DAG-width - Connectivity Measure for Directed Graphs

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Masaryk University, Brno

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xobdrzal@fi.muni.cz

Miami, 23 January 2006

Note aside

The measure was independently proposed by *Berwanger, Dawar, Hunter and Kreutzer* [STACS2006]

That paper also contains the algorithm for solving parity games.

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Or read my thesis.

Tree-width – a primer

Robertson and Seymour (1984)

Measures how close is a given graph to being a tree.

(Trees have tree-width **one**.)

Low tree-width $\Longrightarrow$ decomposition into small number of subproblems

Some NP-complete problems are linear/polynomial on structures of bounded tree-width, e.g.

- graph colouring *(linear)*
- MSOL *(linear)*
- clique, Hamiltonian cycle, ... *(linear)*
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- graph isomorphism *(polynomial)*
- **parity games *(polynomial)* [O.]**

Tree-width – the definition

A *tree decomposition* of an (undirected) graph G is a pair $(T, \mathcal{X})$, where T is a tree (its vertices are called *nodes* throughout this paper) and $\mathcal{X} = \{X_t \mid t \in T\}$ is a family of subsets of $V(G)$ satisfying the following three conditions:

(T1) $V(G) = \bigcup_{t \in V(T)} X_t$,

(T2) for every edge $\{v, w\} \in E(G)$ there exists $t \in V(T)$ s.t. $\{v, w\} \subseteq X_t$, and

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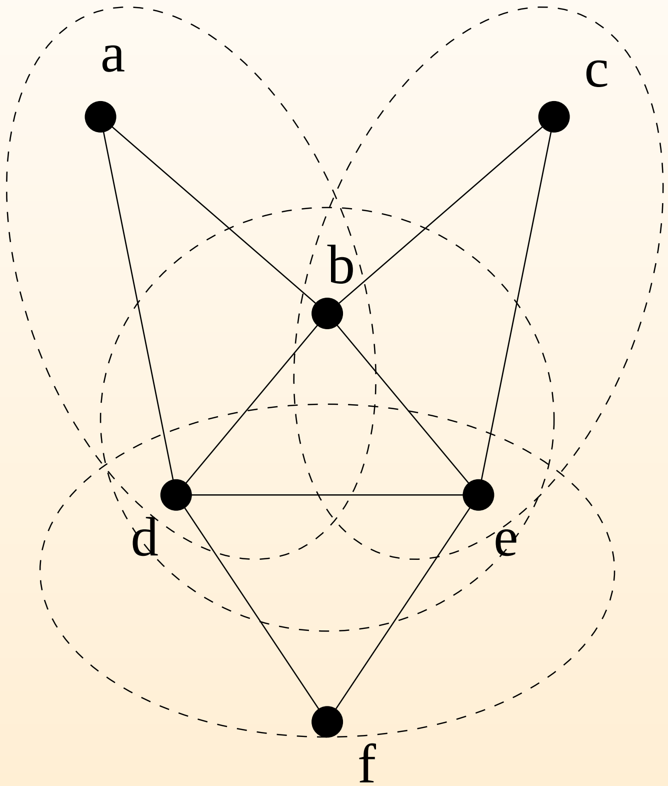
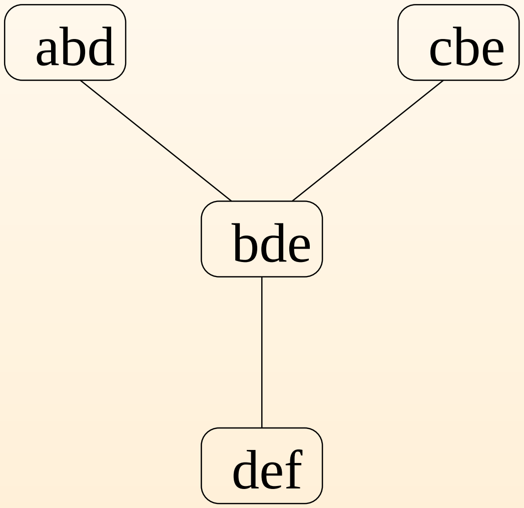
Tree-width – Example





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Objectives:
Robber: to evade capture
Cops: land a cop on the vertex occupied by the robber

More games

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Robber's strategy?

Haven of size k is a function σ assigning to each set $X \in V(G)$, $|X| \leq k$, a *connected component* $\sigma(X)$ of $G \setminus X$ s.t. for any $X, Y \in V(G)$ of size less than k we have $\sigma(X)$ *touches* $\sigma(Y)$.

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Duality theorem [Seymour, Thomas 93]:
The following are equivalent:

1. k cops can search G
2. k cops can monotonely search G
3. G has tree-width at most $k - 1$
4. G has a haven of size k

Tree-width and directed graphs

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If used as measure of directed graphs, we forget the orientation of edges.

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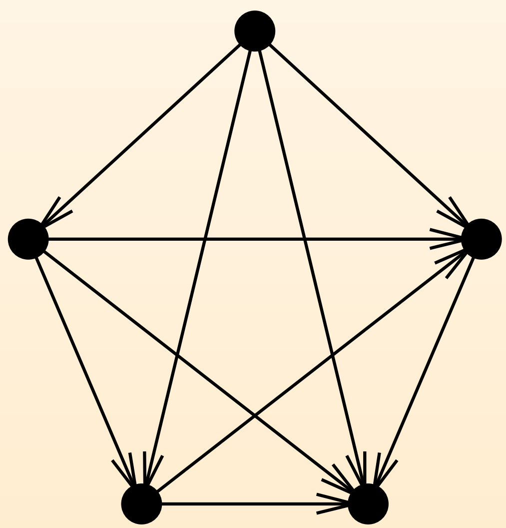

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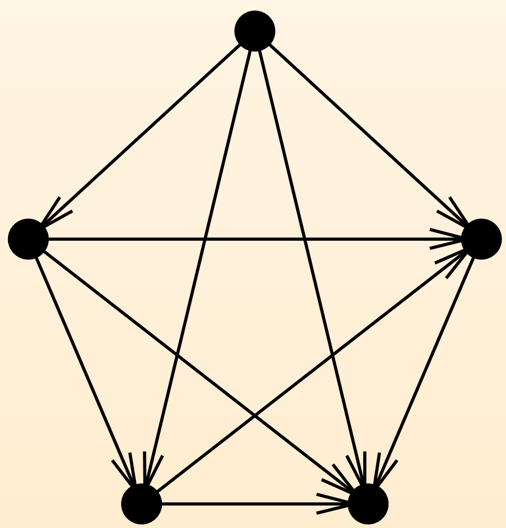

But it is easy to solve parity games on DAGs!

Z normal sets

For a graph G a set $S \subseteq V \setminus Z$ is *Z-normal* if there is no directed path in $G \setminus Z$ with first and last vertices in S that uses a vertex of $G \setminus (S \cup Z)$. I.e. no path can leave S and then return back to S without passing through a vertex in Z.

Directed Tree-width

(Johnson, Robertson, Seymour and Thomas, 2001)

An *arboreal decomposition* of a graph G is a triple $(R, \mathcal{X}, \mathcal{W})$ where R is a directed tree, and $\mathcal{X} = \{X_e \mid e \in E(R)\}$, $\mathcal{W} = \{W_r \mid r \in V(R)\}$ are sets of vertices of G satisfying:

(R1) $\mathcal{W}$ is a partition of $V(G)$ into nonempty sets

(R2) for $e \in E(R)$, $e = (r_1, r_2)$ the set $\bigcup\{W_r \mid r \in V(R)$ and $r \geq r_2\}$ is X_e-normal.

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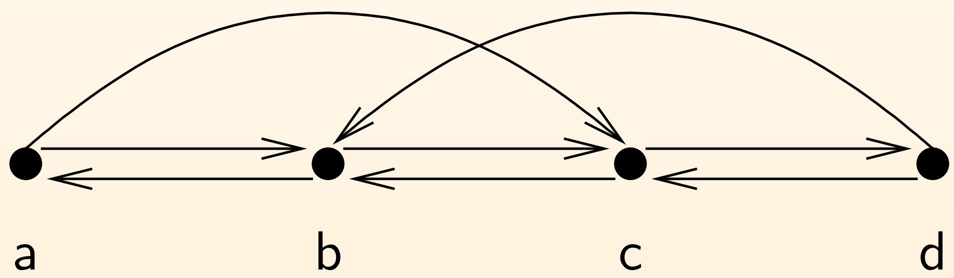

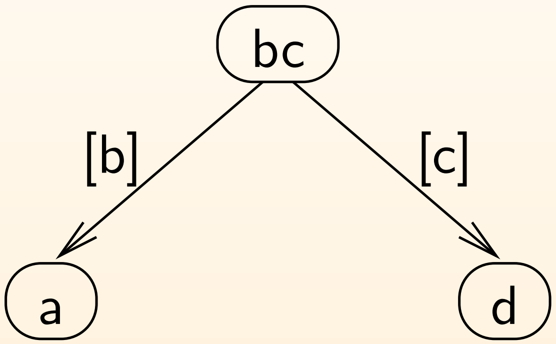

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While (1) is natural, (2) does not "feel right".

DTW - game equivalence

We have:

- $dtw(G) \leq k-1 \implies k$ cops can search G [JRST01]
- Opposite implication **does not** hold. [O.]
- Monotone and non-monotone strategies for cops are not equivalent. [JRST01]

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Strategies for the robber:

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A *DAG-decomposition* of a (directed) graph G is a pair $(D, \mathcal{X})$ where D is a DAG and $\mathcal{X} = \{X_d \mid d \in V(D)\}$ is a family of subsets of $V(G)$ satisfying:

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DAGs have DAG-width zero.

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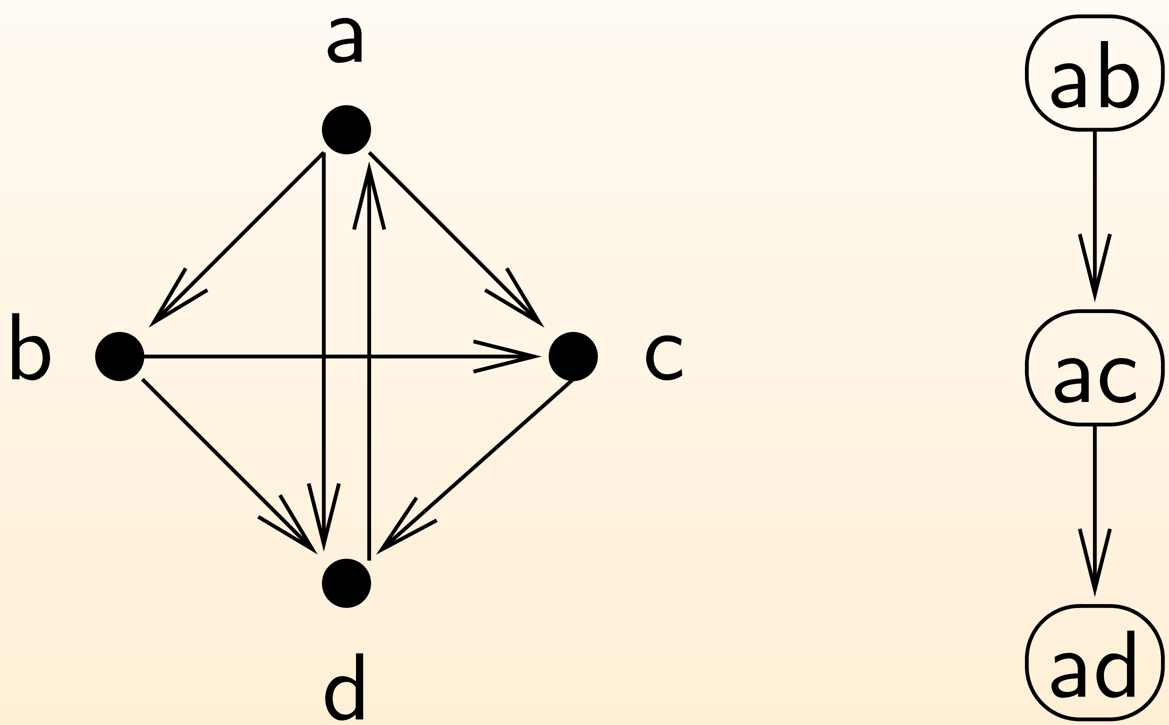

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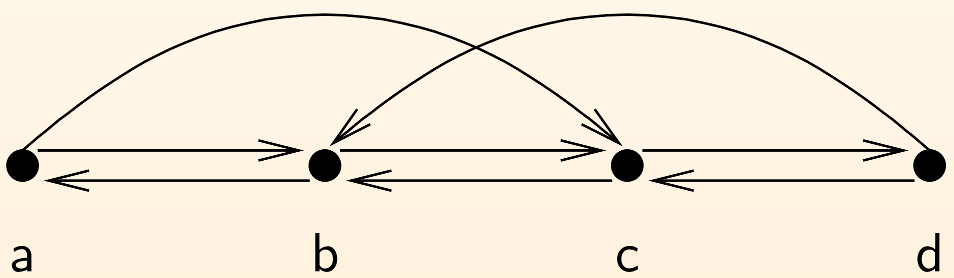

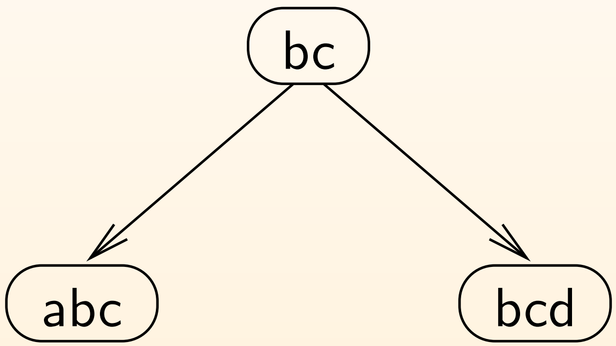

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Havens cannot be connected components.

DAG-width and tree-width

$$\text{DAG-width}(G) \leq \text{tree-width}(G)$$

Easy to prove using games.

The inequivalence may be sharp – e.g. for directed cliques.

DAG-width and directed tree-width

$$\text{directed tree-width}(G) \leq \text{DAG-width}(G)$$

- This is up to a constant factor.
- We cannot use the game characterization directly here, as the cops strategies for directed tree-width are not cop monotone.
- Conversion of a DAG-decomposition into an arboreal decomposition of the same width is not possible in a *strict* sense.
- Inequivalence can be sharp.

Algorithms

The following in general NP-complete problems can be solved in polynomial time on graphs of bounded DAG-width:

- Hamiltonian cycle, Hamiltonian path

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Note: In the case of tree-width the algorithms usually run in linear time.

Future work

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Is there a notion of decomposition linear in the size of G?

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Which otherwise NP-complete problems can be solved in polynomial time on graphs of bounded DAG-width?

Thanks for your attention.