Entrance Examination - Mathematics

Name and Surname – fill in the field	Application No.	Test Sheet No.
		4

Sets, relations, functions, logic

1 Consider an arbitrary relation R that is a partial order on a set A. Suppose that the ordered set (A, R) has exactly two maximal elements. Which of the following statements about the ordered set (A, R) is in general valid?

A The ordered set (A, R) does not have any least element.

***B** The ordered set (A, R) does not have any greatest element.

C The ordered set (A, R) has one greatest element.

D The ordered set (A, R) has one least element.

E The ordered set (A, R) has two greatest elements.

2 Which of the following propositional formulae is a tautology? (Here A, B are distinct propositional variables.)

A $(A \Leftrightarrow B) \Rightarrow (A \wedge B)$

B $(A \Leftrightarrow B) \Leftarrow (A \vee B)$

C $(A \Leftrightarrow B) \Rightarrow A$

D $(A \Leftrightarrow B) \Rightarrow (A \vee B)$

***E** $(A \Leftrightarrow B) \Leftarrow (A \wedge B)$

3 Which of the following functions f, g on the set of rational numbers satisfy $(f \circ g \circ f^{-1})(1) = 1$? (Here $f \circ g$ denotes the composition of functions, i.e. the function $(f \circ g)(x) = f(g(x))$, and f^{-1} denotes the inverse of the function f.)

A $f(x) = x/2,\ g(x) = x - 2$

B $f(x) = x - 1,\ g(x) = 2x$

***C** $f(x) = x + 1,\ g(x) = 2x$

D $f(x) = x,\ g(x) = 0$

E $f(x) = x/2,\ g(x) = x + 2$

4 Which of the following binary relations R on the set of integers is **not** transitive?

A $R(x, y) \iff x < y$

B $R(x, y) \iff x \neq 3$

C $R(x, y) \iff x = y$

D $R(x, y) \iff x = 3$

***E** $R(x, y) \iff x \neq y$

5 Consider a first-order language with a binary predicate Z and an intepretation in which the universe is the set of all people and the relation $Z(x, y)$ is interpreted as *"the person x knows the person y"*. Which of the following first-order formulae corresponds to the statement *"every person is known by someone"*? (Note that the relation $Z(x, y)$ is not symmetric.)

***A** $\forall y\, \exists x\, Z(x, y)$

B $\exists x\, \forall y\, Z(x, y)$

C $\forall x\, \forall y\, Z(x, y)$

D $\forall x\, \exists y\, Z(x, y)$

E $\exists y\, \forall x\, Z(x, y)$

6 How many elements are in the set

$$(\{1,2,3,4\} \cup \{2,4,8\}) \setminus \{1,2,42\}?$$

(Here $A \setminus B$ denotes the set difference of sets A and B.)

A 1
B 2
***C** 3
D 5
E 4

Linear algebra

7 Compute the product $A^{-1} \cdot \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$, where $A = \begin{pmatrix} 5 & -4 & -2 \\ 2 & -1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$.

***A** $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

B $\begin{pmatrix} 28 \\ -22 \\ -10 \end{pmatrix}$

C $\begin{pmatrix} -8 & 14 & 8 \end{pmatrix}$

D $\begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$

E $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

8 Consider the following system of equations over $\mathbb{R}$:

$$\begin{aligned} x + 2y + 3z &= 4, \\ 2x - y - 7z &= 10, \\ x - 2y - 4z &= 9. \end{aligned}$$

Which of the following statements is true?

A All points of $\mathbb{R}^3$ are solutions of the given system.

B The system has infinitely many solutions and all solutions form a line in $\mathbb{R}^3$.

C The system has infinitely many solutions and all solutions form a plane in $\mathbb{R}^3$.

D The system has no solution.

***E** The system has exactly one solution.

9 Determine which of the following matrices corresponds to the linear transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$, where φ is an orthogonal projection onto the plane given by x and z axes.

***A** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

B $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

D $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

E $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

10 Consider the vector $(1, 3, -1)$ in the basis $[(1, 0, 1), (0, 1, 2), (1, 1, 1)]$. Find its coordinates in the basis $[(2, 1, 0), (2, 1, 2), (-1, 0, 1)]$.

***A** $(1, 1, 4)$

B $(-2, 2, 10)$

C $(11, 2, 5)$

D $(3, -2, 6)$

E $(-12, -4, 2)$

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11 Compute $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -2 \\ -2 & -1 & 4 \\ -1 & -1 & 3 \end{pmatrix}$.

A $\begin{pmatrix} 6 & -4 & -15 \\ 0 & 0 & 0 \end{pmatrix}$

B None of the other answers is correct.

C $\begin{pmatrix} -4 & 3 \\ 7 & -5 \\ 6 & -3 \end{pmatrix}$

***D** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

E The product of the given matrices is not defined.

Graph theory

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12 Let G be an arbitrary undirected graph with 8 vertices. What is the smallest number n such the statement *"if the graph G has at least n edges, it contains a cycle"* holds in general?

A 9
B 7
C 36
D 1
***E** 8

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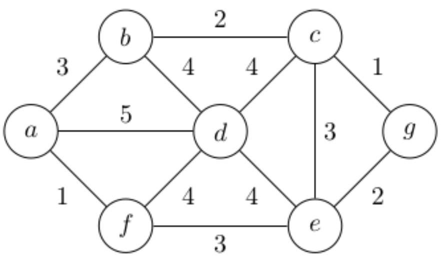
13 Consider the following weighted undirected graph G:



How many distinct minimal spanning trees of G exist?

A 4
B 3
C 2
***D** 8
E 1

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14 How many edges are there in the complete undirected graph with n vertices, i.e. the graph K_n?

A $n - 1$

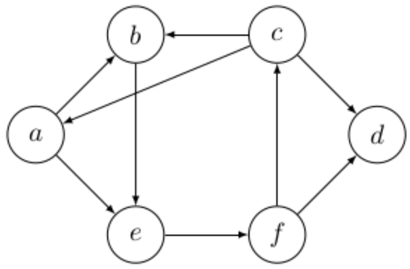
B $n \cdot (n - 1)$

***C** $\frac{n \cdot (n-1)}{2}$

D n^2

E n

15 Consider the following directed graph:



Choose from the following statements the one that is in general valid about the depth-first search of this graph starting in the vertex a. (We do not assume any particular ordering of vertices. The order in which the vertices are discovered during the depth-first search is therefore not uniquely determined.)

A The vertex f can be discovered as the last one.

B The vertex d must be discovered as the last one.

C The vertex b must be discovered before the vertex c.

***D** The vertex f must be discovered before the vertex d.

E The vertex c can be discovered before the vertex e.

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16 Consider the undirected cycle graph with 4 vertices, i.e. the graph C_4. How many pairwise non-isomorphic subgraphs with 4 vertices of C_4 exist?

A 8
B 7
C 5
D 4
***E** 6

Calculus

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17 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function, $a, b \in \mathbb{R}$, $a < b$. Consider the following statements D, I, S:

- D: f has a finite first derivative on $[a, b]$,
- I: the integral $\int_a^b f(x)\,\mathrm{d}x$ exists and it is finite,
- S: the function f is continuous on $[a, b]$.

Which of the following pair of implications is generally valid?

A D $\Rightarrow$ I and I $\Rightarrow$ S

B S $\Rightarrow$ D and D $\Rightarrow$ I

C S $\Rightarrow$ I and I $\Rightarrow$ D

D I $\Rightarrow$ D and D $\Rightarrow$ S

***E** D $\Rightarrow$ S and S $\Rightarrow$ I

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18 The function $f\colon \mathbb{R} \to \mathbb{R}$ given by the formula
$$f(x) = \begin{cases} \ln x & x \geq 1, \\ x - 1 & x \leq 1 \end{cases}$$
is:

A surjective, but not injective

B even or odd

***C** bijective

D injective, but not surjective

E incorrectly defined for $x = 1$

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19 Compute the limit $\lim_{n\to\infty} \dfrac{n}{(\ln n)^3}$.

A 1

B The limit does not exist.

C $\dfrac{1}{6}$

D 0

***E** ∞

20 Consider the function $f(x) = \ln(\cos x)$. Compute $f'(\pi/6)$.

A $\frac{2}{\sqrt{3}}$

B $-\frac{\pi}{12}$

C $\frac{\sqrt{3}}{3}$

***D** $-\frac{\sqrt{3}}{3}$

E $\frac{\pi}{12}$

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21 Compute the integral $\int_1^2 \left(\frac{1}{x^2} + x^3\right) \mathrm{d}x$.

A $\frac{11}{2}$

B $\frac{27}{4}$

***C** $\frac{17}{4}$

D 6

E $\frac{13}{4}$

Probability

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22 Let us have two random events A and B. When are these two events stochastically independent?

A If and only if $P(A \cup B) = 1$.

***B** If and only if $P(A) \cdot P(B) = P(A \cap B)$.

C If and only if $P(A \cap B) = 0$.

D If and only if $P(A) \cdot P(B) = 0$.

E If and only if $P(A) \neq P(B)$.

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23 Consider a random variable X with the range -1 and 1 and the expected value $\frac{1}{2}$. Compute the **variance** of the random variable X.

A The variance cannot be determined unambiguously from the given values.

***B** $\frac{3}{4}$

C $-\frac{1}{2}$

D $\frac{1}{4}$

E $\frac{1}{2}$

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24 A group of 30 athletes arrived at a tournament. How many possibilities are there to divide the group to 3 teams of 10 members each?

A $\frac{30!}{(10!)^3}$

B $30!$

C $\frac{30!}{3!}$

***D** $\frac{30!}{3! \cdot (10!)^3}$

E None of the other answers is correct.

25 Consider a standard six-sided dice for which each result of the toss has the same probability. Further, consider the following game: we toss the dice once; if the result is 5 or 6, the game ends, otherwise we toss again and then the game ends. What is the probability that some toss during the game will come out with value 5 or 6?

A $\frac{4}{9}$

B $\frac{2}{3}$

***C** $\frac{5}{9}$

D $\frac{1}{3}$

E $\frac{1}{2}$