2}, \dots)$.

Proof method - Compressions of Latinons



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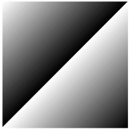

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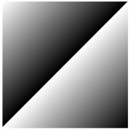

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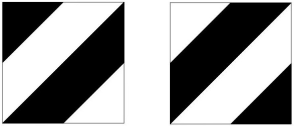

Figure: $W_{1,1}$ and $W_{1,2}$

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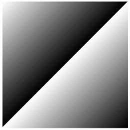

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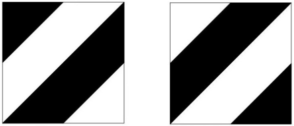

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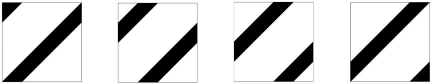

Figure: $W_{2,1}, W_{2,2}, W_{2,3}, W_{2,4}$

Cut-distance for Latinons

Cut-distance for graphons W and U

$\delta_\square(W, U) := \inf_{\varphi \in S_{[0,1]}} \|W - U^\varphi\|_\square$ where
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Cut-distance for Latinons $L_1 = (W, f)$ and $L_2 = (U, g)$

$$\delta_L(L_1, L_2) := \inf_{\varphi,\psi \in S_\Omega} \left(\left\| W - U^{\varphi,\psi} \right\|_L + \left\| O^f - O^{g\circ\varphi} \right\|_\square + \left\| O^f - O^{g\circ\psi} \right\|_\square \right)$$

where $O : \Omega^2 \to [0,1]$ is a graphon s.t. $O(x,y) := \begin{cases} 1, & x < y, \\ 0, & \text{otherwise;} \end{cases}$

$$\left\| W - U^{\varphi,\psi} \right\|_L := \sup_{\substack{R,C \subseteq \Omega, \\ V \subseteq [0,1] \text{ interval}}} \left| \int_{x\in R} \int_{y \in C} W(x,y)(V) - U(\varphi(x),\psi(y))(V) dydx \right|.$$

Motivation for the cut-distance

$$L_n(i,j) := \begin{cases} i+j \mod n & \text{if } i+j \equiv 0 \mod 2, \\ -i-j \mod n & \text{if } i+j \equiv 1 \mod 2. \end{cases}$$

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0	5	2	3	4	1
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2	3	4	1	0	5
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4	1	0	5	2	3
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Equivalence of local and global

Counting Lemma (G., Hancock, Hladký, Sharifzadeh, 20^+)

Let $k, \ell \in \mathbb{N}$. Then there exists a constant $c_{k,\ell}$ such that for every $d \in \mathbb{N}$, Latinons L_1, L_2 and $k \times \ell$ pattern A we have

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For every $\delta > 0$ there exist $k \in \mathbb{N}$ and $\varepsilon > 0$ such that for every two Latinons L_1 and L_2 with $\delta_L(L_1, L_2) > \delta$ there exists a $k \times k$ pattern A such that

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Equivalence

Convergence w.r.t. densities $t(\cdot, \cdot) \iff$ convergence w.r.t. cut-distance δ_L.

Minimality

Approximation (G., Hancock, Hladký, Sharifzadeh, 20^+)

For each Latinon (W, f) there exists a sequence $(L_n)_{n\in\mathbb{N}}$ of finite Latin squares of growing orders such that

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4. Use tools from Keevash's theory about designs to extend the approximate triangle decomposition (partial Latin square) to a triangle decomposition (complete Latin square).

Further Questions

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Thank you for listening.