Entrance Examination - Mathematics

Name and Surname – fill in the field	*Application No.*	*Test Sheet No.*
		6

Sets, relations, functions, logic

1 Consider a relation R on integers such that $(a, b) \in R$ if and only if there is an integer k such that $a - b = 3k$ (i.e. $a - b$ is divisible by 3). The relation is:

A order
B reflexive, but it is neither symmetric nor transitive
C reflexive and transitive, but it is not symmetric
***D** equivalence
E reflexive and symmetric, but it is not transitive

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2 Consider the formula $\forall x \forall y (x * y = z \Rightarrow (x = z \vee y = z))$. Assume that x, y, z are variables interpreted as natural numbers (including zero). The symbol $*$ is interpreted as a standard multiplication of natural numbers. Choose the correct statement.

***A** The formula is true for infinitely many values of z and is also false for infinitely many values of z.
B The formula is true for more than two values of z, but there are only finitely many of such values.
C The formula is false for any value of z.
D The formula is true for exactly two values of z.
E The formula is true for all values of z.

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3 Consider the 2-element set $M = \{a, b\}$. How many distinct injective functions from M to M are there? How many of them are bijective? (By the term function we mean a total function.)

A 4 injective, 2 bijective
***B** 2 injective, 2 bijective
C 1 injective, 1 bijective
D 2 injective, 1 bijective
E 1 injective, none of them bijective

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4 Consider two 2-element sets $A = \{a, b\}$ and $B = \{b, c\}$. How many elements are there in the set $\mathcal{P}(A) \cap \mathcal{P}(B)$? (For an arbitrary set X we denote by $\mathcal{P}(X)$ the set of all subsets of set X.)

A 1
B 4
C 3
***D** 2
E 0

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5 Let $A = \{a, b, c\}, B = \{b, c, d\}$ and $C = \{a, c, d\}$. Which of the following sets is **nonempty**? (By $X \setminus Y$ we denote the set difference of the sets X and Y.)

A $A \setminus (C \cup B)$
B $(A \setminus B) \setminus C$
C $(C \setminus A) \cap (C \setminus B)$
D $(B \setminus A) \setminus C$
***E** $A \setminus (C \setminus B)$

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6 Let A be an arbitrary set, $\preceq$ an arbitrary (partial) order on A and a, b, c arbitrary (not necessarily distinct) elements of A. Which of the following statements is **not** true in general?

A $(a \preceq b \ \wedge \ b \preceq a) \ \Rightarrow \ (a = b)$
B $(a \neq b) \ \Rightarrow \ \neg(a \preceq b \ \wedge \ b \preceq a)$
***C** $(\neg(a \preceq b)) \ \Rightarrow \ (b \preceq a)$
D $a \preceq a \ \wedge \ b \preceq b \ \wedge \ c \preceq c$
E $(a \preceq b \ \wedge \ b \preceq c) \ \Rightarrow \ (a \preceq c)$

Linear algebra

7 Which of the following triples of vectors is linearly independent?

A $(0, 1, 1), (1, 0, 2), (1, 2, 4)$
***B** $(1, 2, 3), (3, 1, 2), (2, 3, 1)$
C $(2, 4, 10), (1, 2, 5), (1, 1, 1)$
D $(3, 3, 4), (2, 2, 1), (3, 3, 3)$
E $(0, 3, 0), (1, 0, 1), (2, 2, 2)$

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8 Calculate the determinant of the following matrix:

$$\begin{pmatrix} 2 & 3 & 2 \\ 5 & 4 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

A 20
B 12
C 19
***D** 16
E -24

9 Which of the following matrices determines a mapping A from $\mathbb{R}^2$ to $\mathbb{R}^2$ which maps vector $(6, 8)$ to vector $(6, 4)$? Consider multiplication by matrix from the left.

***A** $\begin{pmatrix} 0 & \frac{3}{4} \\ \frac{2}{3} & 0 \end{pmatrix}$

B The matrix does not exist.

C $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

D $\begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

E $\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$

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10 Consider the following system of linear equations over $\mathbb{R}$:

$$\begin{aligned} 2x + 3y + 2z &= 3 \\ x - 5y - 3z &= -3 \\ -13y - 8z &= -1 \end{aligned}$$

Which of the following claims holds?

A The system has exactly one solution.

B The system has infinitely many solutions and the set of all solutions is a plane in $\mathbb{R}^3$.

C Every point of $\mathbb{R}^3$ is a solution of the system.

D The system has infinitely many solutions and the set of all solutions is a line in $\mathbb{R}^3$.

***E** The system has no solution.

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11 Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$ be the inverse of A (i.e. $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$). Which of the following is equal to $b_{2,2}$?

A The value is not determined by the given information.

B 1

***C** 2

D 0

E -1

Calculus

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12 We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *even* if $\forall x \in \mathbb{R} : f(-x) = f(x)$ and that f is *odd* if $\forall x \in \mathbb{R} : f(-x) = -f(x)$. Choose the correct statement.

***A** The function $f(x) = 0$ is both even and odd.

B The function $f(x) = \cos x$ is neither even nor odd.

C The function $f(x) = \sin x$ is even.

D The function $f(x) = |x|$ is odd.

E The function $f(x) = x^2 - 3x$ is even.

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13 What is the value of the following limit?
$\lim_{x\to -1} \frac{x^3-2x-1}{x^4+2x+1}$

A ∞

B 2

C 0

***D** $-\frac{1}{2}$

E It is not possible to determine the value because the limit does not exist.

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14 Compute the following integral: $\displaystyle\int_2^4 (x^3 - 4)\mathrm{d}x$

A 44

B 64

C 48

D 56

***E** 52

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15 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that the first derivative of f at 0 is zero. Which of the following statements about f is true in general?

A Function f has a local extremum at 0.

B $\lim_{x\to 0} f(x) = 0$

***C** Tangent to the graph of the function f at 0 is parallel to the x-axis.

D The situation described above cannot occur because continuous functions always have a positive first derivative.

E f is a constant function.

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16 Consider the function $f(x) = 2x^3 e^{\cos x}$. Which of the following functions is equal to the derivative of the function f?

A $6x^2 e^{\cos x} + 2x^3 e^{\cos x} \sin x$

B $6x^2 e^{\cos x} - x^3 e^{\cos x}$

C $6x^2 e^{\cos x} - 2x^3 e^{\cos x}$

D $6x^2 e^{\cos x} + x^3 e^{\cos x} \sin x$

***E** $6x^2 e^{\cos x} - 2x^3 e^{\cos x} \sin x$

Graph theory

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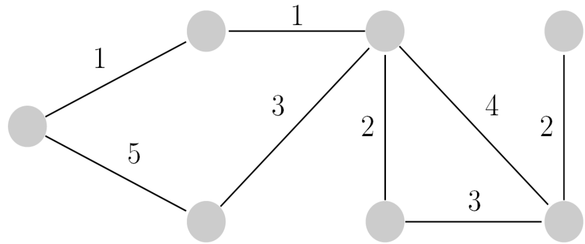
17 Consider the following undirected, edge-weighted graph:



What is the weight (i.e. the sum of edge weights) of its minimal spanning tree?

A 14

B 10

C 11

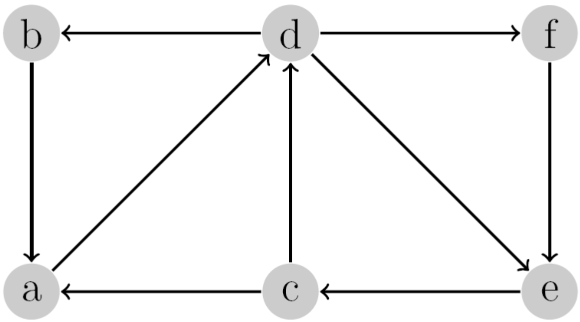
***D** 12

E 13

18 What is the minimal number of vertices in an undirected loopless graph with 8 edges?

A 6
B 8
***C** 5
D 7
E 4

19 Consider the following directed graph:



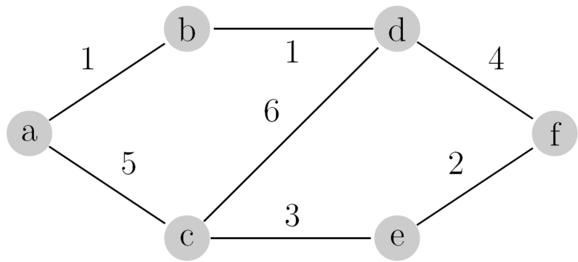
Decide which of the following claims about **breadth-first search** starting from vertex a holds. (We do not assume any ordering on the out-neighbors, i.e. the order in which breadth-first search algorithm visits the neighbors of a vertex is ambiguous.)

A Vertex b will always be visited before vertex e.
B Vertex b will always be the last visited vertex.
C Vertex f will always be visited before vertex e.
D Vertex c will always be visited before vertex e.
***E** Vertex c will always be the last visited vertex.

20 Let G be a connected graph and K its spanning tree. Which of the following claims hold?
(In the following $V(G)$ and $V(K)$ denote the vertex set of graph G and its spanning tree K. Similarly, $E(G)$ and $E(K)$ denote the edge set of G and K.)

A $|E(K)| < |E(G)|$
B $|V(K)| < |V(G)|$
C $|V(G)| < |V(K)|$
D $|V(G)| \leq |E(K)|$
***E** $|E(K)| < |V(G)|$

21 For a graph G and a pair of its vertices u and v we denote by $\delta(u,v)$ the length of the shortest path (with respect to the sum its edge weights) from vertex u to vertex v. The *diameter* of graph G is the number $\max_{u,v\in V(G)} \delta(u,v)$, where $V(G)$ denotes the vertex set of G. What is the diameter of the following graph?



A 6
***B** 8
C 7
D 9
E 5

Probability

22 Students took a test. There were 32 questions in the test, each one with 4 options. In each question, exactly one of the options was correct. For the correct answer a student received 2 points, for incorrect answer they received -1 point. What is the expected amount of points gained by a student who guessed all of the answers randomly?

A 8
***B** -8
C -0.75
D 16
E 0

23 There was an all-play-all tournament and n teams took a part. How many games were played?

A 2^n
***B** $\frac{n(n-1)}{2}$
C n^2
D $n(n+1)$
E $2n$

24 What is the coefficient of the term x^5y^3 in the binomial expansion of $(x+y)^8$?

***A** 56
B 70
C 28
D 8
E 15

25 There are 7 red and 3 blue balls in a box. Two balls are drawn randomly in succession. Before drawing the second ball we return the first ball back into the box. What is the probability that we will draw one ball of each colour?

***A** 42 %
B 58 %
C 50 %
D 29 %
E 21 %