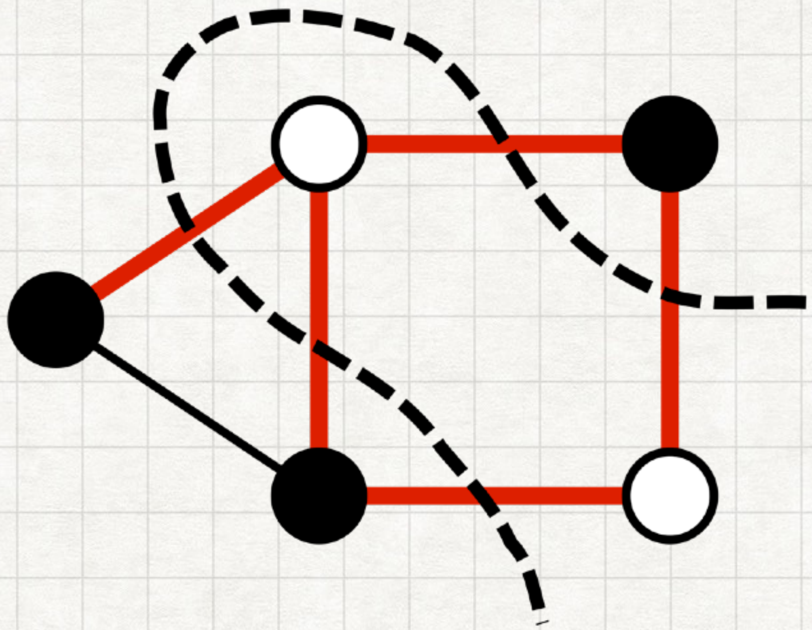

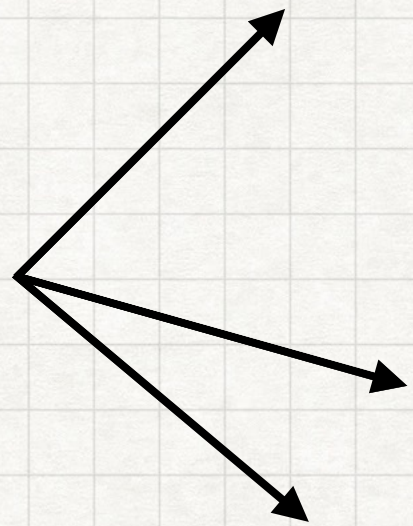

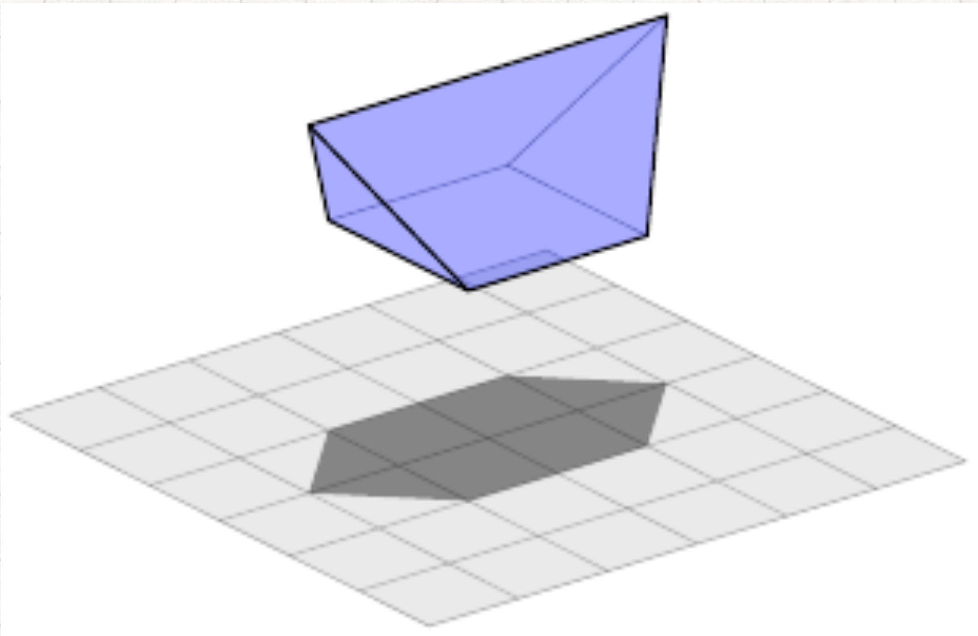

MAXCUT, ORTHONORMAL REPRESENTATIONS, AND EXTENSION COMPLEXITY OF POLYTOPES

By: Igor Balla

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Due to "optimally pseudorandom" H-free graphs which are extremal for the bipartite Turán number problem.

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So $\vartheta(\overline{G}) \geq n/\mathrm{msr}(G)$.

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Conjecture(Alon, Bollobás, Krivelevich, Sudakov 03): For any fixed H, there exists $\varepsilon > 0$, such that the answer is at least $\Omega(m^{3/4+\varepsilon})$.

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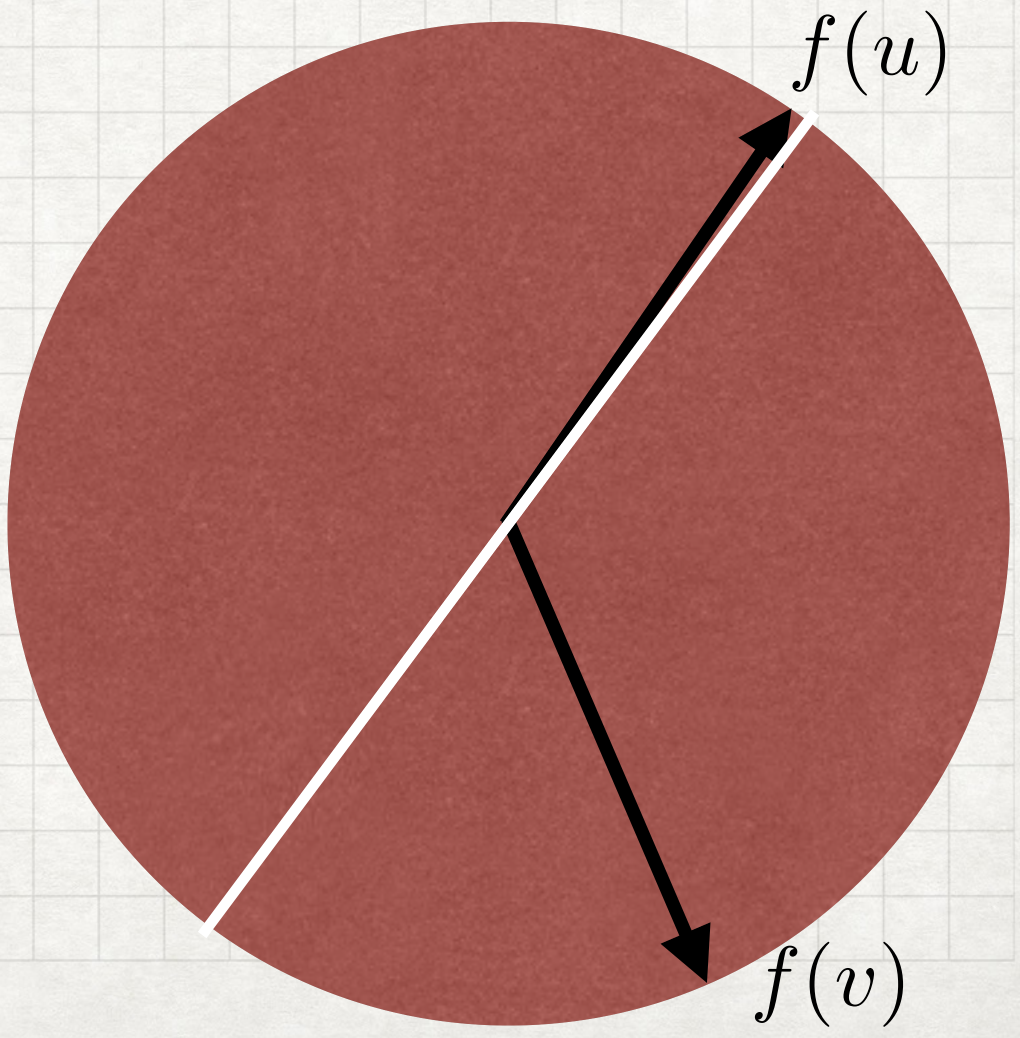

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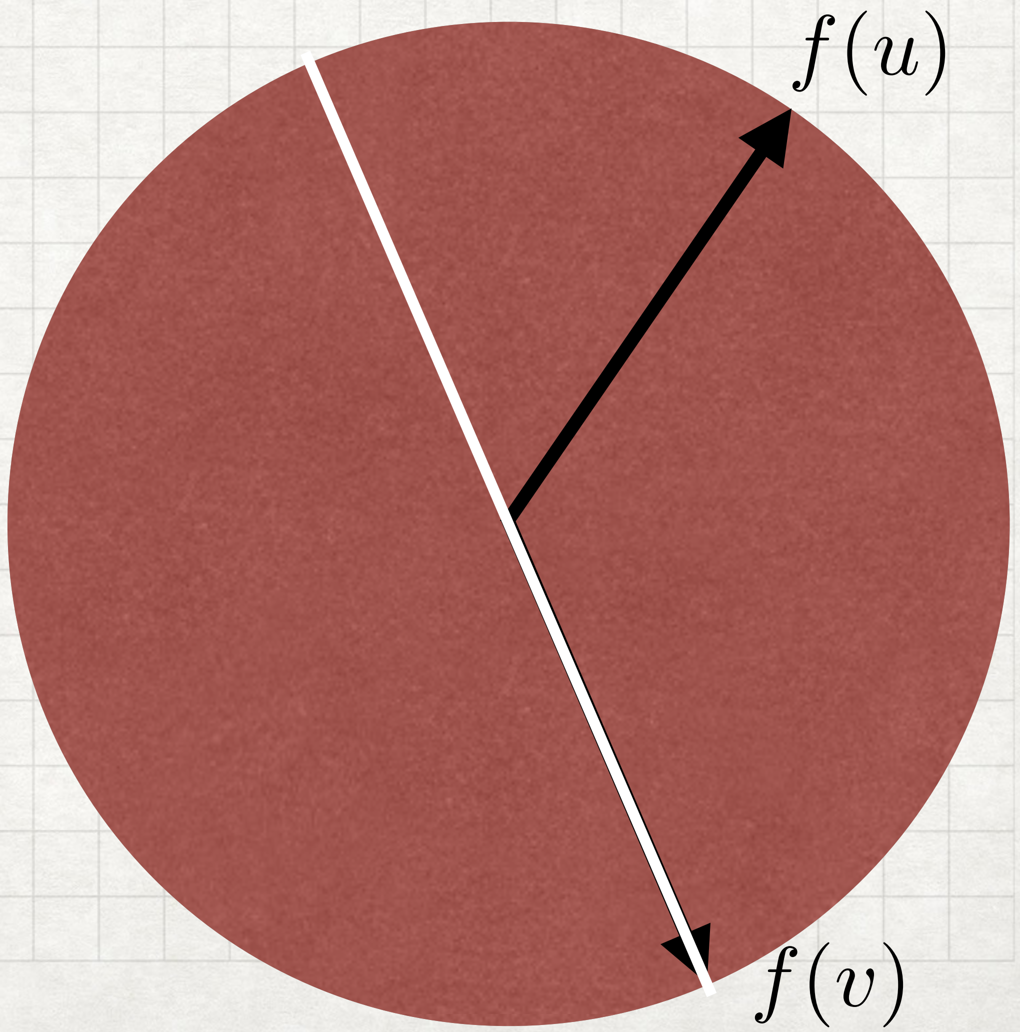

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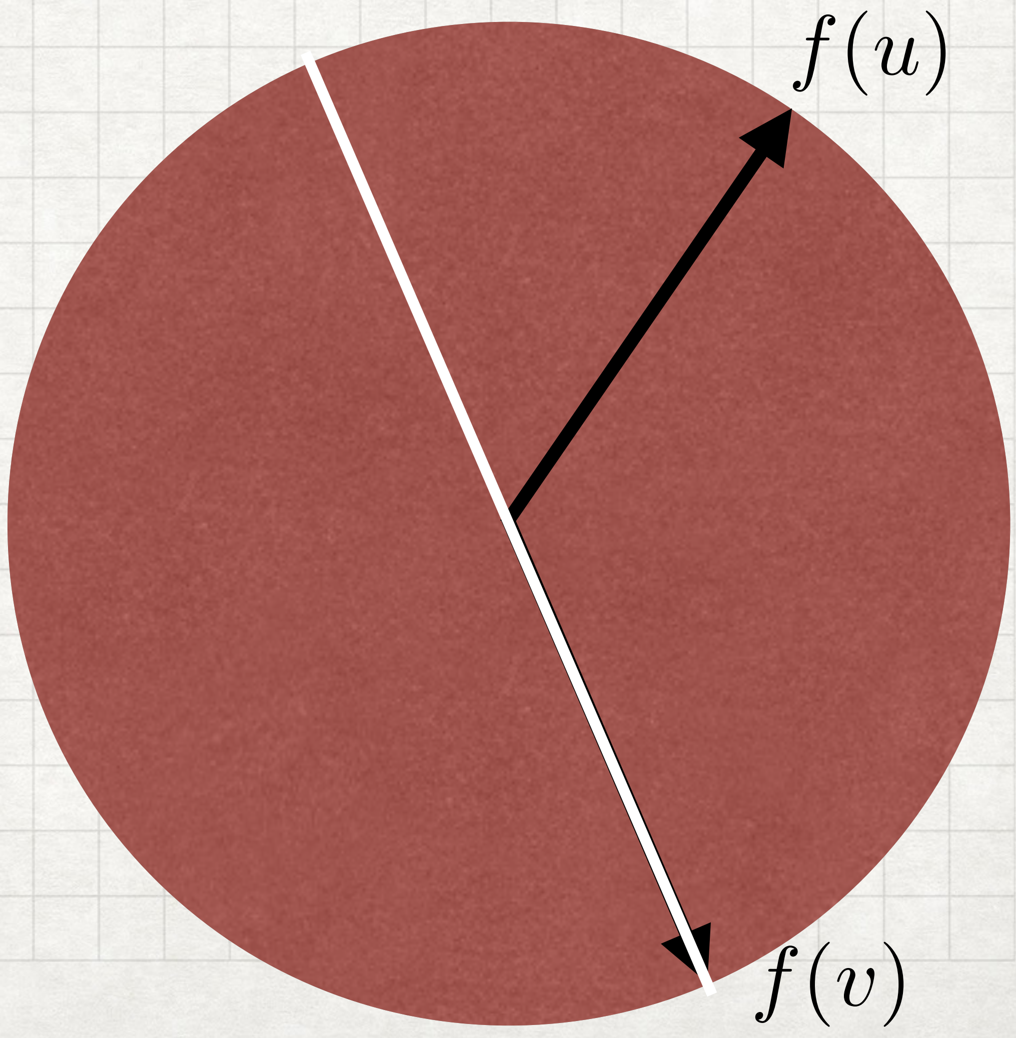

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$$\geq \frac{1}{2} + \frac{-\langle f(u), f(v)\rangle}{\pi}$$

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Conjecture(Elphick 23): $\mathrm{MaxCut}(G) - \frac{m}{2} \geq \frac{1}{3} \cdot \frac{m}{\chi_{\mathrm{vec}}(G)-1}$.

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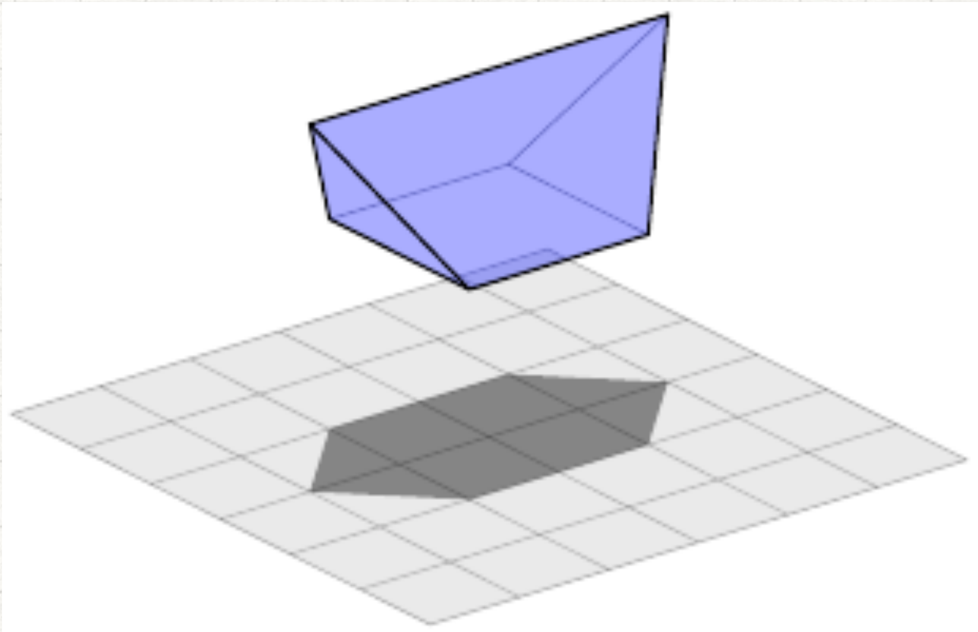

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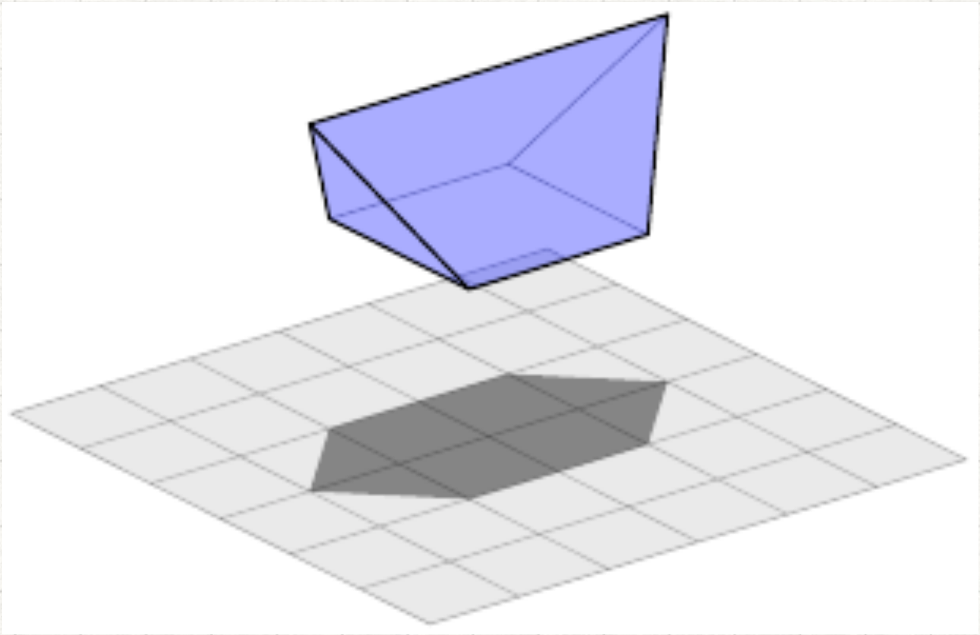
Definition: For a d-dimensional polytope P, its *extension complexity* $\mathrm{xc}(P)$, is the minimum number of facets of a polytope P' such that P is the orthogonal projection of P' onto some d-dimensional subspace.



Theorem(Kwan, Sauermann, Zhao 22): There exists an $n^{o(1)}$-dimensional polytope with at most n vertices and extension complexity at least $n^{1-o(1)}$.

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Any polytope can be described by constraints:

$$P = \{v : b_i - \langle a_i, v \rangle \geq 0, i = 1, \ldots, t\}$$

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Let $\mathcal{F}_n$ be the family of all H-free graphs on n vertices where H is the clique of size k.

$$\Omega\left(n^{3/k}\right) \leq \min\{\mathrm{msr}(G) : G \in \mathcal{F}_n\} \leq n^{O(\log\log k/\log k)}$$

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thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
dziękuję
bedankt
спасибо
go raibh maith agat
mochchakkeram
sukriya
kop khun krap
terima kasih
arigatô
dakujem
grazie
takk
taiku
sagolun
hvala
mauruuru
dank je
tapadh leat
감사합니다
ありがとう
ευχαριστώ
তোমাকে ধন্যবাদ
мерси
рахмат
Баярлалаа
хвала
дякую
спасибі
mersi
kia ora
barka
welalin
tack
spas
mahalo
faafetai lava
vinaka
blagodaram
misaotra
matondo
paldies
grazzi
kiitos
dankie
nanni
nandri
dhanyavad
bayarlalaa
gràçie
köszönöm
enkosi
akun
dankon
aċiū
asante
manana
obrigada
tenki
murakoze
chokrane
mamnun
djiere dieuf
tau
chnorakaloutioun
gratias ago
gràcies
sulpáy
sobodi
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mèsi
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xièxie
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