Entrance Examination - Mathematics

Jméno a příjmení – pište do okénka	*Číslo přihlášky*	*Číslo zadání*
		1

Sets, relations, functions, logic

---

1 For which of the following sets $A \subseteq \mathbb{Q}$ does the standard ordering $\leq$ have the least element? (Here $\mathbb{Z}$ denotes the set of all integers and $\mathbb{Q}$ the set of all rational numbers.)

A $A = \mathbb{Z}$

***B** $A = \{p \in \mathbb{Z} \mid p > 0\}$

C $A = \{p \in \mathbb{Q} \mid p > 0\}$

D $A = \{p \in \mathbb{Q} \mid p < 0\}$

E $A = \{p \in \mathbb{Z} \mid p < 0\}$

---

2 Consider two arbitrary finite sets A and B and a surjective function $f\colon A \to B$. Which of the following statements is generally valid?

A $|A| = |B|$

B $|A| > |B|$

***C** $|A| \geq |B|$

D $|A| < |B|$

E $|A| \leq |B|$

---

3 Consider the predicate logic with equality and one unary function symbol f. Which of the following formulas is logically entailed by the formula $\exists x\,(x = f(f(x)))$?

A $\exists y\,(f(y) = f(f(y)))$

B $\exists y\,(y = f(f(f(y))))$

***C** $\exists y\,(f(y) = f(f(f(y))))$

D $\exists y\,(y = f(y))$

E $\exists y\,(f(f(y)) = f(f(f(y))))$

---

4 What is the number of elements of the set $\mathcal{P}(\{1,2,3\}) \cup \mathcal{P}(\{3,4\})$? (Here $\mathcal{P}(A)$ denotes the set of all subsets of A.)

A 12
B 9
C 8
***D** 10
E 11

---

5 Which of the following propositional formulas is not satisfiable? (Here A, B denote distinct propositional variables.)

A $(A \Rightarrow B) \wedge (B \Rightarrow \neg A)$

B $(A \Leftrightarrow B) \wedge (B \Leftrightarrow A)$

C $(A \Rightarrow B) \wedge (B \Rightarrow A)$

D $(A \Rightarrow B) \wedge (A \Rightarrow \neg B)$

***E** $(A \Leftrightarrow B) \wedge (B \Leftrightarrow \neg A)$

6 Which of the following binary relations R on the set $\mathbb{Z} \times \mathbb{Z}$ of all pairs of integers is not an equivalence? (Equivalence is reflexive, symmetric, and transitive relation.)

A $((a_1, b_1), (a_2, b_2)) \in R$ iff $a_1 + b_1 = a_2 + b_2$

***B** $((a_1, b_1), (a_2, b_2)) \in R$ iff $a_1 = a_2$ or $b_1 = b_2$

C $((a_1, b_1), (a_2, b_2)) \in R$ iff $(a_1)^{10} + 3b_1 = (a_2)^{10} + 3b_2$

D $((a_1, b_1), (a_2, b_2)) \in R$ iff $a_1 = a_2$

E $((a_1, b_1), (a_2, b_2)) \in R$ iff $a_1 = a_2$ and $b_1 = b_2$

Linear algebra

7 Which of the following mappings **is not** linear?

***A** $f(x, y) = x \cdot y$

B $f(x, y) = x$

C $f(x, y) = |-6| \cdot x - y$

D $f(x, y) = 1 \cdot x + 0 \cdot y^2$

E $f(x, y) = 2^2 \cdot x + \sqrt{3} \cdot y$

8 Determine the dimension of the subspace $U \subseteq \mathbb{R}^3$ generated by vectors $\mathbf{u}_1 = (1, -1, 0), \mathbf{u}_2 = (0, 1, -1), \mathbf{u}_3 = (-1, 0, 1)$. If vectors $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ form a basis of $\mathbb{R}^3$, also compute the coefficients of a vector $\mathbf{v} = (2, 1, -3)$ in this basis.

A U has the dimension 3 and the coordinates of $\mathbf{v}$ in the basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ are $(1, 4, -5)$.

B U has the dimension 2 and the coordinates of $\mathbf{v}$ in the basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ are $(1, 2, -1)$.

C U has the dimension 3 and the coordinates of $\mathbf{v}$ in the basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ are $(3, 4, 1)$.

D U has the dimension 3 and the coordinates of $\mathbf{v}$ in the basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ are $(1, 2, -1)$.

***E** U has the dimension 2 and vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ do not form a basis of $\mathbb{R}^3$.

9 Consider the following system of equations over $\mathbb{R}$:

$$\begin{aligned} x - y + 2z &= 1, \\ 3x + 2y - 5z &= 3, \\ 7x - 2y + 3z &= 7. \end{aligned}$$

Which of the following statements is true?

A The system has infinitely many solutions and all solutions form a plane in $\mathbb{R}^3$.

***B** The system has infinitely many solutions and all solutions form a line in $\mathbb{R}^3$.

C The system has no solution.

D The system has exactly one solution.

E All points of $\mathbb{R}^3$ are solutions of the given system.

---

10 Let $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 7 & -2 \\ 1 & 5 & -5 \end{pmatrix}$.

Which of the following statements is true about the elements of the inverse matrix A^{-1} of the matrix A?

***A** All elements of A^{-1} are integers.

B All elements of A^{-1} are rational numbers and at least one element of A^{-1} is not an integer.

C All elements of A^{-1} are complex numbers and at least one element of A^{-1} is not a real number.

D The inverse matrix of A does not exist.

E All elements of A^{-1} are real numbers and at least one element of A^{-1} is not a rational number.

---

11 Compute $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -1 & 4 \end{pmatrix}$.

A $\begin{pmatrix} 7 \end{pmatrix}$

B $\begin{pmatrix} 5 \end{pmatrix}$

C None of the other answers is correct.

D The product of the matrices is not defined.

***E** $\begin{pmatrix} -2 & -1 & 4 \\ 2 & 1 & -4 \\ -4 & -2 & 8 \end{pmatrix}$

Graph theory

---

12 Consider an arbitrary non-empty binary tree, in which each vertex is either a leaf or has exactly two children. Denote as l the number of its leaves and as v the number of its vertices which are not leaves. Which of the following holds in general?

A $v^2 = l$

B $2^v = l$

C $v = l$

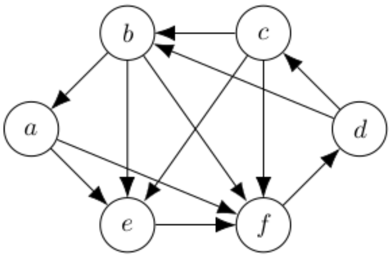
D None of the other answers.

***E** $v + 1 = l$

13 What is the number of pairwise non-isomorphic undirected graphs on 8 vertices, in which each vertex has degree 2?

A 2
B 1
***C** 3
D 4
E 5

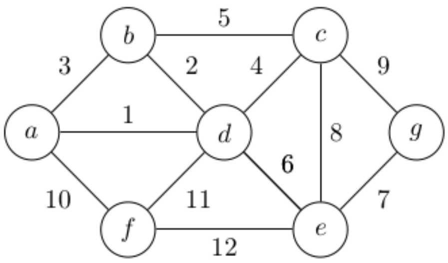
14 Consider the following directed graph:



What is the maximal possible number of vertices including the starting one, which can be discovered before the vertex d during the depth first search from the vertex a? (We do not assume any ordering on the vertices. Therefore, the order in which the vertices are discovered during the search is not uniquely determined.)

A 5
B 2
C 4
D 1
***E** 3

15 Consider the following weighted undirected graph G:



What is the weight (i.e., the sum of weights of all its edges) of the minimal spanning tree of the graph G?

A 28
***B** 30
C 34
D 35
E 32

16 Consider an arbitrary weighted directed graph G that contains at least two vertices between which there are at least two shortest paths (the length of a path is the sum of weights of all its edges). Which of the following holds in general?

A The graph G contains at least 4 distinct vertices.

B The graph G contains at least two distinct edges with the same weight.

***C** The graph G contains at least 3 distinct edges.

D The graph G contains an edge with the weight 0.

E There are at least two shortest paths between all pairs of vertices of the graph G.

Calculus

---

17 Compute the minimal value of the function
$e^x \cdot (x^2 - 5x + 5)$
on the interval $[-1, 4]$.

A $-\frac{5}{4} \cdot e^3$

B 5

***C** $-e^3$

D 3

E 0

---

18 Consider the function $f(x) = (\sin(x^2))^2$. Compute $f'\left(\sqrt{\frac{\pi}{2}}\right)$.

A $\sqrt{2\pi}$

B 2π

C 1

D $2 \cdot \sqrt{\pi}$

***E** 0

---

19 The function $f\colon \mathbb{R} \to \mathbb{R}$ given by the formula $f(x) = e^x - e^{-x}$ is

***A** odd and bijective
B odd, injective but not surjective
C even and bijective
D even, surjective but not injective
E even, injective but not surjective

---

20 Compute the limit $\lim_{n\to\infty} \dfrac{n^2}{n+(\ln n)^2}$.

***A** ∞

B The limit does not exist.

C 0

D 2

E 4

---

21 Compute $\displaystyle\int_{-\pi}^{\pi} x \cdot \cos x \, \mathrm{d}x$.

A -2π

B $2\pi - 2$

C $-2\pi + 2$

***D** 0

E 2π

Probability

22 Consider the random variable X such that $P(X=1)=p$, $P(X=2)=\frac{1}{2}$, $P(X=6)=\left(\frac{1}{2}-p\right)$ and the probability is zero for the other values. For which of the following values of p is the expected value of the random variable X equal to 3?

A $\frac{1}{3}$

B $\frac{1}{4}$

C $\frac{2}{5}$

D $\frac{1}{2}$

***E** $\frac{1}{5}$

---

23 Consider the data sample

$$7, 11, -7, 9, 5, 0, 2, 3, 0.$$

Which of the following statements about the average and a median of this sample is true?

A The average is $\frac{30}{7}$, a median is 3.

B The average is $\frac{10}{3}$, a median is 0.

***C** The average is $\frac{10}{3}$, a median is 3.

D The average is $\frac{30}{7}$, a median is 5.

E The average is $\frac{10}{3}$, a median is 5.

---

24 Consider two discrete random variables X and Y. Their joint probability mass function depends on a parameter $p \in [0,1]$ and is given as follows:

$P(X = 0, Y = 0) = p/2$
$P(X = 0, Y = 1) = (1 - p)/2$
$P(X = 1, Y = 0) = (1 - p)/2$
$P(X = 1, Y = 1) = p/2$

Determine for how many different values of the parameter $p \in [0, 1]$ are the random variables X and Y independent.

A 0, i.e., the random variables X and Y are dependent for any value of $p \in [0, 1]$.

B 3

C ∞, i.e., the random variables X and Y are independent for infinitely many values of p.

***D** 1

E 2

25 The probability that a given person wins a prize in a lottery is 50 %. What is the probability that exactly two people from the family of four win a prize?

A $\frac{5}{8}$

B $\frac{1}{4}$

C $\frac{1}{2}$

***D** $\frac{3}{8}$

E $\frac{1}{8}$