Are there any good digraph width measures? †

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Abstract. Several width measures for digraphs have been proposed in the last few years. However, none of them possess all the "nice" properties of treewidth, namely, (1) being *algorithmically useful*, that is, admitting polynomial-time algorithms for a large class of problems on digraphs of bounded width; and (2) having nice *structural properties* such as being monotone under taking subdigraphs and some form of arc contractions. As for (1), MSO_1 is the least common denominator of all reasonably expressive logical languages that can speak about the edge/arc relation on the vertex set, and so it is quite natural to demand efficient solvability of all MSO_1-definable problems in this context. (2) is a necessary condition for a width measure to be characterizable by some version of the cops-and-robber game characterizing treewidth. More specifically, we introduce a notion of a *directed topological minor* and argue that it is the weakest useful notion of minors for digraphs in this context. Our main result states that any *reasonable* digraph measure that is algorithmically useful and structurally nice cannot be substantially different from the treewidth of the underlying undirected graph.

1 Introduction

An intensely investigated field in algorithmic graph theory is the design of graph *width parameters* that satisfy two seemingly contradictory requirements: (1) graphs of bounded width should have a reasonably rich structure; and, (2) a large class of problems must be efficiently solvable on graphs of bounded width. For undirected graphs, research into width parameters has been extremely successful with a number of algorithmically useful measures being proposed over the years, chief among them being treewidth [17], clique-width [6], branchwidth [18] and related measures (see also [3, 10]). Many problems that are hard on general graphs turned out to be tractable on graphs of bounded treewidth. These results were combined and generalized by Courcelle's celebrated theorem which states that a very large class of problems (MSO_2) is tractable on graphs of bounded treewidth [4].

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However, there still do not exist *directed graph* width measures that are as successful as treewidth. This is because, despite many achievements and interesting results, most known digraph width measures do not allow for efficient algorithms for many problems. During the last decade, many digraph width measures were introduced, the prominent ones being directed treewidth [13], DAG-width [2, 16], and Kelly-width [12]. These width measures proved useful for some problems. For instance, one can obtain polynomial-time (XP to be more precise) algorithms for Hamiltonian Path on digraphs of bounded directed treewidth [13] and for Parity Games on digraphs of bounded DAG-width [2] and Kelly-width [12]. But there is the negative side, too. Hamiltonian Path, for instance, probably cannot be solved [15] on digraphs of directed treewidth, DAG-width, or Kelly-width at most k in time $O(f(k) \cdot n^c)$, where c is a constant independent of k. Note that Hamiltonian Path *can* be solved in such a running time for undirected graphs of treewidth at most k [4].

Moreover, for the measures DAG-depth and Kenny-width[3] which are much more restrictive than DAG-width, problems such as Directed Dominating Set, Directed Cut, Oriented Chromatic Number 4, Max / Min Leaf Outbranching, and k-Path remain NP-complete on digraphs of constant width [8]. In contrast, clique-width and another recent digraph measure bi-rank-width [14] look more promising. A Courcelle-like [5] MSO_1 theorem exists for digraphs of bounded directed clique-width and bi-rank-width, and many other interesting problems can be solved in polynomial (XP) time on these [9, 14]. For a recent exhaustive survey on complexity results for DAG-width, Kelly-width, bi-rank-width, and other digraph measures, see [8].

In this paper, we show that any *reasonable* digraph width measure that is *algorithmically useful* and is closed under a notion of *directed topological minors* upper-bounds the treewidth of the underlying undirected graph. In what follows, we formalize this statement. We start with the notion of algorithmic usefulness and note what is it that makes treewidth such a successful measure. Courcelle's theorem [4] states that all MSO_2-expressible problems are linear-time decidable on graphs of bounded treewidth. To us it seems that an algorithmically useful width measure must admit algorithms with running time $O(n^{f(k)})$, at least, for all MSO_1-expressible problems on n-vertex digraphs of width at most k, where f is some computable function (that is, XP running time). Algorithmically useful digraph width measures do indeed exist. Candidates include the number of vertices in the input graph and the treewidth of the underlying undirected graph. In the latter case we can apply the rich theory of (undirected) graphs of bounded treewidth, but we would not get anything substantially new for digraphs. As such, we are interested in digraph width measures that are *incomparable* to undirected treewidth.

To motivate our discussion of directed topological minors, we note that treewidth has an alternative cops-and-robber game characterization. In fact, several digraph width measures such as DAG-width [2, 16], Kelly-width [12], and DAG-depth [8] admit some variants of this game-theoretic characterization.

[3] Kenny-width [8] is a different measure than Kelly-width [12].

While there is no formal definition of a cops-and-robber game-based width measure, all versions of the cops-and-robber game that have been considered share a basic property that shrinking induced paths does not help the robber. What we actually show is that a directed width measure that is "cops-and-robber game-based" must be closed under directed topological minors. On the other hand, we note that there exist algorithmically useful measures more general than undirected treewidth – digraph clique-width [6] and bi-rank-width [14] – which are not monotone even under taking subdigraphs.

Finally, the notion of a reasonable directed width measure is explained in Section 5 (see Definition 5.1). At this point, it suffices to say that this is simply a technicality that we make use of in the proof of our main theorem (Theorem 5.6). This theorem then states that a digraph width measure that admits XP-time algorithms for all MSO_1-problems wrt the width as parameter and is closed under directed topological minors must necessarily upper bound the treewidth of the underlying undirected graph. This implies that an algorithmically useful digraph width measure that is not treewidth-bounding cannot be characterized by a (version of) cops-and-robber game. We also show with examples that the prerequisities of our theorem cannot be weakened.

The paper is organized in four parts, starting with some core definitions in Section 2. Then in Section 3, we formally establish and discuss the (above outlined) properties an algorithmically useful digraph width measure should have. In Section 4, we introduce the notion of a directed topological minor, and discuss its properties and consider complexity issues. In particular, we show that it is hard to decide for a fixed (small) digraph whether it is a directed topological minor of a given digraph. In the last section, Section 5, we prove our main results which have already been outlined above. Due to lack of space, the proofs of results marked with a star ($\star$) are omitted.

2 Definitions and notation

The graphs (both undirected and directed) that we consider in this paper are *simple*, i.e. they do not contain loops and parallel edges. Given a graph G, we let $V(G)$ denote its vertex set and $E(G)$ denote its edge set, if G is undirected. If G is directed, we let $A(G)$ denote its arc set. Given a directed graph D, the *underlying undirected graph* $U(D)$ of D is an undirected graph on the vertex set $V(D)$; and $\{u, v\}$ is an edge of $U(D)$ if and only if $(u, v) \in A(D)$ or $(v, u) \in A(D)$. A digraph D is an *orientation* of an undirected graph G if $U(D) = G$.

For a vertex pair u, v of a digraph D, a sequence $P = (u = x_0, \ldots, x_r = v)$ is called *directed* (u, v)*-path* of length $r > 0$ in D if the vertices $x_0, \ldots, x_r$ are pairwise distinct and $(x_i, x_{i+1}) \in A(G)$ for every $0 \leq i < r$. We also write $u \to_D^+ v$ if there exists a directed (u, v)-path in D, and $u \to_D^* v$ if either $u \to_D^+ v$ or $u = v$. A *directed cycle* is defined analogously with the modification that $x_0 = x_r$. A digraph D is *acyclic* (a DAG) if D contains no directed cycle.

A parameterized problem Q is a subset of $\Sigma \times \mathbb{N}_0$, where Σ is a finite alphabet. A parameterized problem Q is said to be *fixed-parameter tractable* if there is an algorithm that given $(x, k) \in \Sigma \times \mathbb{N}_0$ decides whether (x, k) is a yes-instance of Q in time $f(k) \cdot p(|x|)$ where f is some computable function of k alone, p is a polynomial and $|x|$ is the size measure of the input. The class of such problems is denoted by FPT. The class XP is the class of parameterized problems that admit algorithms with a run-time of $O(|x|^{f(k)})$ for some computable f, i.e. polynomial-time for every fixed value of k.

Monadic second-order (MSO in short) logic is a language particularly suited for description of problems on "tree-like structured" graphs. For instance, the celebrated result of Courcelle [4], and of Arnborg, Lagergren and Seese [1], states that all MSO_2 definable graph problems have linear-time FPT algorithms when parameterized by the undirected treewidth. The expressive power of MSO_2 is very strong, as it includes many natural graph problems.

Note 2.1. Check this newer description. In this paper we are, however, interested primarily in another logical dialect commonly abbreviated as MSO_1, whose expressive power is noticeably weaker than that of MSO_2. The weaker expressive power is not a handicap but an advantage for our paper since we are going to use it to prove negative results. Similarly to the previous, MSO_1 definable graph problems have FPT algorithms when parameterized by clique-width [5] and, consequently, by rank-width.

Definition 2.2. The language of MSO_1 contains the logical expressions that are built from the following elements:

- variables for elements (vertices) and their sets, and the predicate $x \in X$,
- the predicate $adj(u, v)$ with u and v vertex variables,
- equality for variables, the connectives $\wedge, \vee, \neg, \rightarrow$ and the quantifiers $\forall, \exists$.

Example 2.3. For an undirected graph to have the 3-colorability property is an MSO_1-expression:

$$\exists V_1, V_2, V_3 \big[\forall v\,(v \in V_1 \vee v \in V_2 \vee v \in V_3) \wedge \bigwedge_{i=1,2,3} \forall v, w\,(v \notin V_i \vee w \notin V_i \vee \neg adj(v, w))\big]$$

A decision graph property $\mathcal{P}$ is MSO_1 *definable* if there exists an MSO_1 formula ϕ such that $\mathcal{P}$ holds for any graph G if, and only if, $G \models \phi$, i.e., ϕ is true on the model G. MSO_1 is analogously used for digraphs and their properties, where the predicate $arc(u, v)$ is used instead of $adj(u, v)$.

3 Desirable digraph width measures

A *digraph width measure* is a function δ that assigns each digraph a non-negative integer. To stay reasonable, we expect that infinitely many non-isomorphic digraphs are of bounded width. We consider what properties a width measure is expected to have. Importantly, one must be able to solve a rich class of problems on digraphs of bounded width. But what does "rich" mean?

On one hand, looking at existing algorithmic results in the undirected case, it appears that a *good balance* between the richness of the class of problems we capture and the possibility of positive general algorithmic results is achieved by the class of MSO_1 expressible problems (Definition 2.2). On the other hand, if we consider any logical language $\mathcal{L}$ over digraphs that is powerful enough to deal with sets of singletons (i.e. of monadic second order) and that can identify the adjacent pairs of vertices of the digraph, then we see $\mathcal{L}$ can naturally interpret also the MSO_1 logic of the underlying graph. Hence the following specification appears to be the most natural common denominator in our context:

Definition 3.1. A digraph width measure δ is *powerful* if, for every MSO_1 definable undirected property $\mathcal{P}$, there is an XP algorithm deciding $\mathcal{P}$ on all digraphs D with respect to the parameter $\delta(D)$.

The traditional measures treewidth, branchwidth, clique-width, and more recent rank-width, are all powerful [4, 5] for undirected graphs. For directed graphs, unfortunately, exactly the opposite holds. The width measures suggested in recent years as possible extensions of treewidth – including directed treewidth [13], D-width [20], DAG-width [16, 2], and Kelly-width [12] – all are not powerful.

Another concern is about "non-similarity" of our directed measure δ to the traditional treewidth of the underlying undirected graph; we actually want to obtain and study new measures that significantly differ from treewidth, in the negative sense of the following Definition 3.2. This makes sense because any measure δ which bounds the treewidth of the underlying graph would automatically be powerful but would not help to solve any more problem instances than we already can with traditional undirected measures.

Definition 3.2. A digraph width measure δ is called *treewidth-bounding* if there exists a computable function b such that, for every digraph D, $\delta(D) \leq k$ implies that the treewidth of $U(D)$ is at most $b(k)$.

To briefly outline the current state, we focus in the rest of this section on two of the treewidth-like directed measures which seem to attract most attention nowadays – DAG-width [16, 2] and Kelly-width [12]; and on another two significantly more successful (in the algorithmic sense) measures – directed clique-width [6] and not-so-much-known bi-rank-width [14]. None of these measures are treewidth-bounding.

Since the definitions of DAG-width and Kelly-width are not short, we skip them here and refer to [16, 2, 12] instead. Both DAG- and Kelly-width share some common properties important for us:

- Acyclic digraphs (DAGs) have width 0 and 1, respectively.
- If we replace each edge of a graph of treewidth k by a pair of opposite arcs, then the resulting digraph has DAG-width k and Kelly-width $k+1$.
- Both of the measures are characterized by certain cops-and-robber games.

Proposition 3.3 ($\star$). *If* $\mathrm{P} \neq \mathrm{NP}$*, DAG-width and Kelly-width are not powerful.*

On the other hand, there are measures such as clique-width [6] which was originally defined for undirected graphs, but readily extends to digraphs. Another noticeable directed measure is *bi-rank-width* [14], which is related to clique-width in the sense that one is bounded on a digraph class iff the other one is. Due to restricted space we only refer to [14] or [9] for its definition and properties.

Theorem 3.4 (Courcelle, Makowsky, and Rotics [5]). *Directed clique-width, and consequently bi-rank-width, are powerful measures.*

For a better understanding of the situation, we note one important but elusive fact: Bounding the *undirected* clique-width or rank-width of the underlying undirected graph does not generally help solve directed graph problems.

Proposition 3.5 (⋆). *Undirected clique-width or rank-width are **not** powerful digraph measures unless $P = NP$.*

This is in a sharp contrast to the situation with treewidth where bounding the treewidth of the underlying undirected graph allows all the algorithmic machinery to work also on digraphs. As of now, there is no known non-trivial relationship between undirected measures and their directed generalizations.

After all, comparing Propositions 3.3 and 3.4, we clearly see the advantages of directed clique-width. There is, however, also the other side. Clique-width and bi-rank-width do not possess the nice structural properties common to the various treewidth-like measures, such as being subgraph- or contraction-monotone. This is due to symmetric orientations of complete graphs all having clique-width 2 while their subdigraphs include all digraphs, even those with arbitrarily high clique-width. This seems to be a drawback and a possible reason why clique-width- and rank-width-like measures are, unfortunately, not so widely accepted.

The natural question now is; can we take *the better of each of the two worlds*? In our search for the answer, we will not study specific digraph width measures but focus on general properties of possible width measures. The main result of this paper, Theorem 5.6, then answers this question negatively: One *cannot* have a "nice" digraph width measure which is powerful, not treewidth-bounding and, at the same time, monotone under taking subgraphs and directed topological minors (see in Section 4). This strong and conceptually new result holds modulo technical assumptions which prevent our digraph width measures from "cheating", such as in Theorems 5.7 and 5.8.

4 Directed topological minors

As for the second requirement we impose on a "good" digraph width measure – to possess some nice structural properties similar to those we often see in undirected graph measures – we argue as follows in this section.

Many width measures (e.g. treewidth) for undirected graphs are *monotone* under taking minors. In other words, the measure of a minor is not larger than the measure of the graph itself. Graph H is a *minor* of a graph G if it can be obtained by a sequence of applications of three operations: vertex deletion, edge deletion and edge contraction. (See e.g. [7].) It is therefore only natural to expect that a "good" digraph measure should also be closed under some notion of a directed minor. However, there is currently no widely agreed definition of a directed minor in general. One published, but perhaps too restrictive on subdivisions (as we will see later), notion is the *butterfly minor* [13].

To deal with directed minors, we first need a formal notion of an *arc contraction* for digraphs:

Definition 4.1. Let D be a digraph and $a = (x, y) \in A(D)$ be an arc. The digraph obtained by *contracting arc* a, denoted by D/a, is the digraph on the vertex $(V(D) \setminus \{x, y\}) \cup \{v_a\}$ where v_a is a new vertex, and the arc set A' such that $(u, v) \in A'$ iff one of the following holds

$$(u, v) \in A(D \setminus \{x, y\}), \text{ or}$$
$$v = v_a, \text{ and } (u, x) \in A \text{ or } (u, y) \in A, \text{ or}$$
$$u = v_a, \text{ and } (x, v) \in A \text{ or } (y, v) \in A.$$

See Fig. 1 for an example of a contraction. Note that contraction always produces simple digraphs (that is, no arcs of the form (x, x)). The result of a contraction does not depend on the orientation of the contracted arc, and we treat contraction of a pair of opposite arcs (x, y) and (y, x) as a contraction of a single bidirectional arc.

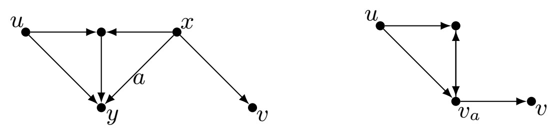

Fig. 1. Arc contraction: digraphs D (left) and D/a.

An important decision point when defining a minor is; which arcs do we *allow to contract*? In the case of undirected graph minors, any edge can be contracted. However, the situation is not so obvious in the case of digraphs. Look again at Figure 1. If we contract the arc a, we actually introduce a new directed path $u \to^+ v$, whereas in undirected graphs no new (undirected) path is ever created by the edge contraction. On the other hand, simply never introducing a new directed path (e.g. the *butterfly minor* of [13]) is not a good strategy either – since one can easily construct, see Figure 2, digraphs in which no arc can be contracted without introducing a new directed path. Yet such digraphs can be "very simple" with respect to usual cops-and-robber games, and arc contractions do not help the robber in the depicted situation.

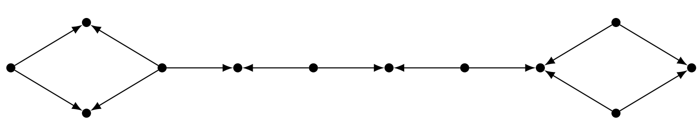

Fig. 2. Any arc contraction in this digraph introduces a new directed path.

In order to deal with the mentioned issue of contractibility of arcs, and to remain as general as possible at the same time, we consider *directed topological* *minors* where we allow only those arc contractions that do not introduce any new directed path between vertices of degree at least three (cf. Figure 2 again). (Note that our definition is different from the one given in Hunter's thesis [11], where an arc is contractible iff at least one endvertex has both out- and in-degree one.) Subsequent claims then justify our choice. For reference we denote by $V_3(D) \subseteq V(D)$ the subset of vertices having at least three neighbors in D.

Definition 4.2. An arc $a = (u, v) \in A(D)$ is *2-contractible* in a digraph D if

- u or v has exactly two neighbors, and
- $(v, u) \in A(D)$ or there is no pair of vertices $x, y \in V_3(D)$ such that $x \to^*_{(D/a)} v$ and $u \to^*_{(D/a)} y$.

A digraph H is a *directed topological minor* of D if there exists a sequence of digraphs $D_0, \dots, D_r \cong H$ such that D_0 is a subgraph of D, and for all $0 \le i \le r-1$, D_{i+1} is obtained from D_i by contracting a 2-contractible arc.

Proposition 4.3 (⋆). *Let D be a digraph and D' be a digraph obtained from D by a sequence of vertex deletions, arc deletions and contractions of 2-contractible arcs. Then D' is a directed topological minor of D.*

A useful notion in reasoning about directed topological minors is that of a 2-path. Let D be a digraph and $P = (x_0, \dots, x_k)$ a sequence of vertices of D. Then P is a *2-path* (of length k) in D if P is a path in the underlying graph $U(D)$ and all internal vertices x_i for $0 < i < k$ have exactly two neighbors in D. Obviously not every 2-path is a directed path. The following lemma explains a key property of our directed topological minors – that they behave analogously to ordinary topological minors, being able to contract any long 2-paths.

Lemma 4.4 (⋆). *Let $S = (x_0, \dots, x_k)$ a 2-path of length $k > 2$ in a digraph D. Then there exists a sequence of 2-contractions of arcs of S in D turning S into*

a) a 2-path of length two, or
b) a single arc if S was a directed path in D.

The obvious question is which of the traditional digraph measures are closed under taking directed topological minors. Here we give the answer:

Proposition 4.5 (⋆). *DAG-width and Kelly-width are monotone under taking directed topological minors unless the width is 0 or 1, respectively. Directed clique-width and bi-rank-width are not.*

We note that the problem of deciding whether a given digraph is a directed topological minor of another digraph is NP-complete.

Theorem 4.6 (⋆). *There exists a digraph H such that the H-*DIRECTED TOPOLOGICAL MINOR *problem (given a digraph D, decide whether H is directed topological minor of D) is* NP*-complete.*

The last result can be generalized to:

Theorem 4.7. *The following decision problem is* NP*-complete: given acyclic digraphs D and H, decide whether H is directed topological minor of D.*

5 An (almost) optimal closure property result for digraph width measures

In this section we finally prove some “almost optimal” negative answers to the questions raised in the Introduction and at the end of Section 3. To recapitulate, we have asked whether it is possible to define a digraph width measure that is closed under some reasonable notion of a directed minor (e.g., Definition 4.2) and that is still powerful (Definition 3.1) analogously to ordinary treewidth. We also recall the property of being treewidth-bounding (which we want to avoid) from Definition 3.2.

Besides the aforementioned several properties we suggest one more technical property that a desired good directed width measure should possess, in order to avoid “cheating” such as in the example of Theorem 5.8. Informally, we do not want to allow the measure to keep “computationally excessive” information in the orientation of edges:

Definition 5.1. A digraph width measure δ is *efficiently orientable* if there exist a computable function h, and a polynomial-time computable function $r : \mathscr{G} \to \mathscr{D}$ (from the class of all graphs to that of digraphs), such that for every undirected graph $G \in \mathscr{G}$, it is $U(r(G)) = G$ and

$$\delta(r(G)) \leq h(\,\min\{\delta(D) : D \text{ a digraph s.t. } U(D) = G\})\,.$$

Proposition 5.2 ($\star$). *DAG-width, Kelly-width, digraph clique-width, and bi-rank-width are all efficiently orientable.*

Our main proof also relies on some deep ingredients from Graph Minors:

Theorem 5.3 ([19]). *If H is a planar undirected graph then there exists a number n_H such that for every G of treewidth at least n_H, H is a minor of G.*

Proposition 5.4 (folklore). *If H is a minor of G and the maximum degree of H is three, then H is a topological minor of G.*

Finally we need the following result whose proof we omit. For a set S of natural numbers, an *S-regular graph* is a graph having every vertex degree in S.

Theorem 5.5 ($\star$). *For any simple undirected graph H and every MSO_1 formula φ, there exist a $\{1,3\}$-regular planar graph G and an MSO_1 formula ψ, such that*

a) $H \models \varphi \iff G \models \psi$, and
b) for every subdivision G_1 of G, we have $G_1 \models \psi \iff G \models \psi$.
c) Moreover, ψ depends only on φ, $|\psi| = \mathcal{O}(|\varphi|)$, and both G and ψ are computable in polynomial time from H and φ, respectively.

With all the ingredients at hand, we state and prove our main result:

Theorem 5.6. *Let δ be a digraph width measure with the following properties*

a) δ is not treewidth-bounding;

b) δ is monotone under taking directed topological minors;
c) δ is efficiently orientable.

Then P = NP*, or δ is* not powerful*.*

Proof. We assume that δ is powerful and show that for every MSO_1 definable property φ of undirected graphs there exists a polynomial-time algorithm that decides, given as input an undirected graph G, whether $G \models \varphi$. Since, by Example 2.3, there are MSO_1 properties φ such that deciding whether $G \models \varphi$ is NP-hard, this would imply that P = NP.

Given an MSO_1-formula φ and an undirected graph H, we construct a $\{1,3\}$-regular planar graph G and an MSO_1-formula ψ as in Theorem 5.5. Let G_1 be the 1-subdivision of G (i.e. replacing every edge of G with a path of length two). We claim that, under assumptions (a) and (b), there exists an orientation D of G_1 such that $\delta(D) \leq k$, for some constant k dependent on δ.

We postpone the proof of this claim, and show its implications first. Since δ is efficiently orientable, by Definition 5.1, we can efficiently construct an orientation D_1 of G_1 such that $\delta(D_1) \leq h(k)$, for some computable function h. Note that since k is a constant, the width of D_1 is at most a constant. Let ψ_1 be the (directed) MSO_1 formula obtained from ψ by replacing $adj(u,v)$ with $(arc(u,v) \vee arc(v,u))$. Then, by Theorem 5.5, $H \models \varphi$ iff $D_1 \models \psi_1$, and hence we have a polynomial reduction of the problems $H \models \varphi$ onto $D_1 \models \psi_1$. Since δ is assumed to be powerful, the latter problem can be solved by an XP algorithm wrt the constant parameter $h(k)$, that is, in polynomial time.

We now return to our claim. Since δ is not treewidth-bounding, there is $k \geq 0$ such that the class $\mathcal{U} = \{U(D)\colon \delta(D) \leq k\}$ has unbounded treewidth. By Theorem 5.3, there exists D_0 such that $\delta(D_0) \leq k$ and $U(D_0)$ contains a G_1-minor. Since the maximum degree of G_1 is three, by Proposition 5.4, G_1 is a topological minor of $U(D_0)$ and hence some subdivision G_2 of G_1 is a subgraph of $U(D_0)$. Therefore there exists D_2, a subdigraph of D_0, with $U(D_2) = G_2$. Finally, by Lemma 4.4 one can contract 2-paths in D_2, if necessary, to obtain a digraph D_3 with $U(D_3) = G_1$. Clearly D_3 is a directed topological minor of D_0 and since δ is closed under taking directed topological minors, we have $\delta(D_3) \leq k$. This completes the proof of our claim and the theorem. □

Due to Theorem 5.6, a powerful digraph width measure essentially "cannot be stronger" than ordinary undirected treewidth, unless P = NP. Our result requires two assumptions about the width measure δ in consideration: δ should be closed under taking directed topological minors, and it should be efficiently orientable. An interesting question is whether these conditions are necessary, or, put differently, whether the result of Theorem 5.6 can be strengthened by weakening these two essential assumptions.

We address this question in the remainder of this section – we show that Theorem 5.6 is almost strongest possible in the following Theorems 5.7 and 5.8. Specifically, we show that if one relaxes either of these two conditions b), c), then one can construct directed measures which definitely do not "look nice". In the first round, we relax the condition b) just to subdigraphs, and get:

Theorem 5.7 (⋆). *There exists a* ***powerful*** *digraph width measure δ s.t.*
a) δ is not treewidth-bounding;
b) δ is monotone under taking subdigraphs;
c) δ is efficiently orientable.
Moreover, the same remains true if we replace b) with b') δ is monotone under taking butterfly minors.

The way the measure of Theorem 5.7 attains its "power" is by subdividing every edge with a tower-exponential number of new vertices. This is certainly not a nice behavior of a desired measure, and hence such a measure δ *should be dismissed*.

In the second round, we take a closer look at the condition that δ is efficiently orientable. It is not unreasonable to assume a digraph width measure to be efficiently orientable since most known digraph measure are, e.g. Proposition 5.2. Furthermore, efficient orientability prevents digraph measures from "keeping excessive information" in the orientation of arcs, such as (Theorem 5.8) the information about 3-colorability of the underlying graph.

Theorem 5.8 (⋆). *There exists a digraph width measure δ such that*
a) δ is not treewidth-bounding;
b) δ is monotone under taking directed topological minors;
c) for every $k \geq 1$, on any digraph D with $\delta(D) \leq k$, one can decide in time $\mathcal{O}(3^k \cdot n^2)$ whether $U(D)$ is 3-colorable, and find a 3-coloring if it exists;
d) and for every 3-colorable graph G there exists an orientation D, $U(D) = G$ such that $\delta(D) = 1$.

6 Conclusions

The main result of this paper shows that an algorithmically useful digraph width measure that is substantially different from treewidth cannot be closed under taking directed topological minors. Since "standard" cops-and-robber games remain invariant on directed topological minors, we can conclude that a digraph width measure that allows efficient decisions of MSO_1-definable digraph properties on classes of bounded width cannot be defined using such games. This gives more weight to the argument [8] that bi-rank-width [14] is the best (though not optimal) currently known candidate for a *good* digraph width measure.

Our main result also leaves room for other ways of overcoming the problems with the currently existing digraph width measures. We have asked for width measures that are powerful, i.e., all MSO_1-definable digraph properties are decidable in polynomial time on digraphs of bounded width. What happens if we relax this requirement? We can ask for more time, like subexponential running time, or we can ask for restricted classes of MSO_1-definable digraph properties. Currently, we are not aware of any noticeable progress in this direction. Another interesting direction for future research is a closer study of efficient orientability and directed topological minors.

Finally, we believe that the results and suggestions contained in our paper will lead to new ideas and research directions in the area of digraph width measures – an area that seems to be stuck at this moment.

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