Faculty of Informatics, Masaryk University

Entrance Examination

Mathematics

June 2012

1 Sets, relations, functions

1. Consider a function F of the type $\mathbb{Z} \to \mathbb{Z}$ defined by $F(n) = n^2$. The function F
   (a) is surjective but not injective
   (b) is neither surjective, nor injective ✓
   (c) is injective but not surjective
   (d) is bijective
   (e) is both injective and surjective
2. Let $S = \{a, b\}$, $T = \{b, c\}$ a $U = \{a, c\}$.
   Which of the following sets is equal to $(R \times S) \cap (S \times U)$?
   (a) $\{(b, c)\}$ ✓
   (b) $\{(b, c), (a, b)\}$
   (c) $\{(b, c), (a, b), (a, c)\}$
   (d) $\emptyset$
   (e) $\{(b, b), (c, c), (a, a)\}$
3. Which of the following sets is equal to $(A \setminus B) \setminus C$?
   (a) $A \setminus (B \setminus C)$
   (b) $A \setminus (B \cap C)$
   (c) $A \setminus (B \cup C)$ ✓
   (d) $(A \cup B) \setminus C$
   (e) $(A \cap B) \setminus C$

   (Here $X \setminus Y$ denotes the difference of the sets X and Y, i.e. the set of all elements of X that do not belong to Y.)
4. The relation $\{(a, a)\}$ on the set $\{a, b, c\}$ is **not**
   (a) symmetric
   (b) transitive
   (c) antisymmetric
   (d) reflexive ✓
   (e) non-empty
5. Let A be a set of size $n \in \mathbb{N}$. What is the size of the set $\mathcal{P}(\mathcal{P}(A))$?
   (Here $\mathcal{P}(A)$ denotes the power set of A, i.e. the set of all subsets of A.)
   (a) 2^{2n}
   (b) n^2
   (c) 2^{n^2}
   (d) 2^{2^n} ✓
   (e) $2^{2^n+2^n}$

2 Logic

6. Let us consider the following statement: **For every** $x > 0$ **we have that** $P(x)$ **holds**. Which of the following statements is obtained by negating the statement above?
   (a) For every $x \leq 0$ we have that $P(x)$ holds.
   (b) For every $x \leq 0$ we have that $P(x)$ does not hold.
   (c) There exists $x \leq 0$ such that $P(x)$ holds.
   (d) There exists $x > 0$ such that $P(x)$ does not hold. ✓
   (e) There exists $x > 0$ such that $P(x)$ holds.

7. Let us assume that all variables are interpreted as natural numbers (with zero). For which given values of the variable x is the following formula true?

$$\exists y\ \exists z\ ((x = y + z) \wedge y \neq 0 \wedge z \neq 0 \wedge y \neq z\,)$$

   (a) 0
   (b) 1
   (c) 2
   (d) 3 ✓
   (e) for no value from the set $\{0, 1, 2, 3\}$

8. Which of the following formulae is **not** in the conjunctive normal form (CNF)?
   (a) $A \wedge B \wedge C$
   (b) $A \vee B \vee C$
   (c) $(A \vee B) \wedge C$
   (d) $\neg(A \wedge B \wedge C)$ ✓
   (e) $\neg A \vee \neg B \vee \neg C$

9. Which of the following formulae is a tautology?
   (A formula is a tautology if it is true for all valuations of the variables.)
   (a) $A \Leftrightarrow C$
   (b) $(A \wedge B) \Leftrightarrow (C \wedge B)$
   (c) $(A \wedge B) \Leftrightarrow (C \vee B)$
   (d) $(A \wedge \neg A) \Leftrightarrow (C \wedge \neg C)$ ✓
   (e) $(A \wedge \neg A) \Leftrightarrow (C \vee \neg C)$

10. Let us assume that all variables are interpreted as integers, the symbol f is interpreted as the function which assigns to each number n the value $2n$, and the symbol c is interpreted as the constant 99. Which of the following formulae is true under these assumptions?
   (a) $\forall x\ \ (f(x) = c)$
   (b) $\forall x\ \ (x = f(c))$
   (c) $\forall x\ \ (x \neq f(c))$
   (d) $\exists x\ \ (f(x) = c)$
   (e) $\exists x\ \ (x = f(c))$ ✓

3 Mathematical analysis

11. Which of the following functions is odd? (A function f is odd if for every $x \in \mathbb{R}$ we have that $f(-x) = -f(x)$.)

    (a) $|x|$

    (b) $\sin(x)$ ✓

    (c) x^2

    (d) e^x

    (e) $1 - x$

12. Consider the function $f(x) = e^x$. Which of the following statements is **false**? (The symbol $\mathbb{R}$ denotes the set of all real numbers.)

    (a) The domain of f is the set $\mathbb{R}$.

    (b) $f'(x) = f(x)$ for every $x \in \mathbb{R}$

    (c) The function f is continuous on $\mathbb{R}$.

    (d) The image of f is the set $\mathbb{R}$. ✓

    (e) $f(0) = 1$

    (f' denotes the derivative of function f.)

13. Consider the sequence of real numbers $(a_n)_{n=0}^{\infty}$ defined as follows:

$$a_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is even} \\ -\frac{1}{2^n} & \text{otherwise} \end{cases}$$

    What is the value of the limit $\lim_{n\to\infty} a_n$?

    (a) -1

    (b) 0 ✓

    (c) 1

    (d) $+\infty$

    (e) The limit does not exist.

14. Which of the following functions is the derivative of $x \cdot e^{3x}$?

    (a) 1

    (b) e^{3x}

    (c) $1 + 3e^{3x}$

    (d) $3 \cdot (1 + e^{3x})$

    (e) $e^{3x} \cdot (1 + 3x)$ ✓

15. Which of the following numbers is equal to $\int_1^2 3x^2 dx$?

    (a) 1

    (b) 3

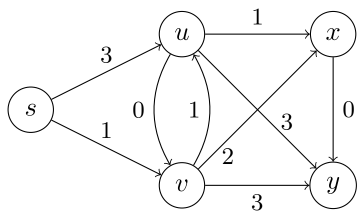
    (c) 7 ✓

    (d) 9

    (e) 21

4 Graphs and graph algorithms

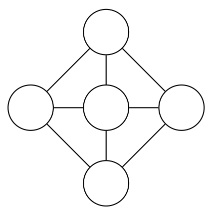
16. Consider the following weighted directed graph:



For every pair of vertices z, z' denote by $\delta(z, z')$ the length (i.e. the sum of edge weights) of the shortest path from the vertex z to the vertex z'. Which of the following equalities is **true**?

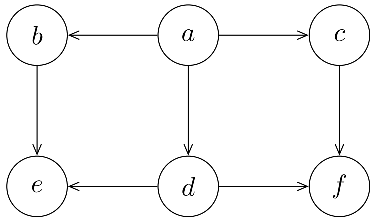
(a) $\delta(s, s) = 1$
(b) $\delta(s, u) = 3$
(c) $\delta(s, v) = 0$
(d) $\delta(s, x) = 4$
(e) $\delta(s, y) = 3$ ✓

17. The *diameter* of an undirected graph $G = (V, E)$ is the number $\max_{u,v \in V} d(u, v)$, where $d(u, v)$ denotes the length (i.e. the number of edges) of the shortest path from a vertex u to a vertex v. What is the diameter of the following graph?



(a) 1
(b) 2 ✓
(c) 3
(d) 4
(e) ∞

18. Consider the following directed graph:



Which of the following sequences may represent an order in which the breadth-first search algorithm discovers new vertices starting from a? (We do not assume any implicit ordering of vertices. Thus, the order in which the breadth-first search algorithm discovers new vertices may not be unique.)

(a) a, b, d, e, c, f

(b) a, c, f, b, d, e

(c) a, b, e, d, c, f

(d) a, b, d, c, e, f $\checkmark$

(e) a, d, e, f, b, c

19. In general, how many edges does a tree with n vertices have? (In the following, $\lfloor x \rfloor$ denotes the largest integer not greater than x.)

(a) $\lfloor \log_2(n) \rfloor$

(b) $n - 1$ $\checkmark$

(c) $\lfloor n \cdot \log_2(n) \rfloor$

(d) $\frac{n \cdot (n-1)}{2}$

(e) n^2

20. For which of the following problems no polynomial time algorithm is known?

(a) Given a weighted graph G, find the shortest paths between all pairs of its vertices.

(b) Given a directed graph G and two of its vertices u and v, decide whether v is reachable from u.

(c) Given an undirected graph G, decide whether G contains a path that visits each vertex exactly once. $\checkmark$

(d) Given an undirected graph G, decide whether G is connected.

(e) Given a weighted undirected graph G, find the minimum spanning tree of G.

5 Linear algebra

21. Which of the following mappings from $\mathbb{R}$ to $\mathbb{R}$ **is not** linear?
(A mapping f is linear if it satisfies $f(x+y) = f(x) + f(y)$ and $f(c \cdot x) = c \cdot f(x)$ for every x, y, c.)

    (a) $f(x) = 0$
    (b) $f(x) = 2x$
    (c) $f(x) = 2x + 3x$
    (d) $f(x) = 2x - 3x$
    (e) $f(x) = 2x \cdot 3x$ ✓

22. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$

    (a) $\begin{pmatrix} 7 & 10 \end{pmatrix}$
    (b) $\begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$
    (c) $\begin{pmatrix} 1 & 4 \\ 3 & 8 \end{pmatrix}$
    (d) $\begin{pmatrix} 5 \\ 11 \end{pmatrix}$ ✓
    (e) $\begin{pmatrix} 3 \\ 14 \end{pmatrix}$

23. Consider the following system of linear equations over $\mathbb{R}$:

$$\begin{array}{rcrcl} x & + & y & = & 5 \\ x & + & 2y & = & 6 \\ 2x & + & y & = & 8 \end{array}$$

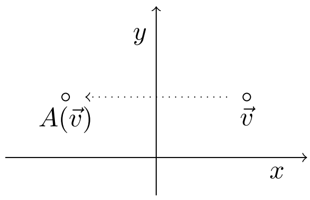
    Which of the following is true?

    (a) The only solution is $x = 3, y = 2$.
    (b) The only solution is $x = 4, y = 1$.
    (c) There is only one solution (but it is neither $x = 3, y = 2$, nor $x = 4, y = 1$).
    (d) There are multiple solutions.
    (e) There is no solution. ✓

24. What is the dimension of the linear span of the set of vectors $\{(1,1,0),(0,0,1),(1,1,1)\}$?
(The linear span of a set of vectors is the space of all linear combinations of these vectors.)

    (a) 0
    (b) 1
    (c) 2 ✓
    (d) 3
    (e) ∞

25. Which of the following matrices determines the linear mapping A from $\mathbb{R}^2$ to $\mathbb{R}^2$ which flips the plane horizontally (as indicated in the figure below)?



(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$

(Every matrix M defines a linear mapping A by $A(\vec{v}) = M\vec{v}$.)