Entrance Examination - Mathematics

Name and Surname – fill in the field	*Application No.*	*Test Sheet No.*
		1

Calculus

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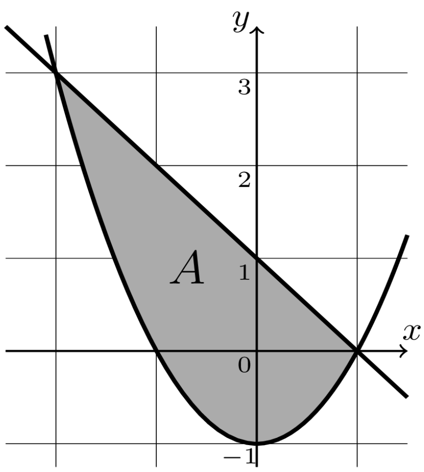
1 Compute the area of a flat shape A which consists of all points (x, y) satisfying the following inequalities:

$$\begin{aligned} y &\geq x^2 - 1 \\ y &\leq 1 - x \end{aligned}$$

(See also the figure.)



***A** $\frac{9}{2}$

B $\frac{11}{2}$

C $\frac{17}{4}$

D 5

E $\frac{19}{4}$

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2 $\lim_{x\to\infty} 3 \cdot e^{-x} =$

A $-\infty$
B 3
***C** 0
D $3 \cdot e$
E ∞

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3 In the following text, replace the terms A, B, C with suitable statements in such a way that the resulting sentence is a correct definition of a limit:

We say that L is the limit of a function f at x_0 if the following holds: .....A..... $\varepsilon > 0$ .....B..... $\delta > 0$ .....C..... every x satisfying $0 < |x - x_0| < \delta$ we have $|f(x) - L| \leq \varepsilon$.

A A: "there exist", B: "and" , C: "such that for"
***B** A: "for every", B: "there exists", C: "such that for"
C A: "for every", B: "it is not true that there exists", C: "such that for"
D A: "for every", B: "and every", C: "and"
E A: "there exists", B: "such that for every", C: "and"

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4 Consider the function

$$f(x) = x^2 + \frac{1}{x}$$

whose domain is $\mathbb{R} \setminus \{0\}$.
Find the set over which the function f is **non-decreasing**.

A $[-1, 0) \cup (0, 1]$

B $(-\infty, -1] \cup [1, \infty)$

C $\emptyset$, i.e. f is strictly increasing over the whole domain

D $(-\infty, -2^{-\frac{1}{3}}]$

***E** $[2^{-\frac{1}{3}}, \infty)$

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5 Consider the function

$$f(x) = \sin(e^x).$$

The derivative of f is:

***A** $\cos(e^x) \cdot e^x$
B $\cos(e^x \cdot e^x)$
C $\cos(e^x)$
D $\sin(e^x) \cdot e^x$
E $\sin(\cos(e^x))$

Sets, relations, functions, logic

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6 Which of the following propositional formulas is a **tautology**? (Capital letters represent propositional variables.)

A $(A \vee B) \rightarrow (A \rightarrow B)$
B $A \rightarrow \neg A$
C $(A \rightarrow B) \rightarrow (A \wedge B)$
***D** $(A \wedge B) \rightarrow (A \rightarrow B)$
E $(A \rightarrow B) \rightarrow (A \vee B)$

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7 Consider the sets $M = \{a, b\}$ and $N = \{a, c\}$. Compute the set $\mathcal{P}((M \times N) \cap (N \times M))$. (Here $\mathcal{P}(X)$ denotes the set of all subsets of X).

A $\{\{(a, a)\}, \{(a, b)\}, \{(b, a)\}, \{(b, b)\}\}$
B $\{\emptyset, a, (a, a)\}$
***C** $\{\emptyset, \{(a, a)\}\}$
D $\{\emptyset\}$
E $\{\emptyset, \{a\}\}$

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8 Which of the following predicate formulas is semantically equivalent to the formula

$\neg\exists x \left((\forall y\, P(y, x)) \wedge (\exists z\, P(x, z))\right)$?

(Here P is a binary predicate and x, y, z are variables.)

***A** $\forall x \left((\exists y\, \neg P(y, x)) \vee (\forall z\, \neg P(x, z))\right)$
B $(\exists x \exists y\, \neg P(y, x)) \wedge (\exists x \forall z\, \neg P(x, z))$
C $(\exists x \exists y\, \neg P(y, x)) \vee (\exists x \forall z\, \neg P(x, z))$
D $\exists x \left((\forall y\, \neg P(y, x)) \vee (\exists z\, \neg P(x, z))\right)$
E $\forall x \left((\exists y\, P(x, y)) \vee (\forall z\, P(z, x))\right)$

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9 Consider the functions F and G of the type $\mathbb{Z} \rightarrow \mathbb{Z}$ (i.e. from integers to integers), defined as follows:
$F(n) \quad = n + 1$
$G(n) \quad = -n.$
Which of the following terms is equal to -10?

A $G(F(G(-10)))$
B $G(G(10))$
***C** $F(G(F(10)))$
D $F(G(F(G(10))))$
E $G(F(F(G(-10))))$

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10 Which of the following statements about partially ordered sets is true?

A Every partially ordered set has either the greatest or the least element.
B A partially ordered set may contain multiple greatest elements.
C Every partially ordered set with a maximal element must also have the greatest element.
D Every partially ordered set has either a minimal or a maximal element.
***E** Every partially ordered set with the least element must also have a minimal element.

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11 Which of the following relations on the set $\{a, b, c\}$ **is not** transitive?

A $\{(a, b), (a, c), (b, c)\}$
B $\emptyset$ (i.e. the empty relation)
C $\{(a, b), (a, c)\}$
D $\{(a, a), (b, b), (c, c)\}$
***E** $\{(a, b), (b, c), (c, a)\}$

Probability

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12 Let us roll a die two times in succession. Compute the conditional probability of the second number rolled being greater than the first number rolled, assuming that the sum of both numbers is even.

A $\frac{15}{18}$
***B** $\frac{1}{3}$
C $\frac{1}{6}$
D $\frac{15}{36}$
E $\frac{2}{3}$

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13 Consider a random variable X such that $P(X = -1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{3}$, and $P(X = 3) = \frac{1}{6}$. Compute the **expected value** of random variable $Y = X^2$. (Here $P(X = a)$ denotes the probability of random variable X attaining the value a.)

A $\frac{16}{3}$
B $\frac{17}{18}$
C $\frac{4}{9}$
***D** $\frac{10}{3}$
E $\frac{1}{18}$

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14 Let us roll a die 20 times in succession, all rolls being independent. Compute the probability that six is rolled **exactly** 8 times.

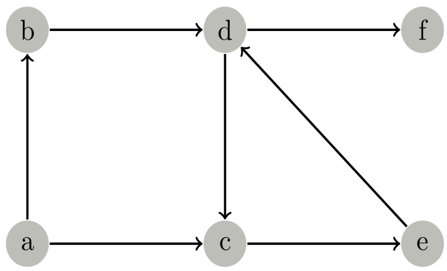
A $\binom{20}{8} \cdot (\frac{1}{6})^7$
B $\binom{20}{8} \cdot 8 \cdot \frac{1}{6} \cdot 12 \cdot \frac{5}{6}$
***C** $\binom{20}{8} \cdot (\frac{1}{6})^8 \cdot (\frac{5}{6})^{12}$
D $(\frac{1}{6})^8$
E $(\frac{1}{6})^8 \cdot (\frac{5}{6})^{12}$

15 Consider the following data sample: $\{1, 3, 3, 3, 5, 5, 9, 12, 22\}$. Denote by m its median and by a its mean. Which of the following holds?

***A** $m = 5$, $a = 7$
B $m = 3$, $a = 7$
C $m = 3$, $a = 5$
D $m = 9$, $a = 7$
E $m = 7$, $a = 5$

Graph theory

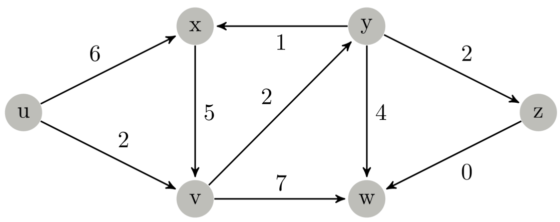
16 Consider the following directed graph:



Which of the following claims about **depth-first search** starting from vertex a is correct? (We do not assume any ordering of the vertices. Thus, the order in which the depth-first search algorithm visits the vertices is ambiguous.)

A Vertex c will always be visited before vertex d.
B Vertex b will always be visited before vertex f.
***C** Vertex e can be the last visited vertex.
D Vertex f will always be the last visited vertex.
E Vertex b will never be the last visited vertex.

17 Consider the following directed edge-weighted graph:



For any pair of its vertices s, s', let $\delta(s, s')$ denote the length (i.e. the sum of edge weights) of the shortest path from s to s'. Which of the following claims holds?

A $\delta(u, w) = 9$
***B** $\delta(u, w) = 6$
C $\delta(u, x) = 6$
D $\delta(x, x) = 8$
E $\delta(v, w) = 7$

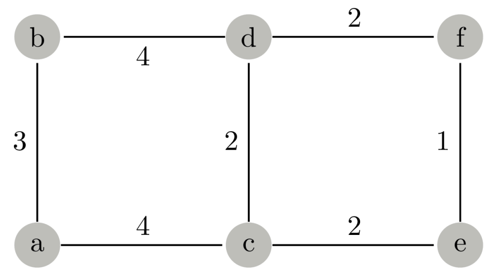
18 An undirected graph is called **complete** if it does not contain loops and there is an edge between every pair of distinct vertices. Which of the following claims about the complete graph on 7 vertices is correct?

A The graph has 42 edges.
B The graph has 28 edges.
C After removing arbitrary 7 edges the resulting graph is always disconnected.
***D** In order to obtain a disconnected graph, it suffices to remove 6 suitably chosen edges.
E In order to obtain a disconnected graph, it is necessary to remove at least 8 edges.

19 What is the least possible number of edges of a connected undirected loopless graph on 103 vertices?

***A** 102
B 205
C 206
D 104
E 103

20 Consider the following undirected edge-weighted graph:



What is the weight (i.e. the sum of weights of edges) of its arbitrary minimal spanning tree?

A 14
B 15
C 7
***D** 12
E 18

Linear algebra

21 Consider a map $\mathbb{R}^2 \to \mathbb{R}^2$ which rotates each vector 180° clockwise around point $(0,0)$. Which of the following is the matrix of this map in the standard basis? (Assume multiplication by a matrix from the left.)

A $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

B $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

C $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

D $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

***E** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

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22 Calculate the determinant of the following matrix:

$$\begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 2 \\ 2 & -3 & 0 \end{pmatrix}$$

A 5
B 12
C -6
***D** -2
E 0

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23 $\begin{pmatrix} 3 & -1 & 2 \\ -2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 \\ 3 & 5 \\ 2 & 1 \end{pmatrix} =$

A $\begin{pmatrix} -3 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 5 & 1 \end{pmatrix}$

B The product is not defined.

C $\begin{pmatrix} 3 & 0 & 2 \\ -2 & -1 & 5 \\ 1 & -3 & 1 \end{pmatrix}$

D $\begin{pmatrix} 2 & 7 \\ 6 & -2 \end{pmatrix}$

***E** $\begin{pmatrix} -2 & 6 \\ 7 & 0 \end{pmatrix}$

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24 Consider the following system of equations over $\mathbb{R}$:

$$\begin{aligned} 3x + y + 2z &= 4 \\ -3x + y - 2z &= 0 \end{aligned}$$

Which of the following claims holds?

A The system has no solution.
B All points of $\mathbb{R}^3$ are solutions of the given system.
C The system has infinitely many solutions and the set of these solutions forms a plane in $\mathbb{R}^3$.
***D** The system has infinitely many solutions and the set of these solutions forms a line in $\mathbb{R}^3$.
E The system has exactly one solution.

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25 Which of the following maps from $\mathbb{R}$ to $\mathbb{R}$ is linear?

***A** $f(x) = \frac{22}{7}x$

B $f(x) = \frac{1}{x}$

C $f(x) = x^3$

D $f(x) = \sin(x)$

E $f(x) = x^2$