Chapter 3

Expectation, inequalities and laws of large numbers

Exercise 3.1

Suppose n balls are distributed at random into r boxes. What is the expected number of empty boxes?

Exercise 3.2

Let X be uniformly distributed on $\{0, 1, \ldots, n\}$. Find the mean and variance of X.

Exercise 3.3

Having two dice, let the random variable X be the outcome of the first die and Y be the maximum of their outcomes. Compute $E(X), E(Y), Var(X), Cov(X, Y)$ and the joint distribution of X and Y.

Exercise 3.4

Consider a group of n people. A *special day* is a day such that exactly k people in the group have a birthday. What is the expected number of special days in a year? (Assume all years are non-leap.)

Exercise 3.5

Consider the same group of n people. What is the expected number of days such that at least two people have a birthday? How large should be n to make this expectation exceed 1?

Exercise 3.6

Let X have the binomial distribution with parameters n and p. Find $E(X)$.

Exercise 3.7

Find $Var(X)$ for X from the previous example.

Exercise 3.8

Suppose X and Y are two independent random variables such that $E\left(X^4\right) = 2$, $E\left(Y^2\right) = 1$, $E\left(X^2\right) = 1$ and $E(Y) = 0$. Compute $\mathrm{Var}(X^2 Y)$.

Exercise 3.9

Suppose a box contains 3 balls labeled 1,2,3. Two balls are selected without replacement from the box. Let X be the number on the first ball and let Y be the number on the second ball. Compute $\mathrm{Cov}(X, Y)$ and $\varrho(X, Y)$.

Exercise 3.10

Let X have a geometric distribution with parameter p. Find $E(X)$.

Exercise 3.11

A p-random graph on n vertices is an unoriented graph where between every distinct vertices $i \neq j$ there is an edge with probability p. Compute the expected value and the variance of the number of all edges in the graph.

Exercise 3.12

Let X_1 and X_2 be two independent random variables with Poisson probability distribution $X_1 \sim p(k; \lambda_1)$ and $X_2 \sim p(k; \lambda_2)$. Calculate

1. Prove that $X_1 + X_2$ has the poisson distribution $p(k; \lambda_1 + \lambda_2)$.

2. Show that the conditional probability distribution of X_1 given $X_1 + X_2$ is binomial, namely

$$\mathcal{P}(X_1 = k | X_1 + X_2 = n) = b\left(k; n, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right).$$