

Qualitative Reachability in Stochastic BPA Games

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BPA

- Basic Process Algebra [concurrency]
- Stateless pushdown automata (PDA)
- Model recursive calls of procedures, without return values.
- Equivalent 1-exit Recursive State Machines (RSM)

2 1/2 player games

- 2 (non-deterministic) players
- 1/2 player – the ‘nature’ – follows prescribed probabilities

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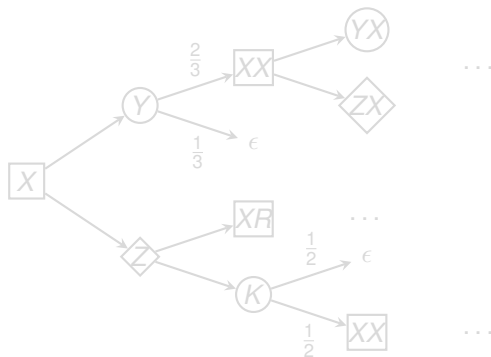
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Stochastic BPA example

Symbols and players: \boxed{X} $\diamond Z$ $\odot Y$ $\odot K$

Rules:

$X \rightarrow Y, X \rightarrow Z, Y \xrightarrow{\frac{2}{3}} XX, Y \xrightarrow{\frac{1}{3}} \epsilon, Z \rightarrow XR, Z \rightarrow K, K \xrightarrow{\frac{1}{2}} XX, K \xrightarrow{\frac{1}{2}} \epsilon$

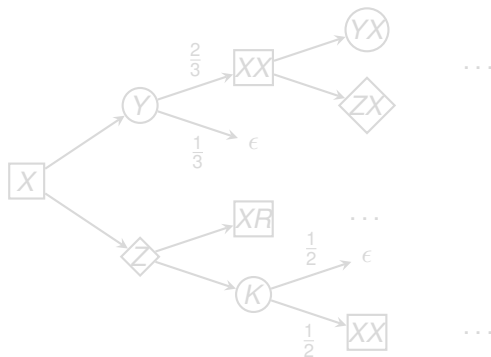


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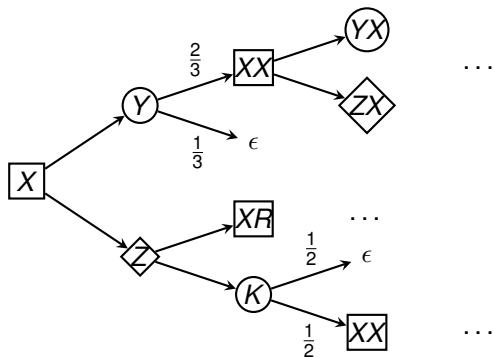


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The problem

Winning

Player \square wins a BPA game iff he has a strategy such that no matter what player \diamond does, the play reaches a configuration in T with probability > 0 ($= 1$) (*qualitative*).

Winning sets of configurations

$$[T]_{\square}^{\equiv 1} \quad [T]_{\diamond}^{\leq 1} \quad \text{or} \quad [T]_{\square}^{\geq 0} \quad [T]_{\diamond}^{\equiv 0}$$

The questions

- Does the player \square (\diamond) have a winning strategy from a given configuration?
- How do the strategies look like?
- Compute the sets of winning configurations for the players.
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What do we know?

- For 1 1/2 player BPA games the problem is solved [BBFK06]
1 1/2 players = Markov Decision Processes (MDP)
- The termination problem (i.e. reaching the empty stack) can be solved in time **$\text{NP} \cap \text{co-NP}$** [EY06]
- Instead of taking a regular set T of configurations, we can restrict ourselves to reaching configurations with a particular symbol (set of symbols) on the top of the stack (product construction).

- Establishing determinacy as a first step towards getting our result.
- Instead for BPA we prove it for games with finite branching.
- *Weak determinacy* (every game has a value) follows from Martin's result (but we can prove it in a much simpler way).
- Moreover, we can prove that (for our class of reachability objectives) one of the players does always have a winning strategy (achieving the value).

The easy case (> 0)

The sets $[T]_{\square}^{\geq 0}$ and $[T]_{\diamond}^{=0}$ are regular, and the membership can be computed in **P**.

$$[T]_{\square}^{\geq 0} = \mathcal{B}^* \mathcal{A} \Gamma^*$$

$$\mathcal{A} = [T]_{\square}^{\geq 0} \cap \Gamma \quad \mathcal{B} = [T_{\epsilon}]_{\square}^{\geq 0} \cap \Gamma$$

- $[T]_{\diamond}^{=0}$ can also be expressed using \mathcal{A} and \mathcal{B}
- Since this is essentially a reachability game, the player \bigcirc – the nature – ‘plays with’ the player \square .
- The sets \mathcal{A} and \mathcal{B} are obtained by a fixed-point computation (which gives us the **P** bound).
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The tricky case (= 1)

The sets $[T]_{\square}^{\equiv 1}$ and $[T]_{\diamond}^{\leq 1}$ are regular, and the membership can be computed in **NP** \cap **co-NP**.

$$[T]_{\square}^{\equiv 1} = B^*C\Gamma^* \quad [T]_{\diamond}^{\leq 1} = B^*\mathcal{A}\Gamma^* \cup B^*$$

$$\mathcal{A} = [T_{\epsilon}]_{\diamond}^{\leq 1} \cap \Gamma \quad B = \dots \quad C = \dots$$

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- \mathcal{A} is the set of symbols from which \diamond can avoid a configuration in T (and ϵ) with a probability >0 .

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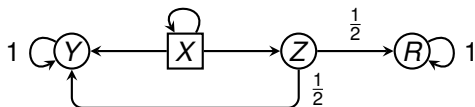
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The issues

- W – the set where \diamond can ‘obviously’ force winning
- \mathcal{A} is *not* an attractor of W .



$$\mathcal{A} = \{X, Y, Z\}$$

$$W = \{Y\}$$

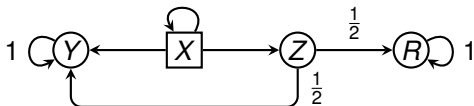
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The solution

- Compute W
- Remove attractors (non-trivial)
- Repeat
- *Can we always do that?*

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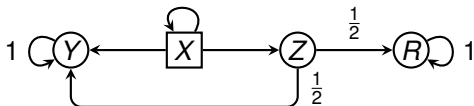
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Terminal sets

- Set $M \subseteq \Gamma$ is *terminal* iff \diamond can force any play to use only symbols from M .
- $T \notin M$ is required.
- Let C be the *largest* terminal set.

Witnesses

$Y \in \Gamma$ is a *witness* if either

- (1) $Y \in [T_\varepsilon]_\diamond^=0$;
 - (2) $Y \in C$ and $Y \in [\varepsilon]_\diamond^{<1}$, where the set $[\varepsilon]_\diamond^{<1}$ is computed in Δ_C .
- W – the set of all witnesses.

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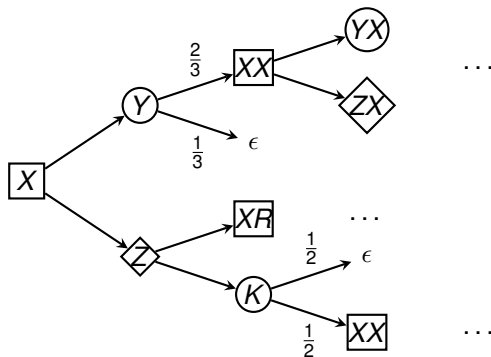
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The core theorem

Theorem

If $W = \emptyset$, then $\mathcal{A} = \emptyset$.

Strategies have to work with the stack:



What happens for

- related classes of transition systems (e.g. one counter automata)
- the quantitative case
- the other winning conditions (Büchi, parity ...)

Thank you

Thank you for your attention