On Digraph Width Measures in Parametrized Algorithmics

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The results

Problem	K-width	DAG-depth	DAG-width	Cycle-rank	DAG	Bi-rank-width
HAM	**FPT**	**FPT**	XP * W[2]-hard	XP *	P	XP W[2]-hard
c-PATH	**FPT**	**FPT**	XP *	XP *	P	**FPT**
k-PATH	**p-NPC**	**p-NPC**	**NPC**	**NPC**	**NPC**	*open*
DIDS	**p-NPC**	**p-NPC**	**NPC**	**NPC**	**NPC**	**FPT**
DISTP	**p-NPC**	**p-NPC**	**NPC**	**NPC**	**NPC**	**FPT**
MAXDICUT	p-NPC	p-NPC	NPC	NPC	NPC	**XP**
c-OCN	**p-NPC**	**p-NPC**	NPC	NPC	NPC	**FPT**
DFVS	*open*	*open*	p-NPC	p-NPC	P	**FPT**
KERNEL	p-NPC	p-NPC	p-NPC	p-NPC	P	**FPT**
φ-MSO$_1$MC	**p-NPH**	**p-NPH**	NPH	NPH	NPH	**FPT**
φ-LTLMC	**p-coNPH**	**p-coNPH**	**coNPH**	**coNPH**	**coNPC**	**p-coNPH**
PARITY	XP	XP	XP *	XP *	P	XP

Tree-width – a success story

Tree-width

- Robertson and Seymour (1984)
- connectivity measure for *undirected* graphs
- measures how close a graph is to being a tree
- cops-and-robber game characterization

The success story

- Many NP-hard become tractable on graphs of bounded tree-width: COLOURING, $MSOL_1$MC, HAMILTONIANPATH, K-CLIQUE . . .
- Some problems of "unknown" complexity become tractable as well: GRAPHISOMORPHISM, PARITYGAMES . . .
- Tree-width (and tree-decomposition) can be computed in FPT.

Directed graph measures I

The motivation

- DAGs are “easy”
- But what about oriented cliques?
- Some problems are naturally defined on directed graphs.

The tree-width like measures

- directed tree-width [JRST98/01]
- D-width [Saf05]
- DAG-width [O.06],[BDHK06]
- Kelly-width [HK07]
- . . .

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Directed graph measures II

Some features

- Cops-and-robber game characterization.
- Coincide with tree-width if we forget the edge orientation.
- Many of these measures are hard to compute.
- Low on DAGs.

Some other measures

- directed path-width [RST05, Bar06]
- cycle-rank [Egg63]
- clique-width [CO00]
- bi-rank-width [Kan08]

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Problems defined on directed graphs

The good . . .

- HAMILTONIANPATH, K-PATH etc. are *polynomial* (XP) on bounded directed tree-width, DAG-width and Kelly-width.
- PARITY is *polynomial* on bounded DAG-width and Kelly-width

. . . the bad . . .

- HAMILTONIANPATH is *W[2]-hard* on DAG-width [LKM08]
- DFVS,KERNEL,MAXDICUT are *NP-hard* on graphs of low directed widths [KO08,LKM08]

. . . and the ugly

- Some problems are hard even on DAGs!
- The decompositions are hard to compute.

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So what?

The ideal measure

- Would be small on many interesting classes of graphs.
- Many important problems should become FPT (or at least XP) when parametrized by this measure.
- Ideally would be FPT to compute.

To find it, we need to

- Find out what makes problems hard on DAGs, and graphs of low width.
- Compare the known measures on a wide range of relevant and interesting problems.

DAG-depth (ddp)

Inspired by tree-depth of Nešetřil and Ossona de Mendez.

$$ddp(G) = \begin{cases} 1 & \text{if } |V(G)| = 1 \\ 1 + \min_{v \in V(G)} ddp(G \setminus v) & \text{if there is a single maximal } G_v \\ \max_{v \in V(G)} ddp(G_v) & \text{otherwise} \end{cases}$$

Where G_v is the subgraph of G reachable from v.

Some features

- game characterization (lift-free strategy for cops)
- at least as big as DAG-width and cycle-rank
- can be arbitrarily large on DAGs
- $\lfloor \log_2 l \rfloor < ddp(G) \leq l$, where l is the length of the longest path in G

K-width

Was inspired by the many *"directed path"* problems.

K-width is the *maximum number* of distinct simple *s-t paths* over all $s, t \in V(G)$

Some features

- at least as big as DAG-width, incomparable with cycle-rank
- can be arbitrarily large on DAGs
- easily computed in time k.poly($|V(G)|$)

“Path” problems

HAM	–	Hamiltonian path
k-PATH	–	given k pairs (s_i, t_i) of vertices, find k vertex-disjoint paths between those pairs
c-PATH	–	as above, but $c \geq 2$ is fixed

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k-PATH	**p-NPC**	**p-NPC**	**NPC**	**NPC**	**NPC**	*open*

Oriented colouring

- The *chromatic number* $\chi(G)$ of G is a minimal c such that G has a homomorphism into K_c.
- The *oriented chromatic number* $\chi_o(G)$ of G is a minimal c such that G has a homomorphism into *some* orientation of K_c.
- Implies that between any two vertices of any two given colours the edges must go in the same direction ($\chi_o(C_5) = 5$).
- The number of colours c is fixed (i.e. not part of the input).

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c-OCN	**p-NPC**	**p-NPC**	NPC	NPC	NPC	**FPT**

Directed Feedback Vertex Set and Kernel

DFVS

- We look for a minimum cardinality set $S \subset V(G)$ such that $G \setminus S$ is acyclic.
- We consider the optimization variant (it's FPT with $k = |S|$).

Kernel

- An independent set $S \subseteq V(G)$ such that for every $x \in V(G) \setminus S$ there is an arc from x into S.

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Maximum Directed Cut

- The problem is to partition $V(G)$ into V_0 and V_1 such that the cardinality of $\{(u, v) \in E(G) \mid u \in V_0, v \in V_1\}$ is maximized.
- Often stated with edge weights.

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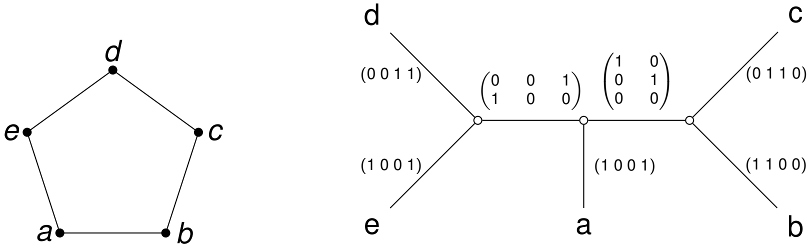
Rank-Width

- Defined by Oum and Seymour, 2004.
- Basically clique-width done right.



- φ-MSO_1MC is FPT.
- Better representation for algorithms: parse trees.

Bi-Rank-Width

- Straightforward generalization of rankwidth.
- Instead of one matrix we use two matrices, one for each direction.
- Defined by Kanté, 2008.
- Parse-trees generalize as well.
- φ-MSO_1MC is FPT.
- For each fixed $t \in \mathbb{N}$ there is an algorithm which, given a graph G, in time $\mathcal{O}(n^3)$ either outputs a bi-rank-decomposition of G of width at most t, or certifies that such decomposition does not exist.

Thank you

Thank you for your attention