Chapter 6

Information Theory

Exercise 6.1

Let us consider a random variable X given by

$$X = \begin{cases} a & \text{with probability } 1/2 \\ b & \text{with probability } 1/4 \\ c & \text{with probability } 1/8 \\ d & \text{with probability } 1/8 \end{cases}.$$

Calculate entropy of X.

Exercise 6.2

Let (X, Y) have the following joint distribution:

	$X = 1$	$X = 2$	$X = 3$	$X = 4$
$Y = 1$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$
$Y = 2$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{32}$
$Y = 3$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
$Y = 4$	$\frac{1}{4}$	0	0	0

Compute $H(X), H(Y)$ and $H(X|Y)$.

Exercise 6.3

Let $A = \{0, 1\}$ and consider two distributions p and q on A. Let $p(0) = 1 - r, p(1) = r$, and let $q(0) = 1 - s$, and $q(1) = s$. Compute $D(p \parallel q)$ and $D(q \parallel p)$ for $r = s$ and for $r = \frac{1}{2}, s = \frac{1}{4}$.

Exercise 6.4

Let (X, Y) have the same distribution as in Exercise 6.2. Compute their mutual information $I(X; Y)$.

Exercise 6.5

Let (X, Y) have the following joint distribution:

	$X = 1$	$X = 2$
$Y = 1$	0	$\frac{3}{4}$
$Y = 2$	$\frac{1}{8}$	$\frac{1}{8}$

Compute $H(X), H(X|Y = 1), H(X|Y = 2)$ and $H(X|Y)$.

Exercise 6.6

Find random variables X, Y and $y \in \mathcal{Y}$ such that $H(X) < H(X \mid Y = y)$.

Exercise 6.7

What is the minimum entropy for $H(p_1, p_2, \ldots, p_n) = H(\vec{p})$ as $\vec{p}$ ranges over all probability vectors? Find all possible values of $\vec{p}$ which achieve this minimum.

Exercise 6.8

What is the general inequality relation between $H(X)$ and $H(Y)$ if

1. $Y = 2^X$
2. $Y = \cos X$

Exercise 6.9

Show that whenever $H(Y \mid X) = 0$, Y is a function of X.

Exercise 6.10

A metric ρ on a set X is a function $\rho\colon X \times X \to \mathbb{R}$. For all $x, y, z \in X$, this function is required to satisfy the following conditions:

1. $\rho(x, y) \geq 0$
2. $\rho(x, y) = 0$ if and only if $x = y$
3. $\rho(x, y) = \rho(y, x)$
4. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

For the metric $\rho(X, Y) = H(X|Y) + H(Y|X)$, show that conditions 1, 3 and 4 hold. Should we define $X = Y$ iff there exists a bijection f such that $X = f(y)$, show that 2 holds as well.